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RICE UNIVERSITY

A RESIDUAL FLEXIBILITY APPROACH FOR DECOUPLED ANALYSIS OF NONLINEAR, NONCLASSICALLY DAMPED SYSTEMS OF COMBINED COMPONENTS

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE DOCTOR OF PHILOSOPHY

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A residual flexibility approach for decoupled analysis of nonlinear, nonclassically damped systems of combined components

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ABSTRACT

A residual flexibility approach for the analysis of systems comprised of multiple components subjected to dynamic loading is presented. In it, the reactive forces at the junctions of the components are computed directly without the synthesis of component modes or the determination of system modes. This is accomplished by expressing the displacements at the junction coordinates of the components in terms of the retained component free-junction normal modes and a first-order account of the residual flexibility of the unretained modes. Once the components are represented in this manner, the requirement of displacement compatibility and force equilibrium at the junction coordinates is enforced. This leads to a set of junction-sized, simultaneous algebraic equations, similar in form to that of the flexibility formulation in statics, in terms of the unknown junction forces. The computed forces at a given time-step then serve to base-drive each component's equations of motion separately, hence the term decoupled analysis. Due to the formulation of the method, the nonlinear, nonclassically damped problem becomes a natural progression. The new method compares well to the traditional method of Component-Mode Synthesis for solutions to a nonclassically damped fixed-fixed beam comprised of two classically damped cantilevered beam components.
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CHAPTER 1

INTRODUCTION

1.1 Motivation

The Finite Element Method has enabled the analyst to formulate high fidelity models of complex structures. Dynamic analyses conducted on these structures represented by finite element equations of motion are often impracticable due to the large order of the matrices. Methods of conducting efficient dynamic analyses by reducing the order of the finite element equations without loss in accuracy have been a subject of interest for some time. These methods rely on representing the system as an assemblage of distinct components. Approximate representations are developed for the individual components which are then synthesized to form an approximate representation for the entire system. The objectives of this dissertation are of two folds. First, to provide a comprehensive review of the methods of Component-Mode Synthesis, a family of reduction techniques for approximating and coupling components for dynamic analyses. Second, to present an efficient alternative to the method Component-Mode Synthesis that treats the substructures as separate bodies yielding a methodology ideal for problems where the combined system is rendered nonlinear and nonclassically damped.

1.2 An Overview of the Method of Component-Mode Synthesis

Przemieniecki (1963) developed the fundamentals of the method of static substructuring. In it, a Ritz transformation was developed in order to formulate the reduced stiffness representation of the components. Guyan (1965) extended the method
to the dynamics problem, formulating the reduced mass matrix. The Guyan reduction
requires the analyst to select the coordinates to which the matrices are reduced to. The
proper selection of these coordinates is imperative to the accuracy and convergence of the
frequencies and therefore the dynamic response (Anderson et. al., 1968). The problem of
substructuring via static reductions is referred to as Static Synthesis. In the problem of
Static Synthesis, each component is statically reduced via the Guyan reduction scheme to
a number of coordinates, a subset of which are the junction coordinates. The components
are then synthesized forming the reduced system matrices. The system eigenvalue
problem is then solved in order to provide a further reduction in the order of the system
matrices. The system modes serve as a basis for the modal superposition method from
which the dynamic response is determined.

The method of Static Synthesis; however, has some serious drawbacks. First, the
static reductions on the components produce appreciable frequency errors for the higher
frequencies, a natural consequence of the Rayleigh-Ritz Bound Theorem. This frequency
error for the higher modes translates into error in dynamic response if the higher modes
have an appreciable participation. Second, important component modes having major
participation in system response may inadvertently be lost in the static reduction due to
the a poor choice of coordinate reduction. Third, the static reduction destroys the
bandedness of sparse matrices leading to expensive eigensolutions. Finally, the
implementation of damping at the component level will become challenging since
damping in usually defined on the component modes.

Due to the shortcomings of the method of Static Synthesis, tremendous research
has been directed towards other methods of dynamic substructuring. Hurty (1965)
developed an ingenious method of dynamic substructuring known as Component-Mode
Synthesis. In it, the component fixed-junction normal modes along with static constraint
modes are used as basis for a Rayleigh-Ritz analysis at the component level. In this
approach, a highly accurate reduction is possible depending on the cut-off frequency of
the fixed-junction normal modes. The system matrices are formulated via synthesis of component modes, giving rise to the methods name. Once the system matrices are formulated, the system level eigenproblem is solved to diagonalize matrices and further reduce basis for a mode superposition analysis. Note that the component damping matrix may be easily defined on the fixed-junction normal modes leading to the nonclassically damped problem at the combined system level, requiring either coupled integration or complex-valued modes. Many authors contributed to variations of Hurty's Fixed-Junction Component-Mode Synthesis, some of which are described in Chapter 2.

The method of Free-Junction Component-Mode Synthesis by MacNeal (1971) and Rubin (1974) is generally regarded as the next major development in this field. These authors proposed using a retained set of free-junction normal modes in a non Rayleigh-Ritz method of deriving the component equations. In the derivation of component equations, the concept of first and second-order residual flexibility is developed. It is noted that historically many practicing engineers in the area of Component-Mode Synthesis have had difficulties with the concept of residual flexibility. A detailed treatment of this concept is presented in Chapter 3.

The two methods of Component-Mode Synthesis have a basic idea in common. That is, the combined system modes are derived from the synthesis of component normal modes. Note that Component-Mode Synthesis inherently leads to the nonclassically damped problem. The system modes are not orthogonal with respect to the system damping matrix, where damping is defined on the individual component modes. Although a variety of iterative methods have been proposed to solve this problem along with complex modal solutions, this dissertation adopts coupled integration algorithms for this solution due to the clarity of approach. Once the integrations of system equations are completed, the primary design variable to be determined is the junction forces between the components. In the method of Component-Mode Synthesis, the junction forces are determined via equilibrium considerations on the component equations of motion. This
phase of the analysis is referred to as the recoveries, where the junction forces are recovered from the generalized response.

1.3 An Overview of the Residual Flexibility Approach to Decoupled Analysis

This dissertation develops an alternative to the traditional method of Component-Mode Synthesis referred to here as decoupled analysis. In this residual flexibility approach to analysis of systems comprised of multiple components, the reactive forces at the junctions are computed directly without the synthesis of component modes or solutions to expensive system eigenproblems. A junction-sized set of simultaneous algebraic equations are derived in terms of the unknown junction forces. The solution to these equations at a time-step is then used to drive the equations of motion of the individual components separately, in a decoupled manner. Since the component's are integrated in their individual coordinate systems, the nonclassically damped problem becomes a trivial case, not requiring coupled integration. Furthermore, the nonlinear, nonclassically damped problem, a formidable task in Component-Mode Synthesis, is treated as a natural progression of the method.

1.4 Outline

In Chapter 1, a brief overview of the objectives of this dissertation are addressed. Furthermore, an overview of the method of Component-Mode Synthesis and residual flexibility approach to decoupled analysis is presented.

Chapter 2 presents a comprehensive study of the method of Fixed-Junction Component-Mode Synthesis. The mathematical formulation of the method is presented. Component damping matrices and the nonclassically damped system are treated. Mode displacement and mode acceleration recoveries of junction forces are addressed.
Furthermore, the junction-loaded component is treated. Numerical studies on the convergence characteristics of component frequencies, system frequencies, and system response are presented. Comparisons between the convergence characteristics of Static Synthesis versus Component-Mode Synthesis are carried. The convergence of the nonclassically damped problem is studied.

In Chapter 3, the method of Free-Junction Component-Mode Synthesis along with a thorough treatment of first and second-order residual flexibility is presented. The component equations of motion are derived. Deliberate emphasis is given to fully developing the concept of residual flexibility. Numerical studies comparing the convergence of component and system frequencies and response to Fixed-Junction Synthesis is presented. The accuracy of solution to the nonclassically damped problem is compared to Fixed-Junction Synthesis.

Chapter 4 presents a Hybrid Method of Component-Mode Synthesis where junction forces are used as generalized coordinates leading to a mixed formulation for the component equations. In this Rayleigh-Ritz analysis, residual flexibility for the components is taken into account. Simple recovery techniques for the component junction forces are presented. Numerical studies comparing this method to Free-Junction Component-Mode Synthesis are presented.

Chapter 5 addresses a method of decoupled analysis based on fixed-junction component representation. The method is fully developed from considerations presented in Chapter 2. The technique is extended to the nonlinear, nonclassically damped problem. Numerical studies conducted on this decoupled approach are compared to Fixed-Junction Component-Mode Synthesis.

In Chapter 6, the residual flexibility approach to decoupled analysis is presented. Mathematics behind deriving the equations leading to direct solution of junction forces are treated. The problem is extended to the nonlinear, nonclassically damped system.
Convergence of this decoupled method is compared to Free-Junction Component-Mode Synthesis for the nonclassically damped system.

Finally, in Chapter 7, pertinent conclusions to the methods of system analysis are presented.
CHAPTER 2

THE METHOD OF COMPONENT-MODE SYNTHESIS - THE FIXED-JUNCTION COMPONENT REPRESENTATION

2.1 Introductory Remarks

The problem of considering a structural system as an assemblage of components or substructures has its origins in statics analysis. In a paper titled "Matrix Structural Analysis of Substructures", Przemieniecki (1963) originated a method of static analysis of complex structural systems by dividing the system into a number of substructures. In this displacement or stiffness method of analysis, each substructure is first analyzed with the displacements of the adjacent substructure boundaries completely restrained. Subsequently, the boundaries are relaxed and the displacements at the boundaries is determined by considering the equilibrium of boundary forces.

Gladwell (1964) is the first author that presented an extended discussion on dynamic analysis of systems of combined components. Referring to components as branches, in a paper titled "Branch Mode Analysis of Vibrating Systems", Gladwell proposed a method of dynamic substructuring limited in scope to components with statically determinate junctions coordinates. In it, Gladwell imposed sets of constraints on the system so that only a few components could vibrate at a time in so called branch modes. Subsequently, branch modes along with system rigid-body modes are used via a Rayleigh-Ritz technique in order to derive the equations of the combined system.

Following the work of Przemieniecki and Gladwell, Hurty (1965) presented a comprehensive method for dynamic substructuring applicable to redundant systems. Referred to as Component-Mode Synthesis, the aim of the method is to derive the normal
modes of the system of combined components from the modes of the individual components. Component-Mode Synthesis is a displacement method in which the displacements of the individual components are described by a distinct set of generalized coordinates that may be classified into three categories: (1) rigid-body generalized coordinates, (2) constraint mode generalized coordinates, and (3) normal mode generalized coordinates.

In the method of Component-Mode Synthesis as derived by Hurty, the physical coordinates of the component are divided into interior physical coordinates and junction physical coordinates. The junction physical coordinates are further divided into statically determinate coordinates and redundant coordinates. The normalized displacement modes from which the generalized coordinates are derived may be described as follows:

1. Rigid-body displacement modes derived from rigid-body translations and small rotations of the component without deformation.

2. Constraint modes derived by enforcing successive unit displacements and rotations at the redundant junction coordinates while all other junction coordinates, both statically determinate and redundant, remain constrained. From this, it is obvious that the number of constraint modes is equal to the number of statically redundant coordinates.

3. Normal modes are derived relative to a constrained set of junction coordinates. They are computed through an eigensolution for the interior coordinates relative to the junction coordinates.

The reduction in the number of component physical coordinates is achieved by retaining a finite number of fixed-junction normal mode generalized coordinates in the Rayleigh-Ritz procedure. The component equations of motion are then derived relative to the specified sets of generalized coordinates via Lagrange's equations. The requirement of displacement compatibility along component boundaries provides a coordinate transformation in order to synthesize the system mass, damping, and stiffness matrices and the system load vector.
Craig and Bampton (1967) reduced the types of generalized coordinates necessary in Component-Mode Synthesis by the following simplification. While Hurty separated the junction coordinates into statically determinate and redundant, Craig and Bampton made no such distinction. The determination of rigid-body modes and relating the rigid-body generalized coordinates to the statically determinate junction degrees of freedom were deemed unnecessary when the all junction coordinates were treated as redundant. The method takes advantage of two types of generalized coordinates. The so called constraint mode generalized coordinates account for the displacements and small rotations at the component junction degrees of freedom. This type of generalized coordinate is related to the constraint modes as described by Hurty. The normal mode generalized coordinates are identical to Hurty's method and are related to the fixed-junction normal modes of vibration. The component equations are formulated through a Rayleigh-Ritz technique employing Lagrange's equations. The system matrices and load vector are formulated through compatibility conditions along component boundaries.

Bajan and Feng (1968) developed a technique of Component-Mode Synthesis that is similar to the method presented by Craig and Bampton independently. In it, the motion of the components is written in terms of fixed-junction normal modes of vibration and constraint modes.

Goldman (1969) presented a method of Component-Mode Synthesis which takes advantage of free-junction component modes of vibration. Goldman notes; however, that large errors occur in some cases due to the fact that constraint modes are not used while the component normal modes are truncated.

The sections in this chapter are devoted to the mathematical treatment of the method of Component-Mode Synthesis. The problem of the nonclassically damped system which results from a Component-Mode Synthesis approach is addressed. The mode-acceleration method and improvements to component modes via junction loading the components are treated. Numerical results considering a system composed of two
cantilever beam components, each uniquely damped thereby resulting in a nonclassical system, is presented. Further results on the topics of convergence of system frequencies as a function of retained component modes and convergence of system response as a function of retained component modes are discussed.

2.2 The Derivation of Displacement Modes for a Component

Assume that the component mass and stiffness matrices have been formulated by the finite element method or some other technique. The constraint modes are defined as displacement mode shapes derived from successive unit displacement and rotation of the junction degrees of freedom, one at a time, while all other junction degrees of freedom are constrained. The interior degrees of freedom are unconstrained. In order to derive the constraint modes, the following problem in statics is considered

\[
\begin{bmatrix}
  F_j \\
  F_a
\end{bmatrix} = \begin{bmatrix}
  K_{jj} & K_{ja} \\
  K_{aj} & K_{aa}
\end{bmatrix} \begin{bmatrix}
  u_j \\
  u_a
\end{bmatrix},
\]

(2.2.1)

In the above equation, the physical coordinates of the component are partitioned into interior and junction degrees of freedom, denoted by the subscripts a and j, respectively. The stiffness matrix is denoted by \( K \), the external forces acting on the component are denoted by the force vector, \( F \), and the component displacements and rotations are denoted by the vector \( u \). Although externally applied, the force vector acting on the component may be conceptualized as the reactions due to successive unit displacements and rotations of a junction coordinate while all other junction coordinates are restrained. It is seen that reactions only occur at the junction coordinates therefore the interior force
vector \( \mathbf{F}_s \), may be considered null. From this, the interior displacements may be expressed in terms of the displacements at the junction coordinates by

\[
\mathbf{u}_a = \Phi_{aj}^c \mathbf{u}_j,
\]

where the constraint modes are defined by

\[
\Phi_{aj}^c = -\mathbf{K}_{aa}^{-1} \mathbf{K}_{aj}.
\]

It is seen that the number of constraint modes is equal to the number of junction coordinates.

The second set of generalized coordinates necessary for component representation in dynamic analysis are the normal mode coordinates related to the junction-constrained normal modes of vibration. These vectors describe the displacements of the interior coordinates relative to the junction coordinates. The following eigenproblem is considered

\[
(K_{aa} - \omega^2 M_{aa}) \Phi_{an}^N = 0.
\]

In the above equation, \( \Phi_{an}^N \) is the component modal matrix and \( \omega^2 \) represents the associated eigenvalues. The subscript q denotes the fact that only q modes of vibration are retained in the modal matrix. The mass matrix \( M_{aa} \) is the interior partition of the component physical mass matrix denoted by
\[ \mathbf{M} = \begin{bmatrix} M_{ij} & M_{ja} \\ M_{aj} & M_{aa} \end{bmatrix}. \] (2.2.5)

Let the component generalized coordinates of a component be denoted by

\[ \mathbf{q} = \begin{bmatrix} u_j \\ q_n \end{bmatrix}. \] (2.2.6)

The Ritz transformation relating the component physical coordinates to the component generalized coordinates may be expressed as

\[ \mathbf{u} = \Psi \mathbf{q} \] (2.2.7)

where

\[ \Psi = \begin{bmatrix} I & 0 \\ \Phi_{aj} & \Phi_{an} \end{bmatrix}. \] (2.2.8)

In the above notation, \( I \) and \( 0 \) denote the identity and the null matrices, respectively. Equation (2.2.8) provides the necessary coordinate transformation to a set of constraint mode and normal mode generalized coordinates. Note that since \( n < a \), the component generalized coordinates are a reduced set.
2.3 Component Mass, Stiffness, Damping, and Loads

The component equation of motion may now be derived via the Rayleigh-Ritz method by considering Equation (2.2.8) as Ritz vectors. From kinetic energy considerations, the transformed component mass matrix may be expressed as

$$ m = \Psi^T M \Psi, $$

(2.3.1)

where the superscript $T$ denotes the operation of matrix transposition. Expanding the above expression, the transformed mass matrix becomes

$$ m = \begin{bmatrix} m_{jj} & m_{jn} \\ m_{nj} & m_{nn} \end{bmatrix}, $$

(2.3.2)

where

$$ m_{jj} = \Phi_{aj}^c (M_{aa} \Phi_{aj}^c + M_{aj}) + M_{ja} \Phi_{aj}^c + M_{jj}, $$

(2.3.3)

$$ m_{nj} = m_{jn}^T = \Phi_{an}^N (M_{aa} \Phi_{aj}^c + M_{aj}), $$

(2.3.4)

and

$$ m_n = \begin{bmatrix} I_n \end{bmatrix}. $$

(2.3.5)
Equation (2.3.3) is the so-called Guyan (1965) mass matrix, a reduced junction-sized mass matrix commonly used in dynamic analyses. Equation (2.3.5) is the result of the orthogonality relations of the eigenvalue problem in Equation (2.2.4). It is assumed here that the eigenvectors are mass normalized.

The transformed stiffness matrix may be derived in a similar fashion from strain energy considerations. The resulting stiffness matrix may be expressed as

\[ k = \Psi^T K \Psi. \]  

(2.3.6)

Expanding the above expression yields

\[ k = \begin{bmatrix} k_{jj} & 0 \\ 0 & k_n \end{bmatrix} \]  

(2.3.7)

where

\[ k_{jj} = K_{jj} + \Phi^{cT} K_{aj} \Phi^T a_j \]  

(2.3.8)

and

\[ k_n = \begin{bmatrix} \omega_n^2 \end{bmatrix} \]  

(2.3.9)

Equation (2.3.8) is the reduced stiffness matrix or boundary stiffness commonly used in static substructuring. Equation (2.3.9) holds true by the virtue of orthogonality of the fixed-junction normal modes of vibration where the normal modes partition of the
stiffness matrix is equal to the fixed-junction eigenvalues. Of interest is the fact that the off-diagonal partitions of the transformed stiffness matrix are null matrices. Physically, this is due to the fact that work done by the forces at the constraints on a normal mode displacement is zero since the normal modes are determined by constraining the junction coordinates.

The definition of the component damping matrix requires certain considerations. It is assumed here that at the component level, the damping is of the Rayleigh, or classical, form. From a practical viewpoint, this assumption makes good sense since often component damping matrices are defined through modal testing. The component damping matrix is defined by

\[
e = \begin{bmatrix} c_{jj} & 0 \\ 0 & c_n \end{bmatrix}, \tag{2.3.10}
\]

where

\[
c_n = \begin{bmatrix} \omega_n \\ 2\zeta_n \omega_n \end{bmatrix}. \tag{2.3.11}
\]

In Equation (2.3.11), \(\zeta_n\) is the modal damping ratio for the fixed-junction normal modes of vibrations as determined by modal testing or judged by the analyst. The damping associated with the junction degrees of freedom may be approximately determined by an eigenproblem of the junction mass and stiffness matrices. Assuming a certain modal damping ratios for these junction modes, \(\zeta_j\), and taking advantage of the orthogonality relations of this eigenproblem, the junction damping is derived to be
\[ c_{j} = m_{j} \Phi_{j} \begin{bmatrix} 2 \zeta \omega_n \xi \end{bmatrix} \Phi_j^T m_j. \] (2.3.12)

The component load vector may be determined via virtual work considerations. The virtual work done by the physical forces in a virtual displacement of the physical coordinates must be the same the virtual work of the transformed forces in the generalized coordinates. From the above statement of virtual work, the forces in the generalized coordinates are

\[ \tilde{\mathbf{F}} = \Psi^T \mathbf{F}. \] (2.3.13)

Expanding Equation (2.3.13), the generalized forces may be expressed in terms of the physical forces acting on the interior of the component and forces acting on the junction coordinates by

\[ \begin{bmatrix} \tilde{\mathbf{F}}_j \\ \tilde{\mathbf{F}}_n \end{bmatrix} = \begin{bmatrix} \Phi_j c^T \mathbf{F}_a + \mathbf{F}_j \\ \Phi_{an} \mathbf{F}_a \end{bmatrix}. \] (2.3.14)

2.4 Derivation of the System Equations of Motion from Component Equations

In this section, synthesis of equations of motion of a system composed of two components is derived. This is done without a loss in generality since synthesis of multiple components follows along the same lines. The important idea here is to enforce geometric compatibility along component boundaries as required by displacement continuity considerations. Note that although the equilibrium of the junction forces is not strictly enforced in Component-Mode Synthesis and most other displacement methods, it is implicit within the derivation.
Consider the equations for dynamic equilibrium for two components, $\alpha$ and $\beta$, in the generalized coordinates discussed in the prior sections. For component $\alpha$, these equations in matrix notation are

$$
\begin{bmatrix}
m_{jj} & m_{jn} \\
m_{nj} & m_n
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_j \\
\ddot{q}_n
\end{bmatrix}
+ \begin{bmatrix}
c_{jj} & 0 \\
0 & c_n
\end{bmatrix}
\begin{bmatrix}
\dot{u}_j \\
\dot{q}_n
\end{bmatrix}
+ \begin{bmatrix}
k_{jj} & 0 \\
k_j & k_n
\end{bmatrix}
\begin{bmatrix}
u_j \\
q_n
\end{bmatrix}
= \begin{bmatrix}
\ddot{f}_j \\
\ddot{f}_n
\end{bmatrix}.
$$

(2.4.1)

Similarly, for component $\beta$, the equations of dynamic equilibrium are

$$
\begin{bmatrix}
m_{jj} & m_{jn} \\
m_{nj} & m_n
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_j \\
\ddot{q}_n
\end{bmatrix}
+ \begin{bmatrix}
c_{jj} & 0 \\
0 & c_n
\end{bmatrix}
\begin{bmatrix}
\dot{u}_j \\
\dot{q}_n
\end{bmatrix}
+ \begin{bmatrix}
k_{jj} & 0 \\
k_j & k_n
\end{bmatrix}
\begin{bmatrix}
u_j \\
q_n
\end{bmatrix}
= \begin{bmatrix}
\ddot{f}_j \\
\ddot{f}_n
\end{bmatrix}.
$$

(2.4.2)

Displacement compatibility along component boundaries requires that

$$
u^\alpha_j = u^\beta_j = u_j.
$$

(2.4.3)

From the condition in Equation (2.4.3), a transformation for component synthesis may be derived. For this case, this transformation is

$$
\begin{bmatrix}
q^\alpha \\
q^\beta
\end{bmatrix} = By.
$$

(2.4.4)
where $B$ is the transformation bearing the information for compatibility of junction displacements and $y$ is the vector of system generalized coordinates. For the case of the two components considered

$$y = \begin{bmatrix} q_n^\alpha \\ q_n^\beta \\ u_j \end{bmatrix}, \quad (2.4.5)$$

The system mass, stiffness, and damping matrices may be derived in a manner consistent with the formulation of the component level matrices. From kinetic, potential and dissipative energy considerations, respectively, the system mass, stiffness, and damping matrices are

$$M^s = B^T \begin{bmatrix} m^\alpha & 0 \\ 0 & m^\beta \end{bmatrix} B, \quad (2.4.6)$$

$$K^s = B^T \begin{bmatrix} k^\alpha & 0 \\ 0 & k^\beta \end{bmatrix} B, \quad (2.4.7)$$

and

$$C^s = B^T \begin{bmatrix} c^\alpha & 0 \\ 0 & c^\beta \end{bmatrix} B. \quad (2.4.8)$$
In the above equations, the superscript $s$ denotes system. The system load vector may be treated by virtual work considerations which yields

$$
\mathbf{F}^s = \mathbf{B}^T \begin{bmatrix} \mathbf{F}^\alpha \\ \mathbf{F}^\beta \end{bmatrix}.
$$

(2.4.9)

Additional insight is gained by explicitly presenting the forms of the system matrices derived in the above equations. Although the above relations present a formal means for deriving the system matrices, these matrices may be derived informally by inspection.

The system mass, stiffness, and damping matrices, stated explicitly, are

$$
\mathbf{M}^s = \begin{bmatrix}
m_n^\alpha & 0 & m_{nj}^\alpha \\
0 & m_n^\beta & m_{nj}^\beta \\
m_{jn}^\alpha & m_{jn}^\beta & m_{jj}^\alpha + m_{jj}^\beta
\end{bmatrix},
$$

(2.4.10)

$$
\mathbf{K}^s = \begin{bmatrix}
k_n^\alpha & 0 & 0 \\
0 & k_n^\beta & 0 \\
0 & 0 & k_{jj}^\alpha + k_{jj}^\beta
\end{bmatrix},
$$

(2.4.11)

and

$$
\mathbf{C}^s = \begin{bmatrix}
c_n^\alpha & 0 & 0 \\
0 & c_n^\beta & 0 \\
0 & 0 & c_{jj}^\alpha + c_{jj}^\beta
\end{bmatrix}.
$$

(2.4.12)
The system load vector, assuming that no external forces act on the junction coordinates is given by

\[ F_s = \begin{bmatrix} \vec{F}_n^\alpha \\ \vec{F}_n^\beta \\ 0 \end{bmatrix}. \]  

(2.4.13)

2.5 The System Eigenproblem and the Reduction to System Modal Coordinates

At this point, the complex structural system is represented by a reduced set of system generalized coordinates. Numerical integration of coupled differential equations may be carried out in order to determine system response from which component response is recovered. The order of the system matrices may be reduced further by considering a system level eigenproblem. The cut-off frequency for system reduction may be determined from the response spectra of the forcing functions. The system eigenproblem is stated as

\[ (K_s - \omega^2 M_s)\Phi_m = 0. \]  

(2.5.1)

In the above equation, the modal matrix contains \( m \) columns up to the cut-off frequency. From the principle of modal superposition, the system generalized coordinates may be expressed in terms of a reduced set of system modal coordinates by

\[ y = \Phi_m z_m. \]  

(2.5.2)
where \( z_m \) is the vector of system modal coordinates, the subscript \( m \) indicating the number of retained modes. Transforming the system equations of motion to system modal coordinates, the following equation of dynamic equilibrium result

\[
\ddot{m}_m \ddot{z}_m + \ddot{c}_m \dot{z}_m + \ddot{k}_m z_m = \ddot{f}_m, \tag{2.5.3}
\]

where

\[
\ddot{m}_m = \Phi_m^T M^s \Phi_m, \tag{2.5.4}
\]

\[
\ddot{c}_m = \Phi_m^T C^s \Phi_m, \tag{2.5.5}
\]

\[
\ddot{k}_m = \Phi_m^T K^s \Phi_m, \tag{2.5.6}
\]

and

\[
\ddot{f}_m = \Phi_m^T F^s. \tag{2.5.7}
\]

Note that this is the so called nonclassically damped problem since the normal modes are not orthogonal with respect to the system damping matrix and Equation (2.5.5) results in a fully populated matrix. Thus the resulting equations are uncoupled by the virtue of orthogonality in the inertia and elastic force terms; however, fully coupled in the damping terms.

Solutions to the nonclassically damped problem have been considered by many authors throughout the years. As recently as 1990, Udwadia and Esfandiari proposed an iterative scheme for solving the nonclassically damped problem by treating the coupling terms in the damping matrix on the right hand side of the equations of motion. Veletsos and Ventura (1986) extended the work of Foss (1957) in solving the complex-valued
eigenproblem in order to uncouple the damping matrix. In practical terms, both approaches to the nonclassically damped problem are in no way more efficient than direct integration of Equation (2.5.3).

2.6 Direct Integration of the System Modal Equations of Motion

It is apparent that the resulting differential equations from Component-Mode Synthesis are coupled. In this section, the Newmark numerical integration method, Newmark (1962), is used in order solve the coupled set of differential equations in Equation (2.5.3). The basic recurrence relations in this single-step, iterative, integration algorithm are

\[
\ddot{z}_{m,i+1} = \ddot{z}_{m,i} + \frac{1}{2}(\dddot{z}_{m,i} + \dddot{z}_{m,i+1})h, \quad (2.6.1)
\]

and

\[
z_{m,i+1} = z_{m,i} + \dot{z}_{m,i}h + \frac{1}{2}(1-\beta)\ddot{z}_{m,i}h^2 + \beta \dddot{z}_{m,i+1}h^2. \quad (2.6.2)
\]

In the above equations, \(\beta\) is the so called Newmark parameter which governs the variation of acceleration within the time-step \(h\) and the subscript \(i\) indicates the discretization of time. It is shown, Newmark (1962), that for a value of \(\beta > 0.25\), the integrator is unconditionally stable. Furthermore, it is shown that the convergence criteria for the integrator as determined by the ratio of the time-step \(h\) to the shortest period of free oscillations is inversely proportional to the square root of Newmark's parameter.

It is possible, through some algebraic manipulation of the above recursive relations (2.6.1) and (2.6.2), to derive a noniterative version of the Newmark integrator.
Solving Equation (2.6.2) for the accelerations at $i+1$ and substituting into Equation (2.6.1), the following recursive relations are derived

$$\ddot{z}_{m,i+1} = a_0 (z_{m,i+1} - z_{m,i}) - a_2 \dot{z}_{m,i} - a_3 \ddot{z}_{m,i},$$

(2.6.3)

and

$$\dot{z}_{m,i+1} = a_0 (z_{m,i+1} - z_{m,i}) - a_2 \dot{z}_{m,i} - a_4 \ddot{z}_{m,i}.$$  

(2.6.4)

In the above equations, the constants of integration are defined as a function of the Newmark parameter and the time-step by

$$a_0 = \frac{1}{\beta h^2}, \quad a_1 = \frac{1}{2\beta h}, \quad a_2 = \frac{1}{\beta h}, \quad a_3 = \frac{1}{2\beta} - 1, \quad a_4 = \frac{h}{2} \left( \frac{1}{2\beta} - 2 \right).$$  

(2.6.5)

Now, consider the equations of dynamic equilibrium for the system modal coordinates, Equation (2.5.3), at $i+1$. This equation is restated by

$$\ddot{\bar{z}}_{m,i+1} + \ddot{\bar{z}}_{m,i+1} + \ddot{\bar{z}}_{m,i+1} = \ddot{\bar{f}}_{m,i+1}.$$  

(2.6.6)

Substituting equations (2.6.4) and (2.6.5) in Equation (2.6.6) results

$$\ddot{\bar{z}}_{m,i+1} = \ddot{\bar{f}}_{m,i+1},$$  

(2.6.7)

where
\[ \ddot{k}_m = a_0 \ddot{m}_m + a_1 \dddot{c}_m + \dddot{k}_m. \]  \hfill (2.6.8)

and

\[ \ddot{f}_{m,i+1} = \ddot{f}_{m,i+1} + \dddot{m}_m (a_0 \dddot{z}_{m,i} + a_2 \dddot{z}_{m,i} + a_3 \dddot{z}_{m,i}) + \dddot{c}_m (a_1 \dddot{z}_{m,i} + a_3 \dddot{z}_{m,i} + a_4 \dddot{z}_{m,i}). \]  \hfill (2.6.9)

In numerical integration terminology, Equation (2.6.8) is known as the effective stiffness matrix, in this case representing \( m \) retained modes. Equation (2.6.9) is the effective load vector that also in this case represents the modal forces acting on \( m \) retained mode shapes. Note that for a constant value of the time-step \( h \), the effective stiffness matrix is a constant and has to be decomposed only once. The solution of system modal displacements from Equation (2.6.7) at a time-step is then substituted into Equations (2.6.3) and (2.6.4) which yield the system modal velocities and accelerations. Hence, the coupled system of modal differential equations is solved in a step-by-step manner for a specified duration of time.

### 2.7 Mode-Displacement Recovery of the Junction Forces

A critical parameter in the design of complex structural system, idealized as an assemblage of components, is the forces at the component junctions. In order to determine this reactive force, it is necessary to first compute the component generalized displacements, velocities, and accelerations from the system modal displacements, velocities, and accelerations. From Equations (2.4.4) and (2.5.2), the displacement generalized coordinates for the components as a function of the system modal displacements are
\[ q^\alpha = \left\{ \begin{array}{c} q_j^\alpha \\ q_n^\alpha \end{array} \right\} = B^\alpha \Phi_\alpha^m z^m, \quad (2.7.1) \]

and

\[ q^\beta = \left\{ \begin{array}{c} q_j^\beta \\ q_n^\beta \end{array} \right\} = B^\beta \Phi_\beta^m z^m. \quad (2.7.2) \]

In the above equations, \( B^\alpha \) and \( B^\beta \) indicate the row partitions of the synthesis coordinate transformation in Equation (2.4.4) related to components \( \alpha \) and \( \beta \), respectively. In the same manner, \( \Phi_\alpha^m \) and \( \Phi_\beta^m \) are row partitions of the system modal transformation in Equation (2.5.2) as related to components \( \alpha \) and \( \beta \), respectively. In a similar manner, the component generalized velocities and accelerations are determined by the following four equations

\[ \dot{q}^\alpha = \left\{ \begin{array}{c} \dot{q}_j^\alpha \\ \dot{q}_n^\alpha \end{array} \right\} = B^\alpha \Phi_\alpha^m \dot{z}^m, \quad (2.7.3) \]

\[ \dot{q}^\beta = \left\{ \begin{array}{c} \dot{q}_j^\beta \\ \dot{q}_n^\beta \end{array} \right\} = B^\beta \Phi_\beta^m \dot{z}^m. \quad (2.7.4) \]

and,

\[ \ddot{q}^\alpha = \left\{ \begin{array}{c} \ddot{q}_j^\alpha \\ \ddot{q}_n^\alpha \end{array} \right\} = B^\alpha \Phi_\alpha^m \ddot{z}^m, \quad (2.7.5) \]

\[ \ddot{q}^\beta = \left\{ \begin{array}{c} \ddot{q}_j^\beta \\ \ddot{q}_n^\beta \end{array} \right\} = B^\beta \Phi_\beta^m \ddot{z}^m. \quad (2.7.6) \]
Owing to the fact that the reactive forces at the junction between the two components occur in equal and opposite pairs

\[ F_{j}^{\alpha} = -F_{j}^{\beta} = F_{j}^{\alpha}, \quad (2.7.7) \]

it is sufficient to consider the equations of equilibrium for either component. Considering the equations of dynamic equilibrium for component \( \alpha \), the generalized junction forces are

\[ \ddot{F}_{j}^{\alpha} = m_{j} \ddot{q}_{j}^{\alpha} + m_{j} \ddot{q}_{n}^{\alpha} + c_{j} \dot{q}_{j}^{\alpha} + k_{j} q_{j}^{\alpha}. \quad (2.7.8) \]

Now, the physical coordinate forces at the junctions may be determined from equations (2.3.14) and (2.7.8) by

\[ F_{j} = \ddot{F}_{j}^{\alpha} - \Phi_{aj}^{\alpha} T_{a}^{\alpha}. \quad (2.7.9) \]

### 2.8 Mode-Acceleration Recovery of the Junction Forces

The mode-acceleration method, Williams (1945), deals with the problem of the slow convergence characteristics of the displacements in the mode-displacement method. It is known that frequently many normal modes at the system level eigensolution are required in order to converge to accurate displacement results by the mode-displacement method. In fact, the mode-displacement method will fail to give accurate displacements even in the case of statically applied loads if the number of retained modes is insufficient.
The mode-acceleration method alleviates the convergence problems of the mode-displacement method and improves the accuracy of the displacements.

Consider the equation of motion in the system generalized coordinates prior to the system level eigensolution where the mass, stiffness, damping, and load are given by Equations (2.4.10), (2.4.11), (2.4.12), and (2.4.13), respectively. The equation of dynamic equilibrium in these coordinates may be written in compact notation by

$$M^s \ddot{y} + C^s \dot{y} + K^s y = F^s. \quad (2.8.1)$$

Restricting our attention to the case where the system has no rigid-body degrees of freedom, the displacements may be solved for from Equation (2.8.1) by the following equation, where the superscript $s$ has been dropped for convenience

$$y = K^{-1}F - K^{-1}Cy - K^{-1}My. \quad (2.8.2)$$

The equation of modal superposition (2.5.2) may be restated as a summation of the eigenvectors multiplied by the system modal coordinates by

$$y = \sum_{k=1}^{m} \Phi_k z_k. \quad (2.8.3)$$

where $\Phi_k$ is the $k$th eigenvector and $z_k$ is $k$th modal coordinate. Substituting the first two derivatives of Equation (2.8.3) in Equation (2.8.2), the following equation is obtained

$$y = K^{-1}F - K^{-1}C \sum_{k=1}^{m} \Phi_k \dot{z}_k - K^{-1}M \sum_{k=1}^{m} \Phi_k \ddot{z}_k. \quad (2.8.4)$$
Taking advantage of the properties of the system eigenproblem in Equation (2.5.1), the above expression may be stated in its final form by

$$y = K^{-1}F - K^{-1}C \sum_{k=1}^{m} \Phi_k \dot{z}_k - \sum_{k=1}^{m} \left( \frac{1}{\omega_k^2} \right) \Phi_k \ddot{z}_k.$$  \hspace{1cm} (2.8.5)

Note that the first term in Equation (2.8.5) is the wholly static response of the system due to the external load. Additionally, the last term converge rapidly due to the presence of $\omega_k$ to the second power in the denominator. Once the improved displacement via the mode-acceleration method is computed, the equations in section 2.7 may be used for the computation of the junction forces.

**2.9 A Modification to Component Mode Representation to Improve System Modes**

The modification to component modes presented in this section is based on the Rayleigh-Ritz concept that the normal modes of vibration of the system of combined components is a linear combination of the component modes. This being true, it is reasonable to assume that a method that can modify the component modes to more closely resemble the system modes would result in greater accuracy at the system level. Benefield and Hruda (1971) introduced the concept that loading the junction of one component with the stiffness and mass of the other component will improve the component mode representation. Component modes modified in this manner become similar to the system modes.

The mathematics used in developing the junction-loaded component rely on energy concepts and constraint modes. Through a modification of Equation (2.2.2), the
displacements of component \( \beta \) in physical coordinates may be written in terms of the junction displacements via the constraint modes

\[
\mathbf{u}_j^\beta = \left\{ \begin{array}{c} u_j^\beta \\ u_a^\beta \end{array} \right\} = \left[ \begin{array}{c} I \\ \Phi_{aj} \end{array} \right] \mathbf{u}_j^\beta. \tag{2.9.1}
\]

In compact notation, the above relation may be rewritten as

\[
\mathbf{u}_j^\beta = \Phi_{fj} \mathbf{c}_j^\beta \mathbf{u}_j^\beta \tag{2.9.2}
\]

where the subscript \( f \) denotes the union of the sets \( a \) and \( j \). The boundary stiffness or junction stiffness matrix commonly used in static substructuring can now be formed by evaluating the components potential energy, written in quadratic form by

\[
U_j^\beta = \frac{1}{2} \mathbf{u}_j^\beta \mathbf{K}_j^\beta \mathbf{u}_j^\beta. \tag{2.9.3}
\]

Substituting Equation (2.9.2) in Equation (2.9.3), the potential energy may be rewritten as

\[
U_j^\beta = \frac{1}{2} \mathbf{u}_j^\beta \mathbf{k}_j^\beta \mathbf{u}_j^\beta, \tag{2.9.4}
\]

where the junction stiffness matrix is given by

\[
\mathbf{k}_j^\beta = \Phi_{fj} \mathbf{K}_j^\beta \Phi_{fj}^\beta. \tag{2.9.5}
\]
Note that the junction stiffness matrix was derived prior to this by Equation (2.3.8) in a different fashion for Component-Mode Synthesis. The current derivation is however necessary in order to formally introduce the concept of the junction loaded component.

In a similar fashion, the junction or Guyan mass matrix may be derived by kinetic energy considerations

\[
\mathbf{m}_j = \Phi_f^T \mathbf{M} \Phi_f. \tag{2.9.6}
\]

Equation (2.9.6) was derived in a slightly different fashion for Component-Mode Synthesis and is given by Equation (2.3.3). Again, the above derivation is necessary for the introduction of the new concept.

The total potential energy of the two components may be expressed in quadratic forms by

\[
U^{\alpha+\beta} = \frac{1}{2} u^\alpha K^\alpha u^\alpha + \frac{1}{2} u^\beta K^\beta u^\beta. \tag{2.9.7}
\]

From the compatibility of displacements at the junction coordinates, the following expression results

\[
u_j = T_{j*} u^\alpha = [I \ 0] \begin{bmatrix} u_j \ u_1 \end{bmatrix}. \tag{2.9.8}
\]

Substituting first Equation (2.9.2) into Equation (2.9.7) and then substituting Equation (2.9.8) into the resulting expression, the potential energy of the combined system becomes
\[ U^a = \frac{1}{2} u^a \hat{K}^a u^a, \]  

(2.9.9)

where the junction loaded component stiffness matrix is

\[ \hat{K}^a = K^a + T_{f_j}^\tau m_{j} \beta T_{f_j}. \]  

(2.9.10)

In a similar fashion, the junction loaded component mass matrix, derived from kinetic energy considerations is given by

\[ \hat{M}^a = M^a + T_{f_j}^\tau m_{j} \beta T_{f_j}. \]  

(2.9.11)

Note that in Equations (2.9.10) and (2.9.11), the stiffness and mass matrices of component \( \alpha \) are augmented by an account of the junction stiffness and mass of component \( \beta \). Employing Lagrange's equations, the equation of motion of component \( \alpha \) may be written, for free vibrations, as

\[ \hat{M}^a \ddot{u}^a + \hat{K}^a u^a = 0. \]  

(2.9.12)

The eigenvectors resulting from an eigenproblem of Equation (2.9.12) contain the approximate dynamic effects of component \( \beta \); hence, component modes of vibration similar to system modes of vibration result. As previously stated, this type of component mode has been shown to improve results at the system level.
2.10 Results and Discussions

This chapter presented a discussion of a displacement method of dynamic substructuring known as Fixed-Junction Component-Mode Synthesis, where the normal modes of vibration of a system of structural components were determined from the modes of the individual components. In it, the components were represented by two types of generalized coordinates. Normal mode generalized coordinates were shown to be related to fixed-junction normal modes of vibration. These generalized coordinates represent a truncated set which provides a reduction in the number of component generalized coordinates. Constraint mode generalized coordinates were said to be related to constraint modes which were determined from statics considerations by successive application of unit displacements and rotations at a junction coordinate while all other junction coordinates remain fixed. Following a Rayleigh-Ritz procedure, the component mass, stiffness, and damping matrices were derived. From Lagrange's equations, the equations of motion of the individual components were formulated and synthesized to form the system matrices. It was shown that this yielded the nonclassically damped problem requiring direct integration of coupled differential equations. Recovery of component junction forces by the mode displacement and mode acceleration method was discussed. An improvement to component mode representation through junction loading the component in order to increase accuracy at the system level was presented.

In the remainder of this chapter and throughout this dissertation, results from a Component-Mode Synthesis of two cantilevered beam components as depicted in Figure (2.1) are presented. As with any Rayleigh-Ritz method, the issue of convergence of component and system frequencies as a function of the number of retained fixed-junction component modes is of importance. Furthermore, convergence of displacement, velocity, acceleration, junction shear, and junction moment response time histories are considered.
Consider two cantilevered beam components as shown in Figure (2.1). Each cantilevered beam is discretized via the finite element method to 12 degrees of freedom, each node having 2 translational degrees of freedom and one rotational degree of freedom. The assembled system is a fixed-fixed beam composed of 21 degrees of freedom. Each beam component is 200 inches long with a modulus of elasticity of 10 million psi and a mass density of 0.1 lb sec$^2$/in$^4$. The area moment of inertia for each component is 0.2 in$^4$. The junction nodes for components $\alpha$ and $\beta$ are nodes 4 and 5, respectively. The system is subjected to a 100 lb suddenly applied force at node 1, dof 2 as depicted in Figure (2.1).

Consider the convergence of the cantilevered beam modes as a function of the number of fixed-junction modes retained. Note that the fixed-junction modes in this case are beam modes with fixed-fixed end conditions. The component cantilevered modes are a linear superposition of the component fixed-junction normal modes in Figure (2.4) and constraint modes, shown in Figures (2.2) and (2.3). The component fixed-junction modes have the frequencies of 1.2606, 3.5402, 6.9485, 13.1460, 21.7415, 24.9797, 35.0311, 49.3124, and 69.5338 cps. The sixth, eighth, and ninth fixed-junction component modes are axial mode of vibration. Table 2.1 lists the cantilevered frequencies of the component as the number of retained fixed-junction modes is reduced from the first four to just the fundamental and second modes. It should not be surprising that the first three natural frequencies related to the flexural modes of the cantilevered beam are approximated.
accurately by both the four-mode and the two-mode approximations. This is due to the fact that the first three cantilevered modes of a beam can be approximated quite accurately with the two static constraint modes and one or two flexural fixed-junction normal mode of vibration. In fact, the fundamental mode of vibration for the beam component may be predicted with sufficient accuracy with the static constraint modes alone since the first constraint mode which is the displacements due to a unit translation of the junction coordinate perpendicular to the axis of the beam with zero slope at the junction resembles the first cantilevered beam mode.

A more complex convergence question involves the system modes of vibration of the two cantilevered beam components. Of interest is the convergence of the system modes, in this case the modes of a fixed-fixed beam, as a function of the number of retained fixed-junction component modes. Note that this is the essence of Component-Mode Synthesis where the system modes are derived from the modes of the individual components. Table 2.2 lists the first ten system modes as a function of retaining four and then two fixed-junction normal mode per component.

In order to gain some insight into the concept of synthesis of component modes, consider the fundamental mode of vibration for the system. This is simply the fundamental mode of a fixed-fixed beam at 0.3147 cps. It is intuitively obvious that the component fixed-junction modes do not contribute to the fundamental mode of the fixed-fixed beam since the junction coordinate is a node point (point of zero deflection) in the component fixed-junction modes. It is the component static constraint modes that are playing a critical role in approximating the fundamental mode of the system and to a great extent the lower modes. The effect of the retaining fixed-junction normal modes becomes pronounced as the deflected mode shape of the fixed-fixed beam becomes more complex.

The response of the system to the suddenly applied force is also affected by the Rayleigh-Ritz approximation. Figures (2.5-2.7) show the displacement, velocity, and
acceleration of the point under the force and in the direction of the force (node 1, degree of freedom 2) as a function of the number of retained fixed-junction modes. The components are undamped and all system modes in the system eigenproblem are retained so that the effect of retained component fixed-junction modes on convergence of response is clearly depicted. It is seen that for this case retaining four fixed-junction modes per component corresponds well with the exact solution where the component matrices are assembled in physical coordinates and directly integrated. Hence it may be deduced that although all flexural fixed-junction modes experience an amplification factor of 2.0 due to the suddenly applied force, the flexural fixed-junction modes after the fourth mode participate little in these response quantities. As for the shear at the junction, contribution of the higher modes become more pronounced due to the sudden application of the force as depicted in the convergence characteristics of the four mode solution in Figure (2.8). The four mode solution for the junction moment in Figure (2.9) continues to agree well with the exact solution indicating that the contribution of the higher fixed-junction normal modes in the junction moment calculation is negligible.

The convergence of system frequencies in Static Synthesis, where the components are statically reduced, versus Component-Mode Synthesis is presented in table 2.3. In this analysis, four component modes are retained in Component-Mode Synthesis and all translational degrees of freedom along with the rotational degrees of freedom at the junctions are retained in the Static Synthesis. It is seen that large errors are incurred in Static Synthesis by the eighth mode. A comparison of accuracy and convergence of response in Static Synthesis and Component-Mode Synthesis is given in Figures (2.10-2.14). It is clearly seen that the method of Component-Mode Synthesis has superior convergence to the exact solution.

Figures (2.15-2.19) present results on a convergence study of the nonclassically damped problem solved via the Component-Mode Synthesis approach. In it, each component is damped at 1% of critical with four fixed-junction normal modes retained
per component. All system level normal modes are retained. It is observed that the four mode Component-Mode Synthesis solution is sufficient to ensure convergence for the nonclassically damped problem.
<table>
<thead>
<tr>
<th>Mode # / Frequency (cps)</th>
<th>Exact</th>
<th>4 - Mode Approximation</th>
<th>2 - Mode Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1978</td>
<td>0.1978</td>
<td>0.1979</td>
</tr>
<tr>
<td>2</td>
<td>1.2413</td>
<td>1.2414</td>
<td>1.2423</td>
</tr>
<tr>
<td>3</td>
<td>3.4985</td>
<td>3.4991</td>
<td>3.5065</td>
</tr>
<tr>
<td>4</td>
<td>6.9019</td>
<td>6.9035</td>
<td>13.6768</td>
</tr>
<tr>
<td>5 (axial mode)</td>
<td>12.4993</td>
<td>12.8373</td>
<td>14.2045</td>
</tr>
<tr>
<td>6</td>
<td>12.8372</td>
<td>13.6768</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>20.6167</td>
<td>33.8018</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>32.6843</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 (axial mode)</td>
<td>37.3418</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>53.6280</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11 (axial mode)</td>
<td>60.3403</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12 (axial mode)</td>
<td>75.7541</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.1 - Convergence of Component Cantilevered Frequencies as a Function of Retained Fixed-Junction Normal Modes

<table>
<thead>
<tr>
<th>Mode # / Frequency (cps)</th>
<th>Exact</th>
<th>4 - Mode Approximation</th>
<th>2 - Mode Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3147</td>
<td>0.3147</td>
<td>0.3147</td>
</tr>
<tr>
<td>2</td>
<td>0.8681</td>
<td>0.8681</td>
<td>0.8682</td>
</tr>
<tr>
<td>3</td>
<td>1.7048</td>
<td>1.7050</td>
<td>1.7088</td>
</tr>
<tr>
<td>4</td>
<td>2.8291</td>
<td>2.8292</td>
<td>2.8372</td>
</tr>
<tr>
<td>5</td>
<td>4.2549</td>
<td>4.2571</td>
<td>4.3679</td>
</tr>
<tr>
<td>6</td>
<td>5.9982</td>
<td>6.0007</td>
<td>9.1882</td>
</tr>
<tr>
<td>7</td>
<td>7.9797</td>
<td>7.9968</td>
<td>13.6777</td>
</tr>
<tr>
<td>8</td>
<td>11.2638</td>
<td>11.3276</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>12.4994</td>
<td>13.6768</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>14.3802</td>
<td>14.5155</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.2 - Convergence of System Modes as a Function of Retained Component Fixed-Junction Modes
<table>
<thead>
<tr>
<th>Mode # /Frequency (cps)</th>
<th>Exact</th>
<th>Component-Mode Synthesis</th>
<th>Static Synthesis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3147</td>
<td>0.3147</td>
<td>0.3148</td>
</tr>
<tr>
<td>2</td>
<td>0.8681</td>
<td>0.8681</td>
<td>0.8682</td>
</tr>
<tr>
<td>3</td>
<td>1.7048</td>
<td>1.7050</td>
<td>1.7068</td>
</tr>
<tr>
<td>4</td>
<td>2.8291</td>
<td>2.8292</td>
<td>2.8405</td>
</tr>
<tr>
<td>5</td>
<td>4.2549</td>
<td>4.2571</td>
<td>4.3346</td>
</tr>
<tr>
<td>6</td>
<td>5.9982</td>
<td>6.0007</td>
<td>6.1095</td>
</tr>
<tr>
<td>7</td>
<td>7.9797</td>
<td>7.9968</td>
<td>8.1268</td>
</tr>
<tr>
<td>8</td>
<td>11.2638</td>
<td>11.3276</td>
<td>15.9361</td>
</tr>
</tbody>
</table>

Table 2.3 - Convergence of System Frequencies in Component-Mode Synthesis and Static Synthesis; Four Component Modes Retained in C.M.S., All Translations Retained in S.S.
Component-Mode Synthesis
Fixed-Junction Approach
Constraint Mode

Figure (2.2) - Component Constraint Mode Due to a Unit Displacement at the Junction Coordinate with Zero Slope Enforced
Component-Mode Synthesis
Fixed-Junction Approach
Constraint Mode

Figure (2.3) - Component Constraint Mode Due to Unit Rotation at the Junction Coordinate with Zero Translation Enforced
Component-Mode Synthesis
Fixed-Junction Approach
Fixed-Junction Normal Modes

Figure (2.4) - Component Fixed-Junction Normal Modes
Component-Mode Synthesis Solution
Fixed-Junction Approach
Node 1, DOF 2

Figure (2.5) - Convergence of Displacement vs Number of Retained Fixed-Junction Normal Modes per Component; Step-Pulse Forcing Function
Component-Mode Synthesis Solution
Fixed-Junction Approach
Node 1, DOF 2

Figure (2.6) - Convergence of Velocity vs Number of Retained Fixed-Junction Normal Modes per Component; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Fixed-Junction Approach
Node 1, DOF 2

Figure (2.7) - Convergence of Acceleration vs Number of Retained Fixed-Junction Normal Modes per Component; Step-Pulse Forcing Function
Component-Mode Synthesis Solution
Fixed-Junction Approach
Junction Shear

2 - modes
4 - modes
exact

Figure (2.8)- Convergence of Junction Shear vs Number of Retained Fixed-Junction Normal Modes per Component; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Fixed-Junction Approach
Junction Moment

Figure (2.9) - Convergence of Junction Moment vs Number of Retained Fixed-Junction Normal Modes per Component; Step-Pulse Forcing Function
Component-Mode Synthesis vs Static Synthesis
Node 1, DOF 2

- static synthesis  - 4 - mode cms  - exact

Figure (2.10) - Comparison of Displacements in Fixed-Junction Component-Mode Synthesis and Static Synthesis; Four Component Modes Retained in CMS, Translational DOF Retained in SS; Step-pulse Forcing Function
Figure (2.11) - Comparison of Velocities in Fixed-Junction Component-Mode Synthesis and Static Synthesis; Four Component Modes Retained in CMS, Translational DOF Retained in SS; Step-pulse Forcing Function
Figure (2.12) - Comparison of Accelerations in Fixed-Junction Component-Mode Synthesis and Static Synthesis; Four Component Modes Retained in CMS, Translational DOF Retained in SS; Step-pulse Forcing Function
Component-Mode Synthesis vs Static Synthesis

Junction Shear

- static synthesis  - 4 - mode cms  - exact

Figure (2.13) - Comparison of Junction Shears in Fixed-Junction Component-Mode Synthesis and Static Synthesis; Four Component Modes Retained in CMS, Translational DOF Retained in SS; Step-Pulse Forcing Function
Component-Mode Synthesis vs Static Synthesis
Junction Moment

Figure (2.14) - Comparison of Junction Moments in Fixed-Junction Component-Mode Synthesis and Static Synthesis; Four Component Modes Retained in CMS, Translational DOF Retained in SS; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Nonclassically Damped Problem
Node 1, DOF 2

Figure (2.15) - Convergence of Displacement for Nonclassically Damped Problem; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Nonclassically Damped Problem
Node 1, DOF 2

Figure (2.16) - Convergence of Velocity for Nonclassically Damped Problem; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Component-Mode Synthesis Solution  
Nonclassically Damped Problem  
Node 1, DOF 2

---

Figure (2.17) - Convergence of Acceleration for Nonclassically Damped Problem; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Nonclassically Damped Problem
Junction Shear

---

Figure (2.18) - Convergence of Junction Shear for Nonclassically Damped Problem; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Nonclassically Damped Problem
Junction Moment

Figure (2.19) - Convergence of Junction Moment for Nonclassically Damped Problem; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
CHAPTER 3

RESIDUAL FLEXIBILITY AND ITS APPLICATION TO COMPONENT-MODE SYNTHESIS - THE FREE-JUNCTION COMPONENT REPRESENTATION

3.1 Introductory Remarks

In Chapter 2, the method of Component-Mode Synthesis with the fixed-junction component representation was discussed in some detail. Specifically, the component was represented by a reduced set of generalized coordinates related to its fixed-junction normal modes of vibration and constraint modes. The stiffness at the junction coordinates was accounted for by the constraint modes which was used in determining the junction stiffness matrix commonly used in static substructuring. In this way, the total flexibility associated with the component junction coordinates was preserved.

Consider the use of a truncated set of free-junction normal modes of vibration for component representation. This method of component representation fails to account for the total flexibility at the component junction coordinates and is therefore susceptible to inaccuracies due to weak convergence at the system level. MacNeal (1971) developed a first-order method of accounting for the lost or residual flexibility at the junction coordinates of a component associated with the flexibility of the unretained modes. The method; however, did not gain wide acceptance in the structural dynamics community since it is written in electrical engineering terminology such as admittance and impedance. Rubin (1974) reworked MacNeal's first-order residual flexibility problem in a fashion attractive to the structural dynamist and further proposed a second-order method which in addition to residual flexibility accounts for residual damping and inertia associated with the free-junction normal modes of vibration not retained.
This chapter presents a non Rayleigh-Ritz treatment of Component-Mode Synthesis based on representing the component by a retained set of free-junction normal modes of vibration and a first and second-order account of the residual flexibility of the unretained modes at the junction coordinates. MacNeal's and Rubin's first and second-order methods are applied here to Component-Mode Synthesis, a development that is not presented in either paper. Furthermore, it is noted that the mathematics behind residual flexibility is somewhat the cause of it remaining a novelty in the practical arena of Component-Mode Synthesis. Therefore, it is the purpose of this chapter to expand upon the main ideas of residual flexibility and provide clarification and consistency to the available theory starting with first-order residual flexibility and advancing to the second-order method. The method of Free-Junction Component-Mode Synthesis presented here is considerably more difficult to grasp and implement than the Fixed-Junction method; however, studies by Chang (1977) indicate that for components having many junction coordinates, the convergence of the method is superior to that of Fixed-Junction. Note that the topics in this chapter form the basic tools necessary for the development of the new method of linear and nonlinear analysis for nonclassically damped systems of combined components presented in Chapter 6.

3.2 A Simplified Explanation of Residual Flexibility

This section serves to familiarize the reader with the concept of residual flexibility before a rigorous treatment is presented in later sections. In order to accomplish this, consider a multi-degree of freedom system subjected to a suddenly applied rectangular load. The vector of displacements at any time is given by
\[
\mathbf{u} = \sum_{k=1}^{N} \left( \frac{\mathbf{\Phi}_k^T \mathbf{P} \mathbf{F}_0}{\omega_k^2 \mathbf{\Phi}_k^T \mathbf{M} \mathbf{\Phi}_k} \right) \mathbf{\Phi}_k \left(1 - \cos \omega_k t\right),
\]

(3.2.1)

where \( \omega_k \) and \( \mathbf{\Phi}_k \) are the kth circular natural frequency and eigenvector, respectively, \( \mathbf{P} \) is the load locator matrix, \( \mathbf{F}_0 \) is the magnitude of the load vector, \( \mathbf{M} \) is the component mass matrix, and the index \( N \) represents the total number of degrees of freedom in the structural system considered. Equation (3.2.1) is the exact solution mode displacement solution to the problem since all modes are retained.

The series in Equation (3.2.1) may be alternatively expressed in terms of a series up to mode \( n \) and a second one from mode \( n+1 \) to \( N \) given by

\[
\mathbf{u} = \sum_{k=1}^{n} \left( \frac{\mathbf{\Phi}_k^T \mathbf{P} \mathbf{F}_0}{\omega_k^2 \mathbf{\Phi}_k^T \mathbf{M} \mathbf{\Phi}_k} \right) \mathbf{\Phi}_k \left(1 - \cos \omega_k t\right) + \sum_{k=n+1}^{N} \left( \frac{\mathbf{\Phi}_k^T \mathbf{P} \mathbf{F}_0}{\omega_k^2 \mathbf{\Phi}_k^T \mathbf{M} \mathbf{\Phi}_k} \right) \mathbf{\Phi}_k \left(1 - \cos \omega_k t\right).
\]

(3.2.2)

Note that in practice, the first series is retained and the second series is neglected altogether. However, in doing so, not only the dynamic but also the static contribution of the second series to the displacements is lost. In other words, even if the load was statically applied, taking into consideration only the first series would not yield the correct solution. The residual displacement vector needed for a statically complete solution is given by

\[
\mathbf{u}_r = \sum_{k=n+1}^{N} \left( \frac{\mathbf{\Phi}_k^T \mathbf{P} \mathbf{F}_0}{\omega_k^2 \mathbf{\Phi}_k^T \mathbf{M} \mathbf{\Phi}_k} \right) \mathbf{\Phi}_k
\]

(3.2.3)
where the subscript \( r \) stands for residual. Equation (3.2.3) requires the explicit determination of the higher modes in order to evaluate the residual displacements. In the following section, first-order residual flexibility is derived in a rigorous fashion without the need for explicit evaluation of the higher modes.

### 3.3 Determination of First-Order Residual Flexibility

Consider the equations of dynamic equilibrium for a component in terms of its junction and interior coordinates

\[
\begin{bmatrix}
M_{jj} & M_{ja} \\
M_{aj} & M_{aa}
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_j \\
\ddot{u}_a
\end{bmatrix}
+ \begin{bmatrix}
C_{jj} & C_{ja} \\
C_{aj} & C_{aa}
\end{bmatrix}
\begin{bmatrix}
\dot{u}_j \\
\dot{u}_a
\end{bmatrix}
+ \begin{bmatrix}
K_{jj} & K_{ja} \\
K_{aj} & K_{aa}
\end{bmatrix}
\begin{bmatrix}
u_j \\
u_a
\end{bmatrix} = \begin{bmatrix} F_j \\
0 \end{bmatrix},
\]  
(3.3.1)

where the forces acting on the interior coordinates are ignored without a loss in generality. The eigenvalue problem for Equation (3.3.1) may be expressed in compact notation by

\[
(K - \omega_k^2 M) \Phi_k = 0.
\]  
(3.3.2)

The mode displacement solution for Equation (3.3.1) in terms of retained and unretained free-junction modes may be expressed in terms of retained and unretained or residual generalized coordinates by

\[
u = \Phi_n q_n + \Phi_r q_r
\]  
(3.3.3)
where the subscript \( n \) denotes the retained and \( r \) denotes unretained or residual. The objective is to express the static contribution of the unretained modes to the displacements without having to actually evaluate these modes. In order to accomplish this task, consider the first-order approximation of the dynamic equations of equilibrium given in Equation (3.3.1) which is accomplished by neglecting the inertia and the damping terms given by

\[
K u^{(1)}_E = F, \tag{3.3.4}
\]

where the superscript \((1)\) on the displacement vector denotes first-order approximation and the subscript \( E \) denotes elastic displacements meaning that the rigid-body has been removed. This notion requires clarification and is discussed later in this section. For the general case of the unconstrained component possessing anywhere up to six rigid-body modes, the evaluation of displacements from the above equation is not possible since the flexibility matrix does not exist. However, note that if the inertial forces associated with the rigid-body accelerations are subtracted from the external force vector, the resulting inertial-relief forces can not produce rigid body motion. Motivated by this, let us begin by expressing the displacements in terms of a set of rigid-body and elastic modes given by

\[
u = \Phi_R q_R + \Phi_E q_E, \tag{3.3.4}\]

where the subscript \( R \) denotes rigid-body and \( E \), as defined previously, denotes elastic-body. Substituting Equation (3.3.4) into the equation of dynamic equilibrium (3.3.1), the rigid-body accelerations may be expressed as
\[ \ddot{q}_R = \mathbf{m}_R^{-1} \Phi_R^\top \mathbf{F}, \]  

(3.3.5)

where the rigid-body mass matrix is given

\[ \mathbf{m}_R = \Phi_R^\top \mathbf{M} \Phi_R. \]  

(3.3.6)

Subtracting from the external forces the inertial forces which result from rigid-body motion, the resulting inertia-relief load vector is given by

\[ \mathbf{F} - \mathbf{M} \ddot{u}_R = \mathbf{F} - \mathbf{M} \Phi_R \ddot{q}_R. \]  

(3.3.7)

Substituting Equation (3.3.5) into Equation (3.3.7), the following relationship results

\[ \left( \mathbf{I} - \mathbf{M} \Phi_R \mathbf{m}_R \Phi_R^\top \right) \mathbf{F} = \mathbf{A} \mathbf{F}, \]  

(3.3.8)

where \( \mathbf{A} \) is referred to in literature as the load projection matrix.

An important conceptual point arises at this juncture. The inertia relief loads are a self-equilibrating load system. Therefore, the imposition of any statically determinate set of constraints on the component stiffness matrix will produce no reactions. Taking full advantage of this fact, a set of statically determinate constraints is imposed on the component stiffness in Equation (3.3.4) which is then inverted to yield

\[ \mathbf{u}_c^{(1)} = \mathbf{G}_c \mathbf{A} \mathbf{F}, \]  

(3.3.9)
where the subscript c denotes that the displacements are with respect to the set of imposed constraints. The elastic displacements differ from the displacements given in Equation (3.3.9) by a rigid-body contribution. The rigid-body contribution may be removed from the displacements in Equation (3.3.9) by making the elastic displacements orthogonal to the set of rigid-body modes. This is accomplished by imposing the following condition on the elastic-body displacements

$$\Phi_R^\top M u_E^{(1)} = 0.$$  \hspace{1cm} (3.3.10)

The first-order elastic displacements may be expressed in terms of the constrained displacements and a certain rigid-body contribution to the displacements given by

$$u_E^{(1)} = u_c^{(1)} + \Phi_R Q_R.$$  \hspace{1cm} (3.3.11)

where the generalized rigid-body displacement vector, $Q_R$, is to be determined via condition in Equation (3.3.10). Substituting Equation (3.3.11) into Equation (3.3.10) and taking advantage of orthogonality condition in Equation (3.3.6) yields

$$Q_R = -m_R \Phi_R^\top M u_c^{(1)}.$$  \hspace{1cm} (3.3.12)

The above generalized rigid-body displacements are now substituted into Equation (3.3.11) which results in

$$u_E^{(1)} = \left( I - \Phi_R m_R^{-1} \Phi_R^\top M \right) u_c^{(1)} = A^\top u_c^{(1)}.$$  \hspace{1cm} (3.3.13)

From Equation (3.3.9), the elastic displacements may be determined by
\[ \mathbf{u}_E^{(i)} = (A^T G_c A) \mathbf{F} = G \mathbf{F}, \quad (3.3.14) \]

where the matrix \( G \) represents the components total flexibility.

Returning to the problem of computing first-order residual flexibility for the component, let us begin by substituting Equation (3.3.3) into the equations of dynamic equilibrium for the component, Equation (3.3.1). Ignoring the 'dynamics' of the unretained modes of vibration, the following equations result

\[ m_n \ddot{q}_n + c_n \dot{q}_n + k_n q_n = \Phi_n^T \mathbf{F}, \quad (3.3.15) \]

and

\[ k_r q_r = \Phi_r^T \mathbf{F}, \quad (3.3.16) \]

where \( \Phi_p \) denotes the junction rows of the modal matrix and

\[ m_n = \Phi_n^T \mathbf{M} \Phi_n, \quad (3.3.17) \]

\[ c_n = \Phi_n^T \mathbf{C} \Phi_n, \quad (3.3.18) \]

\[ k_n = \Phi_n^T \mathbf{K} \Phi_n, \quad (3.3.19) \]

\[ k_r = \Phi_r^T \mathbf{K} \Phi_r. \quad (3.3.20) \]

From Equation (3.3.16), the residual displacements expressed in terms of the unretained modes of vibration are given by
\( \mathbf{u}_r^{(1)} = (\Phi_r \mathbf{k}_r^{-1} \Phi_r^T) \mathbf{F} = \mathbf{G}_r \mathbf{F}, \)  
(3.3.21)

where \( \mathbf{G}_r \) denotes the first-order residual flexibility of the component. By the same token, the first-order retained displacements may be determined by ignoring the dynamics in Equation (3.3.15) given by

\( \mathbf{u}_n^{(1)} = \mathbf{G}_n \mathbf{F} = (\Phi_n \mathbf{k}_n^{-1} \Phi_n^T) \mathbf{F}, \)  
(3.3.22)

where \( \mathbf{G}_n \) is the retained flexibility matrix. From the solution of the special statics problem, the total flexibility as determined by Equation (3.3.14) must be the sum of the retained flexibility in Equation (2.3.22) and the residual flexibility in Equation (3.3.21). Mathematically, this is expressed as

\[ \mathbf{G} = \mathbf{G}_n + \mathbf{G}_r. \]  
(3.3.23)

From Equation (2.3.23), the first-order residual displacement vector is given by

\[ \mathbf{u}_r^{(1)} = \mathbf{G}_r \mathbf{F} = (\mathbf{G} - \mathbf{G}_n) \mathbf{F}, \]  
(3.3.24)

which does not require the explicit evaluation of the unretained modes of vibration. From the above considerations, the displacements of the component in terms of the retained free-junction modes of vibration and first-order residual flexibility may be expressed as

\[ \mathbf{u} = \Phi_n \mathbf{q}_n + \mathbf{G}_r \mathbf{F}. \]  
(3.3.25)
3.4 Component Representation with First-Order Residual Flexibility

The developments in section (3.3) give rise to the following equations for component representation

\[(m_n p^2 + c_n \dot{p} + k_n)q_n = \Phi_{jn}^\top F_j,\] (3.4.1)

and

\[u_j^{(1)} = \Phi_{jn} q_n + G_{jj,r} F_j.\] (3.4.2)

Equation (3.4.1) is Equation (3.3.15) rewritten taking advantage of the derivative operator notation \(p\) and the fact that only forces at the junction coordinates are present. Equation (2.4.2) is the mode displacement recovery of the junction displacements using free-junction normal modes of vibration and a first-order account of the residual flexibility at the junction coordinates where the residual flexibility is given by Equation (3.3.24). Solving Equation (3.4.2) for the junction forces yields

\[F_j = G_{jj,r}^{-1}(u_j - \Phi_{jn} q_n).\] (3.4.3)

Substituting Equation (3.4.3) into Equation (3.4.1) gives

\[(m_n p^2 + c_n \dot{p} + k_n)q_n = \Phi_{jn}^\top G_{jj,r}^{-1}(u_j - \Phi_{jn} q_n).\] (3.4.4)

Equations (3.4.3) and (3.4.4) may now be rearranged to give the following equation for the component representation
\[
\begin{bmatrix}
G_{jj,r}^{-1} & -G_{jj,r}^{-1}\Phi_{jn} \\
-\Phi_{jn}^\tau G_{jj,r}^{-1} & m_n p^2 + c_n p + k_n + \Phi_{jn}^\tau G_{jj,r}^{-1}\Phi_{jn}
\end{bmatrix}
\begin{bmatrix}
u_j \\
q_n
\end{bmatrix}
= \begin{bmatrix}
F_j \\
0
\end{bmatrix}.
\]
(3.4.5)

Equation (3.4.5) is the first-order residual flexibility representation of the component's equations of dynamic equilibrium as given in Equation (3.3.1). Note that the mass and damping matrices are diagonal, assuming classical damping at the component level, and the stiffness matrix is fully populated and symmetric.

### 3.5 Component-Mode Synthesis via Direct Stiffness Method

In this section, the direct stiffness method will be used to couple or synthesize the first-order representation of two components \(\alpha\) and \(\beta\). The objective here is the same as in section (2.4) for fixed-junction components, to derive the system equations of motion. However, for simplification, a direct stiffness method is adopted here instead of the formalized synthesis procedures presented in section (2.4). The first-order residual flexibility equations for dynamic equilibrium of component \(\alpha\) are given by Equation (3.4.5) restated here as

\[
\begin{bmatrix}
G_{jj,r}^{-1} & -G_{jj,r}^{-1}\Phi_{jn} \\
-\Phi_{jn}^\tau G_{jj,r}^{-1} & m_n p^2 + c_n p + k_n + \Phi_{jn}^\tau G_{jj,r}^{-1}\Phi_{jn}
\end{bmatrix}
\begin{bmatrix}
u_j^\alpha \\
q_n^\alpha
\end{bmatrix}
= \begin{bmatrix}
F_j^\alpha \\
0
\end{bmatrix}.
\]
(3.5.1)

In similar fashion, the equations for component \(\beta\) are expressed by
\[
\begin{bmatrix}
G_{jj,r}^{-1} & -G_{jj,r}^{-1}\Phi_{jn} \\
-\Phi_{jn}G_{jj,r}^{-1} & m_n p^2 + c_n p + k_n + \Phi_{jn}^\top G_{jj,r}^{-1}\Phi_{jn}
\end{bmatrix} \begin{bmatrix}
q_n \\
u_j
\end{bmatrix}^\beta = \begin{bmatrix}
F_j \\
0
\end{bmatrix}^\beta.
\] (3.5.2)

The requirement of displacement compatibility and force equilibrium at the junction coordinates is given as

\[
u_j^\alpha = u_j^\beta = u_j,
\] (3.5.3)

and

\[
F_j^\alpha = -F_j^\beta = F_j.
\] (3.5.4)

From Equations (3.5.3) and (3.5.4), the first-order equations for the coupled system of differential equations may be expressed as

\[
\begin{bmatrix}
k_{\alpha \alpha} & 0 & k_{\alpha \beta} \\
0 & k_{\beta \beta} & 0 \\
k_{\beta \alpha} & k_{\beta \beta} & k_{\alpha \alpha + \beta}
\end{bmatrix}
\begin{bmatrix}
q_n^\alpha \\
q_n^\beta \\
u_j
\end{bmatrix} = 0,
\] (3.5.5)

where

\[
k_{\alpha \alpha} = m_n \alpha p^2 + c_n \alpha p + k_n \alpha + \Phi_{jn}^\top G_{jj,r}^{-1}\Phi_{jn},
\] (3.5.6)

\[
k_{\alpha \beta} = m_n \alpha p^2 + c_n \beta p + k_n \beta + \Phi_{jn}^\top G_{jj,r}^{-1}\Phi_{jn},
\] (3.5.7)

\[
k_{\beta \beta} = m_n \beta p^2 + c_n \beta p + k_n \beta + \Phi_{jn}^\top G_{jj,r}^{-1}\Phi_{jn}.
\]
\[ k'_{nj} = k'_{jn} = -\Phi_{jn} G_{jj, r}^{\alpha^{-1}} \]  
\[ k'_{nj} = k'_{jn} = -\Phi_{jn} G_{jj, r}^{\beta^{-1}} \]  
\[ k'_{nj} = k'_{jn} = G_{jj, r}^{\alpha^{-1}} + G_{jj, r}^{\beta^{-1}} \]  

It is apparent from Equation (3.5.5) that the system size is equal to the total number of retained modes of the components plus the number of junction degrees of freedom. A further reduction in coordinates may be obtained by solving the system level eigenproblem for a specified number of modes. This is discussed in detail in section (2.5).

3.6 Determination of Second-Order Residual Flexibility

Section (3.3) presented a detailed derivation of first-order residual flexibility. In it, the static contribution of the unretained modes of vibrations to the displacements were determined without explicitly solving for these modes. A second-order approximation of this static contribution is possible by considering Equation (3.3.1) and the first-order displacements in Equation (3.3.14). Placing the inertial and the damping forces on the right hand side, Equation (3.3.1) may be rewritten as

\[ Ku_E^{(2)} = F - \left( Cp + Mp^2 \right) u_E^{(1)}, \]
where the second-order approximation of the displacements is expressed in terms of the first-order approximation. Substituting Equation (3.3.14) rearranging and solving for the second-order displacements yields

$$\mathbf{u}_E^{(2)} = G \left[ \mathbf{I} - (C_p + M p^2) G \right] \mathbf{F}.$$  \hfill (3.6.2)

The second-order residual displacements are determined from Equation (3.6.2) by comparison of Equations (3.3.14) and (3.3.24) and are given by

$$\mathbf{u}_r^{(2)} = G_r \left[ \mathbf{I} - (C_p + M p^2) G \right] \mathbf{F},$$  \hfill (3.6.3)

where the residual flexibility matrix is given by Equation (3.3.24). Substituting Equation (3.3.23) into Equation (3.6.3) and expanding the resultant equation gives

$$\mathbf{u}_r^{(2)} = G_r \left[ \mathbf{I} - (C_p + M p^2) \left( \mathbf{G}_n + G_r \right) \right] \mathbf{F}
= \left[ G_r - \left( G_r G_{n p} + G_r G_{r p} + G_r M G_{n p^2} + G_r M G_r p^2 \right) \right] \mathbf{F}
= \left[ G_r - \left( G_r G_{r p} + G_r M G_r p^2 \right) \right] \mathbf{F}
= \left( G_r - \mathbf{B}_r p - \mathbf{H}_r p^2 \right) \mathbf{F},$$  \hfill (3.6.4)

where \( \mathbf{B}_r \) is the residual damping matrix and \( \mathbf{H}_r \) is the residual inertia matrix. Equation (3.6.4) took advantage of the following orthogonality relationships involving the residual and retained flexibility matrices

$$G_r G_{n p} = 0,$$  \hfill (3.6.5)
and

$$G_r M G_n = 0,$$  (3.6.6)

where the assumption of classical damping at the component level is required in order to derive Equation (3.6.5). The displacements may now be expressed in terms of the free-junction normal modes and a second order account of the residual flexibility of the unretained modes by

$$u = \Phi_n q_n + (G_r - B_r p - H_r p^2) F.$$  (3.6.7)

### 3.7 Component Representation with Second-Order Residual Flexibility

In order to proceed with the component representation, Equation (3.6.7) is restated in terms of the component junction coordinates as

$$u_j = \Phi_{jn} q_n + (G_{jj, r} - B_{jj, r} p - H_{jj, r} p^2) F_j.$$  (3.7.1)

Solving Equation (3.7.1) for the junction forces yields

$$F_j = (G_{jj, r} - B_{jj, r} p - H_{jj, r} p^2)^{-1}(u_j - \Phi_{jn} q_n).$$  (3.7.2)

The next series of equations involve algebraic manipulations to Equation (3.7.2) in order to eliminate the inverse operation. This is accomplished through a MacLaurin series of $(1 - x)^{-1}$ keeping all second order terms. The algebra involved is as follows
\[ (G_{jj,r^2} - B_{jj,r} p - H_{jj,r} p^2)^{-1} \]

\[ = [G_{jj,r} (I - G_{jj,r} r B_{jj,r} p - G_{jj,r} r H_{jj,r} p^2)]^{-1} \]

\[ = [I - (G_{jj,r} r B_{jj,r} p + G_{jj,r} r H_{jj,r} p^2)]^{-1} G_{jj,r}^{-1} \]

\[ = [G_{jj,r}^{-1} + G_{jj,r}^{-1} B_{jj,r} G_{jj,r}^{-1} p + (G_{jj,r}^{-1} H_{jj,r} G_{jj,r}^{-1} + G_{jj,r}^{-1} B_{jj,r} G_{jj,r}^{-1}) p^2] \]

\[ = \hat{k}_{jj,r} + \hat{c}_{jj,r} p + \hat{m}_{jj,r} p^2. \]

Restating Equation (3.7.2) for the computation of the junction forces and incorporating the above result

\[ F_j = (\hat{k}_{jj,r} + \hat{c}_{jj,r} p + \hat{m}_{jj,r} p^2)(u_j - \Phi_{jn} q_n), \quad (3.7.4) \]

where the second-order residual mass, stiffness, and damping terms in the above equation are defined in Equation (3.7.3). Substituting the second-order expression for junction forces into Equation (3.4.1) results

\[ \left( m_n p^2 + c_n p + k_n \right) q_n = \Phi_{jn}^T (\hat{k}_{jj,r} + \hat{c}_{jj,r} p + \hat{m}_{jj,r} p^2)(u_j - \Phi_{jn} q_n). \quad (3.7.5) \]

From Equations (3.7.4) and (3.7.5), the component second-order equations of dynamic equilibrium may be expressed as
\[
\begin{bmatrix}
Z_n + \Phi_{jn} \Phi_{jn}^T Z_{jj,r} + \Phi_{jn} Z_{jj,r}^T \\
Z_{jj,r} \Phi_{jn}
\end{bmatrix}
\begin{bmatrix}
q_n \\
u_j
\end{bmatrix}
= \begin{bmatrix}
0 \\
F_j
\end{bmatrix},
\] (3.7.6)

where

\[
Z_n = (m_n p^2 + c_n p + k_n)
\] (3.7.7)

and

\[
Z_{jj,r} = (\bar{m}_{jj,r} p^2 + \bar{c}_{jj,r} p + \bar{k}_{jj,r})
\] (3.7.8)

It is seen from Equation (3.7.6) that the second-order representation of the component reduces to the first-order when the residual damping and mass terms in Equation (3.7.8) are ignored.

3.8 Recovery of Junction Forces Using Component Generalized Displacements

In Chapter 2, section (2.7), the equations for recovery of junction forces via equilibrium considerations as related to the fixed-junction method of Component-Mode Synthesis were derived. In it, the component generalized accelerations, velocities, and displacements at the junction coordinates were involved. In the residual flexibility approach, it is also possible to solve for the component junction forces via component equilibrium considerations; however, it is not necessary.

A unique approach to solving for the junction forces involves junction compatibility considerations. The first-order junction displacements for components \(\alpha\) and \(\beta\) are given by
\[ u_j^\alpha = \Phi_{jn}^\alpha q_n^\alpha + G_{jj,r}^\alpha F_j \]  

(3.8.1)

and

\[ u_j^\beta = \Phi_{jn}^\beta q_n^\beta - G_{jj,r}^\beta F_j. \]  

(3.8.2)

From the requirement of compatibility of displacements at the junction coordinates,

\[ F_j = (G_{jj,r}^\alpha + G_{jj,r}^\beta)^{-1}(\Phi_{jn}^\beta q_n^\beta - \Phi_{jn}^\alpha q_n^\alpha), \]  

(3.8.3)

where the component generalized displacements are determined from integration of the synthesized component equations, either in system modal coordinates of combined system generalized coordinates. Equation (3.8.3) provides a convenient way of solving for the junction forces directly from the combined component first-order residual flexibilities and the generalized component displacements.

3.9 Results and Discussions

In this chapter, a non Rayleigh-Ritz approach to representing a component by a retained set of free-junction normal modes of vibration augmented by a first-order and subsequently second-order account of the residual flexibility was presented. It was noted that this type of Component-Mode Synthesis requires the addition of the residual flexibility to the component junction coordinates in order to retain the full flexibility of these coordinates necessary for accurate computation of system response. The equations of the component with both first and second-order residual flexibility added to the
junction coordinates were derived. Furthermore, Component-Mode Synthesis using the residual flexibility approach was discussed.

In this section, the convergence of component and system frequencies as a function of the number of retained free-junction component modes, the first two of which are shown in Figure (3.1), will be studied for a system composed of two cantilevered beam components. Furthermore, the residual flexibility approach to Component-Mode Synthesis will be compared to the traditional fixed-junction approach for the nonclassically damped problem.

Consider the cantilevered beam components shown in Figure (2.1), section (2.10). The objective here is to ascertain the effect of the number of retained cantilevered beam modes on component cantilevered frequencies. The component is first represented by a four-mode approximation and subsequently a two-mode approximation as was the case in table 2.1 for the fixed-junction component. The exact component frequencies are computed via an eigenproblem of the component mass and stiffness matrices (12x12) in physical degrees of freedom. Table 3.1 presents these results with cantilevered frequencies from the fixed-junction approach from table 2.1 in parenthesis.

The results from table 3.1 are easily defended. In the four-mode approximation, the first four cantilevered beam modes are used in the free-junction modal representation. However, these free-junction modes are identical to the first four cantilevered modes of the component resulting in frequencies that match the exact values. The component modes beyond the fourth mode have to be predicted as a linear combination of the first four free-junction modes and a static residual flexibility vectors. These modes will be harder to approximate as evident in the table. The same argument holds for the two-mode approximation.

Next, the convergence of the combined system frequencies (for the first ten system modes) as a function of the retained free-junction component modes is addressed. The results from fixed-junction Component-Mode Synthesis from table 2.2 are restated in
parenthesis. It is seen that the four-mode approximation does quite well up to the eighth system frequency. The two-mode approximation holds well up to the fourth system frequency with the fixed-junction approximation proving more accurate in the fifth system frequency.

Consider the convergence of response time histories for the system of cantilevered beams subjected to a suddenly applied force. Figures (3.2-3.6) present the displacement, velocity, acceleration, junction shear and junction moment for the system solutions involving two free-junction modes per component and four free-junction modes per component. It is seen that the four-mode solution is in excellent agreement with the exact solution for all response items other than the junction shear due to the greater participation of higher modes of vibration.

As a further study, the convergence of response for the nonclassically damped problem comparing Free-Junction Component-Mode Synthesis with residual flexibility to Fixed-Junction Component-Mode Synthesis is presented in Figures (3.7-3.11). Each component is damped at 1% of critical and all system modes are retained. The results are in close agreement, indicating both methods of Component-Mode Synthesis perform equally well in this example.
<table>
<thead>
<tr>
<th>Mode # / Frequency (cps)</th>
<th>Exact</th>
<th>4 - Mode Approximation</th>
<th>2 - Mode Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1978</td>
<td>0.1978 (0.1978)</td>
<td>0.1978</td>
</tr>
<tr>
<td>2</td>
<td>1.2413</td>
<td>1.2413 (1.2414)</td>
<td>1.2413 (1.2423)</td>
</tr>
<tr>
<td>3</td>
<td>3.4985</td>
<td>3.4985 (3.4991)</td>
<td>3.7612 (3.5065)</td>
</tr>
<tr>
<td>4</td>
<td>6.9019</td>
<td>6.9019 (6.9035)</td>
<td>13.6768 (13.6768)</td>
</tr>
<tr>
<td>6</td>
<td>12.8372</td>
<td>13.7358 (13.6768)</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>20.6167</td>
<td>41.7557 (33.8018)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>32.6843</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 (axial mode)</td>
<td>37.3418</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>53.6280</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11 (axial mode)</td>
<td>60.3403</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12 (axial mode)</td>
<td>75.7541</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1 - Convergence of Component Frequencies as a Function of Retained Component Free-Junction Modes; Frequencies from Table 2.1 in parenthesis

<table>
<thead>
<tr>
<th>Mode # / Frequency (cps)</th>
<th>Exact</th>
<th>4 - Mode Approximation</th>
<th>2 - Mode Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3147</td>
<td>0.3147 (0.3147)</td>
<td>0.3147 (0.3147)</td>
</tr>
<tr>
<td>2</td>
<td>0.8681</td>
<td>0.8681 (0.8681)</td>
<td>0.8681 (0.8682)</td>
</tr>
<tr>
<td>3</td>
<td>1.7048</td>
<td>1.7048 (1.7050)</td>
<td>1.7057 (1.7088)</td>
</tr>
<tr>
<td>4</td>
<td>2.8291</td>
<td>2.8291 (2.8292)</td>
<td>2.8913 (2.8372)</td>
</tr>
<tr>
<td>5</td>
<td>4.2549</td>
<td>4.2549 (4.2571)</td>
<td>5.0346 (4.3679)</td>
</tr>
<tr>
<td>6</td>
<td>5.9982</td>
<td>5.9984 (6.0007)</td>
<td>12.4719 (9.1882)</td>
</tr>
<tr>
<td>7</td>
<td>7.9797</td>
<td>7.9832 (7.9968)</td>
<td>13.6768 (13.6777)</td>
</tr>
<tr>
<td>8</td>
<td>11.2638</td>
<td>11.5447 (11.3276)</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>12.4994</td>
<td>13.6768 (13.6768)</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.2 - Convergence of System Frequencies as a Function of Retained Component Free-Junction Modes; Frequencies from Table 2.2 in parenthesis
Component-Mode Synthesis
Free-Junction Approach
Free-Junction Normal Modes

Figure (3.1) - Component Free-Junction Normal Modes
Component-Mode Synthesis Solution
Residual Flexibility Approach
Node 1, DOF 2

Figure (3.2) - Convergence of Displacement vs Number of Retained Free-Junction Normal Modes per Component; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Residual Flexibility Approach
Node 1, DOF 2

Figure (3.3) - Convergence of Velocity vs Number of Retained Free-Junction Normal Modes per Component; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Residual Flexibility Approach
Node 1, DOF 2

Figure (3.4) - Convergence of Acceleration vs Number of Retained Free-Junction Normal Modes per Component; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Residual Flexibility Approach
Junction Shear

Figure (3.5) - Convergence of Junction Shear vs Number of Retained Free-
Junction Normal Modes per Component; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Residual Flexibility Approach
Junction Moment

Figure (3.6) - Convergence of Junction Moment vs Number of Retained Free-Junction Normal Modes per Component; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Nonclassically Damped Problem
Node 1, DOF 2

- - - - free-junction cms  - - - - fixed-junction cms  --- exact

Figure (3.7) - Comparison of Displacements in Free-Junction Component-Mode Synthesis and Fixed-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Nonclassically Damped Problem
Node 1, DOF 2

Figure (3.8) - Comparison of Velocities in Free-Junction Component-Mode Synthesis and Fixed-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Nonclassically Damped Problem
Node 1, DOF 2

Figure (3.9) - Comparison of Accelerations in Free-Junction Component-Mode Synthesis and Fixed-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Nonclassically Damped Problem
Junction Shear

Figure (3.10) - Comparison of Junction Shears in Free-Junction Component-Mode Synthesis and Fixed-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Nonclassically Damped Problem
Junction Moment

Figure (3.11) - Comparison of Junction Moments in Free-Junction Component-Mode Synthesis and Fixed-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
CHAPTER 4

A HYBRID METHOD OF COMPONENT-MODE SYNTHESIS WITH SECOND-ORDER RESIDUAL FLEXIBILITY

4.1 Introductory Remarks

Chapter 3 presented a comprehensive discussion on first and second-order residual flexibility and their application to Free-Junction Component-Mode Synthesis. It was shown that unlike Hurty's Fixed-Junction Component-Mode Synthesis, Rubin's residual flexibility approach to synthesis does not follow a general Rayleigh-Ritz procedure. The component representation was formulated following extensive mathematical considerations. It is however possible to facilitate the application of residual flexibility methods to Component-Mode Synthesis through Rayleigh-Ritz considerations. A literature survey on Rayleigh-Ritz methods in residual flexibility Component-Mode Synthesis indicate that primarily Craig and Chang (1977) conducted active research in this area.

The sections in this chapter present a hybrid method of Rayleigh-Ritz Component-Mode Synthesis that incorporates Rubin's second-order residual correction. The term hybrid refers to using a mixed set of coordinates for component representation; namely, free-junction normal mode generalized coordinates and junction force generalized coordinates. The use of junction forces as generalized coordinates leads to some interesting simplifications producing a synthesized system of components with similar properties to the second-order residual system presented in the prior chapter. In this manner, this chapter attempts to provide a rational and consistent alternative to Rubin's existing theory on non Rayleigh-Ritz residual flexibility approach to Component-Mode
Synthesis. Since all the mathematics of residual flexibility has been developed in Chapter 3, formulations presented in this chapter will be rather brief. Numerical studies will be conducted comparing the convergence of this Rayleigh-Ritz method of Component-Mode Synthesis to Rubin's second-order method.

4.2 Component Representation in Hybrid Coordinates

The first-order displacements of a component may be expressed in terms of a retained set of free-junction modes and a first-order residual flexibility account of the unretained modes by

\[ u = \Phi_n q_n + G_r F. \]  \hspace{1cm} (4.2.1)

Partitioning the above expression in terms of interior and junction displacements results in

\[ \begin{bmatrix} u_a \\ u_j \end{bmatrix} = \begin{bmatrix} \Phi_{an} & G_{aj,r} \\ \Phi_{jn} & G_{jj,r} \end{bmatrix} \begin{bmatrix} q_n \\ F_j \end{bmatrix}, \]  \hspace{1cm} (4.2.2)

where the transformation to the mixed coordinates in terms of free-junction normal modes and first-order residual flexibility is established. In the above Ritz transformation, it can be rigorously shown that the columns of the residual flexibility matrix are linearly independent from the free-junction normal modes of vibration and therefore form an appropriate basis for the Ritz transformation.

The kinetic energy of the component is expressed in terms of the physical coordinates as
\[ T = \frac{1}{2} \begin{bmatrix} \ddot{u}_a & \ddot{u}_j \end{bmatrix} \begin{bmatrix} M_{aa} & M_{aj} \\ M_{ja} & M_{jj} \end{bmatrix} \begin{bmatrix} \ddot{u}_a \\ \ddot{u}_j \end{bmatrix}. \]  

(4.2.3)

Substituting the Equation (4.2.2) in the above expression yields

\[ T = \frac{1}{2} \begin{bmatrix} \ddot{q}_n & \ddot{F}_j \end{bmatrix} \begin{bmatrix} \Phi_n^\top \\ G_{j,r} \end{bmatrix} M \begin{bmatrix} \Phi_n & G_{j,r} \end{bmatrix} \begin{bmatrix} \ddot{q}_n \\ \ddot{F}_j \end{bmatrix}. \]  

(4.2.4)

Simplifying the inner triple product by taking advantage of orthogonality conditions between retained normal modes and residual flexibility modes, the component second-order mass matrix becomes

\[ \hat{m} = \begin{bmatrix} m_n & 0 \\ 0 & H_{jj,r} \end{bmatrix}, \]  

(4.2.5)

where the diagonal generalized mass submatrix is given as before by

\[ m_n = \Phi_n^\top M \Phi_n. \]  

(4.2.6)

and lower right submatrix is Rubin's fully-populated, symmetric second-order residual inertia given as before by

\[ H_{jj,r} = G_{j,r}^\top M G_{j,r}. \]  

(4.2.7)
It may be instructive here to prove that the off-diagonal submatrices in Equation (4.2.5) are null matrices. The typical off-diagonal submatrix has the form

\[ \mathbf{m}_{nj} = \Phi_n^T \mathbf{M}_j \mathbf{G}_{j,r} \]  

(4.2.8)

Expressing the first-order residual flexibility in terms of the unretained modes in the above expression yields

\[ \mathbf{m}_{nj} = \Phi_n^T \mathbf{M} \Phi_r \mathbf{k}_{r,j,\mathbf{r}}^{-1} \Phi_{j,r}^T \]  

(4.2.9)

which results in a null matrix by the virtue of the orthogonality of normal modes given by

\[ \Phi_n^T \mathbf{M} \Phi_r = 0. \]  

(4.2.10)

Following along the same lines, the component hybrid stiffness and hybrid damping matrices, assuming classical damping at the component level, may be derived from potential energy and Rayleigh dissipative energy considerations, respectively. These hybrid matrices are given by

\[ \hat{\mathbf{k}} = \begin{bmatrix} \mathbf{k}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_{jj,r} \end{bmatrix} \]  

(4.2.11)

and
\[
\hat{e} = \begin{bmatrix} e_n & 0 \\ 0 & B_{jj,r} \end{bmatrix},
\]

where the diagonal generalized stiffness submatrix is given as before by

\[
k_n = \Phi_n^\top K \Phi_n,
\]

and the diagonal generalized damping submatrix and second-order residual damping submatrix are given by

\[
c_n = \begin{bmatrix} 2 \xi_n \omega_n \\ 0 \end{bmatrix},
\]

and

\[
B_{jj,r} = G_{j,r}^\top C G_{j,r}.
\]

The hybrid coordinate force vector may be formulated via virtual work considerations. This vector of forces is given by

\[
\hat{F} = \begin{bmatrix} \Phi_n^\top \\ G_{j,r}^\top \end{bmatrix} F.
\]

The component hybrid equations of motion are now formulated formally by Lagrange's equation given by
\[
\begin{bmatrix}
m_n p^2 + c_n \dot{p} + k_n & 0 \\
0 & H_{jj,r} p^2 + B_{jj,r} \dot{p} + G_{jj,r}
\end{bmatrix}
\begin{bmatrix}
q_n \\
F_j
\end{bmatrix} =
\begin{bmatrix}
\Phi_n^T \\
G_{j,r}^T
\end{bmatrix} F_j
\tag{4.2.17}
\]

Again note that all the formal mathematics used in the above derivation were previously derived in a rigorous fashion in Chapter 3 for second-order residual flexibility.

### 4.3 Formulation of the Synthesis Transformation

Consider components $\alpha$ and $\beta$ whose equations of motion are given in hybrid generalized coordinates by Equation (4.2.17). The displacements at the junction coordinates of the components may be expressed from Equation (4.2.1) as

\[
u_j^\alpha = \Phi_{jn}^\alpha q_n^\alpha + G_{jj,r}^\alpha F_j^\alpha,
\tag{4.3.1}
\]

and

\[
u_j^\beta = \Phi_{jn}^\beta q_n^\beta + G_{jj,r}^\beta F_j^\beta.
\tag{4.3.2}
\]

The requirement of compatibility of the displacements at the junction coordinates and equilibrium of junction forces are given by

\[
u_j^\alpha = \nu_j^\beta
\tag{4.3.3}
\]

and
\[ F_j^\alpha = -F_j^\beta = F_j \]  

(4.3.4)

Incorporating the above expressions in Equations (4.3.1) and (4.3.2) yields

\[ F_j = (G_{jj,r}^\alpha + G_{jj,r}^\beta)^{-1}(\Phi_{jn}^\beta q_n^\beta - \Phi_{jn}^\alpha q_n^\alpha). \]  

(4.3.5)

It is possible at this point to develop a synthesis transformation which enforces displacement compatibility and force equilibrium simultaneously at the junction coordinates. From Equation (4.3.5), the synthesis transformations for components \( \alpha \) and \( \beta \) are given by

\[
\begin{bmatrix}
q_n^\alpha \\
F_j
\end{bmatrix}^\alpha = \begin{bmatrix}
I & 0 \\
-G_{jj,r}^{\alpha+\beta^{-1}}\Phi_{jn}^\alpha & G_{jj,r}^{\alpha+\beta^{-1}}\Phi_{jn}^\beta
\end{bmatrix}^\alpha \begin{bmatrix}
q_n^\alpha \\
q_n^\beta
\end{bmatrix}^\alpha = \Pi^\alpha \begin{bmatrix}
q_n^\alpha \\
q_n^\beta
\end{bmatrix},
\]  

(4.3.6)

and

\[
\begin{bmatrix}
q_n^\beta \\
F_j
\end{bmatrix}^\beta = \begin{bmatrix}
0 & I \\
-G_{jj,r}^{\alpha+\beta^{-1}}\Phi_{jn}^\alpha & G_{jj,r}^{\alpha+\beta^{-1}}\Phi_{jn}^\beta
\end{bmatrix}^\beta \begin{bmatrix}
q_n^\alpha \\
q_n^\beta
\end{bmatrix}^\beta = \Pi^\beta \begin{bmatrix}
q_n^\alpha \\
q_n^\beta
\end{bmatrix}.
\]  

(4.3.7)

From the above synthesis transformations, it is observed that the finalized system coordinates are simply the free-junction normal mode coordinates for the combined components, i.e., the junction displacement coordinates have been eliminated from the final system matrices. On the other hand, the junction coordinates are present in the second-order residual flexibility Component-Mode Synthesis in Chapter 3.
4.4 Derivation of the System Matrices

Treating the synthesis transformations as simply another coordinate transformation, the component equations of motion may be transformed into system coordinates in the following fashion

$$\Pi^\alpha \begin{bmatrix} m_n p^2 + c_n p + k_n & 0 \\ 0 & H_{ij,r} p^2 + B_{ij,r} p + G_{jj,r} \end{bmatrix} \Pi^\alpha \begin{bmatrix} q_n^\alpha \\ q_n^\beta \end{bmatrix} = \Pi^\alpha \begin{bmatrix} \Phi_n^T \\ G_j^T \end{bmatrix} F^\alpha, \quad (4.4.1)$$

and

$$\Pi^\beta \begin{bmatrix} m_n p^2 + c_n p + k_n & 0 \\ 0 & H_{ij,r} p^2 + B_{ij,r} p + G_{jj,r} \end{bmatrix} \Pi^\beta \begin{bmatrix} q_n^\alpha \\ q_n^\beta \end{bmatrix} = \Pi^\beta \begin{bmatrix} \Phi_n^T \\ G_j^T \end{bmatrix} F^\beta. \quad (4.4.2)$$

Therefore, the finalized system matrices are given by the addition of the above equations

\[
\left( \Pi^\alpha \begin{bmatrix} m_n p^2 + c_n p + k_n & 0 \\ 0 & H_{ij,r} p^2 + B_{ij,r} p + G_{jj,r} \end{bmatrix} \Pi^\alpha + \Pi^\beta \begin{bmatrix} m_n p^2 + c_n p + k_n & 0 \\ 0 & H_{ij,r} p^2 + B_{ij,r} p + G_{jj,r} \end{bmatrix} \Pi^\beta \right) \begin{bmatrix} q_n^\alpha \\ q_n^\beta \end{bmatrix} = \Pi^\alpha \begin{bmatrix} \Phi_n^T \\ G_j^T \end{bmatrix} F^\alpha + \Pi^\beta \begin{bmatrix} \Phi_n^T \\ G_j^T \end{bmatrix} F^\beta, \quad (4.4.3)
\]
or in compact notation

\[
(\mathbf{m}^{\alpha \beta} p^2 + \mathbf{c}^{\alpha \beta} p + \mathbf{k}^{\alpha \beta}) \begin{pmatrix} q_n^\alpha \\ q_n^\beta \end{pmatrix} = \mathbf{F}^{\alpha \beta},
\]

(4.4.4)

where all system matrices are fully populated.

### 4.5 Transformation to System Modal Coordinates and Recovery of the Junction Forces

The number of system generalized coordinates may be further reduced via a system level eigensolution of the mass and stiffness matrices in Equation (4.4.4). This modal superposition transformation may be expressed as

\[
\begin{pmatrix} q_n^\alpha \\ q_n^\beta \end{pmatrix} = \begin{bmatrix} \Phi^{sa} \\ \Phi^{sb} \end{bmatrix} z_n^s,
\]

(4.5.1)

where the system modal matrix in the above equation contains eigenvectors up to the frequency of interest. Direct integration of the resulting coupled modal equations of motion is necessary since again this is a nonclassically damped problem where the damping matrix is fully populated. From this, the junction forces may be computed via a modification to Equation (4.3.5), namely

\[
\mathbf{F}_j = (\mathbf{G}_{jj}^{\alpha} + \mathbf{G}_{jj}^{\beta})^{-1} (\Phi_{jn}^{\beta} \Phi^{sb} - \Phi_{jn}^{\alpha} \Phi^{sa}) z_n^s.
\]

(4.5.2)
As an alternative to the above formulation for junction forces recovery, consider the fact that the Ritz transformation is to a set of normal mode generalized coordinates and junction force generalized coordinates. From this, it is possible to recover the junction forces in the following simplified manner

\[ \mathbf{F}_j = \Pi^\alpha_j \Phi^S z^S \]  \hspace{1cm} (4.5.3)

where \( \Pi^\alpha_j \) denotes the junction rows of the synthesis transformation for component \( \alpha \).

### 4.6 Results and Discussions

This chapter considered a hybrid method of Component-Mode Synthesis based on a Rayleigh-Ritz approach. In it, the component is expressed in terms of two distinct types of generalized coordinates. The first set of generalized coordinates are based on the component's free-junction normal modes. The second set of generalized coordinates are the component's residual modes. A transformation is developed to simultaneously enforce displacement compatibility and junction force equilibrium. This transformation couples the components, at the same time reducing out the junction forces. Therefore, the pre-eigenproblem coordinates are simply the component's free-junction generalized coordinates. Since the junction forces are used as generalized coordinates in this hybrid formulation, their recovery is a trivial matter.

As in the prior chapters, the convergence of system frequencies as a function of the number of free-junction component normal modes is discussed. In addition, the convergence of system response and junction forces as a function of the number of retained modes is presented. Finally, the convergence characteristics of the response of the nonclassically damped problem are discussed comparing the Hybrid Component-
Mode Synthesis method to the Free-Junction Component-Mode Synthesis with second-order residual flexibility.

Table 4.1 presents a listing of the system frequencies for the Hybrid Component-Mode Synthesis approach for the two mode and four mode component representation. In parenthesis, system frequencies from second-order residual flexibility approach are listed for comparison. Figures (4.1-4.5) present results from a convergence study of the hybrid method for the case of the suddenly applied load. It is observed that the convergence of the kinematic response is superior to the junction forces for the four mode approximation.

The nonclassically damped problem is presented in Figures (4.6-4.10). In these figures, the convergence characteristics of the hybrid method is compared to Free-Junction method for a four mode component representation. The convergence of junction shear and moment in Free-Junction Component-Mode Synthesis is observed to be superior to the hybrid approach.
<table>
<thead>
<tr>
<th>Mode # / Frequency (cps)</th>
<th>Exact</th>
<th>4 - Mode Approximation</th>
<th>2 - Mode Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3147</td>
<td>0.3148 (0.3147)</td>
<td>0.3148 (0.3147)</td>
</tr>
<tr>
<td>2</td>
<td>0.8681</td>
<td>0.8681 (0.8681)</td>
<td>0.8681 (0.8681)</td>
</tr>
<tr>
<td>3</td>
<td>1.7048</td>
<td>1.7048 (1.7048)</td>
<td>1.7105 (1.7057)</td>
</tr>
<tr>
<td>4</td>
<td>2.8291</td>
<td>2.8291 (2.8291)</td>
<td>2.9457 (2.8913)</td>
</tr>
<tr>
<td>5</td>
<td>4.2549</td>
<td>4.2558 (4.2549)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>5.9982</td>
<td>5.9994 (5.9984)</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>7.9797</td>
<td>8.1073 (7.9832)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>11.2638</td>
<td>12.2384 (11.5447)</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1 - Convergence of System Frequencies as a Function of Retained Component Free-Junction Modes; Frequencies from Table 3.2 in parenthesis
Component-Mode Synthesis Solution
Hybrid Residual Flexibility Approach
Node 1, DOF 2

Figure (4.1) - Convergence of Displacement vs Number of Retained Free-Junction Normal Modes per Component; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Hybrid Residual Flexibility Approach
Node 1, DOF 2

Figure (4.2) - Convergence of Velocity vs Number of Retained Free-Junction Normal Modes per Component; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Hybrid Residual Flexibility Approach
Node 1, DOF 2

Figure (4.3) - Convergence of Acceleration vs Number of Retained Free-Junction Normal Modes per Component; Step-pulse Forcing Function
Figure (4.4) - Convergence of Junction Shear vs Number of Retained Free-Junction Normal Modes per Component; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Hybrid Residual Flexibility Approach
Junction Moment

Figure (4.5) - Convergence of Junction Moment vs Number of Retained Free-Junction Normal Modes per Component; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Nonclassically Damped Problem
Node 1, DOF 2

Figure (4.6) - Comparison of Displacements in Free-Junction Hybrid Component-Mode Synthesis and Free-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Nonclassically Damped Problem
Node 1, DOF 2

Figure (4.7) - Comparison of Velocities in Free-Junction Hybrid Component-Mode Synthesis and Free-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Component-Mode Synthesis Solution
Nonclassically Damped Problem
Node 1, DOF 2

Figure (4.8) - Comparison of Accelerations in Free-Junction Hybrid
Component-Mode Synthesis and Free-Junction Component-Mode Synthesis;
Four Component Modes Retained, Components Damped at 1% of Critical;
Step-pulse Forcing Function
Component-Mode Synthesis Solution
Nonclassically Damped Problem
Junction Shear

Figure (4.9) - Comparison of Junction Shears in Free-Junction Hybrid
Component-Mode Synthesis and Free-Junction Component-Mode Synthesis;
Four Component Modes Retained, Components Damped at 1% of Critical;
Step-pulse Forcing Function
Component-Mode Synthesis Solution
Nonclassically Damped Problem
Junction Moment

- - - - free-junction hybrid  - - - - free-junction  - - - - exact

Figure (4.10) - Comparison of Junction Moments in Free-Junction Hybrid Component-Mode Synthesis and Free-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
CHAPTER 5

DECOUPLED DYNAMIC ANALYSIS OF NONLINEAR, NONCLASSICALLY DAMPED SYSTEMS OF COMBINED COMPONENTS - THE FIXED-JUNCTION COMPONENT REPRESENTATION

5.1 Introductory Remarks

The previous chapters of this dissertation were devoted to a thorough treatment of the methods of Component-Mode Synthesis. The Fixed-Junction, Free-Junction, and Hybrid methods of Component-Mode Synthesis were studied. It was demonstrated that the methods of Component-Mode Synthesis provide a reduction of the number of generalized coordinates representing the combined system while preserving the accuracy required for dynamic analyses. Specifically, it was shown that the reduction of combined system generalized coordinates takes place at two different levels. First, there is a reduction in component generalized coordinates via either a fixed-junction or free-junction approach. Second, there is a reduction of the combined system generalized coordinates via the system eigenproblem where the system modes provide a reduced basis.

Spanos et. al. (1990) provided an alternative to the method of Component-Mode Synthesis which achieves the basic objective of reducing the number of system generalized coordinates while maintaining accuracy without having to solve a large system eigenproblem. Starting with the fixed-junction type of component representation, a junction-sized set of differential equations are derived which yield the junction accelerations for the different components in the combined system at a given time-step. In turn, the junction accelerations at a time-step are used to base-drive the equations of
motion of the components separately, in a decoupled manner. This proceeds step-by-step for the duration of integration. Hence, unlike Component-Mode Synthesis, the component modes are not synthesized nor system modes of vibration determined.

In this chapter, the mathematics and numerical implementation of this decoupled method of analysis are presented. It is demonstrated that due to the simplifications resulting from the decoupled nature of the analysis, treatment of the nonclassically damped problem is a trivial case. The method is further extended to components with junction nonlinearities which renders the system nonlinear. This is accomplished by an iterative predictor-corrector methodology that converges on a reliable value of the nonlinear junction force. Numerical results from this method are compared against Fixed-Junction Component-Mode Synthesis for the nonclassically damped system considered in the prior chapters. Furthermore, the nonlinear, nonclassically damped problem is considered through some numerical examples.

5.2 Mathematical Background - Linear Analysis

In chapter 2, the reduction to the fixed-junction component was discussed in some detail. In this section, the decoupled methodology is derived starting with the basic equations of fixed-junction component. The Rayleigh-Ritz transformation required to attain a fixed-junction type of component representation, restated here from Equation (2.2.8), is given by

\[
\begin{bmatrix}
\{u_j\} \\
\{u_a\}
\end{bmatrix}
=\begin{bmatrix}
I & 0 \\
\Phi_c & \Phi_n
\end{bmatrix}\begin{bmatrix}
\{u_j\} \\
\{q_n\}
\end{bmatrix},
\]  

\(5.2.1\)
where a set of retained fixed-junction normal modes of vibration and static constraint modes provide the basis vectors for the transformation to a reduced set of fixed-junction generalized coordinates and physical junction coordinates. From Lagrange's equations, the following equations of dynamic equilibrium result for component $\alpha$

$$
\begin{bmatrix}
  m_{jj} & m_{jn} \\
  m_{nj} & m_n
\end{bmatrix} \begin{bmatrix}
  \ddot{u}_j \\
  \ddot{q}_n
\end{bmatrix} + \begin{bmatrix}
  0 & 0 \\
  c_n & 0
\end{bmatrix} \begin{bmatrix}
  \dot{u}_j \\
  \dot{q}_n
\end{bmatrix} + \begin{bmatrix}
  0 & k_{jj} \\
  0 & k_n
\end{bmatrix} \begin{bmatrix}
  q_j \\
  q_n
\end{bmatrix} = \begin{bmatrix}
  I & 0 \\
  \Phi_j & \Phi_n
\end{bmatrix}^T \begin{bmatrix}
  F_j \\
  F_n
\end{bmatrix},
$$

(5.2.2)

where the notation and the submatrices are defined in section (2.3). Note that in this method, it is assumed for now that the junction damping submatrix is null. The upper partition of the above set of differential equations yields the following set of equations for the physical junction coordinates

$$
m_{jj} \ddot{u}_j + k_{jj} u_j = \Phi^{c}_{aj} F_a + F_{aj} a - m_{jn} \ddot{q}_n.
$$

(5.2.3)

Similarly, the lower partition for the fixed-junction generalized coordinates is expressed by the following set of differential equations

$$
m_n \ddot{q}_n + c_n \dot{q}_n + k_n q_n = \Phi^{N}_{an} F_a - m_{nj} \ddot{u}_j.
$$

(5.2.4)

The equations of dynamic equilibrium for component $\beta$ are expressed in terms of a retained set of fixed-junction generalized coordinates and physical junction coordinates by the following set of differential equations
\[
\begin{bmatrix}
  m_{jj} & m_{jn} \\
  m_{nj} & m_{nn}
\end{bmatrix}
\begin{bmatrix}
  \ddot{u}_j \\
  \ddot{q}_n
\end{bmatrix}
+ \begin{bmatrix}
  0 & 0 \\
  0 & c_n
\end{bmatrix}
\begin{bmatrix}
  \dot{u}_j \\
  \dot{q}_n
\end{bmatrix}
+ \begin{bmatrix}
  k_{jj} & 0 \\
  0 & k_{nn}
\end{bmatrix}
\begin{bmatrix}
  u_j \\
  q_n
\end{bmatrix}
= \begin{bmatrix}
  I & 0 \\
  \Phi_{aj}^c & \Phi_{an}^N
\end{bmatrix}
\begin{bmatrix}
  F_j \\
  F_a
\end{bmatrix}
\]

(5.2.5)

As before, the upper partition for the junction coordinates may be expressed by the following expression

\[
m_{jj} \ddot{u}_j + k_{jj} u_j = \Phi_{aj}^c F_j + F_a - m_{jn} \ddot{q}_n,
\]

(5.2.6)

and the lower partition for the fixed-junction generalized coordinates may be written as

\[
m_{nn} \ddot{q}_n + c_n \dot{q}_n + k_{nn} q_n = \Phi_{an}^N F_a - m_{nj} \ddot{u}_j.
\]

(5.2.7)

The compatibility of junction displacements and equilibrium of junction forces between the two components are governed by the following equations

\[
u_j^a = u_j^\beta = u_j,
\]

(5.2.8)

and

\[F_j^a = -F_j^\beta = F_j.
\]

(5.2.9)

Enforcing Equations (5.2.8) and (5.2.9), the following junction-sized set of differential equations for the junction coordinates results from Equations (5.2.3) and (5.2.6)
\[ (m_{jj}^\alpha + m_{jj}^\beta)\ddot{u}_j + (k_{jj}^\alpha + k_{jj}^\beta)u_j = \tilde{F}_{jj}^{\alpha+\beta} - m_{jn}^\alpha \ddot{q}_n^\alpha - m_{jn}^\beta \ddot{q}_n^\beta, \quad (5.2.10) \]

where the combined external force vector is given by

\[ \tilde{F}^{\alpha+\beta}_{jj} = \Phi_{aj}^c \dot{z}_a + \Phi_{aj}^c \dot{z}_a. \quad (5.2.11) \]

The above junction-sized set of differential equations may be uncoupled via a small, junction-sized eigenproblem. More importantly, as a consequence of the coordinate transformation provided by the eigenproblem, the junction coordinates may be damped to a certain critical damping by simply damping the resulting modes. Mathematically, the coordinate transformation resulting from the eigenproblem transforms the junction coordinates to a set of modal coordinates by the following expression

\[ u_j = \Phi_{jj}z_j. \quad (5.2.12) \]

where all normal modes must be retained in order to preserve the total flexibility of the junction coordinates. Substituting Equation (5.2.12) into Equation (5.2.10) and premultiplying by the transpose of the modal matrix, the following set of modal differential equations result

\[ \ddot{m}_j \ddot{z}_j + \ddot{c}_j \ddot{z}_j + \ddot{k}_j z_j = \Phi_{jj}^T(\tilde{F}_{jj}^{\alpha+\beta} - m_{jn}^\alpha \ddot{q}_n^\alpha - m_{jn}^\beta \ddot{q}_n^\beta) \quad (5.2.13) \]

where for mass normalized eigenvectors, the diagonal mass, damping, and stiffness matrices are given by
\( \hat{\mathbf{m}}_j = [\mathbf{m}] \),

(5.2.14)

\( \hat{\mathbf{c}}_j = [\mathbf{c}] \),

(5.2.15)

and

\( \hat{\mathbf{k}}_j = [\mathbf{k}] \).

(5.2.16)

### 5.3 Numerical Implementation - Linear Analysis

The Newmark algorithm for numerical integration plays a critical role in setting up the decoupled approach. The recurrence relations for the algorithm are restated here as

\[
\ddot{z}_{j,i+1} = \dot{z}_{j,i} + \frac{1}{2} (\ddot{z}_{j,i} + \ddot{z}_{j,i+1}) h
\]

(5.3.1)

and

\[
\mathbf{z}_{j,i+1} = \mathbf{z}_{j,i} + \dot{\mathbf{z}}_{j,i} h + b_1 \ddot{\mathbf{z}}_{j,i} + b_2 \ddot{\mathbf{z}}_{j,i+1}.
\]

(5.3.2)

where the coefficients of integration are defined in terms of the Newmark quadrature constant, \( \beta \), by

\[
b_1 = (\frac{1}{2} - \beta) h^2
\]

(5.3.3)

and
\[ b_2 = \beta h^3. \]  
\hspace{1cm} (5.3.4)

Equation (5.2.13) may be discretized in time by the following index notation

\[
\tilde{m}_{j,j,i+1} + \tilde{c}_{j,j,i+1} + \tilde{k}_{j,j,i+1} = \Phi^{T}_{j,j,i+1}(\tilde{F}_{j,j,i+1} - m_{jn}^{\alpha} \tilde{q}_{n,i+1}^{\alpha} - m_{jn}^{\beta} \tilde{q}_{n,i+1}^{\beta}).
\]  
\hspace{1cm} (5.3.5)

Substituting Newmark's recurrence relations into the above set of differential equations results

\[
\tilde{m}_{j,j,i+1} = -\Phi^{T}_{j,j,i+1}(m_{jn}^{\alpha} \tilde{q}_{n,i+1}^{\alpha} + m_{jn}^{\beta} \tilde{q}_{n,i+1}^{\beta}) + \tilde{F}_{j,j,i+1},
\]  
\hspace{1cm} (5.3.6)

where the effective mass matrix and force vector are defined by

\[
\tilde{m}_{j} = \tilde{m}_{j} + \frac{1}{2} h \tilde{c}_{j} + b_{2} \tilde{k}_{j},
\]  
\hspace{1cm} (5.3.7)

and

\[
\tilde{F}_{j,j,i+1} = \Phi^{T}_{j,j,i+1}(\tilde{F}_{j,j,i+1} - \frac{1}{2} h \tilde{c}_{j} + b_{1} \tilde{k}_{j}) \tilde{z}_{j,i+1} - (\tilde{c}_{j} + h \tilde{k}_{j}) \tilde{z}_{j,i+1} - \tilde{k}_{j} \tilde{z}_{j,i+1}.
\]  
\hspace{1cm} (5.3.8)

In a similar fashion, Equations (5.2.4) and (5.2.7) representing the motion of the fixed-junction generalized coordinates for the components may be discretized via the following index notation

\[
m_{n}^{\alpha} \tilde{q}_{n,i+1}^{\alpha} + c_{n}^{\alpha} \tilde{q}_{n,i+1}^{\alpha} + k_{n}^{\alpha} \tilde{q}_{n,i+1}^{\alpha} = \Phi^{N}_{n} \tilde{F}_{a,i+1}^{\alpha} - m_{nj}^{\alpha} \Phi \tilde{z}_{j,i+1}^{\alpha}.
\]  
\hspace{1cm} (5.3.9)
and

\[ m_n \dot{q}_{n,i+1} + c_n \ddot{q}_{n,i+1} + k_n q_{n,i+1} = \Phi_n^{aT} F_{a,i+1} - m_n \dot{q}_{j,j,i+1}. \] (5.3.10)

where the modal superposition relation in Equation (5.2.12) has been substituted. In a fashion similar to the combined system junction differential equations, the Newmark recurrence relations may be substituted in the above component equations of dynamic equilibrium resulting in

\[ \overline{m}_{n}^{\alpha} \ddot{q}_{n,i+1}^{\alpha} = -m_{nj}^{\alpha} \dot{q}_{j,j,i+1}^{\alpha} + \overline{F}_{n,i+1}^{\alpha} \] (5.3.11)

and

\[ \overline{m}_{n}^{\beta} \ddot{q}_{n,i+1}^{\beta} = -m_{nj}^{\beta} \dot{q}_{j,j,i+1}^{\beta} + \overline{F}_{n,i+1}^{\beta}, \] (5.3.12)

where the effective component mass matrices and the effective component force vectors for components \(\alpha\) and \(\beta\) are defined by the following expressions, respectively

\[ \overline{m}_{n}^{\alpha} = m_{n}^{\alpha} + \frac{1}{2} h c_{n}^{\alpha} + b_{i} k_{n}^{\alpha} \] (5.3.13)

\[ \overline{F}_{n,i+1}^{\alpha} = \Phi_n^{aT} F_{a,i+1}^{\alpha} - (\frac{1}{2} h c_{n}^{\alpha} + b_{i} k_{n}^{\alpha}) \ddot{q}_{n,i}^{\alpha} - (c_{n}^{\alpha} + h k_{n}^{\alpha}) \dot{q}_{n,i}^{\alpha} - k_{n}^{\alpha} q_{n,i}^{\alpha} \] (5.3.14)
\[
\overline{m}_n^\beta = m_n^\beta + \frac{1}{2} \hbar c_n^\beta + b_n k_n^\beta
\]  
(5.3.15)

\[
\overline{F}_{n,i+1}^\beta = \Phi^{N\beta}_{an} \overline{F}_{a,i+1}^\beta - \left( \frac{1}{2} \hbar c_n^\beta + b_n k_n^\beta \right) \overline{q}_{n,i}^\beta - \left( c_n^\beta + \hbar k_n^\beta \right) \overline{q}_{n,i}^\beta - k_n q_{n,i}^\beta.
\]  
(5.3.16)

Solving Equations (5.3.11) and (5.3.12) for the fixed-junction generalized accelerations and substituting the resulting equations in Equation (5.3.6) results

\[
\hat{m}_{j,j,i+1} = \overline{F}_{j,i+1}^\beta - \Phi^{\beta T}_{ij} \overline{F}_{n,i+1}^\beta + m_n \overline{m}^{-1} \overline{F}_{n,i+1}^\beta,
\]  
(5.3.17)

where the above effective mass matrix is given by

\[
\hat{m}_j = [\overline{m}_j - \Phi^{\beta T}_{ij} (m_n^{-1} m_n^\alpha + m_n \overline{m}^{-1} m_n^\beta) \Phi_{ij}].
\]  
(5.3.18)

Note that all vectors on the right-hand side of Equation (5.3.17) are known for a given time-step and the effective mass is a junction-sized constant matrix which has to be inverted only once. Therefore, the junction-sized modal acceleration vector, \( \tilde{z}_{j,i+1} \), may be determined from Equation (5.3.17) from which the equations of dynamic equilibrium for the components, Equations (5.3.11) and (5.3.12), are base-driven, in a decoupled manner, to determine the generalized accelerations for the time-step. Furthermore, the component generalized velocities and displacements and the modal velocities and displacements, \( \tilde{z}_{j,i+1} \) and \( z_{j,i+1} \) are determined from the Newmark recurrence relations for the time-step. In this manner, the decoupled method of analysis can proceed step-by-step for the duration of interest. Once the generalized accelerations, velocities, and displacements for
each component have been determined, the junction forces may be recovered by the following equation

$$F_i = m_{ij} \Phi_i \dot{z}_j + k_{ij} \dot{\Phi}_i \dot{z}_j + m_{jn} \dot{q}_n - \Phi_i \dot{F}_a.$$

(5.3.19)

5.4 Mathematical Background - Nonlinear Analysis

In this section, the decoupled approach to dynamic analyses of combined systems is extended to components with nonlinear junctions which renders the combined system nonlinear. Methods presented by Bathe (1973, 1978) that treat the nonlinearity as a force vector on the right-hand side of the dynamic equations of equilibrium are implemented. Equilibrium iterations are used within a time-step in order to converge to a reliable value for the nonlinear force vector. Bathe and Gracewski (1981) worked extensively on the nonlinear substructuring problem in structural dynamics and in a paper titled "On Nonlinear Dynamic Analysis Using Substructuring and Mode Superposition", nonlinear substructuring is extensively discussed. However, the method presented by Bathe does not take advantage of the method of Component-Mode Synthesis in order to ensure efficient substructuring; instead, statically reduced substructures are used. Additionally, Bathe solves the system eigenproblem in order to provide a coordinate transformation for the nonlinear problem to a reduced subspace. This is not a requirement for efficient analyses in the decoupled approach. Overall, extension of the decoupled analysis scheme to the nonlinear problem as presented in the following sections proves itself as one of the most efficient and versatile means of achieving accurate solutions to the nonlinear system problem.
Consider the set of differential equations for the junction coordinates of the components as given for the linear case in Equations (5.2.3) and (5.2.6). The nonlinear force vector may be added to the right-hand side of the component equations as given by

$$
\mathbf{m}_{jj} \ddot{\mathbf{u}} + \mathbf{k}_{jj} \mathbf{u} = \Phi^{\alpha} \alpha_{Tj} \mathbf{F}^{\alpha}_j + \mathbf{F}^{\alpha}_j + \mathbf{F}_{j}^{NL\alpha} - \mathbf{m}_{jn} \dot{\mathbf{q}}^{\alpha}_{n} \tag{5.4.1}
$$

and

$$
\mathbf{m}_{jj} \dddot{\mathbf{u}} + \mathbf{k}_{jj} \ddot{\mathbf{u}} = \Phi^{\alpha} \beta_{Tj} \mathbf{F}^{\beta}_j + \mathbf{F}^{\beta}_j + \mathbf{F}_{j}^{NL\beta} - \mathbf{m}_{jn} \dot{\mathbf{q}}^{\beta}_{n}. \tag{5.4.2}
$$

Now enforcing displacement compatibility and force equilibrium as given in Equations (5.2.8) and (5.2.9) and noting that

$$
\mathbf{F}_{j}^{NL\alpha} = -\mathbf{F}_{j}^{NL\beta} \tag{5.4.3}
$$

results in the following set of junction-sized differential equations for the combined system

$$
(\mathbf{m}_{jj}^{\alpha} + \mathbf{m}_{jj}^{\beta}) \ddot{\mathbf{u}} + (\mathbf{k}_{jj}^{\alpha} + \mathbf{k}_{jj}^{\beta}) \mathbf{u} = \mathbf{F}_{j}^{\alpha} + \mathbf{m}_{jn} \dot{\mathbf{q}}^{\alpha}_{n} - \mathbf{m}_{jn} \dot{\mathbf{q}}^{\beta}_{n} + \mathbf{F}_{j}^{NL\alpha} + \mathbf{F}_{j}^{NL\beta}. \tag{5.4.4}
$$

Furthermore, solving the associated eigenproblem and transforming coordinates results in

$$
\tilde{\mathbf{m}} \ddot{\mathbf{z}} + \tilde{\mathbf{c}} \dot{\mathbf{z}} + \tilde{\mathbf{k}} \mathbf{z} = \Phi^{\alpha + \beta}_{jj_{Tj}} (\mathbf{F}_{j}^{\alpha + \beta} - \mathbf{m}_{jn} \dot{\mathbf{q}}^{\alpha}_{n} - \mathbf{m}_{jn} \dot{\mathbf{q}}^{\beta}_{n} + \mathbf{F}_{j}^{NL\alpha} + \mathbf{F}_{j}^{NL\beta}) \tag{5.4.5}
$$
which is similar to Equation (5.2.13) not withstanding the addition of the component nonlinear junction forces.

5.5 Numerical Implementation - Nonlinear Problem

Equation (5.4.5) is now discretized in time and the Newmark recurrence relations substituted which results in the following set of differential equations

\[ \mathbf{m} \ddot{\mathbf{x}}_{j,i+1} = \mathbf{F}_{j,i+1} - \Phi_{jj}^T (\mathbf{m} \dot{\mathbf{x}}_{n,i+1} + \mathbf{m} \dot{\mathbf{x}}_{n,i+1} - \mathbf{F}_{j,i+1}^{NL} - \mathbf{F}_{j,i+1}^{NL}) \]  (5.5.1)

where the matrices are defined in the same manner as the linear case. Now substituting the fixed-junction generalized accelerations for the components, Equations (5.3.11) and (5.3.12), into the above equation results in

\[ \mathbf{m} \ddot{\mathbf{x}}_{j,i+1} = \mathbf{F}_{j,i+1} - \Phi_{jj}^T (\mathbf{m} \dot{\mathbf{x}}_{n,i+1} - \mathbf{m} \dot{\mathbf{x}}_{n,i+1} - \mathbf{F}_{j,i+1}^{NL} - \mathbf{F}_{j,i+1}^{NL}) \]  (5.5.2)

where the matrices are defined in the same manner as for the linear problem. With the exception of the nonlinear junction force at i+1, the remainder of the quantities in the right-hand side vector are defined. Now, a predictor step may be implemented in order to provide a first estimate of the nonlinear force vector given by a truncated Taylor series given by

\[ \mathbf{F}_{j,i+1}^{NL} = \mathbf{F}_{j,i}^{NL} (\dot{\mathbf{u}}_{j,i+1} + \dot{\mathbf{u}}_{j,i+1} + \ddot{\mathbf{u}}_{j,i+1} + \mathbf{h}). \]  (5.5.3)
From this estimate, Equation (5.5.2) may be solved for the junction modal accelerations from which Equations (5.3.11) and (5.3.12) may be solved for the fixed-junction generalized accelerations for the components. From these solutions, the Newmark recurrence relations, and the governing nonlinear law, a corrected value for the nonlinear junction force may be determined. This iterative step should continue until the difference in two successive values of the nonlinear force satisfy the following relation

$$\frac{|F_{NL}^{(s+1)}_{j,i+1} - F_{NL}^{(s)}_{j,i+1}|}{F_{NL}^{(s+1)}_{j,i+1}} \leq \delta,$$

(5.5.4)

where \(s\) is the iteration step and \(\delta\) is the constant governing the accuracy of the nonlinear forces.

Once the component displacement, velocities, and accelerations have been determined for the duration of the transients, the linear junction forces may be determined by adjusting Equation (5.3.19) with the nonlinear junction forces as follows

$$F_{j} = m_{jj} \ddot{z}_j + k_{jj} \dot{z}_j + m_{jn} \dot{q}_n - \Phi^a_{aj} F^a_{j} - F_{NL}^{\alpha}. $$

(5.5.5)
5.6 Results and Discussions

This chapter presented a decoupled method of analyzing the response of a system of combined components without having to resort to the determination of system modes. The method derives a set of junction-sized differential equations based on a fixed-junction component representation. The junction accelerations are determined at a time point by numerically integrating this set of junction-sized equations from which the individual components are base-driven separately. Due to the formulation of this method, the nonclassically damped system becomes a trivial problem. Furthermore, the method is easily extended to the nonlinear system via a predictor-corrector approach.

The convergence of the decoupled analysis method as a function of the number of retained fixed-junction modes subjected to two different force time histories is presented. Furthermore, the nonclassically damped problem is solved comparing the convergence characteristics of the decoupled approach to Fixed-Junction Component-Mode Synthesis. The nonlinear, nonclassically damped problem is also treated here, further demonstrating the versatility of this methodology.

Figures (5.1-5.5) present the convergence characteristics of the decoupled approach with a two mode and a four mode component representation of the cantilevered beam components acted upon by a suddenly applied load. The four mode solution approximates the exact solution for the kinematics and with acceptable accuracy for the junction forces the reasons for which were explained in chapter 2. Figures (5.6-5.10) display the nonclassically damped problem with four modes retained per component. In these figures, the convergence characteristics of the decoupled method is compared to Fixed-Junction Component-Mode Synthesis for the kinematics at node 1, dof 2 and the junction forces. It is seen that the methods compare well. Figures (5.11-5.14) are solutions to the nonlinear, nonclassically damped system where a nonlinear spring of stiffness 1000 lb/in is used to connect the two components in shear. These figures present
the differences in kinematics in dof 2 and junction shear due to different junction nonlinearities.
Decoupled Analysis
Fixed-Junction Approach
Node 1, DOF 2

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2-mode 4-mode exact

Figure (5.1) - Convergence of Displacement vs Number of Retained Fixed-Junction Normal Modes per Component; Step-pulse Forcing Function
Decoupled Analysis
Fixed-Junction Component
Node 1, DOF 2

Figure (5.2) - Convergence of Velocity vs Number of Retained Fixed-Junction Normal Modes per Component; Step-pulse Forcing Function
Decoupled Analysis
Fixed-Junction Component
Node 1, DOF 2

Figure (5.3) - Convergence of Acceleration vs Number of Retained Fixed-Junction Normal Modes per Component; Step-pulse Forcing Function
Decoupled Analysis
Fixed-Junction Component
Junction Shear

Figure (5.4) - Convergence of Junction Shear vs Number of Retained Fixed-Junction Normal Modes per Component; Step-pulse Forcing Function
Decoupled Analysis
Fixed-Junction Component
Junction Moment

Figure (5.5) - Convergence of Junction Moment vs Number of Retained
Fixed-Junction Normal Modes per Component; Step-pulse Forcing Function
Decoupled Analysis
Nonclassically Damped Problem
Node1, DOF 2

Figure (5.6)- Comparison of Displacements in Decoupled Analysis and Fixed-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Decoupled Analysis
Nonclassically Damped Problem
Node 1, DOF 2

Figure (5.7) - Comparison of Velocities in Decoupled Analysis and Fixed-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Decoupled Analysis
Nonclassically Damped Problem
Node 1, DOF 2

Figure (5.8) - Comparison of Accelerations in Decoupled Analysis and Fixed-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Decoupled Analysis
Nonclassically Damped Problem
Junction Shear

Figure (5.9) - Comparison of Junction Shears in Decoupled Analysis and Fixed-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Decoupled Analysis
Nonclassically Damped Problem
Junction Moment

Figure (5.10) - Comparison of Junction Moments in Decoupled Analysis and Fixed-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Decoupled Analysis
Nonlinear, Nonclassically Damped System
Node 1, DOF 2

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Figure (5.11) - Effect of Different Junction Spring Nonlinearities on Displacement Response; All Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Decoupled Analysis
Nonlinear, Nonclassically Damped System
Node 1, DOF 2

Figure (5.12) - Effect of Different Junction Spring Nonlinearities on Velocity Response; All Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
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CHAPTER 6

A RESIDUAL FLEXIBILITY APPROACH FOR DECOUPLED ANALYSIS OF NONLINEAR, NONCLASSICALLY DAMPED SYSTEMS OF COMBINED COMPONENTS

6.1 Introductory Remarks

The prior chapter was concerned with a method of decoupled analysis for systems of combined components where the equations of dynamic equilibrium of the component are of the fixed-junction type, derived via a Rayleigh-Ritz procedure. In it, a set of junction-sized differential equations in terms of the unknown junction accelerations were solved in a step-by-step manner. Given the junction accelerations of the different components at a given time, the equations of motion for the components were integrated separately. This was accomplished without having to solve a large system eigenproblem; instead, a junction-sized eigenproblem was solved in order to manifest junction damping. Once the component displacements, velocities, and accelerations were determined, the junction forces were recovered via equilibrium considerations.

This chapter addresses a first-order and second-order residual flexibility approach for decoupled analysis of systems of combined components. This method of decoupled analysis may be formulated directly without having to resort to a Rayleigh-Ritz representation for the component. In it, the reactive forces at the junctions of the components are computed directly without the synthesis of component modes or the determination of system modes. Therefore, recovery of junction forces, a requirement in all methods of Component-Mode Synthesis and the decoupled approach in Chapter 5, is not necessary. This is accomplished by expressing the displacements at the junction
coordinates of the components in terms of the retained component modes and an account of the residual flexibility of the unretracted modes. The residual flexibility ensures that the total flexibility at the junction coordinates is preserved leading to accurate solutions for the junction forces. Once the components are presented in this manner, the requirement of displacement compatibility and force equilibrium at the junction coordinates is enforced. This leads to a set of junction-sized, simultaneous algebraic equations, similar in form to the flexibility formulation in statics, in terms of the unknown junction forces. The computed forces at a given time-step then serve to base-drive each components equations of motion separately, in a decoupled manner. The method in its entirety is cast into a numerical method by invoking the Newmark integration algorithm.

If the component's junctions are such that the combined system is linear, the junction forces are computed directly. If the component's junctions lend the system nonlinear, a predictor-corrector scheme for converging onto a reliable value of the nonlinear force is devised. The computed junction forces are compared to free-junction Component-Mode Synthesis for a nonclassically damped fixed-fixed beam comprised of two classically damped cantilevered beam components. Nonlinear, nonclassically damped problems with quadratic and cubic nonlinearities (Duffing nonlinear equation) are solved with the decoupled method and compared against linear solutions.

6.2 Mathematical Background - Linear Analysis

In this section, some of the basics of residual flexibility, considered in detail in Chapter 3, are outlined in order to facilitate the transition to the decoupled approach and provide completeness. The equations of motion of a component in physical coordinates may be expressed in standard form as
\[ \dot{\mathbf{u}} + C \mathbf{u} + K \mathbf{u} = \mathbf{F}(t). \] (6.2.1)

The solution of a unconstrained eigenvalue problem provides the transformation to the retained component modal coordinates given by

\[ \mathbf{u} = \Phi_n \mathbf{q}_n. \] (6.2.2)

Proceeding in this rudimentary fashion, the retained set of modal equations of motion can now be expressed by substituting Equation (6.2.2) into (6.2.1) and pre-multiplying by the transposed modal transformation. This equation is expressed as

\[ m_n \ddot{q}_n + c_n \dot{q}_n + k_n q_n = \Phi_n^T \mathbf{F}(t), \] (6.2.3)

where the modal mass, damping, and stiffness matrices are diagonal due to the orthogonality of the normal modes and the assumption that the component is classically damped.

The retained set of unconstrained normal modes without residual effects may be deficient in representing the total flexibility present at the junction coordinates of the component. Therefore, in the general case, the second order residual contribution of the unretained modes have to be determined. One can proceed by considering the static first order displacements from Equation (6.2.1). This is accomplished by ignoring the inertia and the damping terms. Since the component is unconstrained, inverting the stiffness matrix in order to determine the flexibility is not possible without additional mathematical manipulation. Specifically, subtracting the rigid-body inertial loads from the externally applied loads yields the inertia relief load vector. This may be expressed as
\[ F - F_1 = F - M\ddot{u}_R, \quad (6.2.4) \]

where the loading vector is self-equilibrating producing no reactions if a set of statically
determinate constraints are imposed on the component stiffness. The rigid-body
accelerations may be expressed in terms of the physical applied force as

\[ \ddot{u}_R = \Phi_R m_R^{-1} \Phi_R^T F. \quad (6.2.5) \]

Combining Equations (6.2.4) and (6.2.5), the inertia relief loads may be expressed as

\[ F - F_1 = AF, \quad (6.2.6) \]

where the transformation between the inertia relief loads and the externally applied loads
is referred to in literature as the projection matrix. Next, imposing a set of statically
determinate constraints on the stiffness produces the following first order constrained
displacement vector

\[ u_c^{(i)} = G_c AF. \quad (6.2.7) \]

Equation (6.2.7) is the relative displacement of the component with respect to the
imposed constraints. It differs from the actual flexible displacements by a rigid body
contribution. It is then necessary to remove this rigid-body contribution. This is
accomplished by adding an arbitrary rigid-body displacement to the constrained
displacements and then orthogonalizing it with respect to all rigid-body modes. This is
accomplished by the equations
\[ u^{(1)} = u_c^{(1)} + \mathbf{\Phi}_R \mathbf{Q}_R, \]  
(6.2.8)

and

\[ \mathbf{\Phi}_R^T \mathbf{M} u^{(1)} = 0. \]  
(6.2.9)

Next, solving for the constant in Equation (6.2.8), the flexible displacements may be expressed in terms of the constrained displacements by the following equation

\[ u^{(1)} = \mathbf{A}^T u_c^{(1)}. \]  
(6.2.10)

Combining Equations (6.2.7) and (6.2.10), the total flexibility matrix for an unconstrained component may now be expressed by the equation

\[ \mathbf{G} = \mathbf{A}^T \mathbf{G}_c \mathbf{A}. \]  
(6.2.11)

The second order quasi-static displacements may now be expressed in terms of the first order displacements as

\[ u^{(2)} = \mathbf{G} (\mathbf{F} - \mathbf{C} u^{(1)} - \mathbf{M} \ddot{u}^{(1)}). \]  
(6.2.12)

Substituting the first order displacements gives

\[ u^{(2)} = (\mathbf{G} - \mathbf{B} \mathbf{p} - \mathbf{H} \mathbf{p}^2) \mathbf{F}, \]  
(6.2.13)
where the \( p \) represents the derivative operator and the total damping and inertia matrices are defined respectively by the equations

\[
B = GCG, \quad H = GMG.
\] (6.2.14)

In order to derive the second order residual flexibility, the quasi-static contribution of the retained modes of vibration have to be subtracted from (6.2.13). To this end, the first order retained displacements may be derived from Equation (6.2.3) as

\[
\mathbf{u}^{(1)}_n = \Phi_n^T \mathbf{k}_n^{-1} \Phi_n^T \mathbf{F} = \mathbf{G}_n \mathbf{F}.
\] (6.2.15)

Following the prior methodology, the second order retained quasi-static displacements may be computed from Equation (6.2.3) as

\[
\mathbf{u}^{(2)}_n = (\Phi_n^T \mathbf{k}_n^{-1} \Phi_n^T \mathbf{F} - \Phi_n^T \mathbf{k}_n^{-1} \mathbf{c}_n \mathbf{k}_n^{-1} \Phi_n^T \mathbf{p} - \\
\Phi_n^T \mathbf{m}_n \mathbf{k}_n^{-1} \Phi_n^T \mathbf{p}^2) \mathbf{F}.
\] (6.2.16)

After some algebraic manipulation, Equation (6.2.16) may be expressed as

\[
\mathbf{u}^{(2)}_n = (\mathbf{G}_n - \mathbf{B}_n \mathbf{p} - \mathbf{H}_n \mathbf{p}^2) \mathbf{F},
\] (6.2.17)

where the retained damping and inertia matrices have the same form as in Equation (6.2.14) but the triple products are with the retained flexibility.

Now, subtracting Equation (6.2.17) from Equation (6.2.13) yields the second order residual displacements. After some algebraic manipulation and noting that the
residual and retained flexibility matrices are mutually orthogonal, the second order residual displacements may be expressed as

\[ u_r^{(2)} = (G_r - B_r p - H_r p^2) F, \]  

(6.2.18)

where the residual flexibility, damping, and inertia are as

\[ G_r = G - G_n, \]  

(6.2.19)

\[ B_r = G_r CG_r, \]  

(6.2.20)

\[ H_r = G_r MG_r. \]  

(6.2.21)

6.3 Direct Solution for the Junction Forces

Consider the unconstrained modal equations of motion for two components \( \alpha \) and \( \beta \). These equations are of the same form as Equation (6.2.3); however, the junction forces which are a partition of the force vector are expressed explicitly. By taking into account that junction forces occur in equal and opposite pairs these equations are written as

\[ m_n^{\alpha} \dot{q}_n^{\alpha} + c_n^{\alpha} \ddot{q}_n^{\alpha} + k_n^{\alpha} q_n^{\alpha} = \Phi^{\alpha T} F_n^{\alpha} + \Phi^{jn \alpha T} F_j, \]  

(6.3.1)

and
\[ m_n \, \beta \dot{q}_n \, \beta + c_n \, \beta \dot{q}_n \, \beta + k_n \, \beta q_n \, \beta = \Phi_a \, \beta r F_a \, \beta - \Phi_j \, \beta r F_j. \] (6.3.2)

The displacements at the junction degrees of freedom for components \( \alpha \) and \( \beta \). Taking into account the second-order residual flexibility in Equation (6.2.18), the junction displacements may be expressed as

\[
\mathbf{u}_j^\alpha = \Phi_j \, \alpha q_n \, \alpha + (G_{j j, r} \, \alpha - B_{j j, r} \, \alpha p - H_{j j, r} \, \alpha p^2) F_j + (G_{j a, r} \, \alpha - B_{j a, r} \, \alpha p - H_{j a, r} \, \alpha p^2) F_a^\alpha,
\] (6.3.3)

and

\[
\mathbf{u}_j^\beta = \Phi_j \, \beta q_n \, \beta - (G_{j j, r} \, \beta - B_{j j, r} \, \beta p - H_{j j, r} \, \beta p^2) F_j + (G_{j a, r} \, \beta - B_{j a, r} \, \beta p - H_{j a, r} \, \beta p^2) F_a^\beta.
\] (6.3.4)

Displacement compatibility at the junction degrees of freedom requires that

\[ \mathbf{u}_j^\alpha = \mathbf{u}_j^\beta. \] (6.3.5)

From Equations (6.3.3), (6.3.4), and (6.3.5), the junction forces may be directly determined by the following expression
\[
(G_{jj, r}^{\alpha+\beta} - B_{jj, r}^{\alpha+\beta} p - H_{jj, r}^{\alpha+\beta} p^2) F_j =
\Phi_{jn}^{\beta} q_n^{\beta} - \Phi_{jn}^{\alpha} q_n^{\alpha} +
\]

\[
(G_{ja, r}^{\beta} - B_{ja, r}^{\beta} p - H_{ja, r}^{\beta} p^2) F_a^{\beta} -
\]

\[
(G_{ja, r}^{\alpha} - B_{ja, r}^{\alpha} p - H_{ja, r}^{\alpha} p^2) F_a^{\alpha}.
\]

Equation (6.3.6) represents a junction-sized set of differential equations in terms of the junction forces. Equations (6.3.6), (6.3.3), and (6.3.4) describe the general formulation for the decoupled analysis of combined components with a second order account of the residual effects.

A further approximation is made in order to facilitate the decoupled approach. Only the contribution of the first-order terms will be considered in the decoupled approach; therefore, the residual damping and inertia terms are dropped owing to their second-order contribution to the residual displacements. The first-order residual flexibility representation of a component reduces Equation (6.3.6) to a set of junction-sized simultaneous algebraic equations in terms of the junction forces given by

\[
G_{jj, r}^{\alpha+\beta} F_j = \Phi_{jn}^{\beta} q_n^{\beta} - \Phi_{jn}^{\alpha} q_n^{\alpha} +
\]

\[
G_{ja, r}^{\beta} F_a^{\beta} - G_{ja, r}^{\alpha} F_a^{\alpha}.
\]

(6.3.7)
6.4 Numerical Implementation - Linear Analysis

A standard Newmark integration algorithm is used in the implementation of the flexibility method. The constants of integration in the Newmark method are defined in terms of the quadrature parameter

\[ a_0 = \frac{1}{\beta h^2}, \quad a_1 = \frac{1}{2\beta h}, \quad a_2 = \frac{1}{\beta h}, \]
\[ a_3 = \frac{1}{2\beta} - 1, \quad a_4 = \frac{h}{2} \left( \frac{1}{2\beta} - 2 \right). \]  
(6.4.1)

The finite difference approximations for the derivatives are

\[ \dot{q}_{n,i+1} = \dot{q}_{n,i} + (\ddot{q}_{n,1} + \ddot{q}_{n,i+1})h, \]  
(6.4.2)

and

\[ \ddot{q}_{n,i+1} = a_0 (q_{n,i+1} - q_{n,i}) - a_2 \ddot{q}_{n,i} - a_3 \dddot{q}_{n,i}. \]  
(6.4.3)

Substituting Equations (6.4.2) and (6.4.3) into (6.3.1) and (6.3.2), the following finite difference approximations can be formulated for the components

\[ \bar{K}_n q_{n,i+1} = \bar{F}_{n,i+1} + \Phi \tau_{jn} \bar{F}_{j,i+1} \]  
(6.4.4)

\[ \bar{K}_n q_{n,i+1} = \bar{F}_{n,i+1} - \Phi \tau_{jn} \bar{F}_{j,i+1}. \]  
(6.4.5)
where the effective stiffness is defined as

$$\bar{k}_n = a_0 m_n + a_1 c_n + k_n,$$  \hspace{1cm} (6.4.6)

and the effective force is

$$\bar{F}_{n,t+i+1} = \Phi_\text{an} \mathbf{F}_{a,t+i+1} + m_n (a_0 q_{n,i} + a_2 \dot{q}_{n,i} + a_3 \ddot{q}_{n,i}) + c_n (a_1 q_{n,i} + a_3 \dot{q}_{n,i} + a_4 \ddot{q}_{n,i}).$$  \hspace{1cm} (6.4.7)

Equation (6.3.7) for time step i+1 can be written as

$$G_{\alpha+\beta}^{\text{ja,r}} F_{t,i+1} = \Phi_{\alpha}^{\text{jn}} q_{n,i+1}^{\beta} - \Phi_{\beta}^{\text{jn}} q_{n,i+1}^{\alpha} + G_{\alpha}^{\text{ja,r}} F_{a,t,i+1}^{\beta} - G_{\beta}^{\text{ja,r}} F_{a,t,i+1}^{\alpha}.$$  \hspace{1cm} (6.4.8)

The solution for the junction forces for time-step i+1 depends on the component modal displacements at i+1. Solving Equations (6.4.4) and (6.4.5) for the modal displacements at i+1 and substituting into (6.4.8) yields

$$\bar{G}_{\alpha+\beta}^{\text{jj}} F_{j,i+1} = X_{j,i+1},$$  \hspace{1cm} (6.4.9)

where
\[ \bar{G}^{\alpha+\beta}_{jj} = G_{jj,r}^{\alpha+\beta} + \Phi_{jn}^{\alpha} k_{jn,n}^{-\alpha} \Phi_{jn}^{\tau} + \Phi_{jn,n}^{\beta} k_{jn}^{-\beta} \Phi_{jn}^{\tau} \] (6.4.10)

and

\[ X_{j,i+1} = \Phi_{jn}^{\beta} f_{jn,n,i+1}^{-\beta} - \Phi_{jn,n}^{\alpha} k_{jn}^{-\alpha} f_{jn,n,i+1}^{\alpha} + G_{Ja,r}^{\beta} F_{a,i+1}^{-\beta} - G_{Ja,r}^{\alpha} F_{a,i+1}^{\alpha}. \] (6.4.11)

Note that the inverses in Equation (6.4.10) involve diagonal matrices and are trivial computationally. Equation (6.4.9) has the same basic form as the flexibility formulation in statics. It is the solution of this flexibility equation that yields the unknown junction forces at a given time-step. Since the flexibility coefficients in Equation (6.4.10) are constant for a constant time-step, the junction-sized matrix has to be inverted only once. From the solution of Equation (6.4.9), Equations (6.4.4) and (6.4.5) are solved for the component modal displacements. The modal velocities and accelerations for each component are then solved by using Equations (6.4.2) and (6.4.3). The effective force in Equation (6.4.7) is then computed for each component from which Equation (6.4.11) is updated for the next time-step. The analysis proceeds for the desired length of time base-driving each component separately by computing the junction forces through a small statics problem.

6.5 Mathematical Background - Nonlinear Analysis

In this section, components with nonlinear junctions which render the combined system nonlinear are treated. First-order residual flexibility of the unretained component modes are added back to the linear and nonlinear coordinates at the junctions of each
component in order to retain the full flexibility of these coordinates which results in accurate junction forces. The thrust of the method relies on formulating a junction-sized set of algebraic equations at time \(i+1\) in terms of the linear and the nonlinear junction-forces. From here, using a predictor-corrector scheme along with equilibrium iterations, highly reliable values of the nonlinear forces are computed at the time-step. Given the linear and the nonlinear forces at a given time-step, the modal equations of dynamic equilibrium for the individual components are integrated separately, in a decoupled manner. As a consequence of the decoupling of the components, the component damping problem which resulted in a nonclassically damped system in Component-Mode Synthesis, presents no additional difficulties or computational expense.

The incremental modal equations for dynamic equilibrium of component \(\alpha\) at time-step \(i+1\) are

\[
(m_n^\alpha p^2 + c_n^\alpha p + k_n^\alpha)q_{n,i+1}^\alpha = \Phi_{an}^{\alpha T}F_{a,i+1}^\alpha + \Phi_{jn}^{\alpha T}F_{j,i+1}^\alpha + \Phi_{jn}^{NL}F_{j,i+1}^{NL} \tag{6.5.1}
\]

where the nonlinearities at the junction coordinates are treated as a nonlinear force vector on the right hand side of the equations. Taking into account the first-order residual flexibility of the unretained modes of vibration at the linear and nonlinear junction degrees of freedom, the junction displacements of the linear coordinates for component \(\alpha\) are

\[
u^L_{j,i+1} = \Phi_{jn}^{L\alpha}q_{n,i+1}^\alpha + G_{jj,r}^{L\alpha}F^L_{j,i+1} + G_{jj,r}^{NL\alpha}F_{j,i+1}^{NL} \tag{6.5.2}
\]
where the superscripts L and NL stand for linear and nonlinear coordinates, respectively.

Similarly, for component $\beta$, the equations for dynamic equilibrium and junction displacements of the linear coordinates are,

\[
(m_n^{\beta}p^2 + c_n^{\beta}p + k_n^{\beta})q_n,i+1^\beta = 
\Phi_{an}^{\beta T}F_{a,i+1}^\beta - \Phi_{jn}^{sT}F_{j,i+1}^\beta - \Phi_{jn}^{NL}F_{j,i+1}^{NL} \tag{6.5.3}
\]

and,

\[
u_{j,i+1}^\beta = \Phi_{jn}^{\beta}q_{n,i+1}^\beta - G_{jj,r}^{L}F_{j,i+1}^\beta -
G_{jj,r}^{NL}F_{j,i+1}^{NL} + G_{ja,r}^{L}F_{a,i+1}^\beta \tag{6.5.4}
\]

Displacement compatibility at the linear junction coordinates requires that for all time,

\[
u_{j,i+1}^\beta = \nu_{j,i+1}^\alpha \tag{6.5.5}
\]

As a consequence of the compatibility of junction displacements at the linear coordinates, the following equation is derived by equating Equations (6.5.2) and (6.5.4),

\[
(G_{jj,r}^{L,\alpha} + G_{jj,r}^{L,\beta})F_{j,i+1}^\alpha +
\Phi_{jn}^{\beta}q_{n,i+1}^\beta - \Phi_{jn}^{\alpha}q_{n,i+1}^\alpha +
G_{ja,r}^{L}F_{a,i+1}^\beta - G_{ja,r}^{\alpha}F_{a,i+1}^\alpha -
(G_{jj,r}^{NL,\alpha} + G_{jj,r}^{NL,\beta})F_{NL}^{NL,j,i+1} \tag{6.5.6}
\]
Equation (6.5.6) is a junction-sized system of equations which expresses the unknown linear junction forces at time i+1 in terms of the component modal displacements at i+1, the component external nodal force vectors at i+1 and the unknown nonlinear junction forces at i+1. In the following section, an efficient numerical algorithm, taking full advantage of equilibrium iterations, is adopted to solve Equation (6.5.6) and converge to highly reliable values of the linear and nonlinear force vectors at the junctions.

6.6 Numerical Implementation - Nonlinear Analysis

Substituting Equations (6.4.2) and (6.4.3) into (6.5.1) and (6.5.3), the following finite difference approximations can be formulated for the component's nodal equations of dynamic equilibrium,

\[ \bar{K}_n^\alpha \bar{q}_{n,i+1}^\alpha = \bar{F}_{n,i+1}^\alpha + \Phi_{jn}^{\text{L}} \bar{T}_j^{\text{L}} \bar{F}_{j,i+1}^\alpha + \Phi_{jn}^{\text{NL}} \bar{T}_j^{\text{NL}} \bar{F}_{j,i+1}^\alpha \]  (6.6.1)

and,

\[ \bar{K}_n^\beta \bar{q}_{n,i+1}^\beta = \bar{F}_{n,i+1}^\beta - \Phi_{jn}^{\text{L}} \bar{T}_j^{\text{L}} \bar{F}_{j,i+1}^\beta - \Phi_{jn}^{\text{NL}} \bar{T}_j^{\text{NL}} \bar{F}_{j,i+1}^\beta \]  (6.6.2)

where the effective stiffness and force are defined in Equations (6.4.6) and (6.4.7).

The solution for the linear and nonlinear junction forces for time step i+1 is depended on the component modal displacements at i+1. Solving Equations (6.6.1) and (6.6.2) for the modal displacements at i+1 and substituting into Equation (6.5.6) results in,
\[
\overline{G}_{jj}^{L,L+\alpha+\beta} 
= X_{j,i+1}^L - \overline{G}_{jj}^{L,NL,\alpha+\beta} F_{j,i+1}^{NL} 
\]

(6.6.3)

where,

\[
\overline{G}_{jj}^{L,L,\alpha+\beta} = (G_{jj,\alpha}^{L,L} + G_{jj,\beta}^{L,L}) +
\Phi_{jn}^{\alpha} \Phi_{jn}^{\alpha} + \Phi_{jn}^{\beta} \Phi_{jn}^{\beta},
\]

(6.6.4)

\[
\overline{G}_{jj}^{L,NL,\alpha+\beta} = (G_{jj,\alpha}^{L,NL} + G_{jj,\beta}^{L,NL}) +
\Phi_{jn}^{\alpha} \Phi_{jn}^{\alpha} \Phi_{jn}^{\alpha} \Phi_{jn}^{\alpha} + \Phi_{jn}^{\beta} \Phi_{jn}^{\beta} \Phi_{jn}^{\beta} \Phi_{jn}^{\beta},
\]

(6.6.5)

and

\[
X_{j,i+1}^L = \Phi_{jn}^{\beta} F_{n,i+1}^{\beta} - \Phi_{jn}^{\alpha} F_{n,i+1}^{\alpha} +
G_{ja,\alpha}^{L} F_{a,i+1}^{\alpha} - G_{ja,\beta}^{L} F_{a,i+1}^{\beta}
\]

(6.6.6)

The nonlinear forces at the junctions of the components at time \( i+1 \) may be predicted by a truncated Taylor series in terms of the displacements, velocities, and accelerations at the nonlinear junction coordinates by

\[
F_{j,i+1}^{NL} = F_{j}^{NL}(u_{j,i}^{NL} + \dot{u}_{j,i}^{NL} h, \dot{u}_{j,i}^{NL} + \ddot{u}_{j,i}^{NL} h)
\]

(6.6.7)

where the component displacements, velocities, and accelerations at the nonlinear junction coordinates are
\[ u_{j,i+1}^{NL} = \Phi_{jn}^{NL} q_{n,i+1} + G_{jj}^{NL} F_{j,i+1} + G_{ja}^{NL} F_{a,i+1} \]

\[ j,i+1 = \Phi_{jn}^{NL} \dot{q}_{n,i+1} \]  

\[ i,i+1 = \Phi_{jn}^{NL} \ddot{q}_{n,i+1} \]  

The equilibrium iterations are initiated by predicting the value of the nonlinear force vector based on the displacements, velocities, and acceleration of the nonlinear coordinates and the appropriate nonlinear law using the truncated Taylor series expansion in Equation (6.6.7). From this, the junction-sized set of algebraic equations, Equation (6.6.3), is solved for the linear forces at the junction coordinates. The predicted value of the nonlinear force and the linear force are then used in Equations (6.6.1) and (6.6.2) to solve for the component modal displacements and then Equations (6.4.2) and (6.4.3), respectively, for the component modal velocities and accelerations. From Equations (6.6.8), (6.6.9), and (6.6.10) the displacements, velocities, and accelerations of the nonlinear coordinates are determined and the nonlinear force vector is then corrected based on the governing nonlinear law. The iterations continue until convergence occurs based on the following criteria for the nonlinear forces

\[ \left| \frac{F_{j,i+1}^{NL}}{(s)_{j,i+1}} - \frac{F_{j,i+1}^{NL}}{(s-1)_{j,i+1}} \right| \leq \delta \]  

where \( s \) is the iteration step and \( \delta \) is the equilibrium iteration tolerance value. This scheme continues step-by-step until the time-history for the linear and nonlinear forces at the junctions have been determined.
6.7 Results and Discussions

This chapter presented an efficient alternative to the method of Component-Mode Synthesis which departs from the notions of synthesizing component modes and determining system modes. A set of junction-sized algebraic equations in terms of the unknown junction forces are derived and solved at a time point from which the equations of motion for the different components are integrated in a decoupled manner. Due to the decoupling formulation, the nonlinear, nonclassically damped problem is a treated without further complications.

In this section, the convergence characteristics of the residual flexibility approach to decoupled analysis for different number of retained component modes versus the exact solution is investigated. Furthermore, a comparison of this decoupled analysis technique and Free-Junction Component-Mode Synthesis for the nonclassically damped problem is conducted for the first-order decoupled approach. Finally, solutions to the nonlinear, nonclassically damped system considering different types of junction nonlinearities are computed.

Figures (6.1-6.5) present the convergence characteristics of the residual flexibility decoupled method subject to a suddenly applied force. The kinematics at node 1, dof 2 and junction forces for the fixed-fixed beam system considered throughout this dissertation are considered. The components are undamped and represented via a two mode and a four mode representation with residual flexibility. It is observed that the four mode component approximation shows adequate convergence to the exact solution with the exception of the junction shear as noted in the prior chapters.

Figures (6.6-6.10) shed some light on to the effect of first-order residual flexibility on system response acted upon by the suddenly applied loading. It is clearly seen that the first-order static correction is necessary in order to provide adequate convergence for the
four mode solution in the very same manner that the mode-acceleration method discussed in chapter 2 provides rapid convergence.

Figures (6.11-6.15) compare the first-order residual flexibility decoupled approach to Free-Junction Component-Mode Synthesis for the suddenly applied force case. Four normal modes are retained per component with a residual flexibility account of the unretained modes at the junction coordinates. The problem is nonclassically damped with each component damped at 1% of critical. The convergence characteristics of the decoupled approach are similar to the Free-Junction Synthesis for the kinematics. The Free-junction Synthesis convergence to the exact solution for junction shear is superior to the first-order decoupled analysis for four modes retained. This is attributed to the retention of residual inertia term in Free-Junction Synthesis. It can be shown that retaining an additional bending mode per component in the first-order decoupled method produces excellent convergence of the junction shear. The junction moment converges adequately with four component modes augmented with first-order residual corrections.

Figures (6.16-6.19) present results from the nonlinear, nonclassically damped problem. In the nonlinear problem, the components are connected via a nonlinear spring in shear, with a spring stiffness of 1000 lb/in. The linear solution is compared to the quadratic and the cubic nonlinearity.
Decoupled Analysis
Residual Flexibility Approach
Node 1, DOF 2

Figure (6.1) - Convergence of Displacement vs Number of Retained Free-Junction Normal Modes per Component; Step-pulse Forcing Function
Decoupled Analysis
Residual Flexibility Approach
Node 1, DOF 2

Figure (6.2) - Convergence of Velocity vs Number of Retained Free-Junction Normal Modes per Component; Step-pulse Forcing Function
Decoupled Analysis
Residual Flexibility Approach
Node 1, DOF 2

Figure (6.3) - Convergence of Acceleration vs Number of Retained Free-Junction Normal Modes per Component; Step-pulse Forcing Function
Decoupled Analysis
Residual Flexibility Approach
Junction Shear

Figure (6.4) - Convergence of Junction Shear vs Number of Retained Free-Junction Normal Modes per Component; Step-pulse Forcing Function
Decoupled Analysis
Residual Flexibility Approach
Junction Moment

Figure (6.5) - Convergence of Junction Moment vs Number of Retained Free-Junction Normal Modes per Component; Step-pulse Forcing Function
Decoupled Analysis
Residual Flexibility Approach
Node 1, DOF 2

Figure (6.6) - Comparison of Displacements in Zeroth-Order and First-Order Decoupled Methods, Four Modes Retained per Component; Step-pulse Forcing Function
Decoupled Analysis
Residual Flexibility Approach
Node 1, DOF 2

- - - - - - zeroth-order  - - - - - - first-order  - - - - - - exact

Figure (6.7) - Comparison of Velocities in Zeroth-Order and First-Order Decoupled Method, Four Modes Retained per Component; Step-pulse Forcing Function
Decoupled Analysis
Residual Flexibility Approach
Node 1, DOF 2

Figure (6.8) - Comparison of Accelerations in Zeroth-Order and First-Order Decoupled Method, Four Modes Retained per Component; Step-pulse Forcing Function
Decoupled Analysis
Residual Flexibility Approach
Junction Shear

Figure (6.9) - Comparison of Junction Shears in Zeroth-Order and First-Order Decoupled Method, Four Modes Retained per Component; Step-pulse Forcing Function
Decoupled Analysis
Residual Flexibility Approach
Junction Moment

Figure (6.10) - Comparison of Junction Moments in Zeroth-Order and First-Order Decoupled Method, Four Modes Retained per Component; Step-pulse Forcing Function
Decoupled Analysis
Nonclassically Damped System
Node 1, DOF 2

Figure (6.11) - Comparison of Displacements in First-order Decoupled Analysis and Free-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Decoupled Analysis - Residual Flexibility Approach
Nonclassically Damped System
Node 1, DOF 2

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Figure (6.12) - Comparison of Velocities in First-order Decoupled Analysis and Free-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Decoupled Analysis
Nonclassically Damped System
Node 1, DOF 2

Figure (6.13) - Comparison of Accelerations in First-order Decoupled Analysis and Free-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Decoupled Analysis
Nonclassically Damped System
Junction Shear

Figure (6.14) - Comparison of Junction Shears in First-order Decoupled Analysis and Free-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Decoupled Analysis
Nonclassically Damped System
Junction Moment

Figure (6.15) - Comparison of Junction Moments in First-order Decoupled Analysis and Free-Junction Component-Mode Synthesis; Four Component Modes Retained, Components Damped at 1% of Critical; Step-pulse Forcing Function
Decoupled Analysis
Nonlinear, Nonclassically Damped System
Node 1, DOF 2

Figure (6.16) - Effect of Different Junction Spring Nonlinearities on Displacement Response; Components Damped at 1% of Critical; Step-pulse Forcing Function
Decoupled Analysis
Nonlinear, Nonclassically Damped System
Node 1, DOF 2

Figure (6.17) - Effect of Different Junction Spring Nonlinearities on Velocity Response; Components Damped at 1% of Critical; Step-pulse Forcing Function
Decoupled Analysis
Nonclassically Damped Problem
Node 1, DOF 2

Figure (6.18) - Effect of Different Spring Nonlinearities on Acceleration Response; Components Damped at 1% of Critical; Step-pulse Forcing Function
Decoupled Analysis
Nonclassically Damped Problem
Junction Shear

Figure (6.19) - Effect of Different Spring Nonlinearities on Junction Shear; Components Damped at 1% of Critical; Step-pulse Forcing Function
CHAPTER 7

SUMMARY AND CONCLUSION

This dissertation considered the problem of dynamic substructuring of complex systems represented by the Finite Element Method. Such systems are represented by nodal finite element dynamic equations of large order. It was noted that efficient analyses of these dynamic problems involves subdividing the complex system into a number of smaller components. Methods of approximately representing the components by reduced order equations of nodal dynamic equilibrium were presented. From these approximate representations of the components, the equations for the combined system were characterized. The methods for reducing the order of component equations and formulating the approximate system representation were referred to as Component-Mode Synthesis. Component-Mode Synthesis was broadly defined as a class of reduction methods in which the normal modes of the combined system of components is derived from the synthesis of the normal modes of its individual parts. In the second part of the dissertation, a new class of methods for efficiently analyzing the dynamics of combined systems of components without having to synthesize component modes or compute expensive system eigenproblems was described. Ideal for the nonlinear, nonclassically damped system, these methods of decoupled analysis provide a versatile and efficient alternative to the method of Component-Mode Synthesis.

Chapter 1 discussed the basic objectives of the dissertation. The method of Static Synthesis, where components are statically reduced then synthesized to form the system representation, was discussed. Emphasis was given to describing the method of Component-Mode Synthesis. It was noted that Component-Mode Synthesis is subdivided into two methods. The Fixed-Junction method incorporates fixed-junction component
normal modes of vibration along with static constraint modes to approximately represent the component equations in a rather straight-forward Rayleigh-Ritz approach. On the other hand, the Free-Junction method is considerably more difficult to formulate. The method uses the component's free-junction normal modes along with a first or second-order account of the residual flexibility of the unretained modes. Chapter 1 also presented an overview of a new method of system analysis referred to here as decoupled analysis. It was noted that these method do not require the solution of a large, expensive system eigenproblem in order to remain efficient.

The method of Fixed-Junction Component-Mode Synthesis was presented in detail in Chapter 2. The Ritz transformation was derived from component eigenproblem and statics analysis. The reduced mass and stiffness matrices for the component were characterized. It was pointed out that the transformed junction mass matrix is commonly referred to as the Guyan mass matrix, a subset of the reduced mass matrix in Static Synthesis. Furthermore, the total flexibility at the junction coordinates was preserved via the reduced stiffness matrix, formulated by using the static constraint modes. The chapter discussed synthesizing the component modes in order to formulate the system equations. Characterizing component damping for the generalized and junction partitions and synthesizing to form the system damping matrix was discussed. In order to further reduce the number of system coordinates and diagonalize the mass and stiffness matrices prior to integration, the system eigenproblem was solved. This gave rise to the nonclassically damped problem, a natural consequence of the method of Component-Mode Synthesis. Coupled integration of the system modal equations was discussed. Recoveries of junction forces via mode displacement and a superior method known as mode acceleration were presented. Finally, a method of mass and stiffness loading one component's junctions with another components junction mass and stiffness prior to the system eigenproblem was considered. This method modifies the component modes to
more closely resemble the system modes. As a consequence of this, a smaller number of component modes will be required in order to achieve accurate system response.

The emphasis in Chapter 3 was on the concept and mathematical derivation of first and second-order residual flexibility and its use in Free-Junction Component-Mode Synthesis. This method of Component-Mode Synthesis is considerably more difficult than the Fixed-Junction approach; however, studies indicate that it has superior convergence when the component has a large number of junction coordinates. First-order residual flexibility was formulated by subtracting the flexibility of the retained modes of vibration from the total flexibility of the component. The contribution first-order residual flexibility was then added back to the junction coordinates of the component, preserving the total flexibility where junction forces are to be computed. The second-order residual flexibility, damping, and mass matrices were also derived from basic considerations. Again, their effect on the junction coordinates was added back to the component equations of motion. Numerical studies were conducted comparing the convergence of the Fixed-Junction approach to Free-Junction for the nonclassically damped problem. For the sample problem chosen involving two cantilevered beam components excited by a suddenly applied load, both methods converged equally well to the exact solution with four normal modes retained per component.

A variation to the method of Component-Mode Synthesis was presented in Chapter 4. A Hybrid formulation, incorporating junction forces as generalized coordinates along with free-junction normal mode coordinates was developed. This mixed formulation took advantage of second-order residual flexibility in order to improve the convergence of the response. A synthesis transformation which simultaneously invoked displacement compatibility and force equilibrium at the junction coordinates was derived based on first-order residual flexibility considerations. The final system coordinates prior to system eigensolution were simply the individual component free-junction generalized coordinates. A direct method of recovering junction forces was
presented based on using a partition of the synthesis transformation and system modes. Numerical studies indicate that the method has adequate convergence characteristics; however, the Free-Junction Component-Mode Synthesis has superior convergence.

A method of decoupled analysis based on a fixed-junction representation of the component was presented in Chapter 5. In it, a junction-sized set of differential equations was derived and numerically integrated to yield the junction accelerations. The junction accelerations were in turn used to base-drive the component's equations for that step independent from one another. The decoupled approach does not require the formulation of system matrices or the solution of a system eigenproblem. The method is designed for the nonclassically damped problem and is further extended to the nonlinear problem. The convergence characteristics of the method are comparable to Fixed-Junction Component-Mode Synthesis.

A residual flexibility approach to decoupled analysis was presented in Chapter 6. Like the decoupled approach in Chapter 5, this method does not require the synthesis of component modes or the solution of system modes. The junction forces are determined directly from the solution of a junction-sized set of algebraic equations. From the knowledge of the junction forces at a given step, the equations of the components are driven in a decoupled manner. The approach lends itself well to the nonlinear, nonclassically damped system problem. In the nonlinear problem, an iterative scheme to converge on a reliable value of the nonlinear force is implemented. This decoupled approach differs from the decoupled approach in Chapter 5 in the following ways,

(i) All integrations are conducted on uncoupled equations, whether the analysis is linear or nonlinear. The equations of motion for the components in the decoupled approach in Chapter 5 are partially coupled in mass and stiffness.

(ii) There is no need to conduct an eigenproblem of any type. The decoupled approach in Chapter 5 requires a small order eigenproblem in order to implement junction damping
since the component normal modes are fixed-junction. The component junctions in the
decoupled approach in Chapter 6 are inherently damped since component level damping
is applied on the free-junction normal modes.

(iii) The junction force are computed directly. In the decoupled approach in Chapter 5,
as well as all methods of Component-Mode Synthesis, the junction forces have to be
recovered once the component response has been determined.

(iv) The formulation does not require a Ritz transformation in order to represent the
component prior to the decoupling formulation. All that is required are the component's
free-junction normal modes and residual flexibility at the junction coordinates.

The method of decoupled analysis presents an efficient, versatile tool for analysis
of nonlinear, nonclassically damped systems of components. Extension of the method
presented in Chapter 6 to the Multibody problem where the components undergo large
rigid-body rotations would be a natural progression. It is believed that due to its many
advantages over the traditional method of Component-Mode Synthesis, the methods of
decoupled analysis deserve serious attention in dynamic analysis of linear and nonlinear
systems.
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