INFORMATION TO USERS

The most advanced technology has been used to photograph and reproduce this manuscript from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI
University Microfilms International
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor MI 48106-1346 USA
313 761-4700  800 521-0600
Filter approaches to stochastic dynamic analysis of compliant offshore platforms

Bhattacharjee, Subir, Ph.D.

Rice University, 1990
NOTE TO USERS

THE ORIGINAL DOCUMENT RECEIVED BY U.M.I. CONTAINED PAGES WITH SLANTED PRINT. PAGES WERE FILMED AS RECEIVED.

THIS REPRODUCTION IS THE BEST AVAILABLE COPY.
RICE UNIVERSITY

FILTER APPROACHES TO STOCHASTIC DYNAMIC ANALYSIS
OF COMPLIANT OFFSHORE PLATFORMS

by

SUBIR BHATTACHARJEE

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

DOCTOR OF PHILOSOPHY

APPROVED, THESIS COMMITTEE:

Dr. Pol D. Spanos, (Advisor)
Professor of Mechanical and Civil
Engineering

Dr. John E. Akin
Professor and Chairman,
Department of Mechanical Engineering

Dr. Ruben D. Cohen
Assistant Professor of Mechanical
Engineering

Dr. Panos C. Dakoulas
Assistant Professor of
Civil Engineering

Houston, Texas
March, 1990
FILTER APPROACHES TO STOCHASTIC DYNAMIC ANALYSIS
OF COMPLIANT OFFSHORE PLATFORMS

SUBIR BHATTACHARJEE

ABSTRACT

The dynamics of offshore structures subjected to random waves is investigated. The load modeling and response analysis are conducted by using a new approach. The problems of the wave kinematics simulation and of the dynamic response determination are treated in an unified approach using filter techniques and the state space analysis.

Some interesting results on water depth characterization in connection with the two parameter spectra are derived. These permit the application of the linear wave theory from monochromatic waves to random seas. Throughout this investigation filters which can generate the relevant wave kinematics or represent the random wave forces are sought. This is achieved by relying on developments in the field of signal processing and filter design.

Novel platform concepts which hold promise for offshore oil production in deep waters are adopted for the dynamic analysis. In this context, attention is focussed on the efficacy of the method of state space analysis in the time domain, using the well known Liapunov equation. The advantages of this technique, particularly when motion control is of interest, is emphasized.

The dynamic response of an idealized guyed tower is obtained in a closed form by using the Liapunov equation. A complete numerical example is included for the guyed tower. An alternate solution in the frequency domain is also obtained for comparison.

Finally, the applicability of the state space method for modeling the wave excitation on an idealized 3 degree of freedom tension leg platform is investigated. The tension leg platform dynamics is examined conceptually, and further research recommended. Some key problems in the use of this powerful method are identified.
To the memory of my father,
a humble teacher,
who had always encouraged me to dream;
and to my mother,
who with constant love and care,
taught me how to persevere
and turn the dream into a reality.
Acknowledgement

Firstly, I would like to express my deep sense of gratitude to Prof. Spanos, the thesis supervisor, for his continued guidance, help and support throughout the research, and the numerous stimulating ideas offered by him. I sincerely thank Prof. Akin, Prof. Cohen and Prof. Dakoulas, my thesis committee members, for their time, help and cooperation in spite of their busy schedules. I am thankful to the Mechanical Engineering Department for providing excellent research facilities, and financial support in way of fellowships. I am also grateful to the NSF and other agencies that provided financial support for various stages of the research.

I thank all my colleagues at Hudson Engineering, and the management for their cooperation. Special thanks are due to Mr. Trevor Mills for his help in allowing me to take time off in the midst of a busy production schedule. I am also specially thankful to my colleague Mr. Sid Sircar for numerous stimulating discussions and practical suggestions.

I have deep appreciation for my research colleagues at Rice University, particularly Dr. Roger Ghanem, Mr. Tein and Mr. Mark Donley for their ready help and cooperation. I would also like to thank the staff of the Mechanical Engineering Department for their cooperation, and particularly Mrs. Lee Richardson, for her voluntary help on numerous occasions.

Finally, I would like to express my deepest gratitude to those who have provided me personal help and emotional support at various stages. I thank all members of my family for their patience and love. Special thanks are due to my mother for all the inspiration, my brother Arun for putting me on the path that brought me here, and Dr. Dipak Bhattacharyya who has deeply influenced my thinking. Last but not the least, I thank Mr. Champak Sadhu and Mrs. Chandra Sadhu for their generous help and emotional support at difficult times. Particular thanks are due to Mrs. Sadhu for not only volunteering her time to type a good part of the thesis, but for taking the trouble of learning a difficult computer language for typing the equations.
# Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF FIGURES</td>
<td>x</td>
</tr>
<tr>
<td>CHAPTER I: Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Overview</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Background</td>
<td>1</td>
</tr>
<tr>
<td>1.2.1 Generation and Propagation of Ocean Waves</td>
<td>4</td>
</tr>
<tr>
<td>1.2.2 Probabilistic Dynamics of Compliant Platforms</td>
<td>7</td>
</tr>
<tr>
<td>1.3 Literature Survey</td>
<td>9</td>
</tr>
<tr>
<td>1.3.1 Stochastic Representation of Waves</td>
<td>10</td>
</tr>
<tr>
<td>1.3.2 Simulation of Irregular Waves</td>
<td>12</td>
</tr>
<tr>
<td>1.3.2.1 Computer Simulation of Random Waves</td>
<td>12</td>
</tr>
<tr>
<td>1.3.2.2 Random Wave Generation in Model Tanks</td>
<td>14</td>
</tr>
<tr>
<td>1.3.3 Stochastic Response of Ships and Compliant Platforms</td>
<td>15</td>
</tr>
<tr>
<td>1.4 Objective</td>
<td>18</td>
</tr>
<tr>
<td>CHAPTER II: Extension of Airy Wave Criteria to Random Waves</td>
<td>20</td>
</tr>
<tr>
<td>2.1 Overview</td>
<td>20</td>
</tr>
<tr>
<td>2.2 Introduction</td>
<td>20</td>
</tr>
<tr>
<td>2.3 Theory of Linearized Gravity Waves</td>
<td>21</td>
</tr>
<tr>
<td>2.3.1 The Bottom Boundary Condition (BBC)</td>
<td>22</td>
</tr>
</tbody>
</table>
2.8.1.3 Total Force ........................................................................................................ 48
2.8.2 Interactive Form of Morison Equation ................................................................. 49
2.8.3 Coefficients In Morison Equation ........................................................................ 50
2.8.4 Wave Force Regimes ........................................................................................... 52
2.9 Critical Member Dimension For Use of Morison Equation ..................................... 53
2.10 Conclusions ........................................................................................................... 54

CHAPTER III : Simulation Through Harmonic Superposition .................................... 56
3.1 Overview ................................................................................................................ 56
3.2 Introduction .............................................................................................................. 56
3.3 Representation of a Random Sea ............................................................................. 57
  3.3.1 Random Phase Model ......................................................................................... 57
  3.3.2 Random Amplitude Model ................................................................................ 58
  3.3.3 Comparison of DSA and NSA Models ................................................................. 59
3.4 Advantages and Drawbacks of Harmonic Superposition ......................................... 60
3.5 The Fast Fourier Transform (FFT) Algorithm ......................................................... 61
3.6 Conclusions ............................................................................................................. 62

CHAPTER IV : Filter Theory and Wave Simulation ....................................................... 63
4.1 Overview ................................................................................................................ 63
4.2 Introduction ............................................................................................................. 63
4.3 Transforms Relating the Time and Frequency Domains ......................................... 64
  4.3.1 Laplace Transforms ......................................................................................... 65
CHAPTER V: Filter Design and Implementation ................................................. 87

5.1 Overview ......................................................................................... 87

5.2 Introduction .................................................................................... 88

5.3 Propagation of Kinematics in Shallow Water ........................................ 88

5.3.1 Horizontal Propagation ................................................................. 89

5.3.2 Vertical Propagation ..................................................................... 90

5.3.2.1 Horizontal Component .............................................................. 91

5.3.2.2 Vertical Component ................................................................. 92

5.4 Propagation of Kinematics in Intermediate Water Depth ....................... 92

5.4.1 Vertical Propagation ..................................................................... 93

5.4.1.1 Horizontal Velocity Component ............................................... 93

5.4.1.2 Vertical Velocity Component ................................................... 95

5.4.1.3 Limitations of Cascaded Oscillator Models ............................... 96

5.4.1.4 Symmetric MA Filter For Vertical Propagation ....................... 101

5.5 Horizontal Propagation in Intermediate Water Depth ........................... 103

5.5.1 Horizontal Propagation Through M-A Filters .................................. 104

5.5.2 Horizontal Propagation with All-Pass Filters ................................... 105

5.6 Relation Between Vertical and Horizontal Components ....................... 109

5.7 Filter Implementation and Related Issues ............................................. 110
5.7.1 Alternate Choice of Simulation and Propagation Schemes ............................................. 110

5.7.2 Non-Dimensionalization ...................................................................................... 112

5.8 Conclusion ............................................................................................................. 113

CHAPTER VI : Theory of State-Space Analysis of Dynamic Systems ........................................ 116

6.1 Overview .............................................................................................................. 116

6.2 Introduction ......................................................................................................... 116

6.3 State Space Applications in Random Vibration ..................................................... 117

6.4 Concept of State Variables .................................................................................. 118

6.4.1 Controllability .................................................................................................. 121

6.4.2 Observability ................................................................................................... 122

6.5 State Space Realization and Transfer Functions ................................................... 124

6.5.1 Transfer Functions .......................................................................................... 124

6.5.2 Impulse response Function .............................................................................. 125

6.5.3 State Space Representation from Transfer Functions ....................................... 126

6.6 Stochastic Processes ............................................................................................ 127

6.6.1 Gaussian Process ............................................................................................. 128

6.6.2 Markov Process ............................................................................................... 129

6.6.3 Gauss-Markov Process and White Noise ......................................................... 130

6.6.4 Markovianization of a Non-Markovian System .............................................. 132

6.7 Formulation of the Covariance Equation ............................................................... 133

6.7.1 Single Degree of Freedom with White Noise Input ......................................... 134
6.7.2 Filter Augmentation for Non-White Input ........................................... 135
6.7.3 Multi Degree Freedom System with Non-White Input .......................... 136
6.8 Conclusions ......................................................................................... 139

CHAPTER VII : State-Space Analysis of the Guyed Tower Dynamics ............... 140
7.1 Overview ............................................................................................. 140
7.2 Introduction and Background ............................................................... 141
7.3 Guyed Tower Concept .......................................................................... 142
7.4 Model Idealization for Analysis ............................................................. 144
7.4.1 Assumptions ...................................................................................... 144
7.4.2 Mooring System Stiffness ................................................................. 145
7.4.3 Equation of Motion of Rigid Tower .................................................. 146
7.4.4 Excitation Model ............................................................................... 147
7.5 State Space Equations .......................................................................... 148
7.6 Second Moment Statistics .................................................................... 151
7.6.1 General Procedure ........................................................................... 151
7.6.2 Direct Determination of Covariance ................................................ 152
7.7 Stationary Response of Guyed Towers .................................................. 153
7.7.1 Solution of the Liapunov Equation .................................................... 154
7.8 Frequency Domain Solution .................................................................. 155
7.9 Numerical Example .............................................................................. 158
7.9.1 Tower Particulars .............................................................................. 158
7.9.2 Mooring System ................................................................. 158
7.9.3 Excitation Modeling .......................................................... 160
7.10 Conclusions ........................................................................ 163

CHAPTER VIII: State Space Analysis Concepts For TLP Dynamics .............. 164
8.1 Overview ............................................................................. 164
8.2 Introduction ......................................................................... 164
8.3 Literature Survey ............................................................... 165
8.4 Excitation Forces on TLP ..................................................... 167
  8.4.1 Surge Force .................................................................... 169
    8.4.1.1 Surge Force on Columns ........................................... 169
    8.4.1.2 Surge Force on Hulls ............................................... 170
    8.4.1.3 Total Surge Force ................................................... 170
  8.4.2 Heave Excitation ........................................................... 171
    8.4.2.1 Column Force ......................................................... 172
    8.4.2.2 Vertical Hull Force .................................................. 172
    8.4.2.3 Total Heave Force ................................................... 173
  8.4.3 Pitch Response .............................................................. 174
    8.4.3.1 Column Force Effects .............................................. 175
    8.4.3.2 Vertical Acceleration Effects ................................. 175
    8.4.3.3 Dynamic Pressure Contribution .............................. 176
    8.4.3.4 Total Pitch Moment ............................................... 177
8.5 Approximation for the Excitation ................................................................. 177
8.6 Conclusion ........................................................................................................ 179

CHAPTER IX: Summary and Conclusions ............................................................ 181
  9.1 Overview ........................................................................................................ 181
  9.2 Conclusions .................................................................................................... 181
  9.4 Areas Requiring Further Research ............................................................... 183

APPENDIX A: Equivalent Linearization Procedure ............................................... 185
APPENDIX B: Cross-Covariance of Input and Output ........................................... 188
APPENDIX C: Response Statistics of Guyed Tower ............................................. 189
APPENDIX D: Derivation of TLP Column Forces ................................................. 197
FIGURES ............................................................................................................... 199
REFERENCES ......................................................................................................... 238
List of Figures

Fig. 2.1(a) Effect of Wind Velocity on Pierson-Moskowitz Spectrum of Horizontal Velocity at SWL (Depth = 300 mt) ................................................................. 199

Fig. 2.1(b) Effect of Wind Velocity on Pierson-Moskowitz Spectrum of Horizontal Velocity at SWL (Depth = 150 mt) ................................................................. 200

Fig. 2.2(a) Effect of Local Water Depth on Pierson-Moskowitz Spectrum of Horizontal Velocity at SWL (Wind vel = 60 Knots) ....................................................... 201

Fig. 2.2(b) Effect of Local Water Depth on Pierson-Moskowitz Spectrum of Horizontal Velocity at SWL (Wind vel = 30 Knots) ....................................................... 202

Fig. 2.3 Critical Water Depth for Validity of "Deep Water Assumption" For Various Fractional Energy Contributions ................................................................. 203

Fig. 2.4 Critical Water Depth for Validity of "Shallow Water Assumption" For Various Fractional Energy Contributions ................................................................. 204

Fig. 5.1 Impulse Response for Vertical Propagation of Horizontal Velocity ........................................ 205

Fig. 5.2 Impulse Response for Vertical Propagation of Vertical Velocity ........................................... 206

Fig. 5.3 Phase Response of Oscillator Model for Vertical Propagation of Horizontal Velocity ................................................................. 207

Fig. 5.4 Phase Response of Oscillator Model for Vertical Propagation of Vertical Velocity ................................................................. 208

Fig. 5.5(a) M-A Filter Approximation of Vertical propagation of Horizontal Velocity Component ( MA Filter Order = 5 ) ................................................................. 209

Fig. 5.5(b) M-A Filter Approximation of Vertical propagation of Horizontal Velocity Component ( MA Filter Order = 7 ) ................................................................. 210
Fig. 5.5(c) M-A Filter Approximation of Vertical propagation of Horizontal Velocity Component (MA Filter Order = 10) ................................................................. 211

Fig. 5.6 Moving Average Approximation of Horizontal Propagation of Wave Kinematics ................................................................................................................. 212

Fig. 5.7 Horizontal Propagation for Variable Location Through Moving Average Filtering ........................................................................................................ 213

Fig. 5.8 Phase Response of the Moving Average Filter for Horizontal Propagation ......................................................... 214

Fig. 5.9 Fourier Series Expansion with truncation ($\mu^2$ neglected after 5 terms) ................................................................. 215

Fig. 5.10 Phase Approximation of 2nd Order All-Pass Filter for Horz Propagation of Wave Kinematics (Wind = 30 Kn, d= 2.83 mt) ................................................................................................. 216

Fig. 5.11(a) Phase Approximation of 2nd Order All-Pass Filter for Horz Propagation of Wave Kinematics (Wind = 50 Kn, d= 7.85 mt) ................................................................................................. 217

Fig. 5.11(b) Phase Approximation of 2nd Order All-Pass Filter for Horz Propagation of Wave Kinematics (Wind = 50 Kn, d=9.81 mt) ................................................................................................. 218

Fig. 5.11(c) Phase Approximation of 2nd Order All-Pass Filter for Horz Propagation of Wave Kinematics (Wind = 50 Kn, d=13.1 mt) ................................................................................................. 219

Fig. 5.12(a) Phase Approximation of 2nd Order All-Pass Filter for Horz Propagation of Wave Kinematics (Wind = 70 Kn, d= 15.39 mt) ................................................................................................. 220

Fig. 5.12(b) Phase Approximation of 2nd Order All-Pass Filter for Horz Propagation of Wave Kinematics (Wind = 75 Kn, d= 17.67 mt) ................................................................................................. 221

Fig. 5.13(a) Coefficients of 2nd Order All-Pass Filters For Different Propagation Distances (Wind = 50 Kn) ................................................................................................................................. 222

Fig. 5.13(b) Coefficients of 2nd Order All-Pass Filters For Different Propagation Distances (Wind = 70 Kn) ................................................................................................................................. 223

Fig. 5.14 Fourth Order All-Pass Filter for Horizontal Propagation (Wind = 30 kn) ................................................................. 224
Fig. 5.15 Filtering of Horizontal Velocity to obtain The Vertical Velocity with proper Phase ............................................................. 225

Fig. 5.16 Filtering of Horz. Vel. with "Lanczos Window" to obtain the Vertical velocity with proper Phase ............................................................. 226

Fig. 7.1 Model of Idealized Guyed Tower ............................................................. 227

Fig. 7.2 Stiffness Properties of 20-Line Mooring System ........................................ 228

Fig. 7.3 Verification of Radial Symmetry in System Restoring Forces ...................... 229

Fig. 7.4 Polynomial Approximation of Horizontal Restoring Force ....................... 230

Fig. 7.5 Polynomial Approximation of Vertical Restoring Force ............................ 231

Fig. 8.1 Tension Leg Platform Concept ............................................................. 232

Fig. 8.2 Tension Leg Platform Geometry ............................................................ 233

Fig. 8.3 TLP Heave Excitation Force RAO ......................................................... 234

Fig. 8.4 TLP Pitch Excitation Moment RAO ....................................................... 235

Fig. 8.5 TLP Surge Excitation Force RAO .......................................................... 236

Fig. 8.6 TLP Excitation For P-M Spectrum (50 Kn. Wind) .................................... 237
CHAPTER I
Introduction

1.1. Overview

This chapter introduces the general topic of simulation and propagation of random ocean waves and the response of compliant offshore platforms subjected to wave induced loads. To provide a proper perspective of the different aspects of the problem, a pertinent background is presented first. This is followed by a qualitative description of the underlying mechanisms of ocean wave generation and propagation. The associated problem of the response of offshore platforms to random seas is then considered. Special emphasis is given to compliant deepwater platforms and the use of an unconventional technique of analysis in the time domain. The method provides an unified approach to the problems of wave simulation and random vibration analysis, and provides the link between these two areas.

The published literature with significant contributions to the problem is reviewed. In carrying out the review process, a classification of the articles is made based on the issues addressed. Thus, the literature is classified under the broad categories of wave kinematics simulation, and platform response analysis. Finally, the aims and objectives of the investigation are summarized.

1.2. Background

The oceans have been an important part of man’s life for centuries, providing various natural resources. They have also provided a continuous challenge to man by the enormous forces contained in the waves. The waves in a severe storm may devastate coastal villages and wreck ships, harbors, lighthouses and other important structures, causing great damages. However, the same waves if harnessed properly, can be an everlasting source of energy. Attempts to fully understand the complex phenomena of wave dynamics have been only partially successful.
An accurate and improved analysis of the highly irregular wave dynamics has become even more important with the increased interest in natural resources offshore.

A significant part of the world oil and natural gas reserves lies beneath the sea-bed. The drilling and production operation to exploit these offshore oil and gas supplies is generally done from offshore platforms. The first offshore platform was installed in Louisiana in 1947 to stand in 20 feet of water. Since that time the offshore industry has seen many challenging developments. The bottom supported fixed platforms had to be replaced by novel concepts like compliant platforms, as operations moved into deeper waters. These platforms ride or "comply" with the waves, like sea weeds in moving water, rather than resist the forces due to wave action. The tension leg platform and the guyed tower are two proven designs of compliant platforms, in successful operation. These new generation of platforms, have great economic potential for deeper waters, but they also pose some extremely complex problems to the designer. The importance of the dynamic response is one of the main problems that characterizes these deep water platforms. Fatigue and second order response are some other issues that need increased attention.

To date about 10,000 platforms have been built offshore, of which about 2,000 are major platforms. There are several platforms built for water depths over 1000 feet, such as the Cognac Platform, the Lena Guyed Tower and the Bullwinkle Platform, all in the Gulf of Mexico waters. Last summer, Conoco has installed a tension leg platform to operate at a water depth of 1760 feet in the Green Canyon Field, again in the Gulf of Mexico. Currently, Shell Oil is finalizing the design of a tension leg platform for even deeper waters in the Gulf of Mexico. They are designed to withstand the 100 year worst hurricane, and the associated waves resulting from stormy seas. In the North sea area where there is another large concentration of platforms, the waves during an winter storm could be as high as 100 feet. Analyzing the loads arising from such hostile environ-
ments, and designing structures that are economic yet reliable, are proving to be extremely difficult tasks.

Though most of these platforms have successfully withstood the forces of nature, there have been major damages and some catastrophic failures. Off the Louisiana coast alone, in the two-year period between 1957-59, there was an estimated damage of about 200 million dollars in losses to drilling, production and pipeline facilities. These resulted from two major hurricanes, the Hilda and Betsy. In 1980 the Alexander Kielland, a semisubmersible rig, failed and sank in the Norwegian sector of the North sea, killing the entire crew of 123 aboard. Another accident involving the floating platform, Ocean Ranger, killed all the crew on board, in 1982. These accidents reflect inadequacy in the design and analysis or in the operational procedures adopted. The rules for the design and analysis have been continually modified, to reflect increased understanding of the situation. However, there are several areas that still require more understanding and need more reliable analytical models. The assessment of hydrodynamic loads is one such area. These loads result primarily from the effects of water currents and wave actions. The effect of current is relatively easy to assess. However, the action of waves is significantly more complex, and difficult to analyze. The difficulty lies both in modeling the wave motion itself, and in accounting for the irregular nature of the waves in a real ocean.

The irregular nature of the waves calls for a probabilistic analysis of the waves and their effects. As no two wave records in a sea will be exactly alike, the only way to represent them is by statistical means. Consequently, deterministic methods are being increasingly replaced by probabilistic approaches while analyzing wave and platform dynamics. A large body of literature is available on the stochastic representation of sea waves. However, before surveying this literature, the mechanism of the generation and propagation of ocean waves is briefly described.
1.2.1. Generation and Propagation of Ocean Waves

Surface waves on any large expanse of water are caused by winds blowing over them. The fact is not so obvious as waves can be present even without the presence of the wind. The phenomenon of ocean waves attracted considerable attention from scientists, mathematicians, and later, from engineers. Although satisfactory models for describing the waves were developed in the last century, the exact mechanism of their generation remained unexplained. It is only in the last two decades or so that some reliable theory, accounting for all the observed phenomena associated with waves, has emerged. The following section attempts to give a very simple, qualitative description of the complex process of generation and propagation of these waves.

The difference in thermal inertia of the land mass and the water, and other factors causing pressure differences in the atmosphere, set up a wind field over the water. Unlike the winds blowing over the land, there is very little hindrance to the winds over the open ocean, and they continue to blow uninterrupted. There are three factors related to the wind, that affect the waves. They are the mean velocity of the wind, the duration or time for which it is blowing, and the fetch or distance over the water that the wind field extends. These three parameters are adequate to predict the wave conditions with reasonable accuracy.

The mechanism of wave generation is best described by considering the energy transfer. The friction between the moving air particles and the water particles on the surface initiates the process. The surface layer of the water start moving with the velocity of the adjacent wind layer. However, the motion of the moving air mass is far from being streamlined. It is highly turbulent and gusty, generating random pressure fluctuations, causing surface irregularities on the calm water. The wind starts acting immediately on the projected surfaces by pressing on them directly. This creates an excellent opportunity for the transfer of energy from the wind to the
water. As a result the small surface irregularities continue to grow in size and travel in the direction of the wind propagation. The rate of transfer of energy is directly proportional to the relative velocity between the wind and the propagating waves.

The waves so formed, do not keep growing for ever, but reach a limiting state, when the wave phase velocity equals the mean velocity of the wind. The parameters that best describe the waves are the crest to trough height, the wave length, and the wave period which is directly related to wave frequency. Though the length and height of waves are relatively independent quantities there is a limiting relationship between them. Waves become unstable and break when their height exceeds one-seventh the length, or the angle at the crest becomes smaller than 120 degrees. Thus, small steep waves are constantly breaking, dissipating part of their energy as foam, and transferring the rest to longer waves as they overtake them. This overtaking occurs because waves travel at speeds that depend on their respective frequencies. Thus, the longer waves continue to grow at the expense of the smaller waves. Since the process starts with high frequency small waves, the spectrum of a developing sea will have a peak of smaller magnitude, occurring at a higher frequency, as compared to a fully developed sea. The long waves are also capable of building up to a larger height without breaking. Their size grows until an equilibrium state is reached, where no more energy can be transferred from the wind to the waves.

Both the wind and the waves contribute to the randomness. Though the wind has a predominant direction and a mean velocity, it has considerable local fluctuations in magnitude and direction of flow. The pressure fluctuations are one source of randomness in the waves. The other source of randomness is in the interaction between waves, and the process of wave breaking. The waves overlap each other because of the varying propagation speeds. This overlapping often is at some angle, due to the variability in the direction of wind flow about the mean direc-
tion. The intersection of waves traveling in different directions results in a complex interaction between them introducing further randomness in amplitudes and direction of propagation. The resulting waves are random, traveling in different directions with their respective speeds, intersecting and overlapping each other, and being modified by mutual interactions and breaking waves. Thus, it becomes extremely difficult to describe these waves by deterministic mathematical models. Hence, statistical description of the waves is chosen. The common form of representation is the power spectrum of the surface elevations, and sometimes, the probability distributions of the different wave parameters.

The behavior of waves as they travel outside the generating area changes significantly. The sharp crests become more rounded, the symmetry of the profile about the mean waterline is also restored partially. They form into groups of similar frequencies, and in this form they can travel almost unaltered for thousands of miles to distant shores. Interestingly, the nature of these waves is very close to that predicted by classical theory of linearized waves. These waves are called swells, and persist even in the absence of any wind.

The traveling waves are transformed significantly on approaching the shallow waters of the shoreline. They approach with the crests almost parallel to the shore, the waves become steeper, and some of them start breaking. The waves traveling at an angle to the shore get refracted and the crestline deforms, adapting the contour of the shore. This happens as the part of the wave reaching shallow water gets slowed down. Different sections of the same wave get slowed down at different stages to finally approach the shore being parallel locally. The breaking is related to the speed of propagation, which in shallow water depends on the water depth, and gets reduced progressively. The wavelength also decreases, increasing the height, until they finally break by becoming unstable.
1.2.2. Probabilistic Dynamics of Compliant Platforms

The wind generated waves described in the previous section form a primary source of dynamic excitation on offshore platforms. The random nature of these waves result in a random excitation, and a probabilistic analysis is required in order to obtain meaningful results for the structural response. For the early offshore platforms in relatively shallow waters, the dynamic effects are relatively insignificant. The short lengths make these structures relatively stiff, and the natural frequencies are generally well above that of the predominant wave frequencies. Thus, dynamic amplification is very small, and a quasi-static analysis is usually considered adequate.

In contrast, the present day exploration and production from the continental slopes are in considerable water depths. Structures in larger water depths tend to be relatively flexible, and more easily excited by the predominant wave frequencies. In an attempt to have economic recovery from deeper waters, the compliant platforms are proving to be the most promising concepts. By their very nature, they "comply" with the dynamic forces rather than resisting their actions. Thus, the dynamic analysis of these platforms is essential to the design process. The dynamic analysis so performed has to take into account the random nature of the wind and wave forces, and a probabilistic analysis is required.

There are two different concepts of compliant platforms in current use. They are the guyed tower, and the tension leg platform. The former is a slender structure, resting on the sea bottom on piles, and held in place by a radial array of catenary guy lines or cables. The latter is a positively buoyant floating structure held in location by a set of taut mooring lines, also known as the tendons or the tension legs, from which its name derives. The dynamics of both of these concepts are investigated in the present research. To date, the probabilistic analyses of these platforms have invariably followed one of two different approaches. The first approach analyzes the
dynamics in the frequency domain, after suitable linearization of all associated non-linearities. The second approach utilizes numerical integration of the dynamic equations of motion in the time domain. The latter approach, can account for non-linearities directly, but is only pseudo-random in nature. The random nature of the excitation in this case is handled by repeatedly generating realizations of the random excitation time series by the Monte-Carlo technique. The response statistics are obtained by taking the statistics across the sample realizations.

An alternate approach for the random vibration, the state space analysis in the time domain, has been well established in the fields of system identification and control. The method has also been successfully used to analyze building response to earthquakes, vehicle response to road surface irregularities, and structural response to turbulent wind forces. There has been some limited use of the method for the time history prediction of platform motions based on measurements contaminated by noise.

To the best of the author's knowledge, this technique of state space analysis has not been used for the random vibration analysis of offshore platforms. One of the aims of the present work is to investigate the feasibility of this technique for single and multiple degree freedom models of offshore systems, and to identify areas requiring further research.

The seemingly unrelated areas of wave kinematics simulation and that of random vibration analysis by the state space method are treated in parallel in the present investigation. This is done in order to demonstrate the versatility of the method, whereby a unified approach can be used both for the simulation and the dynamic analysis. The filter approaches developed for wave field simulation form a natural basis for representing the random excitation for the state space analysis. Thus, the two apparently disjoint problems can be treated identically by this more general approach.
1.3. Literature Survey

The research in the area of random waves can be divided into two broad categories. These are the representation of the random waves, and their simulation based on a target spectrum. The following survey covers both areas. The earlier body of research was devoted primarily towards the analysis of the random wave field, and its proper representation. These investigators were concerned with identifying the probabilistic properties of random waves, their spectral representation, and long term prediction of extremes. As these areas became more established, the research attention started shifting more and more towards the problem of simulating the wave fields with known statistical properties.

In the dynamic studies of ocean platforms it is not always possible to obtain analytically the response statistics, given the wave statistics. In random vibration problems of this kind, simulation studies are often the only available method of obtaining solutions. This is particularly true for systems excited by non-Gaussians random inputs, or nonlinear systems subjected to random excitations. Thus, studies of offshore platforms often require time records of various wave kinematics. Such records are especially important for determining flow induced structural responses. In general, time records of various kinematics of interest are not available for an arbitrary location and a specific set of conditions. Simulation of these kinematics based on more easily obtainable information and some idealized model is a natural choice. And these kinematics can then be used to describe a particular realization of the load time history. By generating several such records, and analyzing the structural response in each case, response statistics can be obtained. Some of the earlier papers on Monte Carlo simulation techniques applied to general problems in structural dynamics relied on the harmonic superposition method. Notable among them are the works of Shinozuka [1970, 1972], who considered the multivariate and multidimen-
sional simulation problems.

1.3.1. Stochastic Representation of Waves

The most widely used model of sea surface elevations has its origin in the analysis of random noise [Rice 1945]. The wave prediction methods were first developed by Sverdrup and Munk during World War II and the results remained classified until 1947. Later these methods were substantially improved by Bretschneider [1952, 1957]. The concept of spectral representation of ocean waves in the frequency domain was introduced by Newman [1953]. Based on the works of Rice, it became possible through spectral analysis to evaluate various parameters commonly used in characterizing sea severity such as average and significant heights. The probabilistic prediction of wave heights, including extreme values, in random seas was first introduced by Longuet-Higgins [1952]. He investigated the concepts of non-narrowband waves, joint distribution of wave height and period, and non-linear non-Gaussian waves.

The validity of the application of the Rayleigh probability distribution in predicting wave heights observed in the ocean has been discussed by Goda [1974], Chakrabarti and Cooley [1977]. Based on observed data, all these investigators confirm that the Rayleigh model provides satisfactory predictions of various statistical quantities.

The engineering application of stochastic waves was introduced by St. Dennis and Pierson [1953] in predicting ship motions in a seaway. These concepts were soon extended for application to offshore structures. For these random waves, Pierson and Holmes [1965] derived the probability distribution of the horizontal force on a pile, and Borgman [1967] derived the spectral density for the same force.

Longuet-Higgins [1952] applied the results of Rice [1945] to ocean waves possessing a
narrow-band spectrum and found that the wave heights possess a Rayleigh distribution. In general, the elevation is considered to be a narrow-band Gaussian process that is spatially homogeneous and temporally stationary. The wave peaks maxima are considered statistically independent. However, knowledge of the surface elevation alone is not adequate for dynamic studies. In addition, time records of particle velocity and acceleration are required at several locations of interest to evaluate hydrodynamic forces. This can be pursued only after satisfactory models for the wave and the hydrodynamic forces are selected. The theory of monochromatic Airy waves is most commonly used for the evaluation of the wave kinematics based on the wave amplitude and frequency [Sarpkaya and Issacson, 1981]. Usually, the random surface profile is represented as the result of the superposition of a large number of sinusoidal waves. The amplitudes of the waves are considered to be deterministic, compatible with the sea spectrum. The phases are considered to be random, and uniformly distributed in the interval \([0, 2\pi]\).

The most widely used model for computing hydrodynamic forces on small diameter members are due to Morison, et al. [1950]. Combining the spectrum of surface elevation, the linear wave theory and the Morison model, it is possible to predict the spectrum of wave force on a structural member with certain approximations [Borgman, 1967]. Validation of these wave force predictions were attempted by a joint industry study (Standard Oil, Shell Development and Exxon Production Research). The findings were released in a series of articles by Thrasher and Aagaard [1970], Dean and Agaard [1970], Wheeler [1970] and Evans [1970]. The project measured the actual forces on structures in 30-ft. and 100-ft waters in the Gulf of Mexico during several hurricanes and periods of high wave activity. These data provided a sound basis for the present method of wave force computation, and also provided a calibration for the coefficients of the Morison model for random waves.
1.3.2. Simulation of Irregular Waves

The nature of the problem of irregular wave simulation varies slightly, depending on whether numerical simulation by a digital computer or generation of real waves in a model tank is sought. Generation of a surface elevation is all that is sought in the wave tank. The kinematics need not be explicitly computed, and the system response to these irregular waves is measured directly. In the wave tank simulation, it is assumed that the simulation of the surface elevation automatically generates the appropriate random wave kinematics at different spatial locations under the waves.

In contrast, the numerical simulation based on the elevation spectrum provides no explicit information about the wave kinematics. The simulation of kinematics at different spatial location requires that a particular wave theory be assumed, and the kinematics propagated in a consistent manner that maintains the cross-correlations between them. This is achieved by considering the transfer functions between the various kinematics.

1.3.2.1. Computer Simulation of random Waves

Numerical simulation on the digital computer is done by using one of two basic approaches. These are superposition of harmonics and digital filtering of band-limited white noise. The first and more established method involves the superposition of a finite number of harmonics with different amplitudes, frequencies, and a random phase [Borgman 1969, Shinozuka and Wai, 1979, Shinozuka, Fang and Nishitani, 1979]. The latter two investigators used the FFT algorithm in conjunction with the above model. The same approach of harmonic superposition using the FFT algorithm is used by Hudspeth [1975], and Hudspeth and Chen [1979] to introduce some non-linearities in the simulated waves. The method of harmonic superposition has been the most
widely used because of its simplicity and the capability of maintaining spatial correlations. Critical discussion of these methods are provided by Tucker et. al. [1984] and Tuah and Hudspeth [1984].

The second method, formally proposed by Borgman [1969] uses digital filters to generate the surface profile or the wave kinematics from a white noise input. Three different algorithms have been developed in connection with kinematics simulation through digital filtering. A moving-average (MA) algorithm synthesizes the current value of the time series as a linear combination of white noise deviates [Borgman, 1969, and Spanos, 1983]. An autoregressive (AR) algorithm [Spanos and Hansen, 1981, Lin and Hartt, 1984] predicts the present value as a linear combination of the past values of the process and a white noise deviate. Autoregressive methods in conjunction with Maximum Entropy Concepts have also been used for the estimation of sea spectra [Holm and Hoven, 1979, Houmb and Overvik, 1981]. The third, and the most versatile method is the autoregressive moving-average (ARMA) algorithm [Spanos, 1983, Flower and Vijeh, 1983] which involves a combination of the previous two methods. The ARMA modeling done by Spanos [1983] and Flower and Vijeh [1983] is an analog filter approximation. The same simulation problem has been solved for the discrete case by Samii and Vandiver [1984] and by Spanos and Minolet [1986], this time for the vertical velocity component at the free surface.

The time histories of wave kinematics at several space locations can be obtained by simultaneous simulation [Samaras, Shinozuka and Tsurui, 1985, Minolet and Spanos, 1987]. Alternatively, this can be achieved by spatial propagation of kinematics simulated at one location [Groves, 1960, Samii and Vandiver, 1984, Spanos, 1986, Bhattacharjee and Spanos, 1987]. Considerable simplification in many of the expressions representing the kinematics can be brought about by invoking the assumption of Airy’s linear, small amplitude waves in deep water. How-
ever, the above assumption is valid only under certain conditions. The criteria for using the deep water or shallow water expressions are well established for monochromatic waves.

In either case, the kinematics at a particular location of interest can be simulated by following one of two alternatives. Firstly, a direct simulation based on the appropriate spectrum of the kinematics can be performed for the location of interest. Alternatively, the kinematics at the surface can be propagated by using filters representing the appropriate transfer functions.

1.3.2.2. Random Wave Generation in Model Tanks

In principle, there are three ways of simulating irregular waves in a laboratory [Svendsen, 1985]. The difference between these three methods hinges upon the way the control signal for the wavemaker is produced. The three ways of producing the control signals are superposition of a finite number of uncoupled sine-generator signals, filtered white noise and reproduction of an actual wind wave time series record.

There are some fundamental difference in the scenario being represented. The first two methods try to satisfy the condition that the energy spectrum of the generated waves matches a target measured or theoretical spectrum. Thus, the first two methods can generate waves that reproduces only the information stored in the wave spectrum. The wave amplitude spectrum contains information on the amplitudes alone. The information regarding the phase is completely lost in the spectral representation, and thus cannot be reproduced.

The third method is more complicated than the first two, and is relatively recent, made possible by the developments in fast and inexpensive computers. The control signal for the wavemaker in this method may be determined by the solution to the inverse problem [Lundgren and Sand, 1978]. This method aims at reproducing one particular time series of surface eleva-
tions including the phases between the components. It is claimed [Svendsen, 1985] that the correct phase-reproduction is important in the testing of moored floating structures, ships in harbors, harbor resonance and other resonance dominated problems. Wave groups cannot be represented by spectral information alone. The phase information has to be included [Sand, 1982].

1.3.3. Stochastic Response of Ships and Compliant Platforms

Another closely related problem is the motion prediction of an offshore platform in a random sea. This problem is attracting renewed interest with very novel type of platforms that are being installed currently or being contemplated for the near future. Besides the conventional ship shape, there is a wide range of platform concepts in offshore operation. These include the catenary moored semisubmersible, the guyed tower platform, and the tension leg platform. An accurate response analysis of these platforms in random seas is essential to the design and estimation of their operational capabilities under various weather conditions. The pioneering work in this area is by St. Dennis and Pierson [1953]. They investigated the motion of a ship in a random sea and set the stage for later applications. Subsequently, there has been a number of papers in the area of ship and platform motion prediction by using filtering techniques. Some of the recent publications are by Yumori [1981], Triantafyllou, Bodson and Aihans [1983], Jefferys and Samra [1985], and Lin [1987].

The motion prediction and simulation of offshore platforms has primarily been motivated by the requirements of aircrafts landing on deck. An accurate motion prediction can greatly reduce the occurrences of abortive landing attempts. This problem has been investigated for over two decades by various researchers. Dalzel [1965] was one of the earliest investigators to focus attention on this topic. His approach was to identify the ship response by the moving average
(MA) filter, a technique well known for its simplicity and robustness. The MA technique is described in detail in Chapters 4 and 5 in the context of wave simulation. Yumori [1981], and Lin [1987] used the more general auto regressive moving average technique for the prediction based on measurements. In contrast, Sidar and Doolin [1975], and Triantafyllou et. al. [1983] used the more modern Kalman filter approach for the prediction based on measurements.

The investigations reported in the last two paragraphs aim to perform a time series analysis by designing an optimum filter that relates the measured surface elevation to the motion of the platform. In general, such filter design does not necessarily require precise knowledge of the nature of the excitation force or the response. The filter can be designed based purely on input output measurements in the laboratory, or on the prototype. Also, the emphasis is on prediction of future response based on past and present observations.

In recent years, interest in active control of the marine platform response has increased considerably. The idea of controlling the response is not new, and attempts are made at different stages of the design to incorporate passive control by careful selection of the design variables. The judicious choice of stiffness and mass distribution to avoid resonance is a practice common to all fields of dynamics. The proper balance of displacements of the underwater pontoons and the water piercing columns in a semisubmersible or TLP to cancel the vertical force at a selected wave frequency is yet another example of this practice. The dynamic positioning of drill ships operating in deep waters is one very sophisticated example of the practice. However, to meet the more demanding requirements of current offshore operations, the idea of motion control is being exploited in an ever increasing manner. The interest has been renewed with wave power devices [Jefferys, 1984], and compliant platforms which exhibit large motions. Addition of a passive control device to minimize fatigue damage to the TLP tethers has been investigated by Kitami, et.
al. [1982], Nordgren [1987], and White et. al. [1989]. Active control devices on the compliant platforms are just beginning to draw the attention of researchers [Reinhorn et. al., 1987, Yoshida, Suzuki and Mishima, 1989]. Judging by the experiences in other fields, active control is likely to receive increasing attention for its great potential to minimize platform motions.

The analytical tools used for the active control of dynamic systems has undergone a dramatic swing over the last two decades. Originally based on the frequency domain methods developed by Wiener [Wiener and Masani, 1958], the algorithm for modern control system design are invariably based on the theory developed by Kalman [Kalman and Bucy, 1961] in the time domain, using the state space analysis technique.

The increasing popularity of the state space method can be attributed to several appealing features. Firstly, it provides a clearer insight to some previously unexplained phenomena [Fossard, 1977] through the new concepts of controllability and observability. Secondly, it is more suited to digital computation. Finally, this method is capable of handling multi-variable problems, where frequency domain methods prove inadequate. In view of these above advantages the later sections of the present investigation explores the feasibility of the state space method for the dynamic analysis of compliant platforms.

The natural development of the topic from that of wave simulation to platform response analysis is covered in the course of the nine chapters. Chapter 1 explores the area in very general terms, and reviews the pertinent literature. Chapter 2 covers the theory of gravity waves, their spectral representation, and the forces they exert on offshore structures. Some of the concepts of deterministic wave theory are extended for the random wave case. Chapter 3 critically examines the most well established method of wave simulation through harmonic superposition, and identifies the limitations. The theory of digital filters in the context of random waves is
investigated in Chapter 4. Some theoretical issues that are not encountered commonly are discussed, and their relevance to wave kinematics simulation explained. Chapter 5 demonstrates the successful implementation of filter approaches for wave kinematics simulation making best use of the filter theory. Some new algorithms are developed and presented. A review of the theory of state space analysis is provided in Chapter 6. This is followed by example applications in Chapters 7 and 8. In Chapter 7 the global response of the guyed tower is addressed. Whereas in Chapter 8, the rigid body motions of a tension leg platform is conceptually investigated. The first example demonstrates, the application of the method to a single variable case. The second example explores the applicability of the method to a multivariate problem of a single input and multiple outputs, and identifies the areas requiring further research.

1.4. Objective

The present research investigation has several aims and objectives. A review of the existing methods of random wave kinematics simulation, wave force estimation, and the limitations of the present theory are provided in the first two chapters. Further, an attempt is made to generalize some of the concepts of deterministic wave theory in the context of water depth classification, and wave force regime identification in Chapter 2. The relevant literature in signal processing and filter design are reviewed in Chapter 4. Furthermore, interpretations of the concepts in the context of wave kinematics simulation are provided, and some of the ideas not yet common to the ocean engineering field are introduced. The applicability of various filter types to the wave kinematics simulation problem are investigated in Chapter 5. In addition, improved methods for the vertical and horizontal propagation of wave kinematics over a spatial grid by using the most appropriate filter are developed and some of these concepts demonstrated by suitable implementation. The fundamental concepts of the Covariance analysis through the state space method are
reviewed, and the associated theory of Gauss-Markov processes discussed in Chapter 6. The feasibility of the state space method for the single and multiple degree of freedom problems in offshore engineering, with special emphasis on the global response analysis of compliant platforms is explored. Idealized models of the guyed tower and the tension leg platform are used for the investigation in Chapters 7 and 8 respectively. Finally, the findings of the present investigation are summarized in Chapter 9, which also aims to state the limitations, and identify areas requiring further research.
CHAPTER II

Extension of Airy Wave Criteria to Random Waves

2.1. Overview

Two new ideas are presented in this chapter. The first idea will help to characterize the critical water depth, for a given sea-state, where simplifications of deterministic linear wave theory can be applied. The second idea, uses a similar concept to characterize critical member dimensions for the use of Morison equation in a random sea. The remaining parts of the chapter develops the background for these two ideas by an extensive review of the theory. The more important theories for deterministic gravity waves and the representation of nondeterministic waves are reviewed over the first few sections of this chapter. The remaining part of the chapter is devoted to the review of the method for practical computation of wave forces on slender offshore structures.

2.2. Introduction

This chapter begins with a review of the well-established theory of deterministic linear gravity waves. The complex nature of the gravity waves and the difficulty in obtaining an exact analytical solution is explained. The assumptions under which a simple solution is possible are discussed, and the mathematical equations based on these assumptions are outlined. This is followed by a critical review of the probabilistic theory of sea wave representation. The review covers the fundamental principles of probabilistic analysis with emphasis on spectral description of irregular waves. Some of the common spectral representations are examined and the situations where they are usually applied are indicated.
The classification of water depths for the deterministic waves, is reviewed. These are well documented in the literature (Sarpkaya and Issacson, 1981) as the relevant expressions can be considerably simplified by invoking these assumptions. However, no rational criteria are available for the application of these concepts to the case of random waves. A rational method is proposed in this chapter to extend these concepts of water depth classification to the situation of random waves. This classification is done by using the Pierson-Moskowitz (P-M) spectrum, and by relating critical water depth to wind velocity. This investigation is relevant as it critically assesses the accuracy of using the simplified version of the wave equations for the confused seas.

The most common model for evaluating hydrodynamic forces on small diameter structural members, the Morison equation, is reviewed. The modification to this model to predict the forces on a vibrating structure is briefly described. This is followed by a discussion on the relative importance of the drag, inertia and diffraction forces for different structural dimensions. Finally, a method is proposed for identifying the critical structural dimension for applying the Morison equation in the case of a wave spectrum.

2.3. Theory of Linearized Gravity Waves

The motion of water waves is a complex phenomenon. Even in its most simple form, after numerous simplifying assumptions, only an approximate solution can be obtained (Chakrabarti, 1987, Dean and Dalrymple, 1984). In a strict sense, water waves propagate in a viscous fluid medium, over an irregular bottom of varying permeability. Fortunately, the main body of the fluid motion is nearly irrotational, except for a "thin boundary layer" near the bottom and the surface. Also, the water can be considered incompressible for all practical purposes. These conditions imply that a velocity potential exists for the main body of the fluid, and the objective of any wave theory is to solve for this potential function.
The linear wave theory has been developed for long-crested waves. These are deterministic waves that are propagating in one direction, and the wave crests are "long enough" so that the fluid has no motion in the direction of the wave crest. Thus the flow can be considered essentially two-dimensional, and restricted to the x–z plane of propagation. The conditions of irrotationality and incompressibility when substituted in the continuity equation (in two-dimensional flow) leads to the 2-D Laplace equation

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0. \tag{2.1}$$

The governing equation for the gravity wave is obtained as a solution for $\Phi$ in eqn. (2.1), subject to the appropriate boundary conditions. In the above, $x$ denotes the positive direction of wave propagation, $z$ is positive upward, and the origin is fixed at the still water level (SWL).

2.3.1. The Bottom Boundary Condition (BBC)

The sea bottom is assumed to be a rigid, impenetrable, horizontal boundary for the mathematical derivation. This results in the condition of "no flow across the rigid boundary,"

$$\frac{\partial \Phi}{\partial z} = 0 \quad \text{at} \quad z = -d. \tag{2.2}$$

2.3.2. The Kinematic Free Surface Boundary Condition (KFSBC)

This boundary condition is more complex in nature, and most texts present only the final equation, without explaining how it is arrived at. The KFSBC states that a particle lying on the free surface at one instant of time would continue to remain on the free surface. Mathematically it implies that if the free surface of a wave is described by $F(x,z,t) = z - \eta(x,t) = 0$, where $\eta(x,t)$ is the displacement of the free surface about the SWL ($z = 0$). Then,

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u \frac{\partial F}{\partial x} + w \frac{\partial F}{\partial z} = 0 \tag{2.3}$$
or,

\[ \frac{\partial F}{\partial t} + \vec{u} \cdot \nabla F = 0. \quad (2.4) \]

Thus,

\[ \vec{u} \cdot \vec{n} = \frac{-\frac{\partial F}{\partial t}}{\sqrt{\nabla F}} \quad (2.5) \]

Where, \( \vec{n} \) is the unit normal vector associated with the gradient function \( \nabla F \). On substituting \( F(x, z, t) = z - \eta(x, t) = 0 \) in eqn. (2.5), the kinematic boundary condition at the free surface is

\[ \vec{u} \cdot \vec{n} = \frac{\frac{\partial \eta}{\partial t}}{\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + 1}} \quad \text{on} \quad z = \eta(x, t), \quad (2.6) \]

where, the unit normal to the surface \( F(x, z, t) \) is given by

\[ \vec{n} = \frac{-\left( \frac{\partial \eta}{\partial x} \right) \hat{i} + 1 \hat{k}}{\sqrt{\left( \frac{\partial \eta}{\partial x} \right)^2 + 1}}. \quad (2.7) \]

Thus,

\[ \vec{u} \cdot \vec{n} = \frac{-u \left( \frac{\partial \eta}{\partial x} \right) + w}{\sqrt{\left( \frac{\partial \eta}{\partial x} \right)^2 + 1}} = \frac{\frac{\partial \eta}{\partial t}}{\sqrt{\left( \frac{\partial \eta}{\partial x} \right)^2 + 1}} \quad (2.8) \]

which leads to

\[ w = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} \quad \text{on} \quad z = \eta(x, t). \quad (2.9) \]

Substituting \( w = \frac{\partial \Phi}{\partial z} \) in the above, the KFSBC can be represented as

\[ \frac{\partial \Phi}{\partial z} = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} \quad \text{at} \quad z = \eta(x, t). \quad (2.10) \]

2.3.3. Dynamic Free Surface Boundary Condition

The free surface, such as the water-air interface cannot support pressure variations, neglecting surface tension effects, across the interface. Thus, the free surface responds in order to maintain the uniformity in pressure. A complicating factor in the wave equation is that the upper boundary is not known apriori. Applying the Bernoulli equation, with \( p_u = \text{constant} \), on the free
surface, \( z = \eta(x, t) \), yields

\[
- \frac{\partial \Phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 \right] + \frac{p_n}{\rho} + gz = C(t) .
\]

(2.11)

Usually, the \( p_n/\rho \) term is absorbed in the constant \( C(t) \), replacing the latter by a new constant \( C_1(t) \). Thus, the problem is to find a solution to \( \Phi(x, z, t) \) which satisfies eqn. (2.1) and satisfies the boundary conditions specified by eqns. (2.10) and (2.11), at \( z = \eta \).

2.3.4. Approximations For Linearization

By invoking further assumptions of small amplitude and slope, and looking for only a first order solution, the second order terms are set equal to zero that is

\[
u \frac{\partial \eta}{\partial x} = 0,\]

(2.12)

and

\[
\left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 = 0.
\]

(2.13)

Next, the boundary conditions are applied at \( z = 0 \) instead of \( z = \eta \), simplifying the boundary conditions to

\[
\frac{\partial \Phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{at} \; z = 0
\]

(2.14)

and

\[
- \frac{\partial \Phi}{\partial t} + g\eta = C_1(t) \quad \text{at} \; z = 0.
\]

(2.15)

The solution to this simplified boundary value problem is [Dean and Dalrymple, 1984]

\[
\Phi = \frac{\pi H}{\kappa T} \frac{\cosh [\kappa(z + d)]}{\sinh(\kappa d)} \sin \Theta
\]

(2.16)

Where \( H \) is the wave height, \( T \) is the wave period, \( \kappa = 2\pi/\lambda \) is the wave number, \( \lambda \) is the
wave length, $\omega$ is the wave frequency, $d$ is the local water depth, and $\Theta = (\kappa x - \omega t)$ is the wave phase. The frequency and wave number are related for linear waves by the dispersion equation

$$\omega^2 = g\kappa \tanh(\kappa d).$$

(2.17)

The independent wave parameters are the local water depth, $d$, wave height, $H$, and any one of the following four parameters: $\omega$, $\kappa$, $\lambda$ or $T$. Furthermore, waves become unstable and break [Stokes, 1847] when either the crest angle exceeds $120^0$ or the following ratio of wave height to length is exceeded:

$$\frac{H}{\lambda} \geq 0.142 \tanh(\kappa d)$$

(2.18)

Based on the solution for $\Phi$, and the relations listed earlier, the following expressions for the various wave kinematics are obtained.

The horizontal and vertical particle displacements are

$$\xi(x,z,t) = -\frac{H}{2} \frac{\cosh [\kappa(z + d)]}{\sinh(\kappa d)} \sin\Theta$$

$$\eta(x,z,t) = \frac{H}{2} \frac{\sinh [\kappa(z + d)]}{\sinh(\kappa d)} \cos\Theta.$$  

(2.19)  

(2.20)

The corresponding velocity components are

$$u_x(x,z,t) = \frac{\omega H}{2} \frac{\cosh [\kappa(z + d)]}{\sinh(\kappa d)} \cos\Theta$$

$$u_z(x,z,t) = \frac{\omega H}{2} \frac{\sinh [\kappa(z + d)]}{\sinh(\kappa d)} \sin\Theta.$$  

(2.21)  

(2.22)

Finally, the corresponding accelerations components are

$$u''_x(x,z,t) = \frac{\omega^2 H}{2} \frac{\cosh [\kappa(z + d)]}{\sinh(\kappa d)} \sin\Theta.$$  

$$u''_z(x,z,t) = -\frac{\omega^2 H}{2} \frac{\sinh [\kappa(z + d)]}{\sinh(\kappa d)} \cos\Theta.$$  

(2.23)  

(2.24)

Further, the dynamic pressure under a surface wave at a depth $z$ below the SWL is
\[ p = - \rho gz + \frac{\rho g H}{2} \frac{\cosh [\kappa(z + d)]}{\cosh(\kappa d)} \cos \Theta \]  

(2.25)

where the first term on the right hand side denotes the hydrostatic pressure due to the water head up to the still water, and the second term represents the dynamic pressure due to wave motion.

The dynamic pressure term has the same sign as the hydrostatic pressure under a trough and the opposite one under a crest. It is interesting to note the action of surface waves on a fully submerged, neutrally buoyant body. Due to this dynamic pressure effect it would experience a net downward force under a wave crest, and an upward force under the trough. This fact is used to advantage in the design of semisubmersibles, where the change in the vertical forces on the columns due to variable submergence is canceled by the opposing force on the submerged hulls, at the design wave frequency.

To summarize, the assumptions of the linear wave theory are stated. The amplitude \( H/2 \) is small, relative to the wave length, \( \lambda \). The pressure contribution from the term \( (u^2 + w^2)/2g \) is negligible, the local water depth, \( d \), is uniform. The fluid is considered inviscid and irrotational. In addition, the fluid is incompressible and unstratified or homogeneous. The deflection force associated with the earth's rotation, the Coriolis force, is negligible. Surface tension effects are negligible. The bottom is smooth and impermeable, and the sea level atmospheric pressure can be considered uniform.

2.4. Non-Linear Wave Theory

Observation of waves at sea reveal that the wave crests are often sharp, and the troughs are relatively flat. This indicates an asymmetry of real waves not captured by the idealized model of the small amplitude linear gravity waves. Scientists and mathematicians have attempted to represent this situation by developing more rigorous, finite amplitude models that can reflect
these realities. Notable among them is the model due to Stokes [1847], which provides a higher order wave theory. Instead of linearizing the non-linear problem, he presented a perturbation solution to the problem. The model provides better approximations to the real problem with increasing number of terms in the expansion. Interestingly, the first term in the Stokes expansion provides the linear Airy wave model. Thus, the higher order terms, neglected in the linear theory, help to provide a more accurate representation. Common among the Stokes model are the second order and the fifth order Stokes model, which solve for the first two and first five terms, respectively, in the expansion for the potential function, $\Phi$. The phenomenon of breaking waves, not explained by the Airy model is very satisfactorily explained by Stokes theory. Thus, the Stokes model provides a quite accurate representation of real waves in deep waters. However, the improved nonlinear model is obtained only by sacrificing the simplicity of the linear model. The higher order Stokes models are very cumbersome to use.

In shallow waters, the Stokes theory becomes inadequate. This happens as the coefficients in the higher order terms "blow up" for shallow waters, by becoming large compared to the lowest order terms in the expansion. The cnoidal wave theory, [Korteweg and de Vries 1895] was proposed for shallow waters based on a non-linear model. More recently, Fenton [1979] presented a cnoidal wave theory which is capable of extension to any desired order, and is suited for engineering calculations.

Another model, the trochoidal wave theory was presented by Gerstner [1802], based on the assumptions of a rotational fluid motion. Though this model represents the crest-trough asymmetry better than linear waves, the model has not found widespread use.
2.5. Wave Environment for Design

A realistic description of the sea waves is required to design and operate ocean platforms that are structurally reliable, yet economically designed. The random nature of the waves in a real sea has been briefly described in the introductory chapter. The randomness is primarily due to the effects of wind turbulence, directional spreading of waves and interaction between, and breaking of, waves. The complex nature of the theory of gravity waves was explained in the last section and the theory for the simplest case, the deterministic linear gravity wave model, was developed. The more sophisticated nonlinear models of the gravity wave [Stokes, 1847, Fenton 1979], provide a more realistic representation of ocean waves. However, to this date there is no satisfactory probabilistic model for the non-linear waves.

Thus, when computing wave forces, the designer can choose between one of two alternative methods. The first, and the more traditional approach is that of using the concept of a single design wave, represented by its period and height. This single wave is chosen to be one that causes the most unfavorable response. The most severe wave is associated with a certain return period which exceeds the design life of the structure. This deterministic wave can be represented by the linear or a non-linear model (Stoke’s higher order model, cnoidal wave etc.). The key advantage of the design wave method is the simplicity of the analysis. However, identifying the wave that would cause the most severe response is not an easy task in most situations, even for an experienced designer. Frequently, the structural response to a number of competing loading situations have to be compared to determine the extreme wave condition. It is not necessarily the largest wave that causes the most unfavorable response. Furthermore, the method may prove inadequate for some important analysis procedures such as fatigue calculations, where the small waves with higher frequency have significant contributions to cumulative damage.
The second and more sophisticated method uses the wave spectrum. A very powerful yet simple method of probabilistic analysis is possible by using the power spectrum and assuming that the surface elevations are Gaussian and that the system being excited by the waves is linear. This method recognizes the simultaneous presence of waves of different frequencies and amplitudes in the real ocean. Thus, platform response to a particular sea state, as opposed to a single design wave, is sought. The sea state is represented by some important parameters such as the mean wind velocity, or some characteristic wave statistics. The spectrum contains second moment information on the distribution of surface elevations, for various wave frequencies. For the Gaussian process the second moment description contains all the probability information. Also, any linear transformation on a Gaussian process produces another Gaussian process. Thus, the elevation spectrum is commonly used with the linear wave model, to derive the spectrum of wave kinematics describing the excitation to the structural system. If the structural system is linear, the spectrum of the structural response is directly obtained, and various quantities of interest evaluated subsequently.

The most suitable wave spectrum for design purposes is one measured at the site. Often, this information is not available at the design stage. As an alternative a theoretical spectrum model is usually chosen, based on data that are more readily available. The important parameters governing a wave spectrum are the mean wind speed, $U$, the duration for which the wind is blowing, $T_d$, and the fetch, $F$. The fetch is defined as the extent of the open ocean over which the wind blows to produce waves, and refers to an oceanic region over which wind speed and direction are reasonably constant. Berteaux[1976] provides a very informative table on this topic indicating the expected relations between the various parameters, such as the wind speed, fetch, duration, wave height, period etc. for a fully developed sea.
2.6. Fourier Analysis of Stationary Random Seas

A simple physical interpretation of the wave spectrum is provided by the concept of wave energy and the Fourier analysis technique. The surface elevation is treated as a random process. The power spectrum of this random process is derived under certain simplifying assumptions of stationarity and homogeneity of the sea waves.

Stationarity of the surface elevation process is a common assumption. In a strict sense, stationarity implies that all the statistics of the process are independent of the choice of the time origin. A stationary sea would ideally be a fully developed sea for a given wind speed, and possibly exists for a few hours at most. Similarly, a homogeneous sea is one in which the statistics of the elevation process is independent of the choice of the space location. Thus, the measured elevation spectra for a homogeneous sea would be the same everywhere in the spatial region of interest.

As opposed to the strict sense stationarity, the idea of second moment stationarity is very commonly used for engineering purposes. Specifically, only the stationarity of the first two moments as opposed to moments of all order are considered adequate for representation purpose. In practice, the elevation may be assumed stationary, even in the second moment sense, only for a certain duration and over a certain region. Thus, the sea during a developing storm is non-stationary, and the region of the sea over which the wind or the bottom conditions have significant variations represents a non-homogeneous sea.

The assumption that the sea is ergodic, implies that the conditions of stationarity and homogeneity are automatically satisfied. The ergodic theorem, associated with this assumption, states that the temporal statistics adequately reflect the ensemble statistics. This assumption simplifies the computation procedure significantly. The statistics of the entire process is evaluated from the
temporal statistics of a sample run of data that is "sufficiently long."

The energy per unit area under a long-crested wave is

$$ E = \frac{\rho g H^2}{2}. $$  \hspace{1cm} (2.26)

The fundamental idea is to consider the irregular sea resulting from the superposition of a number of simple harmonic waves with different amplitudes and frequencies, and having some phase lag between the components. Thus, a Fourier series representation of the irregular surface profile is

$$ \eta(t) = \sum_{n=1}^{N} A_n \cos(n\omega t + \phi_n), $$ \hspace{1cm} (2.27)

which may also be written as

$$ \eta(t) = \sum_{n=1}^{N} \{ a_n \cos(n\omega t) + b_n \sin(n\omega t) \}, $$ \hspace{1cm} (2.28)

where $N$ is the number of Fourier components. This representation is based on the linear superposition of the different harmonics, and is valid for only the linear wave model. Assuming the length of the wave record is $\hat{T}$, and relying on help of Fourier analysis, the coefficients, $a_n$, and $b_n$ are computed from the integrals

$$ a_n = \frac{2}{\hat{T}} \int_{0}^{\hat{T}} \eta(t) \cos(n\omega t) \, dt $$ \hspace{1cm} (2.29)

and

$$ b_n = \frac{2}{\hat{T}} \int_{0}^{\hat{T}} \eta(t) \sin(n\omega t) \, dt. $$ \hspace{1cm} (2.30)

The random sea state on a short term basis maintains certain identifiable statistical properties and is best represented by its energy spectrum. The total energy of a surface wave, $E$, per unit area in the wave record between infinite time limits is given by the integral
\[ E = \frac{1}{2} \rho g \int_{-\infty}^{\infty} \eta(t)^2 \, dt. \] (2.31)

The surface elevation can be expressed by its Fourier transform representation

\[ \eta(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[ a(\omega) \cos(\omega t) + b(\omega) \sin(\omega t) \right] \, d\omega, \] (2.32)

where

\[ a(\omega) = \int_{-\infty}^{\infty} \eta(t) \cos(\omega t) \, dt \] (2.33)

\[ b(\omega) = \int_{-\infty}^{\infty} \eta(t) \sin(\omega t) \, dt. \] (2.34)

Thus, it can be shown that [Chakrabarti, 1987]

\[ E = \frac{\rho g}{2} \int_{-\infty}^{\infty} \eta(t)^2 \, dt = \frac{\rho g}{2\pi} \int_{-\infty}^{\infty} [a^2(\omega) + b^2(\omega)] \, d\omega. \] (2.35)

On using

\[ (A(\omega))^2 = \{ a^2(\omega) + b^2(\omega) \}, \] (2.36)

one obtains

\[ \int_{-\infty}^{\infty} |\eta(t)|^2 \, dt = \frac{1}{\pi} \int_{-\infty}^{\infty} (A(\omega))^2 \, d\omega, \] (2.37)

which is a restatement of the Parseval’s theorem. Let \( \overline{\eta(t)^2} \) be the mean square value or variance of \( \eta(t) \) over a specified record length, \( \hat{T} \). Replacing ensemble averages by the sample average and invoking the ergodic theorem one obtains

\[ \overline{\eta(t)^2} = \frac{1}{\hat{T}} \int_0^{\hat{T}} [\eta(t)^2] \, dt. \] (2.38)

This can be related to the mean energy per unit area

\[ \overline{E} = \frac{\rho g}{2\pi} \int_{-\infty}^{\infty} \frac{(A(\omega))^2}{\hat{T}} \, d\omega. \] (2.39)
Defining the spectral energy density as

$$S(\omega) = \frac{|A(\omega)|^2}{\pi T}$$  \hspace{1cm} (2.40)

the total energy per unit area is obtained by integrating the energy density curve as a function of frequency

$$E = \frac{\rho g}{2} \int_{-\infty}^{\infty} S(\omega) \, d\omega$$  \hspace{1cm} (2.41)

Note that $S(\omega)$ has units of $L^2 T$.

2.6.1. Power Spectrum Computation

The computation of energy spectrum based on surface elevation data is commonly done by one of two methods. The first and the older method utilizes the auto-correlation method. The second and relatively recent method uses the Fast Fourier Transform technique.

2.6.1.1. The Autocorrelation Method

This is the older method and first computes the autocorrelation, $R(\tau)$, of the surface profile record of $\eta(t)$, for time lag, $\tau$, through a convolution operation,

$$R_{\eta \eta}(\tau) = \frac{1}{T - \tau} \int_{0}^{T - \tau} \eta(t) \eta(t + \tau) \, dt$$  \hspace{1cm} (2.42)

This computation is repeated for various values of $\tau$. The next step is to transform the autocorrelation function in the time domain to the spectral density function in the frequency domain, by the Wiener-Khintchine relation

$$S(\omega) = \frac{\dot{\tau}}{\pi} \int_{-\tau}^{\tau} R_{\eta \eta}(\tau) \cos(\omega \tau) \, d\tau$$  \hspace{1cm} (2.43)

Here, the symmetry property of an autocorrelation function, about $\tau = 0$ is utilized, and $\dot{\tau}$
represents the maximum value of the time lag, \( \tau \), permitted by the run of data.

2.6.1.2. Fast Fourier Transform (FFT) Method

The more efficient and relatively recent method uses the FFT algorithm [Cooley and Tukey, 1965]. Here the data are transformed directly into the frequency domain by using this computationally efficient technique. Convolution of two functions in the frequency domain is equivalent to a multiplication in the frequency domain. This fact is used by this method to obtain the elevation spectrum, as

\[
S(\omega) = \frac{1}{T} \left[ \sum_{k=1}^{N} \eta(k\Delta t) e^{i2\pi (k\omega) \Delta t} \right]^2. \tag{2.44}
\]

Here, the discrete version of the continuous time is used with \( \Delta t \) as the sampling interval on the time axis. A more complete discussion of the discretization process, and smoothing of the spectrum so computed is provided by Kinsman [1965], and Chakrabarti [1987].

2.6.2. Mathematical spectrum Models

The mathematical spectrum models are quite useful for evaluating the wave force spectrum for a site for which adequate wave data are not available. Oceanographers have directed considerable research efforts towards the development of such models based on meteorological data, which are more easily available. Several mathematical models have been proposed to represent ocean wave spectra. These models are generally based on one or more important parameters, such as the mean wind speed, significant wave height, significant wave period and spectral shape factors. The mathematical form for these models are proposed by consideration of the wave generation and decay process. The numerical coefficients for these empirical models are determined by extensive analysis of wave data.
The equilibrium range of the spectrum of a fully developed sea has been proposed by Phillips [1958] as

\[ S(\omega) = \frac{\alpha g^2}{\omega^5}, \]  

(2.45)

where \( \alpha \) and \( g \) are the Phillips constant, and the gravitational constant, respectively. This expression was arrived at by dimensional analysis combined with a hypothesis of wave steepness limitation of a fully developed sea, to obtain the asymptotic high frequency relation. This relation provides an upper bound on the energy spectrum, and is independent of the wind speed or the fetch. It provides the basis for many of the other spectral models in common use.

### 2.6.2.1. General Form For Spectral Models

A general form of the expressions that can be used to describe most mathematical models is written as

\[ S(\omega) = \frac{A}{\omega^p} \exp \left[ -\frac{B}{\omega^q} \right] \]  

(2.46)

In which \( S(\omega) \) is the ordinate of the one-sided energy density spectrum dimension \( L^2 T \), and \( A \), \( B \), \( p \) and \( q \) are four parameters of the spectrum. It can be shown that the peak frequency, \( \omega_0 \), at which the spectral peak occurs is given by

\[ \omega_0 = \left[ \frac{Bq}{p} \right]^{1/q}. \]  

(2.47)

The two common wave parameters used to describe the spectrum are the significant wave height, \( H_s \), and the mean wave period, \( \bar{T} \). These can be described by spectral moments. Defining the \( n \)-th spectral moment as

\[ m_n = \int_0^\infty \omega^n S(\omega) \, d\omega. \]  

(2.48)
the quantities, $H_z$ and $\bar{T}$ are given by

$$H_z = 4 \sqrt{m_0}$$

(2.49)

and

$$\bar{T} = \frac{2\pi}{\omega}.$$  

(2.50)

where $\bar{\omega}$ is the mean wave frequency given by

$$\bar{\omega} = \frac{m_1}{m_0}.$$  

(2.51)

The mean zero-upcrossing period, $T_z$, on the other hand, is the average period between successive zero-upcrossings, and is associated with the mean zero upcrossing frequency, expressed as

$$\bar{\omega}_z = \sqrt{\frac{m_2}{m_0}}.$$  

(2.52)

An important parameter associated with any spectrum is the bandwidth parameter, $\epsilon$, defined by

$$\epsilon^2 = \frac{m_0 m_4 - m_2^2}{m_2 m_4}.$$  

(2.53)

The value of this parameter lies in the range $0 < \epsilon < 1$. When $\epsilon$ is close to zero, the spectrum consists of a narrow band of frequencies and the spectrum is termed narrow-band. The last quantity of interest is $T_p$, the expected time between successive peaks or maxima in the random process $\eta(t)$. For a stationary zero-mean Gaussian process, $T_p$ can be shown to be related to $m_2$ and $m_4$ by

$$T_p = 2\pi \left[ \frac{m_2}{m_4} \right]^{1/2}.$$  

(2.54)

The degree of irregularity of the random process is given by the ratio of the number of peaks to the number of zero crossings,

$$r = \frac{T_m}{T_z} = \frac{m_2}{\left[ m_0 m_4 \right]^{1/2}}.$$  

(2.55)

Note that $0 < r < 1$. For narrow-band spectra $r$ is close to unity and the number of maxima is
not significantly greater than the number of zero up-crossings.

2.6.2.2. Some Common Spectral Models

The most common spectral model is the single-parameter spectrum proposed by Pierson and Moskowitz [1964]. The model is based on the significant wave height or the wind speed, and represents a fully developed sea. Departures from fully developed seas can be due to limitations on fetch or wind duration. To account for these limitations, more than one parameters are required. There are several two-parameter spectrum models available, that can account for developing, or partially developed seas. Some of the common ones are Bretschneider [1969], Scott [1965], ISSC [1964] and ITTC [1966]. A five parameter spectrum is provided by the JONSWAP spectrum [Hasselman, 1973, 1976] to account for fetch limited seas. Usually three of these five parameters are held constant. A six parameter spectrum model is proposed by Ochi and Hubble [1976], which is capable of describing spectrum with double peaks, one due to sea and other due to swell. This spectrum also provides more insight into the variability of the spectrum for any given set of parameters.

2.6.2.2.1. Pierson-Moskowitz (P-M) Spectrum

This is an one-parameter spectrum for fully developed seas. A fully developed sea is one in which each spectral component has reached its maximum amplitude for a given wind speed. In a strict sense, the fetch and the duration are considered infinite in this model. Pierson and Moskowitz [1964] proposed this model based on a similarity theory and the analysis of wind and wave observations taken in the North Atlantic Ocean.

In spite of the strong underlying assumptions of infinite fetch and duration, the P-M spectrum is most commonly used by ocean engineers as representative for water waves all over the
world. The P-M spectrum model is expressed as

\[ S(\omega) = \frac{\alpha g^2}{\omega^5} \exp \left[ -\beta \left( \frac{\omega_0}{\omega} \right)^4 \right] \]  

(2.56)

where

- \( S(\omega) = \) spectral density (\( L^2 T \) units)
- \( \alpha = 0.0081 \)
- \( g = 9.81 \text{ m/s}^2 \): acceleration due to gravity
- \( \beta = 0.74 \)
- \( \omega_0 = g/U_{19.5} \): a constant for a given wind speed
- \( U_{19.5} = \) wind speed in m/s measured at 19.5m above sea level.

The constants \( \alpha, \beta \) and \( \omega_0 \) were empirically determined from wind and wave observations taken in the North Atlantic Ocean. The peak frequency can be shown to be

\[ \omega_p = \left[ \frac{4\beta}{5} \right]^{1/4} \omega_0 = 0.877 \omega_0 \]  

(2.57)

The power spectrum of the surface elevation (eqn. 2.20 evaluated at \( z = 0 \)) is represented by eqn. (2.56). This is the usual variable of interest for oceanographic analysis. However, offshore engineers are often interested in the wave kinematics, rather than the surface elevations. Of particular interest are the horizontal components of the kinematics. Combining the linearized wave theory eqn. 2.21 with the equation for the P-M Spectrum, eqn. 2.56, the spectrum of horizontal velocity component at any vertical location \( z \), measured positive upwards from the SWL, is given by

\[ S_{\omega_x}(\omega) = \omega^2 \frac{\cosh \kappa(z + d)}{\sinh(\kappa z)} S_{\eta_x} \]  

(2.58)

Figures 2.1 (a) – (b) illustrates how the shape of the horizontal velocity spectra at the SWL
depends on the single parameter of the P-M spectrum, the wind velocity.

2.6.2.2. Bretschneider Spectrum

This is a two-parameter spectrum model which describes the spectrum based on the information about significant wave height, $H_s$, and significant wave period, $T_s$. Based on the assumption that the spectrum is narrow-banded and the individual wave height and period follow the Rayleigh distribution, Bretschneider [1969] derived the following spectrum model

$$S(\omega) = \frac{A}{\omega^5} \exp \left[ -\frac{B}{\omega^4} \right],$$  

(2.59)

where

$$A = 0.1687 H_s^2 \omega_s^2$$

$$B = 0.675 \omega_s^4$$

$$\omega_s = \frac{2\pi}{T_s}$$

and the significant wave period, $T_s$, is defined as the average period of the significant waves. It can be shown that [Chakrabarti, 1987]

$$T_s = 0.946 T_0$$  

(2.60)

where $T_0$ is the peak period. The model, though formulated for fully developed seas, provides satisfactory representation of partially developed seas as well. Bretschneider [1959] provides relationships between mean wind speed and the statistical values of wave height and period. For fully developed seas the relations are

$$\frac{gH_s}{U_w} = 0.282$$  

(2.61)

and
\[ \frac{gT_s}{U_w} = 6.776 \]  
(2.62)

For partially (80% and 90%) developed seas the relations are

\[ \frac{gH_s}{U_w} = \begin{cases} 0.254 \text{ (90\%)} \\ 0.226 \text{ (80\%)} \end{cases} \]  
(2.63)

and

\[ \frac{gT_s}{U_w} = 4.764. \]  
(2.64)

A detailed description of the other two-parameter spectra is provided by Chakrabarti [1987], where a systematic comparison of the parameters of these two-parameter spectra with the P-M spectrum is also made.

2.6.2.2.3. JONSWAP Spectrum

Frequently, the development of waves and the associated spectra is limited by the presence of coast-lines, resulting in fetch-limited spectra. The JONSWAP spectrum can adequately represent such fetch-limited seas. Observations indicate that the spectrum is more peaked than the P-M or Bretschneider class of spectra. The JONSWAP spectrum resulted from the Joint North Sea Wave Project [Hasselmann et al., 1973] and incorporates the fetch directly, through the non-dimensional parameter

\[ \bar{F} = \frac{F}{U_{10}}, \]  
(2.65)

where \( F \) is the fetch length, and \( U_{10} \) is the mean wind speed at a height of 10 meters above SWL.

The spectrum is expressed as

\[ S(\omega) = \frac{\alpha g^2}{\omega^5} \exp \left[ -\frac{5}{4} \left( \frac{\omega}{\omega_p} \right)^{-4} \right] \gamma^\alpha. \]  
(2.66)

In this equation
\( \alpha = 0.076 \bar{F} = 0.0081 \)

the shape factor, \( \gamma \), is given by

\[
\gamma = \begin{cases} 
7.0 & \text{for very peaked spectra} \\
3.3 & \text{mean of selected JONSWAP data,} \\
1.0 & \text{for P-M spectrum}
\end{cases}
\]

and the peak frequency is related to the fetch through the relation

\[ \omega_p = 20 \bar{F}^{-0.33} \]

The constants \( \alpha \) and \( \sigma \) are given by

\[
a = \exp \left[ -\frac{1}{2} \left( \frac{\omega - \omega_p}{\sigma \omega_p} \right)^2 \right]
\]

and

\[
\sigma = \begin{cases} 
0.07 & \text{for } \omega \leq \omega_p \\
0.09 & \text{for } \omega \geq \omega_p
\end{cases}
\]

The shape factor \( \gamma \) represents the ratio of the maximum spectral ordinate of the JONSWAP spectrum to the corresponding P-M spectrum, and is related to the spectral bandwidth.

2.6.2.3. Variation In Spectral Models

A critical analysis of the variety of spectral models reveals a wide variation in their shape and magnitude. This is because the theoretical expressions are nothing but "best fit curves," representing the mean of a family of spectra. A variation of 30% or more between measured and theoretical values have been reported. Ochi and Hubble [1976] have explained some of these discrepancies and introduced the concept of "families of wave spectra." Using the two-parameter spectra for a given wave height, they have presented a family of spectra, along with their probability of occurrence and confidence limits. This clearly indicates the need for introduction of more parameters to define a spectrum more accurately. The six-parameter spectrum proposed by
Ochi is expected to represent this variation satisfactorily.

2.7. Contributions Towards Characterization of Water Depth

This section proposes a method for the water depth characterization. The criteria associated with the deterministic linear Airy waves are extended for the case of irregular waves represented by the P-M spectrum.

Transfer functions encountered in the simulation and propagation of wave kinematics usually involve the wave number, $\kappa$, related to the frequency, $\omega$, through the dispersion equation (2.17)

$$\omega^2 = g \kappa \tanh(\kappa d)$$

where $d$ and $g$ are the water depth and the acceleration due to gravity, respectively [Sarpkaya and Issacson, 1981]. For monochromatic waves, the above dispersion equation can be simplified to two different expressions based on the asymptotic values of $\tanh(\kappa d)$. For a fixed frequency, or wave-number, shallow water is defined by

$$\kappa d \leq \frac{\pi}{10}$$  \hspace{1cm} (2.67)

when

$$\tanh(\kappa d) = \kappa d.$$  \hspace{1cm} (2.68)

In this case the dispersion equation (2.17) reduces to

$$\omega^2 = gd \kappa^2$$  \hspace{1cm} (2.69)

which, when substituted into equation (2.67) yields

$$d_{sh} \leq \frac{\omega}{g} \left[ \frac{\pi}{10 \omega} \right]^2.$$  \hspace{1cm} (2.70)

Similarly, deep water is defined by
\[ \kappa d \geq \pi, \quad (2.71) \]

when

\[ \tanh(\kappa d) = 1. \quad (2.72) \]

Then, the dispersion equation reduces to

\[ \omega^2 = g\kappa, \quad (2.73) \]

which, on substitution into equation (2.71), yields

\[ d_{dp} \geq \frac{\pi g}{\omega^2} \quad (2.74) \]

There is no ambiguity regarding the use of the above expressions for deterministic analysis, using monochromatic waves. However, when trying to classify the water depth in an irregular sea state, one is faced with the problem of the presence of not a unique frequency, \( \omega \), but a range of frequencies. A simplified version of the dispersion equation corresponding to deep water is generally used in literature without specifically addressing the issue of the water depth for which the results are valid.

For shallow waters or fetch-limited seas the JONSWAP spectrum is more appropriate. The Wallops spectrum [Huang et al., 1981] and its extension [Huang et al. 1983] have also been proposed for very shallow waters like the great lakes. However, these spectra have not found as widespread acceptance as the P-M spectrum. Thus, the P-M spectrum, with the wind velocity providing all the required information, remains the most commonly used model. In this regard, the effect of finite water depth should at least be incorporated at the level of kinematics calculation. Ideally, it should be accounted for simultaneously in evaluating the kinematics as well as the spectrum.

The identification of proper "water depth regime" is quite important in kinematics calculation, as the error involved in making the "deep water" assumption can sometimes be significant.
As an example, consider the peak value of the vertical velocity spectrum at the surface based on the P-M spectrum, with wind velocity 50 knots (25.7 m/sec) and how this magnitude decays at 50 meters below the surface. According to "deep water" assumption this decay is independent of water depth and is 47.9% of the surface value. According to the exact theory, the same figure is 37.4% and 45.3% for total water depth of 100 and 150 meters respectively. Thus, the velocity at 50 meters below the surface in total water depth of 100 meters can be in error by 28%, on using deep water approximation.

Similarly, the horizontal velocity spectrum at the SWL depends on the water depth. The dependence of the exact horizontal spectrum on the water depth is displayed in Fig. 2.2 (a)-(b). Under the assumption of "deep water" all these curves are represented by the outermost curve.

As no guideline was found in the existing literature as to what depths qualified for "deep" or "shallow" water, when dealing with a spectrum, an attempt is made here to develop a rational procedure for the above, using linearized wave theory.

The idea proposed here consists of selecting a characteristic frequency, \( \omega_{ch} \), which is specified by the energy content in the spectrum below or above that value of frequency. If this value of frequency is used in equations (2.70) and (2.74), one obtains the limiting value of the water depth, such that a fraction, p, of the total "energy" of the spectrum, is contributed by waves which satisfy the "deep" or "shallow" water conditions. The concept is best illustrated through its application.

Consider the expression of the one-sided P-M spectrum of sea wave elevations

\[
S(\omega) = \frac{A}{\omega^5} \exp \left( -\frac{B}{\omega^4} \right)
\]  

(2.75)

where
\[ A = 8.2 \times (10)^{-3} \ g^2; \quad B = 0.74 \left[ \frac{g}{U} \right]^4 \]  
\text{(2.76)}

and \( g \) and \( U \) represent acceleration due to gravity and wind velocity, in consistent units. The expression in equation (2.76) admits a first integral in a simple form,

\[ E(\omega) = \int_0^\omega S(\Omega) \ d\Omega = \frac{A}{4B} \exp \left[ -\frac{B}{\omega^4} \right] \]  
\text{(2.77)}

This integral is proportional to the energy density of the surface waves. Denoting by \( p \) the fraction of energy contained in frequencies lower than and equal to \( \omega \), it is readily seen that

\[ p = \frac{E(\omega)}{E(\infty)} \]  
\text{(2.78)}

Substituting equations (2.75) and (2.76) into equation (2.77) and (2.78), and solving for \( \omega \), yields

\[ \omega_u = \left[ \frac{0.74}{-\ln(p)} \right]^{\frac{1}{4}} \left( \frac{g}{U} \right) \]  
\text{(2.79)}

Similarly, if \( p \) denotes the fraction of energy lying above frequency \( \omega \), then it is readily seen that

\[ \omega_l = \left[ \frac{0.74}{-\ln(1-p)} \right]^{\frac{1}{4}} \left( \frac{g}{U} \right) \]  
\text{(2.80)}

where the subscript \( u \) and \( l \) have been added to correspond to upper and lower bound values of \( \omega \). To be substituted into equations (2.70) and (2.74), respectively. Carrying out these substitutions and simplifying, the following expressions are obtained

\[ d_{dp} = \pi \frac{U^2}{g} \left[ \frac{-\ln(1-p)}{0.74} \right]^{\frac{1}{2}} \]  
\text{(2.81)}

\[ d_{sh} = \frac{(\pi U)^2}{100g} \left[ \frac{-\ln(p)}{0.74} \right]^{\frac{1}{2}} \]  
\text{(2.82)}

where \( p \) represents the fractions of energy contributed by "deep" or "shallow" water waves. Thus, for a given wind velocity, one has to choose a fractional value \( p \) that is considered satisfactory,
and the limiting values of water depth are obtained directly from equation (2.81) and (2.82). Figs. 2.3 and 2.4 represent plots of deep and shallow water depths against wind velocity using $p$ as a parameter. These graphs can be used either to determine the water depths that permit a simplification of the dispersion equation, or for a given water depth, the degree of confidence that should be assigned to the assumption. Once it has been established that the wind velocity and local water depth is such that it permits one of the two possible simplifications of the dispersion equation (2.69) or (2.73), a numerical scheme can be developed that exploits the simplification so obtained.

2.8. Wave Forces on Slender Offshore Structures

The problem of predicting the total force on a cylinder, subjected to fluid motions resulting from wave action, is extremely complex. Analytic solution is possible only in some idealized cases. The source of the difficulty arises from several factors. These factors include the flow separation taking place due to the viscous nature of the fluid, the oscillatory nature of the flow, and the effect of the orbital motion of the particles under a wave.

When a cylinder is subjected to a harmonic flow normal to its axis, the flow not only accelerates and decelerates, but changes direction as well during each cycle. This produces a reversal of the wake, or flow of separated streamlines, from the downstream side to the upstream side whenever the velocity changes sign. This causes large excursion of the separation points in the flow field. The boundary layer over the cylinder may be anything from fully laminar to fully turbulent states and the Reynold number may change from sub-critical to post-critical over a given cycle.

The most general approach to evaluating the hydrodynamic force on a structure is to
integrate the dynamic pressure over the surface of the body. For evaluating wave forces on structural members it is somewhat difficult to obtain analytical expressions for the pressure distribution for real fluids. The difficulty is associated with the "ideal fluid assumption" in deriving the wave equations, whereby the fluid viscosity is neglected. However, water has a finite viscosity and there is an appreciable contribution from viscous effects to the total force in some situations. The viscous boundary layer effects, as well as the effects of flow separation produces vortices with associated pressure field modifications. The proper viscous forces along with the ideal fluid pressure effects need to be incorporated. This requires a combination of experimental and theoretical methods to predict the forces in a realistic manner.

The flow due to waves is more complex than the harmonically oscillating flow. Even if the surface effects are ignored, the orbital motion of the particles cause three-dimensional flow effects. The complexities of the two-dimensional case discussed earlier are now extended to a three-dimensional flow situation. Theoretical analysis of the separated flow is extremely difficult and much of the desired information is obtained by a combination of numerical and experimental investigations.

2.8.1. Morison Equation

A very convenient empirical model for predicting the hydrodynamic forces on slender structural members has been proposed by Morison et. al. [1950]. The formulation is based on experimental studies of wave forces on a pile and is a heuristic approximation of the measured forces. This empirical model is most appropriate for slender members, and accounts for the viscous as well as the inertia forces in an unsteady flow. Originally formulated for predicting forces on a rigid pile, the model is extensively used for evaluating wave and current forces on various submerged structural elements of offshore platforms. According to this model the total
force on the structure can be considered to be the algebraic sum of a drag force and an inertia force.

2.8.1.1. Drag Force

The drag force represents the contributions from viscous effects to the total force and attempts to incorporate the boundary layer and flow separation effects. It depends on the fluid velocity in a quadratic manner, and linearly on the projected surface area. The drag force per unit length of a cylinder is defined as

\[ F_d = \frac{1}{2} \rho c_d D \| u \| u \]  \hspace{1cm} (2.83)

where \( \rho \) is the fluid density, \( D \) is the diameter (or some characteristic dimension), \( u \) is the undisturbed fluid velocity, and \( c_d \) is the drag coefficient which is determined from experiments.

2.8.1.2. Inertia Force

The inertia force represents the contributions due to fluid acceleration and is present even under ideal fluid assumptions. The added mass effect and the Froude Krylov force are direct consequences of this force. It depends on the fluid acceleration and the cross-sectional area of the member. The inertia force per unit length is expressed as

\[ F_l = c_m \rho A_s \dot{u} \]  \hspace{1cm} (2.84)

where \( A_s \) is the cross-sectional area, \( \dot{u} \) is the acceleration of the fluid and \( c_m \) is the inertia coefficient associated with the geometrical shape of the structure. The coefficient, \( c_m \) is usually derived from experiments, and can be theoretically obtained only in some very special cases.

2.8.1.3. Total Force

According to this model the total force on the structure is the vectorial sum of these two
forces. Thus the total hydrodynamic force per unit length on a slender structural member, subjected to an unsteady flow of a real fluid around it is given by

\[ F_T = F_d + F_I = \frac{1}{2} \rho c_d D |u| u + \rho c_m A_T \ddot{u} \]  

(2.85)

The values of the velocity and acceleration in this equation (2.85) have to be known before the force can be evaluated. For computing forces due to waves or currents, the velocity and acceleration in the flow field in the absence of the structural member is usually substituted. The true values of these kinematics in the presence of the solid body can be solved for only by considering the relatively complex fluid-structure interaction problem. This simplification implies that the structural member does not affect the flow field significantly. This assumption is reasonably valid only for slender structural members, and hence, the restriction of this model to small diameter members. For structural members of relatively larger dimension the diffraction and reflection effects play an increasingly important role. An assessment of the situations under which the different components play important roles are discussed in more detail in a following section.

2.8.2. Interactive Form of Morison Equation

Morison equation was derived for rigid piles, and due consideration is required when analyzing the force on moving structures, especially if the structure is very flexible. In a more sophisticated formulation the forces are considered to depend not on the absolute velocity and acceleration of the water particles, but on their relative magnitudes with respect to the structure of interest. That is,

\[ F_T = \frac{1}{2} c_d \rho D |u - \dot{x}| (u - \dot{x}) + c_m \rho A_T (\ddot{u} - \ddot{x}) + \rho A_T \dddot{x} \]  

(2.86)

where \( x \) represent the displacement of the moving structure, and the dots represent derivatives with respect to time. The applicability of the model for rigid piles to the case of a moving struc-
ture is an empirical extension of this model. The same coefficients are used but in conjunction with the relative motions. Systematic experimental values of the coefficients to cover the general case of a moving structure are not available, and the above model is accepted by the engineering community as a logical extension of the theory.

2.8.3. Coefficients In Morison Equation

The accuracy of the force prediction depends, firstly, on applying the equation to the situation where they are applicable. Secondly, the accuracy depends on assessing the correct values of the variables and parameters of the equation. An accurate representation of the force coefficients, $c_d$ and $c_m$ are considered the most challenging problem for the application of this model. Attempts at quantifying the values of these coefficients have only been partially successful, and research is still being continued to this effect. Simplistic analyses treat these coefficients as constants. There is extensive scatter of the data on these coefficients [Sarpkaya, 1981]. A more involved analysis of these data try to identify the parameters on which the coefficients depend. Some of the more important parameters are the cylinder surface roughness, the Reynold number, $R_e$, and the Keulegan-Carpenter number, $K_e$. The Reynold number represents the ratio of inertia force to the viscous force

$$R_e = \frac{\rho u D}{\mu} \quad (2.87)$$

The Keulegan-Carpenter number, on the other hand, represents the ratio of the orbital diameter to the structural diameter in a harmonically oscillating flow. Interestingly, this non-dimensional number can be used as an indicator for the relative importance of the drag force with respect to the inertia force.

$$K_e = \frac{u_0 T}{D} \quad (2.88)$$
where, \( \rho \) is the density of the fluid, \( \mathbf{u} \) is the instantaneous local velocity of the particle, \( \omega_0 \) is the amplitude of this harmonically varying velocity, \( T \) is the time period of the harmonic flow, \( D \) is the diameter of the cylinder, \( \mu \) coefficient of viscosity. Thus, according to this relatively sophisticated formulation the coefficients are analyzed as

\[
c_d = c_d (R_e, K_c, \varepsilon)
\]

(2.89)

and

\[
c_m = c_m (R_e, K_c, \varepsilon)
\]

(2.90)

where \( \varepsilon \) is another non-dimensional parameter representing the surface roughness of the cylinder.

For rough cylinders, \( c_d \) and \( c_m \) have values which become nearly independent of Reynolds' number for values of \( R_e \) in excess of \( 4 \times 10^5 \), but which still depend on the \( K_c \) and the relative roughness, \( \varepsilon \). Experiments performed on smooth cylinders in oscillating flows resulted in approximately same values of the coefficients as for steady flows. However, for rough cylinders undergoing oscillations, these values for oscillating cylinders were as high as 1.2 to 1.8 and considerably higher than the value of 0.7, commonly used in design.

Even with this sophisticated analysis, where all the identifiable variables are considered, the observed values show a wide scatter that remains unexplained [Sarpkaya, 1981]. The scatter is usually attributed to one or more of the several factors. They are, irregularity of the ocean waves, effect of the free surface on the force computation, 3-dimensional nature of the actual flow represented by a 2-dimensional model, inadequacy of the averaged resistance coefficients to represent the actual variation of a non-linear drag force and omission of some important parameters in this form of modeling.

Typically, the value of \( c_d \) can range between 0.5 to 2.0 depending on the flow situation and surface roughness. The value of \( c_m \) can vary between 0.6 to 2.0 depending on the geometry of the
member. Commonly used values for these coefficients for a circular cylinder are $c_d = 0.60$ to $1.0$, and $c_m = 2.0$.

In contrast to the numerous laboratory tests, some full-scale force measurements and analysis were done by Exxon on a 325 millimeter diameter pile exposed to wave loading in the open ocean [Kim and Hibbard, 1975]. This permitted an assessment of morison's coefficients, independent of any wave theory. The mean value obtained for $c_d$ was 0.61 with a coefficient of variation of 0.24, whereas for $c_m$, the corresponding mean and coefficient of variation were 1.20 and 0.22. No significant differences were found for seas with significant wave heights ranging from 0.8 to 3.0 meters.

2.8.4. Wave Force Regimes

Wave forces on structures are computed by a variety of methods. The choice of a particular method depends on the dimension of the structure relative to the characteristic dimensions of the wave.

The parameters used to define these regimes for force computation purposes are cylinder diameter, $D$, peak to trough wave height, $H$, and the wavelength, $\lambda$.

For $D/\lambda > 1$ : Condition approximates pure reflection of the waves by the structure.

For $D/\lambda > 0.2$ : Diffraction forces need to be considered.

For $D/\lambda \leq 0.2$ : Morison equation is valid in this region.

For $0.5 \leq DI/H \leq 1$ : Inertia forces can be used to represent the total force on the structure, for example, the large diameter structures like the columns supporting the decks of gravity type structures.
For $0.1 \leq D/H \leq 0.5$: both inertia and drag force need to be considered.

For $D/H = 0.2$: Drag and inertia forces are comparable.

For $D/H \leq 0.1$: Drag forces can be used to represent the total force. An example being small diameter members like conductor tubes etc., where viscous effects provide the primary force contributions.

It is also worth noting at this point that the ratio $D/H$ can be related to $D/\lambda$ based on the limiting heights of breaking waves. Waves become unstable and break when $H/\lambda \geq 1/7$. Thus for stable waves, $D/H \geq 7D/\lambda$. In a strict sense, the concept of orbital width should be used, instead of the wave height for identifying the force regimes. The orbital width is defined as

$$W = 2 \left( \frac{H \cosh \kappa d}{2 \sinh \kappa d} \right) \cos(\Theta) \mid_{\Theta=0} = \frac{H}{\tanh \kappa d}$$

(2.91)

Note that in deep water, $\tanh (\kappa d) \to 1$, whereby $W = H$ in the above equation.

A very important assumption of Morison equation is that the waves are unaffected by the presence of the structure. This is justified only in case of relatively small diameter members, as indicated by the preceding relations. For members of larger diameters the diffraction and reflection effects become increasingly important, and Morison equation has to be replaced by diffraction theory that accounts for this [Chakrabarti, 1987].

2.9. Critical Member Dimension For Use of Morison Equation

The reliable use of Morison equation for wave force computation is governed by the relative dimension of the member with respect to the wave ($D/\lambda \leq 0.2$). First, this relation is rewritten by replacing the wave length by a characteristic frequency

$$\frac{\kappa D}{2\pi} \leq 0.2$$

(2.92)
Which yields

\[ D \leq \frac{0.4\pi}{\kappa} \]  \hspace{1cm} (2.93)

where, \( \kappa \) is the wave number which is a function of the frequency. Using the dispersion relation for deep water

\[ \omega^2 = g\kappa \]  \hspace{1cm} (2.94)

and using \( \omega_{ch} \) as a characteristic value of the frequency,

\[ D \leq \frac{0.4g\pi}{\omega_{ch}^2} \]  \hspace{1cm} (2.95)

Substituting \( \omega_{ch} \) for \( \omega \) for a chosen percentage value of \( p \) in the P-M spectrum, one obtains

\[ \omega_{ch} = \omega_x = \left[ \frac{0.74}{-\ln(p)} \right]^{\frac{1}{4}} \frac{g}{U} \]  \hspace{1cm} (2.96)

On substitution of eqn. (2.96) in (2.95), one obtains

\[ D \leq \frac{0.4\pi}{g} \left[ \frac{-\ln(p)}{0.74} \right]^{\frac{1}{2}} U^2 \]  \hspace{1cm} (2.97)

Here, \( U \) is the wind velocity which must have units consistent with \( D \) and \( g \), and \( p \) is the fraction of waves satisfying eqn. (2.95).

2.10. Conclusion

A rational method is developed for identifying critical water depths for use with linear Airy waves and the PM spectrum. This proposed method forms a logical extension of the concepts used for deterministic wave theory to the case of irregular waves. Closed-form expressions relating wind velocity to critical water depth are derived. It is shown that for a majority of the offshore platforms on site, neither the deep water nor the shallow water approximations are justified. The error involved in making the "deep water" assumption is indicated.
A similar concept is proposed for identifying critical member dimensions for the reliable use of Morison equation for computing member forces in a given sea state. Again, the method relates the critical member dimensions with the wind velocity using the one parameter PM spectrum. Hopefully, these concepts will eliminate some of the "guesswork" in using these models in the computation of irregular wave forces.
CHAPTER III
Simulation Through Harmonic Superposition

3.1. Overview

The present chapter critically assesses a well established concept in the field of Monte Carlo Simulation. Specifically, the concept of simulation through harmonic superposition is extensively reviewed in the context of offshore structures. The primary merits and demerits of the method are investigated, and the need for research to develop an alternate approach is pointed out.

3.2. Introduction

The response analysis of ocean structures and platforms to irregular waves frequently require a Monte-Carlo solution approach. In this procedure the irregular waves and the associated kinematics, specified by the spectrum, need to be simulated. There are two different approaches for the simulation procedure; through harmonic superposition, and by digital filtering of a white noise series. The present chapter concentrates on the former method, whereas the latter method is dealt with in the following two chapters.

The harmonic superposition model generates the irregular record by the superposition of a finite number number of discrete harmonic components. This popular and well-established procedure for simulation is based on a model proposed by Rice [1954]. The harmonic superposition model can be subdivided into the random phase deterministic spectral amplitude (DSA) model, and the non-deterministic spectral amplitude (NSA) model. Both of these models are critically reviewed in this section. Their merits and demerits are analyzed and the need for the relatively new method of digital filtering is emphasized.
3.3. representation of a Random Sea

The random sea surface, \( \eta(t) \), is commonly represented by the superposition of harmonics with random phases [Pierson, Newman, and James, 1955]. The model is based on a Fourier-Stieltjes form of integral representation [Pierson, 1955; Kinsman, 1965; Phillips, 1969], and the concept of an infinitesimal random wave [Kree, 1986]. That is,

\[
\eta(t) = \int_{-\infty}^{\infty} dB(\omega) \, e^{-i\omega t}
\]  

(3.1)

The integral is usually replaced by an infinite summation, truncated for practical purposes, to a finite summation. There are two versions of this model. The first involves deterministic amplitudes and random phases. The second involves non-deterministic amplitude. Both are based on the model first proposed by Rice [1954]. The former model seems to be the more popular one with the offshore industry, and only rarely is the latter model used. A comparison of the two methods is reported by Tuah and Hudspeth [1982], as applied to the prediction of hurricane waves for which measured data were available to compare the validity of the predictions.

3.3.1. Random Phase model

This is a very convenient and conceptually simple model, where one can visualize the random surface to be the result of superposition of an infinite number of harmonics with random phases. The amplitudes of the harmonics are considered fully deterministic, specified by the spectral magnitudes at those discretized frequencies. Thus the surface elevation process at a point on the still water level can be represented as,

\[
\eta(t) = \sum_{n=1}^{N} C_n \cos(\omega_n t - \phi_n),
\]  

(3.2)

where the deterministic amplitude is
\[ C_n = \sqrt{2S(\omega_n)\Delta \omega_n} \]  
(3.3)

and \( \phi_n \) is the random phase angle. Thus, the model is also known as the deterministic spectral amplitude (DSA) model. The random phases are considered to be uniformly distributed in the interval \([0, 2\pi]\).

The validity of simulations based on this popular model has been criticized by Tucker et al., [1984]. This model can simulate time histories which can satisfy the conditions of a Gaussian sea only in the limiting case of \( N \to \infty \). Another criticism of this model is that the true "randomness of the real wave system will be lost" [Tucker, et. al., 1984], due to the deterministic nature of the amplitudes. Jefferys [1988] reports that the time series so generated are too "typical". In addition, the time series fails to represent the fourth order cumulants, extreme value statistics and groupiness effects.

Groupiness relates to the presence of "a finite run of higher-than-normal waves" [Wayne and Sobey, 1987] frequently observed in a real sea and wave records. These wave groups can have serious impact on a wide range of coastal and offshore activities. This is specially important in the context of analysis of second order response of moored ships or tension leg platforms (TLP). The natural periods of these systems in surge, sway or yaw motions, usually are well above that of the energy intensive waves in a real sea. However, these motions are easily excited by the second order effects of the waves and can dominate the dynamic response of the system.

3.3.2. Random Amplitude Model

Some of the limitations of the DSA model can be overcome by considering the spectral amplitudes to be random, according to the second model of Rice [1954]. This is expressed as

\[ \eta(t) = \sum_{n=1}^{N} \left[ A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right] \]  
(3.4)
The coefficients $A_n$ and $B_n$ are independent random variables with zero mean and a standard deviation given by

$$
sigma_{A_n} = \sigma_{B_n} = \left[ 2S(\omega_n) \Delta \omega_n \right]^{\frac{1}{2}} \tag{3.5}
$$

This model has randomness built into the spectral amplitudes and is known as the non-deterministic spectral amplitude model (NSA). This model is superior in representing groupiness effects [Tucker, et. al., 1984], and has also proved more accurate in simulating realistic wave trains for hurricane storms [Tuah and Hudspeth, 1982]. Thus, the NSA model can be recommended without reservations over the DSA model, for random wave simulation through superposition of harmonics.

3.3.3. Comparison of DSA and NSA Models

Each of the above simulation methods has some advantages and disadvantages. In the DSA method the sample spectrum will be identical to the target spectrum for every simulation. On the other hand, the sample spectrum of time histories simulated by the NSA method will have a random fluctuation about the target spectrum, each sample spectrum being different from the other. Theoretically, an infinite sample sum must be generated in order for the statistics of the process simulated by either of these methods to be Gaussian. However, for discrete band-limited white noise, the infinite sample sum must be truncated to some finite sample length that is determined by the Nyquist cut-off frequency. It is reported [Hudspeth, Nath and Sollitt, 1985] that band-limited white noise applications which use the NSA simulation method will yield discrete time sequences that display better Gaussian properties than will the DSA method.
3.4. Advantages and Drawbacks of Harmonic Superposition

The primary advantage of simulation by harmonic superposition is its conceptual clarity. Also, the correlation between the various kinematics is naturally maintained, as the kinematic parameters are essentially computed for each harmonic and then superimposed. For these reasons, it is popularly used in industry.

Among the disadvantages, the most important is the computational cost. A large number of harmonics have to be considered to get a good approximation of the spectrum. The generated process tends to a zero mean Gaussian process only in the limit $N \to \infty$. The number of harmonics used by Tucker et. al. [1984] is 900, and that used by Elgar et. al. [1985] is 2130. The latter authors claim to have obtained satisfactory agreement of groupiness representation between the DSA and NSA methods, by using these higher number of harmonics. The computational costs are reduced considerably by using the FFT algorithm, which is the current practice of implementing this model.

Furthermore, the spectrum corresponding to the generated time series is discontinuous. This is because the theoretical form of the spectrum based on the harmonic superposition model is a series of Dirac delta functions, at the discrete frequencies. The effect is somewhat masked in the spectrum computed by Fourier transforming the generated time records. The finite length of the time record corresponds to a multiplication in the time domain by a rectangular window, which results in a convolution in the frequency domain, causing a "smearing" effect. This is due to "leakage," or presence of high frequency side lobes introduced by this finite length time window. Usually, the resulting spectrum is "smoothed" by using some window in the frequency domain. Thus, the delta functions are "smoothed out" to sharply peaked triangle-like functions. The resulting discontinuities in the spectrum could result in significant errors especially when
second order response from slow drift forces are sought, and a very large number of harmonics need to be used for such problems. At least 50 harmonics are recommended in any case, and about 200 or more when very accurate representation of the spectrum is sought [Chakraborti, 1987].

Finally, the simulated time record repeats itself after a generation time $T_{\text{max}} = 1/(\Delta f)$ due to the inherent periodicity of the model. This can be overcome by selecting unequal intervals on the frequency axis [Borgman, 1969]. However, the unequal discretization of the frequency axis excludes the use of the highly efficient FFT [Cooley and Tukey, 1965] algorithm for the computation.

3.5. The Fast Fourier Transform (FFT) Algorithm

The development of the Fast Fourier Transform (FFT) algorithm [Cooley and Tukey, 1965] was a significant break-through in signal processing. It has replaced the computation of the spectrum through the auto-correlation technique, by direct Fourier transformation to the frequency domain. The FFT algorithm has also been implemented to perform the computations of the harmonic superposition model more efficiently. In its most efficient form, the method can reduce the number of operations from $N^2 \text{to } 2N \text{log} N$.

The harmonic superposition model using an FFT algorithm is the currently accepted simulation procedure in the offshore industry [Shinouzuki and Wai, 1979, Hudspeth and Chen, 1979, Hudspeth and Borgman, 1979]. Note that the simulated records have the same advantages and disadvantages of the harmonic superposition model, except for a significant reduction of computational costs. The time series is periodic, as the frequency intervals need to be equally spaced to implement the FFT algorithm, and the spectrum is discontinuous.
3.6. Conclusion

The harmonic superposition method is the most convenient model for the representation of random seas. It also provides a conceptually simple model for simulating wave kinematics compatible with a given target spectrum. However, the method has some serious limitations in terms of computational cost, repetitive nature of the generated time series, and discontinuous nature of the spectrum. This clearly indicates the need for a method that overcomes these drawbacks. The next chapter explores an alternative scheme of wave simulation through filtering of band-limited white noise.
CHAPTER IV
Filter Theory and Wave Simulation

4.1. Overview

The present chapter aims to serve two purposes. Firstly, it aims to provide a brief yet self-contained introduction to the emerging field of signal processing, as applied to the ocean wave simulation problem. Secondly, some useful interpretations of theorems from the literature on filter design are provided. These interpretations, which are relevant to the wave simulation problem, are not readily available in the existing literature on the topic.

An attempt is made to provide, in very simple terms, a broad perspective of these developments and their impact on computer simulation of ocean waves. It is expected that these interpretations would provide a deeper understanding of the filter approaches to the wave simulation and propagation problems, along with their inherent limitations.

4.2. Introduction

The field of signal processing has experienced rapid developments in the last two decades. Simultaneously, significant advances have been made in computer hardware, primarily in terms of speed and storage capacities. New methods, which exploit these current developments, have replaced older methods which relied on limited computational facilities. These developments have brought about major changes in most fields of engineering. The area of Monte-Carlo simulation is no exception. The method of harmonic superposition used to be the predominant method for simulation of random processes [Borgman 1969, Shinozuka, Fang and Nishitani 1979, Shinozuka and Wai, 1979,]. Recently, a number of interesting papers have focussed on the alternate method of simulation by filtering a white noise process [Spanos and Hansen, 1981,

A good part of the material discussed here is available in a scattered form in the literature on signal processing and filter design. Moreover, most of the literature in the area of signal processing is written for the specialists in that field. Unfortunately, the material is usually not presented in a manner comprehensive enough to a non-specialist in the field. Engineers engaged in design and analysis of offshore platforms and structures frequently need to perform simulation studies, and often do not have enough exposure to the above fields of specialization, developed primarily as an area of electrical engineering. One of the aims of this work is to bridge this gap.

In this regard, the following section introduces some of the important concepts in filter theory, and briefly explains the properties of different filter types, and their advantages and disadvantages in general use. A critical analysis of their applications in ocean wave simulation is included whenever possible. Also, some useful interpretations are provided and some conclusions are drawn regarding the capabilities of certain filter types for specific offshore applications.

4.3. Transforms Relating the Time and Frequency Domains

In random vibration studies, which has close links to the theoretical developments in the signal processing area, there are two approaches that one can take. The theory can be developed in the time or the frequency domain. The common link between the two domains is provided by the Wiener-Khintchine relationship. Thus, the transfer function in the frequency domain is related to the impulse response in the time domain through one of the following transform techniques. They all serve the same basic purpose. However, each has its own merits and demerits.
4.3.1. Laplace Transforms

In the study of analog filters one uses either the Laplace transform or the Fourier transform.

The Laplace transform can be one-sided or two-sided, and is defined by

\[ H(s) = L \left[ h(t) \right] = \int_{0}^{\infty} h(t) e^{-st} dt \]  (4.1)

where \( L \) and \( L^{-1} \) denotes the Laplace transform and its inverse. The lower limit of integration is 0 for the one-sided Laplace transform, and is \(-\infty\) for the two-sided Laplace transform. The corresponding inverse Laplace transform from the frequency to the time domain is given by

\[ h(t) = L^{-1} \left[ H(s) \right] = \frac{1}{2\pi j} \int_{c} H(s) e^{st} ds \]  (4.2)

where \( c \) represents a contour integration, and \( j \) represents \( \sqrt{-1} \).

4.3.2. Fourier Transforms

The Fourier transform is more common in engineering use, and can be shown to be a special case of the more general two-sided Laplace transform. The Fourier transform is defined by

\[ H(j\omega) = F \left[ h(t) \right] = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \]  (4.3)

\[ h(t) = F^{-1} \left[ H(j\omega) \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) e^{j\omega t} d\omega \]  (4.4)

The Fourier transform can be obtained as a special case of the two-sided laplace transform, by putting \( s = j\omega \), provided the region of convergence includes the imaginary axis.

4.3.3. Z-Transform

While studying digital filters one encounters the z-transform of a sequence, \( h_n \), defined as

\[ H(z) = \sum_{n=0}^{\infty} h_n z^{-n} \]  (4.5)
The inverse $z$-transform is given by

$$h_n = \frac{1}{2\pi j} \int_{C} H(z) z^{-n-1} \, dz \quad (4.6)$$

The symbol $c$ denotes the closed contour, counter-clockwise, in the region of convergence of $H(z)$ and encircling the origin in the $z$-plane, and $z$ is defined as

$$z = e^{iT} = e^{j\omega T} \quad (4.7)$$

with $T = \pi/\omega_b$ being the sampling interval. The $z$-transform reduces to the Fourier transform on substituting $|z| = 1$, again provided that the unit circle lies in the region of convergence, which implies that all the poles are within the unit circle.

A potential source of confusion for the non-specialist in the area is the completely opposite conventions adopted by researchers in mathematics and economics, as opposed to those in the signal processing and digital control. The former group uses a positive exponent of $z$ in the $z$-transform equation, obtaining a characteristic function which is a polynomial in positive powers of $z$. The latter group uses a negative exponent in the $z$-transform, to obtain a polynomial in negative powers of $z$. Thus, stability in signal processing requires the poles of the transfer function to be inside the unit circle, whereas for the other group, the poles need to be outside the unit circle. Throughout this presentation the convention of the signal processing group is followed.

4.4. Causality

A linear time-invariant filter is said to be causal or realizable if the output depends only the past values of the input. Most of the theory in filter design centers around causal filters, but often non-causal filters are encountered, such as ideal low-pass filters and ideal differentiators. Usually, an attempt is made to use causal filter approximations to these non-causal filters. The primary motivation of using causal filters is firstly to avoid the undesirable situation where the
present value of the system response depends on the to future values of the inputs. For the IIR filters this could sometimes lead to the highly undesirable situation of the present output depending on the future values of the inputs at infinite time. Mathematically, causality implies

\[ h(t) = 0 \quad \text{for} \ t < 0 \quad \text{(4.8)} \]

or

\[ h_n = 0 \quad \text{for} \ n < 0 \quad \text{(4.9)} \]

Furthermore, for a causal or purely non-causal system, the specification of the transfer function alone, without a specified region of convergence, is sufficient to recover a unique \( h(t) \) given \( H(s) \). For a non-causal system, \( H(s) \) along with a region of convergence needs to be specified, to uniquely determine \( h(t) \).

4.5. Stability

A causal linear time-invariant filter is said to be stable if bounded inputs produce bounded outputs. This is equivalent to the impulse response of the filter being finitely summable. For the analog filter, the usual norm of stability is

\[ \int_{0}^{\infty} |h(t)| \, dt < \infty \quad \text{(4.10)} \]

and for the discrete case,

\[ \sum_{n = 0}^{\infty} |h_n| < \infty \quad \text{(4.11)} \]

For analog filters the transfer function cannot have poles with positive real parts, that is all poles are in the left half of the \( s \)-plane. For digital filters this is equivalent to having all poles within the unit circle.
4.6. Phase Response and Delay

Given a complex transfer function $H(\omega)$, having both real and imaginary parts, it is usually represented by the magnitude function defined as

$$M(\omega) = \left[ | \text{Re} \, H(\omega) |^2 + | \text{Im} \, H(\omega) |^2 \right]^{1/2}$$  \hfill (4.12)

and the phase is defined as

$$\theta(\omega) = \tan^{-1} \left( \frac{\text{Im} \, H(\omega)}{\text{Re} \, H(\omega)} \right)$$  \hfill (4.13)

According to the above definition, the magnitude is not an analytic function, and the phase is a discontinuous function of $\omega$, giving rise to mathematical problems. It is common practice to redefine the real-valued amplitude function $A(\omega)$ that can take positive and negative values, and a continuous version of the phase $\phi(\omega)$. Thus,

$$A(\omega) = \pm M(\omega)$$  \hfill (4.14)

and

$$\phi(\omega) = -\theta(\omega)$$  \hfill (4.15)

The phase delay is defined as the time by which the output lags the input for a given frequency. Mathematically it can be expressed as

$$\tau_d = \frac{\phi(\omega)}{\omega}$$  \hfill (4.16)

The group delay is a measure of the average delay of the filter as a function of the frequency. It represents the delay of the envelop of the signal, for closely spaced frequency bands. The group delay is defined as

$$\tau_g = \frac{d\phi(\omega)}{d\omega}$$  \hfill (4.17)

Similar concepts in the theory of gravity waves are phase velocity and group velocity.
A highly desirable property in many signal transmission is a linear phase, when the phase delay equals the group delay and this delay is constant for all frequencies. There is no dispersion, or propagation of different frequency components at different speeds in this case. This is the primary motivation behind linear phase, obtained through FIR filters. It is interesting to note that the horizontal propagation of Airy waves in shallow water reduces to a constant delay, non-dispersive transfer function, due to the linear relation of the wave number and frequency in this regime. In general, the transfer function for horizontal propagation of wave kinematics has a non-linear phase due to the non-linear, dispersive, relation between the wave number, k, and frequency, ω, provided by the non-linear dispersion relation.

4.7. Aliasing and the Nyquist relation

Use of sampled data to represent a continuous signal raises the natural question about the validity of the representation. Do the sampled data match the continuous signal only at the sample points? It can be shown that the sampled data can be used to reconstruct the continuous signal exactly provided the Nyquist criterion is satisfied. It is expressed by the equation

$$T = \frac{\pi}{\omega_c}$$  \hspace{1cm} (4.18)

where ω_c is the cutoff frequency, the largest value of frequency present in the signal, and T is the sampling interval. In this case, the continuous signal x_d(t) can be recovered exactly by interpolation between the sample points using the relation

$$x_d(t) = \sum_{k=-\infty}^{\infty} x_d(kT) \frac{\sin[(\pi/T)(t-kT)]}{(\pi/T)(t-kT)}$$  \hspace{1cm} (4.19)

In simple terms, the Nyquist criterion ensures that there is at least two samples per period for all frequency components present with any appreciable energy contribution. If this is not ensured then the frequency response of the discrete system will no longer match that of the continuous
system, due to aliasing errors. Higher frequencies beyond the cutoff frequency, if present, would
be aliased into the lower frequencies, yielding an erroneous frequency response function espe-
cially near the cutoff frequencies. For many signals the spectrum at the high frequency range
decay in some asymptotic manner. Strictly speaking, such spectra go to zero only as \( \omega \to \infty \).
Using a cut-off frequency at some finite value of \( \omega \) does introduce some aliasing error. However,
this error is usually acceptable, as long as the magnitude of the error is insignificant. Frequently,
the cut-off frequency is determined by comparing the spectral magnitude at the high frequency
end with that at the spectral peak. A commonly used value is one-hundredth of the peak value.

4.8. Kramers-Kronig Relationship

For any causal system there is a unique relationship between the real and imaginary parts of
the transfer function. They form a Hilbert transform pair [Oppenheim and Schafer, 1975].

\[
\begin{align*}
\text{Re } H(j\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im } H(j\sigma)}{\omega - \sigma} \, d\sigma \\
\text{Im } H(j\omega) &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Re } H(j\sigma)}{\omega - \sigma} \, d\sigma
\end{align*}
\] (4.20) (4.21)

This property, eqns. 4.20 - 4.21, is also known as the Kramers-Kronig relationship. A necessary
condition for the existence of a causal filter is provided by these relationships. Careful inspection
of these relations reveals that the vertical propagation of wave kinematics cannot be achieved by
using any causal filter. The transfer function for vertical propagation being purely a real function
of \( \omega \), the Kramers Kronig relations are not satisfied by this transfer function. Thus, a non-causal
filter has to be used to approximate this zero phase-lag transfer function.

4.9. Paley-Weiner Condition
The Paley-Wiener criterion imposes a fundamental restriction on all physical transfer functions \cite{Zadeh_and_Desoer_1963}. It states that a transfer function \( M(\omega) \) can be realized by a causal, stable filter if and only if the inequality

\[
\int_{-\infty}^{\infty} \frac{1}{1 + \omega^2} \ln |M(\omega)| \, d\omega < \infty
\]  

is satisfied. This implies that the transfer function \( H(\omega) \) cannot vanish over any band of frequencies on the real frequency axis, though it may have a countable number of zeros on the \( j\omega \) axis. Further, \( H(\omega) \) cannot fall off to zero more rapidly than exponential order. That is,

\[
|M(\omega)| < N \alpha^{|\omega|}
\]  

where \( N \) and \( \alpha \) are constants.

The Paley-Wiener condition is not satisfied by the P-M spectrum, and this was the source of numerical instabilities in an AR approximation of this spectrum \cite{Spanos_and_Hansen_1981}. The problem in this context is discussed by Kree \cite{Kree_1986} and by Spanos and Minolet \cite{Spanos_and_Minolet_1986}.

The transfer function for vertical propagation in deep waters is described by a lowpass Gaussian filter \cite{Samii_and_Vandiver_1984}. This transfer function cannot be realized by a causal filter, as the Paley Wiener conditions is not satisfied in this case because the transfer function dies off faster than the exponential order.

A simple technique to make possible the use of a causal filter for a transfer function that does not satisfy the Paley-Wiener condition is to approximate it by a new function, with the desirable properties. The selected function should have a mathematical form that satisfies the Paley-Wiener condition and should approximate the magnitudes satisfactorily \cite{Zadeh_and_Desoer_1963}. However, filters based on fairly close approximations of the transfer functions may yield completely different impulse responses, a fact that is not always obvious.
The Paley-Wiener condition also ensures the existence of a solution to the spectral factorization problem [Robinson and Silvia, 1981, pp-59]. However, the uniqueness of the spectral factorization problem is not guaranteed by this condition.

4.10. Spectral Factorization

The problem of spectral decomposition, or spectral factorization, is a highly complex problem. Generally, an unique solution to the spectral decomposition problem is sought. As a mathematical tool, the spectral factorization was introduced by Wiener [1949] to obtain a frequency domain solution to optimal filtering problems in univariate analysis. The extension of the concept to cover the multi-input-multi-output problem is relatively recent and came with developments in the fields of system theory, network theory and optimal control. Given a spectral matrix, the problem involves finding a stable and minimum phase transfer function matrix. Analytical methods for the solution of this problem can be found in Youla [1961], Davis [1963]. The use of these analytical techniques presents some practical problems when the order of the matrix is large. Numerical schemes using iteration are more suitable for these situations. Tuel [1968], and Anderson [1974] appeal to matrix Riccati equations. Rissanen [1973], Youla and Kazanian [1978] and Kucera [1979] perform a triangular factorization. Wilson [1969, and 1972] used the Newton method of iteration. A highly efficient method, superior to the ones mentioned earlier is reported by Jezek and Kucera [1985]. The technique uses the Newton method but at the same time makes full use of the special structure of the equations to be solved in each iteration. The algorithm is outlined for both the discrete and continuous versions.

4.11. Minimum Phase Filters

The minimum phase filter provides a unique solution to the spectral factorization problem
[Robinson and Silvia, 1981, p-59]. For these filters, the phase spectrum can be uniquely determined from the magnitude spectrum, through the use of the Bode relationship [Bode, 1940]

$$\theta(\omega) = \frac{2\omega}{\pi} \int_0^\infty \frac{\ln M(\sigma) - \ln M(\omega)}{\sigma^2 - \omega^2} d\sigma$$  \hspace{1cm} (4.24)

The minimum phase condition requires that all poles or zeros of the transfer function be in the left half of the s-plane for analog filters, or inside the unit circle in the z-plane for digital filters.

In system identification problems, this concept plays an important role. There are many alternate and equivalent representations of a system, of which the minimum phase representation is unique. Another situation where minimum phase filters are very useful is in inverse filtering [Oppenheim and Schafer, 1975, p-345] where it is necessary to obtain an appropriate phase curve given only an autocorrelation function. This concept may provide some clues to the problem of representation of groupiness through reconstruction of phase information, in wave records.

In many filter applications a zero phase lag is desired, but cannot be obtained. A minimum phase filter is the best choice in that situation [Parks and Burrus, 1987, p-111]. Finally, these filters automatically ensure a minimum filter order, or length over which the impulse response is non-zero [Robinson, 1981, Jefferys, 1988], yielding the most economical filter implementation.

4.12. All Pass Filters

All pass filters are a very special type of IIR filter. They have magnitude of unity over the entire frequency range, $\omega$, and bring about a phase modification alone. The transfer function of these filters are of the form

$$H_{ap}(s) = \frac{(s + \bar{p}_1)(s + \bar{p}_2) \cdots (s + \bar{p}_n)}{(s - p_1)(s - p_2) \cdots (s - p_n)}$$  \hspace{1cm} (4.25)

Where the bar denotes complex conjugate. To ensure stability, $\text{Re} \ p_j < 0$. These all-pass filters
serve to bring about a frequency-dependent phase shift, and are also known as phase or delay equalizers. The zeros and poles of an all pass filter are symmetric with respect to the $j\omega$ axis and differ only in the sign of their real parts. A number of these filters can be combined in cascade to build up a desired phase response curve.

They are frequently used to equalize the phase delay, leaving the transfer function magnitude unaffected, of an output obtained through filters which do not have good phase characteristics, but are otherwise very efficient. This situation is common, especially as most of the proven efficient filters are designed from magnitude considerations alone [Parks and Burrus, 1987]. However, the alternative of modifying the phase of the response obtained through minimum phase filters [Jefferys, 1988], should be critically assessed. This option should be chosen only when the minimum phase filter gives a significant economy in filter order, or impulse response duration. The advantage of the minimum phase filter is likely to be outweighed by the additional computation needed for the all-pass filter added. Thus, a direct design of an FIR filter, can be a better choice in these situations.

The use of an all pass filter for the frequency dependent dispersion of gravity waves as they propagate horizontally, seems to be an attractive idea, suggested by Jefferys [1988]. This propagation problem has so far been handled through FIR filters, by Samii and Vandiver [1984] for the case of "deep water" and by Bhattacharjee and Spanos [1987] for the general case of any arbitrary water depth. The design of second and fourth order all-pass filters are investigated in the present study and satisfactory approximations are obtained. This investigation is reported under the section on horizontal propagation, discussed in the next chapter.
4.13. White noise process

In the simulation of random processes one frequently uses the white noise process as the input to a shaping filter to obtain a random process with desired spectral characteristics. The term "white noise" refers to second order stationary stochastic processes with constant power spectral density. The name has its origin in optics, used to define "white" light as a spectrum with all spectral components being present. For practical purposes one uses a band-limited white noise, which has a finite power, as opposed to the infinite power in the case where all frequencies are present. A complete description of the white-noise process and its properties are provided by Grigoriu [1987], and by Clough and Penzien [1975]. Here, \( w_n \) denotes a bandlimited white-noise process being a sequence of independent and identically distributed random numbers, with auto-correlation function

\[
E \left[ w_i w_j \right] = \omega_b \delta_{ij}
\]

where, \( \delta_{ij} \) is the Kronecker delta function.

4.14. Filter Types

The filters can be broadly classified into two categories, each category having its digital and analog versions. The first kind are called Finite duration Impulse Response (FIR) filters. They are also known as moving average (MA) filters for discrete time, digital version, and convolution filters for continuous time, analog version. The second kind are the Infinite duration Impulse Response (IIR) filters. They are commonly referred to as the auto-regressive (AR) and the auto-regressive-moving-average (ARMA) as the discrete time, digital versions, and the differential equation form in continuous time, analog version. The mathematical forms of these filters are discussed under the respective filter headings.
4.14.1. Moving Average (MA) Filters

The MA filter belongs to the FIR class of filters. By definition, the impulse response function has finite duration, and decays out after a certain time lag, or number of filter terms. This filter generates the current value of the output as a linear, weighted combination of the white noise inputs \( w(t) \). When the impulse response is zero for \( t < 0 \), then the system is causal, the output depends only on the past inputs, and the filter is called the one-sided MA or a causal MA filter. Specifically, the filter output can be expressed as

\[
x_n = \sum_{l=0}^{q} b_l w_{n-l} \quad n = q + 1, q + 2, \ldots \quad (4.27)
\]

The analog version of the same filter is

\[
x(t) = \int_{0}^{t} h(\tau) \, \hat{w}(t-\tau) \, d\tau \quad (4.28)
\]

When the output depends on the past, present, and the future values of the inputs, the filter is called a two-sided MA or a non-causal MA filter. The discrete and analog version of the input-output relation for the non-causal system are

\[
x_n = \sum_{l=-q}^{q} b_l w_{n-l} \quad (4.29)
\]

and

\[
x(t) = \int_{-\infty}^{\infty} h(\tau) \, \hat{w}(t-\tau) \, d\tau \quad (4.30)
\]

Here one must recognize that \( h(\tau) \) becomes negligible beyond certain range of \( \tau \). Thus, the integration, in practice, needs to be carried out only within this range. Clearly, a non-causal filter cannot be used for online filtering, for example for control purposes, as it requires the future values of inputs. However, it is frequently used in simulation procedures for its capability to represent purely real transfer functions, through a symmetric filter, centered at zero. The finite
duration of the impulse response function is crucial while using a non-causal model.

The transfer function of the causal filter in the z-plane is a polynomial in \( z^{-1} \). That is,

\[
H(z) = \sum_{i=0}^{q} b_i z^{-i}
\]  

(4.31)

It is also known as an all-zero filter, as the zeros in the z-plane completely define the frequency response of the filter which has only one pole at the origin. The analog version of this filter generates the output as a linear convolution of the input with the impulse response function of the filter. The duration time of the impulse response of this filter is always finite, hence the name FIR. The filter operates by producing an weighted average of the input, and moving over the input series to produce successive values of the output. Since the output depends only on the inputs there is no recursion, and the filter is also known as an non-recursive filter.

The absence of any recursive relation, or feedback, makes the filter always stable, a highly desirable property. The filter can have transfer functions with exactly linear phase, whereby the group delay is the same as the phase delay, and phase distortions are minimized. However, the all-zero form results in a transfer function of polynomial form. Thus, transfer functions which can be easily approximated by polynomials can be represented by a MA filter of reasonably small order. Transfer functions which are sharply peaked may need an MA filter of high order for satisfactory approximation.

Convolution filters were used by Groves [1960] for computing wave kinematics. The application of digital MA filters to ocean wave simulation was first suggested by Borgman [1969], who provides an excellent description of the procedure to evaluate the filter coefficients. Spanos [1983] applied the method to the simulation of surface elevations based on the Pierson-Moskowitz spectrum, and also provided some insight into the optimality criterion for this type of
filter design. The method is also implemented in model tank simulation of random waves [Bryden and Greated, 1984].

4.14.2. Auto Regressive (AR) Filters

The AR filter belongs to the IIR class of filters. This filter generates the output as a linear combination of previous values of the output, and a current value of the input. This properly justifies the name auto-regressive or linear predictive coding. The filter output can be expressed as

$$x_n = - \sum_{k=1}^{p} a_k x_{n-k} + b_0 w_n$$ (4.32)

The continuous time version of this filter is a differential equation, where the non-homogeneous part provides the input. This is,

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \ldots + a_0 x = w(t)$$ (4.33)

where $x$ denotes $x(t)$ and the superscript $(n)$ denotes the $n$-th derivative of the variable with respect to time, $t$. The pulse transfer function has only poles in the $z$-plane, giving it the name all-poles method. The $z$-transform of the filter is given by the equation

$$H(z) = \frac{b_0}{1 + \sum_{k=1}^{p} a_k z^{-k}}$$ (4.34)

When used for spectral analysis, it is commonly referred to as the Maximum Entropy Method (MEM). The name IIR only implies that these filters are capable of having an infinite duration impulse response, not that it will necessarily be so. Typically the response dies off exponentially for high values of $a_k$ or time, $t$.

The presence of a recursion relation in the filter, eqn. 4.32, implies feedback, and an AR
filter can be unstable. Stability has to be ensured in the design by making sure that all the poles lie within the unit circle in the z-plane, or the left half of the s-plane.

The recursive relation based on the filter coefficients can be used to predict future values of the auto-correlation based on known values through the Yule-Walker equations. This capability of extrapolation of the auto-correlation function gives the method an edge over the periodogram analysis using the FFT, in some situations. The latter method can only use computed values of the auto-correlation function based on the measured run of the data. On the other hand, the AR method can extrapolate the auto correlation sequence beyond the run of measured data.

The existence of poles in the transfer function implies that very high peaks in the spectrum can be economically represented by the AR method. This method has been used for estimating ocean wave spectrum by Holm and Hoven [1979], and Houmb and Overvik (1981), and for simulation of random waves by Spanos and Hansen [1981].

4.14.3. Auto-Regressive Moving-Average (ARMA) Filter

The most versatile of all is the ARMA filter which combines the MA and the AR filters, and belongs to the class of IIR filters. The present value of the output is generated as a linear combination of the past values of the output, and the past and the present inputs. The filter output can be expressed as

\[ x_n = - \sum_{k=1}^{p} a_k x_{n-k} + \sum_{l=0}^{q} b_l w_{n-l} \]  

(4.35)

The continuous time version of this model is a differential equation involving the derivatives of both the input and output

\[ a_m x^{(m)} + a_{m-1} x^{(m-1)} + \ldots + a_0 x = b_n x^{(n)} + b_{n-1} x^{(n-1)} + \ldots + b_0 x \]  

(4.36)

The transfer function in the present case is a rational function, being a ratio of polynomials, and
has both poles and zeros. It is given by the equation

$$
H(z) = \frac{\sum_{l=0}^{g} b_l z^{-l}}{1 + \sum_{k=1}^{p} a_k z^{-k}}
$$

(4.37)

The transfer function in the case of the MA filter is a polynomial and for the AR filter it has a polynomial as the denominator.

A rational function is considerably more versatile than a polynomial, or its inverse, in approximating an arbitrary function. This makes the ARMA filter more powerful and economical in general. An ARMA filter can generally achieve a sharper transition between band edges, at pass-band and stop-band, than an MA filter with the same number of coefficients. The reason is that the ARMA filter has a pole near the edge of the pass band and a nearby zero near the edge of the stop band. However, it cannot achieve linear phase when it is causal, unlike the MA filter. Also, the design problem is significantly more complex, due to the the approximation of a rational polynomial instead of a simple polynomial. There is the usual problem of stability due to the recursive nature of the algorithm and associated feedback. Moreover, there is the delicate task of implementing a recursive realization in a fixed-point arithmetic [Parks and Burrus, 1987]. For some problems, the filter coefficients exhibit a strong sensitivity to the chosen value of cut-off frequency [Spanos, and Mignolet, 1986].

The differential equation approximation for wave simulation based on the Pierson-Moskowitz spectrum has been reported by Spanos [1983], and Flower and Vijeh [1983]. In a strict sense, this approximation is the analog version of a degenerate ARMA model, as there is only one term in the numerator polynomial. However, the numerator term being the derivative of the input series does not qualify it as an AR model where the derivatives are on the output series.
only, and not on the input. In both of the above references, the P-M spectrum is approximated by
the modulus square of the transfer functions of two SDF oscillators in cascade. The numerical
schemes for the minimization of the error function is different for the two investigators. The
digital version of the same problem has been solved by Samii and Vandiver [1984], and Spanos
and Minolet [1986]. The number of coefficients required for a good approximation by the
ARMA model is about 15 [Spanos and Minolet, 1986], which is approximately half of that by
the MA filter [Spanos, 1983].

It needs to be realized that the fewer number of coefficients of the IIR filter does not neces-
sarily translate into a direct saving of computational time. A specific comparison reported by
Parks and Burrus [1987; p-273] indicates that the FIR filter takes only about one fifth the time of
an IIR filter, with the same number of coefficients. This is because of the regular structure of
implementing the non-recursive filter as compared to the irregular structure required for the recu-
srve filter.

4.15. Design of Digital Filters From Analog Filters

The most popular approach to the design of digital IIR filters is to digitize an analog filter.
There are several well-established methods available for this transformation procedure. They
include, the method of mapping of differentials, the impulse invariance transformation, the bil-
inear transformation and the matched Z-Transform technique.

There are two desirable properties of any mapping from continuous to discrete space. First,
the $j\Omega$ axis in the $s$ plane should be mapped on to the unit circle in the $z$ plane. Second, points in
the left-half of the $s$ plane (Re $[s] < 0$) should be mapped inside the unit circle $|z| < 1$. The first
property preserves through the uniformity of the mapping the frequency selective properties of
the continuous system, whereas the second property ensures that stable continuous system, or filters, are mapped into stable discrete systems.

4.15.1. Mapping of Differentials

In this method the system is first represented in the time domain. For IIR filters this would be the differential equation form. The next step is to replace the differential equation

\[ \sum_{i=0}^{N} a_i \frac{dy(t)}{dt^i} = \sum_{i=0}^{M} b_i \frac{dx(t)}{dt^i} \] (4.38)

as a difference equation, by using finite difference formulation. The simplest approach is to use a forward or backward difference scheme

\[ \sum_{i=0}^{N} a_i \Delta_i[y(n)] = \sum_{i=0}^{M} b_i \Delta_i[x(n)] \] (4.39)

where \( \Delta_i[w(n)] \) is the \( ith \) difference defined by the equation

\[ \Delta_{i+1}[w(n)] = \Delta_i(\Delta_i[w(n)]) \] (4.40)

It can be shown that the digital transfer function can be obtained directly from the analog transfer function through the substitution

\[ s = \frac{1 - z^{-1}}{T} \] (4.41)

for the case of backward difference formula. Similar formulas for the forward and central difference schemes are available in the open literature [Rabiner and Gold, 1975]. Thus the process of replacing derivatives by differences does indeed correspond to a mapping of the \( s \) plane to the \( z \) plane. It can further be shown [Oppenheim and Schaffer, 1975] that this transformation maps the imaginary axis to a circle contained by the unit circle. Thus the stability requirements are satisfied.
4.15.2. Impulse Invariant Transformation

The impulse invariant transformation makes the digital filter a sampled version of the corresponding analog filter

$$h(n) = h_d(nT).$$  \hspace{1cm} (4.42)

The frequency response of the digital filter is an aliased version of the frequency response of the analog filter

$$H(z)|_{z=e^{j\omega}} = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_d(s + j\frac{2\pi k}{T})$$  \hspace{1cm} (4.43)

This transformation does not correspond to any simple algebraic transformation but maps horizontal strips from the s-plane into the entire z-plane. However, the left half of the s-plane is always mapped inside the unit circle in the z-plane. This ensures that stable analog systems are mapped into stable digital systems. The introduction of the aliasing effect requires caution in the use of these filters.

4.15.3. The Bilinear Transformation

The bilinear transformation eliminates the two limitations of the method of mapping of differentials. It maps the \( j\omega \) axis into the unit circle, and transforms stable analog filters into stable digital filters. The mapping is brought about by the following set of transformations.

$$s = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}$$  \hspace{1cm} (4.44)

This value of \( s \) is substituted in the analog filter to obtain the digital filter

$$H(z) = H_d(s)$$  \hspace{1cm} (4.45)

The present transformation also eliminates the problem of aliasing encountered with the use of impulse invariant transformation, but introduces a phase distortion. The relationship between the frequency variable \( \omega \) of the digital filter and \( \Omega \) of the analog filter are given by
\[
\frac{T\Omega}{2} = \tan \left[ \frac{\omega T}{2} \right]
\] (4.46)

This relationship is approximately linear for small values of \(\omega\), but is highly non-linear for the remaining range of \(\omega\), imposing a strong restriction on the use of the method. It implies that only analog systems with piecewise constant amplitude response can be satisfactorily transformed. Fortunately, for lowpass, bandpass and bandstop filters, this frequency warping can be compensated for. The procedure, in principle, maps the required cut-off frequencies in the \(\omega\) variable to that in the \(\Omega\) variable, designs an analog filter to match the new conditions and transforms it to a digital filter.

In summary, the bilinear transformation provides a simple mapping between continuous and digital filters which is algebraic in nature, and preserves the two most important requirements of stability and proper mapping of the imaginary axis. It introduces a frequency warping which can be compensated for in some situations. The method does not preserve the impulse response or the phase response of the analog filter.

4.15.4. Matched Z Transforms

This transformation directly maps the poles and zeros in the \(s\)-plane to poles and zeros in the \(z\)-plane. The mapping has the property that an \(s\)-plane pole or zero at \(s = -a\) is mapped to a \(z\)-plane pole or zero at \(z = e^{-aT}\) where \(T\) is the sampling period. Incorporating this relationship, if the frequency response of the analog systems is described as

\[
H_a(s) = \frac{\prod_{k=1}^{M} (s - a_k)}{\prod_{k=1}^{N} (s - b_k)}.
\] (4.47)

Then the matched \(Z\)-transform of \(H_a(s)\) is defined to be
\[ H(j\omega) = \frac{\prod_{k=1}^{M} (1 - e^{a_k T z^{-1}})}{\prod_{k=1}^{N} (1 - e^{b_k T z^{-1}})}. \quad (4.48) \]

An inspection of the Z-transform representation reveals that the denominator has the same form as the usual frequency response of a digital system with the corresponding poles. However, the numerator has a different form from that of the filter with zeros at \( a_k \).

The matched Z-transform method results in adequate digital filter design for relatively simple narrowband filters. However, the method is inadequate for designing wideband filters. The method does not preserve the characteristics of typical prototype analog filters such as equal ripple amplitude response or linear phase response.

4.16. Equivalence of Digital and Analog Filters

A thorough mathematical treatment of the equivalence of analog and digital filters is presented by Steiglitz [1965]. The issue has considerable importance because the theory of digital filters have drawn heavily from the more well-established field of analog filters. The equivalence is often used to obtain the optimum solution in the discrete case, by solving the optimization problem for the continuous case. Objective comparison of the two methods are provided by Parks and Burrus [1987]. The most important relation is that of the mapping of the \( i\omega \) axis in the \( s \)-plane on to the unit circle in the \( z \)-plane by the bilinear transformation

\[ s = \frac{z - 1}{z + 1} \quad (4.49) \]
\[ z = \frac{1 + s}{1 - s} \quad (4.50) \]

The equivalence of analog and digital versions of filter impulse response and the excitation for convolution purposes are given by the equations
\[ h_n = h(nT) = \int_{nT - \frac{T}{2}}^{nT + \frac{T}{2}} h(t) \, dt \]  
\[ w_n = w(nT) = \int_{nT - \frac{T}{2}}^{nT + \frac{T}{T}} w(t) \, dt . \]

Here \( w_n \) and \( w(t) \) represent the discrete and continuous version of the white noise process, and \( T = \pi/\omega_b \) is the sampling interval as before. The discrete Fourier Transform of a signal, \( X(e^{j\omega T}) \), is related to the Fourier transform in the analog case, \( X_a(\omega) \), by the relation

\[ X(e^{j\omega T}) = \frac{1}{T} \sum_{m=-\infty}^{\infty} X_a(\omega + \frac{2\pi}{T} m) \]  

If the analog frequency response is bandlimited to the range \( \omega_c \leq \pi/T \), i.e., \( X_a(\omega) = 0 \), \( |\omega| > \pi T \), then in the frequency range \( |\omega| \leq \pi/T \)

\[ X(e^{j\omega T}) = \frac{1}{T} X_a(\omega) \]  

In this case the digital frequency response is related in a straightforward manner to the analog frequency response. If the analog frequency response is not limited to these cutoff frequency bands, no such simple relations exist, and aliasing effects come into play.

**4.17. Summary**

The essentials of filter design and analysis are summarized in this chapter. Some very useful concepts like Paley-Wiener condition, Kramers-Kronig relationship, all pass filters, and minimum phase filters are discussed. The above concepts play a very important role for the simulation of wave kinematics. However, unlike the more common topics, these are not covered in most text books, and must be assimilated from various sources. It is hoped that the present chapter brings together the existing theory in terms of proper explanations and interpretations, in a language that can be followed by the ocean engineer.
CHAPTER V

Filter Design and Implementation

5.1. Overview

Some direct applications of the theory developed in the last chapter, is presented here. The applications relate directly to the problem of wave kinematics propagation, which is crucial to the successful use of digital filters for simulating wave kinematics over a spatial grid. This chapter includes a critical analysis of some techniques along with some original contributions to the field of wave kinematics simulation. The main emphasis is on the kinematics propagation, and forms a complement to the more thoroughly investigated problem of kinematic simulation at one point on the spatial grid.

Two entirely different formulations for the horizontal propagation are presented, one based on the M-A and the other based on the ARMA analog version algorithm. The ARMA formulation for this problem is an entirely new concept, which can improve simulation efficiency.

Similarly, the vertical propagation problem is investigated with the objective of developing both the M-A and the ARMA algorithms. A clear explanation is provided as to why the ARMA approach is difficult to apply to this problem. Thus, subsequent efforts have been directed to the development of the more familiar M-A algorithm.

Finally, some of the options available for the filter implementations are discussed. Options that have a clear advantage over the others are identified whenever possible. In situations where a clear advantage is not obvious, alternatives are discussed with the relative merits and demerits.
5.2. Introduction

The problem of wave kinematic simulation has received considerable attention over a period of time [Borgman, 1969, Shinozuka, 1979, Spanos, 1983, Samii and Vandiver, 1984]. The earlier contributions to the problem relied heavily on the method of superposition of harmonics. In this approach, the propagation problem is taken care of automatically, and does not pose any particular challenges. The introduction of filter approaches to the problem, in an attempt to improve efficiency, brought to surface this new problem of kinematic propagation. In this approach, the propagation problem has to be given special attention in order to ensure the proper spatial correlations.

This problem has been considered recently by Jefferys [1988], Spanos [1986], and Samii and Vandiver [1984]. The work of Samii concentrated on the special case of "deep water," with its associated simplifications. The problem of propagation in any arbitrary water depth has been investigated by Spanos [1986], and also discussed by Jefferys [1988]. The present investigation draws from all of the above and systematically investigates the different formulation possibilities. The investigation begins with the simplest case of propagation in "shallow waters," and subsequently proceeds to the more difficult problem of propagation in water of arbitrary depth.

5.3. Propagation of Kinematics in Shallow Water

Waves in shallow water tend to behave more and more differently from the idealized linear Airy waves, or their superposition. This is due to more pronounced nonlinear effects and a tendency for the wave-form to "break." The use of a shallow water wave theory like the cnoidal wave theory is more appropriate in this case. However, the present day state of art on the analysis of irregular seas is limited to linear waves only. Recognizing the limitations of the
assumptions, one usually has to make a choice between a deterministic non-linear analysis and a non-deterministic linear analysis. Frequently, the latter option is selected, in keeping with the probabilistic analysis of wave excitation. In these cases where the theory of linear Airy waves is applied to the propagation of irregular waves in shallow water, the approach outlined in this section can be adopted.

A brief discussion of phase and group velocity is appropriate in this connection. The phase velocity is the speed of phase propagation, defined as

\[ V_p = \frac{\omega}{\kappa} = \sqrt{\frac{g}{\kappa}} \tanh(\kappa d) \]  \hspace{1cm} (5.1)

Group velocity carries a clear meaning when there is a finite band of frequencies, and indicates the velocity at which energy is transferred. It is defined as

\[ V_g = \frac{d\omega}{d\kappa} = \frac{V_p}{2} \left[ 1 + \frac{2\kappa d}{\sinh(2\kappa d)} \right] \]  \hspace{1cm} (5.2)

It is to be noted that in deep water, based on equations (5.1-5.2) one obtains

\[ V_g = \frac{V_p}{2} = \frac{1}{2} \sqrt{\frac{g}{\kappa}} \]  \hspace{1cm} (5.3)

and in shallow water it reduces to

\[ V_g = V_p = \sqrt{gd} \]  \hspace{1cm} (5.4)

Thus, in shallow water waves of different frequencies travel at the same speed, which is a function of the water depth. The simplification of the dispersion equation in shallow water significantly reduces the complexity of the propagation problem.

5.3.1. Horizontal propagation

The generation of wave kinematics over a spatial grid requires the propagation of the kinematics in both vertical and horizontal directions. The horizontal propagation problem is
considered in this section and the vertical propagation problem in the next section. The transfer function for horizontal propagation can be expressed as

$$H_{ax} (\omega; \Delta x) = H_{ax} (\omega; \Delta x) = \exp (-jk\Delta x)$$

(5.5)

In the above equations, the first subscript refers to the kinematics of interest and the second subscript refers to the direction of propagation. Substituting equation (2.68) in equation (5.5), one can have alternative forms for the transfer function in shallow water. Specially,

$$H_x (\omega; \Delta x) = \exp \left( -j\frac{\omega \Delta x}{\sqrt{gd}} \right)$$

(5.6)

or

$$H_x (\omega; \Delta x) = \exp ( -j\omega \tau_0 )$$

(5.7)

where

$$\tau_0 = \frac{\Delta x}{\nu_p}$$

(5.8)

The inverse Fourier transform based on equation (4.3-4.4) yields the impulse response in the form of a Dirac delta function,

$$h_x (\tau) = \delta (\tau - \tau_0 )$$

(5.9)

Thus, the kinematics simulated at one location propagate to another point situated in the direction of propagation after a time lag $\tau_0$ with a delay, but without any attenuation. This is expected since according to linear theory, waves of different frequencies travel at the same speed in shallow water.

5.3.2. Vertical Propagation

The kinematics propagation in the vertical direction needs a slightly different treatment from the horizontal propagation problem. The transfer function for vertical attenuation in shallow water is different for the vertical and the horizontal component, and are dealt with separately.
5.3.2.1. Horizontal Component

Having simulated the horizontal component at the still water level \((z = d)\) the transfer function required to propagate the kinematics to a point at a distance \(z\) measured from the bottom is given by

\[
H_{\omega\omega}(\omega; z, d) = \frac{\cosh(\kappa z)}{\cosh(\kappa d)}
\]  
(5.10)

Considering a Taylor series expansion,

\[
\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots
\]
(5.11)

one obtains from eqn (5.10)

\[
H_{\omega\omega}(\omega; z, d) = \frac{1 + \kappa z^2/2 + \cdots}{1 + \kappa d^2/2 + \cdots}
\]
(5.12)

This expression is not too useful in bringing about any simplification. Retaining only the first order term makes the transfer function unity over the entire range of interest. Thus, according to first order approximation the horizontal velocity component in shallow water remains unattenuated with depth. Instead, one can write the transfer function between the elevation at the free surface and the horizontal velocity at location \(z\), as

\[
H_{\omega z}^H(\omega; z, d) = \omega \frac{\cosh(\kappa z)}{\sinh(\kappa d)}
\]
(5.13)

Using the Taylor series expansion

\[
\sinh(x) = x + \frac{x^3}{3!} + \cdots
\]

and retaining only first order terms, one obtains

\[
H_{\omega z}^H(\omega; z, d) = \frac{\omega d}{\kappa}
\]
(5.14)

The simplification assumes that for small \(x\), \(\cosh(x) \approx 1\), \(\sinh(x) \approx x\). Substituting the shallow water dispersion equation (2.68) to eliminate \(\omega/\kappa\) one obtains
\[ H_{\alpha}^{\omega} (\omega ; z, d) = \sqrt{\frac{\kappa}{d}} \] (5.15)

which is independent of frequency and wave number, and is a function of only the local water depth.

5.3.2.2. Vertical Component

The transfer function between the vertical velocity component at the free surface \( z = d \) and that at a vertical location \( z \), both measured from the seabed, is given by

\[ H_v(\omega; z, d) = \frac{\sinh (\kappa z)}{\sinh (\kappa d)} = \frac{z}{d} \] (5.16)

This transfer function is independent of frequency and is a linearly varying function of the vertical location.

The impulse response functions corresponding to the transfer functions given by eqns (5.15) and (5.16) are Dirac delta functions multiplied by the corresponding transfer function magnitudes. Thus, to make the propagation in shallow water most efficient, one should simulate the time histories of the elevation, the horizontal and vertical velocity components simultaneously and select the appropriate input for the different propagation problems.

5.4. Propagation of Kinematics in Intermediate water Depth

Propagation of kinematics in intermediate water depths involves the equations pertaining to the linearized waves in the most general form. Algorithms based on these expressions are more involved, but automatically take care of the deep and shallow water situation. Thus, the generality is obtained at the expense of increased complexity.
5.4.1. Vertical Propagation by Cascade of Oscillators

The following section explores an attractive model for the kinematics propagation, and analyzes the limitation of the method.

Based on approximations presented by Spanos [1986], the velocities at a fractional depth $\alpha$, measured from the bottom can be simulated using the time histories of the surface, and a convolution with the appropriate set of impulse response functions. The auto-spectrum of the velocity, after a change of variable and a suitable modification, is approximated by a cascade of filters each representing the squared modulus transfer function of an SDF oscillator. The impulse response is then obtained by taking an inverse Laplace transform of the transfer function.

5.4.1.1. Horizontal velocity component

The spectrum of horizontal velocity of particles at any location $z$ measured from the bottom is given by

$$S_{\omega u}(\omega) = \left[ \frac{\omega \cosh(\kappa z)}{\sinh(\kappa d)} \right]^2 S_{\eta \eta}(\omega)$$

(5.17)

By making the following substitutions to normalize the variables [Spanos, 1986],

$$b = \kappa d$$

(5.18)

$$\omega^2 \frac{d}{g} = \frac{\omega^2}{b} = b \tanh(b)$$

(5.19)

$$\alpha = \frac{z}{d}$$

(5.20)

one obtains

$$\frac{\tilde{S}_{\omega u}(\omega)}{\tilde{S}_{\omega u}(\omega)} = \frac{d}{g} \frac{S_{\omega u}(\omega)}{S_{\eta \eta}(\omega)} = b \tanh(b) \left[ \frac{\cosh(\alpha b)}{\sinh(b)} \right]^2$$

(5.21)

which can be approximated by

$$\tilde{S}_{\omega u}(\omega) = \tilde{S}_{\omega u}(\omega) = \frac{\omega_1^4 \omega_2^4}{\left[ (\omega^2 - \omega_1^2)^2 + (2\zeta_1 \omega_1 \omega)^2 \right] \left[ (\omega^2 - \omega_2^2)^2 + (2\zeta_2 \omega_2 \omega)^2 \right]}$$

(5.22)
where \( \omega_1, \omega_2, \zeta_1, \zeta_2 \) are coefficients dependent on \( \alpha \). The right-hand side of equation (5.22) is then decomposed

\[
\tilde{S}_u(\overline{\omega}) = H_{u_2}(i\overline{\omega}) H_{u_2}(-i\overline{\omega})
\]  

(5.23)

On replacing the complex variable \( i\overline{\omega} \) by \( s \), and taking inverse Laplace transform of \( H_{u_2}(s) \), one obtains the impulse response function

\[
h_{u_2}(t) = sgn(R_1) \frac{2\omega_1^2\omega_2^2}{\sqrt{R_1^2 + I_1^2}} e^{-\alpha\zeta_1^2} \cos(\omega_1\sqrt{1-\zeta_1^2}t + \phi_1)
\]

\[
+ sgn(R_2) \frac{2\omega_1^2\omega_2^2}{\sqrt{R_2^2 + I_2^2}} e^{-\alpha\zeta_2^2} \cos(\omega_2\sqrt{1-\zeta_2^2}t + \phi_2)
\]

(5.24)

where \( sgn(R_i) \) represents sign of expression for \( R_i \), and

\[
R_1 = 4\omega_1^2 \left[ (\omega_1\zeta_1 - \omega_2\zeta_2) (1 - \zeta_1^2) \right]
\]

(5.25a)

\[
I_1 = (2\omega_1\sqrt{1-\zeta_1^2}) \left[ (\omega_1^2 - \omega_2^2) - 2\zeta_1^2\omega_1^2 + 2\omega_1\zeta_1\zeta_2\omega_2 \right]
\]

(5.25b)

\[
\tan (\phi_1) = \frac{I_1}{R_1}; \quad -\pi/2 \leq \phi_1 \leq \pi/2
\]

(5.26)

\[
R_2 = 4\omega_2^2 \left[ (\omega_2\zeta_2 + \omega_1\zeta_1) (1 - \zeta_2^2) \right]
\]

(5.27)

\[
I_2 = (2\omega_2\sqrt{1-\zeta_2^2}) \left[ (\omega_2^2 - \omega_1^2) - 2\zeta_2^2\omega_2^2 + 2\omega_2\zeta_1\zeta_2\omega_2 \right]
\]

(5.28)

\[
\tan (\phi_2) = \frac{I_2}{R_2}; \quad -\pi/2 \leq \phi_2 \leq \pi/2
\]

(5.29)

Fig. 5.1 shows plots of \( h_{u_2}(t) \) against non-dimensional time \( t \), for values of \( \alpha = 0.20, 0.40, 0.60, 0.80, \) and \( 0.90, \) respectively. The convolution integral, assuming a causal system \( i.e. h(t) = 0 \) for \( t<0 \), is

\[
V(t) = \int_0^t h(\tau) x(t-\tau) d\tau
\]

(5.30)

where \( x(t) \) is the appropriate input for the convolution. Noting that beyond a certain value of \( t \) the absolute value of \( h(t) \) becomes negligible, the above integral can be replaced by a numerical
integration scheme through rectangular or trapezoidal rule.

\[ V(n\Delta t) = \sum_{k=1}^{j} h(k\Delta t) w(k) x(n-k\Delta t) \]  \hspace{1cm} (5.31)

where \( w(k) \) represent the appropriate weighting factors for the selected integration scheme, and \( h(t) = 0 \) for \( t > j(\Delta t) \). A suitable choice of cut-off value based on an acceptable level of error for the entire propagation range needs to be chosen. Note that this error depends not only on the propagation distance, but also on the time step of integration. Note that the spectrum used is the true one divided by \((g/d)\) and the frequency is a non-dimensional one. Thus, the impulse response so obtained needs to be multiplied by \( \sqrt{g/d} \), and the real time step is \( \sqrt{d/g} \) times the non-dimensional time step. The input to the convolution is the surface elevation.

5.4.1.2. Vertical Velocity component

Following exactly the same procedure, the spectrum to be approximated, after a suitable change of variables, is

\[ \tilde{S}_v(\bar{\omega}) = \frac{S_v(\omega)}{S_\eta(\omega)} \left[ \frac{d}{\alpha^2 g \bar{\omega}^2} \right] = \left[ \frac{\sinh(\alpha b)}{\alpha \sinh(b)} \right]^2 \]  \hspace{1cm} (5.32)

which is approximated as

\[ \tilde{S}_v(\bar{\omega}) \approx \tilde{S}_v(\bar{\omega}) = \frac{v^4}{\left( \bar{\omega}^2 - v^2 \right)^2 + (2\beta \bar{\omega}v)^2} \left[ 1 + (\bar{\omega}v)^2 \right] \]  \hspace{1cm} (5.33)

where \( v, \beta \) and \( c \) are functions of \( \alpha \) as before. The impulse response, corresponding to the transfer function of this spectrum can be shown to be

\[ h_v(t) = sgn(R) \frac{2v^2 e^{-v\sqrt{t}}}{\sqrt{R^2 + t^2}} \cos(v_d t + \phi) + \frac{v^2 c^2 e^{-t/c}}{1 - 2v \beta v + v^2 c^2} \]  \hspace{1cm} (5.34)

where

\[ v_d = v \sqrt{1 - \beta^2} \]  \hspace{1cm} (5.35)
\[ I = 2 \left( \sqrt{\beta c} - 1 \right) \]
\[ R = 2cv_d \]
\[ \phi = \tan^{-1} \left( \frac{I}{R} \right) ; \quad -\pi/2 \leq \phi \leq \pi/2 \]

Fig. 5.2 shows plots of \( h_{nl}(t) \), against non-dimensional time, for \( \alpha = 0.2, 0.4, 0.6, 0.8, 0.9 \). The impulse response is to be multiplied by \( \sqrt{g/d} \) and the time step is scaled by the factor \( \sqrt{d/g} \) as before. However the input to the convolution in this case is the vertical velocity at the still water level.

The cut-off value of \( h(t) \) and the time steps in the convolution are selected as before. It is easily shown that the integral of the impulse response function from \( t = 0 \) to \( \infty \) is the value of the Laplace transform evaluated at \( s = 0 \), and turns out to be unity in both cases, providing a simple check on the accuracy of numerical integration.

5.4.1.3. Limitations of Cascaded Oscillator Models

The approximation of the transfer functions of the vertical propagation of wave kinematics by those of cascaded S-D-F oscillator models [Spanos,1986] has some very desirable qualities. Firstly, representations in this form allows the evaluation of integrals required to obtain response statistics quite readily, due to the rational function form of the expressions. Secondly, the time domain representation of these transfer functions are differential equations, which can be used very efficiently for the purpose of numerical simulation. Also, the impulse response obtained by inverse Laplace transformation of the transfer function can be used in the simulation for convolution. However, the method suffers from a serious limitation. This representation is based on the approximation of the auto-spectrum of the process, and ignores the phase information associated with the transfer function. The phase response of the approximating transfer function is allowed to be arbitrary, governed by the characteristics of the oscillator model, which in turn is designed
purely from magnitude consideration. This introduces a phase lag between the input or excitation, and the output or response, over which the design does not have any control. The phase response of this approximate transfer function is

\[ \phi (\omega) = \tan^{-1} \left[ \frac{IM \{ H(\omega) \}}{RE \{ H(\omega) \}} \right] \]  

(5.39)

This situation, though undesirable, is acceptable as far as the response statistics of an SDF model is sought. The response spectrum of the output is unaffected by the introduction of the phase, though the cross-spectrum of input-output signals is affected.

Unfortunately, when a Multi-Degree-Freedom system is being analyzed, the approximation based purely on the auto-spectrum is no longer adequate. The autospectral terms form only the diagonal terms of the spectral matrix. The approximation of the auto-spectral terms alone, can generate time records with the desired auto-spectrum for each component of the vector random process. However, the simulated vector process would fail, in general, to produce the correct cross-spectral terms, which may be just as important for fully or partially correlated components of the vector process. In this situation the approximation has to consider the complete spectral matrix.

The general approach to this problem is to follow the theory of multivariate simulation. The problem is that of designing filters for a multi-input-multi-output (MIMO) system. The problem of filter design in this case becomes extremely complex as numerical coefficients in the univariate case have to be replaced by matrices in the present situation. Obtaining these matrices describing the digital filter is an arduous task [Spanos and Minolet, 1987, Samaras and Shinozuka, 1985], no matter what approach is adopted.

The procedure is different for the MA and the AR method. In the MA scheme the first step
in the design is a factorization or decomposition of the spectral matrix of the vector random process, to obtain the transfer function matrix. This is popularly known as the Spectral Factorization problem, discussed earlier. A triangular or Cholesky decomposition is suggested by Borgman [1969]. The AR method does not need this decomposition, but requires manipulation of systems of matricial equations [Samaras and Shinozuka, 1985, Spanos and Minolet, 1987].

However, the special nature of the random gravity wave model, allows a relatively simple approach that avoids analysis of the multivariate problem. Thus, the general procedure is not the best approach to the problem of wave kinematics simulation at different space locations. This is possible because of the deterministic model, implying complete coherency, for the vertical propagation of the random wave kinematics. The components of the vector random process, that is the horizontal velocity or the vertical velocity at different space locations, are not independent. They are related by the deterministic relations of linear wave theory. This, in turn, makes the spectral matrix of this vector random process singular. The singularity arising from the linear dependence between rows, or columns, of the spectral matrix. Consequently, the Cholesky decomposition of this spectral matrix yields a transfer matrix with zeros everywhere except one column, as opposed to a triangular non-zero part of the general multivariate problem. As a result, all components of the output vector are generated by a single component of the input vector. Usually, the kinematics at the surface is chosen as the input. This explains how the problem degenerates into a single-input-multi-output (SIMO) problem, even when formulated as an MIMO problem, due to the deterministic nature of the spatial propagation problem. It should not come as a surprise, as the kinematics at the different vertical locations result from the same generating mechanism, the waves at the surface. If this was not the case, the different components of the vector would have been generated by the combined action of several independent white noise processes, which is the
case for the general MIMO system.

Specifically, consider the simulation of the wave kinematics at different locations along a vertical line. This simulation problem can be looked upon as an SIMO problem. The single input is either the kinematics at the surface or a white noise series depending on the choice of transfer function, vertical propagation of simulated kinematics or direct simulation of kinematics at different locations. The approximation of auto-spectral terms generates time histories at the different points, which have a frequency-dependent phase-shift between the spatially separated points. The transfer function for the vertical propagation of the horizontal velocity component is approximated as

$$
\hat{H}_v (\bar{\omega}) = \frac{\omega_1^2 \omega_2^2}{(\omega_1^2 - \bar{\omega}^2) + j(2\zeta_1 \omega_1 \bar{\omega})} \left(\omega_2^2 - \bar{\omega}^2\right) + j(2\zeta_2 \omega_2 \bar{\omega})
$$

(5.40)

where \(\omega_1, \omega_2, \zeta_1, \zeta_2\) are coefficients dependent on the fractional depth, \(\alpha\). The vertical velocity component is approximated as

$$
\hat{H}_v (\bar{\omega}) = \frac{\nu^2}{\left(\bar{\omega}^2 - \nu^2\right) + j2\beta \nu \bar{\omega}} \left[1 + j \frac{\omega}{c}\bar{\omega}\right]
$$

(5.41)

where \(\nu, \beta\) and \(c\) are functions of \(\alpha\) as before [Spanos, 1986]. Based on these relationships the phase response of the filters are given by eqns (5.42) and (5.43).

$$
\theta_v (\omega) = \text{Arg} \left[ \hat{H}_v (\bar{\omega}) \right] = \tan^{-1} \left[ \frac{2(\zeta_1 \omega_1 + \zeta_2 \omega_2) \omega^3 - 2\omega_1 \omega_2 (\zeta_1 \omega_2 + \zeta_2 \omega_1) \bar{\omega}}{\omega^4 - \omega_1^2 \omega_2^2 + 4\zeta_1 \zeta_2 \omega_1 \omega_2 \omega^2 + \omega_1^2 \omega_2^2} \right]
$$

(5.42)

$$
\theta_v (\bar{\omega}) = \text{Arg} \left[ \hat{H}_v (\bar{\omega}) \right] = \tan^{-1} \left[ \frac{c \omega^3 - (2\beta \nu + c \nu^2) \bar{\omega}}{-2\beta \nu \nu + 1) \omega^2 + \nu^2} \right]
$$

(5.43)

Equations (5.42) and (5.43) display a phase response which is a non-linear function of the frequency. However, the vertical propagation of wave kinematics requires a pure attenuation, mag-
magnitude decay only, with zero phase-shift. The corresponding target transfer functions in both cases are purely real functions given by eqns (5.44) and (5.45)

\[ H_{\omega} (\omega ; \alpha) = \frac{\cosh (\alpha \omega d)}{\cosh (\alpha d)} \quad \text{(5.44)} \]

\[ H_{\alpha} (\omega ; \alpha) = \frac{\sinh (\alpha \omega d)}{\sinh (\alpha d)} \quad \text{(5.45)} \]

Note that in the above equations for the transfer function of vertical propagation the input and output are the kinematics at the SWL and that at the required location, respectively. The phase-shifts given by eqns. (5.42) and (5.43) would still be acceptable if they were same for each and every location, in which case the phase lag between the kinematics at different points would again be zero.

This is difficult to achieve in practice. What is possible, on the other hand, is to design filters that approximate the appropriate transfer functions, instead of the modulus square of the transfer functions. For a SIMO problem, the transfer function matrix degenerates into a vector. The individual matching of the components of the transfer function vector, independently, is sufficient for the matching of the auto as well as cross-spectral terms of the spectral matrix. However, the special nature of the transfer function for the vertical propagation problem, pure attenuation without phase lag, requires some investigation, in terms of the realizability of this transfer function by the assumed oscillator model.

The oscillator model is an IIR filter. In order to use such a filter for simulation purpose, one needs to impose the condition of causality, which ensures that the response depends only on the past values of the input, and not on the future values. A necessary condition for the realizability of a transfer function by a a causal filter is that the real and imaginary parts of the transfer function must form a Hilbert Transform pair [Zadeh and Desoer, 1963]. This property is also known
as the Kramers Kronig relationships, eqns. 4.20-4.21.

The transfer functions for the vertical propagation of kinematics, eqns. 5.44-5.45, have only real parts and are pure attenuators. The imaginary parts being zero, these transfer functions fail to satisfy the relations specified by eqns. 4.20-4.21. Thus, such transfer functions cannot be realized by the use of causal filters. A non-causal, symmetric, FIR filter is the only design option allowed in such a case. Thus the use of oscillator models is not permissible, if simultaneous simulation at more than one location is desired.

The natural question that follows is what is the penalty of using this model of cascaded oscillators. To this effect, the phase response of the approximate transfer function, based on the oscillator model, is evaluated through eqns. 5.42-5.43. These phase spectra for different vertical locations are plotted in Figs 5.3 and 5.4, for the vertical and the horizontal component, respectively, based on the approximations suggested by Spanos [1986]. Examination of these plots reveal that the phase lag between the kinematics near the surface and that close to the sea-bed can be as large as 180 degrees or more. In physical terms, this implies that the simulated particle motions at two vertical locations can be completely out of phase. For a vertical element of any offshore structure, this could result in underprediction of the force due to the cancellation of forces through the opposing effects at different locations.

5.4.1.4. Symmetric MA Filter For Vertical Propagation

The special nature of the transfer functions for vertical propagation require that only an FIR filter be used for the approximation. Furthermore, the zero-phase nature of the transfer function requires the use of a symmetric, non-causal filter. Design of an MA filter satisfying the above properties is considered in this section. The filter can be designed only for a specific case,
corresponding to given values of wind velocity, water depth and propagation distance. The design is based on the procedure reported by Borgman [1969]. The coefficients of the two sided moving average (MA) filter are obtained through equations (5.46-5.50)

\[ A_n = \frac{1}{\omega_c} \int_0^{\omega_c} \text{Re} \left[ H_\omega (\omega ; \Delta x ) \right] \cos\left( \frac{n \pi \omega}{\omega_c} \right) d\omega \] (5.46)

\[ B_n = \frac{1}{\omega_c} \int_0^{\omega_c} \text{Im} \left[ H_\omega (\omega ; \Delta x ) \right] \sin\left( \frac{n \pi \omega}{\omega_c} \right) d\omega \] (5.47)

\[ a_0 = A_0 \] (5.48)

\[ a_n = A_n + B_n \] (5.49)

\[ a_{-n} = A_n - B_n \] (5.50)

And the simulation scheme is given by

\[ y_k = \sum_{n=-N}^{N} a_n x_{k-n} ; \quad k = N + 1, N + 2, \ldots \] (5.51)

where subscript \( n \) stands for the \( n \)-th time step, \( t = n \Delta t \), and the time step \( \Delta t \) is related to the cut-off frequency, \( \omega_c \), through the Nyquist relation

\[ \Delta t = \frac{\pi}{\omega_c} \] (5.52)

It is observed that when \( N \) represents the number of terms in the Fourier expansion of the transfer function, the total order of the non-causal M-A filter is \( 2N+1 \). The filter coefficients for the causal and the non-causal parts are symmetric in the present case, as the imaginary part of the transfer function is zero. Relatively few coefficients are required to approximate the transfer function, as it is a bell-shaped curve. The approximations for the different cases are shown in Figs 5.5(a)-(c).

Note that the purely real transfer function results in a symmetric filter, \( B_n = 0 \), and the phase response is zero over the entire frequency range.
5.5. Horizontal Propagation in Intermediate Water Depth

The transfer function for horizontal propagation is the same for any kinematics of interest and is given by

\[ H_x(\omega; \Delta x) = \exp(-j\kappa \Delta x) \] (5.53)

Introducing the new variable

\[ \mu = \frac{\Delta x}{d} \] (5.54)

and substituting eqns (5.18) and eqn. (5.54) simultaneously into eqn. (5.53), one obtains

\[ H_x(\omega; \Delta x) = \exp(-j\mu b) = \cos(\mu b) - j\sin(\mu b) \] (5.55)

As the transfer function is an explicit function of wave number, a change of variable is necessary to express it as a function of the frequency. A convenient approach would be to work with non-dimensional frequency. The real time step is \(\sqrt{d/g}\) times the non-dimensional time step.

Inspection of equations (5.46-5.52) reveal that the M-A coefficients depend directly on the horizontal propagation distance and the wave number. The change of variable from wave number to frequency involves the water depth. Also, the cut-off frequency for the P-M spectrum is determined by the wind velocity, \(U\). Thus the M-A coefficients for the horizontal propagation are a function of the propagation distance, the local water depth and the wind velocity. Fig. 5.6 represents a plot of the absolute value of the transfer function and its real and imaginary parts based on the approximation by two-sided MA filter. For this specific case the values of the parameters are \(\mu = 0.20\), \(d = 150\) meters and \(U = 50\) knots, and the filter order is 31.

Analysis of fluid-structure interaction or similar problems would require the kinematics at different horizontal locations. This requires that the filter coefficients be evaluated for any arbitrary location within a specified region. This can be achieved by direct evaluation at locations of
interest or at closely-spaced points over the region, and interpolating in between. Thus, an attractive alternative would be to have the M-A filter coefficients expressed as an analytic function of \( \mu \). This idea was explored in the present study and a procedure for implementing the same has been developed.

### 5.5.1. Horizontal Propagation Through M-A Filters

The real and imaginary part of the transfer function as given by equation (5.54), are expressed as Fourier series expansion of non-dimensional wave number, \( b \) [Spiegel, 1968]. The coefficients of the series are functions of \( \mu \). Making a change of variable

\[
\mu b = \hat{\mu} \hat{b}
\]

such that \(-\pi < b < \pi\), where

\[
b = \frac{\pi b}{b_{\text{max}}}
\]

and

\[
\hat{\mu} = \frac{\mu b_{\text{max}}}{\pi}
\]

one obtains, for \( \hat{\mu} \neq \text{integer} \)

\[
\cos(\hat{\mu} \hat{b}) = \frac{\sin(\hat{\mu} \pi)}{\pi} \left[ \frac{1}{\mu} + 2 \hat{\mu} \sum_{n=1}^{NF} (-1)^n \frac{\cos(n \hat{b})}{\mu^2 - n^2} \right]
\]

\[
\sin(\hat{\mu} \hat{b}) = \frac{2 \sin(\hat{\mu} \pi)}{\pi} \sum_{n=1}^{NF} (-1)^n \frac{n \sin(n \hat{b})}{\hat{\mu}^2 - n^2}
\]

The moving average filter coefficients are obtained through the use of equations (5.46-5.52) where the real and imaginary parts of the transfer function are replaced by the series representations of equations (5.59-5.60). It needs to be mentioned that the integrations required by equations (5.46-5.47) are evaluated numerically. Thus, there is one numerical integration for each of
the $NF$ terms of the series. With $NM$ representing the order of the M-A filter, the total number of numerical integration is of the order of $(2NF+1)(2NM+1)$, as compared to $(2NM+1)$ for one fixed location. In addition, the series expansion involves some algebraic operations to obtain the M-A coefficients. In the special case where $\hat{\mu}^2$ can be neglected in comparison to $n^2$ in equations (5.59-5.60), the algebraic computations are significantly reduced. Fig. 5.7 shows the approximation of the transfer function for the case where the M-A coefficients are obtained through the series expansion. Figure 5.8 displays the approximation of the phase spectrum. Figure 5.9 represents the comparison of the Fourier series expansions of the sine and cosine terms (equations 5.59-5.60) neglecting $\hat{\mu}^2$ beyond the fifth term of the series, for $\hat{\mu} = 1.50$.

5.5.2. Horizontal propagation with All-Pass Filters

The all-pass (all pass) filters provide a very attractive alternative for the horizontal propagation problem. A procedure for this purpose is developed in this section. The all pass filters have unit magnitude for all frequencies, and a phase that is a non-linear function of the frequency. Thus, they introduce dispersion without magnitude gain. This is exactly what is required for the horizontal propagation of wave kinematics. These filters are investigated here, and some design procedures developed.

The all pass filters have the poles and zeros symmetric with respect to the imaginary axis. Advantage can be taken of this symmetry, in computing the phase. Although the transfer function is a ratio of polynomials, the phase response can be computed from either the numerator or the denominator polynomial.

$$H_{ap}(s) = \frac{(s + \bar{p}_1)(s + \bar{p}_2) \cdots (s + \bar{p}_n)}{(s - p_1)(s - p_2) \cdots (s - p_n)}$$ (5.61)

Where the bar denotes complex conjugate. To ensure stability, $\text{Re } p_j < 0$ These filters serve to
bring about a frequency-dependent phase shift, and are also known as phase or delay equalizers.

As opposed to the usual practice of approximating the magnitudes of the transfer function, the design needs to match the target phase. The phase response of an all pass filter is given by

\[
\tan \left[ \theta(\omega) \right] = 2 \left[ \frac{\text{IM} \left[ D(\omega) \right]}{\text{RE} \left[ D(\omega) \right]} \right]
\]

(5.62)

where \( D(\omega) \) is the complex denominator polynomial. This design problem not covered by the common texts in signal processing and filter design, and well-developed algorithms are not readily available.

The problem is that of the phase approximation, as opposed to the more common problem of magnitude approximation. The optimization problem that needs to be solved is highly non-linear. The design procedure proposed in this section is formulated by the author, and the design problem is solved by using a non-linear optimization routine based on Levenberg-Marquardt algorithm [Levenberg,1944, Marquardt,1963]. The transfer function of the all pass filter is cast in the form of the ratio of transfer functions of simple harmonic oscillators. Using the conventional expression for the SDF oscillator, the transfer function of a second order all pass filter is

\[
H(\omega)=\frac{\left(\omega_0^2 - \omega^2\right) - j \left(2\zeta \omega_0\right) \omega}{\left(\omega_0^2 - \omega^2\right) + j \left(2\zeta \omega_0\right) \omega}
\]

(5.63)

where \( \omega_0, \zeta \) and \( \omega \) are the natural frequency, the critical damping ratio and the independent frequency variable, respectively. The phase of this second order all pass system can be expressed as

\[
\theta(\omega) = -2\tan^{-1} \left[ \frac{\left(2\zeta \omega_0\right) \omega}{\omega_0^2 - \omega^2} \right]
\]

(5.64)

The design variables are the critical damping ratio and the system natural frequency. The phase response for the filter is evaluated over the range \([0, 2\pi]\) and the filter coefficients chosen to make this phase as close as possible to the target phase. The target phase, based on the transfer func-
tion for the horizontal propagation is

$$\theta_\varepsilon(\omega) = -\kappa(\omega) \Delta x$$  \hspace{1cm} (5.65)

where $\kappa(\omega)$ is the frequency dependent wave number, and $\Delta x$ is the horizontal propagation distance. Thus, the minimization problem is to reduce the error $\epsilon$ in a least square sense at a finite number of points over the interval $[0, \omega_c]$.

$$\epsilon = \kappa(\omega) \Delta x - 2 \tan^{-1} \left( \frac{2 \zeta \omega_0}{\omega_0^2 - \omega^2} \right)$$  \hspace{1cm} (5.66)

The magnitude response is always unity by the special structure of these filters. Certain comments on the success of the approximation and the performance of the filters are appropriate here.

Firstly, the distance over which the filter can successfully propagate the kinematics depends on the filter order. A simple harmonic oscillator can have a maximum phase lag of $\pi$. A second order all pass filter can have a maximum phase lag of twice that of the simple oscillator, i.e., $2\pi$. Contributions to the phase shifts come from both the denominator and the numerator polynomial.

The maximum phase lag that can be achieved with a chosen model puts a restriction on the maximum distance through which the kinematics can be propagated. The phase of the target transfer function is given by the product of the wave number and the propagation distance, $\kappa \Delta x$. The cutoff frequency is governed by the wind velocity. For a given water depth, the maximum value of wave number is indirectly governed by the wind velocity. Thus, for a given wind velocity, the allowable maximum propagation distance is restricted. The product $\kappa \Delta x$ must lie within the maximum phase delay of the filter.

The above relation specifies the maximum allowable propagation distance, but does not guarantee that the filter performance would be satisfactory over this range. A cursory investigation reveals that the all pass filters have the same shape as the target transfer function only over
the frequency range from zero to the "natural frequency" of the all pass system, Figures 5.10-12. Beyond this region the two phase curves diverge from one another, for all values of the filter parameters. Thus, only this portion of the all pass filter can be effectively utilized for simulation.

This information is incorporated into the design procedure by introducing a coefficient, \( \omega_c \), which ensures that \( \omega_c \) corresponds to approximately the natural frequency of the all pass system. In this manner, the part of the all pass filter phase response, which diverges from the target response is eliminated from the design. Varying this fraction indicates that the approximation becomes poorer as the fraction approaches unity.

The approximation error is minimized in a least square sense at discrete points on the frequency interval. The sum of the square of the errors at these discrete points is the objective function that is optimized. As this optimization routine gives a local minimum for the vector, different initial guesses were tried. This was an attempt to avoid the algorithm from locking on to a local minimum close to the initial guess. The solutions were found to be relatively insensitive to initial guesses in most cases. Only positive values for the parameters were accepted, to rule out instability. Figures 5.10-12 represent the phase matching for different conditions of wind velocity and propagation distances.

Some repeated runs were also made for the same conditions, varying the propagation distance within the maximum permissible value. The filter coefficients show a smooth variation with propagation distance as indicated by Figs. 5.13(a)-(b). This variation can be readily expressed as polynomial equations, and incorporated in simulation algorithms.

The maximum allowable propagation distance exhibits a direct relation with the wind velocity. For small wind velocity, the cutoff frequency is relatively large, and the propagation distance has to be small, to keep the product constant, as dictated by the choice of the filter order.
To overcome this problem for smaller wind velocities, a fourth order all pass filter was also tried. The fourth order all pass filter can propagate the kinematics to twice that of the second order system. The properties are quite similar to the second order system, with comparable accuracy. This is shown in Figure 5.14.

5.6. Relation Between Vertical and Horizontal Components

An inspection of the expressions for the horizontal and vertical components of velocity based on deterministic theory (eqns. 2.21-2.22), reveals that there is a phase lag between the two components

\[ u = \frac{H}{2} \frac{\cosh(kz)}{\sinh(kz)} \frac{\cos(\omega x - \omega t)}{\sinh(kz)} \]  \hspace{1cm} (5.67)

\[ v = \frac{H}{2} \frac{\sinh(kz)}{\sinh(kz)} \frac{\sin(\omega x - \omega t)}{\sinh(kz)} \]  \hspace{1cm} (5.68)

The simultaneous simulation of the two components for random seas as developed to this point generate records which represent the proper auto-spectra but not the cross-spectral relations. Thus, the records preserve the proper magnitudes, but not the phase relationship between them. One way of maintaining the proper phase would be to obtain one component from the other through filtering.

As the vertical velocity at the free surface is independent of water depth, it is of interest to simulate the vertical velocity based on the P-M spectrum, independent of water depth, and to obtain the horizontal component for a specific water depth. However, numerical problems arise from the behavior of this transfer function

\[ |H(\omega)| = \left| \frac{\cosh(\omega d)}{\sinh(\omega d)} \right| \]  \hspace{1cm} (5.69)

The source of this numerical difficulty arises from the singular behavior of the transfer function at
\( \kappa = 0 \). Instead, if the horizontal component is simulated first then the transfer function to obtain the vertical velocity is given by

\[
H(\omega) = \tanh(\kappa \omega) e^{-i \pi^2} \tag{5.70}
\]

using equations (5.46) through (5.52) the two-sided M-A coefficients are obtained. The transfer function corresponding to these coefficients is compared to that given by equation (5.69) in Fig. 5.14. There is a considerable level of fluctuation which does not necessarily die out with increasing number of terms. This problem, associated with Gibb's phenomenon, is to some degree suppressed by use of windowing. Figure 5.16 shows the same transfer function based on coefficients that are modified by using Lanczos smoothing factors, which in principle averages out the ripples. Thus, following this procedure, one is capable of simulating records of vertical and horizontal velocity components simultaneously, maintaining the proper cross-spectral relations.

5.7. Filter Implementation And Related Issues

The implementation of these filters can be done in one of several ways. Firstly, they can be used in the differential equation form, with numerical integration. This will be appropriate when one is interested in the kinematics at the instantaneous position of a moving structure, acted upon by hydrodynamic loads. The time steps need to be consistent for the two differential equations; that describing the structural response, and that specifying the kinematics propagation.

The filter can also be implemented by obtaining the equivalent digital filter or computing the associated impulse response function and carrying out a convolution.

5.7.1. Alternate Choice of Simulation and Propagation Schemes
The simulation and propagation of wave kinematics with complete coherency between
kinematics at different points is based on an one-point simulation and subsequent propagation to
different spatial points based on frequency dependent transfer functions provided by the linear
Airy wave theory. There are several options for the choice of the kinematics to be simulated ini-
tially. This could be the surface elevation, a component of the velocity at the still water level
(SWL), or a component of the acceleration at the SWL.

The simulation of surface elevation would imply successive differentiation to obtain the
velocity and acceleration. Two successive levels of differentiation on randomly fluctuating data,
should be avoided if possible. The other alternative is to simulate a component of the acceleration
at the SWL. There is a serious problem associated with the acceleration spectrum based on
the P-M formulation, as it decays as $\omega^{-1}$ and there is infinite area under this spectral curve. The
only way to ensure that the area is finite is to truncate the spectrum at some frequency, $\omega_c$ and
impose the condition that $S(\omega) = 0$, for $\omega > 0$. Inspection reveals that $\omega_c$ has to be large in order
to ensure that the errors introduced by the above approximation is acceptable. Thus, the choice of
a velocity component for the initial simulation seems to be a good compromise.

To implement the propagation scheme to the n-th discrete point starting from the SWL
there are two options. First, one can propagate the kinematics with respect to the previous point.
Second, one can propagate the kinematics with respect to the SWL for every point of interest.
The former approach results in a loss of generated data at each level of convolution. In addition,
the approximation errors multiply at each stage. For direct propagation to the point of interest the
transfer function may have a relatively sharper transition. The latter option is chosen for this
investigation.
When both the vertical and horizontal components of the kinematics are of interest, there is another decision to be made. Either they can be simulated independently, maintaining the phase relationship between them, and propagated simultaneously. Only one kinematic can be simulated and the other obtained by filtering this simulated kinematics. In the latter case, either the vertical or the horizontal component has to be chosen for the initial simulation. The choice of the vertical component is more attractive as this is independent of the water depth, and can be made independent of the wind velocity by suitable transformation. However, the singular nature of the transfer function for obtaining the horizontal component from the vertical one rules out this choice. The other option is to simulate the horizontal velocity component first, and to obtain the vertical one by subsequent filtering.

5.7.2. Non-Dimensionalization

The filters for the surface elevation or the vertical velocity component at the surface is independent of the local water depth, and is a function of the wind velocity only, for the one-parameter family of wave spectra. This makes it possible to express the spectrum in a non-dimensional form [Chakrabarti, 1987]. The maximum value of the spectral magnitude is unity and the non-dimensional frequency is the actual frequency divided by the peak frequency. The peak frequency is an explicit function of the wind velocity. The ARMA coefficients obtained by Spanos [1983] and Spanos and Minolet [1986] are for these non-dimensional elevation and vertical velocity spectra. The horizontal velocity spectrum for intermediate water depth does not allow a similar non-dimensionalization. The spectrum in this case is a function of both the wind velocity and the water depth. The variation with wind velocity and that with water depth was discussed in chapter 2 (see Fig. 2.1(a)-(b)). It is not possible to express this spectrum in a non-dimensional form, and one has to design filters for particular combinations of water depth and
wind velocity. This makes the implementation somewhat involved as the filters need to be
designed for each problem of interest. However, the proposed design procedure using the Mov-
ing Average scheme, is conceptually very simple, and the coefficients can be evaluated quite
efficiently.

5.8. Conclusion

Some useful solutions to the problem of wave kinematics propagation are developed. The
general case of kinematics propagation in water of arbitrary depth, is addressed. In addition,
simplifications associated with "deep" and "shallow" water waves are also considered. An
entirely new concept of using all pass filters for wave kinematics propagation is developed and
found to be extremely attractive. A thorough investigation is carried out to identify the suitability
of the different filter types for the wave kinematics propagation problem. Some new interpreta-
tions are provided and it is shown why some techniques are inapplicable for the present problem.

Filters are obtained for the vertical and horizontal propagation of wave kinematics simu-
lated at a point on the SWL. The investigation reveals that the complex problem of kinematics
propagation cannot be efficiently handled by the use of one particular type of filter. A synthesis
of the different filter kinds is required. Thus, the vertical propagation is best achieved by an M-A
filter, whereas the horizontal propagation is efficiently done through the All-Pass (all pass)
ARMA type of filter. An alternate formulation for the horizontal propagation, using the simpler
Moving Average (M-A) formulation is also obtained. In this connection, it is shown why the
vertical propagation problem cannot be satisfactorily achieved by the use of non-causal filters.

The importance of the phase information, frequently ignored in the univariate simulation
procedures, is emphasized for the kinematics propagation propagation problem. The procedure
for all pass filter design, which focuses on phase approximation, is developed. This filter is proposed for the horizontal propagation problem. The nonlinear nature of the optimization problem in this case is pointed out. Also, there is a fundamental limitation on the horizontal propagation distance. The parameters that govern this distance are identified and the associated expressions developed.

The more conventional M-A filter is recommended for the vertical propagation problem. The M-A filters are conceptually simple but suffer from the Gibbs phenomenon, which causes undesirable ripples in the transfer function obtained through the filter. It is noted that using Lanczos smoothing window reduces the ripples considerably in some cases. Unfortunately, this success in suppressing the ripples is not universal. In some situations it is remarkable while in other situations it does not exhibit the same efficiency. It needs to be pointed out that the smoothing, though desirable in some respects, destroys the optimality condition of the approximation errors on which the M-A algorithms are based. The error, integrated over the existing range of frequencies, is no longer minimum. However, the undesirable oscillations around the cut-off frequencies are eliminated.

Attempts were made to obtain filter coefficients which have minimum dependence on the parameters such as wind velocity and local water depth. Unfortunately, this is not possible in general. The filter coefficients depend on the wind velocity and local water depth in general. This is due to the fact that the wind velocity determines the cut-off frequency and the water depth relates the wave number to frequency through the dispersion equation.

For simulation of wave kinematics, the state-of-art in the industry is to use harmonic superposition with random phase. The present work focuses on the development of the filter approaches to wave simulation. However, a great majority of the published literature on the topic
concentrates on the surface elevation alone. The present work extends the boundaries by concentrating on the equally important problem of kinematics propagation. The solutions presented are likely to have some direct impact. Furthermore, it is expected to draw attention to this area which has not been explored as thoroughly.
CHAPTER VI
Theory of State Space Analysis of Dynamic Systems

6.1. Overview

The present chapter introduces some of the important concepts of state space analysis, considered necessary for a proper understanding of the literature in the field. Many of these ideas are unique to the field of system identification and control theory, and are not common knowledge to the ocean engineer or the structural dynamicist. It is attempted to provide a good overview of the topics that would be used directly, or indirectly, in the subsequent application of the method. The central ideas of Controllability and Observability, Minimal Realization through Canonical representation, and the relation between the time and frequency domain are some of the topics covered. This is followed by a brief introduction to the theory of stochastic processes. The commonly encountered random processes such as the Normal, Markov and purely random processes are discussed. The idea of the white noise as a limiting case of the Gauss-Markov process is emphasized. The idea of Markovianization of non-Markovian processes is outlined. Finally, the advantages of the state space analysis are summarized.

6.2. Introduction

The application of the State-Space method for the analysis of random vibration problems is not as common as the frequency domain method. The reason is mainly historical. The frequency domain, or spectral density method, is based on the works of Wiener's filter theory in the frequency domain, developed around 1950. The further development of filter theory in the time domain by Kalman since 1960 found little application in the study of vibrations. Only very recently some break-through in the use of the covariance method to random vibration problems
has been made. The trends in the other engineering fields indicate that there is likely to be a steady increase in the use of this relatively new method.

A dramatic swing in the pendulum from frequency domain to time domain occurred in the general field of electrical and electronics engineering, and particularly in the area of control theory [Fossard, 1977]. As a result of the major developments based on the works of Kalman [1960] the frequency domain methods have almost completely been replaced by the state-space methods for control problems. There are several reasons for this change in approach. Firstly, the state-space methods proved more versatile to solve a wider range of problems, including multivariate problems. Secondly, the new approach gave an increased understanding of some of the more complex problems in control, primarily through a better insight into "controllability" and "observability". Finally, this method proved better suited for computer applications, a factor which did not play any significant role at the stage of the developments of the frequency domain method.

6.3. State-Space Applications in Random Vibration

In the field of random vibration the state-space method has proved more powerful in handling non-stationary problems [Deb-Chaudhury, 1988]. In addition, the method is reported to be significantly more efficient computationally [Schiehlen, 1984, 1985, Gossman and Waller, 1985]. One of the earliest applications of the method in random vibration is due to Lin [1967]. Subsequently, the method has found some application in most areas of random vibration. Very recently, there has been a series of papers on the application of the method for earthquake response of buildings. Gasparini and Deb-Chaudhury have been pioneers in this area. Their work includes non-stationary, multivariate, colored excitation in the context of structural response to earthquake excitation.
In the field of aerospace engineering, it has seen some application through the random vibration analysis of the helicopter rotor blades [Lakshikant and Wan, 1977].

In the field of wind engineering there have been some recent publications reporting the successful application of this method [Gossman and Waller, 1984, Muscolini, 1983].

In the field of random vibration analysis of vehicles moving on irregular surfaces, there have been a number of papers published recently [Schiehlen, 1983, 1984, 1987].

To the best of the author's knowledge, the method has so far not been applied to the random vibration analysis of offshore structures. However, there has been some indirect use of the state-space method through Kalman filters for the purpose of prediction rather than random vibration analysis. Some of the recent publications on the topic are by Yumori [1981], Triantafyllou and Bodson [1982], Triantafyllou, Bodson and Athans [1983], Jefferys and Samra [1985], and Lin [1987]. Kalman Filters are being used to drive the controller for the dynamic positioning system for floating drilling platforms. Kalman filters have also been applied to the motion prediction of ships based on instantaneous measurements on board, particularly in connection with landing of aircrafts on board the carrier ship.

The use of the state-space method to random vibration, with its numerous advantages, is expected to grow steadily as the method becomes more well-known. Also, with an increasing interest in the active control of dynamic systems, the method is likely to be a natural choice whenever control is of interest.

6.4. Concept of State Variables

The state of a system is defined as the smallest set of numbers that must be known in order that its future response to any given input can be calculated from the dynamic equations. Thus,
the state is a very compact representation of the history of the system, which can be utilized for the prediction of its future behavior in response to any external stimulus. Since the complete solution of a differential equation of order \( n \) requires precisely \( n \) initial conditions, it follows that the state of such a system will be specified by the values of \( n \) variables, called the state variables. Thus, a second order oscillator with question of the motion

\[
m\ddot{x} + c\dot{x} + kx = F(t),
\]

is represented as a vector equation in the state vector \( \mathbf{x} = [x, \dot{x}]^T \) in the form

\[
\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{u},
\]

where \( A \) is the transition matrix, and \( \mathbf{u} \) is the input or excitation vector. It is worth mentioning that the state is not necessarily a directly measurable quantity. It is a mathematical concept introduced to solve the dynamic problem. The state is a description of the internal structure of the dynamic system, not carrying any physical meaning in general. The quantities that carry physical meaning are the ones that can be generated or observed. They are the inputs and the outputs. In many simple situations, the state vector can be directly observed. However, as that is not always the case, the theory of state-space analysis has been derived with generality by the system analysts. In order to take full advantage of the results obtained in the above field, and to understand some important concepts, the general form of the equations are presented. The general form of the dynamic equations used in the field of linear system theory are

\[
\begin{align*}
\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\
\mathbf{y} &= C\mathbf{x} + \mathbf{v}
\end{align*}
\]

For time-invariant dynamic systems, \( A, B, \) and \( C \) are constant matrices. The vectors \( \mathbf{u} \) and \( \mathbf{x} \) represent the stochastic input and the state, respectively. Finally, the vector \( \mathbf{y} \) represents the observation, or the physically measurable quantity, contaminated by the additive measurement noise vector \( \mathbf{v} \). Interestingly, the above set of equations are used parallelly for system
identification and control problems.

The above equations (6.3a - 6.3b) were encountered in physics [Wang and Uhlenbeck, 1945] and have been investigated for the last few decades in the context of a wide variety of problems. However, new insight was obtained with the contributions of Kalman and Bucy [1961]. The problem addressed by Kalman can be stated in simple words as follows. Given the corrupted observations $Y(t)$ over an interval $0 \leq t \leq t_1$ and the statistics of the noise processes $U$ and $V$ what is the "best" estimate of the current state $x(t_1)$? The Kalman Filter [Kalman and Bucy, 1961] provides an answer to the above problem by minimizing the mean square deviations of the true state from the estimate over the interval $0 \leq t \leq t_1$. The complete treatment of the topic is not presented in the present review. However, the important concepts of Controllability and Observability are reviewed here. These are fundamental concepts of the theoretical developments of state space analysis and are used extensively for various investigations relating to the system. The notions of Controllability and Observability play an essential role in the design of control systems and provide valuable insight into the physical problems.

There is a fundamental difference between classical dynamics and modern mathematical system theory. The difference lies in the approach to the problem. The classical dynamicist assumed the passive role of an observer who analyzed the system and was happy if a solution to the dynamic equations of motion could be obtained. The role of the present day system analyst is that of an activist. The system behavior is regarded as modifiable by means of inputs that are at a decision maker's disposal. Thus, the focus is on being able to control the system behavior subject to constraints on the manner in which interactions are allowed. In this context, one can wonder under what conditions can the system be controlled, or when adequate information about the system can be deduced based on external observations. The answers to the above questions are pro-
vided by the theorems of Controllability and Observability.

6.4.1. Controllability

The property of Controllability characterizes the possibility to influence the behavior of the dynamic system by control and disturbance input functions. The condition of Controllability indicates whether it is possible to control all the states of a system completely by suitable choice of input functions.

Mathematically speaking, a system is completely controllable if it is possible to find an input \( U(t) \) that will transfer the system, eqn. 6.3, from any given state \( X(t_0) \) to any given final state \( X(t_f) \) over a specified interval of time \( (t_f - t_0) \). It is common practice to use the term "uncontrollable" for a system that is not completely controllable.

In very simplistic terms, the Controllability or the ability to influence the state by suitable choice of inputs is governed by the nature of the matrix \( B \) which introduces the inputs to the system equations. If all terms of the i-th row of the matrix \( B \) are identically zero, then the i-th state of the system remain uninfluenced by the inputs and the system is not completely controllable. This concept plays a very important role not only in the electrical and mechanical systems, but also in many environmental and socio-economic problems, including economic planning. In the present investigation, the concept would be used indirectly through the concept of minimal realization of a given transfer function. The concept of controllability was introduced by Kalman [1960] who provided a mathematical criterion for complete controllability. Thus, the dynamic system (eqns 6.3a - 6.3b) is completely controllable if and only if

\[
\text{rank} \left[ B \mid AB \mid A^2B \mid \ldots \mid A^{n-1}B \right] = n
\] (6.4)

The uncontrollable states of the system are characterized by the condition
\[ X^T \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = 0 \quad (6.5) \]

6.4.2. Observability

The property of Observability leads to conditions about the information on the state of the system that can be obtained by measurements. For the system of equations (6.3a-6.3b) consider the situation where the additive noise vector, \( V \) is non-existent. This system is completely observable if for each arbitrary initial condition \( X(0) \) there exists a finite time \( t_1 > 0 \) such that the initial condition \( X(0) \) can be determined by the knowledge of the control function \( U(t) \) and the measurement function \( Y(t) \) on the interval \( [0, t_1] \). Intuitively, we can see that the matrix \( C \) dictates whether the states of the system contribute to the observation vector \( Y \). If for example, the \( j \)-th column of the matrix \( C \) is a null vector, then the \( j \)-th state of the system is unobservable. In terms of the matrix equation, the dynamic system is completely observable if and only if

\[ \text{rank} \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T & \cdots & (A^T)^{n-1} C^T \end{bmatrix} = n. \quad (6.6) \]

If all the states are not observable, then the system is termed as unobservable. The unobservable states of the system are given by the equation

\[ X^T \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T & \cdots & (A^T)^{n-1} C^T \end{bmatrix} = 0 \quad (6.7) \]

The simple concept of Controllability and Observability has a profound impact on the understanding of the behavior of dynamical systems. These concepts clearly indicate that there is something "pathological" about the states of a system which are either uncontrollable or cannot be controlled by choosing a suitable input, or unobservable, indicating the presence of motions which do not show up in the observations and remain undetected.

A simple example [Power and Simpson, 1977] illustrates the various combinations of the above properties of Controllability and Observability. The equations of motion of a system under
investigation in normal co-ordinates are associated with the matrices

\[
A = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & -4 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -3
\end{bmatrix}
\]  
\hspace{1cm} (6.8a)

\[
B = \begin{bmatrix}
0 & 1 \\
-1 & 0 \\
0 & 0 \\
1 & 2 \\
-3 & 1 \\
0 & 0
\end{bmatrix}
\]  
\hspace{1cm} (6.8b)

\[
C = \begin{bmatrix}
0 & -1 & 0 & 0 & -3 & 1 \\
1 & 0 & 0 & 0 & 4 & 0 \\
0 & 1 & 0 & 0 & 2 & 1
\end{bmatrix}
\]  
\hspace{1cm} (6.8c)

The four groups into which the states \(X_i, i = 1, 2, ..., 6\) of the system can be classified are as follows:

a) Controllable and observable: \(X_1, X_2, X_5\)

b) Controllable but unobservable: \(X_4\)

c) Uncontrollable but observable: \(X_6\)

d) Uncontrollable and unobservable: \(X_3\)

The variables \(X_1, X_2\) and \(X_5\) can be "controlled" or influenced by the inputs, and the effects "observed" at the output. However, the state variable \(X_4\) can be controlled, but no indication of its behavior appears at the output, as it is unobservable. The state variable \(X_3\) is something to be concerned about. It is unobservable, implying its behavior remains undetected by the observer. It is also uncontrollable, whereby it cannot be controlled by a suitable choice of inputs. If, in addition the response corresponding to such a state is unstable, it can destroy the system by building up excessive motions, without giving any observable indications. Thus, the notions of Con-
trollability and Observability are not mere mathematical abstractions, but important concepts related to physical systems.

6.5. State Space Realization and Transfer Functions

The state space equations and the transfer function, which is scalar for univariate and matrix for multivariate case, both represent dynamical systems. It is natural to expect that the above two representations are related to each other. The relation between the time and frequency domains are straightforward when one is dealing with the impulse response function and the transfer function. It is common knowledge that they form a Laplace (or Fourier) transform pair. Similarly, the transfer function can be obtained by taking transforms of the state-space equations. However, the passage from the transfer function to the state-space equation is considerably more involved.

6.5.1. Transfer Functions

Consider the state-space system of equations (6.3a-6.3b)

\[ \dot{X} = AX + BU \]
\[ Y = CX \]

in the equations \( \dot{X} = AX + BU \), and \( Y = CX \) introduced earlier. Let the input \( U \) be bounded, that is,

\[ \int_0^\infty |U(t)| e^{-\sigma t} dt < \infty, \quad i = 1, \ldots, m, \]

where, \( \sigma < \infty \). Let \( X(s) \), \( Y(s) \) and \( U(s) \) denote the Laplace transform of \( X(t) \), \( Y(t) \) and \( U(t) \), respectively. Then, taking the Laplace transform of equations (6.3a-6.3b) one obtains

\[ sX(s) - X(0) = AX(s) + BU(s) \]  \hspace{1cm} (6.9a)
\[ Y(s) = CX(s) \]  \hspace{1cm} (6.9b)

Thus,
\[ Y(s) = C \left( sI - A \right)^{-1} \left[ X(0) + B U(s) \right] \] (6.10)

Or, if \( X(0) = 0 \), then

\[ Y(s) = C \left( sI - A \right)^{-1} B U(s) \] (6.11)

The transfer function, defined as the ratio of the transforms of the output and input is given by

\[ H(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B \] (6.12)

Thus, the above equation provides the procedure for obtaining the transfer function form the state-space equations.

6.5.2. Impulse Response Function

The conventional theory of vibration analysis in the time domain utilizes the concept of the impulse response function. It is only natural to expect that there is some relation between the state-space equations and the impulse function (or matrix). Referring back to the system of equations (6.3a-b) with the initial condition \( X(t) = 0 \), it is easily verified that [Casti, 1987]

\[ X(t) = \Phi(t, t_0) X_0 + \int_{t_0}^{t} \Phi(t, s) B(s) U(s) \, ds, \] (6.13)

where the matrix \( \Phi \), known as the state transition matrix satisfies the linear equations

\[ \frac{\partial \Phi(t, s)}{\partial t} = A(t) \Phi(t, s), \quad s < t \] (6.14a)

\[ \Phi(s, s) = I \] (6.14b)

Thus,

\[ Y(t) = C(t) \Phi(t, t_0) X_0 + \int_{t_0}^{t} C(t) \Phi(t, \tau) B(\tau) U(\tau) \, d\tau \] (6.15)

Note, that the above contains the effect of the initial state and the excitation, homogeneous solution and the particular solution, respectively. If the initial state is \( X_0 = 0 \), then one has the "zero state".
\[ Y(t) = \int_{t_0}^{t} C(t) \Phi(t, s) B(s) U(s) \, ds = \int_{t_0}^{t} W(t, s) U(s) \, ds, \quad (6.16) \]

where \( W(t, s) \) is the so-called impulse response, since it measures the observed output due to a "delta function" input. On the other hand, when \( U(t) = 0 \) for every \( t \geq t_0 \), one has the "zero-input" response. Thus, the output of the time-varying system may also be decomposed into two distinct parts corresponding to the zero initial state and the zero input state. These are given by the second and the first part, respectively, of the R.H.S. of eqn. 6.16.

6.5.3. State Space Representation From Transfer Functions

The derivation of the state-space equations from a given transfer function, scalar or matrix form, is a non-trivial problem of system theory and control engineering. Popularly known as the realization problem, this has been a topic of active research for the last two decades. The problem is complicated by the fact that we are interested in the "simplest" form of the representation called the minimal realization. The minimal realization contains no "extraneous information". The motivation for minimal realization is not only computational economy, but desirable mathematical properties that help avoid theoretical and computational difficulties. The solution to this inverse problem is non-unique. In other words, different equivalent state-space realizations can have the same transfer function. However, one is often interested in a representation that is "basic" in nature, and the "structural" properties of the system are invariant with respect to the choice of co-ordinate system. It turns out that the Canonical representation have been universally accepted for the realization problem.

Canonical forms have considerable practical value, particularly in system identification problems. Interestingly, Canonical forms represent the system with a minimum number of parameters. Naturally, it is an attractive choice for system identification, whereby the number of
unknown parameters characterizing the system is reduced to a minimum.

There are several alternate canonical forms of representation. Some of the more well known ones are control canonical or the 1st companion form, observer canonical or the 2nd companion form and Jordan canonical or the diagonal companion form.

For random vibration studies, the Control Canonical form is of maximum interest. This is simply because it uses the familiar procedure of converting an n-th order differential equation to a system of n first order differential equations. The state space form can usually be obtained by inspection, for a single variable problem. Unfortunately, for the multivariate case, the realization problem is extremely complex. Typically, the theory of matrix series expansion and Hankel matrix representation [Casti, 1987] is applied to the transfer function matrix in turn. As mentioned earlier, the interest is in obtaining minimal realizations. The special interest in minimal realization is because of the fact that besides being minimal, they are also unique up to a coordinate transformation in the state space.

To this point our attention was focussed on the state space analysis. To apply the method to random vibration problems some understanding of the theory of stochastic processes is also required. Particular interest lies in Gauss-Markov processes, as such processes are naturally suited for state space analysis.

6.6. Stochastic Process

Almost all the natural processes we encounter in everyday life, exhibit some degree of randomness. This is particularly true of dynamical systems. The surface roughness of a pavement, the instantaneous elevation of the sea, the local velocity of a gusty wind, or the flutter of a leaf are all characterized by their randomness. These are examples of random processes, which can be
accurately described only by statistical means.

A random process, \( x(t) \), also known as a stochastic process, can be viewed as an indexed family of functions of time. Each individual function of time is then called a realization \( x(t) \) of the process. For any given instant \( t = t_1 \), the stochastic process is characterized by a random variable \( x(t_1) \). A standard procedure for describing the above process is through the concepts of probability theory. It is common practice to indicate the probability that \( x \) assumes the value \( a \) as \( p(a) \), for the discrete case, and the probability that \( x(t_1) \leq a \) as \( P(x \leq a) \). For the continuous case, one uses the probability density function \( f(x) \) and the probability distribution function \( F(x) \). Thus, the boundary values are given by

\[
P(-\infty) = 0
\]

and

\[
P(+\infty) = 1
\]

(6.17a)

(6.17b)

The distribution function \( F(x) \) is related to the density function \( f(x) \) through the relation

\[
f(x) = \frac{dF(x)}{dx}
\]

(6.18)

The boundary values for the continuous case are similarly specified as \( F(-\infty) = 0 \), and \( F(+\infty) = 1 \).

6.6.1. Gaussian Processes

Normal processes, also known as Gaussian processes, are of special interest because they are frequently encountered in nature. This is particularly true of the case when the process results from the combined effect of a number of independent random phenomena. The uniqueness of the Gaussian distribution lies in the fact that the process is completely described by its first two moments. That is, the mean and the covariance. For a general stochastic process, one needs an
infinite hierarchy of moments to completely describe the process. Another interesting property is that the Gaussian nature is retained by linear operations on the process. The two properties mentioned above greatly simplify the probabilistic analysis of linear systems subject to Gaussian excitation [Nigam, 1983].

A random process \( x(t) \) is called Gaussian if the random variables \( x_1 = x(t_1), x_2 = x(t_2), \ldots, x_n = x(t_n) \) are jointly normal. That is,

\[
P_{x_1, x_2, \ldots, x_n} = P_X = [2\pi]^{-n/2} |K_{XX}|^{-1/2} \exp \left[ -\frac{1}{2}(X - \mu)(K_{XX})^{-1}(X - \mu)^T \right]
\]  

(6.19)

6.6.2. Markov Process

For many physical systems the probabilistic law governing the future state of the system depends only on the present state, irrespective of how the system arrived at its present state [Nigam 1983]. In other words, the future behavior of the process depends only on the most recent known state of the process. All relevant predictions of the future can completely neglect the past. This special property is known as the Markov Property, and processes with such property are known as Markov Processes. Mathematically speaking, a random process \( X(t) \) is said to be Markov if

\[
P \left[ X(t_n) \leq x_n \mid X(t_1) = x_1, \ldots, X(t_{n-1}) = x_{n-1} \right] = P \left[ X(t_n) \leq x_n \mid X(t_{n-1}) = x_{n-1} \right].
\]  

(6.20)

for any \( n \) and for any \( t_n \) where, \( t_1 < t_2 < \ldots < t_n \). The symbol \( x \) denotes the state of the process \( X(t) \). The set of all possible values of \( x \) is called the state space. Based on the nature of the state space and that of the indexing set, being discrete or continuous, there are four possible types of Markov processes. A thorough treatment of this classification is provided by Nigam [1983].

A very fundamental concept in the study of Markov Processes is the transition probability function \( P[ x(t) \in E \mid X(s) = x] \), which is the conditional probability that the state of the system...
belongs to the set $E$ given that the system is in state $x$ at $s < t$. Thus, a Markov process has one step memory.

A very similar concept in the theory of state-space analysis of dynamical systems is that of the transition matrix. This matrix relates any two consecutive states of the system. Thus, given the initial state and the input (excitation), the response at any future time can be obtained through the transition matrix.

6.6.3. Gauss-Markov Process and White Noise

White noise is an idealized process which is "purely random". In the light of the Markov property, the future state of such a process is completely unpredictable. For such a process

$$P \left[ X(t_n) \leq x_n \mid X(t_{n-1}) = x_{n-1} \right] = P \left[ X(t_n) \leq x_n \right]$$

(6.21)

A stochastic process $W(t)$ is called a stationary white noise if it is an idealized Gauss-Markov process with independent increments for $(t_2 - t_1) \to 0$. Such a process has the properties

$$E \left[ W(t) \right] = 0$$

(6.22a)

$$R_{ww}(t, \tau) = Q_w \delta(t - \tau). \quad Q_w \geq 0$$

(6.22b)

where $Q_w$ is the intensity of the white noise, and $\delta(t - \tau)$ is the Dirac Delta function. It has to be noted that white noise, as indicated above is not a real process. However, some processes closely resemble white noise, when the time scale of measurement is very small. Examples involve the instantaneous deviation of the thrust of a rocket from the mean thrust; the deviation, from the mean value, of the instantaneous current in an electrical wire or an electron beam.

The "purely random" nature of the white noise process indicates very poor correlation of the process even at small intervals. The auto-covariance, auto correlation at zero lag, of the white
noise process tends to infinity. Strictly speaking, white noise is a mathematical abstraction and not a physically realizable process. However, integrated white noise is physically more meaningful and is called a Wiener Process.

Considerable amount of practical interest lies in this idealized white noise process. The spectral density of such a process is constant in magnitude. For several complex dynamical systems subjected to random excitation, analytical solutions are possible only when the excitation is a white noise. Such approximation of a random excitation by a white noise is quite reasonable when the characteristic response time of the dynamical system being excited is much greater than that of the excitation. Mathematically speaking, if \( p[ W(t) \mid W(\tau) ] = p[ W(t) ] \), for \( |t - \tau| > T \), and \( T \) is much smaller than the characteristic response time of the system, the white noise assumption for the random input provides fairly accurate results.

A Gauss-Markov random process is a Markov random process with the added restriction that \( p[ x(t) ] \) and \( p[ x(t) \mid x(\tau) ] \) are Gaussian density functions for all \( t, \tau \) in the interval \([ t_0, t_f ]\).

The density function \( f[ X(t) ] \) of a Gauss-Markov process is completely described by specifying the mean value of the vector random process, \( \mu(t) = E[X(t)] \), and the covariance matrix

\[
C_{XX} = E \left\{ \begin{bmatrix} X(t) - \mu(t) \\ X(t) - \mu(t) \end{bmatrix} \begin{bmatrix} X(t) - \mu(t) \\ X(t) - \mu(t) \end{bmatrix}^T \right\}.
\]

Several natural and man-made dynamical phenomena can be approximated quite accurately by Gauss-Markov random processes. Moreover, when the information available on the nature of the random process is limited only to the first two moments, it is often expedient to represent it as a Gauss-Markov process even if it is not.

A stochastic process \( x(t) \), \( t \geq t_0 \), is said to have independent increments, if

\[
[ x(t_0) ] = 0
\]

\[ (6.23a) \]
\[ E \left[ x(t_2) - x(t_1) \right] = E \left[ x(t_4) - x(t_3) \right] = 0 \]  \hspace{1cm} (6.23b)

\[ E \left\{ (x(t_2) - x(t_1))^T (x(t_4) - x(t_3)) \right\} = 0 \]  \hspace{1cm} (6.23c)

for every sequence of instants \( t_0 \leq t_1 \leq t_2 \ldots \leq t_4 \). If in addition to having independent increments, each of the increments \( [x(t_2) - x(t_1)] \) is a Gaussian random vector with vanishing mean value and a covariance matrix \( C_{xx} = Q(t_2 - t_1) \), where \( Q \) is a constant, non-negative definite intensity matrix, then the process is called a Wiener process.

Since linear transformations of a Gaussian vector preserves its Gaussian character, a Gauss-Markov random process can always be represented by the state vector of a continuous dynamic system forced by a Gaussian, purely random process, where the initial state vector is Gaussian. This concept will be exploited extensively by designing "shaping filters" to model non-white random excitations.

### 6.6.4. Markovianization of a Non-Markovian System

The Markov property implies that given a random sequence \( s_i, \ i = 0, 1, 2, \ldots, n \), the initial density function \( f(x_0) \) and the transitional density function, \( f \{ x_{k+1} | x_k \} \) the process is completely described. Thus, it becomes possible to relate the finite past with the finite future. This property, in an indirect way, can be utilized to solve the complex problems involving vibrating systems subjected to random non-white excitation.

For illustration, consider the equation [Bryson and Ho, 1967] governing the cumulative charge in an electronic beam

\[ \dot{x}_1 = C(t) x_1 + W(t) \]  \hspace{1cm} (6.24)

where \( C(t) \) is a deterministic function of time, \( W(t) \) is a purely random process of white noise resulting in the scalar Markov process \( x(t) \). As opposed to the above, consider the system
defined by
\[ \dot{x}_2 = C_2(t) \dot{x}_2 + C_1(t) x_2 + W(t). \] (6.25)

Clearly, \( x_2(t) \) is not Markovian as it depends not only on its first derivative immediate past but also on the second derivative. However, this non-Markov scalar process \( x_2(t) \) can be cast into the form of a Markov vector process \( Y(t) \) as follows. Define \( y_1 = x_2 \), and \( y_2 = \dot{x}_2 \) and the vector \( y^T = [y_1, y_2] \). Then, eqn. 6.25 can be rewritten as
\[ \dot{Y} = AY + BW \] (6.26)
which is a matrix equation for the second order system written as
\[ \frac{d}{dt} \begin{bmatrix} x_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ C_2(t) & C_1(t) \end{bmatrix} \begin{bmatrix} x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} W(t). \] (6.27)

The above procedure can readily be extended to higher order systems. Thus, given any random process involving a finite number of time derivatives, an equivalent Markov vector representation can always be found by increasing the dimension of the state vector.

6.7. Formulation of the Covariance Equation

A great majority of random vibration problems have a zero mean Gaussian excitation. If the system being excited is linear, the output is also a zero mean Gaussian process, and evaluation of the covariance of the response provides the complete solution to the problem. In the following sections the formulation of the covariance equation, also known as the Liapunov equation is outlined. This is done in three stages. Firstly, a single degree of freedom (SDF) system subjected to Gaussian white noise excitation is considered. Next, the same system subjected to a non-white excitation is investigated. Finally, a multi-degree of freedom (MDF) system subjected to a non-white excitation is examined.
6.7.1. Single Degree Freedom with White Noise Input

Let the governing differential equation of motion for the SDF system be

\[ m\ddot{x} + c\dot{x} + kx = W(t). \]  

(6.28)

Define a vector \( \mathbf{Y} \), such that \( \mathbf{Y}^T = [x, \dot{x}] \). Then, the above equation can be rewritten as

\[ \dot{\mathbf{Y}} = A\mathbf{Y} + B \]  

(6.29a)

where,

\[ A = \begin{bmatrix} 0 & 1 \\ m^{-1}K & m^{-1}C \end{bmatrix} \]  

(6.29b)

and

\[ B^T = [0, m^{-1}W(t)]. \]  

(6.29c)

Note that the above transformation converts a scalar non-Markovian system to vector Markovian system. Post multiplying equation (6.29) by the vector, \( \mathbf{Y}^T \) one obtains,

\[ \dot{\mathbf{Y}}\mathbf{Y}^T = A\mathbf{Y}\mathbf{Y}^T + B\mathbf{Y}^T, \]  

(6.30a)

or, taking the transpose of the above equation,

\[ \mathbf{Y}\ddot{\mathbf{Y}} = \mathbf{Y}\mathbf{Y}^TA^T + \mathbf{Y}B^T. \]  

(6.30b)

Adding the two equations and taking expectations of both sides, one obtains,

\[ E\left[ \dot{\mathbf{Y}}\mathbf{Y}^T + \mathbf{Y}\ddot{\mathbf{Y}} \right] = A E\left[ \mathbf{Y}\mathbf{Y}^T \right] + E\left[ \mathbf{Y}\mathbf{Y}^T \right]A^T + E\left[ B\ddot{\mathbf{Y}} + \dot{\mathbf{Y}}B^T \right]. \]  

(6.31)

or

\[ E\left[ \dot{\mathbf{Y}}\mathbf{Y}^T + \mathbf{Y}\ddot{\mathbf{Y}}^T \right] = A E\left[ \mathbf{Y}\mathbf{Y}^T \right] + E\left[ \mathbf{Y}\mathbf{Y}^T \right]A^T + E\left[ B\ddot{\mathbf{Y}} + \mathbf{Y}B^T \right]. \]  

(6.32)

Recalling that \( E[\mathbf{Y}\mathbf{Y}^T] \) is the covariance matrix of the vector \( \mathbf{Y} \), and denoting it by \( C_{\mathbf{YY}} \), eqn. 6.32 can be rewritten as

\[ \frac{d}{dt}C_{\mathbf{YY}} = A C_{\mathbf{YY}} + C_{\mathbf{YY}}A^T + G. \]  

(6.33)

In the above, \( G = E[BY^T + \dot{\mathbf{Y}}B^T] \), and the symmetry of the covariance matrix has been used.
Solution of the above matrix differential equation provides the non-stationary covariance of the
response. For the stationary case, the covariance is independent of time. Hence, its time deriva-
tive is zero. Thus, the stationary response of the system is governed by the equation

\[ A C_{YY} + C_{YY} A^T = -G. \]  
(6.34)

This is the well known Liapunov equation encountered in control theory. The solution of the
matrix, \( C \) provides the statistics of response.

6.7.2. Filter augmentation For Non-White Input

When the excitation process is a colored or non-white random process, a Markovian
representation can still be obtained by augmenting the state with a shaping filter. Let the dynamical
system be represented by

\[ \dot{Y} = AY + BU \]  
(6.35)

where, \( U \) is non-white in nature. Usually, the process \( U \) can be represented as the output of
another dynamical system excited by white noise, having a state space representation of

\[ \dot{U} = FU + GW. \]  
(6.36)

Combining equations (6.35-36) one obtains,

\[
\begin{bmatrix}
\dot{Y} \\
U
\end{bmatrix} =
\begin{bmatrix}
A & B \\
0 & F
\end{bmatrix}
\begin{bmatrix}
Y \\
U
\end{bmatrix} +
\begin{bmatrix}
0 \\
G
\end{bmatrix} W(t)
\]  
(6.37a)

which can be viewed as an augmented state space equation in the Markovian form

\[ \dot{Y}_a = A_a Y_a + B_a W. \]  
(6.37b)

Thus, a cascaded set of dynamical systems can be represented by an increased order of the state
vector. If the order of the state vector \( Y \) and \( U \) were 2 in each case, the state of the augmented
vector \( Y_a \) is \( 2 + 2 = 4 \), and it represents a Markovian system as before. The covariance matrix
of the augmented state vector can be formulated as before using eqns. 6.31 - 6.34.
6.7.3. Multi Degree Freedom System With Non-White Input

The solution of the multi degree freedom system with non-white input represents the more general problem of random vibration analysis. Common examples involve multiply supported structures subjected to earthquake excitation, multi-storey building subjected to gusty winds, and offshore platform subjected to random waves.

Here, it will be assumed that a differential equation representation of the non-white excitation is available, without being concerned about how it is obtained. A Control Canonical representation of the augmented system will be obtained, and subsequently, the pertinent Liapunov equation will be defined. The derivation of the state space representation of this case will closely follow the developments of Lin [1967]. As before, let the $n$ degree of freedom system be represented by the matrix differential equation

$$M \ddot{X} + C \dot{X} + K X = F(t), \quad (6.38a)$$

and

$$Q \begin{bmatrix} F(t) \end{bmatrix} = W(t). \quad (6.38b)$$

In the above equations, $M$, $C$ and $K$ are square matrices of order $n$, $X$ and $F$ are vectors of order $n$, $W$ is white noise vector also of order $n$, and $Q$ represents a linear operator. Let the form of the operator, $Q$ be given by

$$Q = \sum_{i=0}^{N} L_i \frac{d^{i}}{dt^{i}} \quad (6.38c)$$

where, $L_i$ are constant matrices of order $m \times n$. Combining eqns. 6.38a - 6.38c, one obtains,

$$Q M \ddot{X} + Q C \dot{X} + Q K X = Q F(t) = W(t). \quad (6.39)$$

Carrying out the linear operation of $Q$ on the three terms of the equation (39), the expanded form can be represented as
\[ L_N M X^{(N+2)} + L_{(N-1)} M X^{(N+1)} + L_{(N-2)} M X^{(N)} + \cdots + L_0 M X^{(2)} + L_N C X^{(N+1)} + L_{(N-1)} C X^{(N)} + \cdots + L_1 C X^{(2)} + L_0 C X^{(1)} \]
\[ + L_N K X^{(N)} + \cdots + L_2 K X^{(2)} + L_1 K X^{(1)} + L_0 K X = W(t). \]  
(6.40)

Collecting terms of the same power, the equation can be rewritten as

\[
\sum_{i=0}^{m} P_i \frac{d^i X}{dt^i} = W(t). 
\]  
(6.41a)

where, \( m = N + 2 \) and the coefficient matrices are

\[
P_0 = L_0 K \quad (6.41b)
\]

\[
P_1 = L_1 K + L_0 C \quad (6.41c)
\]

\[
P_2 = L_2 K + L_1 C + L_0 M \quad (6.41d)
\]

\[
P_i = L_i K + L_{(i-1)} C + L_{(i-2)} M \quad (6.41e)
\]

\[
P_N = L_N K + L_{(N-1)} C + L_{(N-2)} M \quad (6.41f)
\]

\[
P(N+1) = L_N C + L_{(N-1)} M \quad (6.41g)
\]

\[
P_m = L_N M. \quad (6.41h)
\]

Define the state vectors as

\[
Y_j = X_j \quad (6.42a)
\]

\[
Y_{(j+n)} = \frac{d}{dt} Y_j = \dot{X}_j \quad (6.42b)
\]

\[
Y_{(j+2n)} = \frac{d}{dt} Y_{(j+n)} = \ddot{X}_j \quad (6.42c)
\]

\[
Y_{j+(m-1)n} = \frac{d}{dt} Y_{j+(m-2)n} = \frac{d^{(m-1)} X_j}{dt^{(m-1)}} \quad (6.42d)
\]

where \( j = 1, 2 \cdots n ; \ m = N + 2 \). Thus, equation (6.41) can be rewritten as

\[
P_m \frac{d}{dt} \begin{bmatrix} Y_{1+(m-1)n} \\ Y_{2+(m-1)n} \\ \vdots \\ Y_m \end{bmatrix} + P_{m-1} \begin{bmatrix} Y_{1+(m-1)n} \\ Y_{2+(m-1)n} \\ \vdots \\ Y_m \end{bmatrix} + P_{m-2} \begin{bmatrix} Y_{1+(m-2)n} \\ Y_{2+(m-2)n} \\ \vdots \\ Y_{(m-1)n} \end{bmatrix}
\]
\[ Y_1 
\begin{bmatrix} Y_2 \\ \vdots \\ Y_n \end{bmatrix} + \ldots + P_0 \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix} \] (6.43e)

At this stage, the dynamical equation is ready to be cast into the familiar first order matrix differential equation form. Premultiplying equation (6.43) throughout by \( P_m^{-1} \) and rearranging, one obtains

\[ \dot{Z} = AZ + G \] (6.44a)

where,

\[ Z^T = \begin{bmatrix} Y_1, Y_2, \ldots, Y_N, Y_{N+1}, Y_{N+2}, \ldots, Y_{2N}, \ldots \\ \ldots \\ Y_{1+(m-1)n}, Y_{2+(m-1)n}, \ldots, Y_{mn} \end{bmatrix} \] (6.44b)

\[ G_T = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} (P_m^{-1}W) \] (6.44c)

and

\[ A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ -P_m^{-1}P_0 -P_m^{-1}P_1 -P_m^{-1}P_2 - \cdots - (P_m)^{-1}P_{m-1} & \end{bmatrix} \] (6.44d)

Thus, \( A \) is a square matrix of order \( (mn \times mn) \). The corresponding Liapunov equation is again

\[ AC_T + C_T A^T = -B. \] (6.45)

The matrix \( B \) contains the excitation in an indirect way:

\[ B = E \left[ GZ^T + ZG^T \right] \] (6.46)

where the symbol \( E[.] \) represents the expected value. By using the properties of a white noise process and its cross-correlation with the various derivatives of the response, Appendix B, the final expression for this matrix is [Spanos, 1987, Lin 1976].
\[
B = \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & \left( P_{\alpha}^{-1} J P_{\alpha}^{-T} \right)^{-1} \\
0 & 0 & \ldots & 0 & \left( P_{\alpha}^{-1} J P_{\alpha}^{-T} \right)^{-1}
\end{bmatrix}
\]  

(6.47)

In the above, \( J \) is the intensity matrix of the white noise vector, \( W(t) \).

6.8. Conclusion

The theory of state space analysis of dynamic systems is reviewed comprehensively. The application of the method to single degree and multi degree freedom systems is examined. The procedure of treating a non-white excitation through filter augmentation is also illustrated. The theory provides a very attractive and economic alternative for solving random vibration problems in the time domain. The method is more powerful than the frequency domain method for obtaining non-stationary solutions, and is easily extended to analyze time-dependent systems. The method is naturally suited for digital computation. The transition matrix operates on any given state to yield the future state through matrix multiplication. The stationary solution is obtained through algebraic manipulation only, bypassing the need for complex integrations. Finally, the method has great appeal because of its generality, whereby it can be used directly for system identification and control problems.
CHAPTER VII
State Space Analysis of Guyed Tower Dynamics

7.1. Overview

The guyed tower which is a deep water compliant platform, is selected for the demonstration of the method of the state space analysis. Its predominant response is a rigid body rotation about the bottom support, which can be considered to be a hinge joint. Thus, it becomes an ideal candidate for the application of the state space method to a single degree of freedom system. The restoring force, provided by the guylines, exhibits some degree of non-linearity. The method of equivalent linearization is used to linearize the stiffness. The well established results for the Duffing oscillator subjected to Gaussian excitation are used. Similarly, an equivalent linear model for the nonlinear drag term in the Morison equation is assumed, ensuring a spectral representation of the random excitation. From published literature, an approximation for the excitation in the form of a rational function, which enables simple filter representation of the excitation is used. The Liapunov equation for the covariance matrix of the state vector of the response is solved directly. The simplicity of the method is illustrated through this application. The same response statistics are derived by use of the frequency domain method, to provide a comparison of the two procedures. Finally, an example problem is solved for a guyed tower in 1425 feet of water. The system stiffness for the selected mooring pattern is computed numerically, and approximated by polynomial functions. The response statistics are computed in a close-form by using the Liapunov equation. The results are compared with published literature.
7.2. Introduction and Background

As the drilling and production of offshore oil moves from the continental shelf to the continental slope, increasing attention is focussed on deep-water structures. The environmental loads on these structures become extremely large. In addition, these taller structures tend to have natural periods closer to the predominant wave frequencies. This results in higher dynamic amplification of the response and requires further increase in the structural strengths. The Bullwinkle platform, installed recently by Shell Offshore in the Gulf of Mexico is the tallest rigid platform installed to date. It stands in 1353 feet of water depth, and is 1615 feet tall. It uses over 75,000 tons of steel for the entire structure.

In contrast, the compliant platforms provide an attractive alternative for these deeper waters. Instead of resisting the action of the waves like the rigid structures, they "comply with" the force, like a blade of grass in the wind. The environmental forces are considerably reduced because of this behavior of "yielding to the force". The guyed tower is one such compliant platform concept which has proved its success in the field [Glasscock and Finn, 1984].

In 1983, the first commercial guyed tower was installed by Exxon to operate in 1,000 feet of water, in Gulf of Mexico [Mississippi Canyon Block 280]. The tower has a total height of 1,305 feet and is designed to withstand the 100 year hurricane in the region. It is expected to have a total sway displacement of about 40 to 50 feet in the extreme storm. However, for moderate weather, it will provide a stable, relatively motionless, working platform.

It is a very tall, slender structure supported at the sea bed by a joint that permits small rotations about the base and provide a moment-free bottom support. The tower is maintained in its vertical position by a set of cables, or guy-lines attached near the top end. These cables are symmetrically placed around the tower and spread out radially to pre-installed piled anchors on the
sea-bed. The upper part is a lead cable which acts as a stiff spring in moderate seas. The middle portion is a heavy segment with clump weights which are lifted off the bottom during severe storms. The overall behavior of the guylines is that of a multi-segment catenary.

The dynamic response of the guyed tower plays a critical role in its design. The dynamic response can be classified into two main categories. Firstly, there is the rigid body rotations of the tower about its base. Next, there are the flexural vibration modes of the tower. The former effect produces very large displacements at the deck level, 40 feet or more, and controls the overall design of the system. The latter causes relatively small deflections in the structure, but govern the structural design of the tower. It is the first of these two effects that will be treated here. The dynamic analysis of the guyed tower has been investigated by several researchers, [Finn, 1976, Basu and Dutta, 1982, Hanna et. al., 1982, Brinkman 1983, and Kanegaonkar and Haldar, 1987]. The present investigation uses the same model proposed by Kanegaonkar and Haldar [1987], but applies a different methodology for the solution procedure.

The primary differences of the present investigation from the above are two-fold. Firstly, the Liapunov equation for the covariance is formulated and solved here as opposed to the non-gaussian closure technique applied by Kanegaonker and Haldar [1987]. Secondly, the present investigation assumes linearization of both the excitation and the system stiffness, in a consistent manner. In contrast, the former investigators, apply linearization to the excitation, but proceeds with a rigorous technique of non-linear analysis for the system. This inconsistency does not appear justifiable.

7.3. Guyed Tower Concept

The tower is a lattice type truss structure which supports the platform deck at the top. The
bottom end of the structure rests on the sea-bed. However, the foundation is specially designed to behave differently from conventional bottom-supported rigid platforms. Instead of the 'cantilever' type of the connection to the sea bed, small rotations are permitted, thereby making the connection moment-free. Two alternate concepts have been proposed for the bottom connection. One concept proposes the use of a spud-can at the bottom, resting on a specially prepared foundation [Basu and Dutta, 1982]. The other concept uses an unique connection through centrally located, ungrouted piles [Glasscock and Finn 1984].

The tower is held upright by an array of mooring lines. The top end of each mooring line is connected to the tower, a little below the free surface, where wave activity is pronounced. The other end of the mooring line is connected to piled anchors. A segment of the mooring line consists of a heavy clump-weight with distributed weights. The proper choice of the parameters of the clump weight, such as unit weight and length, and its location on the catenary, affects the stiffness properties significantly. The clump weight is located so that under pretension condition, most of it rests on the bottom. For operational weather, when the tower displacements are small, the catenary acts as a hardening spring, restricting the displacements within a few feet. However, for storm weather, the clump-weight is lifted completely off the bottom and the system stiffness is that of a softening spring. This gives the tower the required "compliance" of having large excursions without increasing line tensions significantly.

The hardening and softening behavior of a single catenary line is somewhat masked by the combined action of all the catenary lines. Furthermore, the vertical force arising from the mooring lines, and the action of the platform weight under a displaced position acts counter to the restoring moments. This effect, usually referred as the "P-Delta effect", considerably reduces the system stiffness. Thus, the overall behavior of the system stiffness is non-linear, and "softening"
in nature.

7.4. Model Idealization for Analysis

The rigid body rotation of the tower about its base is analyzed here. The dynamic behavior of the tower in this mode dominates the global response. The primary assumptions used in the modeling procedure are discussed below [Basu and Dutta, 1982, Kanegaonkar and Haldar 1987].

7.4.1. Assumptions

The tower is assumed to behave as a rigid body and the flexural deflection modes are not considered. Further, the tower is assumed to be hinged at the bottom sea-bed support. The restoring moment from the foundation is considered negligible, compared to that arising from the guy lines. As far as the cable arrays are concerned, they are assumed to be radially symmetric with respect to the tower. The excitation is taken to be unidirectional in nature. In terms of loading, the hydrodynamic forces on the guylines are negligible compared to that acting on the tower. Further, the cables behave as springs only and their inertia is ignored. Finally, the angular deflections of the tower can be considered to be small; thus,

\[ \sin(\theta) \approx \theta = \tan(\theta) \]

The first three assumptions are generally accepted, based on model test observations. The radial symmetry of the system stiffness is substantiated later in the investigation. Numerical computation of the horizontal restoring force corresponding to excursions at 0, 5, and 10 degrees for a system where the adjacent mooring lines are at 20 degrees indicates that the variation is insignificant. The assumption that the mooring system behaves statically has been questioned by some investigators. Their dynamic response and the interaction with the tower dynamics, has been investigated by Brinkman [1983], and Morrison and Will [1983]. A hysteretic behavior is
displayed by the mooring line tensions. The overall effect of considering line dynamics is two-
fold. The line tensions are amplified, and the tower response is reduced significantly by the guy-
line dynamic. For simplicity, both these effects are neglected in the present analysis.

7.4.2. Mooring System Stiffness

Each of the guylines form a non-linear spring. The nature of the force-deflection relation-
ship depends on the value of the pretension chosen. The governing equations for a system of
multicomponent catenary lines can be obtained by writing the equilibrium equations for each seg-
ment in turn, and then satisfying the continuity of the tension and the slope at the boundaries
between adjacent segments [Chakrabarti, 1987]. These equations turn out to be hyperbolic tri-
gonometric functions. The force deflection relationship of the combined system can be obtained
by vectorial addition of the contributions from all the lines. Wilson [1984] has reported the suc-
cessful approximation of total effective stiffness as a cubic nonlinearity. A similar procedure was
followed by Kanegaonkar and Haldar [1987]. They report successful approximations of the hori-
zontal force and the vertical force by cubic and quadratic expressions, respectively. Unfortu-
nately, they do not report the numerical results of these approximations. Thus, the system stiff-
ness and the associated approximations were performed independently in the present investiga-
tion, and confirmed the claims of the above investigators.

Following the same convention, the horizontal restoring force as a function of the rigid
body rotation angle, $\theta$, measured from the vertical, is assumed to be

$$ R_x = c_1 \theta + c_2 \theta^3, \quad (7.1) $$

where, $c_1$ and $c_2$ are regression coefficients. Similarly, the vertical reaction at the attachment
point, as a function of the angular deflection, $\theta$ is assumed to be
\[ R_z = \dot{\theta}_1 + \dot{\theta}_2 \theta^2 \]  

(7.2)

where, \( \dot{\theta}_1 \) and \( \dot{\theta}_2 \) are regression coefficients as before. Note the choice of the different expressions, eqns 7.1-7.2, for the total horizontal and vertical forces. The horizontal force \( R_x \) is zero for zero rotation, and changes sign at the origin. Thus, it is represented by a linear and a cubic term. In contrast, the total vertical force has an initial value even for zero rotation, and is symmetric about the origin. Thus, it is appropriately represented by a constant term plus a quadratic term for the symmetric non-linearity.

7.4.3. Equation of Motion of Rigid Tower

The governing equation of motion of the tower considers the rigid body motions of the guyed tower about the base "hinge". The motion in a plane can be obtained by writing the equilibrium equation for the moment about the base. Thus, the governing equation of motion is [Kane-gaonkar and Haldar, 1987],

\[ J_0 \ddot{\theta} + c d^2 \dot{\theta} + R_x z_{ca} + (F_b z_b - W_p D - z_T DW_T - R_z z_{ca}) \theta = F(t) h . \]  

(7.3)

Note, that the above equation considers the nonlinearity of the mooring system stiffness introduced through \( R_x \) and \( R_z \), but linearizes the geometric nonlinearity arising from the \( \sin(\theta) \) and \( \cos(\theta) \) terms. The same model is retained for the present investigation. In the above equation, \( J_0 \) is the mass moment of inertia of the entire platform about the base including added mass effects, \( d \) is half the depth of the idealized tower, \( W_p \) and \( DW_T \) are the weight of the platform deck and the tower. Further, \( F_b \) is the buoyancy of the platform and \( z_T \) and \( z_b \) are the distances from the base of the centers of gravity and buoyancy of the tower alone, without the deck. The symbol \( z_{ca} \) denotes the distance from the base of the attachment point of the cables. All of the above distances are measured parallel to the tower centerline. The symbol \( F(t) \) is the total wave force acting on the tower and \( h \) is the center of action of this force. On
substitution of the nonlinear force displacement relation for the restoring forces, eqn. (7.3) can be rewritten as

\[ J_0 \ddot{\theta} + cd^2 \dot{\theta} + (z_{ca} c_1 - W z_{cg} - z_{ca} \dot{\epsilon}_1 + F_b z_b) \theta + z_{ca} (c_2 - \dot{\epsilon}_2) \theta^3 = F(t) \ h. \quad (7.4) \]

Note that the individual weights of the tower and the deck has been replaced by the combined weight, \( W \), and its center of action is \( z_{cg} \).

7.4.4. Excitation Model

Under realistic conditions, the total environmental force on the tower is due to wind, current and waves. In general, the waves are the primary source of dynamic excitation. Standard practice is to compute the mean offset of the tower due to the wind and the current. The dynamics of the tower is then analyzed about this mean offset position. For the present analysis only the effects of the waves are considered. The waves are considered random in nature specified by the elevation spectrum. Choice of the waves as the only excitation source is motivated by the desire to keep the model simple and to illustrate the use of the state space method with clarity.

The wave elevation is considered to be a random process, stationary in time and homogeneous in space. In addition, it is considered to be a zero-mean, Gaussian process, specified by the Pierson-Moskowitz spectrum. The wave field kinematics are assumed to follow the linear Airy Wave theory. The force on the structure is estimated by the use of the Morison equation. Using all of the above assumptions and the equivalent linearization technique for the non-linear drag term, an approximate model for the spectrum of the wave-induced moment about the base has been presented as [Kanegaonkar and Halder, 1987]

\[ \tilde{\delta}_{MM}(\omega) = \frac{G_1 \omega^2}{(\omega^2 - K_0)^2 + (C_0\omega)^2} \quad (7.5) \]
The same model for the excitation has been retained in the present investigation. This approximation for the spectrum of the wave induced moment is derived by minimizing the error between the true spectrum and the approximate spectrum. The coefficients $G_1$, $K_0$, and $C_0$ are the variables in the optimization process, the least square error being the objective function. The present optimization problem is solved using the Levenberg and Marquardt algorithm [Levenberg, 1944, Marquardt, 1963]. An exactly analogous model had been obtained by Spanos [1983] for approximating the Pierson Moskowitz spectrum, and again for the vertical propagation of wave kinematics [Spanos, 1986].

Note the specific choice of a rational function for the approximation. This ensures the existence of a shaping filter which can be used to produce this spectrum as its output to white noise input. Thus, the governing differential equation for the filter can be written as

$$\ddot{M} + c_0 \dot{M} + K_0 M = \sqrt{G_1} \dot{\hat{W}}(t) \ .$$ (7.6)

The fact that eqns. (7.5) and (7.6) are equivalent representations of the process $M(t)$, can be verified by taking the Laplace transform of eqn. (7.6) to obtain $H_M(s)$, and then computing $\hat{S}_{MM}$ by using the relation $\hat{S}_{MM}(\omega) = H_M(\omega)\overline{H}_M(\omega)$, where $\overline{H}(\omega)$ represents the complex conjugate of $H(\omega)$. The special interest in a rational function approximation of the spectrum is motivated by the relative ease of constructing the equivalent filter representation in the time domain. In the present case, a control Canonical form of representation can be obtained very easily.

### 7.5. State Space Equations

The governing equations of the dynamic system can now be written as

$$\ddot{\theta} + 2\beta_0 \omega_0 \dot{\theta} + \omega_0^2 \theta = \ddot{M}(t) \ .$$ (7.7)
\[
\ddot{M} + C_0 \dot{M} + k_0 M = \sqrt{G_0} W(t),
\]

where, \( \sqrt{G_0} = \sqrt{G_1/J_0} \) and \( \dot{M} = MJ_0 \). In the above, the first equation is obtained by dividing eqn. (7.4) throughout by \( J_0 \), and introducing an "equivalent linear stiffness," Appendix A, for the non-linear restoring force represented by the last two terms on the L.H.S. of eqn. (7.4). The random excitation \( \dot{M} \) is viewed as the output of another linear system excited by a Gaussian white noise process. The dynamic system subjected to a random excitation, when expressed in the above form, can be viewed as two dynamic systems in cascade. Alternatively, the second order dynamic system subjected to non-white excitation can now be represented by a higher order system subjected to a Gaussian white noise excitation. The system can now be easily recast in the form of a first order vector differential equation with white noise input. Thus, the theory of Markov vector processes is applicable, and an algebraic solution for the stationary response is directly obtainable by solving the covariance equation.

To this point, the present investigation has closely followed the model for the idealized tower proposed by Kanegaonkar and Haldar [1987]. From this point onwards the approach is different for the two investigations. In the above reference, a non-Gaussian closure technique is followed, resulting in differential equations for higher order moments. Thus, for the second, third and fourth moments, ten, twenty and thirty-five differential equations need to be solved, respectively. In contrast, the present investigation proceeds to solve for only the second order moments for the response of the "equivalent linear system". The logic of linearizing the excitation but retaining the non-linearity in the stiffness, as reported in the above reference is questionable. The present analysis proposes linearization at both stages for consistency. In the solution process, the present method provides a systematic demonstration of the covariance analysis procedure in the state space, by solving the Liapunov equation. The procedure for Markovianization of a non-
Markovian system subjected to a colored random inputs is also demonstrated. Differentiating equation (7.7) twice with respect to time, and substituting the results in eqn. (7.8), one obtains

\[ \Theta^{IV} + a_4 \Theta^{III} + a_3 \Theta^{II} + a_2 \Theta' + a_1 \Theta = \sqrt{G_0} \hat{W}(t) \]  

(7.9)

where the Roman superscripts I - IV represent the time derivatives of the corresponding order, and the coefficients \( a_i \) are given by the equations

\[ a_4 = 2 \beta_0 \omega_0 + C_0 \]

(7.10)

\[ a_3 = \omega_0^2 + K_0 + 2 \beta_0 \omega_0 \omega_0 \]

(7.11)

\[ a_2 = 2 \beta_0 \omega_0 K_0 + C_0 \omega_0^2 \]

(7.12)

\[ a_1 = K_0 \omega_0^2 \]

(7.13)

An attempt to solve eqn. (7.9) directly in the present form involves taking expectation of terms involving the derivative of the white noise process. That is, one needs to solve for terms like \( E \left[ \frac{d^n \Theta}{dt^n} \hat{W}(t) \right] \). To avoid this problem, a change of variables is made. Let eqn. (7.9) be integrated once with respect to time, yielding

\[ \Theta^{III} + a_4 \Theta^{II} + a_3 \Theta' + a_2 \Theta + a_1 \Theta dt + \hat{C} = \sqrt{G_0} W(t) \]

(7.14)

where, \( \hat{C} \) is a constant of integration. Now, define

\[ y = \frac{\hat{C}}{a_1} + \int \Theta \, dt \]

(7.15)

Substitution of eqn. (7.15) in eqn. (7.14) leads to

\[ Y^{IV} + a_4 Y^{III} + a_3 Y^{II} + a_2 Y' + a_1 Y = \sqrt{G_0} W(t) \]

(7.16)

which is the desired differential equation driven by white noise, as opposed to the derivative of white noise. A first order vector differential equation, in the "control canonical form" can be directly derived for eqn. (7.16). Let

\[ y_1 = y = \frac{\hat{C}}{a_1} + \int \Theta \, dt \]

(7.17)

\[ y_2 = \dot{y}_1 = \Theta \]

(7.18)
\[ y_3 = \dot{y}_2 = \dot{\theta} \]  
(7.19)

\[ y_4 = \dot{y}_3 = \ddot{\theta} \]  
(7.20)

\[ Y^T = [ y_1 \ y_2 \ y_3 \ y_4 ] \]  
(7.21)

Then, eqn. (7.16) can be rewritten as

\[ \dot{Y} = AY + G \]  
(7.22)

In its expanded form, this matrix equation is

\[
\frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a_1 & -a_2 & -a_3 & -a_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{G_0 W} \end{bmatrix} \]  
(7.23)

Note that in the above system of equations the variables with physical meaning are the angular displacement, velocity and acceleration, i.e. \( y_2, y_3, \) and \( y_4 \). No physical meaning can be attached to the state variable \( y_1 \). However, it should not be the cause of major concern. The state variable \( y_1 \) is the output of a system of two stable linear filters in cascade, excited by a zero mean Gaussian process. The output is also expected to be a zero mean Gaussian process. During the solution of the covariance equation, one needs to focus attention on the state variables of specific interest.

7.6. Second Moment Statistics

The response of a linear system subjected to a Gaussian excitation is also Gaussian. Therefore, the response statistics are completely specified by the first and second moments, or mathematical expectations. Thus, the mean vector \( \mu_Y \) and the covariance matrix \( C_{YY} \) contain all the necessary information about the system response.

7.6.1. General Procedure

The general solution of the system response is obtained directly in the state space. The
non-stationary response is usually obtained by writing the general solution of eqn. (7.22) in terms of the state transition matrix \( \Phi(t) = e^{At} \) and the initial state, \( Y(0) = Y_0 \). Thus, the stochastic response of the system described by eqn. (7.22) is given by

\[
Y(t) = \Phi(t) Y_0 + \int_0^t \Phi(t - \tau) G(\tau) \, d\tau .
\]

(7.24)

The non-stationary mean response of \( Y(t) \) in eqn. (23) in given by

\[
\mu_Y(t) = \Phi(t)\mu_0 ,
\]

(7.25)

and the non-stationary covariance matrix is given by

\[
C_{YY}(t) = \Phi(t) C_0 \Phi^T(t) + \int_0^t \Phi(t - \tau) Q \Phi^T(t - \tau) \, d\tau .
\]

(7.26)

In the above equations, \( \mu_0 \) is the mean of the initial state of the system, \( C_0 \) is the covariance of the initial state, and \( Q \) is the covariance of the excitation vector, \( G \). The last equation can be reduced to the following form [Muller and Schiehlen, 1987]

\[
C_{YY}(t) = \Phi(t) (C_0 - C) \Phi^T(t) + C
\]

(7.27)

where, the constant matrix, \( C \), satisfies the Liapunov equation given by

\[
AC + CA^T = -B
\]

(7.28)

and represents the stationary value of the covariance matrix \( C_{YY}(t) \) as \( t \to \infty \). Note that the above procedure derives the stationary statistics as a limit of the general non-stationary solution.

7.6.2. Direct Determination of Covariance

A relatively simple procedure to derive the above results directly is also available [Lin 1967, Spanos 1981]. Post multiplying eqn. (7.22) by \( Y^T \) and adding the resulting equation to its transposed version, one obtains, upon taking a mathematical expectation,

\[
\frac{d}{dt} C_{YY} = A C_{YY} + C_{YY} A^T + <GY^T + YG^T >
\]

(7.29)
In the above, $C_{YY}$ represents the symmetric covariance matrix of the state vector, $Y$, and $<>$ represents the expectation operator as before. The stationary value of the covariance is independent of time. Hence, the derivative term drops out, and $C_{YY}$ becomes a constant matrix in the stationary case. A procedure for evaluating the last term on the R.H.S. of eqn. (7.30) is outlined by Lin [1967] and more systematically derived by Spanos [1987]. Thus, the stationary solution is given by

$$AC + CA^T = -<GY^T +YG^T>$$  (7.30)

7.7. Stationary Response of Guyed Towers

The stationary response of the guyed tower is obtained as a solution of the Liapunov equation (7.30), where the matrix $A$ and vectors $Y$ and $G$ are specified by eqns. (7.22 - 7.24). The assumption of stationarity, combined with the fact that the process is Gaussian, permits a number of simplifications. For the present case where the number of state variables $y_i$ is 4, a close-form solution of the covariance matrix is easily obtained. For higher order systems with relatively larger number of states, a numerical computational scheme may prove more efficient [Muller and Schiehlen, 1987].

First, recall that the covariance matrix is symmetric. That is

$$C_{YY} = \begin{bmatrix}
<y_1^2> & <y_1y_2> & <y_1y_3> & <y_1y_4>\\
<y_2^2> & <y_2y_3> & <y_2y_4>\\
<y_3^2> & <y_3y_4>\\
sym
<y_4^2>
\end{bmatrix}$$  (7.31)

A number of terms of the above matrix can be simplified or eliminated by exploiting the properties of a stationary Gaussian process. Using the definition of the state variables as specified by eqns. (7.16 - 7.21), and the orthogonality relationship of a stationary Gaussian process with its
derivative process, the following simplifications for the non-diagonal terms can be obtained

\[ <y_1 y_2> = <y_1 y'_1> = \frac{1}{2} \frac{d}{dt} <y_1^2> = 0 \]  
(7.32)

\[ <y_1 y_3> = <y_1 y'_2> = \frac{d}{dt} <y_1 y_2> - <y_2^2> = - <y_2^2> \]  
(7.33)

\[ <y_2 y_3> = <y_2 y'_2> = \frac{1}{2} \frac{d}{dt} <y_2^2> = 0 \]  
(7.34)

\[ <y_2 y_4> = <y_2 y'_3> = \frac{d}{dt} <y_2 y_3> - <y_3^2> = - <y_3^2> \]  
(7.35)

\[ <y_3 y_4> = <y_3 y'_3> = \frac{1}{2} \frac{d}{dt} <y_3^2> = 0 \]  
(7.36)

\[ <y_1 y_4> = <y_1 y'_3> = \frac{d}{dt} <y_1 y_3> - <y_2 y_3> = 0 \]  
(7.37)

Thus, the covariance matrix simplifies to

\[ C_{yy} = \begin{bmatrix}
<y_1^2> & 0 & -<y_2^2> & 0 \\
0 & <y_2^2> & -<y_3^2> & 0 \\
-sym & -<y_3^2> & 0 & -<y_2^2> \\
< y_3^2 > & 0 & -<y_2^2> & < y_1^2 > \\
\end{bmatrix} \]  
(7.38)

When \( G_1 \) is the intensity of the excitation in the equation (7.11), the matrix on the R.H.S. of eqn. (7.30) can be shown to be, Appendix B,

\[ -<GY^T + YG^T> = - \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\pi G_1 \\
\end{bmatrix}. \]  
(7.39)

### 7.7.1. Solution of the Liapunov Equation

An interesting feature in obtaining response statistics via the state space method is that a set of algebraic equations need to be solved, instead of the integrals in the frequency domain. In addition, use of the symmetry of the covariance matrix and the null terms of the \( A \) matrix help to reduce the number of variables considerably in the Liapunov equation. The following long-hand solution illustrates the fact. For the present case, the Liapunov equation given by eqn.
(7.30) is

\[ AC_{YY} + C_{YY}A^T = -< GY^T + YG^T > \]

where matrix \( C \) is to be solved for. Inspection of eqn. (7.38) reveals that the 4X4 covariance matrix has only 4 non-zero variables that need to be solved for. That is the auto-covariances of the states \( y_1, y_2, y_3 \) and \( y_4 \). Also the 4X4 matrix on the R. H. S. of the above equation has only one non-zero entry. Thus, combining eqns. (7.22, 7.23, 7.30, 7.38 and 7.39) and rearranging the terms, one obtains,

\[
\begin{bmatrix}
-a_1 & a_3 & -1 & 0 \\
0 & -a_2 & a_4 & 0 \\
0 & -a_1 & -a_3 & 1 \\
0 & 0 & 2a_2 & -2a_4
\end{bmatrix}
\begin{bmatrix}
<y_1^2 > \\
<y_2^2 > \\
<y_3^2 > \\
<y_4^2 >
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
-2\pi G_0
\end{bmatrix}
\]

(7.40)

The solution to this matrix equation, where the \( <y_i^2> \) are the unknowns, are obtained by direct solution of the set of simultaneous linear algebraic equations. Thus, the final solution, in terms of the coefficients \( a_i \) and the intensity \( G_1 \) are given by the equations

\[
<y_1^2> = \frac{\pi G_0 (a_3a_4 - a_2)}{a_1(a_2a_3a_4 - a_2^2 - a_1a_3^2)}
\]

(7.41)

\[
<y_2^2> = \frac{\pi G_0 a_4}{a_2a_3a_4 - a_2^2 - a_1a_3^2}
\]

(7.42)

\[
<y_3^2> = \frac{\pi G_0 a_2}{a_2a_3a_4 - a_2^2 - a_1a_3^2}
\]

(7.43)

\[
<y_4^2> = \frac{\pi G_0 (a_2a_3 - a_1a_4)}{a_2a_3a_4 - a_2^2 - a_1a_3^2}
\]

(7.44)

7.8. Frequency Domain Solution

Solution in the frequency domain is obtained through the method of spectral analysis.

Given a linear dynamic system with frequency response function \( H(j\omega) \), and an excitation pro-
vided by $S_{FF}(\omega)$ the spectral matrix of the output is given by the equation

$$S_{XX}(\omega) = H(j\omega) \ S_{FF}(\omega) \ H^T(j\omega), \quad (7.45)$$

where the bar over the matrix $H$ represents complex conjugation, and the superscript $T$ represents the matrix transpose operator. For a single degree freedom system, eqn. (7.45) simplifies to a scalar equation,

$$S_{xx}(\omega) = H(j\omega) \ S_{ff}(j\omega) \ H(j\omega) = |H(j\omega)|^2 \ S_{ff}(\omega) \quad (7.46)$$

The standard deviation of the response is obtained by integrating the power spectrum of the response,

$$\sigma_{xx}^2 = \int_{-\infty}^{\infty} S_{xx}(\omega) \ d\omega. \quad (7.47)$$

Considering the governing equation of the guyed tower dynamics, eqn. (7.7) and the approximate model for the excitation spectrum, eqn. (7.6), the power spectrum of the response can be expressed as

$$S(\omega) = \frac{G_0 \ \omega^2}{\left[ (\omega_{o_c}^2 - \omega^2)^2 + (2 \ \beta_o \omega_{o_c} \omega)^2 \right] \left[ (K_0 - \omega^2)^2 + (C_0 \omega)^2 \right]} \quad (7.48)$$

To obtain the standard deviation of the response, one needs to integrate the above function with respect to $\omega$. In general, such operations involve contour integrations utilizing the theory of complex variables. However, for the special case of rational functions, the above integral can be obtained in closed form by algebraic operations [Gradshteyn and Ryzhik, 1965]. The results of the integration are summarized below.

$$I_n = \int_{-\infty}^{\infty} \frac{g_n(x)}{h_n(x) \ h_n(-x)} \ dx = \frac{\pi i}{d_0} \ \frac{M_n}{\Delta_n}, \quad (7.49)$$

where,

$$g_n(x) = b_0 \ x^{2n-2} + b_1 \ x^{2n-4} + \ldots + b_{n-1} \quad (7.50)$$
\[ h_n(x) = d_0 x^n + d_1 x^{n-1} + \ldots + d_n \]  
(7.51)

and \( M_n \) and \( \Delta_n \) are determinants of the coefficient matrices. Results are available for \( n = 1, 2, \ldots, 5 \) in the above reference. Referring to the specific case when \( n = 4 \), one obtains for \( g_4(x) \) by comparing eqn. (7.49-7.51)

\[ b_0 = b_1 = b_3 = 0 \]  
(7.52)

\[ b_2 = G_0 \]  
(7.53)

Similarly, by inspecting the numerator function,

\[ h_4(\omega) = d_0 \omega^4 + d_1 \omega^3 + d_2 \omega^2 + d_3 \omega + d_4. \]  
(7.54)

Comparing eqn. (7.54) with the Fourier transform of the L.H.S. of eqn. (7.14)

\[ d_0 = 1 \]  
(7.55)

and,

\[ d_1 = (-j)(C_0 + 2\beta_0 \omega_0) = (-j)a_4 \]  
(7.56)

\[ d_2 = -(K_0 + \omega_{0e}^2 + 2\beta_0 \omega_0 C_0) = -a_3 \]  
(7.57)

\[ d_3 = +j(2\beta_0 \omega_0 K_0 + C_0 \omega_{0e}^2) = (+j)a_2 \]  
(7.58)

\[ d_4 = K_0 \omega_{0e}^2 = a_1 \]  
(7.59)

The solution to eqns. (7.49-51), where the coefficients are given by eqn. (7.52-59), is expressed as

\[ I_4(\omega) = \frac{\pi j}{d_0 (d_0 d_3^2 + d_1^2 d_4 - d_1 d_2 d_3)} \begin{bmatrix} b_0 \left(-d_1 d_4 + d_2 d_3\right) - d_0 d_3 b_1 + d_0 d_1 b_2 + \frac{d_0 b_3}{d_4} \left(d_0 d_3 - d_1 d_2\right) \end{bmatrix} \]  
(60)

By virtue of the fact that \( b_0 = b_1 = b_3 = 0 \), the numerator reduces to \( \pi G_1 (C_0 + 2\beta_0 \omega_0) \).

Comparison of the numerators obtained by the two procedures is best performed by substituting the coefficients \( d_k \) in eqn. 35 by the corresponding \( a' \)s from eqns. (7.34a-e):

\[ \Delta_4 = d_0 (d_0 d_3^2 + d_1^2 d_4 - d_1 d_2 d_3) \]

\[ = (1) \left[(1)(j a_2)^2 + (-j a_4)^2 a_1 - (-j a_4)(-a_3)(j a_2)\right] \]

\[ = \left[a_2 a_3 a_4 - a_2^2 - a_1 a_4^2\right] \]  
(7.61)
Thus, the present result compares exactly with the denominator obtained in eqn. (7.43), indicating the equivalence of the state space method with the conventional frequency domain method.

7.9. Numerical Example

The present section illustrates the use of the covariance approach by solving a numerical example, for an idealized guyed tower. The selected system parameters are those reported by Kanegaonkar and Halder [1987], with a few exceptions. The mooring system has been changed to 20 lines instead of 16. Similarly, the mass moment of inertia of the present example tower is considerably greater, in order to have a reasonable value for the natural period. The particulars, Fig 7.1, are listed below.

7.9.1. Tower Particulars

Water depth; \( D_0 = 1425 \text{ ft} \)
Height of guyline attachment; \( Z_{ca} = 1300 \text{ ft} \)
Height of platform deck; \( D = 1500 \text{ ft} \)
Weight of platform deck; \( W_p = 20,000 \text{ Kips} \)
Unit weight of tower; \( W_T = 4.25\text{Kips/ft} \)
Mass moment of inertia of tower
(\text{including added mass}); \( J_0 = 2.176 \times (10)^{10} \text{kips-sec}^2 \)
Nett buoyancy of modules; \( F_B = 12,000 \text{ Kips} \)
Point of action of buoyancy; \( Z_B = 1,000 \text{ ft} \)
Equivalent Diameter of tower; \( D_e = 60 \text{ ft} \)

7.9.2. Mooring System

The particulars of the mooring system are summarized below. The total number of guylines was changed from 16 to 20. This was done to obtain a system stiffness comparable to other publications.
Length of catenary line; \( L_1 = 3560 \text{ ft} \)
Unit weight of catenary line; \( \omega_1 = 40 \text{ lb/ft} \)
Length of distributed clump-weight; \( L_2 = 140 \text{ ft} \)
Unit weight of clump-weight; \( \omega_2 = 1920 \text{ lb/ft} \)
Length of trailing line to anchor; \( L_3 = 6900 \text{ ft} \)
Unit weight of trailing line; \( W_3 = 40 \text{ lb/ft} \)

The mooring system pretension was selected to be 570 kips. This results in an initial active catenary length of 3,552 feet. The associated total vertical force from the mooring system is 5,340 kips. The force-displacement relationship for the entire system is obtained as a vectorial sum of the contributions from the individual mooring lines. These are obtained through a numerical solution of the multi-segment catenary equations. The governing equations involve hyperbolic terms and are difficult to solve without the help of a computer.

The variation of the total restoring force exhibits a weak form of non-linearity. Noting that the horizontal restoring force is antisymmetric, changing sign at origin, and that the vertical restoration force is symmetric about the origin, the approximating polynomials are chosen accordingly. The variation of the line tension, and the total horizontal and vertical force with horizontal excursions of the attachment point is shown in Fig. 7.2. Note that at zero displacement, the total horizontal force is zero because of the radial symmetry. In contrast, the vertical force and the line tensions are non zero.

The assumption of radial symmetry is also verified by comparison of the horizontal restoring force for excursions in directions of 0, 5 and 10 degrees. The difference in the restoring force for these three directions is almost negligible Fig. 7.3.

A regression analysis is performed for the restoring force over the range of 0 to 120 feet of horizontal excursion to obtain the coefficients for the polynomial approximation. This results in the expression for the vertical restoring force as
\[ R_Z = 5340 + 0.0191068 X^2. \]  \hspace{1cm} (7.62)

Similarly, the total horizontal restoring force is obtained as

\[ R_X = 36.7347919 X - 0.0002121 X^3. \]  \hspace{1cm} (7.63)

The comparisons of the approximating polynomial with the target function is shown in Fig. 7.4 -7.5. The magnitude of the error at any point on the curve is less than 0.3%.

7.9.3. Excitation Modeling

The response analysis of a system subjected to non-white excitation requires a filter representation for the "colored noise" excitation. Such filter representations are very efficiently obtained by using non-linear optimization routines for least square approximations. A filter representation for the Pierson Moskowitz spectrum was provided by Spanos [1983]. Similar filter representation for vertical propagation of wave kinematics was also considered by Spanos [1986]. Earlier in this investigation, Chapter 5.5.2 a similar filter approximation was successfully derived for the all-pass filters for horizontal propagation. The numerical algorithm for this approximation is due to Levenberg [1944] and Marquardt [1963].

The above technique has been subsequently used by Kanegaonkar and Haldar [1987] for the approximation of the spectrum of the total overturning moment of an idealized guyed tower model. They have used the Morison equation with equivalent linearization of the drag term. The coefficients for the filter approximation, eqn. (7.5), are presented for wind velocities of 30, 40, 70 and 80 feet/sec.

The present investigation uses the same model for the excitation. To enable this, the tower dimensions, and water depth have been selected to be the same. However, some departure from the system described in the above reference was necessary. Calculations of the stiffness based on their description of the mooring system resulted in unrealistic values, which did not compare
with their numbers. Subsequently, checks with the results presented by Wilson [1984], for an almost identical system provided an independent verification. The present investigator feels that some computational error has been made in the stiffness computation by Kanegaonkar and Haldar [1987]. The system restoring force and inertia being different, the responses obtained in the present investigation does not compare exactly with the above reference. The same trend in the response is noticed, however. Table 7.1 summarizes the response statistics for the deck displacement of the tower for different sea states. The detailed calculations for each of these cases are provided in Appendix C.
Table 7.1 Filter Parameters and Response Statistics

<table>
<thead>
<tr>
<th>Wind (ft/s)</th>
<th>$H_s$ (ft)</th>
<th>$K_0$</th>
<th>$C_0^2$</th>
<th>$G_1$</th>
<th>$\sigma_{X_0}$ (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>5.81</td>
<td>0.839552</td>
<td>0.1077623</td>
<td>0.442 (10)$^{11}$</td>
<td>1.45</td>
</tr>
<tr>
<td>40</td>
<td>10.38</td>
<td>0.5173579</td>
<td>0.8762989</td>
<td>0.168 (10)$^{12}$</td>
<td>4.45</td>
</tr>
<tr>
<td>70</td>
<td>31.84</td>
<td>0.2055000</td>
<td>0.06171161</td>
<td>0.141 (10)$^{13}$</td>
<td>29.73</td>
</tr>
<tr>
<td>80</td>
<td>41.58</td>
<td>0.1642096</td>
<td>0.0415229</td>
<td>0.155 (10)$^{13}$</td>
<td>39.02</td>
</tr>
</tbody>
</table>

$H_s$ = significant wave height for the given sea state  
$\sigma_{X_0}$ = standard deviation of (horizontal) deck displacement

It must be mentioned at this stage that depending on the form assumed for the Morison equation, the response can be coupled with the excitation. Thus, if the interactive form of Morison equation is used, the excitation will depend on the response. In this case, the excitation cannot be modeled separately and an iterative scheme needs to be adopted. However, if the non-interactive form of Morison equation is used, then the response and excitation can be separated. The equivalent linearization for these two parts can be done independently of each other.

The present investigation assumes the non-interactive form of the Morison equation. Equivalent linearization of the nonlinearity in the stiffness requires an iteration to obtain the response. Typically, convergence of the response statistics is obtained after three to four iterations, for all the four cases investigated here.
7.10. Conclusions

The successful use of the state space method for the solution of the predominant motion of the guyed tower has been demonstrated. The advantages of this method, so powerfully used in several other fields, is emphasized. The present chapter also provides a clear example of the concept of state augmentation to represent colored random excitation. Though the present investigation limits itself to the stationary response statistics alone, the method is easily extended to the case of non-stationary response analysis.

The complete solution for an example guyed tower is provided. The response statistics are computed for different levels of excitations dictated by the wind. Wind velocities in the range of 30 to 80 ft/sec are considered here. The results are compared with similar cases from published literature, and are found to compare favorably. The suitability of the method for digital computation via the state transition matrix, and its readiness for implementing control algorithms, strongly indicates the potential of the method to solve a variety of dynamic problems.
CHAPTER VIII
State Space Analysis Concepts For TLP Dynamics

8.1. Overview

The present chapter investigates the feasibility of applying the state space method to the
dynamic analysis of the tension leg platform (TLP). The background and development of the
TLP concept is first outlined. This is followed by a brief review of some of the key references on
the topic.

The simplified model selected for the present analysis is an idealized, 3 degree of freedom
(DOF) TLP in unidirectional random sea. The motions being considered are confined to a verti-
cal plane. The excitation force is computed through the Morison equation. The random sea is
characterized by the Pierson Moskowitz spectrum. The primary focus of the chapter is on the
wave load modeling for the TLP in the context of the state space analysis. Some of the
difficulties which need to be resolved before the method can be applied to general problems are
discussed. Finally, the results of the investigation are summarized and areas requiring further
research identified.

8.2. Introduction

The development of the TLP concept and its successful design and operation in the Hutton
Field, North Sea, is undoubtedly one of the landmarks in the offshore industry. The concept
holds great promise for deep waters and for marginal fields. It is a unique concept in which the
buoyancy force exceeds the weight of the platform. The platform is held on location by a set of
vertical taut mooring lines, Fig. 8.1. These mooring lines, or tethers, are designed to have a ten-
sile force under all operating conditions, hence the name "tension leg platform".
Like the guyed tower, the TLP is a compliant platform with large horizontal motions in waves. However, unlike other buoyant platforms such as the semisubmersible or the conventional ship form, the vertical motions are restrained by the tethers. The primary advantage of the TLP over the guyed tower is that the great mass of the supporting under water jacket structure is replaced by the tethers, and is practically eliminated. This brings about significant savings in the structural cost.

The platform consists of an above-water deck supported by surface piercing columns, which in turn are connected to under-water pontoons or hulls. The configuration provides ample opportunities for optimizing the design by careful choice of the geometric parameters, and the tether pretensions.

8.3. Literature Survey

The TLP has been extensively investigated over the last two decades. Publications on the topic are numerous, and no attempt is made to review all the articles. Instead, only a few representative articles are reviewed here.

Some of the aspects of the TLP that have received considerable attention are the global design of the platform geometry, the dynamic response of the TLP to environmental loads, and response analysis of the tethers to the combined action of the environment and platform motion.

One of the earliest descriptions of the TLP concept is provided by Paulling and Horton [1970]. Investigations on the platform geometry including parametric studies are reported by Angelides, Chen and Will [1982]. The design experience for the successfully installed Hutton TLP is provided by Mercier and Leverette [1982].

The TLP hydrodynamics problem is addressed by Wybro [1980], Kirk and Etok [1979], and
Faltinsen, Fylling and Van Hooff [1982]. Verification of the theory with suitable model tests is reported by Faltinsen et. al. [1982], Yashima [1976], and Lyons and Patel [1984].

The dynamic response of the TLP to environmental loads has been investigated by a large body of researchers. The linear analysis in the frequency domain is reported by Kirk and Etok [1979], Botelho et. al. [1984]. Non-linear analysis in the time domain is investigated by Finnigan et. al. [1984] and Albrecht and Koenig [1978]. Slow drift motion of the TLP, a second order effect of the waves, which plays an important role in TLP dynamics, is investigated by DeBoom, Pinkster and Tan [1983], and Faltinsen et. al. [1982]. Denise and Heaf [1979] provide a comparison of the linear with the non-linear analysis. Spanos and Agarwal [1984] have investigated the non-linearity resulting from the computation of the force at a displaced position.

The dynamic response of the TLP tethers is investigated by Larsen [1984], Patel and Sarohia [1984], and Patel and Lynch [1984]. High frequency response of the tethers is investigated by Beynet et. al. [1984], and Nordgren [1987], with special emphasis on tether fatigue.

Parametric response of the TLP, resulting from time dependent tension variation in the tethers is extensively treated by Rainey [1977], and has subsequently been further investigated by Strickland and Mason [1981], and Muhuri and Gupta [1983].

Motion control of the TLP response in order to reduce tether fatigue is investigated by Nordgren [1987], and Kitami [1982]. Both of them have explored the feasibility of using an artificial damper at the connection of the hull and tethers. In contrast, the concept of an active control device to reduce TLP surge response has been proposed by Reinhor and Manolis [1987]. The first two investigator use a frequency domain approach to implement passive control in heave motion. The investigation by Reinhor et. al. focuses on control algorithm for non-linear structures and selects the surge response of the TLP for an example application, along with other civil
engineering structures.

The present investigation focuses on the feasibility of the state space modeling of the TLP response, considering the surge, heave and pitch motions in an undirectional random sea.

8.4. Excitation Forces On TLP

The following section presents the excitation forces on an idealized 3 degree of freedom TLP, subjected to unidirectional random sea. The degrees of freedom being considered are surge, heave and pitch, i.e. the motions in a vertical plane. The excitation forces are derived in the frequency domain, in the from of Response Amplitude Operators (RAO). This is defined as the response of the TLP, corresponding to a particular degree of freedom (DOF), to a wave of unit height and monochromatic frequency, ω. This method of analysis depends on the assumption of system linearity.

The mathematical derivation centers primarily around the computation of inertia related forces or dynamic pressure effects. The drag force is neglected from the excitation. This is a reasonable approximation as the drag force is expected to be only a very small fraction of the total excitation force. These large diameters fall primarily in the “inertia force regime” in the Morison model. Also, the natural frequencies of the TLP corresponding to all DOFs fall well outside the predominant wave excitation. For example, the heave, pitch and roll natural periods are of the order of 3 seconds, and the surge, sway and yaw natural periods are of the order of 50 seconds or more. Typical wave spectral peak periods are between 6 to 15 seconds. Thus, both groups of motion responses fall outside the wave spectrum period range, the former group at the lower end, and the latter group on the upper end of the energy spectra plotted against wave periods. Consequently, direct resonance, a situation where damping effects control the response are unlikely.
Hence, the overall penalty of not incorporating damping is considered insignificant.

The expressions for the RAO's are derived from first principles, and verified with final expressions presented by Kirk and Etok [1979]. To make the task of verification simple, the notations are kept as close as possible to the above reference.

There are several assumptions made in deriving the mathematical model for the TLP. Thus, the wave forces are calculated at the static equilibrium position, as opposed to the instantaneous position [Spanos and Agarwal, 1983]. The effects of wave diffraction and sheltering are negligible in calculating wave forces. Current and Wave induced drag forces in Morison's equation are taken to be negligible in dynamic response calculations. Tension variation effects in the tethers are also considered negligible. The above DOF's are considered uncoupled. The theory of linear waves in deep water is used to compute the forces. Integration of fluid inertial forces on the surface piercing columns is carried out to the still water level instead of the instantaneous free surface. The waves are considered long-crested and unidirectional. The direction of wave propagation coincides with one of the TLP axes.

Typical TLP designs attempt to have a zero in the RAO around the spectral peak frequency at site. This is brought about by proper optimization of the dimensions of the hulls and columns to have a cancellation of vertical forces around a selected frequency. Thus, the tension variations produced by design spectrum frequencies will be small compared to the pretension.

Justification for the uncoupling of the excitation forces are provided by Kirk and Etok [1979]. It is claimed that heave and pitch uncoupling is fully justified for a TLP. However, surge and pitch are in general coupled. Fortunately, the degree of coupling is small when the wave period, $T$, is less than 25 seconds, justifying the assumption of uncoupling at wave frequencies.
8.4.1. Surge Force

The horizontal surge force on the TLP arises from fluid acceleration forces in the horizontal direction. Contributions to the surge force are from horizontal forces on the hulls, as well as the columns, Figure 8.2.

8.4.1.1. Surge Force on Columns

The surge force on a single column, located at \( x_i \) from the wave crest, is obtained by integrating the horizontal particle acceleration from the column base to the SWL. Let the horizontal particle acceleration at location \( y \) below the SWL be \( \dot{U}(\omega; y) \). Then, the force on a single column is

\[
F_i = a_i \int_{-h}^{0} \dot{U}(\omega; y) \, dy,
\]

where

\[
a_i = \frac{\pi}{4} \rho C_m D_i^2,\]

and \( \rho \) is the density of water, \( C_m \) the inertia coefficient is taken to be 2.0, and \( D_i \) is the diameter of the ith column. Thus, the total horizontal force on all four columns is

\[
F_s = 4 \, b_i \, H \left[ 1 - e^{-\kappa h} \right] \cos(\kappa a) \sin(\omega t),
\]

where \( b_i = a_i \, g/2 \). The above expression for horizontal force on the columns is similar to that derived by Spanos and Agarwal [1983] except for the term involving \( \cos(\kappa a) \). This term results from the horizontal separation of the columns not included in the above reference, and introduces zeros in the RAO function.
8.4.1.2. Surge Force on Hulls

Horizontal inertia forces are experienced only by the hulls aligned normal to the direction of wave propagation. Thus, surge forces arise only from wave action on hulls 3 and 4. Let the vertical location of the hull centerline be \( h_c \) from the SWL. Then, the horizontal force on the hull located at \( x = x_i \) is

\[
F_{H_i} = \int_{-b}^{b} a_x \omega^2 \sin(\kappa x_i - \omega t) \, dy ,
\]

where,

\[
a_x = \frac{\pi}{8} \rho C_m D_h^2 e^{-kh}.
\]

On carrying out the integration and summing the forces on both hulls, located at \( x_i = \pm a \), one obtains

\[
F_{H_k} = a_x \left( 2 \hat{b} \right) \omega^2 \left[ -2 \cos(\kappa a) \sin(\omega t) \right]
\]

Collecting the frequency dependent terms leads to the equation

\[
F_{H_k} = -\frac{\pi}{2} \rho C_m D_h^2 \hat{b} H \left[ \omega^2 e^{-kh} \cos(\kappa a) \right] \sin(\omega t).
\]

8.4.1.3. Total Surge Force

Combining the surge force on the hulls and the columns the total surge force is given by the equation

\[
F_x = F_c + F_{H_k} = -\left( \frac{\pi}{2} \rho C_m \right) H \left[ g D_h^2 \left( 1 - e^{-kh} \right) + \hat{b} D_h^2 \omega^2 e^{-kh} \right] \cos(\kappa a) \sin(\omega t)
\]

Thus, the surge transfer function, defined as surge force amplitude per unit wave amplitude is given by

\[
|H(\omega)|_x = -\pi \rho C_m \left[ g D_h^2 \left( 1 - e^{-kh} \right) + \hat{b} D_h^2 \omega^2 e^{-kh} \right] \cos(\kappa a).
\]
A qualitative analysis of the transfer function yields some insight into its behavior. Consider the 3 sets of frequency dependent terms two inside the parenthesis and one outside. Their behavior with increasing value of \( \omega \) are as follows. The first term decays asymptotically from 1 to 0. The second term tends to zero for \( \omega \to 0 \) by virtue of the \( \omega \) term. For large values of \( \omega \), on the other hand, the exponential term dominates and the product tends to zero again. The function reaches a maximum, somewhere in between. The third term is oscillatory in nature, and introduces zeros in the transfer function and changes the sign of the magnitude function. The zeros occur for \( \kappa a = (\pi/2)(2n + 1) \). Thus, the overall behavior of the transfer function can be expected to be somewhat like a decaying oscillation with amplitude modulation.

Some attempt is made at the design stage to derive advantage from these zeros in the transfer function. By careful selection of the platform geometry, the first zero can be chosen to coincide with predominant wave frequency at the site.

The phase of the transfer function can be obtained by inspection by substituting \( x = 0 \) in the expression for wave phase, and comparing with the surge excitation force. Thus, a wave input of \( \cos(\omega t) \) causes an output in the form of \( |H(\omega)|\sin(\omega t) \) indicating a phase shift of \( \pi/2 \). Thus, the excitation can be written as

\[
H(\omega) = |H(\omega)|\cos(\omega t + \pi/2)
\]  

for an input of the form \( \eta(t) = \cos(\omega t) \)

8.4.2. Heave Excitation

The vertical wave force on the TLP results from several effects. They are the effect of the dynamic pressure at the bottom of the columns, the inertia forces on the hulls obtained by integrating the dynamic pressures around the submerged hulls, and finally, the effect due to the
change of the instantaneous waterline on the columns. This effect is usually neglected when using the linearization implied by the small amplitude and small slope assumption, in the wave equation.

8.4.2.1. Column Force

The dynamic pressure at the bottom of the i-th column is obtained from the expression for the wave potential

\[ p_i(t) = \rho \frac{d\theta}{dt} \bigg|_{y=k} = -\rho \frac{g}{2} \frac{H}{e^{-\kappa h_c}} \cos(\kappa x_i - \omega t). \]  

(8.9)

With the further assumption that the column diameter is small compared to the wave length, so that the pressure can be assumed to be constant across the bottom surface, the vertical column force is

\[ F_c = \sum_{i=5}^{8} A_i p_i = \frac{\pi}{4} \sum_{i=5}^{8} p_i D_i^2 = \frac{\pi}{4} \rho \frac{g}{2} \frac{H}{e^{-\kappa h_c}} \sum_{i=5}^{8} D_i^2 \cos(\kappa x_i - \omega t). \]  

(8.10)

Using the identity

\[ \cos(A - B) - \cos(A + B) = 2 \cos A \cos B, \]

one obtains, on substituting \( x_i = \pm a \)

\[ F_c = \frac{\pi}{2} \rho \frac{g}{2} H D^2 \left[ e^{-\kappa h_c} \cos(\kappa a) \right] \cos(\omega t). \]

(8.11)

8.4.2.2. Vertical Hull Force

The vertical component of fluid acceleration at the center line of the submerged hulls is

\[ \dot{V} = -\omega^2 \frac{H}{2} e^{-\kappa h_c} \cos(\kappa x_i - \omega t). \]  

(8.12)

The magnitude of fluid acceleration on hulls 3 and 4 is constant along the hull length and the integration reduces to a simple multiplication. The force on each hull is
\[ F_{H} = \left( \frac{\pi}{2} \rho C_m D_h^2 \right) (2 \delta) \left[ -\frac{\omega^2}{2} \frac{H}{b} e^{-\delta_1} \cos(\kappa x_i - \omega t) \right] \]  

(8.13)

On substituting \( x_i = \pm a \), and summing up the forces on hulls 3 and 4, one obtains

\[ F_{H_{c_{3,4}}} = -\frac{\pi}{2} \rho C_m D_h^2 \delta H \left( \omega^2 e^{-\delta_1} \right) \cos(\kappa a) \cos(\omega t). \]  

(8.14)

Similarly, the vertical force on hulls 1 or 2 is given by the equation

\[ F_{H_{c_{1,2}}} = -\left( \frac{\pi}{4} \rho C_m D_h^2 \right) \left[ \omega^2 \frac{H}{2} e^{-\delta_1} \right] \int_{-d}^{d} \cos(\kappa x - \omega t) \, dx. \]  

(8.15)

Carrying out the integration yields

\[ F_{H_{c_{1,2}}} = -\left( \frac{\pi}{4} \rho C_m D_h^2 \right) \left[ \omega^2 \frac{H}{2} e^{-\delta_1} \right] \left( \frac{2}{\kappa} \sin(\kappa a) \sin(\omega t) \right). \]  

(8.16)

Invoking the wave dispersion equation for deep water, and summing the forces on the two hulls, one obtains

\[ F_{H_{c_{1,2}}} = -\left( \frac{\pi}{2} \rho g C_m D_h^2 \right) \left[ \frac{H}{2} e^{-\delta_1} \right] \sin(\kappa a) \cos(\omega t). \]  

(8.17)

8.4.2.3. Total Heave Force

The total heave force on the TLP, obtained by summing all the hull and column forces is given by the equation

\[ F_v = \pi \rho \frac{H}{2} \left[ A - B \right] \cos(\omega t), \]  

(8.18)

where,

\[ A = g D_c^2 \left\{ e^{-\delta_1} \cos(\kappa a) \right\} \]

and

\[ B = C_m D_h^2 \left\{ \beta \omega^2 \cos(\kappa a) + \frac{g}{2} \sin(\kappa a) \right\} e^{-\delta_1}. \]
Note that the heave force is in phase with the wave elevation. However, the magnitude depends on opposing contributions from the columns and the hulls. Inspection reveals that under a crest the heave force on the column is upwards while that on the hulls is downwards. This downward force on a submerged body located under the crest is somewhat misleading. This contribution arises only from the dynamic pressure effects. The static pressure contribution on a fully submerged body, on the other hand, is insensitive to the passage of wave. The net upward force due to static pressure is our familiar buoyancy force according to Archimedes Principle.

8.4.3. Pitch Response

The pitching behavior of a TLP is somewhat different from that of a free floating body like a ship or a semi-submersible. In the latter case the restoring moment is provided by the action of emerged and immersed volumes around the water plane. In contrast, for a TLP, the pitch restoring moment is provided primarily by changes in tether tension due to elastic deformation. This effect outweighs, by far, the contributions due to changes in column submergence. In a strict sense, the pitching axis is one about which moment of all the forces (including elastic tether deformation) are zero. The tether deformation, in turn, depends on the location of the pitching axis. Thus, explicit solution for the location of the pitch axis is not possible. Here, it would be assumed that the pitch axis is at the level of the connection of the tethers to the columns [Kirk & Etok 1979]. The contributions to pitching moment come from four sources. Firstly, there is the horizontal acceleration on all vertical columns. Secondly, there is the horizontal acceleration on hulls aligned normal to the direction of wave propagation. Thirdly, there is the vertical acceleration of the wave particles on hulls 1, 2, 3, & 4. Finally, there is the dynamic pressure variation on the bases of the four corner columns.
8.4.3.1. Columns Force Effects

The horizontal inertia force on the i-th column, multiplied by the lever distance from axis through \( K \)

\[
M_c = a_i \omega^2 \frac{H}{2} \sin(\kappa x_i - \omega t) \int_{-h}^{0} \left( y + h_i \right) e^{xy} dy,
\]

(8.19)

where, \( a_i = \frac{\pi}{4} \rho C_m D_i^2 \). On substituting \( \omega^2 = g \kappa \), and summing the moments arising from all four columns, one obtains

\[
M_c = -H \left[ \frac{\pi}{2} \rho g C_m D_i^2 \right] \left[ \left( \frac{h_i}{\kappa} - \frac{1}{\kappa} \right) + \frac{e^{-xh_i}}{\kappa} \right] \cos(\kappa y) \sin(\omega t)
\]

(8.20)

The horizontal acceleration on hulls 1 and 2 has no contribution to the pitch moment. However, the horizontal force on hulls 3 and 4 produces a pitch moment. This is obtained by multiplying the horizontal force with the lever, measured form the pitching axis. Thus, the pitching moment from hulls 3 and 4 combined is given by the equation

\[
M_{H(3+4)} = -\frac{\pi}{4} H b \rho C_m D_h^3 \omega^2 e^{-xh} \cos(\kappa y) \sin(\omega t)
\]

(8.21)

8.4.3.2. Vertical Acceleration Effects

First, consider hulls 1 and 2, aligned along the direction of wave propagation. The pitching moment on an infinitesimal strip of length \( dx \), located at a distance \( x \) from the origin is given by

\[
dM_{H(1,2)} = \left[ \frac{\pi}{4} \rho C_m D_h^2 \right] \left[ -\frac{H}{2} \omega^2 e^{-xh} \right] \int_{-d}^{d} x \cos(\kappa x - \omega t) dx
\]

(8.22)

On carrying out the integration by parts, and combining the contributions from both the hulls, one obtains

\[
M_{H(1,2)} = \pi \rho C_m D_h^2 H \omega^2 e^{-xh} \left[ \frac{\cos(\kappa d)}{\kappa^2} - \frac{\bar{d} \cos(\kappa d)}{\kappa} \right] \sin(\omega t)
\]

(8.23)
Note that the maximum occurs when the crest has a phase of $\pi/2$. That is, the zero-crossing point of the wave profile is at the centerline of the TLP. Thus, the pitch moment is maximum when the wave profile assumes an antisymmetric position about the TLP centerline.

Next, consider the effect of the vertical accelerations on hulls 3 and 4. These hulls are parallel to the crestline, and the force per unit length is constant along the length of the hull. The pitch moment is obtained by multiplying the vertical force, derived for heave excitation, by the lever distance form the centerline. Noting that the length of each hull is $2b$ and that the two hulls are located at $\pm a$ from the centerline, the combined contribution to the pitch moment is

$$M_{Z(H3, H4)} = \left( \pi \rho C_m D_h^2 \right) \left( a b \omega^2 H e^{-\kappa h} \right) \sin(\kappa a) \sin(\omega t). \quad (8.24)$$

Again, this moment has the maximum value when $\omega t = \pi/2$. That is, for the antisymmetric position of the wave profile with respect to the TLP centerline.

8.4.3.3. Dynamic Pressure Contributions

The exposed base of the corner columns of the TLP experience a hydrostatic pressure which has no contributions to the pitch moment, when the integration is taken to the still waterline. However, there is a dynamic pressure variation due to the passage of the waves, whose net contribution is non-zero. The dynamic part of the pressure is given by, see equation (2.25),

$$p_d = -\frac{1}{2} \rho g H e^{-\kappa h} \cos(\kappa x - \omega t) \quad (8.25)$$

The associated pitching moment arising from this contribution is given by

$$M_{p_d} = -\rho g \left( \frac{\pi}{4} D_c \right) e^{-\kappa h} \sum_{i=5}^{8} (-1)^i a \cos(\kappa x_i - \omega t), \quad (8.26)$$

where, $a$ is the lever distance, and $\xi$ assumes the value of $\pm a$ depending on the column of interest. On expanding the summation term and simplifying, one obtains,
\[ M_p = \rho g \frac{H}{2} \pi D_c^2 a e^{-\kappa h} \sin(\kappa z) \sin(\omega t) \]  
(8.27)

### 8.4.3.4. Total Pitch Moment

Thus, the total pitch moment arising from all sources is

\[ M = \left( C_1(\kappa) + C_2(\kappa, \omega) \cos(\kappa z) + C_3(\kappa, \omega) \cos(\kappa d) \right. \]
\[ \left. + C_4(\kappa, \omega) + C_5(\kappa) \sin(\kappa z) \right) \sin(\omega t), \]  
(8.28)

where,

\[ C_1(\kappa) = -\left( \frac{\pi}{2} \rho g C_m D_c^2 \right) \left( h_i - \frac{1}{\kappa} + \frac{e^{-\kappa h}}{\kappa} \right) \]  
(8.29)

\[ C_2(\kappa, \omega) = -\frac{\pi}{4} \rho C_m b D_h^2 \omega^2 e^{-\kappa h} \]  
(8.30)

\[ C_3(\kappa, \omega) = \pi \rho C_m D_h^2 e^{-\kappa h} \left( \frac{1}{\kappa} - \hat{a} \right) \]  
(8.31)

\[ C_4(\kappa, \omega) = (pi \rho C_m a \hat{b} D_h^2) \left( \omega^2 e^{-\kappa h} \right) \]  
(8.32)

\[ C_5(\kappa) = -\frac{\pi}{2} \rho g a D_c^2 e^{-\kappa h} \]  
(8.33)

### 8.5. Approximation for the Excitation

The transfer functions derived in the previous section relate the wave elevation to the TLP excitation. In terms of the three degrees of freedom being considered in the present analysis, the excitation force spectrum on the TLP is

\[ S_{FF}(\omega) = H(j\omega) S_{\eta\eta}(\omega) H^T(-j\omega). \]  
(8.34)

In this equation, \( H(j\omega) \) is the complex transfer function matrix of order 3 x 1, \( H^T(-j\omega) \) is the complex conjugate of \( H(j\omega) \) transposed, and \( S_{\eta\eta}(\omega) \) is the wave elevation spectrum. The resulting excitation \( S_{FF}(\omega) \) is the 3 x 3 spectral matrix of the excitation force. The excitation is a vector of order 3, and it is generated by the effects of the univariate random process, the sur-
face wave elevation. This permits a slightly different approach to the problem. The approximation to the spectral matrix, by stochastic realization, can be replaced by the approximation to the vector transfer function \( H(j\omega) \sqrt{S_{\eta\eta}(\omega)} \). Denoting this transfer function as \( H_F(j\omega) \), a filter representation is sought for this vector transfer function. Note, that if the phase and amplitude of each component of the vector is properly approximated, the filter realization will produce an exactly equivalent representation. However, if the terms are approximated in magnitude alone, the resultant representation will generate a spectral matrix that will have correct auto-spectral terms, the cross-spectral, or the off-diagonal terms, will have the proper magnitude but not necessarily the proper phase. The motivation for this approach is the enormous saving in computational effort. Instead of the general equation for the multi-input-multi-output systems, a set of equivalent single input and single output systems can be analyzed. If such a representation is possible then the remaining solution procedure is exactly identical to the guyed tower response.

Comparison of the surge motion with the heave and pitch indicates that surge natural period is one order of magnitude greater than heave and pitch. Also, the effect of the heave and pitch is to cause vertical motions of high frequency and that of surge is to cause horizontal motions of low frequency. Thus, the inter-relationship of the phase between heave and pitch is more important than that between surge and the other two motions. This importance is purely in terms of the effects caused by the motions. Consequently, the components of the \( H_F(j\omega) \) matrix are examined in an attempt to obtain economic representations with an acceptable approximation error. The variation of the transfer function with frequency is an approximate indicator of the filter order required. This is done for various sea state conditions and similar trends are noticed. The response amplitude operator (RAO) for the various degrees of freedom for different frequencies is presented in Figures 8.3 to 8.5. For each degree of excitation the contributions from the columns
and hulls are presented, along with the total force.

The plots reveal that there are sharp peaks and valleys in the RAO for each of the three degrees of freedom. Also there are several zeros in the transfer function with associated changes in sign. This clearly indicates that simple filters of the harmonic oscillator type, successfully used for the guyed tower, will not be adequate in the present situation. This calls for the use of higher order filters of more complex nature.

Figure 8.6 presents the excitation transfer function including $\sqrt{S_\eta \eta}$, for wind velocity of 50 knots. Interestingly, the transfer function of the P-M spectrum acts as a "band-pass filter" and the resulting transfer function is somewhat modified. However, the shape is still far from that of simple oscillators, and more complex filters need to be designed.

8.6. Conclusion

The present chapter derives the excitation force and moment response amplitude operators (RAO) for an idealized TLP with three degrees of freedom. Similarly, the transfer function for the excitation, to be generated as filter output to white noise are also derived by including the P-M wave elevation spectrum.

The transfer functions display a number of poles and zeros with multiple peaks and eliminate the possibility of using simple filter with white noise inputs for the excitation. Thus, the technique of state augmentation used for the formulation of the Liapunov equation cannot be used as easily in this case. A high order filter would be required and the associated number of states would also have to increase.

The present investigation of the TLP dynamics in the state space has been pursued only conceptually, and is only a very preliminary investigation. More research is required to apply the
method with the same degree of success as in the case of the guyed tower. In particular, the
direct application of the stochastic realization algorithm needs to be investigated in this context
by using numerical methods.
CHAPTER IX

Conclusions

9.1. Overview

The present chapter serves two main functions. Firstly, it summarizes the conclusions derived from the present research. Secondly, it identifies areas requiring further research.

9.2. Conclusions

The deterministic wave theory criteria for classifying water depths need no longer be arbitrarily used for spectral analysis. A rational procedure is developed in the present investigation for such classification using the concepts of probability theory, in Section 2.7. Similarly, the validity of using the Morison equation in connection with random waves is rationalized by a similar procedure, in Section 2.9.

The simulation of the surface elevation process resulting from a random wave field can be achieved very economically using filter techniques. This is particularly so for long simulations. For digital filter methods, the periodicity of the simulated time series is dictated only by the periodicity of the random number generator. In contrast, for the more popular harmonic superposition method, the periodicity of the simulated record is dictated by the frequency discretization intervals.

The kinematics propagation problem, in contrast to the simulation problem, is more difficult to handle by the filter approach. The problem is in maintaining the correct cross-correlation between kinematics at different points. For the harmonic superposition, this is naturally incorporated. The nature of the wave propagation problem is such that one uniform filter approach cannot be recommended unreservedly to cover all situations.
The vertical propagation of wave kinematics can not be achieved by any causal filters. The only realistic solution is the use of a symmetric, non-causal filter. Moving average filters are found to be quite efficient in this case. In contrast, auto-regressive AR type of filters are inappropriate. They introduce an undesirable phase shift which is unacceptable in this case.

The horizontal propagation problem for gravity waves is complicated by the dispersive nature of the gravity waves. Only in the special case of linear waves in shallow water, a situation considered unrealistic and only a mathematical abstraction, is the behavior non-dispersive. The wave kinematics at two different points on a vertical plane are related by a transfer function of unit magnitude and a non-linear phase response. Two different solution procedure are provided by the present investigation. The first involves the use of the well established Moving Average (MA) algorithm. The second and more novel approach developed in the present research exploits the special nature of the transfer function and optimizes the phase error by designing an all pass filter. This approach can provide very economic computation.

The state space method, so powerfully used in other engineering disciplines, has hardly received attention for random vibration analysis of offshore structures. The applicability of the method for solving the offshore structural dynamics problems, is investigated. The first application involves the dynamic analysis of the guyed tower response. Based on a rational function approximation for the excitation, closed form expressions for the response statistics are derived. The results are compared with the frequency domain solution and are shown to be more general in their scope. In particular, non-stationary solutions can be obtained directly. Also, the form is readily usable for implementation of control algorithms. A complete numerical example is provided using an idealized guyed tower model. The response statistics are obtained for four different sea-states.
The feasibility of applying the state-space method to the TLP dynamics is also explored in concept. Simple solutions could not be obtained in this case. The primarily difficulty was found to be adequate modeling of the excitation.

9.3. Areas Requiring Further Research

For the simulation of wave Kinematics, the knowledge of the exact transfer function for the propagation problem is required. In the present study, the waves are assumed to be unaffected by wave-wave interactions, and propagation distance. Experimental verification is required to validate this assumption in a real sea under storm conditions.

The model involving complete coherence as indicated above, results in algorithmic problems when the theory of multivariate simulation procedure is considered. This results from the special physical nature of the wave kinematics problems, whereby the spectral matrix becomes singular. The theory of multivariate simulation generally assumes the spectral matrix to be nonsingular. Further investigation is required to adapt the theory to the case of singular matrices.

The phase response is normally given less importance than the amplitude. Thus, research pertaining to proper phase approximation is relatively scarce. For gravity waves, and platform response analysis, phase plays an important role. Similarly, the importance of phase in the formation of wave groups is being recognized more and more in the context of second order, slow drift response, of moored and compliant platforms. More research is required in the area of wave phase effects.

There are two major stumbling blocks in the application of the state space theory to the case of MDF system subjected to non-white excitation. The first is the spectral factorization problem, and the second is the stochastic realization problem. Both topics are being actively researched at
present in the field of system theory and control. However, the algorithms available to date are extremely complex in nature. More research is required to develop algorithms that are conceptually simple and conveniently applicable. A relatively simple solution procedure for these two tasks would be quite effective in popularizing the method for general random vibration analysis problems.

Finally, the application of the state space method to investigate time-dependent systems, such as TLP response with time-varying stiffness, and non-stationary aspects, such as offshore platform response to earthquake excitation, need to be explored. Similarly, the applicability of the method for non-linear analysis of the dynamic response of moored tankers and tension leg platforms need to be investigated.
Appendix A

The Equivalent Linearization Procedure

The method of stochastic linearization, also known as statistical or equivalent linearization, has proved to be a very powerful tool for engineering analysis of non-linear dynamic systems subjected to random excitation. Compared to other more rigorous methods of dealing with non-linearities, it is conceptually simple and provides results with acceptable level of accuracy when the non-linearity is small.

Here, a general treatment of the problem [Spanos, 1981, Roberts and Spanos, 1990] is outlined first. Subsequently, the specific case of nonlinearity in the stiffness term alone is discussed.

Let the dynamical system be represented as

\[ M \ddot{X} + C \dot{X} + KX + g(X, \dot{X}) = F(t), \]  \hspace{1cm} (A.1)

where, \( g(X, \dot{X}) \) represents the non-linear term. The primary objective is to replace the non-linear term \( g(X, \dot{X}) \) by an "equivalent linear expression" \( g(X, \dot{X}) = C_e \dot{X} + K_e X \) to obtain the system

\[ M \ddot{X} + (C + C_e) \dot{X} + (K + K_e) X = F(t) \]  \hspace{1cm} (A.2)

Define the error term to be minimized as

\[ e = g(X, \dot{X}) - C_e \dot{X} - K_e X \]  \hspace{1cm} (A.3)

The optimal values of \( C_e \) and \( K_e \) are obtained by minimizing the error in a least square sense. That is,

\[ \frac{\partial \langle e^2 \rangle}{\partial C_e} = 0, \]  \hspace{1cm} (A.4)

and

\[ \frac{\partial \langle e^2 \rangle}{\partial K_e} = 0. \]  \hspace{1cm} (A.5)
The solution of these set of simultaneous equations (A.3-A.5) lead to

\[ C_e = \frac{\langle X^2 \rangle \langle \dot{X} \dot{g}(X, \dot{X}) \rangle - \langle X \dot{X} \rangle \langle X \dot{g}(X, \dot{X}) \rangle}{\langle X^2 \rangle \langle \dot{X}^2 \rangle - [\langle X \dot{X} \rangle]^2} \]  \hspace{1cm} (A.6)

and

\[ K_e = \frac{\langle \dot{X}^2 \rangle \langle X \dot{g}(X, \dot{X}) \rangle - \langle X \dot{X} \rangle \langle \dot{X} \dot{g}(X, \dot{X}) \rangle}{\langle X^2 \rangle \langle X^2 \rangle - [\langle X \dot{X} \rangle]^2} \]  \hspace{1cm} (A.7)

Note that in general, knowledge of the joint probability density of the response, \( \langle X \dot{X} \rangle \) is required to solve for equations (A.6 - A.7). However, considerable simplification is possible if one assumes that the response is a zero mean stationary process. Specifically, \( \langle X \dot{X} \rangle = 0 \) for such a process. Thus, eqns. (A.6) and (A.7) simplify to

\[ C_e = \frac{\langle \dot{X} \dot{g}(X, \dot{X}) \rangle}{\langle \dot{X}^2 \rangle} \]  \hspace{1cm} (A.8)

\[ K_e = \frac{\langle X \dot{g}(X, \dot{X}) \rangle}{\langle X^2 \rangle} \]  \hspace{1cm} (A.9)

For the special case where \( X \) is a Gaussian process, eqns. (A.6) and (A.7) simplify to

\[ K_e = \frac{\partial g(X, \dot{X})}{\partial X} \]  \hspace{1cm} (A.10a)

and

\[ C_e = \frac{\partial g(X, \dot{X})}{\partial \dot{X}} \]  \hspace{1cm} (A.10b)

For a system with a nonlinearity in the stiffness described by

\[ g(X) = b X^3, \]  \hspace{1cm} (A.11)

Where \( b \) is a constant, the equivalent linear stiffness for Gaussian \( X \) is

\[ K_e = 3b \langle X^2 \rangle = 3b \sigma_X. \]  \hspace{1cm} (A.12)

The same result can be obtained from eqn. (A.9) under the assumption of stationarity and
normality. Specifically,

\[ K_e = \frac{(b<X^4>)}{<X^2>} \quad (A.13) \]

Further, using the recursive relations for a Gaussian process one has,

\[ <X^n> = \mu <X^{n-1}> + (n-1) \sigma^2 <X^{n-2}> \quad (A.14) \]

where \( \mu_X \) and \( \sigma_X \) are the mean and standard deviation of the Gaussian process \( X \). For the zero-mean case, when \( n = 4 \)

\[ <X^4> = 3 \sigma^2 <X^2> \quad (A.15) \]

Substituting eqn. (A.15) in (A.13), the equivalent linear stiffness is given by the equation

\[ K_e = 3 b \sigma \quad (A.16) \]
Appendix B
Cross-Covariance of Output and Input

Systematic procedure for the evaluation of the cross-covariance matrix involving the state vector and the white noise excitation is given by Spanos [1983]. For the equation given by \( \dot{Y} = AY + G \), the matrix \( <GY^T + YG^T> \) is obtained in the following manner. Recalling that \( Y^T = [Y_1 Y_2 Y_3 Y_4] \) and that \( G^T = [0 0 0 (\sqrt{G_1} W_T)] \)

\[
<GY^T + YG^T> = \sqrt{G_1} \begin{bmatrix}
0 & 0 & 0 & <W_{Y_1}>
0 & 0 & 0 & <W_{Y_2}>
0 & 0 & 0 & <W_{Y_3}>
<W_{Y_1}> & <W_{Y_2}> & <W_{Y_3}> & 2<W_{Y_4}>
\end{bmatrix}
\] (B.1)

Based on the above reference, these expectations in eqns (B.1) are

\[
<W_{Y_1}> = 0 \quad \text{(B.2)}
\]
\[
<W_{Y_2}> = 0 \quad \text{(B.3)}
\]
\[
<W_{Y_3}> = 0 \quad \text{(B.4)}
\]
\[
<W_{Y_4}> = \pi G_1 \quad \text{(B.5)}
\]

The general result derived in the above reference can be summarized as follows: For a \( n-th \) order dynamical system, involving \( n \) derivatives of response, excited by a white noise of intensity \( G_1 \), the zero-lag correlations for the input with the derivatives of the response are given by

\[
R_{y^i w}(0) = 0 \quad ; \quad i = 0, 1, \ldots, n - 2 \quad \text{(B.6)}
\]
\[
R_{y^{n-1} w}(0) = \frac{\pi}{a_0} \quad , \quad \text{(B.7)}
\]

Where, \( a_0 \) is the coefficient associated with the \( n-th \) derivative of the response.
APPENDIX-C

Response Statistics of Guayed Tower

<table>
<thead>
<tr>
<th>Numerical Example (Wind = 30 ft/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>G1  Numerator of approximating function  4.42E+10</td>
</tr>
<tr>
<td>K0  stiffness term of approximating function  0.839552</td>
</tr>
<tr>
<td>C0  Square of the damping term in approx spectrum  0.1077623</td>
</tr>
<tr>
<td>C0  Damping term of approximating function  0.3282717</td>
</tr>
</tbody>
</table>

**Tower and Platform Parameters**

| D  Water depth (ft)  1425 |
| Zca height of cable attachment points  1300 |
| Fb  amount of buoyancy (kips)  12000 |
| Zb  height of center of buoyancy (ft)  1000 |
| Wp  Weight of Deck Platform  20000 |
| Zp  height of Deck (ft)  1500 |
| Wt  Unit weight of tower (kips/ft)  4.25 |
| h1  Height of tower truss (ft)  1450 |
| Z1  height of center of tower structure (ft)  725 |
| D0  Equivalent Diam of tower (ft)  60 |
| J0  Mass momt of Inertia ab base, incl added mass  2.176E+10 |
| Cm  Coefficient of Added Mass  1.5 |
| Cd  Drag Coeff  1 |
| cs  Structural Damping (% critical)  5 |

**Catenary Mooring Parameters**

<p>| L1  length of leading catenary segment (ft)  3560 |
| w1  unit weight of leading segment (Lb/ft)  40 |
| L2  length of clump weight segment (ft)  140 |
| w2  unit weight of clump weight segment (Lb/ft)  1920 |
| L3  length of trailing catenary segment (ft)  6900 |
| w1  unit weight of trailing segment (Lb/ft)  40 |
| c1  linear coef of horz. restoring force = a1<em>Zca  47755.229 |
| c2  cubic coef of horz. restoring force= a3</em>Zca<strong>3  -46583.7 |
| c2  constant coef of vert. restoring force = a0  5340 |
| c2  quad coef of vert. restoring force= a2*Zca</strong>2  32290.492 |</p>
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega ) square of natural Frequency</td>
<td>0.0015015</td>
</tr>
<tr>
<td>( \omega ) Approximate linear natural frequency</td>
<td>0.0387488</td>
</tr>
<tr>
<td>T Linearized Natural period</td>
<td>162.17274</td>
</tr>
<tr>
<td>( \varepsilon ) ratio of non-linear to linear stiffness</td>
<td>-19.826051</td>
</tr>
<tr>
<td>( \text{equiv omsq} ) square of the equivalent linear natural frequency</td>
<td>0.0015014</td>
</tr>
<tr>
<td>( \beta ) structural damping</td>
<td>0.0038749</td>
</tr>
<tr>
<td>( a_1 ) coef of zero-th derivative of ( y )</td>
<td>0.0012605</td>
</tr>
<tr>
<td>( a_2 ) coef of first derivative of ( y )</td>
<td>0.003746</td>
</tr>
<tr>
<td>( a_3 ) coef of second derivative of ( y )</td>
<td>0.8423254</td>
</tr>
<tr>
<td>( a_4 ) coef of third derivative of ( y )</td>
<td>0.3321466</td>
</tr>
<tr>
<td>( G_0 ) Numerator term of moment spectrum/ ( (J_0*J_0) )</td>
<td>9.335E-11</td>
</tr>
<tr>
<td>denom ( a_2<em>a_3</em>a_4-a_2<em>a_1</em>a_2*a_2 )</td>
<td>0.001034</td>
</tr>
<tr>
<td>( \text{var y1} ) variance of ( y_1 )</td>
<td>0.0006157</td>
</tr>
<tr>
<td>( \text{var y2} ) variance of ( y_2 )</td>
<td>9.338E-07</td>
</tr>
<tr>
<td>( \text{var y3} ) variance of ( y_3 )</td>
<td>1.053E-08</td>
</tr>
<tr>
<td>( \text{var y4} ) variance of ( y_4 )</td>
<td>7.694E-09</td>
</tr>
<tr>
<td>s.d. ( x ) Standard Deviation of Deck Displacement</td>
<td>1.4495133</td>
</tr>
</tbody>
</table>
TABLE C-2

<table>
<thead>
<tr>
<th>Numerical Example (Wind = 40 ft/sec)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>G1</td>
<td>1.68E+11</td>
</tr>
<tr>
<td>K0</td>
<td>0.517358</td>
</tr>
<tr>
<td>C0SQR</td>
<td>0.08763</td>
</tr>
<tr>
<td>C0</td>
<td>0.296023</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Tower and Platform Parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>1425</td>
</tr>
<tr>
<td>Zca</td>
<td>1300</td>
</tr>
<tr>
<td>Fb</td>
<td>12000</td>
</tr>
<tr>
<td>Zb</td>
<td>1000</td>
</tr>
<tr>
<td>Wp</td>
<td>20000</td>
</tr>
<tr>
<td>Zp</td>
<td>1500</td>
</tr>
<tr>
<td>Wt</td>
<td>4.25</td>
</tr>
<tr>
<td>h1</td>
<td>1450</td>
</tr>
<tr>
<td>Zt1</td>
<td>725</td>
</tr>
<tr>
<td>D0</td>
<td>60</td>
</tr>
<tr>
<td>J0</td>
<td>2.176E+10</td>
</tr>
<tr>
<td>Cm</td>
<td>1.5</td>
</tr>
<tr>
<td>Cd</td>
<td>1</td>
</tr>
<tr>
<td>cs</td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Catenary Mooring Parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>3560</td>
</tr>
<tr>
<td>w1</td>
<td>40</td>
</tr>
<tr>
<td>L2</td>
<td>140</td>
</tr>
<tr>
<td>w2</td>
<td>1920</td>
</tr>
<tr>
<td>L3</td>
<td>6900</td>
</tr>
<tr>
<td>w1</td>
<td>40</td>
</tr>
<tr>
<td>c1</td>
<td>47755.229</td>
</tr>
<tr>
<td>c2</td>
<td>-465983.7</td>
</tr>
<tr>
<td>c1</td>
<td>5340</td>
</tr>
<tr>
<td>c2</td>
<td>32290.492</td>
</tr>
<tr>
<td>Parameter</td>
<td>Description</td>
</tr>
<tr>
<td>--------------------</td>
<td>-------------------------------------------------------</td>
</tr>
<tr>
<td>$\omega^2$</td>
<td>square of natural frequency</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Approximate linear natural frequency</td>
</tr>
<tr>
<td>$T$</td>
<td>Linearized Natural period</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>ratio of non-linear to linear stiffness</td>
</tr>
<tr>
<td>$\text{eqv},\omega^2$</td>
<td>square of the equivalent linear natural frequency</td>
</tr>
<tr>
<td>$\beta$</td>
<td>structural damping</td>
</tr>
<tr>
<td>$a_1$</td>
<td>coef of zero-th derivative of $y$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>coef of first derivative of $y$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>coef of second derivative of $y$</td>
</tr>
<tr>
<td>$a_4$</td>
<td>coef of third derivative of $y$</td>
</tr>
<tr>
<td>$G_0$</td>
<td>Numerator term of moment spectrum/ $(J_0^2 J_0)$</td>
</tr>
<tr>
<td>$\text{denom}$</td>
<td>$a_2 a_3 a_4 - a_2 a_3 - a_1^2 a_2$</td>
</tr>
<tr>
<td>$\text{var},y_1$</td>
<td>variance of $y_1$</td>
</tr>
<tr>
<td>$\text{var},y_2$</td>
<td>variance of $y_2$</td>
</tr>
<tr>
<td>$\text{var},y_3$</td>
<td>variance of $y_3$</td>
</tr>
<tr>
<td>$\text{var},y_4$</td>
<td>variance of $y_4$</td>
</tr>
<tr>
<td>s.d. $x$</td>
<td>Standard Deviation of deck displacement</td>
</tr>
</tbody>
</table>
### TABLE C-3

<table>
<thead>
<tr>
<th>Numerical Example (Wind = 70 ft/sec)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>G</strong></td>
<td>Numerator of approximating function</td>
</tr>
<tr>
<td><strong>K0</strong></td>
<td>stiffness term of approximating function</td>
</tr>
<tr>
<td><strong>C0SQR</strong></td>
<td>Square of the damping term in approx spectrum</td>
</tr>
<tr>
<td><strong>C0</strong></td>
<td>Damping term of approximating function</td>
</tr>
</tbody>
</table>

---

**Tower and Platform Parameters**

| **D** | Water depth (ft) | 1425 |
| **Zca** | height of cable attachment points | 1300 |
| **Fb** | amount of buoyancy (kips) | 12000 |
| **Zb** | height of center of buoyancy (ft) | 1000 |
| **Wp** | Weight of Deck Platform | 20000 |
| **Zp** | height of Deck (ft) | 1500 |
| **Wt** | Unit weight of tower (kips/ft) | 4.25 |
| **ht** | Height of tower truss (ft) | 1450 |
| **Zt** | height of center of tower structure (ft) | 725 |
| **D0** | Equivalent Diam of tower (ft) | 60 |
| **J0** | Mass moment of inertia ab. base, incl added mass | 2.176E+10 |
| **Cm** | Coefficient of Added Mass | 1.5 |
| **Cd** | Drag Coeff | 1 |
| **cs** | Structural Damping (% critical) | 5 |

---

**Catenary Mooring Parameters**

<p>| <strong>L1</strong> | length of leading catenary segment (ft) | 3560 |
| <strong>w1</strong> | unit weight of leading segment (Lb/ft) | 40 |
| <strong>L2</strong> | length of clump weight segment (ft) | 140 |
| <strong>w2</strong> | unit weight of clump weight segment (Lb/ft) | 2120 |
| <strong>L3</strong> | length of trailing catenary segment (ft) | 6900 |
| <strong>w1</strong> | unit weight of trailing segment (Lb/ft) | 40 |
| <strong>c1</strong> | linear coef of horz. restoring force = a1Zca | 47755.229 |
| <strong>c2</strong> | cubic coef of horz. restoring force = a3Zca<strong>3 | -465983.7 |
| <strong>c1</strong> | constant coef of vert. restoring force = a0 | 5340 |
| <strong>c2</strong> | quad coef of vert. restoring force = a2Zca</strong>2 | 32290.492 |</p>
<table>
<thead>
<tr>
<th>Response Parameters (wind=70 ft/s)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>omega sq</td>
<td>square of natural Frequency</td>
</tr>
<tr>
<td>omega</td>
<td>Approximate linear natural frequency</td>
</tr>
<tr>
<td>T</td>
<td>Linearized Natural period</td>
</tr>
<tr>
<td>epsilon</td>
<td>ratio of non-linear to linear stiffness</td>
</tr>
<tr>
<td>eqv omncr</td>
<td>square of the equivalent linear natural frequency</td>
</tr>
<tr>
<td>beta</td>
<td>structural damping</td>
</tr>
<tr>
<td>a1</td>
<td>coef of zero-th derivative of y</td>
</tr>
<tr>
<td>a2</td>
<td>coef of first derivative of y</td>
</tr>
<tr>
<td>a3</td>
<td>coef of second derivative of y</td>
</tr>
<tr>
<td>a4</td>
<td>coef of third derivative of y</td>
</tr>
<tr>
<td>G0</td>
<td>Numerator term of moment spectrum/ (J0*J0)</td>
</tr>
<tr>
<td>denom</td>
<td>a2<em>a3</em>a4-a2<em>a2-a1</em>a2*a2</td>
</tr>
<tr>
<td>var y1</td>
<td>variance of y1</td>
</tr>
<tr>
<td>var y2</td>
<td>variance of y2</td>
</tr>
<tr>
<td>var y3</td>
<td>variance of y3</td>
</tr>
<tr>
<td>var y4</td>
<td>variance of y4</td>
</tr>
<tr>
<td>s.d. x</td>
<td>Standard Deviation of deck displacement</td>
</tr>
</tbody>
</table>
### TABLE C-4

<table>
<thead>
<tr>
<th><strong>Numerical Example (Wind = 80 ft/sec)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>G1</td>
</tr>
<tr>
<td>K0</td>
</tr>
<tr>
<td>C0 SQR</td>
</tr>
<tr>
<td>C0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Tower and Platform Parameters</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
</tr>
<tr>
<td>Zca</td>
</tr>
<tr>
<td>Fb</td>
</tr>
<tr>
<td>Zb</td>
</tr>
<tr>
<td>Wp</td>
</tr>
<tr>
<td>Zp</td>
</tr>
<tr>
<td>Wt</td>
</tr>
<tr>
<td>ht</td>
</tr>
<tr>
<td>Zt</td>
</tr>
<tr>
<td>D0</td>
</tr>
<tr>
<td>J0</td>
</tr>
<tr>
<td>Cm</td>
</tr>
<tr>
<td>Cd</td>
</tr>
<tr>
<td>cs</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Catenary Mooring Parameters</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
</tr>
<tr>
<td>w1</td>
</tr>
<tr>
<td>L2</td>
</tr>
<tr>
<td>w2</td>
</tr>
<tr>
<td>L3</td>
</tr>
<tr>
<td>w1</td>
</tr>
<tr>
<td>c1</td>
</tr>
<tr>
<td>c2</td>
</tr>
<tr>
<td>c1</td>
</tr>
<tr>
<td>c2</td>
</tr>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>------------------------</td>
</tr>
<tr>
<td>( \omega^2 )</td>
</tr>
<tr>
<td>( \omega )</td>
</tr>
<tr>
<td>( T )</td>
</tr>
<tr>
<td>( \epsilon )</td>
</tr>
<tr>
<td>( \text{equiv om } )</td>
</tr>
<tr>
<td>( \beta )</td>
</tr>
<tr>
<td>( a_1 )</td>
</tr>
<tr>
<td>( a_2 )</td>
</tr>
<tr>
<td>( a_3 )</td>
</tr>
<tr>
<td>( a_4 )</td>
</tr>
<tr>
<td>( G_0 )</td>
</tr>
<tr>
<td>( \text{denom} )</td>
</tr>
<tr>
<td>( \text{var y1} )</td>
</tr>
<tr>
<td>( \text{var y2} )</td>
</tr>
<tr>
<td>( \text{var y3} )</td>
</tr>
<tr>
<td>( \text{var y4} )</td>
</tr>
<tr>
<td>( \text{s.d. x} )</td>
</tr>
</tbody>
</table>
Appendix D

Derivation of TLP Column Force

This section provides the derivation of the TLP RAO’s from first principles. The origin for the wave system is directly under the wave crest and at the SWL. The Y axis coincides with the center-line axis of the TLP. The expressions for the wave kinematics used in the derivation are presented in eqns. (D.1 - D.6).

\[ \eta(x,z,t) = \frac{H}{2} \frac{\sinh \left[ \kappa(z + d) \right]}{\sinh(\kappa d)} \cos \Theta \]  \hspace{1cm} (D.1)

The corresponding velocity components are given by

\[ u_x(x,z,t) = \frac{\omega H}{2} \frac{\cosh \left[ \kappa(z + d) \right]}{\sinh(\kappa d)} \cos \Theta \]  \hspace{1cm} (D.2)

\[ u_z(x,z,t) = \frac{\omega H}{2} \frac{\sinh \left[ \kappa(z + d) \right]}{\sinh(\kappa d)} \sin \Theta \]  \hspace{1cm} (D.3)

The corresponding acceleration components are

\[ u_x(x,z,t) = \frac{\omega^2 H}{2} \frac{\cosh \left[ \kappa(z + d) \right]}{\sinh(\kappa d)} \sin \Theta \]  \hspace{1cm} (D.4)

\[ u_z(x,z,t) = -\frac{\omega^2 H}{2} \frac{\sinh \left[ \kappa(z + d) \right]}{\sinh(\kappa d)} \cos \Theta \]  \hspace{1cm} (D.5)

And the dynamic pressure under a surface wave at a depth \( z \) below the SWL is given by:

\[ p = -\rho g z + \frac{\rho g H}{2} \frac{\cosh \left[ \kappa(z + d) \right]}{\cosh(\kappa d)} \cos \Theta \]  \hspace{1cm} (D.6)

In the above, \( \Theta = (\kappa X - \omega t) \), is the wave phase. These equations are a simplified (deep water) version of the linear wave theory presented earlier in Chapter 2. Using only the inertia term of the Morison equation the horizontal force on a single column is given by the equation

\[ F_c = a_i \int_{-A}^{0} \omega^2 \frac{H}{2} e^{\kappa y} \sin(\kappa x - \omega t) dy \]  \hspace{1cm} (D.7)

where, \( a_i = \frac{\pi}{4} \rho C_m D_c^2 \) \( \rho \) is the water density, \( C_m \) is the inertia coefficient for the column,
and is taken to be 2.0, and \( D_c \) is the diameter of the i-th column. All terms except the exponential one can be taken out of the integration operation, yielding

\[
F_c = a_i \omega^2 \frac{H}{2} \sin(\Theta_i) \left[ \frac{e^{\kappa y}}{\kappa} \right]_{-h}^{0}
\]

On substituting \( \frac{\omega^2}{\kappa} = g \) from the dispersion equation

\[
F_c = a_i g \frac{H}{2} \sin \Theta_i \left[ 1 - e^{-\kappa h} \right]
\]

Noting that there are four columns, two at \( x_i = +a \), and the other two at \( x_i = -a \), and using the identity

\[
\sin(A - B) = \sin A \cos B - \cos A \sin B
\]

one obtains

\[
F_{ci} = -4 b_i H [1 - e^{-\kappa h}] \cos(\kappa a) \sin(\omega t)
\]

where, \( b_i = a_i \frac{g}{2} = \frac{\pi}{8} \rho g C_m D_c^2 \)
Fig. 2.1(a) Effect of Wind Velocity on Pierson-Moskowitz Spectrum of Horizontal Velocity at SWL (Depth = 300 m)
Fig. 2.1(b) Effect of Wind Velocity on Pierson-Moskowitz Spectrum of Horizontal Velocity at SWL (Depth = 150 m)
Fig. 2.2(a) Effect of Local Water Depth on Pierson-Moskowitz Spectrum of Horizontal Velocity at SWL (Wind vel = 60 Knots)
Fig. 2.2(b) Effect of Local Water Depth on Pierson-Moskowitz Spectrum of Horizontal Velocity at SWL (Wind vel = 30 Knots)
Fig. 2.3 Critical Water Depth for Validity of "Deep Water Assumption" For Various Fractional Energy Contributions
Fig. 2.4 Critical Water Depth for Validity of "Shallow Water Assumption" For Various Fractional Energy Contributions
HORZ VEL VERT PROPAGATION
ALPHA = .2, .4, .6, .8, .9

Fig. 5.1 Impulse Response for Vertical Propagation of Horizontal Velocity
Fig. 5.2 Impulse Response for Vertical
Propagation of Vertical Velocity
Vertical Propagation of Horizontal Component of Kinematics

(Depth = 150 m  Wind = 50 Kn)

LEGEND:
- alpha = 0.95
- alpha = 0.85
- alpha = 0.75
- alpha = 0.50
- alpha = 0.25

Fig. 5.3 Phase Response of Oscillator Model for Vertical Propagation of Horizontal Velocity
Fig. 5.4 Phase Response of Oscillator Model for
Vertical Propagation of Vertical Velocity
Fig. 5.5(a) M-A Filter Approximation of Vertical propagation of Horizontal Velocity Component (MA Filter Order = 5)
Two-Sided MA Filter
Vertical Propagation of Horizontal Velocity
Wind = 50 knots; D = 150 m  N = 7

LEGEND
- - - Alpha = 0.95
- - Alpha = 0.80
- - Alpha = 0.65
- - Alpha = 0.50
- - Alpha = 0.25

Fig. 5.5(b) M-A Filter Approximation of Vertical propagation
of Horizontal Velocity Component (MA Filter Order = 7)
Fig. 5.5(c) M-A Filter Approximation of Vertical propagation of Horizontal Velocity Component (MA Filter Order = 10)
HORZ PROPAGATION
D=150(M) MU=.2 U=50(KN)
WMAX=1.05*PI (N-D P-M SPEC)
NT(2*N+1)=31

Fig. 5.6 Moving Average Approximation of
Horizontal Propagation of Wave Kinematics
Fig. 5.7 Horizontal Propagation for Variable Location Through Moving Average Filtering
Depth = 150.0 m
h/d = 0.2, V = 50 kn

Fig. 5.8 Phase Response of the Moving Average Filter
for Horizontal Propagation
Fig. 5.9 Fourier Series Expansion with truncation

($\mu$ neglected after 5 terms)
Fig. 5.10 Phase Approximation of 2nd Order All-Pass Filter for Horz Propagation of Wave Kinematics (Wind = 30 Kn, d= 2.83 m)
Fig. 5.11(a) Phase Approximation of 2nd Order All-Pass Filter for Horz Propagation of Wave Kinematics (Wind = 50 Kn, d = 7.85 m)
Fig. 5.11(b) Phase Approximation of 2nd Order All-Pass Filter for Horz Propagation of Wave Kinematics (Wind = 50 Kn, d=9.81 m)
Fig. 5.11(c) Phase Approximation of 2nd Order All-Pass Filter for Horz Propagation of Wave Kinematics (Wind = 50 Kn, d=13.1 m)
Fig. 5.12(a) Phase Approximation of 2nd Order All-Pass Filter for Horz Propagation of Wave Kinematics (Wind = 70 Kn, d= 15.39 m)
Fig. 5.12(b) Phase Approximation of 2nd Order All-Pass Filter for Horz Propagation of Wave Kinematics (Wind = 75 Kn, d= 17.67 m)
Fig. 5.13(a) Coefficients of 2nd Order All-Pass Filters
For Different Propagation Distances (Wind = 50 Kn)
Fig. 5.13(b) Coefficients of 2nd Order All-Pass Filters
For Different Propagation Distances (Wind = 70 Kn)
Fig. 5.14 Fourth Order All-Pass Filter for Horizontal Propagation (Wind = 30 kn)
HORZ TO VERT FILTERING

$D = 150 \text{ (M) } U = 50 \text{ (KN) } \alpha = 0.75$

NO OF COEF $(2N+1) = 31$

---

**Fig. 5.15** Filtering of Horizontal Velocity to obtain

The Vertical Velocity with proper Phase
HORZ TO VERT FILTERING
WITH LANCZOS SMOOTHING
D=150(M) U=50(KN) ALPHA=.75
NO OF COEF(2*N+1)=31

Fig. 5.16 Filtering of Horz. Vel. with "Lanczos Window"
to obtain the Vertical velocity with proper Phase
Fig. 7.1 Model of Idealized Guyed Tower
Fig. 7.2 Stiffness Properties of 20-Line Mooring System
Fig. 7.3 Verification of Radial Symmetry in System Restoring Forces
Fig. 7.4 Polynomial Approximation of Horizontal Restoring Force
Fig. 7.5 Polynomial Approximation of Vertical Restoring Force
Fig. 8.1 TENSION LEG PLATFORM
YAW
Y (HEAVE)

PITCH
Z (SWAY)

ROLL
X (SURGE)

Fig. 8.2 TENSION LEG PLATFORM GEOMETRY
Fig. 8.3 TLP Heave Excitation Force RAO
Fig. 8.4 TLP Pitch Excitation Moment RAO
Fig. 8.5 TLP Surge Excitation Force RAO
Fig. 8.6 TLP Excitation For PM Spectrum (50 Kn Wind)
REFERENCES


Gerstner, F., Theorie der Wellen, Prague, 1802.


Goda, Y., Simulation in Examination of Directional Resolution, pp. 387-407., 1982. Univ. of California at Berkeley


Yoshida, K., Motion and Leg Tensions of Tension Leg Platforms, pp. 75-81, Proceedings of Thirteenth Annual OTC, 1981.

Tokyo, March 1989.


