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STUDIES OF DYNAMIC RESPONSE OF LIQUID STORAGE TANKS

by

YU TANG

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

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Chapters 2, 7 and 8 are the same as the papers on these topics prepared for publication with Dr. Veletsos.
ABSTRACT

STUDIES OF DYNAMIC RESPONSE
OF
LIQUID STORAGE TANKS

by

YU TANG

This dissertation consists of four parts. The first part deals with the response of liquid storage tanks to a vertical component of ground motion. Galerkin's method is used to solve this problem approximately, and its accuracy is established by comparing the results with those obtained by the exact modal superposition method. The effects of soil structure interaction are examined, and a simple practical design procedure is proposed for providing for these effects.

The second part deals with the response of liquid storage tanks to a horizontal component of ground shaking. The problem is analyzed by application of the Rayleigh-Ritz procedure in combination with Lagrange's equation. The details of the method of analysis are presented along with comprehensive numerical data which may be used readily in design applications.

The third part of the dissertation deals with the response of liquid storage tanks, both rigid and flexible, to a rocking component of ground shaking. Emphasis is placed on understanding the behavior of the tank-liquid system and establishing the interrelationship of the responses of the system to rocking and to a lateral, translational motion.

The fourth part deals with the harmonic response of massless ring foundations supported on a homogeneous elastic halfspace and subjected either to a vertical force or an
overturning moment. The theory of elastic wave propagation is used to obtain the impedance functions. The method of analysis takes due account of the mixed boundary conditions at the surface of the halfspace. The effect of foundation mass on the response is also studied for the vertically excited system. The reported data are believed to be of high accuracy.
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CHAPTER 1

INTRODUCTION

Liquid storage tanks are important components of modern industrial facilities and, when located in earthquake prone regions, should be designed to survive the earthquakes to which they may be subjected.

The construction in recent years of large capacity storage tanks, with diameters of the order of about 300 ft or more, has increased the importance of these structures and the need to understand better their performance during earthquakes so that rational methods may be formulated for their design. Of greatest interest in practice is the response of circular cylindrical tanks.

1.1 BACKGROUND

The response of such tanks to earthquake-induced ground motions has been the subject of numerous studies in recent years. A brief account of this effort is given in a recent publication of the National Research Council (33) and a more detailed account is given in (43), which also includes an extended list of relevant references.

By contrast to early studies of this problem in which the tank wall was presumed to be rigid (24,25), the more recent studies (3,15,18,42,47) have considered the important effect of tank flexibility. Primary attention has been given to the effects of the horizontal component of ground shaking.

The latter studies have shown that the flexibility of the tank wall may affect significantly the magnitude of the resulting hydrodynamic forces, and have led to the formulation of simplified design procedures for evaluating these forces. Two methods of analysis are now in use for design purposes: (a) The Veletsos-Yang procedure (43,47); and (b) the Haroun-Housner procedure (15,43).

Following the approach recommended by Jacobsen (25) and Housner (24) for rigid-
wall tanks, the hydrodynamic effects in these methods are expressed as the sum of two components: (a) a convective component, which is associated with the sloshing action of the liquid, and (b) an impulsive component, which represents the effects of the part of the liquid that may be considered to move in synchronism with the tank wall.

The convective effects in both procedures are considered to be unaffected by the flexibility of the tank wall. The two procedures differ in the way in which the impulsive effects are evaluated. In the Veletsos-Yang procedure, the maximum values of the impulsive effects are computed from the corresponding solution for a rigid tank simply by replacing the peak value of the ground acceleration in all expressions for the dynamic response by the pseudoacceleration value corresponding to the fundamental natural frequency of the tank-liquid system. In the Haroun-Housner procedure, a two-mass mechanical model is used which effectively considers the response of the system in its fundamental mode of vibration. There is evidence to the effect that the latter procedure is unconservative for slender tanks.

The studies referred to above have been concerned primarily with the effects of the horizontal component of ground shaking. Although some attention has been given in recent years to the effects of the vertical component of excitation (19,20,27,32,44), this problem requires additional research. Also requiring detailed study is the response of flexible tanks to a rocking component of base excitation.

In the studies referred to above, the base motion experienced by the tank was presumed to be the same as the free-field ground motion. The effects of coupling or interaction between the vibrating tank-liquid system and the supporting medium to date have not received the attention they deserve. Only exploratory studies have been made of this problem (9,17,43).

1.2 OBJECTIVES
The objective of this research is three-fold:

1. To assess the accuracy of available seismic design procedures for tanks that are rigidly supported at the base (i.e. tanks for which the soil-structure interaction effects are not important) and, where necessary, to propose requisite refinements.

2. To make a critical evaluation of the response of tanks to a rocking component of base motion. In addition to being of interest in its own right, this information is needed in analyses of the response of elastically supported tanks. The base of such tanks may experience substantial rocking motion even under a purely horizontal excitation of the ground.

3. To provide improved insight into the effects of soil-structure interaction on the dynamic response of tanks and to formulate rational procedures for providing for these effects.

1.3 SCOPE OF WORK

The tanks are presumed to be cylindrical, of circular cross section, uniform wall thickness and anchored at the base. The effects of both the horizontal and vertical components of shaking are considered. The detailed scope is outlined below:

   
a. Response to vertical component of shaking (Ch.2 and Ch.3).

   In this part, first, the tank is assumed to be rigidly supported at the base; then, this assumption is relaxed, and the effects of soil structure interaction are studied. Two levels of approximation are employed for the flexible tank, including the approximation in which the tank is presumed to response as a single-degree-of-freedom system in its fixed-base condition. Due provision is made for the energy dissipated into the supporting medium by the radiation of waves.
b. **Response to horizontal component of shaking (Ch.4).**

In this part, the tank is assumed to be rigidly supported at the base. Lagrange’s equation is used to derive the equations of motion, and comprehensive numerical data for base shear and overturning moment are presented.

c. **Response to prescribed rocking excitation of the base (Ch.5 and Ch.6).**

In this part, the tank is again assumed to be rigidly supported. The responses of rigid tanks are studied extensively first; then, the effects of the flexibility of the tank wall are assessed by making use of a method of analysis similar to that used in Ch. 4.

The broad objectives of these studies are to gain improved understanding of the nature of the response in each case, and to formulate methods of analysis that may be implemented readily and accurately in design applications.

2. **Studies of Foundation Impedance Functions (Ch.7, Ch.8 and Ch.9).** In the soil-structure interaction studies referred to under item 1, the base of the tank is presumed to be supported on a rigid circular base. As a step toward the analysis of tanks that are supported on a ring foundation, the impedance functions for harmonically excited rings supported on an elastic halfspace are evaluated, and numerical data are generated over a wide range of exciting frequencies and ring dimensions. Rings in both vertical and rocking motions are examined.

**1.4 NOTATION**

The letter symbols used are defined where they are first introduced in the text, and the notation is not necessarily the same in all chapters.
CHAPTER 2

DYNAMICS OF VERTICALLY EXCITED LIQUID STORAGE TANKS

2.1 INTRODUCTION

Recent studies of the dynamic response of liquid storage tanks (7, 21, 22, 27, 30, 32, 43, 44) have revealed that the hydrodynamic effects induced by a vertical component of ground shaking may be quite important, reaching the level of the hydrostatic effects (44) for a ground motion with a peak acceleration of about one-third the acceleration of gravity. These studies referred to rigidly supported tanks for which the motion experienced at the base is the same as the free-field ground motion. Although it has been recognized that, because of the large radiational damping capacity of vertically excited foundations, soil-structure interaction may significantly reduce the design forces for tanks supported on relatively soft soils (43, 44), the consequences of such interaction have not to date been adequately assessed.

The objectives of this chapter are: (1) to identify the principal effects of soil-structure interaction on the dynamic response of vertically excited, upright, circular cylindrical liquid storage tanks; and (2) to present a simple, practical procedure with which the critical design forces for both rigidly and flexibly supported tanks may be evaluated readily.

The tanks are presumed to be supported through a rigid circular mat at the surface of a homogeneous elastic halfspace, and they are analyzed approximately by Galerkin’s method considering the tank-liquid system to response as a single-degree-of-freedom system in its fixed-base condition. Comprehensive numerical data are included which elucidate the actions of rigidly and flexibly supported systems, and the effects and relative importance of the numerous parameters that influence their response. The maximum hydrodynamic effects are related simply to the corresponding hydrostatic effects.

2.2 SYSTEM CONSIDERED
The system investigated is shown in Fig. 2.1. It is a vertically excited upright, circular cylindrical tank of radius, $a$, height, $H$, and constant wall thickness, $h$, which is supported through a rigid base at the surface of a homogeneous elastic halfspace and is filled with liquid of mass density $\rho_f$. The liquid is considered to be incompressible and inviscid and free at its upper surface. The tank wall is presumed to be either clamped or hinged at the base; its mass density is denoted by $\rho$; the modulus of elasticity and Poisson's ratio for the tank material are denoted by $E$ and $\nu$, respectively, and the corresponding quantities for the supporting soil are denoted by $\rho_s$, $E_s$ and $\nu_s$. The excitation is a vertical component of ground shaking with a free-field acceleration $\dddot{z}_g(t) = \dddot{x}_g f(t)$, in which $\dddot{z}_g$ is its maximum or peak value and $f(t) = a$ dimensionless function of time, $t$.

Points for the tank or the contained liquid are defined by the cylindrical coordinate system, $r, \theta, z$, the origin of which is taken at the center of the base, and the positive directions of the coordinates are shown in Fig. 2.1. The radial displacement of the tank wall, measured from the equilibrium position of the full tank, is denoted by $w = w(z,t)$, and the hydrodynamic pressure exerted by the liquid on the tank wall is denoted by $p = p(z,t)$. These quantities are uniformly distributed in the circumferential direction and are taken positive in the direction of the corresponding coordinate axes.

2.3 Method of Analysis

There are two aspects of interaction that must be considered: the interaction between the tank and the contained liquid; and the interaction between the tank-liquid system and the supporting medium. In the spirit of the substructuring approach, the two effects are considered sequentially. First, the response of the tank-liquid system to an arbitrary vertical excitation of its base is evaluated. Then, the relationship of the motion of the tank base to the free-field motion of the ground is established from an analysis of the foundation-soil system. The response of the coupled system is finally obtained by an
appropriate synthesis of the component solutions.

2.4 ANALYSIS OF TANK-LIQUID SYSTEM

Fundamental Relations for Tank and Liquid.-- On the assumption that the inertia forces of the tank wall in the axial direction have negligible influence on its response in the radial direction, the motion of the tank-liquid system is governed by the differential equation

\[ D \frac{\partial^4 w}{\partial z^4} + \frac{Eh}{a^2} w + \rho h \frac{\partial^2 w}{\partial t^2} = p(z, t) \] (2.1)

in which \( D = \frac{Eh^3}{12(1-\nu^2)} \) is the flexural rigidity per unit circumferential length of the tank wall, and \( p(z, t) \) is the hydrodynamic wall pressure which remains to be determined.

If \( \phi(r,z,t) \) is a velocity potential function that satisfied Laplace's equation

\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0 \] (2.2)

and appropriate boundary conditions, then the velocity of the liquid at an arbitrary point along the \( n \)-direction, \( v_n \), is given by

\[ v_n(r,z,t) = -\frac{\partial \phi}{\partial n} \] (2.3)

and the hydrodynamic pressure at that point is given by

\[ p(r,z,t) = \rho \frac{\partial \zeta}{\partial t} \] (2.4)

The boundary conditions for the liquid are:

1. The vertical velocity of the liquid at \( z=0 \) must equal the velocity of the tank base, \( \dot{x}_0(t) \), i.e.,

\[-\frac{\partial \phi}{\partial z} \bigg|_{z=0} = \dot{x}_0(t) \] (2.5a)
The quantity \( \dot{z}_0(t) \) is generally different from the free-field velocity of the ground, \( \dot{x}_g(t) \), and should not be confused with the latter.

2. The radial velocity of the liquid adjacent to the tank wall must equal the velocity of the tank wall, i.e.,

\[
- \frac{\partial \phi}{\partial r} \bigg|_{r=a} = \frac{\partial w}{\partial t}
\]  

(2.5b)

3. The hydrodynamic pressure at \( z=H \) must be zero, i.e.,

\[
\partial \phi \bigg|_{z=H} = 0
\]  

(2.5c)

The contribution of the small pressure increment associated with the rigid body, vertical sloshing action of the liquid is neglected in Eq. 2.5c.

On solving Eq. 2.2 subjected to the boundary conditions defined by Eqs. 2.5a through 2.5c and making use of Eq. 2.4, the following expression is obtained for the hydrodynamic pressure induced at an arbitrary point of the liquid:

\[
p(r, \xi, t) = -2 \partial \phi H \sum_{n=1}^{\infty} \frac{1}{\alpha_n} \frac{I_0(\lambda_n \xi)}{I_1(\lambda_n)} \left[ \int_0^1 \tilde{w} \cos(\alpha_n \xi) \, d\xi \right] \cos(\alpha_n \xi)
\]

\[
+ \partial \phi H (1-\xi) \ddot{z}_0(t)
\]

(2.6)

in which \( \xi = z/H \); a dot superscript denotes differentiation with respect to time;

\[
\alpha_n = (2n-1)\pi/2 \quad \text{and} \quad \lambda_n = (a/H)\alpha_n
\]

(2.7)

and \( I_0 \) and \( I_1 \) are modified Bessel functions of the first kind of order zero and one, respectively. The pressure exerted against the tank wall is obtained from Eq. 2.6 by letting \( r=a \). Note that, in addition to the tank dimensions and the characteristics of the foundation motion, \( p(r, \xi, t) \) depends on the radial acceleration of the tank wall, \( w=w(\xi, t) \). It is through the latter term that the two members of Eq. 2.1 are coupled.
When expressed in terms of the dimensionless coordinate, $\xi$, employed in Eq. 2.6, Eq. 2.1 may be rewritten as

\[ \left( -\frac{1}{\delta^4} w^{\cdot\cdot\cdot} + w \right) \frac{Eh}{a^2} + \rho h \ddot{w} = p(\xi, t) \]  \hspace{1cm} (2.8)\]

in which

\[ \delta = \sqrt[4]{\frac{12(1-\nu^2)}{12}} \frac{H}{a^2h} \]  \hspace{1cm} (2.9)\]

Solution for Tank-Liquid System.-- The radial displacement of the tank wall, $w(\xi, t)$, is taken in the form

\[ w(\xi, t) = \psi(\xi) q(t) \]  \hspace{1cm} (2.10)\]

in which $\psi(\xi)$ is a prescribed dimensionless function of $\xi$ that satisfies the boundary conditions at $\xi = 0$ and $\xi = 1$, and $q(t)$ is a time function with units of displacement which is determined by application of Galerkin's method by requiring that the weighted residual

\[ \int_0^1 \left[ \left( -\frac{1}{\delta^4} w^{\cdot\cdot\cdot} + w \right) \frac{Eh}{a^2} + \rho h \ddot{w} - p(\xi, t) \right] \psi(\xi) \, d\xi = 0 \]  \hspace{1cm} (2.11)\]

The wall pressure, $p(\xi, t)$, in this expression is obtained from Eq. 2.6 by letting $r = a$ and replacing $w$ by the right hand member of Eq. 2.10.

On substituting the resulting expression for $p(\xi, t)$, along with Eq. 2.10, into Eq. 2.11 and performing the indicated integration, one obtains the differential equation:

\[ m^* \dddot{q} + k^* q = m_e^* \dddot{\delta_0}(t) \]  \hspace{1cm} (2.12)\]

in which $m^*$ is the effective mass of the tank-liquid system that participates in the response; $k^*$ is the effective stiffness of the tank, and $m_e^*$ is the effective liquid mass that participates in the excitation. These quantities may conveniently be expressed as
\[ m^* = A_w \rho h + A_\xi \rho \xi H = A \rho h \]  
(2.13)

\[ k^* = B (Eh/a^2) \]  
(2.14)

\[ m_e^* = \Gamma \rho \xi H \]  
(2.15)

in which the first term on the right hand member of Eq. 2.13 represents the part of \( m^* \) that is contributed by the tank wall; the second term represents that part contributed by the liquid, and \( A, A_w, A_\xi, B \) and \( \Gamma \) are dimensionless factors defined by

\[ A = A_w + A_\xi (\rho_\xi / \rho) H/h \]  
(2.16a)

\[ A_w = \frac{1}{\pi} \int_0^1 \psi^2(\xi) \, d\xi \]  
(2.16b)

\[ A_\xi = \frac{4}{\pi} \sum_{n=1}^\infty \frac{1}{2n-1} \frac{I_0(\lambda_n)}{I_1(\lambda_n)} \frac{d_n}{d_0} \]  
(2.16c)

\[ d_n = \frac{1}{\pi} \int_0^1 \psi(\xi) \cos((2n-1)\frac{\pi}{2} \xi) \, d\xi \]  
(2.17)

\[ B = \int_0^1 \chi(\xi) \psi(\xi) \, d\xi \]  
(2.18)

\[ \chi(\xi) = \psi(\xi) + \frac{1}{\sigma \xi} \psi'''(\xi) \]  
(2.19)

and

\[ \Gamma = \frac{1}{\pi} \int_0^1 (1-\xi) \psi(\xi) \, d\xi \]  
(2.20)

The function \( \chi(\xi) \) in Eq. 2.19 defines the heightwise distribution of the static pressure corresponding to the assumed displacement configuration (the right hand member of Eq. 2.10, with \( q(t) \) taken as unity). Denoted by \( p(\xi) \), this pressure is obtained from Eq. 2.8 by deleting the inertia term and making use of Eq. 2.19: the resulting expression is

\[ p(\xi) = \chi(\xi) Eh/a^2 \]  
(2.21)
With \( k^* \) and \( m^* \) determined, the circular natural frequency of axisymmetric, breathing mode of vibration of the tank-liquid system, \( \omega \), may be computed from \( \omega = \sqrt{k^*/m^*} \), which, on making use of Eqs. 2.14 and 2.13, may also be written as

\[
\omega = \sqrt{\frac{B}{A}} \omega_o \tag{2.22}
\]

The quantity \( \omega_o \) in the latter expression represents the circular natural frequency of uniform, breathing mode of vibration for a ring having the cross sectional dimensions of the tank wall, and is given by

\[
\omega_o = \frac{1}{a} \sqrt{\frac{E}{\rho}} \tag{2.23}
\]

Equation 2.12 is of the same form as the differential equation governing the deformation, \( D(t) \), of a similarly excited, undamped, simple oscillator with a circular natural frequency \( \omega \), and its solution is determined by analogy to be

\[
q(t) = - C_D D(t) \tag{2.24a}
\]

in which \( C_D \) is a dimensionless factor defined by

\[
C_D = \frac{m^*}{m} = \frac{\Gamma}{A} \frac{\rho H}{\rho h} \tag{2.25}
\]

Equation 2.24a can also be written as

\[
q(t) = C_A \rho H \frac{a^2}{\varepsilon h} A(t) \tag{2.24b}
\]

in which

\[
C_A = \Gamma/B \tag{2.26}
\]

and \( A(t) \) is the instantaneous value of the pseudoacceleration of the simple oscillator, defined by
\[ A(t) = -\omega^2 D(t) \]  

(2.27)

Equation 2.24b is obtained from Eq. 2.24a by multiplying and dividing its right hand member by \( \omega^2 = K^*/m^* \), and making use of Eqs. 2.13 and 2.14. with \( q(t) \) established. the instantaneous value of the radial displacement of the tank wall may be determined from Eq. 2.10; the associated hoop force, \( N_\theta \), may be determined from

\[ N_\theta = (Eh/a) w \]  

(2.28)

the bending moments and transverse shear may be determined by appropriate differentiations of \( w \), and the maximum values of these effects may be determined by use of the maximum values of \( D(t) \) and \( A(t) \) in the relevant expressions.

Equivalent Static Radial Pressure.-- The interrelation of the maximum hydrodynamic effects induced in the tank wall and the corresponding hydrostatic effects may better be appreciated by use of the concept of equivalent radial static pressure introduced in this section. This concept also offers the key to a further simplification in the method of dynamic analysis.

Let \( \overline{p}(\xi,t) \) the equivalent radial static pressure which at any time \( t \) induces the same effects in the tank wall as those actually induced by the ground shaking. Defined by the product of \( \overline{p}(\xi) \) and \( q(t) \), this pressure is determined from Eqs. 2.21 and 2.24b, to be

\[ \overline{p}(\xi,t) = C_A x(\xi) \rho \chi \overline{A}(t) \]  

(2.29)

and its maximum value is obtained by replaced \( A(t) \) by its maximum or spectral value, \( A \). For comparison, the hydrostatic pressure exerted by the liquid on the tank wall is given by

\[ p_{st}(\xi) = (1-\xi) \rho \chi Hg \]  

(2.30)

in which \( g \) is the acceleration of gravity. Note that \( A(t) \) in Eq. 2.29 is the counterpart of
the gravitational acceleration in Eq. 2.30, and the product $C_{AX}(\xi)$ is the counterpart of the linear function $(1-\xi)$.

Evaluation of $A(t)$—The pseudoacceleration function in Eqs. 2.24b and 2.29 is for a single-degree-of-freedom system subjected to the excitation actually experienced by the tank base. For a rigidly supported tank, $\ddot{x}_0(t)$ is the same as the free field ground acceleration, $\ddot{x}_g(t)$, and $A(t)$ may be defined directly from the characteristics of the prescribed free-field ground motion. For an elastically supported tank, however, $\ddot{x}_e(t)$ and $\ddot{x}_g(t)$ are generally different, and it is first necessary to evaluate $\ddot{x}_0(t)$. The interrelationship of $\ddot{x}_e(t)$ and $\ddot{x}_0(t)$ is examined after considering the application of the method to rigidly supported tanks.

2.5 RIGIDLY SUPPORTED TANKS

The response of rigidly supported tanks is examined on the assumption that the tank wall is either clamped or hinged at the base.

Tank Wall Clamped at Base.—The heightwise variation of the radial displacement in this case is taken in the form

$$\psi(\xi) = \cos \frac{\pi \xi}{2} - \sqrt{2} e^{-\delta \xi / \sqrt{2}} \cos \left( \frac{\delta \xi}{\sqrt{2}} - \frac{\pi}{4} \right)$$  \hspace{1cm} (2.31)

This configuration is the same as that of the radial displacement induced in an infinitely long tank that has the same cross sectional dimensions and the same conditions of support at the base as the actual tank and is subjected to a static radial pressure that varies as a half-cosine wave with a maximum value at the base. The dimensionless function $\chi(\xi)$ in the expression for the latter pressure (see Eq. 2.21) is determined from Eq. 2.19 to be

$$\chi(\xi) = \left[ 1 + \left( \frac{\pi}{2 \xi} \right)^4 \right] \cos \frac{\pi}{2} \xi$$  \hspace{1cm} (2.32)

This pressure distribution is the same as that of the hydrodynamic wall pressure induced
in a vertically excited tank that acts as a membrane without any bending resistance \( \delta \). It should be noted that the quantities \( H/a, h/a \) and \( \nu \) do not appear independently in the expressions for \( \psi(\xi) \) and \( \chi(\xi) \), but enter only through the dimensionless factor \( \delta \), representative of those encountered in practice, are displayed in Fig. 2.2.

The displacement function defined by Eq. 2.31 satisfies the conditions of zero displacement and zero slope at the base, but violates the conditions of no moment and no shear at the top. As a result, Eq. 2.11 must be augmented by the addition of terms that represent the work done by the unsatisfied boundary forces acting through the corresponding displacements of the tank wall. However, for the tank proportions considered herein, these terms are negligible because both the shear itself and the associated displacement (see Fig. 2.2) are quite small. The ratios of the end moments, \( M(1)/M(0) \), and end shear, \( V(1)/V(0) \), are given by

\[
\frac{M(1)}{M(0)} = - \frac{\sqrt{2} e^{-\delta/\sqrt{2}} \sin(\frac{\delta}{\sqrt{2}} - \frac{\pi}{4})}{1 - (\pi/2\delta)^2}
\]  

(2.33a)

and

\[
\frac{V(1)}{V(0)} = \frac{1}{\sqrt{2}} \left(\frac{\pi}{2\delta}\right)^3 - e^{-\delta/\sqrt{2}} \cos\frac{\delta}{\sqrt{2}}
\]  

(2.33b)

The quantities \( A_w, A_\xi, B \) and \( \Gamma \) corresponding to the assumed configuration \( \psi(\xi) \) are given approximately by the following expressions, in which small terms involving the factor \( e^{-\delta/\sqrt{2}} \) have been deleted:

\[
A_w = \frac{1}{2} - \frac{5\sqrt{2}}{4\delta}
\]  

(2.34a)

\[
A_\xi = \frac{4}{\pi} \sum_{n=1}^\infty \frac{1}{2n-1} \frac{i_0(\lambda_n)}{i_1(\lambda_n)} \left[ e_n - \frac{\sqrt{2}}{[1 + (2n-1)^4(\pi/2\delta)^4]^{\delta}} \right]
\]  

(2.34b)
\[
\begin{align*}
\text{where} \quad e_n &= \begin{cases} 
\frac{1}{2} & \text{for } n = 1 \\
0 & \text{for } n \neq 1
\end{cases} \\
B &= \left[ \frac{1}{2} - \frac{\sqrt{2}}{\delta} \right] \left[ 1 + \left( \frac{\pi}{2\delta} \right)^4 \right] \\
\Gamma &= \frac{4}{\pi^2} - \frac{\sqrt{2}}{\delta} + \frac{1}{\delta^2}
\end{align*}
\]

(2.35)  \hspace{1cm} (2.36)  \hspace{1cm} (2.37)

One observes that \( A_w, B \) and \( \Gamma \) depend only on \( \delta \), whereas \( A_x \) also depends, through the factor \( \lambda_n \), on \( H/a \). Representative values of these quantities are listed in Table 2.1 for two groups of tanks filled with water: (1) steel tanks with \( h/a = 0.001, \nu = 1/3 \) and \( \rho_x/\rho = 0.127 \). and (2) concrete tanks with \( h/a = 0.01, \nu = 0.17 \) and \( \rho_x/\rho = 0.4 \). Several different values of \( H/a \) in the range between 0.3 and 5 are considered. The corresponding values of \( \delta \) are also identified in the table.

The frequency ratios, \( \omega/\omega_0 \), for these tanks are listed in Table 2. In additional to the results obtained from Eq. 2.22 using the complete expression for \( A_x \) presented in Eq. 2.34b, approximate values, obtained by considering only the first term of the series in the latter expression, are included. Also given are the results of previously reported nearly exact solutions (27,44) and of solutions obtained by considering the tank wall to act as a membrane without any bending resistance. It can be seen that the accuracy of the proposed solution, even in its simplified version, is indeed excellent and clearly superior to that of the membrane solution.

In column 6 of Table 2.2 are given the values of the coefficient \( C_{AX}(0) \) in the expression for the equivalent static radial pressure induced at the base of the tank wall. These values vary within a rather narrow range and may be approximated with good accuracy by the constant value of 0.8. Since the heightwise variation of the pressure is a half-cosine wave, the instantaneous value of the pressure at any level \( \xi \) is given by
\[ \bar{p}(\xi, t) = 0.8 \rho \frac{A(t)}{A} \cos(\pi \xi / 2) \]  \hspace{1cm} (2.38)

and the maximum value of the pressure at that level is obtained by replacing \( A(t) \) by its maximum or spectral value, \( A \). This approximation is the same as that proposed in (30) for concrete tanks that are free to expand radially at the base.

Tank Wall Hinged at Base.-- The response of the tank-liquid system was also evaluated considering the tank wall to be hinged at the base. The configuration of the radial displacement in this case was taken in the form

\[ \psi(\xi) = \cos \frac{\pi \xi}{2} - e^{-5 \xi / \sqrt{2}} \left[ \cos \frac{\delta \xi}{\sqrt{2}} + \left( \frac{\pi}{2} \right)^{2} \sin \frac{\delta \xi}{\sqrt{2}} \right] \]  \hspace{1cm} (2.39)

which, as before, corresponding to a radial pressure that varies as a half-cosine wave from base to top.

Bypassing the details of analysis, in Table 2.3 are given the resulting values of the frequency ratio, \( \omega / \omega_{n} \), and of the coefficient \( C_{A} \chi(0) \) in the expression for the equivalent static pressure at the junction of the tank wall and the base. Comparison of these data with those for the clamped-base tanks presented in Table 2.2 reveals that the frequency values for the two boundary conditions differ only slightly and that the pressure intensities are practically the same. It follows that the hydrodynamic effects are insensitive to the condition of support at the base of the tank wall, and that, provided \( A(t) \) is evaluated for the natural frequency of the system under consideration, Eq. 2.38 may be used irrespective of the degree of base constraint. Reinforcing this conclusion are the results of the studies reported in (30) which have led to essentially the same approximation even for tanks that are free to expand radially at the base.

Simplified Design Procedure.-- In columns (2) and (3) of Table 2.4 are listed the values of \( C_{p} \) in the expression for the total equivalent static force exerted against the tank wall per unit of circumferential length. Denoted by \( P_{w} \), this force is given by
\[ P_w = C_p \rho \chi H^2 A(t) \quad (2.40) \]

in which
\[ C_p = C_A \frac{1}{A} \int_0^1 x(\xi) \, d\xi \quad (2.41) \]

It can be seen that, for both the clamped-base and hinged-base conditions, \( C_p \) is close to 1/2, the value obtained for a pressure that increases linearly from top to bottom.

Since the maximum response of the tank wall is not particularly sensitive to the detailed distribution of the forces acting on it, the equivalent static pressure may further be simplified by use of the linear approximation
\[ \bar{P}(\xi, t) = (1-\xi) \rho H A(t) \quad (2.42) \]

This approximation is particularly convenient for design applications, as it permits the maximum hydrodynamic effects to be determined from the corresponding hydrostatic effects simply by multiplying the latter by the factor \( A/g \), in which \( A \) is the spectral value of the pseudoacceleration corresponding to the natural frequency of the tank-liquid system and \( g \) is the gravitational acceleration.

Fundamental to the solution presented so far has been the assumption that the tank-liquid system is undamped. The effect of damping may be incorporated by interpreting \( A \) to be the pseudoacceleration of a simple oscillator with a damping factor equal to that of the fundamental, axisymmetric mode of vibration of the tank-liquid system.

2.6 FLEXIBLY SUPPORTED TANKS

Before proceeding with the analysis of flexibly supported tanks, it is desirable to evaluate the total hydrodynamic force induced at the base of the tank-liquid system.

Hydrodynamic Force on Tank Base.-- Denoted by \( P_O \), this force is determined from
\[ P_0(t) = 2\pi \int_{0}^{a} p(r,0,t) r \, dr \]  
\[ (2.43) \]

and may conveniently be expressed as

\[ P_0(t) = m_0 \ddot{x}_0(t) + m_1 A(t) \]  
\[ (2.44) \]

in which \( m_0 + m_1 = m_f \). The quantity \( m_0 \) represents the part of the liquid mass that moves in synchronism with the tank base as a rigidly attached mass, whereas \( m_1 \) represents the part that moves as an elastically supported mass with a pseudoacceleration \( A(t) \). The latter quantity is the same as that which appears in Eqs. 2.38, 2.40, 2.41. The values of \( m_1/m_f \), for the clamped-base tanks considered in previous sections are plotted in Fig. 2.3 and are also listed in Table 2.4 along with those for the hinged-base tanks.

Modeling of System.-- It should be clear from Eq. 2.44 that, insofar as its effects on \( P_0(t) \) is concerned, the action of the tank-liquid system may be modeled by the two-mass system shown in part (a) of Fig. 2.4, in which \( m_0 \) is rigidly attached to a base oscillating with an acceleration \( \ddot{x}_0(t) \), and \( m_1 \) is elastically supported by a spring of stiffness

\[ k_1 = m_1 \omega^2 \]  
\[ (2.45) \]

The acceleration of the top mass is then \( A(t) \), and \( P_0(t) \) is the hydrodynamic force induced at a section immediately beneath the bottom mass.

For a rigidly supported tank, \( \ddot{x}_0(t) \) is the same as the free-field ground acceleration, \( \ddot{x}_g(t) \), but for an elastically supported tank, the two accelerations are different and their interrelationship depends on the characteristics of the excitation and of the system itself. The action of the system in the latter case may be modeled by the two-degree-of-freedom shown in Fig. 2.4b, in which the base mass, \( m_0' \), is equal to the sum of the foundation mass, \( m_f \), the tank mass, \( m_s \), and of the rigidly attached liquid mass, \( m_0 \). The bottom spring and dashpot provide for the effects of soil flexibility and capacity of the soil to dissi-
pate energy by radiation of waves. The characteristics of these elements depend on the properties of the soil as well as the characteristics of the excitation.

Details of Model for Harmonic Motion.— For a system in harmonic motion, the complex-valued amplitude of the restraining force exerted by the supporting medium for a unit displacement amplitude (the foundation impedance function) may be expressed as

\[ Q = K_{st} \left[ a + i a_0 \beta \right] \]  \hspace{1cm} (2.46)

in which \( K_{st} \) is the static stiffness of the foundation, given by

\[ K_{st} = \frac{4}{(1 - \nu_s)} G R \]  \hspace{1cm} (2.47)

\( R \) is the radius of the foundation; \( \nu_s \) is Poisson's ratio for the supporting medium; \( i = \sqrt{-1} \); \( a_0 \) is a dimensionless frequency parameter defined by

\[ a_0 = \Omega R/\nu_s \]  \hspace{1cm} (2.48)

\( \Omega \) is the circular frequency of the excitation and of the resulting steady-state response; \( \nu_s = \sqrt{G_s/\rho_s} \) is the velocity of shear wave propagation in the supporting medium, and \( a \) and \( \beta \) are dimensionless functions of \( a_0 \) and \( \nu_s \) which may be approximated by the simple expressions presented in (45). In the spring-dashpot representation of the supporting medium, the stiffness of the spring, \( k_0 \), is given by

\[ k_0 = \alpha K_{st} \]  \hspace{1cm} (2.49)

and the damping coefficient, \( c_0 \), is given by

\[ c_0 = \frac{K_{st} R}{\nu_s} \]  \hspace{1cm} (2.50)

Harmonic Response of System.— The steady-state response of the interaction model shown in Fig. 2.4b is first evaluated for a harmonic free-field acceleration, \( \ddot{x}_g(t) = \ddot{x}_g e^{i \Omega t} \).
The equation of motion for the model are

\[ m_1 \ddot{D}(t) + k_1 D(t) = -m_1 \ddot{x}_0(t) \]  \hspace{1cm} (2.51a)

\[ m_0 \ddot{x}_0(t) - k_1 D(t) + Q[x_0(t) - x_g(t)] = 0 \]  \hspace{1cm} (2.51b)

in which \( Q \) is defined by Eq. 2.46. On noting that \( D(t) = \Omega^2 D(t) \) and making use of Eqs. 2.27 and 2.45, one obtains the following expression for \( A(t) \) from Eq. 2.51a:

\[ A(t) = -\omega^2 D(t) = \frac{\ddot{x}_0(t)}{1 - \phi^2} \]  \hspace{1cm} (2.52)

in which \( \phi = \) the frequency ratio, \( \Omega/\omega \). The interrelationship of \( \ddot{x}_0(t) \) and \( \ddot{x}_g(t) \) is then determined from Eq. 2.51b by expressing \( x_0(t) \) and \( x_g(t) \) in terms of their corresponding accelerations, eliminating \( D(t) \) by use of Eq. 2.52, and replacing \( k_1 \) by the right hand member of Eq. 2.45. On substituting the resulting expression for \( \ddot{x}_0(t) \) into Eq. 2.52, one obtains:

\[ A(t) = \frac{\ddot{x}_g(t)}{1 - \phi^2 - \frac{m_1\omega^2}{k_1} \left[ 1 + (1 - \phi^2) \frac{m_0}{m_1} \right]} \]  \hspace{1cm} (2.53)

The ratio of the pseudoacceleration amplitude of the free-field ground motion, \( \ddot{x}_g \), is plotted in Fig. 2.5 as a function of \( \Omega/\omega \) for the clamped-base tanks examined earlier. The flexibility of the supporting medium in these plots is specified by the ratio of wave propagation velocity, \( v/v_s \), in which \( v = \sqrt{E/\rho} \) and \( v_s = \sqrt{G/\rho_s} \). Several different values of \( v/v_s \), including the zero value which refers to rigidly supported tanks, are considered. The corresponding values of \( v_s \) are also identified in parentheses. For the concrete tanks, the latter values were computed taking \( E = 3.3 \times 10^6 \) psi and \( \gamma = \rho g = 150 \) psf. The mass of the foundation mat and the mass of the tank in these solutions were presumed to be negligible.
and the radius of the foundation was taken equal to the radius of the tank (i.e., $m_0 = m_o$ and $R = a$).

As demonstrated in previous studies of building structures (26,41), the effects of soil-structure interaction are prominently reflected in the magnitudes and locations of the peaks of the frequency response curves in Fig. 2.5. Because of the capacity of the supporting medium to dissipate energy by radiation of waves, the effective damping of a flexibly supported tank is greater than of the associated rigidly supported tank, and this leads to a reduced resonant peak for the flexibly supported system. The reduction in the resonant peak increases in importance with increasing flexibility of the supporting medium, and it is generally greater for broad tanks than for slender tanks.

Increasing the flexibility of the supporting medium also reduces the fundamental natural frequency of the system, and this reduction might be expected to lead to a corresponding shift in the location of the resonant peak. It is noteworthy, however, that whereas the resonant frequencies of tall tanks do in fact decrease with increasing soil flexibility (decreasing value of $v_s$), the opposite is true of broad tanks, the difference being particularly prominent for the group of concrete tanks considered. The explanation for this difference is that the elastically supported system has two degrees of freedom and hence two resonant peaks, one on either side of the resonant peak for the fixed-base system. Because of the large radiational energy dissipating capacity of the supporting medium, only one of these peaks is generally prominent in the frequency response curves. For tall tanks, the controlling peak corresponds to the fundamental resonant frequency, whereas for broad tanks, it corresponds to the second resonant frequency.

Transient Response of System.— With the harmonic response of the tank-liquid-foundation system established, the response to an arbitrary transient excitation may be evaluated by Fourier transform techniques. A reasonable approximation to the absolute maximum value of the response may also be obtained by the simple procedure formulated
in the following paragraphs.

Several of the frequency response curves presented in Fig. 2.5 are compared in Fig. 2.6 and 2.7 with those obtained for a single-degree-of-freedom (SDF) system. The natural frequency and damping of which have been adjusted so that the absolute maximum or resonant value of the pseudoacceleration and the associated frequency are in each case identical to the corresponding values of the actual system. If $A_{\text{max}}$ is the absolute maximum pseudoacceleration and $f_r$ is the associated resonant frequency, the damping factor of the replacement SDF system, $\zeta$, is determined from

$$\frac{A_{\text{max}}}{x_g} = \frac{1}{2\zeta \sqrt{1 - \zeta^2}} \quad (2.54)$$

and its natural frequency, $\tilde{f}$, is determined from

$$f_r = \frac{\tilde{f}}{\sqrt{1 - 2\zeta^2}} \quad (2.55)$$

The quantities $\tilde{f}$ and $\zeta$ will also be referred to as the effective natural frequency and the effective damping factor of the actual system. While they differ at high values of the abscissa, the two sets of curves in Figs. 2.6 and 2.7 are generally in excellent agreement over wide frequency ranges on either side of the resonant peak. Furthermore, where the differences are substantial, the response values are generally quite small.

In the context of Fourier analysis, any transient excitation may be viewed as a linear combination of harmonic motions of different amplitudes and frequencies. In as much as the components of the excitation with frequencies close to that of the resonant peak are likely to be the dominant contributors to the response, the peak responses of the actual system and the replacement oscillator can be expected to be in satisfactory agreement for arbitrary transient excitations as well. It follows that the flexibly supported system in this approach may be analyzed as a SDF oscillator with a natural frequency $\tilde{f}$ and
damping factor \( \tilde{\zeta} \) subjected to the prescribed free-field ground motion.

Effective Natural Frequency and Damping.-- The values of \( \tilde{\omega} \) and \( \tilde{\zeta} \) for the clamped-base tanks examined are given in Fig. 2.8. Note that the maximum damping factor is obtained for tanks with values of \( H/a \) close to unity, and that the effective natural frequency of such tanks are generally quite sensitive to variations in \( H/a \). This sensitivity stems from the fact that the two resonant peaks are nearly equal in this case.

The damping values displayed in Fig. 2.8 are due exclusively to the radiational energy dissipation capacity of the supporting medium. The tank-liquid system was presumed to be undamped in these solutions, and no material damping was considered for the supporting medium. The effects of structural damping may be accounted for readily, and the effects of soil material damping may be evaluated by use of the impedance functions for a viscoelastic medium presented in (45). However, in view of the already large radiational damping present, the maximum response is expected to be generally insensitive to these additional sources of damping.

2.7 SUMMARY OF PROPOSED DESIGN PROCEDURE

For design purposes, the maximum axisymmetric response of a vertically excited liquid containing tank may be evaluated as follows:

1. By application of Eq. 2.22 or the data presented in Table 2.2 and 2.3, evaluate the fundamental natural frequency of axisymmetric vibration of the tank-liquid system in its rigidly supported condition.

2. Using soil properties that are compatible with the stipulated design ground motion, evaluate the relative flexibility parameter for the soil and tank, \( v/v_s \), and from plots such as those presented in Fig. 2.8, determine the effective natural frequency and effective damping factor of the system, \( \tilde{\omega} \) and \( \tilde{\zeta} \), respectively.

3. From the response spectra for SDF systems subjected to the stipulated free-field
ground motion, evaluate the pseudoacceleration, A (Ref. 34), corresponding to the values of \( \bar{f} \) and \( \bar{\zeta} \).

4. The equivalent static radial pressure that defines the maximum hydrodynamic effects is then determined from Eq. 2.38, or more conveniently but somewhat less accurately, from Eq. 2.42. With the latter approximation, the maximum hydrodynamic effects for the tank wall are computed from the corresponding hydrostatic effects merely by multiplying the latter effects by the ratio \( A/g \), in which \( g \) = the gravitational acceleration.

If the response of the rigidly supported tank is desired, the pseudoacceleration value in step 4 should correspond to the natural frequency and damping of axisymmetric mode of vibration of the rigidly supported tank-liquid system. The damping factor in this case is likely to be of the order of about 2 percent critical.

2.8 CONCLUSION

With the method of analysis and the information presented herein, the axisymmetric response of both rigidly and flexibly supported, circular cylindrical, liquid containing tanks to a vertical component of ground shaking may be evaluated readily and with good accuracy. The maximum hydrodynamic effects in the tank wall may be considered to be proportional to the hydrostatic effects, the proportional factor being \( A/g \), in which \( A = \) the spectral value of the pseudoacceleration for a replacement simple oscillator subjected to the prescribed free-field ground motion, and \( g = \) the gravitational acceleration. Soil-structure interaction reduces the maximum hydrodynamic effects. The consequences of such interaction may be approximated by a change in the fundamental natural frequency of the tank-liquid system and by an increase in damping. The maximum value of effective damping for the system examined herein was 35 percent of the critical value.
CHAPTER 3

VERTICAL VIBRATION OF TANKS - MULTIMODE ANALYSIS

3.1 INTRODUCTION

The purpose of this chapter is twofold: (a) to present a more accurate method of analyzing the axisymmetric, vertical response of a flexibly supported tank than the one presented in the preceding chapter; and (b) to assess the accuracy of the one-mode solution presented in that chapter.

Since the method of analysis utilizes the substructuring approach, the response of the rigidly supported tank-liquid system is considered first, and then the effects of soil-structure interaction are examined. The system analyzed is the same as that considered in the preceding chapter and shown in Fig. 2.1.

Although the response of the rigidly supported tank-liquid system to a vertical excitation has been studied by Kumar (27) and Veletsos and Kumar (44) using the modal expansion method, and by Haroun and Tayel (19) using the finite element method, its solution is reconsidered for the sake of completeness and to provide the information needed for the analysis of the soil-structure interaction effects.

The method of analysis employed here is fundamentally the same as that used by Kumar, except that it is carried out in the time domain rather than the frequency domain.

3.2 METHOD OF SOLUTION

The differential equation governing the motion of the vertically excited tank-liquid system, and the hydrodynamic pressure induced at an arbitrary point of the liquid are given by Eqs. 2.1 and 2.6 of the preceding chapter, respectively.

The radial displacement, \( w(\xi, t) \), in this case is expressed as a linear combination of the natural modes of lateral vibration of a uniform cantilever beam, \( w_j(\xi) \), i.e.,

\[
 w(\xi, t) = \sum_{j=1}^{\infty} R_j(t; \gamma_j(\xi)) 
\]  

(3.1)
in which the time-dependent coefficients, $R_j(t)$, represent a set of generalized coordinates.

The natural modes, $w_j(\xi)$, satisfy the differential equation

$$\frac{d^4w_j(\xi)}{d\xi^4} = \epsilon_j^4 w_j(\xi)$$  \hspace{1cm} (3.2)

in which $\epsilon_j$ is the eigenvalue corresponding to the $j$th natural mode of the beam. The first three values of $\epsilon$ are (4)

$$\epsilon_1 = 1.8751 \hspace{1cm} \epsilon_2 = 4.6941 \hspace{1cm} \epsilon_3 = 7.8547$$

Incidentally, $w_j(\xi)$ is also equal to the $j$th axisymmetric mode of vibration of the empty tank.

If $p_m$ be the $m$th natural frequency of the empty tank, it has been shown (27) that it is related to the value $\epsilon_m$ by the equation

$$p_m \omega_o = \sqrt{1 + (\epsilon_m/\delta)^4}$$  \hspace{1cm} (3.3)

in which $\delta$ and $\omega_o$ are defined by Eqs. 2.9 and 2.23 of chapter 2, respectively.

Substituting Eq. 2.6 of chapter 2 and Eq. 3.2 into Eq. 2.1 of chapter 2; then, by making use of the orthogonal property of the beam modes, one obtains the following equation

$$\ddot{R}_j(t) + P_j R_j(t) = -\frac{\rho}{\rho} \frac{1}{\int_0^l w_j^2(\xi) d\xi} \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{I_0(\lambda_n)}{\lambda_n I_0(\lambda_n)} d_{mn} d_j \ddot{R}_m$$

$$+ \frac{\rho H}{\rho} \frac{1}{\int_0^l w_j^2(\xi) d\xi} \int_0^l (1-\xi) w_j(\xi) d\xi j = 1, 2, \ldots, \infty$$  \hspace{1cm} (3.4)

where $\lambda_n$ is defined by Eq. 2.7 of chapter 2 and

$$d_{jn} = \int_0^l w_j(\xi) \cos[(2n-1)\pi/2\xi] d\xi$$
Eq. 3.4 represents an infinite set of simultaneous ordinary differential equations in the unknown functions, \( R_j(t) \). An approximate solution can be obtained by truncating the series expansion defined by Eq. 3.1 to a finite number of terms, \( M_1 \). Numerical studies carried out by Kumar (27) have shown that excellent accuracy to the first few natural frequencies can be obtained by considering \( M_1 = 25 \) for \( H/a \leq 1 \) and \( M_1 = 30 \) for \( H/a > 1 \). In the present study, \( M_1 = 30 \) was used for all \( H/a \) values. Thus Eq. 3.4 leads to a system of thirty simultaneous equations with thirty unknown functions, which can be expressed in matrix form as

\[
[A][\ddot{R}] + [B][R] = [C]\dot{\phi}_0(t)
\]

(3.5)

in which \([A]\) and \([B]\) are square matrices of size 30x30, with the \((m,k)\) element of \([A]\) given by

\[
a_{mk} = \delta_{mk} + \frac{2a}{h} \int_0^1 \frac{1}{W_m(\xi)} \sum_{n=1}^{\infty} \frac{I_0(\lambda_n)}{\lambda_n I_1(\lambda_n)} d_{mn} \phi_{kn} \]

(3.6a)

and the corresponding element of \([B]\) given by

\[
b_{mk} = \rho_m \delta_{mk}
\]

(3.6b)

where

\[
\delta_{mk} = \begin{cases} 
1 & m = k \\
0 & m \neq k 
\end{cases}
\]

The \( m \)th element of the column matrix \([C]\) is given by

\[
c_m = \frac{\frac{p H}{p h} \frac{1}{W_m(\xi)}}{\int_0^1 W_m^2(\xi) d\xi} \int_0^1 (1 - \xi) W_m(\xi) \phi_0 d\xi
\]

(3.7)

The method of modal analysis is employed to solve Eq. 3.5. Let \( \omega_j \) and \( \phi_j \) be the
jth eigenvalue and jth eigenvector of the associated homogeneous equation of Eq. 3.5, respectively; then, it can be shown that the solution to Eq. 3.5 is

$$\{R(t)\} = \sum_{j=1}^{M_1} C^U_j \left(A_j(t)\omega_j^2\right)\{\phi\}$$  (3.8)

in which $A_j(t)$ is the pseudoacceleration function of the jth mode. defined as

$$A_j(t) = \omega_j \int_0^t x_o(\tau) \sin\omega_j(t-\tau) \, d\tau$$  (3.9)

and $C^U_j$ is the participating factor which is given by the equation

$$C^U_j = \frac{\{\phi\}^T_j \{c\}}{\{\phi\}^T_j [A] \{\phi\}_j} = \frac{\{\phi\}^T_j \{c\}}{\{\phi\}^T_j [B] \{\phi\}_j} \omega_j^2$$  (3.10)

It can be shown that the second time derivative of $A_j(t)$ is related to $A_j(t)$ by the equation

$$\ddot{A}_j(t) = \omega_j^2 \left(x_o(t) - A_j(t)\right)$$  (3.11)

Taking the second derivative of Eq. 3.8 and making use of Eq. 3.11, one obtains

$$\ddot{\{R(t)\}} = \sum_{j=1}^{M_1} C^U_j \left(\ddot{x}_o(t) - A_j(t)\right)\{\phi\}_j$$  (3.12)

Further, on introducing Eq. 3.8 into Eq. 3.1, one obtains

$$W(\xi, t) = \sum_{j=1}^{M_1} \sum_{i=1}^{M_1} C^U_j \frac{A_j(t)}{\omega_j^2} \phi_{i,j} W_i(\xi)$$  (3.13)

and its second time derivative is

$$\ddot{W}(\xi, t) = \sum_{j=1}^{M_1} \sum_{i=1}^{M_1} C^U_j \left(\ddot{x}_o(t) - A_j(t)\right) \phi_{i,j} W_i(\xi)$$  (3.14)

In Eqs. 3.13 and 3.14, $\phi_{i,j}$ is the ith element of the jth eigenvector, $\{\phi\}_j$. 
The following expression for the hydrodynamic pressure may now be obtained by substituting Eq. 3.14 into Eq. 2.6 of chapter 2:

\[
P(r, \xi, t) = -2 \int_0^a \sum_{j=1}^{M_1} \sum_{n=1}^{\infty} C_j \left( \frac{r}{\lambda_n a} \right) \cos \left( (2n-1) \frac{\pi}{a} \xi \right) \frac{d\xi}{\lambda_n I_n(\lambda_n)} \frac{\partial}{\partial \xi} \frac{d_{jn} \phi_j}{ \lambda_n I_n(\lambda_n)}
\]

\[
\left( \ddot{\xi}_o(t) - A_j(t) \right) + \int_0^\infty \rho H \left( (1-\xi) \ddot{\xi}_o(t) \right)
\]

With the pressure at an arbitrary point evaluated, the total hydrodynamic force exerted on the tank base, \( P_o(t) \), may be obtained by setting \( \xi = 0 \) in Eq. 3.15 and substituting the resulting expression into Eq. 2.43 of chapter 2:

\[
P_o(t) = m_o \ddot{\xi}_o(t) + \sum_{j=1}^{M_1} m_j A_j(t)
\]

(3.16)

in which

\[
m_j = \left\{ \int_0^\infty \frac{H}{a} \left[ \sum_{j=1}^{M_1} \phi_j \sum_{n=1}^{\infty} \frac{1}{\lambda_n I_n(\lambda_n)} \frac{d_{jn}}{n^2} \right] C_j \right\} m_j
\]

(3.17)

and

\[
m_o = m_L - \sum_{j=1}^{M_1} m_j
\]

(3.18)

The quantities \( m_o \) and \( m_j \) have the same physical meaning as those presented in the preceding chapter.

The pressure acting on the tank wall is obtained from Eq. 3.15 by letting \( r = a \). The result can be expressed as

\[
P(\xi, t) = \int_0^a \rho H \left( 1-\xi \right) \ddot{\xi}_o(t) - \int_0^\infty \sum_{j=1}^{M_1} C_j \left[ \ddot{\phi}_j(t) - A_j(t) \right]
\]

(3.19)

in which
\[ P_x(\xi) = \sum_{i=1}^{M_1} \left\{ \frac{A_i}{(2\pi)^2 I_0(\lambda_i)} \right\} \frac{d^n}{d\lambda^2} W(\lambda_i) \cos\left\{ \frac{(2n-1)\pi}{2} \frac{\lambda_i}{\xi} \right\} \]  
\hspace{1cm} (3.20)

Having determined the hydrodynamic wall pressure, one may proceed to compute the equivalent static radial pressure, \( p(\xi, t) \). In Chapter 2, this pressure was computed from the equation

\[ \tilde{p}(\xi, t) = D \frac{\partial W}{\partial Z^2} + \frac{E h}{a^2} W \]  
\hspace{1cm} (3.21)

For the approximating deflection function employed, this approach is not satisfactory, however, it leads to a zero pressure at \( z = 0 \), which is obviously not correct. Alternatively, \( \tilde{p}(\xi, t) \) can be computed from the equation

\[ \tilde{p}(\xi, t) = p(\xi, t) - \int_{0}^{\xi} \frac{d^2 W}{d\xi^2} \]  
\hspace{1cm} (3.22)

which on making use of Eqs. 3.14 and 3.19 may be expressed as

\[ \tilde{p}(\xi, t) = P_x + \sum_{j=1}^{M_1} P_{\psi_j}(\xi) A_j(t) \]  
\hspace{1cm} (3.23)

where

\[ P_{\psi_j}(\xi) = C_j \left[ P_{\psi_j}(\xi) + (\rho h \rho_x H \psi_j(\xi)) \right] \]  
\hspace{1cm} (3.24)

and

\[ \psi_j(\xi) = \sum_{i=1}^{M_1} \phi_{ij} w_i(\xi) \]  
\hspace{1cm} (3.25)

In the derivation of Eq. 3.23, the following identity has been used

\[ 1 \cdot \xi = \sum_{\alpha=1}^{M_1} P_{\psi^* \alpha}(\xi) \]  
\hspace{1cm} (3.26)

the proof of which is given by Kumar (27). On making use of this identity, Eq. 3.19 can
also be written as

$$P(\xi, t) = \rho_r H \sum_{j=1}^{M_1} C_{p,j}^U \left[ \Delta\psi_j(\xi) A_j(t) + \frac{\rho_h}{\rho_r H} \psi_j(\xi) \ddot{x}_o(t) \right]$$

(3.27)

With \(\bar{p}(\xi, t)\) defined by Eq. 3.23, the total equivalent static force exerted against the tank wall per unit of circumferential length, \(P_w\) is given by

$$P_w = \rho_r H^2 \sum_{j=1}^{M_1} C_{p,j} A_j(t)$$

(3.28)

in which

$$C_{p,j} = \int_0^1 \bar{P}_{\psi,j}(\xi) d\xi$$

(3.29)

It may be of interest to note that if the values of \(A_j(t)\) in Eqs. 3.16, 3.23, 3.27 and 3.28 are taken equal to the ground acceleration, i.e., \(A_j(t) = \ddot{x}_o(t)\) for all \(j\), the results should reduce to those induced in a very stiff tank. Indeed, Eqs. 3.23 and 3.27 reduce to

$$\bar{p}(\xi, t) = \ddot{x}_o(t) = \rho_r H (1 - \xi) \ddot{x}_o(t)$$

(3.30)

and Eq. 3.16 reduces to

$$P_o(t) = m \ddot{x}_o(t)$$

(3.31)

Similarly, Eq. 3.26 reduces to

$$P_w = \frac{1}{2} \rho_r H^2 \ddot{x}_o(t)$$

(3.32)

These expressions are identical to those for rigid tanks.

3.3 MECHANICAL MODEL FOR TANK-LIQUID SYSTEM

As noted in the chapter 2, a mechanical modal is helpful in the analysis of soil-
structure interaction effects. The mechanical modal for the total force exerted on the tank base is shown in Fig. 3.1. It consists of a base mass, \( m'_0 \), and M1 sprung masses, \( m_i \). The base mass is equal to the sum of the foundation mass, \( m_F \), and of the rigidly attached liquid mass, \( m_o \). The spring constant, \( k_i \), for the \( i \)th mass is chosen such that the natural frequency of the \( i \)th mass is the same as that of the \( i \)th axisymmetric mode of vibration of the actual tank liquid system, i.e.,

\[
k_i = m_i \omega_i^2
\]

(3.33)

where \( \omega_i \) is the \( i \)th circular frequency.

This model does not consider the effect of the tank inertia in the axial direction. However, this effect is believed to be small and can be accounted for approximately by adding the tank mass to the base mass. This approach, which effectively considers the tank to behave as a rigid body in the vertical direction is acceptable as axial vibration of the tank wall is much higher than that for the radial mode of vibration of the tank-liquid system.

3.4 PRESENTATION OF DATA

The values of the first three sprung masses, \( m_1 \) through \( m_3 \), are listed in columns 4 through 6 of Table 3.1 for two groups of tanks filled with water: (1) steel tanks with \( h/a = 0.001 \), \( \nu = 1/3 \) and \( \rho_f/\rho = 0.127 \); (2) concrete tanks with \( h/a = 0.01 \), \( \nu = 0.17 \) and \( \rho_f/\rho = 0.4 \). Several different values of \( H/a \) in the range between 0.3 and 5 are considered. Also listed in columns 1 through 3 of the table are the corresponding values of \( C_{pj} \).

On comparing the first mode data listed in Table 3.1 with the corresponding data presented in chapter 2, one concludes that, excepting the results for concrete tank with \( H/a = 0.3 \), the solutions presented in Ch. 2 is indeed a good approximation to the first mode solution.
In Fig. 3.2, the equivalent static wall pressures for the first mode of tanks with different values of $H/a$ are compared with those computed in Ch. 2 by the Galerkin method. It can be seen that the agreement between the two sets solutions is quite good. Also included for comparison is the triangular distribution which is strictly valid for rigid tanks.

3.5 FLEXIBLE SUPPORTED TANKS

In this section, the equations of motion of the tank-liquid-foundation system are established, and they are solved approximately by considering an increasing number of natural modes for the superstructure.

The system is modeled in the manner shown in Fig. 3.3a, in which $k_o$ and $c_o$ represent the spring and dashpot of the supporting medium. The latter quantities are defined by Eqs. 2.49 and 2.50, respectively.

The equations of motion are formulated by considering free body diagrams defined by cuts through the foundation-halfspace interface and the structure-foundation interface, as shown in Fig. 3.3b. The total force transmitted from the sprung masses to the base, $F_T$, is given by

$$F_T = \sum_{i=1}^{m_l} A_i(t)$$  \hspace{1cm} (3.34)

and the restraining force exerted by the supporting medium, $V_b$, is given by

$$V_b = Q(x_0 - x_g)$$  \hspace{1cm} (3.35)

in which $Q$ for a harmonic motion is defined by Eq. 2.46. Finally, the dynamic equilibrium of the forces acting on the foundation mass requires that

$$m_o\ddot{x}_0(t) + F_T + V_b = 0$$  \hspace{1cm} (3.36)
For a harmonic free-field acceleration, \( \ddot{x}_g(t) = \dot{x}_g e^{i\Omega t} \), substitution of Eqs. 3.34 and 3.35 into Eq. 3.36 yields
\[
-\Omega^2 m'_o x_o + \sum_{\lambda=1}^{N} (AF)_{\lambda} (-\Omega^2 x_\lambda) m_\lambda + Q(x_o - x_d) = 0
\]  
(3.37)

in which \( A_\lambda(t) \) has been replaced by
\[
A_\lambda(t) = (AF)_{\lambda} (-\Omega^2 x_\lambda) e^{i\Omega t}
\]
(3.38)

where
\[
(AF)_{\lambda} = \frac{1}{1 - \phi_i^2}
\]
(3.39)

\( \phi_i = \) the frequency ratio, \( \Omega/\omega_i \)

Note that the upper limit of the summation in Eq. 3.37 has been changed from \( M_1 \) to \( N \). Theoretically, \( N = M_1 \); however, as demonstrated in Ref. 8 for a related problem, good accuracy can be achieved by considering only the first few modes of the superstructure, i.e., by taking \( N \) much smaller than \( M_1 \). That this is true may be appreciated from Eq. 3.34 by comparing the relative magnitudes of the masses \( m_i \) listed in Table 3.1. If all the values of \( A_\lambda \) fall in the constant pseudoacceleration region of the response spectrum, the relative magnitude of the various terms will be proportional to the relative value of \( m_i \), and it is clear from Table 3.1 that \( F_T \) may be approximated with good accuracy even by considering only the first mode. It should further be noted that since the total masses have to be equal to \( m_f \), in Eq. 3.18 \( m_0 \) has to be interpreted as
\[
m_0 = m_f \cdot \sum_{\lambda=1}^{N} m_i
\]
(3.40)

Eq. 3.40 implies that for the modes higher than \( N \), the natural frequencies are so high that they can be treated as rigid modes.

The ratio of \( x_o \) to \( x_g \) is determined from Eq. 3.37 to be
\[
\frac{x_o}{x_g} = 1 - \frac{\alpha^2}{Q} \left[ m'_o + \sum_{i=1}^{N} (AF)_i m_i \right]
\]  

(3.41)

and the ratio of \(A_i\) to \(x'_g\) is related \(x_o/x_g\) by the following equation:

\[
A_i/x'_g = (AF)_i (x_o/x_g)
\]  

(3.42)

No damping has been considered for the superstructure. The effects of structural damping may be examined by introducing the damping ratio, \(\xi_i\), into each vibration mode, which is a well accepted procedure for building type structures. By doing so, the Eq. 3.37 has to be interpreted as:

\[
(AF)_i = \frac{1}{(1 - \phi^2) + 2\xi_i \phi^2}
\]  

(3.43)

Since it is believed that the structure damping comes from the friction between the molecules of the tank wall, the damping force doesn't get transmitted to the foundation; therefore, the force transmitted by the sprung masses is still given by Eq. 3.34, in which \(A_i\) is defined by Eqs. 3.38 and 3.43.

### 3.6 PRESENTATION OF RESULTS

The force \(P_w\), defined by Eq. 3.28, is chosen as a measure of the effect of the soil-structure interaction. There are two reasons for this choice: (1) \(P_w\) is a measure of the total hydrodynamic force exerted on the tank wall, and (2) it is easy to calculate.

The variation of the amplitude of this force, \(|P_w|\), with \(\phi_1\) is displayed in Figs. 3.4 through 3.9. The results were computed by considering a progressively larger number of natural modes of vibration, and are displayed in dimensionless form, in term of the amplification factor, \(AF^*\), defined as:

\[
AF^* = \frac{|P_w|}{P_{w_0}}
\]  

(3.44)
in which \( P_{wo} = 0.5 \rho \xi H^2 \gamma \) is the total equivalent static force per unit of circumferential length exerted against the wall of a rigid tank. Three different values of \( H/a \) and two values of \( v/v_s \) are considered. The curves on the left of each figure were obtained assuming no damping for the superstructure and the curves at right correspond to a modal damping factor of 2 percent of the critical value for each mode of vibration of the superstructure. The mass of the foundation mat in these solutions was presumed to be negligible, and the radius of the foundation was taken equal to the radius of the tank (i.e., \( m'_o = m_o \) and \( R = a \)).

Examination of the results obtained considering damping in the superstructure reveals that the one-mode and multi-mode solutions are generally in very good agreement. Since the response for any transient excitation can be expressed as a linear combination of the responses to harmonic excitation of different amplitude and frequencies, it should be clear that the transient responses calculated using a single mode and many modes should be in satisfactory agreement as well. Also, because the approximate solution based on the Galerkin method presented in the preceding chapter is a good approximation of both the first mode solution and the solution based on the SDF replacement oscillator, it can be concluded that the latter solution is a good approximation to the true response. Expressed differently, the transient response of the liquid-tank-foundation system subjected to vertical excitation can be computed with good accuracy by replacing the system with SDF oscillator considered in the preceding chapter as long as the dominant frequency of the excitation is smaller than the second natural frequency of the tank-liquid system.

3.7 CONCLUSION

The conclusions of this chapter are summarized below:

1. As long as the dominant frequency of the excitation is smaller than the second natural frequency of the tank-liquid system, one mode solution is a good approximation to
multi-mode solution.

2. The response obtained by Galerkin's method presented in the preceding chapter is a good approximation to one mode solution presented in this chapter.

3. The response of any transient excitation may be computed with good accuracy by replacing the system with SDF oscillator considered in the preceding chapter.
CHAPTER 4
LATERAL VIBRATION OF TANKS

4.1 INTRODUCTION

This chapter deals with the lateral response of the tank-liquid system considered in Chapters 2 and 3. The tank is assumed to be fixed to a rigid foundation that experiences the same motion as the free-field ground motion. The method of analysis employed is the same as that described by Yang (50) and Veletsos and Yang (46,47). Specifically, the displacements of the tank are assumed to be linear combinations of the natural vibration modes of a uniform cantilever beam or the derivatives of those modes: the differential equations of motion are then obtained by making use of Lagrange's equations. The same approach was also used in Ref.(2) to evaluate the natural frequencies of the tank-liquid system, but the added liquid mass in the latter study was evaluated approximately from Ref.(4) using the solution for a tank hinged at the base.

The objectives of this study are: (a) to present more comprehensive numerical data for natural frequencies, base shear and overturning moment than those reported in Refs.(43), (47) and (50); and (b) to assess the accuracy and range of applicability of a simple approximate procedure proposed by Veletsos and Yang (47) and further discussed by Veletsos in Ref. (43).

4.2 SYSTEM AND ASSUMPTIONS

The tank-liquid system considered is shown in Fig. 4.1. It is the same as that shown in Fig. 2.1 except that the tank is considered to be fixed to a rigid base and to be excited horizontally by a ground motion with an acceleration, \( \ddot{x}_g(t) \). Only the impulsive component of the response is investigated. For the reasons elaborated in Refs. (43) and (47), it is believed that the convective part of the solution can be evaluated with reasonable accuracy from the corresponding solution for a rigid tank.
4.3 METHOD OF ANALYSIS

The analysis is carried out by use of the Rayleigh-Ritz method and Flugge's theory of cylindrical shells (13). The axial, tangential and radial components of the displacement of a point on the shell, \( u, v \) and \( w \), are expressed in the truncated series form as

\[
u(z, \theta, t) = \sum_{i=1}^{N_1} U_i(t) \chi_i(z) \cos \theta \tag{4.1}
\]

\[
v(z, \theta, t) = \sum_{i=1}^{N_2} V_i(t) \psi_i(z) \sin \theta \tag{4.2}
\]

\[
w(z, \theta, t) = \sum_{i=1}^{N_3} W_i(t) \psi_i(z) \cos \theta \tag{4.3}
\]

where \( \psi_i(z) \) is the \( i \)th natural mode of the lateral vibration of a uniform cantilever beam, and \( \chi_i(z) \) is its derivative. \( N_1, N_2 \) and \( N_3 \) are the number of functions used to approximate the displacements, and \( U_i(t), V_i(t) \) and \( W_i(t) \) are time-dependent coefficients with units of length indicating the degree of participation of each modal shape. The functions \( \psi_i(z) \) and \( \chi_i(z) \) are defined by the equations

\[
\psi_i(z) = \cosh(\beta_i \xi) - \cos(\beta_i \xi) - a_i[\sinh(\beta_i \xi) - \sin(\beta_i \xi)] \tag{4.4}
\]

\[
\chi_i(z) = \sinh(\beta_i \xi) + \sin(\beta_i \xi) - a_i[\cosh(\beta_i \xi) - \cos(\beta_i \xi)] \tag{4.5}
\]

where \( \beta_i \) and \( a_i \) are constants given in Table 4.1. These functions and the associated constants are reproduced from Young and Felgar (12). The displacement functions \( u(z, \theta, t), v(z, \theta, t) \) and \( w(z, \theta, t) \) defined by Eqs. 4.1 through 4.3 are geometrically admissible since they satisfy the boundary conditions at the base of the tank, i.e., \( u=0, v=0, w=0 \) and \( \partial w/\partial z=0 \).

The coefficients \( U_i, V_i \) and \( W_i \) of Eqs. 4.1-3 can be obtained by solving the following set of ordinary differential equations
\[
\begin{pmatrix}
A & 0 & 0 \\
0 & B & 0 \\
0 & 0 & C
\end{pmatrix}
\begin{bmatrix}
\ddot{U} \\
\ddot{V} \\
\ddot{W}
\end{bmatrix}
+ \begin{pmatrix}
D & E & F \\
G & H & I \\
\text{Sym.} & I & I
\end{pmatrix}
\begin{bmatrix}
U \\
V \\
W
\end{bmatrix}
= - \begin{pmatrix}
P_U \\
P_V \\
P_W
\end{pmatrix} \ddot{x}_o(t)
\]  

(4.6a)

which may be written in a more compact form as

\[
[M][\dot{q}] + [K][q] = - [P][\ddot{x}_o(t)]
\]  

(4.6b)

In this expressions, \(u, v\) and \(w\) are the column matrices of \(U_i, V_i\) and \(W_i\), respectively; \(q\) is a column matrix defined by

\[
(q) = (U_1(t), U_2(t), ..., U_{N_1}(t), V_1(t), V_2(t), V_{N_2}(t), W_1(t), W_2(t), ..., W_{N_3}(t))^T
\]  

(4.7)

in which the superscript \(T\) denotes the transpose of the row matrix; \(A\) through \(I\) are square submatrices, and \(P_U, P_V\) and \(P_W\) are column matrices. These matrices are given by Yang (50) and are reproduced in Appendix A. Eq. 4.6a was obtained by making use of Lagrange's equation. Eqs. 4.1 through 4.3 were used to express the kinetic energy and strain energy of the shell, and the hydrodynamic wall pressure and tank inertia for a rigid tank were used to express the virtual work done by the system. The detailed derivation of Eq. 4.6a can be found in Ref. (50).

The impulsive component of the hydrodynamic pressure, \(P(r, \xi, t)\), is related only to the acceleration in the radial direction. In particular, if the radial acceleration is given by

\[
\dddot{w}(z, \theta, t) = \sum_{\ell=1}^{N_3} \dddot{W}_\ell(t) \psi_\ell(z) \cos \theta
\]  

(4.8)

then, the impulsive component of the hydrodynamic pressure is given by (Ref. 50)

\[
P(r, \xi, t) = \left\{ \frac{H}{a} \sum_{\ell=1}^{N_3} \dddot{W}_\ell(t) \sum_{n=1}^{a} \frac{4}{(2n-1)\pi} \frac{I_n[(2n-1)\pi \xi]}{I_n[(2n-1)\pi a \frac{H}{2}]} \right\} d_n \cos \left\{ \frac{(2n-1)\pi \xi}{2} \right\} 
\]  

\[
\int_{0}^{1} \rho \alpha \cos \theta
\]  

(4.9)
in which \( W_i(t) \) is an arbitrary time-dependent function.

\[
d_{in} = \frac{1}{H} \int_0^H \psi_i(x) \cos \left[ \frac{2\pi n - 1}{2} \frac{x}{H} \right] dx
\] (4.10)

and \( \xi = x/H \).

### 4.4 FREE VIBRATION

The equations for free vibration can be obtained from Eq. 4.6b by deleting the

[\mathbf{M}] \{\ddot{q}\} + [\mathbf{K}] \{q\} = 0

(4.11)

Note that the presence of the liquid has no effect on the stiffness matrix, \([\mathbf{K}]\), but does influence the mass matrix, \([\mathbf{M}]\).

The solution of Eq. 4.11 is obtained in the usual manner by letting

\[
\{q\} = (\phi) \sin(\omega t + \alpha)
\] (4.12)

and solving the resulting eigenvalue problem

\[
[K]\phi = \omega^2[M]\phi
\] (4.13)

In particular, if

\[
u(z, \theta, t) = \sum_{i=1}^{N^2} \hat{U}_i \chi_i(z) \cos \theta \sin(\omega t + \alpha)
\]

\[
v(z, \theta, t) = \sum_{i=1}^{N^2} \hat{V}_i \psi_i(z) \sin \theta \sin(\omega t + \alpha)
\] (4.14)

\[
w(z, \theta, t) = \sum_{i=1}^{N^3} \hat{W}_i \psi_i(z) \cos \theta \sin(\omega t + \alpha)
\]

Then

\[
\{\phi\} = (\hat{U}_1, \hat{U}_2, ..., \hat{U}_{N_1}, \hat{V}_1, \hat{V}_2, ..., \hat{V}_{N_2}, \hat{W}_1, \hat{W}_2, ..., \hat{W}_{N_3})^T
\] (4.15)
The natural frequencies are determined by evaluating the roots of the determinental equation $[K] - \omega^2[M] = 0$. Once these have been determined, the vector $\{\phi\}$ corresponding to these values are determined from Eq. 4.13. The $ith$ circular natural frequency of the system, $\omega_i$, may conveniently be expressed in the form

$$\omega_i = \frac{C_i}{H} \sqrt{\frac{E}{\rho}}$$

(4.16)

in which $C_i$ is a dimensionless coefficient. Values of $C_1$ through $C_3$ were obtained using different combinations of $N1$, $N2$ and $N3$ for steel tanks filled with water ($\nu = 0.3$, $\rho_\lambda/\rho = 0.127$ and $h/a=0.001$) and values of $H/a=0.5$, 1 and 3. The results are listed in Table 4.2, from which it should be clear that the combination of $N1=10$, $N2=15$ and $N3=20$ gives satisfactory accuracy up to the third mode. All solutions presented herein were obtained using this combination of terms.

Numerical data for $C_1$ through $C_3$ are presented in Table 4.3 for steel tanks with several different values of $h/a$ in the range between 0.0005 and 0.01 and values of $H/a$ in the range between 0.3 and 3.0. The results for $C_1$ are also displayed graphically in Fig. 4.2. Fig. 4.3 shows the variation of $C_1$ through $C_3$ for steel tanks with $h/a=0.001$, and Fig. 4.4 shows the corresponding curves for concrete tanks ($\nu = 0.17$, $\rho_\lambda/\rho = 0.4$ and $h/a=0.01$). The data used to prepare the latter curves are listed in Table 4.4. From these data and those listed in Table 4.3, it is concluded that the values of $C$ corresponding to two different values of $h/a$, identified with the subscripts $i$ and $j$, are interrelated approximately by the equation

$$\frac{C_i}{C_j} = \left[\frac{\left(\frac{h}{a}\right)_i}{\left(\frac{h}{a}\right)_j}\right]^{1/2}$$

(4.17)

The modal displacements, $u$, $v$ and $w$, are computed from Eq. 4.14, and the radial component of the modal acceleration for the tank wall, $\ddot{w}(z,\theta,t)$, is related to the corre-
sponding displacement by the equation

\[ \ddot{w}(z, \theta, t) = -\omega^2 w(z, \theta, t) \]

\[ = \sum_{\lambda=1}^{N^3} \hat{W}_\lambda \left[ -\omega^2 \sin(\omega t + \alpha) \right] \psi_\lambda(x) \cos \theta \]

The radial acceleration defined by Eq. 4.18 is the same form as that defined by Eq. 4.8. Therefore, the modal hydrodynamic pressure can be obtained from Eq. 4.9 by interpreting the quantity \( \hat{W}_1(t) \) in Eq. 4.8 as the quantity \( -\omega^3 \hat{W}_1 \sin(\omega t + \alpha) \) in Eq. 4.18. The result is given by

\[ P(r, \xi, \tau) = \left\{ \frac{M}{\underline{A}} \sum_{\lambda=1}^{N^3} \hat{W}_\lambda \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi^2} \frac{r}{I_1}\left[ \frac{2n-1}{H} \right] \frac{\rho}{\underline{A}} \cos \theta \left[ -\omega^2 \sin(\omega t + \alpha) \right] \right\} \]

In Fig. 4.5 are shown the distributions of the hydrodynamic wall pressure corresponding to the first mode of vibration of steel tanks with \( H/a = 0.5, 1, 3 \), along with the radial component of the associated displacement, \( w \). Fig. 4.6 is the counterpart of Fig. 4.5 for concrete tanks. It can be seen that the peak ordinates for both the pressure and the displacement shift from the bottom to the top as \( H/a \) increases; also, for the tall tanks, the displacement mode tends to that of a cantilever beam.

4.5 FORCED VIBRATION OF TANK-LIQUID SYSTEM

The primary objective of a forced vibration analysis is to evaluated the base shear and moments induced in the tank by the ground shaking. These quantities are affected by both the hydrodynamic pressure and the inertia effects of the tank itself.

The solution of Eq. 4.6b is obtained by the modal superposition method, which enables one to consider only those modes that contribute significantly to the response of the system. The procedure is first used to evaluate the spatial and temporal variations of the hydrodynamic pressure for each mode of vibration; then the modal components of the
base shear and overturning moment are computed, and the total effects are finally obtained by superposition of the component effects.

Application of Modal Superposition Method:-- The displacement vector, \( \{q(t)\} \), in this case is expressed in the form

\[
\{q\} = \sum_{k=1}^{N} \{q\}_k = \sum_{k=1}^{N} \eta_k(t) \{\phi\}_k
\]

(4.20)
in which \( k=1,2,\ldots, N=N1+N2+N3; \{\phi\}_k \) is the \( k \)th mode of vibration; \( \eta_k(t) \) is the \( k \)th new coordinate. Following the standard procedure, i.e., substituting Eq. 4.20c into Eq. 4.6b. and making use of the orthogonal property of the vibration modes, one obtains \( N \) uncoupled differential equations in which the \( k \)th differential equation is given by

\[
m_k^* \dddot{\eta}_k + k_k^* \ddot{\eta}_k = -m_{ek}^* \ddot{z}_g(t)
\]

(4.21)

where

\[
m_k^* = \{\phi\}_k^T [M] \{\phi\}
\]

\[
k_k^* = \{\phi\}_k^T [K] \{\phi\}
\]

\[
m_{ek}^* = \{\phi\}_k^T [P]
\]

and \( m_k^* \) is related to \( k_k^* \) by the equation

\[
k_k^* = \omega_k^2 m_k^*
\]

(4.22)
in which \( \omega_k \) is the \( k \)th natural circular frequency of the system. Eq. 4.21 can also be written as

\[
\dddot{\eta}_k + \omega_k^2 \dot{\eta}_k = -k_k \ddot{z}_g(t)
\]

(4.23)
in which
\[ \Gamma_k = m_{ek}^* m_k^* \] (4.24)

Eq. 4.23 represents the motion of a single-degree-of-freedom system with a mass, \( m_k^* \), stiffness, \( k_k^* \), and a base acceleration \( \Gamma_k \ddot{x}_g(t) \). The total acceleration of the system, \( \ddot{\eta}_k(t) \), is then given by

\[ \ddot{\eta}_k(t) = \ddot{\eta}_k + \Gamma_k \ddot{x}_g(t) \] (4.25)

which by virtue of Eq. 4.23 can also be written as

\[ \ddot{\eta}_k(t) = -\omega_k^2 \eta_k \] (4.26)

Note that \( \ddot{x}_g(t) \) in Eq. 4.25 is not directly additive to the relative acceleration, \( \ddot{\eta}_k \). The total acceleration, \( \ddot{\eta}_k(t) \), is obtained from Eq. 4.26 by first solving Eq. 4.21 for \( \eta_k \). The solution to Eq. 4.21 is given by

\[ \eta_k(t) = e_k^A A_k(t) \] (4.27)

in which

\[ e_k^A = \frac{1}{\omega_k^2} \left\{ \phi \right\}_k^T \left\{ \phi \right\}_k \] (4.28)

and \( A_k(t) \) is the pseudoacceleration function, defined by

\[ A_k(t) = \omega_k \int_0^t \ddot{\chi}_k(\tau) \sin \omega_k (t-\tau) d\tau \] (4.29)

On substituting Eq. 4.27 into Eq. 4.26, one obtains

\[ \ddot{\eta}_k(t) = e_k^A A_k(t) \] (4.30)

in which
\[ e_k = \omega_k^2 A_k = \text{a participation factor} \quad (4.31) \]

With \( \ddot{\tilde{n}}_k(t) \) determined, the corresponding accelerations of the tank wall in the axial, tangential and radial directions, \( \ddot{U}_k(z,\theta,t), \ddot{V}_k(z,\theta,t), \text{and} \ddot{W}_k(z,\theta,t) \), may now be obtained from Eqs. 4.20, 4.7 and 4.1 to 4.3. The results are

\[
\ddot{U}_k(z,\theta,t) = \left[ \sum_{\lambda=1}^{N^1} \phi_{k\lambda} \chi_{\lambda}(z) \cos \theta \right] \ddot{\tilde{n}}_k(t) \quad (4.32a)
\]

\[
= \left[ \sum_{\lambda=1}^{N^2} \phi_{k\lambda} \chi_{\lambda}(z) \cos \theta \right] e_k^\ell A_k(t)
\]

\[
\ddot{V}_k(z,\theta,t) = \left[ \sum_{\lambda=1}^{N^2} \phi_{k\lambda} \psi_{\lambda}(z) \sin \theta \right] \ddot{\tilde{n}}_k(t) \quad (4.32b)
\]

\[
= \left[ \sum_{\lambda=1}^{N^3} \phi_{k\lambda} \psi_{\lambda}(z) \sin \theta \right] e_k^\ell A_k(t)
\]

\[
\ddot{W}_k(z,\theta,t) = \left[ \sum_{\lambda=1}^{N^3} \phi_{k\lambda} \psi_{\lambda}(z) \cos \theta \right] \ddot{\tilde{n}}_k(t) \quad (4.32c)
\]

\[
= \left[ \sum_{\lambda=1}^{N^3} \phi_{k\lambda} \psi_{\lambda}(z) \cos \theta \right] e_k^\ell A_k(t)
\]

in which \( \phi_{ik} \) is the \( i \)th element of the \( k \)th vibration mode, \( \{ \phi \}_k \).

Eq. 4.32c is again of the same form as Eq. 4.8. Accordingly, the impulsive component of hydrodynamic pressure can be obtained from Eq. 4.9 by interpreting the quantity \( \ddot{\tilde{W}}_i(t) \) in Eq. 4.8 as the quantity \( e_k^\ell \phi_{ik} A_k(t) \) in Eq. 4.32c. The result is

\[
P_k(r,z,\theta,t) = c_{nk}(r,z) A_k(t) \rho \frac{e_k^\ell}{\ell} a \cos \theta \quad (4.33a)
\]

in which

\[
c_{nk}(r,z) = e_k^\ell \frac{\pi}{2} \frac{a}{2}\frac{\pi}{2} \int \left[ \left( \frac{2n-1}{2} \right) \cos \frac{\pi}{2} \left( \frac{r}{H_i} \right) \right] d_i
\]

\[
\quad \cos \left[ \left( \frac{2n-1}{2} \right) \frac{z}{H_i} \right] \quad (4.34)
\]
The pressure exerted against the tank wall can be obtained by evaluating Eq. 4.34 at \( r = a \).

It is given by

\[
P_k(a, z, \theta, t) = c_{o_k}(z) A_k(t) \rho g a \cos \theta \quad (4.35a)
\]

in which \( c_{o_k}(z) = c_{o_k}(a, z) \)

With the \( k \)th modal component of the hydrodynamic pressure established, the total pressure may be obtained by superposition. Specifically,

\[
P(r, z, \theta, t) = \sum_{k=1}^{N} P_k(r, z, \theta, t) \quad (4.33b)
\]

and the pressure on the wall is given by

\[
P(a, z, \theta, t) = \sum_{k=1}^{N} P_k(a, z, \theta, t) \quad (4.35b)
\]

The pressure distributions, \( c_{o_1}(z) \), for the first mode of steel tanks with \( H/a = 0.5 \), 1.0 and 3.0 and \( h/a = 0.001 \) are shown in solid lines in Fig. 4.7. The corresponding distributions for rigid tanks are also identified in dotted line. It can be seen from this figure that while the pressure distribution for broad tanks, \( H/a \leq 1.0 \), is not sensitive to the flexibility of the wall, for tall tanks it is fairly sensitive. In Table 4.5 the functions \( c_{o_1}(z) \) and \( c_{o_2}(z) \) corresponding to the first and second modes of vibration are tabulated for steel tanks with \( h/a = 0.001 \) and concrete tanks with \( h/a = 0.01 \) for \( H/a = 0.5 \), 1 and 3. It should be clear that whereas the second mode contributes importantly to the response of tall tanks, it has a negligible effect for broad tanks.

Base Shear.-- The component of the instantaneous value of the base shear contrib-
uted by the kth mode, $Q_k(t)$, consists of two parts: the first, denoted by $Q_{wk}(t)$, is obtained by integrating the horizontal component of the hydrodynamic wall pressure for the kth mode; and the second, denoted by $Q_{sk}(t)$, is obtained by integrating the corresponding component of the inertia force for the tank itself. The result is

$$Q_k(t) = Q_{wk}(t) + Q_{sk}(t)$$  \hspace{1cm} (4.36)

in which

$$Q_{wk}(t) = \int_0^H \int_0^{2\pi} P_k(a,z,\theta,t) \cos \theta \, d\theta \, dz$$  \hspace{1cm} (4.37)

and

$$Q_{sk}(t) = \int_0^H \int_0^{2\pi} \rho H (\ddot{\mathbf{W}}_k \cos \theta - \ddot{\mathbf{V}}_k \sin \theta) \, d\theta \, dz$$  \hspace{1cm} (4.38)

In Eq. 4.38 ($\ddot{\mathbf{W}}_k \cos \theta - \ddot{\mathbf{V}}_k \sin \theta$) is the horizontal component of the acceleration, and $\ddot{\mathbf{W}}_k$ and $\ddot{\mathbf{V}}_k$ are given by Eq. 4.32b and 4.32c, respectively. After integration, one obtains

$$Q_{wk}(t) = m_{ok} A_k(t)$$  \hspace{1cm} (4.39)

and

$$Q_{sk}(t) = m_{sk} A_k(t)$$  \hspace{1cm} (4.40)

or

$$Q_k(t) = (m_{ok} + m_{sk}) A_k(t)$$  \hspace{1cm} (4.41a)

in which

$$m_{ok} = \left\{ \frac{H}{a} \sum_{k=1}^{N^3} \frac{\phi_{lk}}{\pi \left( (2n-1)\pi \right)^2} \frac{I_n((2n-1)\frac{a}{H})}{I_n(\frac{a}{H})} \right\} m_L$$  \hspace{1cm} (4.42)
\[ m_{sk} = \left[ \frac{1}{2} \rho \sum_{k=1}^{N_2} \left( \sum_{i=1}^{N^3} \phi_{2,i} < \psi_i > - \sum_{i=1}^{N^2} \phi_{2,i} < \psi_i > \right) \right] m_s \]  

(4.43)

and the symbol \(< >\) in Eq. 4.43 stands for

\[ < > = \frac{1}{H} \int_0^H \phi(z) \, dz \]  

(4.44)

Of course, the total base shear is given by

\[ Q(t) = \sum_{k=1}^{N} Q_k(t) = \sum_{k=1}^{N} \left( m_{0k} + m_{sk} \right) A_k(t) \]  

(4.41b)

The quantities \( m_{0k} \) and \( m_{sk} \) are the effective masses of the liquid and shell for the \( k \)th mode of vibration, respectively, and \( m_\ell \) and \( m_s \) are the total liquid mass and shell mass, respectively. The values of \( m_{0k} \) and \( m_{sk} \) for \( k = 1 \) and 2 are listed in Table 4.6 for steel tanks with \( h/a = 0.001 \) and \( H/a \) in the range from 0.3 to 3.0. Also, listed for comparison in the last column of the Table are the values of \( m_o \), the so-called impulsive liquid mass for rigid tanks. These values are reproduced from Ref. 43. The results are also compared graphically in Fig. 4.8, in which the solid line defines \( m_{01} \), and the dashed line defines the sum of \( m_{01} \) and \( m_{02} \). Note that for \( H/a \approx 1.0 \), \( m_{01} \) is in excellent agreement with \( m_o \); this is the basis of the approximation proposed by Veletsos and Yang (43,47) for the impulsive component of the base shear in broad tanks. The approximation is effectively obtained from Eq. 4.41b by deleting the terms corresponding to the higher modes of vibration and the inertia of the tank wall, then replacing \( m_{01} \) by \( m_o \). This leads

\[ Q(t) = m_o A_1(t) \]  

(4.45)

For tall tanks, the following two-mode approximation was proposed by Veletsos (43):

\[ Q(t) = m_{01} A_1(t) + m_{02} A_2(t) \]  

(4.46)

It should be clear from Fig. 4.8 that this approximation is indeed quite good for all practi-
cal values of $H/a$.

Hydrodynamic Moment.-- A clear distinction must be made between the hydrodynamic moment, $M(t)$, induced on a section of the tank immediately above the base, and the moment, $M'(t)$, induced on the foundation itself. The former is due to the pressure exerted on the tank wall, whereas the latter also includes the moment, $\Delta M(t)$, contributed by the pressure on the tank base. Therefore,

$$M'(t) = M(t) + \Delta M(t) \quad (4.47)$$

Wall Moment:-- The moment due to the pressure on the wall consists of two parts. One is due to the hydrodynamic pressure, and the other is due to the tank inertia. Denoted by $M_{wk}(t)$ and $M_{sk}(t)$, the $k$th components of these moments may be determined from

$$M_{wk}(t) = \int_0^H \int_0^{2\pi} P_k(z, \theta, t) z \cos \theta \, a \, d\theta \, dz \quad (4.48)$$

and

$$M_{sk}(t) = \int_0^H \int_0^{2\pi} f (\ddot{U}_k z \cos \theta - \ddot{V}_k z \sin \theta + \dddot{W}_k z s \sin \theta) \, a \, d\theta \, dz \quad (4.49)$$

and may be expressed as

$$M_{wk}(t) = m_{ok} h_{ok} A_k(t) \quad (4.50)$$

and

$$M_{sk}(t) = m_{sk} L_k A_k(t) \quad (4.51)$$

in which
\[ m_{ok} h_{ok} = \left\{ \frac{\pi}{2} \right\} \sum_{n=1}^{\infty} \frac{8 (-1)^{n+1}}{(2n-1)^2 \pi^3} \frac{1}{I_n(2 \pi H)} \frac{n^2}{2 \pi H} d_{in} \]

(4.52)

and
\[ m_{sk} L_k = \left\{ e_k \left[ \frac{A}{H} \sum_{i=1}^{N} \Phi_{ik} \langle x_i \rangle - \sum_{i=1}^{N} \Phi_{ii} \langle z \psi_i \rangle \right. \right. \]
\[ \left. \left. + \sum_{i=1}^{N} \Phi_{ii} \langle z \psi_i \rangle \right] \right\} n_{ik} H \]

(4.53)

The total moment for the kth mode is then
\[ M_k(t) = (m_{ok} h_{ok} + m_{sk} L_k) A_k(t) \]

(4.54a)

and the total moment contributed by all the modes is given by
\[ M(t) = \sum_{k=1}^{N} M_k(t) = \sum_{k=1}^{N} (m_{ok} h_{ok} + m_{sk} L_k) A_k(t) \]

(4.54b)

The values of \( m_{ok} h_{ok} \) and \( m_{sk} L_k \) for \( k=1 \) and \( 2 \) are listed in Table 4.7 for steel tanks with \( h/a = 0.001 \) and \( H/a \) in the range of 0.3 to 5.0. Also listed for comparison in last column of the Table are the corresponding values of \( m_0 h_0 \) for the rigid tank solutions. The latter data are reproduced from Ref. 43. The results are also compared graphically in Fig. 4.9, in which the solid line defines \( m_0 h_0 \) and the dotted line defines \( m_0 h_0 \). The good agreement of the two curves confirms the adequacy of the simple approximation proposed in Ref. 43 and 47 for tanks that are not too tall. This approximation is
\[ M(t) = m_0 h_0 A_1(t) \]

(4.55)

For tall tanks, the following improved approximations had been proposed by Veletsos (43):
\[ M(t) = m_{o1} h_{o1} A_1(t) \]

(4.56)

or
\[ M(t) = m_{o1} h_{o1} A_1(t) + m_{o2} h_{o2} A_2(t) \]

(4.57)
It should be clear from Fig. 4.9 and Table 4.7 that these approximations give good results over the entire range of \( H/a \) values considered. Of course, Eq. 4.55 has the advantage of simplicity.

**Base Moment**-- The base moment, \( \Delta M(t) \), induced by the hydrodynamic pressure exerted on the tank base can be determined from

\[
\Delta M(t) = \int_0^\pi \int_0^a P(r, z, \theta, t) \left| r^2 \cos \theta \right| dr d\theta
\]

(4.58)

where \( P(r, z, \theta, t) \) is given by Eq. 4.33b. Substituting Eq. 4.33b into Eq. 4.58 and performing the integration, one obtains

\[
\Delta M(t) = \sum_{k=1}^{\infty} M_{ok} \Delta h_{ok} A_k(t)
\]

(4.59)

in which

\[
m_{ok} \Delta h_{ok} = \left\{ \phi_k \sum_{n=1}^{\infty} \frac{8}{(2n-1)\pi} \frac{I_2((2n-1)\frac{r}{H})}{I_1((2n-1)\frac{r}{H})} \right\} d \ln \left\{ \frac{A}{H} \right\}
\]

(4.60)

The values of \( m_{ok} \Delta h_{ok} \) for \( k = 1 \) and 2, are tabulated in Table 4.8 for the same tanks as those considered in Table 4.7. Also listed are the corresponding rigid tank solutions obtained from Ref. 43. The results are also plotted in Fig. 4.10, in which the solid line represents \( m_{o1} \Delta h_{o1} \) and the dashed line represents the sum of \( m_{o1} \Delta h_{o1} \) and \( m_{o2} \Delta h_{o2} \) and the dotted line represents the rigid tank solutions. The approximations recommended in Ref. 43 and 47 for this case are

\[
\Delta M(t) = m_0 \Delta h_0 A_1(t)
\]

(4.61)

and

\[
\Delta M(t) = m_{o1} \Delta h_{o1} A_1(t) + m_{o2} \Delta h_{o2} A_2(t)
\]

(4.62)
It can be seen that Eq. 4.61 gives conservative results, but Eq. 4.62 should be used for the higher values of $H/a$.

4.6 SENSITIVITY OF RESULTS TO $H/a$

The data in the preceding sections were for steel tanks with $h/a = 0.001$ only. Additional data for values of $h/a = 0.0005$ and 0.005 are presented in Table 4.9 in which also listed the corresponding data for concrete tanks. It can be seen that the response quantities considered are not sensitive to variations in $h/a$. The reported data may, therefore, be used for other values of $h/a$ as well.

4.7 CONCLUSION

With the numerical data presented herein, the hydrodynamic pressure, base shear, and base moments in laterally excited liquid storage tanks may be evaluated readily. The validity of the simple approximate expressions proposed by Veletsos et al (43,47) has been confirmed by the data presented herein.
CHAPTER 5
ROCKING RESPONSE OF RIGID TANKS

5.1 INTRODUCTION

The seismic response of liquid storage tanks has been the subject of several studies during the past twenty years (3, 15, 18, 20, 27, 44, 46, 50). Most of studies were made under the assumption that the tank foundation is rigid and experiences either a horizontal or vertical component of motion.

If the flexibility of the supporting soils is considered, it is well-known that the tank base will experience a rocking component of motion even for a purely translational free field motion. This problem has not been studied systematically to date. With an objective of analyzing these effects critically, in this chapter, a study is carried out for the response of a rigid tank undergoing rocking motion at the base. In addition, an approximate solution is presented for the response of flexible tanks to a similar excitation.

In previous studies of the response of a laterally excited rigid tank, the hydrodynamic pressure was divided into two parts: (a) the impulsive part, which is computed by neglecting the effect of the surface wave, i.e., assuming the pressure at the surface to be zero, and (b) the convective part, which is associated with the sloshing action of the fluid in the tank.

This division of the hydrodynamic pressure into the impulsive and convective parts has proved of great value in the analysis of both rigid and flexible tanks excited laterally, and it is desired to extend it to the analysis of tanks subjected to a rocking base motion.

In the literature, the liquid rigid tank subjected to steady-state rocking motion was studied by Bauer (5) in connection with the stability analysis of space craft fuel containers. However, in Bauer's solution, the impulsive and convective parts cannot be identified readily. The same system subjected to transient rocking motion was first studied by Haroun and Housner (17) in their paper dealing with the effects of the soil-structure interac-
tion; however, the problem was solved approximately. Later on, Haroun studied the response of a flexible tank subjected to rocking motion, but, only the impulsive component was considered. The corresponding solution to a rigid tank can be deduced from his solution to the flexible tanks. Unfortunately there are not any data corresponding to rigid tanks presented in that paper.

From the above review, one can see that there are some solutions for the rigid tank-liquid system to rocking motion existing, but those solutions are either not in the form that one can use directly in the transient analysis or are not studied completely. Therefore, it is necessary to re-examine the problem.

To have this chapter self-contained the problem will be solved from scratch, and the emphasis will be placed on the understanding the behavior of the response of the system and the relation between the rocking motion and the lateral vibration; furthermore, an approximate approach to evaluate the response of the flexible tank to rocking motion is proposed.

The objective of this chapter is two fold: First, to formulate the exact solution for the problem in such a way that the impulsive and convective parts of the solution can be identified. This is desirable because this separation is fundamental to the approach used in the analysis of the flexible tank. The second objective is to present numerical data, which elucidate the nature of the impulsive and convective effects.

5.2 SYSTEM AND GOVERNING EQUATIONS

The system considered is shown in Fig. 5.1. It is the same system considered in Ch. 4 except that the base experiences the rocking motion.

For the irrotational flow of an impressible inviscid liquid, the velocity potential, \( \phi(r, \theta, z, t) \), must satisfy the Laplace equation,

\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0
\]

(5.1)
along with the appropriate boundary conditions.

The velocity of an arbitrary point of the liquid in an arbitrary n-direction, \(v_n\), is related to the velocity potential by

\[
V_n(r, \theta, z, t) = -\frac{\partial \phi}{\partial n}
\]

and the hydrodynamic pressure at that point is given by

\[
P(r, \theta, z, t) = \rho \frac{\partial \psi}{\partial t}
\]

The boundary conditions for the liquid are

1. The vertical velocity of the liquid at \(z=0\) must equal the velocity of the velocity of the tank base, i.e.,

\[
-\frac{\partial \phi}{\partial z} \bigg|_{z=0} = r \cos \theta \dot{e}_g
\]

2. Along the tank wall, \(r=a\), the radial velocity of the fluid and the tank wall must be the same; therefore,

\[
-\frac{\partial \phi}{\partial r} \bigg|_{r=a} = -z \cos \theta \dot{e}_g
\]

3. At the liquid surface, the following linearized free surface boundary condition is used

\[
\left(\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z}\right) \bigg|_{z=H} = 0
\]

in which \(g\) is the gravitational acceleration.

As noted previously, the solution of Eq. 5.1 is expressed as the sum of a so-called impulsive component, \(\phi_i\), and a convective component, \(\phi_c\), as

\[
\Phi(r, \theta, z, t) = \Phi_i(r, \theta, z, t) + \Phi_c(r, \theta, z, t)
\]

The impulsive component of the solution satisfies the actual boundary conditions
along the tank bottom and wall, and the condition of zero hydrodynamic pressure at z=H. As a result, it does not provide for the effects of the surface waves associated with the sloshing action of the liquid. The convective component of the solution effectively corrects for the difference between the actual boundary condition at z=H and the one considered in the development of the impulsive solution.

With the velocity potential function thus established, the motion of the fluid at any location and time, and the associated hydrodynamic pressure, may be determined from Eqs. 5.2 and 5.3. In particular, the hydrodynamic pressure exerted by the liquid against the tank wall is determined from Eq. 5.3 by letting r=a. The details of the solution for rigid container are given in Appendix B.

5.3 PRESENTATION OF RESULTS FOR RIGID TANKS

Hydrodynamic Wall Pressure.-- The hydrodynamic pressure exerted against the tank wall, p(z,θ,t), is defined by the sum of Eqs. B.12 and B.28 evaluated at r=a, and may be expressed as

\[
P(z,\theta,t) = \left[ C_0^r(z) \ddot{y}(t) + \sum_{j=1}^\infty C_j^r(z) A_j(t) \right] \rho \frac{a}{k} \cos \theta
\]

in which

\[
\ddot{y}(t) = H \ddot{\theta}_g(t)
\]

denotes an equivalent linear acceleration.

The first term of Eq. 5.8 represents the impulsive part of the solution, and the series represent the convective part. The individual terms of the series represent the modal contributions of the portion of the liquid that participates in sloshing motion. Note that the pressure varies as a cosine function in the circumferential direction, and that while the impulsive part is proportional to the ground acceleration, \( \ddot{y}(t) \), the convective components
are proportional to the functions $A_j(t)$. The latter functions represent the instantaneous values of the response pseudoacceleration of a series of single-degree-of-freedom linear oscillators the natural frequencies and damping of which are the same as those for the sloshing modes of vibration of the liquid. All oscillators are presumed to be excited by the same base motion as the tank, $\dddot{y}(t)$. The $j$th sloshing frequency, in cycles per unit of time, is given by

$$f_j = \frac{1}{2\pi} \sqrt{\frac{\lambda_j}{\lambda_j}} \sqrt{\frac{g}{a}} \tanh \left( \frac{\lambda_j}{a} \right)$$

(5.10)

in which $g$ is the gravitational acceleration, and $\lambda_j$ are the zeros of the first derivative of the Bessel function of the first kind and first order. In particular,

$$\lambda_1 = 1.8412 \quad \lambda_2 = 5.3314 \quad \lambda_3 = 8.5363$$

The quantities $C_0(r)$ and $C_j(r)$ are dimensionless functions that define the height-wise distribution of the various components of the hydrodynamic pressure. In addition to the dimensionless distance parameter, $z/H$, these quantities are function of the height-to-radius ratio, $H/a$.

The function $C_0(r)$ is displayed in Fig. 5.2 for several different values of $H/a$. For broad tank, $H/a = 0.5$, the pressure increases almost linearly, while for the tall tank, $H/a = 3$, the pressure distribution is similar to that for a horizontally excited flexible tank (see Fig. 4.5).

In order to fully understand the distribution of the pressure, $p_j(r, \theta, z, t)$, the rocking motion of the tank, shown in Fig. 5.3a, is split into two parts: (1) a rocking motion of the tank wall, with the tank base remaining horizontal (see Fig. 5.3b), and (2) a rocking motion of the tank base, with the tank wall remaining vertical (see Fig. 5.3c).

Let $p_{1w}(r, \theta, z, t)$ represent the impulsive component of hydrodynamic pressure for
the motion shown in Fig. 5.3b, and \( P_{lb}(r, \theta, z, t) \) represents the corresponding component of the pressure for the motion shown in Fig. 5.3c. These pressures may be expressed in the form
\[
P_{lw}^r (r, \theta, z, t) = C_{ow}^r (r, z) \frac{\rho_{l} A}{l_{e}} \ddot{y}(t) \cos \theta
\]
(5.11)

and
\[
P_{lb}^r (r, \theta, z, t) = C_{ob}^r (r, z) \frac{\rho_{l} A}{l_{e}} \ddot{y}(t) \cos \theta
\]
(5.12)
in which \( C_{ow}^r (r, z) \) is given by Eq. B.8 and \( C_{ob}^r (r, z) \) is given by Eq. B.9. Then
\[
P_{li}^r (r, \theta, z, t) = P_{lw}^r (r, \theta, z, t) + P_{lb}^r (r, \theta, z, t)
\]
(5.13)

and
\[
C_{o}^r (r, z) = C_{ow}^r (r, z) + C_{ob}^r (r, z)
\]
(5.14)

The function \( C_{ow}^r (a, z) \) is displayed in Fig. 5.4a for several different values of \( H/a \).

For \( H/a = 3 \), the pressure distribution is similar to that obtained for the fundamental mode of lateral vibration of a flexible tank (see, for example, Fig. 4.5 in chapter 4). This is easy to understand since fundamental-mode configuration of the flexible tank in this case is similar to the motion considered in Fig. 5.3b.

The function \( C_{ob}^r (a, z) \) is displayed in Fig. 5.4b for several different values of \( H/a \).

For tall tank with \( H/a = 3 \), the pressure is concentrated near the bottom; this is easy to understand since the upper part of the liquid in this case is not affected significantly by the motion at bottom. For broad tank with \( H/a = 0.5 \), the pressure increases almost linearly from zero at the top to a maximum at the bottom; this is the same pressure distribution as that for a rigid tank subjected to vertical excitation. This result is reasonable since, in so far as the response of the liquid adjacent to the wall is concerned, there is no essential dif-
ference between a base rocking motion and a base vertical motion.

Examination of Figs. 5.4a and 5.4b reveals that for tall tanks the pressure is dominated by the rocking motion of the wall, whereas for board tanks it is dominated by the rocking motion of the base.

The functions \( C_1^r(z) \) and \( C_2^r(z) \) are plotted in Fig. 5.5 for tanks of different properties. It can clearly be seen from Fig. 5.5 that the convective pressure decays much more rapidly with depth for tall slender tanks than for shallow broad tanks. These trends are the same as those determined previously for laterally excited tanks.

Base Shear.-- The hydrodynamic base shear for the structure, representing the total hydrodynamic force exerted by the liquid on the tank wall, may be expressed as

\[
Q^r(t) = m_o^r \ddot{y}(t) + \sum_{j=1}^{\infty} m_j^r A_j(t)
\]  

(5.15)

in which \( m_o^r \), the impulsive mass of the liquid, denotes the portion of the total mass that may be considered to be rigidly attached to the tank, and \( m_j^r \), the convective liquid mass, denotes the part of the liquid associated with the \( j \)th sloshing mode of the vibration.

Examination of the numerical values of \( m_o^r \) and \( m_j^r \) reveals that these quantities are related to the corresponding quantities in the expression for the foundation moment of a laterally excited tank. Specifically,

\[
m_o^r = m_o \frac{h_o'}{H} \quad \text{and} \quad m_j^r = m_j \frac{h_j'}{H}
\]  

(5.16)

in which \( m_o \) and \( m_j \) are, respectively, the impulsive and convective masses of the liquid for a tank subjected to horizontal base motion, and \( h_o' \) and \( h_j' \) are the heights at which \( m_o \) and \( m_j \) must be concentrated to yield the actual values of the corresponding foundation moments. These results may be interpreted to be a consequence of Betti's principle. The values of \( m_o, m_1, m_2, h_o', h_1' \) and \( h_2 \) are tabulated in Ref. (43). Accordingly, Eq. 5.15 can be rewritten as
\[ Q^r(t) = m_0 \frac{h_0}{H} \ddot{\gamma}(t) + \sum_{j=1}^{\infty} m_j \frac{h_j}{H} \dot{A}_j(t) \]  

(5.17)

Base Moment.-- A clear distinction must be made between the hydrodynamic moment, \( M^r(t) \), induced on a section of the tank immediately above the base, and the moment, \( M^F(t) \), induced on the foundation itself. The former is given by the sum of Eqs. B.18 and B.32 in the Appendix B and can be expressed as

\[ M^r(t) = I_o^r \ddot{\Omega}_d + \sum_{j=1}^{\infty} m_j^r h_j \dot{A}_j(t) \]  

(5.18)

in which \( I_o^r \) and \( h_j \) are given in the Appendix B.

Let \( \Delta M^r(t) \) denote the component of the base moment induced by the hydrodynamic pressure exerted on the tank base. This moment is given by sum of Eqs. B.19 and B.34, and may be expressed as

\[ \Delta M^r(t) = \Delta I_o^r \ddot{\Omega}_d + \sum_{j=1}^{\infty} m_j^r \Delta h_j \dot{A}_j(t) \]  

(5.19)

in which \( \Delta I_o^r \) and \( \Delta h_j \) are given in the Appendix B. In Eqs. 5.18 and 5.19, the first term represents the contribution of the impulsive pressure, and the infinite series represents the contribution of the convective pressure. The quantities \( I_o^r \) and \( \Delta I_o^r \) are listed in Table 5.1, and also are displayed graphically in Fig. 5.6. The quantities \( h_j \) and \( \Delta h_j \) which are given by Eqs. B.33 and B.35 in the Appendix B. They represent the heights at which the convective masses must be put in order to yield the correct moments at the base, and, incidentally, are identical to the corresponding quantities for a laterally excited tank. Numerical values for these quantities can be found in Ref. (43).

As noted previously, the convective masses, \( m_j^r \), for the rocking motion are related to the corresponding masses for lateral motion by Eq. 5.16. Accordingly, Eqs. 5.18 and 5.19 can be rewritten as

\[ M^r(t) = I_o^r \ddot{\Omega}_d + \sum_{j=1}^{\infty} m_j \frac{h_j}{H} \dot{A}_j(t) \]  

(5.20)
\[ \Delta M^r(t) = \Delta I^r \vec{g} + \sum_{j=1}^{\infty} m_j \frac{h_j^r}{H} \Delta h_j^r A_j^r(t) \]  

(5.21)

Further insight into the component of the foundation moment induced by the hydrodynamic base pressure may be obtained by examining the radial distribution of this pressure. The impulsive component of this pressure is shown in Fig. 5.7 for several different values of \( H/a \). As expected the broad tank has the higher pressure than that of the tall tank. The distributions of pressure are significant different from those of laterally excited tanks presented in Ref. (43).

Surface Displacement of Liquid: The vertical displacement, \( d(r, \theta, t) \), of an arbitrary point on the liquid surface is given by the following equation.

\[ d(r, \theta, t) = \left( \sum_{j=1}^{\infty} d_j^r(r) \frac{A_j^r(t)}{g} \right) \cos \theta \]  

(5.22)

in which \( d_j^r(r) \) is the the same as the quantity \( C_j^r(r, H) \) in Eq. B.37. The values of quantities \( d_1^r(r) \) and \( d_2^r(r) \) at \( r=a \) are given in Table 5.2 for several different values of \( H/a \).

5.4 APPROXIMATE APPROACH FOR FLEXIBLE TANKS

For the reasons noted in Refs. (43) and (47), the influence of the tank flexibility in the convective components of the hydrodynamic effects will be neglected, and only the impulsive parts will be presumed to be affected by the flexibility of tank.

For a flexible tank subjected to lateral excitation, Veletsos and Yang (43, 47) have proposed a simple concept for assessing the effect of the tank flexibility. The procedure consists in replacing the ground acceleration in the impulsive part of the rigid tank solution with the pseudoacceleration function corresponding to the fundamental natural frequency of the tank-fluid system. Applying the same concept to the case of rocking motion, considered here, the ground acceleration \( y(t) \) in the rigid tank solution must be replaced by the
pseudoacceleration of the first mode, \( A_1(t) \), this leads to the following results.

For the impulsive component of the hydrodynamic pressure, one obtains

\[
P^r_\perp(z, \theta, t) = C^r_\perp(\theta) \int_\perp \rho \, A_1(t) \, \cos \theta
\]  

(5.23)

For the base shear, one obtains

\[
Q^r_\perp(t) = m_0 \frac{h_0^r}{\lambda} \, A_1(t)
\]  

(5.24)

This expression also follows from Betti's principle applied to the approximate expression for the foundation moment of a laterally excited tank presented in Ref. (43).

For the base moment induced on a section of the tank immediately above the base, one obtains

\[
M^r_\perp(t) = I_\perp \frac{A_1(t)}{\lambda}
\]  

(5.25)

In order to evaluate \( A_1(t) \), one must first determine the fundamental natural frequency of the tank-liquid system. The natural frequencies and vibration modes of the tank in rocking motion must be the same as those for horizontal motion. Accordingly the fundamental natural frequencies of the liquid tank system can be obtained from the information presented in Ch. 4.

5.5 CONCLUSION

A detailed study has been made of the dynamic response of a rigid tank subjected to rocking base motion. The results of this study provided the basis for the analysis of flexible tanks presented in the following chapter.

An extremely simple, approximate procedure has been presented for evaluating the response of flexible tanks. The accuracy of this approach is examined in next chapter.
CHAPTER 6
ROCKING MOTION OF FLEXIBLE TANKS

6.1 INTRODUCTION

In this chapter, the effects of flexibility of the tank wall on the response of a tank-liquid system subjected to a base rocking motion are studied. Recently, the same topic has been studied approximately by Haroun (16) assuming the tank to behave as a shear beam and considering only the first mode of vibration. In this chapter, the more rigorous approach employed in Ch. 4 is used. The objectives of the chapter are to report the numerical data obtained using this approach, and to propose simple means of computing the base shear and moments.

6.2 SYSTEM CONSIDERED AND ASSUMPTION

The system considered is the same as that considered in Ch. 5 except that now the tank wall is considered to be flexible. The sloshing motion is assumed not to be affected by the flexibility of the tank wall, and the amplitude of the base rotational angle, $\theta_g$, is assumed to be small.

6.3 METHOD OF ANALYSIS

The method of analysis is similar to that employed in Ch. 4. Since the free vibrational problem is independent of whether the liquid tank system is excited by a lateral or rocking base motion, the response of the system considered in this chapter can be expressed in terms of the vibration modes used in Ch. 4. The difference between the system excited laterally and in rocking is in the magnitude and distribution of the hydrodynamic pressure exerted on the wall. Additionally, the rocking excitation induces a moment at the base of the tank which will be referred to as the "instantaneous base moment". In other words, the natural frequencies and the vibration modes are the same for a tank subjected to either lateral or rocking base excitation. The different base excitations on the
system change the external lateral forces acting on the tank wall only. Hence, one only has to replace the forcing term in the equation of motion defined by Eq. 4.6b in Ch. 4 with the forcing term defined by Eqs. 6.2a through 6.2c given below. If \( \{ \mathbf{f}^r \} \) denotes the lateral force induced by the rocking excitation, Eq. 4.8b can be rewritten as

\[
[M_i] \{ q \} + [K_i] \{ q \} = - \{ \mathbf{f}^r \} \ y(t)
\]  

(6.1)

in which \( y(t) = H \hat{\mathbf{f}}(t) \) and the components of \( \{ \mathbf{f}^r \} \) are given by

\[
P_{u_k}^r = 0
\]  

\( k = 1, 2, \cdots, N_1 \)

(6.2a)

\[
P_{v_k}^r = 0
\]  

\( k = 1, 2, \cdots, N_2 \)

(6.2b)

and

\[
P_{w_k}^r = \frac{m_{r_k}}{2} \langle \psi_{k}^r \rangle + m_{o_k}^r
\]  

\( k = 1, 2, \cdots, N_3 \)

(6.2c)

where \( m_{o_k}^r \) is given by

\[
m_{o_k}^r = \int_0^{\pi} \int_0^{2\pi} \hat{P}(\theta, \phi) C_{o_k}^r(\theta, \phi, \psi) \cos \theta \ \  \ d\theta \ d\phi
\]

(6.3)

Eqs. 6.2a through 6.2c are obtained from the expression for the virtual work done by the external loads due to the assumed virtual displacement field given by Eqs. 4.1 to 4.3 in Ch. 4. The derivation is similar to that presented in Ref. (50) for laterally excited tanks. The function \( C_{o_k}^r(\theta, \phi, \psi) \) in Eq. 6.3 is defined by B.11 in Appendix B, and represents the distribution of the hydrodynamic pressure exerted on the tank wall when the tank is assumed to be rigid. The procedure of solving Eq. 6.1 is the same as that used in Ch. 4: hence the detailed description of it is omitted here, and the solutions are given in the following section.

Let \( e_k^r \) be the kth participation factor, given by
\[ C^r_K = \frac{\{\phi\}_K^T \{p^r\}}{\{\phi\}_K^T [M] \{\phi\}_K} \quad (6.4) \]

and let \( A^r_K(t) \) be the kth pseudoacceleration function, given by

\[ A^r_K(t) = \omega_K \int_0^t \dot{y}(\tau) \sin \omega_K (t-\tau) d\tau \quad (6.5) \]

Then, the hydrodynamic pressure at an arbitrary point is given by

\[ p^r(r, \theta, z, t) = p^r_L a \cos \theta \left[ \sum_{k=1}^{N} C^r_{ok}(r, z) A^r_k(t) \right] \quad (6.6) \]

and the hydrodynamic pressure exerted against the wall is given by

\[ p^r(a, \theta, z, t) = p^r_L a \cos \theta \left[ \sum_{k=1}^{N} C^r_{ok}(z) A^r_k(t) \right] \quad (6.7) \]

in which

\[ C^r_{ok}(z) \equiv C^r_{ok}(a, z) \]

The impulsive component of the base shear, \( Q^r_i(t) \), and of the base moments, \( M^r_i(t) \) and \( \Delta M^r_i(t) \), are given by the expressions

\[ Q^r_i(t) = \sum_{k=1}^{N} (m^r_{ok} + m^r_{sk}) A^r_k(t) \quad (6.8) \]

\[ M^r_i(t) = \sum_{k=1}^{N} (m^r_{ok} L^r_k + m^r_{sk} L^r_k) A^r_k(t) \quad (6.9) \]

\[ \Delta M^r_i(t) = \sum_{k=1}^{N} m^r_{ok} \Delta h^r_{ok} A^r_k(t) \quad (6.10) \]

The quantities \( C^r_{ok}(r, z) \), \( m^r_{ok} \), \( m^r_{sk} \), \( h^r_{ok} \), \( L^r_k \) and \( \Delta h^r_{ok} \) may be obtained by replacing the participation factor in the counterparts of these expressions in Ch. 4 with the participation factor defined by Eq. 6.4.

### 6.4 Instantaneous Base Moment

As mentioned in the previous section, the difference between the the tank subjected
to lateral excitation and the tank subjected to rocking excitation is that the rocking excitation induces an instantaneous base moment proportional to \( \dot{\theta}_b(t) \). This instantaneous moment is denoted by \( \Delta M_r(t) \), and is determined by solving the following governing differential equation

\[
\frac{\partial^2 \phi_b}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_b}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi_b}{\partial \theta^2} + \frac{\partial^2 \phi_b}{\partial z^2} = 0
\]  

(6.11)

along with the boundary conditions

\[\text{at } z=H \quad \frac{\partial \phi_b}{\partial t} = 0 \]  

(6.12a)

\[\text{at } z=0 \quad \frac{\partial \phi_b}{\partial z} = -\frac{r}{H} \cos \theta \ \dot{\theta}(t) \]  

(6.12b)

\[\text{at } r=a \quad \frac{\partial \phi_b}{\partial r} = 0 \]  

(6.12c)

in which \( \phi_b \) is the velocity potential function corresponding to the instantaneous base rocking motion. Eq. 6.12a states that there is no pressure at the liquid surface; Eq. 6.12b states that at tank base the liquid velocity is equal to the velocity of the foundation and Eq. 6.12c states that there is no wall pressure for the solution considered. The solution of Eq. 6.11 subjected to the boundary conditions 6.12a through 6.12c, is given by

\[
\phi_b = \alpha \cos \theta \left[ \sum_{n=1}^{\infty} \frac{8}{(2n-1)^2 \pi^2} \frac{I_n((2n-1)^2 \pi r H)}{I_n((2n-1)^2 z H)} \cos \left( (2n-1)^2 \pi \frac{z}{H} \right) + \frac{r}{a} \left( 1 - \frac{r}{H} \right) \right] \dot{\theta}(t) \]

and the corresponding hydrodynamic pressure is given by

\[
P_b = \rho \frac{\partial \phi_b}{\partial t} = \rho \alpha \cos \theta \left[ \sum_{n=1}^{\infty} \frac{8}{(2n-1)^2 \pi^2} \frac{I_n((2n-1)^2 \pi r H)}{I_n((2n-1)^2 z H)} \cos \left( (2n-1)^2 \pi \frac{z}{H} \right) + \frac{r}{a} \left( 1 - \frac{r}{H} \right) \right] \dot{\theta}(t) \]  

(6.13)

The instantaneous base moment, \( \Delta M_r(t) \), can then be determined from
\[ \Delta M_r(t) = \int_0^{2\pi} \int_0^a \int_{z=0} \rho_o \frac{r^2 \cos \theta \, dr \, d\theta \, d\phi}{z=0} \]  
(6.14)

Substituting Eq. 6.13 into Eq. 6.14, and performing the integration, one obtains

\[ \Delta M_r(t) = m_2 \Delta h_r(t) \]  
(6.15)

where

\[ \Delta h_r = \left\{ \frac{a}{H} \sum_{n=1}^{\infty} \frac{(-1)^n}{((2n-1)\pi)^3} \frac{I_2 \left[ \frac{(2n-1)\pi a}{H} \right]}{I_1 \left[ (2n-1)\frac{\pi a}{H} \right]} + \frac{1}{4} \left( \frac{a}{H} \right)^2 \right\} H \]  
(6.16)

The total base moment acting on the tank base is the sum of Eqs. 6.10 and 6.15; it is given by

\[ \Delta M_T = \Delta M_r(t) + \Delta M_s(t) \]

\[ = \sum_{k=1}^{N} m_k \Delta h_{r_k} A_k^r(t) + m_2 \Delta h_r(t) \]  
(6.17)

Note that the first part of this expression is a function of the pseudoacceleration functions, \( A_k(t) \), whereas the second part is function of the base acceleration.

### 6.5 PRESENTATION OF DATA

The distributions of the hydrodynamic pressure exerted on the walls of steel tanks with \( H/a = 0.5, 1.0 \) and 3.0 are shown with solid lines in Fig. 6.1. The corresponding pressure distributions for rigid tanks are also shown. It can be seen that, as is the case with lateral vibration, the base ordinates of the diagrams for the flexible tanks are generally smaller than those for the rigid tanks. However, the difference are generally not significant, and the pressure distributions may be considered to be independent of the wall flexibility. This leads to the following approximate equation

\[ P_r^f(z, \theta, t) = C_0(z) \rho \cos \theta A_1^r(t) \]  
(6.18)
The data used to prepare Fig. 6.1 are also tabulated in Table 6.1.

The values of \( m_{01}^r \) and \( m_{01}^r h_{01}^r \) are presented graphically in Fig. 6.2 and Fig. 6.3 where they are compared with the corresponding quantities for a rigid tank, \( m_o \) and \( m_o h_o \). The results are for steel tanks with \( \nu = 0.3, \rho_f/\rho = 0.127, h/a = 0.001 \) and values of \( H/a \) in the range from 0.3 to 3.0. From these two figures one can conclude that \( m_{01}^r \) and \( m_{01}^r h_{01}^r \) can be replaced by \( m_o^r \) and \( m_o h_o^r \). Hence, the following approximate equations can be obtained

\[
Q^r(t) = m_o^r A_1^r(t)
\]  
(6.19)

and

\[
M^r(t) = m_o^r h_o^r A_1^r(t)
\]  
(6.20)

The data used to prepare Figs. 6.2 and 6.3 are tabulated in Tables 6.2 and 6.3, respectively. Included in these tables are also the results for the second mode and the effects of the tank inertia.

A dimensionless distribution function, \( C_b(r,z) \), is defined as the ratio of \( P_b \), given by Eq. 6.13, to \( \rho_f a \cos \theta \hat{y} \); therefore, \( C_b(r,z) \) is given by

\[
C_b(r, z) = \frac{\sum_{n=1}^{\infty} \frac{(z n - 1) \frac{\pi}{2} \frac{C_n}{I_n} \cos \left( (z n - 1) \frac{\pi}{2} \frac{H}{H} \right) + \frac{C_n}{\alpha} \left( I - \frac{z}{H} \right)}{(z n - 1) \alpha H \pi I_n}}{\frac{2}{2} \frac{\pi}{2} \frac{C_n}{I_n} \cos \left( (z n - 1) \frac{\pi}{2} \frac{H}{H} \right) + \frac{C_n}{\alpha} \left( I - \frac{z}{H} \right)}
\]  
(6.21)

The distribution of \( C_b(r, z) \) at \( z = 0 \), \( C_b(r, 0) \), is plotted in Fig. 6.4 for \( H/a = 0.5, 1.0 \) and 3.0. It can be seen from the figure that the pressure decreases as \( H/a \) increases; as a result, the instantaneous base moment will decrease as \( H/a \) increases. The data associated with Fig. 6.4 are listed in Table 6.4.

The coefficients in Eq. 6.15, \( \Delta h_r \), are tabulated in Table 6.5 and plotted in Fig. 6.5 for \( H/a \) from 0.4 to 5.0 on log-log scale. The curve shown in Fig. 6.5 is almost a straight
line; hence, one can derive an empirical equation to calculate $\Delta h_r$. The equation is given by

$$\Delta h_r = \left\{ \frac{1}{\log \left( \frac{a}{a^2 + 342} \right)} \right\} H \quad \text{for} \quad \frac{H}{a} > 0.7$$

(6.22)

In Table 6.5, the data for the quantity $m_{ok}^r \Delta h_{ok}^r$, the base moment defined by Eq. 6.10, and the corresponding solution to rigid tanks are also listed. From Table 6.5 it can be seen that the solutions to rigid tanks give an upper bound; therefore, for all practical purposes one can use an approximate equation to calculate the base moment, which is given by

$$\Delta M_T = (m_o^r \Delta h_o^r \cdot m_x^r \Delta h_x^r) A_1^r(t) + m_x^r \Delta h_x^r \dot{y}(t)$$

(6.23)

6.6 CONCLUSION

The conclusions of this Chapter are summarized below:

1. The natural frequencies and natural modes of the tank wall for rocking base excitation are the same as those for the lateral base excitation.

2. The hydrodynamic pressure, the base shear, the wall moment and the base moment can be estimated conservatively by the simple procedure proposed in Ch. 5.

3. For tanks with $H/a \geq 0.7$, the instantaneous base moment can be evaluated accurately by use of the empirical formula given by Eq. 6.23.
CHAPTER 7
VERTICAL VIBRATION OF RING FOUNDATIONS

7.1 INTRODUCTION

A fundamental step in the analysis of the dynamic response of any foundation and superimposed structure is the computation of the dynamic force-displacement relationship for the supporting medium. This relationship should be determined with due regard for the conditions of contact at the foundation-medium interface. Of special interest in the analysis of linear system is the harmonic response of the foundation computed by considering for the mass of the supporting medium but neglecting the mass of the foundation itself.

Whereas the harmonic response of massless disk foundation has been studied extensively (14) the corresponding response of ring foundation, of the type employed for cooling towers, radar stations and liquid containment structures, has received comparatively little attention. The only known study of the latter problem is the one reported recently by Tassoulas and Kausel (40). who have made a comprehensive evaluation of the response of rigid, circular rings supported at the surface of an elastic stratum overlying a rigid base. Foundations in vertical, torsional, horizontal and rocking modes of vibration have been analyzed, and the foundation impedance for each mode of vibration has been evaluated over a wide range of frequencies and for several different thickness-to-radius ratios for the foundation. However, only relative small stratum depths, of one and one and a half times the outer diameter of the ring, were examined.

The objective herein is to present and interpret corresponding data for the harmonic response of ring foundations supported at the surface of a homogeneous linear half-space, i.e., a stratum of infinite depth. While only vertically excited foundations are considered, their response is studies critically over wide ranges of the parameters involved. The response quantities examined include the stiffness and damping coefficients of the foundation in an equivalent spring-dashpot representation of the supporting medium: the
surface displacements of the foundation and of points on the medium away from the foundation, and the normal pressure at the foundation-medium interface. The results in each case are compared with those obtained for a rigid disk foundation having the same radius as the outer radius of the ring.

It is shown that the dynamic stiffness and damping coefficients for ring foundation may differ significantly from those for the associated disk foundations. Not only are the magnitudes of the results generally different but their variations with frequency also are different. These trends are at variance with those reported previously for foundation on an elastic stratum (40). An approximate, physically motivated, simple model is used to explain these trends and to provide insight into the action of the system.

The data reported herein are obtained by a method of analysis which takes due account of the mixed boundary conditions at the surface of the medium, i.e., the conditions of traction-free boundary away from the foundation and constant vertical displacement beneath the foundation. The analysis is highly efficient, and the reported data are believed to be of high accuracy. The response of the foundation is also evaluated approximately by assuming the distribution of the normal pressure at the foundation-medium interface, and the range of validity of this simpler approach is identified.

7.2 SYSTEM CONSIDERED

The system investigated is shown in Fig. 7.1(a). It is a massless, rigid, circular ring foundation which is supported at the surface of a homogeneous, isotropic, linearly elastic halfspace and is excited by an axisymmetric vertical harmonic force. The inner and outer radii of the ring are denoted by \( R_i \) and \( R_o \), respectively, and the width of the ring is denoted by \( \Delta R \). The interface between the foundation and supporting medium is presumed to be smooth so that it cannot sustain any horizontal stresses. The region of the surface outside the contact area is, of course, stress free. Points on the foundation and in the sup-
porting medium are defined by the cylindrical coordinate system \((r, \theta, z)\), the positive
direction of which are identified on the figure.

The excited force, \(P(t)\), is of the form

\[
P(t) = P_0 e^{i\omega t}
\]  \(\text{(7.1)}\)

in which \(P_0\) = a real-valued quantity representing the peak value of the force; \(t = \text{time}\);
\(i = \sqrt{-1}\); and \(\omega\) = the circular frequency of the force and of the resulting motion. The uni-
form vertical displacement of the foundation, \(w(t)\), may then be expressed as

\[
w(t) = W e^{i\omega t}
\]  \(\text{(7.2)}\)

in which the displacement amplitude, \(W\), is generally a complex-valued quantity. The posi-
tive direction of \(P(t)\) and \(w(t)\) are the same as for the corresponding coordinate axis. The
supporting medium is characterized by its mass density, \(\rho\), shear modulus of elasticity, \(G\),
and Poisson's ratio, \(\nu\).

7.3 METHOD OF ANALYSIS

Excepting details of implementation which significantly affect the accuracy of the
solution, the method of the analysis employed is similar to that used by Lysmer (31) in the
analysis of vertically excited rigid disk foundations. Specifically, the foundation is con-
ceived to be made up of a series of annular ring elements acted upon by uniformly distrib-
uted normal pressures at their interface with the supporting medium as shown in Fig.
9.1(b), and the magnitudes of these pressures are adjusted so that the displacements of all
the elements are the same. Displacements for both the foundation and the supporting
medium are defined along the centerlines of the ring elements; the latter are numbered
from 1 to \(n\), starting with the element closest to the center. The radii of the inner and
outer boundaries of the \(i\)th element are \(r_{i-1}\) and \(r_i\), respectively, and the radius to the cen-
The centerline of the ith element, \( \bar{r}_i \), is given by

\[
\bar{r}_i = \frac{r_{i-1} + r_i}{2}
\]  
(7.3)

Let \( W(r) \) be the amplitude of the surface displacement at an arbitrary radial distance, \( r \), due to the uniform normal pressure acting on the jth element, \( \sigma_j \). A complex valued quantity, this amplitude is given by

\[
W(r) = \frac{\sigma_j}{G} \int_0^\infty \frac{\phi(\xi)}{F(\xi)} M(\xi) J_0 \left( r \xi \right) d\xi
\]  
(7.4)

in which \( \xi \) is a dummy variable with units of one over length;

\[
\phi(\xi) = h_s^2 \sqrt{\xi^2 - h_c^2}
\]  
(7.5a)

\[
F(\xi) = (2 \xi^2 - h_s^2)^2 - 4 \xi^2 \sqrt{\xi^2 - h_c^2} \sqrt{\xi^2 - h_s^2}
\]  
(7.5b)

\[
M(\xi) = r_{j-1} J_1'(r_{j-1} \xi) - r_j J_1'(r_j \xi)
\]  
(7.5c)

\( J_0 \) and \( J_1 \) are Bessel functions of the first kind of order zero and one, respectively;

\[
h_c = \frac{\omega}{v_c} \quad \text{and} \quad h_s = \frac{\omega}{v_s}
\]  
(7.6)

and \( v_c \) and \( v_s \) are the speeds of wave propagation for the compressional and shear waves in the halfspace, respectively. The ratio of the latter quantities is given by

\[
\frac{v_c}{v_s} = \sqrt{\frac{2(1 - \nu)}{1 - 2\nu}}
\]  
(7.7)

Equation 7.4 is deduced from Reissner's solution for a normal pressure uniformly distributed over a circular area (36) by subtracting the effects of pressures covering areas of radii \( r_j \) and \( r_{j-1} \). An error in sign in Reissner's solution, first pointed out in (1), was duly corrected.
Let \(d_{ij}\) be an influence coefficient representing the value of \(W(r)\) at the centerline of the \(i\)th element due to a normal pressure at the interface of the \(j\)th element. This quantity may be determined from Eq. 7.4 by letting \(\sigma_j = 1\) and \(r=r_i\). Accurate integration of the expression for \(d_{ij}\) is crucial to the accuracy of the desired solutions, and its implementation is described in detail in Appendix C.

With the influence coefficients \(d_{ij}\) for all combinations of \(i\) and \(j\) established, the displacement amplitude along the centerline of the \(i\)th element, \(W_i\), is given by

\[
W_i = \sum_{j=1}^{n} d_{ij} \sigma_j
\]  
(7.8)

and the vector of the displacement amplitudes is given by

\[
\{W\} = [d] \{\sigma\}
\]  
(7.9)

in which \([d]\) is a square matrix of the \(n\times n\) influence coefficients \(d_{ij}\), and \(\{\sigma\}\) is a column vector of the interface pressure.

The contact pressure corresponding to any prescribed set of surface displacements may now be determined from Eq. 7.8 by inversion:

\[
\{\sigma\} = [d]^{-1} \{W\}
\]  
(7.10)

in which the \(-1\) superscript denotes the inverse of the matrix to which it is attached. For uniform displacements of amplitude \(W_0\),

\[
\{W\} = \{1\} W_0
\]  
(7.11)

and Eq. 7.10 reduced to

\[
\{\sigma\} = [d]^{-1} \{1\} W_0
\]  
(7.12)

The amplitude of the associated total exciting force, \(P\), is a complex-valued quantity.
given by

\[ P = (A^T [d]^{-1} \{1\} \ W_0 \]  \tag{7.13} \]

in which the superscript matrix represents the transpose of a matrix, and \( \{A\} \) is a column matrix, the \( i \)th element of which represents the area of the \( i \)th ring

\[ A_i = \pi (r_i^2 - r_{i-1}^2) \]  \tag{7.14} \]

The impedance of the foundation, \( K \), is finally determined from

\[ K = \frac{P}{W_0} = (A^T [d]^{-1} \{1\} \]

\[ (7.15) \]

7.4 STATIC SOLUTIONS

The static stiffness of the ring foundation, \( K_{st} \), may conveniently be expressed in terms of the corresponding stiffness of a solid disk foundation having a radius equal to the outer ring radius, as

\[ K_{st} = \alpha (K_{st})_0 \]  \tag{7.16} \]

in which \( (K_{st})_0 \) = the static stiffness of the reference disk foundation, defined by

\[ (K_{st})_0 = \frac{4 GR_0}{1 - \nu} \]  \tag{7.17} \]

\( \alpha \) = a dimensionless coefficient that depends on the ratio of width to outer radius for the ring, \( \Delta R / R_0 \). Poisson's ratio for the supporting medium, \( \nu \), appears only in the expression for \( (K_{st})_0 \).

The variation of \( \alpha \) with \( \Delta R / R_0 \) is shown in Fig. 7.2, and the numerical data used to prepare this plot are listed in Table 7.1. As would be expected, the stiffness of the ring foundation is less than that of the associated disk foundation. However, the difference is
significantly only within an extremely small range of $\Delta R/R_0$ values, and generally not as large as might be anticipated from a mere comparison of the contact areas involved. Note that even for $\Delta R/R_0 = 0.1$, for which the area of the ring is only 19 percent of the corresponding disk area, the stiffness of the foundation. Note further that for $\Delta R/R_0$ in excess of 0.5, the stiffnesses of the ring and disk foundations are for all practical purpose the same. These trends are in general agreement with those reported previously by Dhawan (10), although a precise comparison is not possible because no numerical data were reported in the latter reference. The reliability and range of convergence of the data presented in Table 7.1, as well as of the additional data presented in subsequent sections, are examined later.

The reasons for the relative insensitivity of the foundation stiffness to variation in $\Delta R/R_0$ may be appreciated by reference to Fig. 7.3, which shows the radial distributions of the normal pressure at the foundation-medium interface, $\sigma(r)$. These results are normalized with respect to the normal mean pressure for the reference disk foundation,

$$
\sigma_o = \frac{\rho_o}{A_0}
$$

(7.18)

in which $A_0$ = the area of the disk. It is well known that the central region of the solid disk foundation resists only a small fraction of the total load. Accordingly, the addition of a hole in this region eliminates a part of the foundation that is ineffectively utilized and, unless the hole is extremely large, it affects only slightly the mean value of the contact stress and the corresponding stiffness of the foundation. Furthermore, as a result of the additional singularity that develops around the boundary of the central hole, the ring foundation is more efficiently utilized than the corresponding region of the disk foundation, and this leads to a comparatively increased effective stiffness for that region.

7.5 Dynamic Solutions

Stiffness and Damping Coefficients. -- The foundation impedance defined by Eq.
7.15 may be expressed in a generalized form of Eq. 7.16 as

\[ K = (K_{st})_0 \left[ \alpha + i \beta \right] \]  \hspace{1cm} (7.19)

in which \( a_0 \) is a dimensionless frequency parameter defined by

\[ a_0 = \frac{\omega R_0}{v_s} \]  \hspace{1cm} (7.20)

and \( \alpha \) and \( \beta \) are dimensionless coefficients that depend on \( \Delta R/R_0 \), \( a_0 \) and \( \nu \). The real part of Eq. 7.19 represents the contribution of the force component that is in phase with the motion, whereas the imaginary part represents the contribution of the component that is 90° out of phase. In an equivalent spring-dashpot representation of the supporting medium, the stiffness of the spring, \( k \), is given by

\[ k = \alpha (K_{st})_0 \]  \hspace{1cm} (7.21)

and the damping coefficient for the dashpot, \( c \), is given by

\[ c = \beta \frac{(K_{st})_0 R_0}{v_s} \]  \hspace{1cm} (7.22)

The coefficients \( \alpha \) and \( \beta \) will be referred to as the stiffness and damping coefficients for the foundation, respectively.

Numerical data for \( \alpha \) and \( \beta \) are presented in Table 7.2 for foundation with values of \( \Delta R/R_0 \) in the range between 0.02 and 1 and values of \( a_0 \) up to 6. Poisson's ratio for the halfspace in these solutions is taken as \( \nu = 1/3 \). Some of these data are also displayed graphically in Fig. 7.4, in which the following trends may be observed:

1. The impedances of ring and disk foundations are generally significantly different. Note in particular that, whereas the stiffness coefficient for the disk foundation varies smoothly with \( a_0 \), the corresponding variation for a ring is characterized by high-amplitude oscillations the magnitude and location of which depend on the value of \( \Delta R/R_0 \) involved.
Even for $\Delta R/R_0 = 0.5$, for which the static value of $a$ is practically the same as for the associated disk, the corresponding dynamic values are materially different. These results are in sharp contrast to those reported for foundations on an elastic stratum (40), for which no significant change in trend was detected with changes in $\Delta R/R_0$.

2. The thinner the ring, the smaller is generally the damping coefficient, $\beta$. By contrast, the stiffness coefficient for rings is significantly greater than for the associated disk at large values of $a_0$.

Amplitude of Foundation Impedance.-- It is instructive to examine also the amplitude of the total force necessary to induce a foundation displacement of unit amplitude. Denoted by $|K|$, this amplitude is given by

$$|K| = (K_{st})_0 \sqrt{\alpha^2 + (a_0^2)^2}$$  \hspace{1cm} (7.23)

and it is plotted in Fig. 7.5 as a function of $a_0$ for selected values of $\Delta R/R_0$; as before, $\nu = 1/3$. As might be expected, $|K|$ generally increases with increasing $\Delta R/R_0$ and $a_0$, and its value at high frequencies is dominated by the contribution of the damping term. Note should also be taken of the fact that the undulations of the curves for $|K|$ are more orderly and not nearly as large as those for the stiffness coefficient, $a$, displayed in Fig. 7.4(a).

Foundation Flexibility.-- With the foundation impedance established, the flexibility or compliance of the foundation can be determined by inversion. Specifically, the deflection due to the force $P_0 e^{i\omega t}$ may be expressed as

$$w(t) = (w_{st})_0 [f + ig] e^{i\omega t}$$  \hspace{1cm} (7.24)

in which $(w_{st})_0 = P_0/(K_{st})_0 = \text{the static deflection of the associated disk foundation}$; and $f$ and $g$ are dimensionless flexibility coefficients which are related to $a$ and $\beta$ by
\[ f + ig = \frac{1}{\alpha + 1a_0^6} \]  
\[ (7.25) \]

accordingly,

\[ f = \frac{\alpha}{\alpha^2 + (a_0^6)^2} \quad \text{and} \quad g = -\frac{a_0^6}{\alpha^2 + (a_0^6)^2} \]  
\[ (7.26) \]

The dependence of \( f \) and \( g \) on \( \Delta R/R_0 \) and \( a_0 \) is shown in Fig. 7.6. The real part of Eq. 7.24 represents the component of \( w(t) \) that is in phase with the excited force, and the imaginary part represents the component that is 90° out of phase.

Distribution of Surface Displacement.-- The vertical surface displacement at an arbitrary distance \( r \) may conveniently be expressed in a generalized form of Eq. 7.24 as

\[ w(r,t) = (w_{st})_0 [ f(r) + ig(r) ] e^{i\omega t} \]  
\[ (7.27) \]

in which \( f(r) \) and \( g(r) \) are dimensionless functions which depend on the normalized distance, \( r/R_0 \), and on \( \Delta R/R_0, a_0 \) and \( \nu \). The radial distributions of \( f(r) \) and \( g(r) \) are shown in Figs. 7.7 for several different combinations of \( \Delta R/R_0 \) and \( a_0 \); Poisson's ratio for the halfspace is taken as \( \nu = 1/3 \) in these solutions. The following trends are worth noting:

1. The distributions of \( f(r) \) and \( g(r) \) depend importantly on the frequency parameter, \( a_0 \); the greater the \( a_0 \), the greater is the frequency of the wave involved. As an example, note that in the region of the central hole, the curves for \( f(r) \) have no zero crossing for \( a_0 = 0 \), one such crossing for \( a_0 = 3 \), and two crossing for \( a_0 = 6 \).

2. There is a high concentration or focusing of waves in the central region of the hole, with the result that the peak displacement amplitudes in this region are much larger than \( fcr \) the foundation itself on its outer region of the central hole may have a dominating influence on the stiffness of ring foundations.

**7.6 MODELING OF FOUNDATION-SOIL SYSTEM**
Based on the observations referred to above, it has been hypothesized that the foundation and its supporting medium may be modeled approximately by a combination of two plates: an inner circular plate of radius $R_i$, and an outer infinite plate with a central hole of radius $R_o$, as shown in Fig. 7.7. The two plates are presumed to be of unit thickness, to be excited along their boundaries, and to deform only in shear, as for the plane strain models used by Novak and his associates in studies of piles and embedded foundations (14.35). The bearing resistance of the soil immediately beneath the foundation is neglected, but viscous dampers are added beneath the inner plate to approximate the energy dissipating capacity of the soil in that region. The base of the dampers is presumed to be attached to the moving foundation so that the damping forces are proportional to the relative velocities of the plate and foundation. The damping coefficient, $c_m$, is taken in the form

$$c_m = 2 \rho \omega c_m$$  \hspace{1cm} (7.28)$$

in which $\rho$ = the mass density of the plate and of the soil that it represents, and $\omega$ = a dimensionless damping factor. No dampers are considered for the plate in the outer region as that plate is by itself capable of dissipating energy by radiation.

The impedances for the inner and outer components of the model, $K_{mi}$ and $K_{mo}$, may be expressed as (see Appendix C):

$$K_{mi} = G [\alpha_{mi} + i \omega \beta_{mi}] \quad \text{and} \quad K_{mo} = G [\alpha_{mo} + i \omega \beta_{mo}]$$  \hspace{1cm} (7.29)$$

in which $G$ = the shear modulus of elasticity for the plates and the halfspace material; $\alpha_{mi}$ and $\alpha_{mo}$ are dimensionless stiffness coefficients for the inner and outer plates; and $\beta_{mi}$ and $\beta_{mo}$ are the associated damping coefficients. The total stiffness for the model, $K_m = K_{mi} + K_{mo}$, may then be expressed as
\[ k_m = G \left[ \alpha_m + i \beta_m \right] \] (7.30)

in which \( \alpha_m = \alpha_{mi} + \alpha_{mo} \) and \( \beta_m = \beta_{mi} + \beta_{mo} \).

The variation of \( \alpha_m \) with \( a_o \) is shown in Fig. 7.9 for several different values of \( \Delta R/R_0 \) taking \( \xi_m = 0.25 \). Comparison of these data with those for the exact solutions presented in Fig. 7.4(a) reveals that, while there are differences in detail, the general trends for the two sets of results are similar. Probably the most important difference between the approximate and exact solutions occurs in the neighborhood of \( a_o \). Whereas the exact curves tend to finite values at the origin, the approximate curves tend to zero because the model offers no resistance to static loads.

The dashed line curve in Fig. 7.9 defines the part of \( \alpha_m \) contributed by the outer plate. This result is independent of the value of \( \Delta R/R_0 \) involved and is, therefore, applicable to all cases considered. The relatively small ordinates of this curve confirm the suggestion to the effect that the undulations in the curves for the stiffness coefficient stem from the response of the soil in the region of the central hole, and that this response dominates the behavior of the foundation.

The damping factor for the model, \( \xi_m \), was considered to be the same for all values of \( \Delta R/R_0 \) examined. While a strong argument could be made for varying \( \xi_m \) with the dimensions of the foundation, such a refinement is unnecessary, as the intent of the model has not been to offer a substitute for the exact method of analysis, but rather to provide a simple framework for visualizing the action of the system and for interpreting the salient features of the reported data.

The values of the damping coefficient for the model, \( \beta_m \), do not generally agree as well with the exact values as do the corresponding values of \( \alpha_m \). This is probably due to the approximate manner in which the effects of foundation damping were accounted; the relative sensitivity of the results to the damping mechanism assumed; and for the larger
values of \( \Delta R/R_0 \), to the neglect of the energy dissipating capacity for the part of the supporting medium beneath the foundation.

A Word of Caution.-- It is desirable to evaluate also the amplitude of the difference in the displacement of the ground surface at the center of the foundation and the displacement of the foundation itself. Denoted by \( |Z| \), this amplitude is given by

\[
|Z| = (w_{st})_0 \sqrt{[f(0) - f]^2 + [g(0) - g]^2}
\] (7.31)

in which \( f(0) \) and \( g(0) \) are the values of \( f(r) \) and \( g(r) \) at \( r=0 \).

The values of \( |Z| \) for selected foundation are displayed in Fig. 7.10 along with those of \( |W| \), the displacement amplitude of the foundation itself. Note that \( |Z| \) can be substantially greater than \( |W| \), a result that was also suggested by the distributions of surface displacements presented in Fig. 7.7.

Central to the solution presented herein has been the assumption that the soil in the part of the central hole is unconstrained at the surface. If the clearance between the ground surface and the underside of a superimposed structure is not sufficiently large in that region to accommodate the calculated values of \( |Z| \), then the solutions presented are strictly not valid.

### 7.7 CONTACT PRESSURE

The normal pressure at the interface of the foundation and the supporting medium may be expressed as

\[
\sigma(r,t) = \sigma_{av} \left[ \gamma(r) + i \delta(r) \right] e^{i\omega t}
\] (7.32)

in which \( \sigma_{av} \) should not be confused with the quantity \( \sigma_o \) defined by Eq. 7.17. Whereas \( \sigma_{av} \) stands for the mean contact pressure for the particular ring foundation under consideration, \( \sigma_o \) represents the mean pressure for the associated disk foundation.
The radial distributions of $\gamma(r)$ and $\delta(r)$ are shown in Fig. 7.11 for foundations
having several different combinations of $\Delta R/R_0$ and $a_0$ and a value of $\nu = 1/3$. The follow-
ing trends should be observed:

1. Both $\gamma(r)$ and $\delta(r)$ exhibit singularities along the inner and outer boundaries of
   the ring.

2. Whereas the distributions of these functions are insensitive to variations in $a_0$
   for thin rings, those for thick rings may be quite sensitive, particularly as the limiting case
   of a disk foundation is approached. Increasing $a_0$ in the latter case increases the values of
   $\gamma(r)$ in the central region, and this leads to a more nearly uniform, although somewhat
   undulatory, distribution of pressure than for the corresponding statically loaded foundation.

7.8 APPROXIMATE SOLUTION

The pressure distribution presented in part (a) of Fig. 7.11 suggest that a reason-
able approximation to the dynamic impedance of thin ring foundations might be obtained
by neglecting the imaginary part of the pressure and considering its real part to be uni-
formly distributed. This approach leads to a force-boundary value problem which is much
simpler to handle than the mixed-boundary value problem that has been considered.

The amplitudes of the vertical surface displacements in the simpler approach may
be determined from Eq. 7.4 by letting $\sigma_j = P_0/A$: and the average amplitude of the dis-
placements beneath the foundation may be determined from

$$w_{av} = \frac{1}{A} \int w(r) \, dA$$  \hspace{1cm} (7.33)

in which the integration extends over the area of the foundation, $A$. Determined as the
ratio $P_0/w_{av}$, the foundation impedance may then be expressed in the form,

$$K_a = \frac{P_0}{w_{av}} [ a_a + i a_0 \beta_a ]$$  \hspace{1cm} (7.34)
in which the subscript 'a' identifies the approximate nature of the quantities to which it is attached.

The values $a_a$ and $b_a$ for several different combinations of $\Delta R/R_o$ and $a_o$ are listed in Table 7.3. Comparison of these data with the corresponding exact solutions given in Table 7.2 reveals that the simplified analysis provides a reasonable approximation to the impedance of foundations with $\Delta R/R_o < 0.2$, particularly for the low values of $a_o$. The approximation deteriorates rapidly; however, for the larger values of $\Delta R/R_o$, particularly when $a_o$ also is large.

7.9 EFFECTS OF POISSON'S RATIO

Poisson's ratio for the halfspace affects the dynamic impedance of the foundation in two ways: Though its static value, $(K_{st})_o$, and through the dimensionless parameters $a$ and $b$. The values of $a$ and $b$ presented so far are for $\nu = 1/3$. Supplementary data for three additional values of $\nu$ are presented in Table 7.4.

Comparison of these data with those reported in Table 7.2 reveals that the values of $a$ and $b$ are fairly sensitive to variations in $\nu$, particularly when both $\Delta R/R_o$ and $a_o$ are large. Note in particular that for large $a_o$, the stiffness coefficient becomes negative when $\nu$ is equal to or close to 0.5 and $\Delta R/R_o$ is close to unity. By contrast, the amplitude of the foundation impedance, $|K|$, is relatively insensitive to variations in $\nu$, even when $a_o$ is large. This is demonstrated by the data displayed in Fig. 7.12.

7.10 CONVERGENCE AND ACCURACY OF SOLUTIONS

In Table 7.5 are listed the values of $a$ and $b$ obtained using a progressively larger number of ring elements for foundations having several different combinations of $\Delta R/R_o$ and $a_o$; the locations of the elements considered are identified in Table 7.6. Analysis of these data leads to the following conclusion:

1. The number of ring elements required for a solution of a prescribed accuracy
generally increases with increasing values of $\Delta R/R_o$ and $a_o$.

2. Closer spaced elements are needed around the edges of the foundation where the contact pressure exhibits singularities than for its central region where the pressure is comparatively low and varies gradually.

3. Closely spaced elements also are required for the central region of disk foundations having high values of $a_o$. This is due to the increased intensity and oscillatory character of the contact pressure in this case, and to the fact that the effects of a high-frequency dynamic excitation do not generally decay as rapidly with distance as those of a static excitation of the same magnitude.

4. Solutions of good accuracy can be obtained even with a relative small number of elements.

The numerical data reported herein were obtained with due regard for the conclusions referred to above. The number of ring elements employed ranged from 7 for $\Delta R/R_o = 0.02$, to 11 for $\Delta R/R_o = 0.1$, to 15 for $\Delta R/R_o = 0.5$ and to 20 for disk foundations. The reported data are believed to be generally of high accuracy. Confidence in the accuracy of the data is provided by the consistency of the results obtained with increasing number of elements, and the excellent agreement obtained for the limiting case of a disk foundation with the static solution, and with the more accurate of the previously reported dynamic solutions. A complementary comparison of solutions for a disk foundation is presented in Table 7.7.

7.11 CONCLUSION

Comprehensive numerical data have been presented for the harmonic response of a vertically excited massless, rigid, ring foundation supported at the surface of an elastic halfspace. The response quantities examined include the stiffness and damping coefficients of the foundation in an equivalent spring-dashpot representation of the supporting medium;
the surface displacement of the foundation and of points away from the foundation, and
the normal pressure at the foundation-medium surface interface. The results in each case
have been compared with those obtained for a solid disk having the same radius as the
outer radius of the ring, and it has been shown that the two sets of results may differ sig-
nificantly. A simple approximate model has been used to interpret the results and to pro-
vide insight into the action of the system. The reported data were computed by a procedure
that takes due cognizance of the mixed boundary conditions at the surface of the halfspace,
and they are believed to be of high accuracy.
CHAPTER 8
VERTICAL VIBRATION OF RING FOUNDATIONS WITH MASS

8.1 INTRODUCTION

In the preceding chapter, a comprehensive evaluation was made of the harmonic response of vertically excited, massless, circular ring foundations supported at the surface of a homogeneous, linear halfspace. The objective of this chapter are: (1) to elucidate the corresponding response of foundations with mass, and (2) to present an approximate, simple procedure for evaluating their response to arbitrary transient excitations.

The displacement amplitudes of ring foundations in harmonic motion are evaluated over wide ranges of the parameters involved, and the results are displayed graphically or in tabular form. The parameters examined include the thickness-to-radius ratio for the foundation, dimensionless measures of the foundation mass and of the frequency of the excitation and resulting motion, and Poisson's ratio for the supporting medium. It is shown that, despite the complexity of the foundation impedance functions, both the harmonic and transient responses of foundations with mass may, over a wide range of conditions, be evaluated with good accuracy by considering the foundation-soil system to response as a single-degree-of-freedom system. Plots are included from which the natural frequency and damping of the equivalent simple oscillator may be determined readily.

8.2 SYSTEM CONSIDERED

The system investigated is shown in Fig. 8.1. It is the same as that considered in chapter 8 except that in this chapter the ring foundation has a mass, m, and is excited by an axisymmetric vertical force, F(t), in which t = time.

8.3 HARMONIC RESPONSE OF FOUNDATION

For a harmonically excited foundation, P(t) = P_o e^{i \omega t}, in which P_o = the maximum value of the force; i = \sqrt{-1}; and \omega = the circular frequency of the excitation and of the resulting motion. The steady-state displacement of the foundation, w(t), in this case may
be expressed in the form $w(t) = We^{i\omega t}$, in which $W$ is a complex-valued quantity the real part of which represents the component that is in phase with the exciting force, and the imaginary part represents the component that is $90^\circ$ out of phase.

The steady-state value of the reaction at the foundation-soil interface may similarly be expressed as $Q(t) = Qe^{i\omega t}$, in which $Q$ is a complex-valued quantity given by Eq. 8.1

$$Q = (K_{st})_0[a + ia_0\beta]W \quad (8.1)$$

in which $(K_{st})_0$ is the static stiffness of a solid disk foundation with a radius equal to the outer ring radius, given by

$$K_{st} = \frac{4GR_o}{1-\nu} \quad (8.2)$$

$a_0$ is a dimensionless frequency parameter defined by

$$\alpha_o = \frac{\omega R_o}{v_s} \quad (8.3)$$

and $\alpha$ and $\beta$ are dimensionless coefficients which depend on $\Delta R/R_o$, $a_o$ and $\nu$, and may be determined from Ch. 7. The quantity $v_s = \sqrt{G/\rho}$ in Eq. 8.3 represents the velocity of shear wave propagation in the medium.

**8.4 METHOD OF ANALYSIS**

Equilibrium of the forces acting on the foundation requires that

$$Q - m\omega^2W = P_o \quad (8.4)$$

which, on making use of Eqs. 8.1,8.2,8.3 and of the expression $v_o = \sqrt{G/\rho}$, and introducing the dimensionless mass ratio
\[ B = \frac{1 - \nu}{4} \frac{m}{\rho R_0^3} \]  
(8.5)

leads to

\[ W = \frac{1}{(α - a_o^2 B) + i a_o \beta} \frac{P_0}{(K_{st})_0} \]  
(8.6)

The following expression for the displacement amplitude of the foundation, \(|W|\), is finally obtained by taking the square root of the sum of squares of the real and imaginary parts of Eq. 8.6:

\[ |W| = \frac{1}{\sqrt{(α - a_o^2 B)^2 + (a_o \beta)^2}} \frac{P_0}{(K_{st})_0} \]  
(8.7)

8.5 PRESENTATION OF RESULTS

The variation of \(|W|\) with \(a_o\) is shown in Fig. 8.2 for four different values of \(\Delta R/R_o\) in the range between 0.02 and unity and several different values of \(B\), including the limiting case of a massless foundation (\(B = 0\)). Poisson's ratio for the supporting medium in these solution is taken as \(\nu = 1/3\). The results are displayed in dimensionless form in terms of the amplification factor, \(AF\), defined as

\[ AF = \frac{|W|}{(W_{st})_0} \]  
(8.8)

in which \((W_{st})_0 = the maximum static displacement of the ring foundation under consideration. As a result, all curves start from unity. The normalizing displacement is given by
\[ \left( w_{st}\right)_0 = \frac{1}{a(0)} \frac{P_0}{(K_{st})_0} \]  

(8.9)

in which \( a(0) \) = the value of \( a \) corresponding to \( a_0 = 0 \). The latter quantity depends on \( \Delta R/R_0 \) only and may be determined from Table 7.1 in Ch. 7.

As would be expected from the results for the massless foundations presented in Ch. 7, the frequency response curves for systems with low values of \( B \) (of the order of 0.5 or less) are generally irregular and, depending on the value of \( \Delta R/R_0 \) involved, exhibit one or two prominent peaks within the frequency range examined. By contrast, the curves for the higher values of \( B \) are smooth, have a single prominent peak, and are generally similar to the frequency response curves for viscously damped, single-degree-of-freedom (SDF) system. This similarity suggests that it should be possible in the latter case to model the foundation-soil system by an equivalent, simple oscillator in the spirit of the approach used in Ref. (37) and (45) for the analysis of solid disk foundations.

### 8.6 EQUIVALENT SIMPLE OSCILLATOR

Several of the frequency response curves presented in Fig. 8.2 are compared in Fig. 8.3 with those obtained for SDF system the natural frequency and damping of which have been adjusted so that the absolute maximum, or resonant, value of the response curve and the associated frequency are, in each case, identical to the corresponding values of the actual system. The results are displayed on logarithmic plots so as not to obscure differences between small response values.

If \(|W|\) is the absolute maximum displacement amplitude of the system and \( (a_0)_r \), is the associated value of \( a_0 \), the damping factor of the replacement oscillator, \( \hat{\xi} \), may be determined from
\[
\frac{|W_{\text{max}}|}{(w_{\text{st}})_0} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}
\]

and its circular natural frequency, \( \omega \), is given by

\[
\omega = \frac{(a_0)_r}{\sqrt{1-2\zeta^2}} \frac{v_s}{r}
\]  

(8.11)

The quantities \( \omega \) and \( \zeta \) will be referred to as the effective natural frequency and damping factor of the foundation, respectively.

Excepting system with small values of \( B \) (the precise values are identified in the next section), the two sets of results in Fig. 8.3 are in very good agreement, confirming that the equivalent simple oscillator is indeed an acceptable idealization of the foundation-soil system.

### 8.7 TRANSIENT RESPONSE OF FOUNDATION

Since any transient excitation and its associated response may be expressed as linear combinations of harmonic functions of different amplitudes and frequencies, it should be clear that, for the combinations of parameters for which the harmonic response of the actual and the replacement system are in good agreement, the responses of these systems to an arbitrary transient excitation should be in satisfactory agreement as well. Expressed differently, the transient response of a foundation under these conditions, like the corresponding harmonics response, can be computed with good accuracy by considering the foundation-soil system to behave as a SDF system with a circular frequency \( \omega \) and damping factor \( \zeta \).

For the combinations of the parameters for which the SDF idealization is not sufficiently accurate, the transient response of the foundation must be evaluated by application of Fourier transform techniques.
8.8 NATURAL FREQUENCY AND DAMPING OF EQUIVALENT OSCILLATOR

The values of \((a_o)_{r}\) and \(\tilde{\zeta}\) for the replacement oscillator are plotted in Fig. 8.4 for the values of \(\Delta R/R_0\) and \(B\) for which the SDF idealization of the foundation-soil system is deemed to be adequate. As before, Poisson's ratio for the halfspace in these solutions was taken as \(\nu = 1/3\). Selected values of the damping factor, \(\tilde{\zeta}\), are also listed in table 8.1, where they are compared with those obtained for the limiting values of \(\nu\). To ensure satisfactory accuracy, the frequency response curves in these solutions were evaluated at increments of \(a_o\) of 0.02, and the relevant foundation impedance functions were determined from the data listed in Table 7.2 by parabolic interpolation. Also listed in the table for purposes of comparison are the values of \(\tilde{\zeta}\) computed from the approximate expression

\[
\tilde{\zeta} = \frac{0.425}{\sqrt{B}}
\]

(8.12)

proposed in Ref. 37 for solid disk foundations.

8.9 CONCLUSION

The following conclusion may be drawn from Fig. 8.4:

1. The single-degree-of-freedom representation of the foundation-soil system is valid over rather wide ranges of the parameters \(B\) and \(\Delta R/R_0\).

2. The effective damping for a ring foundation is generally smaller than for the associated disk foundation, the difference increasing with decreasing \(\Delta R/R_0\). By contrast, the effective natural frequency of the foundation is insensitive to the value of \(\Delta R/R_0\).

3. For ring foundations with \(\Delta R/R_0 \geq 0.5\) and values of \(B\) within the range considered in the plots, the responses of ring and disk foundations are essentially the same.
CHAPTER 9
ROCKING VIBRATION OF RING FOUNDATIONS

9.1 SYSTEM CONSIDERED

The system investigated is the same as that shown in Fig. 1a of Ch. 7. However, instead of a vertical force, it is assumed to be excited by a harmonic moment which induces a rocking motion about a horizontal axis normal to the \( \theta = 0 \) plane. The exciting moment is given by

\[
M(t) = M_0 e^{i\omega t}
\]  

(9.1)

in which \( M_0 \) = its peak value, and the resulting vertical surface displacement is given by

\[
w(r, \theta, t) = w(r)\cos \theta \ e^{i\omega t}
\]  

(9.2)

9.2 METHOD OF ANALYSIS

The method of analysis is similar to that employed in Ch. 7, except that the normal pressure at the interface of the foundation and the supporting medium is considered to vary linearly within each annular element rather than being constant. As before, the displacement of a ring element is defined at its centerline. With \( r_{i-1} \) and \( r_i \) denoting the radii of the inner and outer boundaries of the \( i \)th element, the radius to the centerline of the \( i \)th element, \( \bar{r}_i \), is given by

\[
\bar{r}_i = \frac{(r_{i-1} + r_i)}{2}
\]  

(9.3)

Similarly, the normal contact pressure for the \( j \)th element is given by

\[
\sigma_z = -\frac{\sigma_j}{\partial} \frac{r}{R_e} \cos \theta \quad \text{for} \quad r_{j-1} \leq r \leq r_j
\]  

(9.4)

in which \( \sigma_j \) is the unknown constant that remains to be determined.

The vertical surface displacement induced by the stress defined by Eq. 9.4 is given
by

\[ W(r, \theta) = \frac{Q_i}{G R_o} \int_{\xi_0}^{\xi_0} \frac{\Phi(\xi)}{F(\xi)} M(\xi) J_1(\xi r) \, d\xi \cos \theta \]  \hspace{1cm} (9.5) \]

in which \( \xi \) is a dummy variable with units of one over length:

\[ \Phi(\xi) = h_s^2 \sqrt{\xi^2 - h_c^2} \]  \hspace{1cm} (9.6a) \]

\[ F(\xi) = (2^2 - h_s^2)^2 - 4^2 \sqrt{\xi^2 - h_c^2} \sqrt{\xi^2 - h_s^2} \]  \hspace{1cm} (9.6b) \]

\[ M(\xi) = \frac{2}{\xi} \left[ J_1(\xi r_{j-1}) - J_1(\xi r_j) \right] \]  \hspace{1cm} (9.6c) \]

\[ J_0 \text{ and } J_1 \text{ are Bessel functions of the first kind of order zero and one, respectively;} \]

\[ h_c = \omega/v_c \text{ and } h_s = \omega/v_s \]  \hspace{1cm} (9.7) \]

and \( v_c \) and \( v_s \) are the speeds of wave propagation for the compressional and shear waves
in the halfspace, respectively. The derivation of Eq. 9.5 is given in the Appendix C.

As in Ch. 7, let \( d_{ij} \) represent the quantity obtained from Eq. 9.5 for \( \sigma_j = 1, r = r_i \) and \( \theta = 0 \). Then the displacement amplitude along the centerline of the \( i \)th element, \( W_i \), is given
by

\[ W_i = \sum_{j=1}^{n} d_{ij} \sigma_j \cos \theta \]  \hspace{1cm} (9.8) \]

On applying Eq. 9.8 to each element, the displacements at the centerlines of the various
elements can be expressed as

\[ \{W\} = [d] \{\sigma\} \cos \theta \]  \hspace{1cm} (9.9) \]

in which \( \{W\} \) is a column matrix with \( W_i \) as its \( i \)th element; \( [d] \) is a square matrix of the
nxn influence coefficients \( d_{ij} \), and \( \{\sigma\} \) is a column matrix of the interface pressure coefficients, \( \sigma_j \).

For a rigid body rotation of the foundation, the displacement vector, \( \{W\} \), can be expressed as

\[
\{W\} = \{\bar{r}\} \phi \cos \theta
\]

in which \( \{\bar{r}\} \) is the column matrix for which the \( i \)th element is \( \bar{r}_i \), and \( \phi \) is angle of rotation of the foundation.

On substituting Eq. 9.10 into Eq. 9.9 and solving for \( \{\sigma\} \), one obtains

\[
\{\sigma\} = [d]^{-1} \{\bar{r}\} \phi
\]

in which the -1 superscript denotes the inverse of the matrix. The exciting moment, \( M \), associated with the rotation \( \phi \) is given by

\[
M = \{A\}^T [d]^{-1} \{\bar{r}\} \phi
\]

in which the T superscript represents the transpose of a matrix and \( \{A\} \) is a column matrix, the \( j \)th element of which is given by

\[
A_j = \frac{\pi}{4R_o} \left( r_i^\theta - r_{i-1}^\theta \right)
\]

which is the moment induced at \( j \)th element due to the stress distribution on that element which is given by Eq. 9.4 with \( \sigma_{j-1} \)

The rocking impedance of the foundation, \( K \), is finally determined from

\[
K = M/\phi = \{A\}^T [d]^{-1} \{\bar{r}\}
\]

9.3 STATIC SOLUTIONS

The static stiffness of the ring foundation, \( K_{st} \), is given by
\[ K_{st} = a (K_{st})_0 \]

in which \((K_{st})_0\) = the static stiffness of a solid disk foundation with a radius equal to the outer ring radius, given by

\[ (K_{st})_0 = \frac{8}{3(1-\nu)} \mu R_0^3 \]

and \(a\) = a dimensionless ratio that depends on the ratio of width to outer radius for the ring, \(\Delta R/R_0\). Note that Poisson's ratio for the supporting medium, \(\nu\), appears only in the expression for \((K_{st})_0\).

The variation of \(a\) with \(\Delta R/R_0\) is shown in Fig. 9.1, and the numerical data used to prepare this plot are listed in Table 9.1. Fig. 9.1 displays the same trends as those obtained for the vertically loaded vibration, and are in general agreement with those reported previously by Dhawan (11). Note, in particular, that for values of \(\Delta R/R_0\) in excess of 0.5, the stiffness of the ring and disk foundations are almost the same, and for \(\Delta R/R_0 = 0.1\), the stiffness of the ring foundation is 86 percent of that for the associated disk foundation.

The reasons for the relative insensitivity of the foundation stiffness to variations in \(\Delta R/R_0\) may be appreciated by reference to Fig. 9.2, which shows the radial distributions of the normal pressure at the foundation-medium interface, \(\sigma(r)\), along the \(\theta = 0\) and \(\theta = \pi\) axies. These curves were obtained from the stresses at the center points of the ring elements, and the results are normalized with respect to the pressure defined as

\[ \sigma_0 = \frac{4M_0}{\pi R_0^3} \]

It is well known that the central region of the solid disk foundation resists only a small fraction of the total moment not only because of the smaller stress acting on it but also because of the shorter moment arms. Accordingly, the addition of a hole in this region
eliminates a part of the foundation that is ineffectively utilized and, unless the hole is extremely large, it affects only slightly the resulting moment that the foundation resists. Furthermore, as a result of the additional singularity that develops around the boundary of the central hole, the ring foundations is more efficiently utilized than the corresponding region of the disk foundation, and this leads to a comparatively increased effective stiffness for that region.

9.4 DYNAMIC SOLUTIONS

Stiffness and Damping Coefficients.— The foundation impedance defined by Eq. 9.14 may be expressed in a generalized form of Eq. 9.15 as

$$K = (K_{st})_0 (\alpha + i a_0 \beta)$$  \hspace{1cm} (9.17)

in which $a_0$ is a dimensionless frequency parameter defined by

$$a_0 = \omega R_0 / v_s$$  \hspace{1cm} (9.18)

and $\alpha$ and $\beta$ are dimensionless coefficients that depend on $\Delta R / R_0$, $a_0$ and $\nu$.

Numerical data for $\alpha$ and $\beta$ are presented in Table 9.2 for foundations with values of $\Delta R / R_0$ in the range between 0.02 and 0.5 and values of $a_0$ up to 6. Poisson’s ratio for the halfspace in these solutions is taken as $\nu = 1/3$. Some of these data are also displayed graphically in Fig. 9.3, in which the following trends may be observed:

1. The impedances of ring and disk foundations differ in general significantly. One observes in particular that, whereas the stiffness coefficient for the disk foundation decreases monotonically with $a_0$, the corresponding variation for a ring is characterized by high-amplitude oscillations the magnitude and location of which depend on the value of $\Delta R / R_0$ involved.

2. The thinner the ring, the smaller is the damping coefficient, $\beta$. By contrast, the
stiffness coefficient for rings is significantly greater than for the associated disk at large values of \(a_0\).

**Amplitude of Foundation Impedance.**-- The amplitude of the total moment necessary to induce unit rotation in the foundation is denoted by \(|K|\) and is given by

\[
|K| = (K_{st}) \sqrt{a^2 + (\omega a_0 \beta)^2}
\]

(9.19)

This quantity it is plotted in Fig. 9.4 as a function of \(a_0\) for selected values of \(\Delta R/R_0\) and \(\nu = 1/3\). As expected, \(|K|\) generally increases with increasing \(\Delta R/R_0\) and \(a_0\), and its value at high frequencies is dominated by the contribution of the damping term. For \(\Delta R/R_0 = 0.5\) the value of \(|K|\) is for all practical purposes the same as that for \(\Delta R/R_0 = 1\), the solid foundation.

**Foundation Flexibility.**-- With the foundation impedance established, the flexibility or compliance of the foundation can be determined by inversion. The dimensionless flexibility coefficients, \(f\) and \(g\), are defined by

\[
f + ig = \frac{1}{\alpha \pm \omega a_0 \beta}
\]

(9.20)

Accordingly,

\[
f = \frac{\alpha}{\alpha^2 + (\omega a_0 \beta)^2} \quad \text{and} \quad g = \frac{-\omega a_0 \beta}{\alpha^2 + (\omega a_0 \beta)^2}
\]

(9.21)

The dependence of \(f\) and \(g\) on \(\Delta R/R_0\) and \(a_0\) is shown in Fig. 9.5. Note that for \(\Delta R/R_0 = 0.5\) the values of \(f\) and \(g\) are almost the same as those for \(\Delta R/R_0 = 1\).

**9.5 CONTACT PRESSURE**

The normal pressure at the interface of the foundation and the supporting medium along the \(\theta = 0\) axis may be expressed as
\[ \sigma(r,t) = \tilde{\sigma}_0[\gamma(r) + \delta(r)]e^{j\omega t} \] (9.22)

in which \( \tilde{\sigma}_0 \) is defined by

\[ \tilde{\sigma}_0 = MR_0/I_0 \] (9.23)

and

\[ I_0 = \frac{\pi}{4} \left( R_0^4 - R_o^4 \right) \]

The quantities \( \gamma(r) \) and \( \delta(r) \) are dimensionless factors that depend on \( r/R_0, \Delta R/R_0, a_o \) and \( \nu \).

The radial distribution of \( \gamma(r) \) and \( \delta(r) \) are shown in Fig. 9.6 for foundations having several different combinations of \( \Delta R/R_0 \) and \( a_o \) Poisson’s ratio for the halfspace in these solutions is considered to be \( \nu = 1/3 \).

### 9.6 EFFECTS OF POISSON’S RATIO

The values of the dimensionless stiffness and damping coefficients, \( a \) and \( \beta \), are presented in Table 9.3 for two different additional values of \( \nu \). Comparison of these data with those reported in Table 9.2 reveals that the values of \( a \) and \( \beta \) are fairly sensitive to variations in \( \nu \). Note, in particular, that for large \( a_o \), the stiffness coefficient becomes negative when \( \nu = 0.5 \) and \( \Delta R/R_0 = 0.5 \).

The number of elements used to compute the data presented so far are the same as that used in Ch. 7.

Convergence and Accuracy of Solutions.-- Table 9.4 is a similar table to Table 7.5 summarizing the convergence and accuracy of the presented method; the locations of the elements considered are identified in Table 7.6. Analysis of these data leads the same conclusion found as those indicated in Ch. 7 for the vertically excited ring foundations.

### 9.7 CONCLUSION
Comprehensive numerical data have been presented for the harmonic response of a massless, rigid, ring foundation supported at the surface of an elastic halfspace subjected to a rocking moment. The response quantities examined include the stiffness and damping coefficients of the foundation in an equivalent spring-dashpot representation of the supporting medium, and the normal pressure at the foundation-medium surface interface. The results in each case have been compared with those obtained for a rigid disk having the same radius as the outer radius of the ring, and it has been shown that the two sets of results may differ significantly. The reported data were computed by a procedure that takes due account of the mixed boundary conditions at the surface of the halfspace, and they are believed to be of high accuracy.
REFERENCES


APPENDIX A

ELEMENTS OF MASS, STIFFNESS AND LOADING MATRICES

The mass, stiffness and loading matrices for Eq. 4.6a are reproduced from Ch. 4 of Ref. (50). The expression $< >$ represents

$$< > = \frac{1}{H} \int_{0}^{H} ( \ldots ) \, d \xi$$  \hspace{1cm} (A.1)

The mass submatrices are

$$A_{ij} = \frac{m_{ij}}{2} < \chi_i \chi_j >$$  \hspace{1cm} (A.2)

$$B_{ij} = \frac{m_{ij}}{2} < \psi_i \psi_j >$$  \hspace{1cm} (A.3)

$$C_{ij} = \frac{m_{ij}}{2} < \psi_i \psi_j > + m_{ij}$$  \hspace{1cm} (A.4)

where $m_{ij}$ is the added mass contributed by liquid. It is given by

$$m_{ij} = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \frac{H}{\alpha} \left[ \frac{1}{I_{H}^{n}} \left[ \frac{\alpha}{H} \right] \right] d_{in} d_{jn} \psi_{n}$$  \hspace{1cm} (A.5)

The stiffness submatrices are

$$D_{ij} = \frac{\pi E H}{2(1-\nu^2)} \frac{h}{\alpha} \left[ 2 \left( \frac{h}{\alpha} \right)^2 < \chi_i \chi_j > + (1-\nu) < \chi_i \chi_j > \right]$$

$$+ \frac{1}{12} \left( \frac{h}{\alpha} \right)^2 \left[ (1-\nu) < \chi_i \chi_j > \right]$$  \hspace{1cm} (A.6)

$$E_{ij} = \frac{\pi E H}{2(1-\nu^2)} \frac{h}{\alpha} \left[ 2 \nu \frac{H}{\alpha} < \chi_i \psi_j > - (1-\nu) \frac{H}{\alpha} < \chi_j \psi_i > \right]$$  \hspace{1cm} (A.7)

$$F_{ij} = \frac{\pi E H}{2(1-\nu^2)} \frac{h}{\alpha} \left[ 2 \nu \frac{H}{\alpha} < \psi_i \psi_j > + \frac{1}{12} \left( \frac{h}{\alpha} \right)^2 \left[ -2 \left( \frac{H}{\alpha} \right)^2 < \psi_i \psi_j > \right. \right. \right.$$  

$$\left. \left. + (1-\nu) \frac{H}{\alpha} < \psi_i \psi_j > \right] \right]$$  \hspace{1cm} (A.8)

$$G_{ij} = \frac{\pi E H}{2(1-\nu^2)} \frac{h}{\alpha} \left[ 2 < \psi_i \psi_j > + (1-\nu) \left( \frac{h}{\alpha} \right)^2 < \psi_i \psi_j > \right. \right.$$  

$$\left. + \frac{1}{12} \left( \frac{h}{\alpha} \right)^2 \left[ 3(1-\nu) \left( \frac{H}{\alpha} \right)^2 < \psi_i \psi_j > \right] \right]$$  \hspace{1cm} (A.9)
\[ H_{xj} = \frac{\pi E H}{2(1-\nu^2)} \frac{h}{a} \left\{ 2 \langle \psi_x \psi_j \rangle + \frac{i}{2z} \frac{h}{a} \left[ \frac{3}{2} (1-\nu) \left( \frac{a}{H} \right)^2 \langle \psi_x \psi_j \rangle \right. \right. \]
\[ - 2\nu \left( \frac{a}{H} \right)^2 \langle \psi_x \psi_j'' \rangle \left. \right\} \]
(A.10)

\[ I_{xj} = \frac{\pi E H}{2(1-\nu^2)} \frac{h}{a} \left\{ 2 \langle \psi_x \psi_j \rangle + \frac{i}{2z} \frac{h}{a} \left[ \frac{3}{2} (1-\nu) \left( \frac{a}{H} \right)^2 \langle \psi_x \psi_j \rangle \right. \right. \]
\[ + 4(1-\nu) \left( \frac{a}{H} \right)^2 \langle \psi_x \psi_j'' \rangle - 2\nu \left( \frac{a}{H} \right)^2 \langle \psi_x'' \psi_j \rangle - \langle \psi_x' \psi_j' \rangle \left. \right\} \]
(A.11)

The loading submatrices are

\[ P_u = 0 \]
(A.12)

\[ P_v = - \frac{m_s}{2} \langle \psi_v \rangle \]
(A.13)

\[ P_w = \frac{m_s}{2} \langle \psi_w \rangle + m_{o,i} \]
(A.14)

in which \( m_{o,i} \) is given by

\[ m_{o,i} = \int_{-H}^{H} \int_{0}^{2\pi} S_x(z) \psi_x(z) \cos \theta \, a \, d\theta \, dz \]  
(A.15)

and \( S_x(z) \) is defined by

\[ S_x(z) = \left\{ \sum_{n=1}^{\infty} \frac{g_{(-n)}^{(-n+1)}}{(2n-1)^2} \frac{I_1(2n-1) \frac{\pi}{2} \frac{z}{H}}{I_1(2n-1) \frac{\pi}{2} \frac{a}{H}} \cos \left[ (2n-1) \frac{\pi}{2} \frac{z}{H} \right] \right\} \rho \frac{h}{a} \cos \theta \]  
(A.16)

Eq. (A.15) represents the contribution of liquid.
APPENDIX B
SOLUTIONS FOR ROCKING RESPONSE OF RIGID TANKS

The impulsive component of the velocity potential function in Eq. 5.7, \( \phi_i = \phi_i(r, \theta, z, t) \), must satisfy the following boundary conditions:

at

\[
\begin{align}
    & z=0 \quad \frac{\partial \phi_z}{\partial z} = -r \cos \theta \dot{\theta} \\ 
    & z=H \quad \frac{\partial \phi_z}{\partial t} = 0 \\ 
    & r=a \quad \frac{\partial \phi_z}{\partial r} = z \cos \theta \dot{\theta} 
\end{align}
\]

(B.1)

and the convective component, \( \phi_c = \phi_c(r, \theta, z, t) \), must satisfy the relations:

at

\[
\begin{align}
    & z=0 \quad \frac{\partial \phi_c}{\partial z} = 0 \\ 
    & z=H \quad \frac{\partial^2 \phi_c}{\partial t^2} + g \frac{\partial \phi_c}{\partial z} = -g \frac{\partial \phi_z}{\partial z} \\ 
    & r=a \quad \frac{\partial \phi_c}{\partial r} = 0
\end{align}
\]

(B.2)

The sum of the corresponding parts of these equations are naturally the same as the boundary conditions defined by Eqs. 5.4, 5.5 and 5.6 in Ch. 5.

Impulsive component:-- The function \( \phi_i \) is further divided into \( \phi_{iw} \) and \( \phi_{ib} \) as

\[
\phi_i = \phi_{iw} + \phi_{ib}
\]

(B.3)

Both \( \phi_{iw} \) and \( \phi_{ib} \) are harmonic functions. The boundary conditions for \( \phi_{iw} \) are

\[
\begin{align}
    & z=0 \quad \frac{\partial \phi_{iw}}{\partial z} = 0 \\ 
    & z=H \quad \frac{\partial \phi_{iw}}{\partial t} = 0
\end{align}
\]

(B.4)
\[ r = a \quad \frac{\partial \phi_{ib}}{\partial r} = \bar{r} \cos \theta \dot{\theta}_i \]  \hspace{1cm} (B.4c)

The boundary conditions for \( \phi_{ib} \) are

\[ z = 0 \quad \frac{\partial \phi_{ib}}{\partial z} = -r \cos \theta \dot{\theta}_i \]  \hspace{1cm} (B.5a)

\[ z = H \quad \frac{\partial \phi_{ib}}{\partial t} = 0 \]  \hspace{1cm} (B.5b)

\[ r = a \quad \frac{\partial \phi_{ib}}{\partial r} = 0 \]  \hspace{1cm} (B.5c)

It should be clear that \( \phi_{iw} \) is the impulsive component of the velocity potential function for the motion shown in Fig. 5.3b, and \( \phi_{ib} \) is the component for the motion shown in Fig. 5.3c.

The solutions for \( \phi_{iw} \) and \( \phi_{ib} \), obtained by the method of separation of variables, can be expressed in the form

\[ \phi_{iw} = C_{ow}^r (r, \bar{z}) a_H \dot{\theta}_j (t) \cos \theta \]  \hspace{1cm} (B.6)

and

\[ \phi_{ib} = C_{ob}^r (r, \bar{z}) a_H \dot{\theta}_j (t) \cos \theta \]  \hspace{1cm} (B.7)

in which

\[ C_{ow}^r (r, \bar{z}) = \frac{H}{a} \sum_{n=1}^{\infty} \left[ (-1)^{n+1} \alpha_n^2 \frac{r}{\alpha_n} - \frac{2}{\alpha_n^3} \right] I_1 \left[ \frac{\alpha_n r}{H} \right] \cos \left( \alpha_n \frac{\bar{z}}{H} \right) \]  \hspace{1cm} (B.8)

and

\[ C_{ob}^r (r, \bar{z}) = \frac{H}{a} \sum_{n=1}^{\infty} \frac{\alpha_n^2}{\alpha_n^3} \frac{r}{I_1 \left[ \frac{\alpha_n r}{H} \right]} \cos \left( \alpha_n \frac{\bar{z}}{H} \right) + \frac{r}{\alpha} \left( 1 - \frac{\bar{z}}{H} \right) \]  \hspace{1cm} (B.9)

\( I_1 \) is the modified Bessel function of the first kind, and \( I_1' \) is its first derivatives, and \( a_n = \)
\[(2n - 1)\pi/2.\]

If the function \( \phi_i \) is expressed as

\[
\phi_i = C_i^r (r, z) \left( \frac{1}{H} \frac{\partial}{\partial \theta} \right) a \cos \theta
\]

and use is made of Eq. B.3, one obtains the following expression for \( C_i^r (r, z) = C_{ow}(r, z) + C_{ob}(r, z) \):

\[
C_i^r (r, z) = \frac{H}{a} \sum_{n=1}^{\infty} \left( \frac{(-1)^{n+1}}{2} \frac{4}{\alpha_n^2} - \frac{4}{\alpha_n^3} \right) \frac{I_1 (\alpha_n \frac{r}{H})}{I_1 (\alpha_n \frac{a}{H})} \cos (\alpha_n \frac{z}{H}) + \frac{r}{a} \left( 1 - \frac{z}{H} \right)
\]

The impulsive component of the hydrodynamic pressure, \( p_i^r = p_i^r (r, \theta, z, t) \), is determined from Eq. 5.3 to be

\[
p_i^r = C_i^r (r, z) \frac{H}{a} \left( \frac{1}{H} \frac{\partial}{\partial \theta} \right) a \cos \theta
\]

and the corresponding wall pressure is determined by evaluating \( C_i^r (r, z) \) at \( r = a \).

The base shear, \( Q_i^r (t) \), corresponding to an arbitrary wall pressure, \( p_i^r = a \), is given by

\[
Q_i^r (t) = \int_0^{2\pi} \int_0^H p_i^r \left|_{r=a} \right. \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \cos \theta \, d\theta \, dz
\]

and its impulsive component, \( Q_i^r (t) \), is obtained by replacing for \( p_i^r \) the expression given in Eq. B.12. This leads to

\[
Q_i^r (t) = m_i^r \left( \frac{1}{H} \frac{\partial}{\partial \theta} \right) \frac{\partial}{\partial \theta} (t)
\]

in which
\[ m_0^r = \left\{ \frac{1}{2} + 2 \frac{H}{\alpha} \sum_{n=1}^{\alpha} \frac{1}{\alpha_n} \left[ 1 - \frac{2(-1)^{n+1}}{\alpha_n} \right] \frac{I_1(\alpha_n \frac{a}{H})}{I_1'(\alpha_n \frac{a}{H})} \right\} m_l \]  

(B.15)

The base moment, \( M^r(t) \), induced by the wall pressure, \( p^r_{r=a} \), is given by

\[ M^r(t) = \int_{0}^{2\pi} \int_{0}^{H} p^r \left| r \cos \theta \right| d\theta \ d\z \]  

(B.16)

and the moment increment, \( \Delta M^r(t) \), induced by the pressure on the tank base, \( p^r_{|z=0} \), is given by

\[ \Delta M^r(t) = \int_{0}^{2\pi} \int_{0}^{H} p^r \left| r \cos \theta \right| dr \ d\z \]  

(B.17)

The impulsive components of these moment, \( M^r(t) \) and \( \Delta M^r(t) \), are obtained by substituting for \( p \) the expression given in Eq. B.12. The results are

\[ M^r(t) = I_0^r \dot{\Theta} \]  

(B.18)

and

\[ \Delta M^r(t) = \Delta I_0^r \ddot{\Theta} \]  

(B.19)

in which

\[ I_0^r = \left\{ \frac{1}{2} + \frac{H}{\alpha} \sum_{n=1}^{\alpha} \frac{2}{\alpha_n^3} \left[ 1 - (-1)^{n+1} \right] \frac{3}{\alpha_n} + \frac{2}{\alpha_n^2} \right\} \frac{I_1(\alpha_n \frac{a}{H})}{I_1'(\alpha_n \frac{a}{H})} m_l H^2 \]  

(B.20)

and

\[ \Delta I_0^r = \left\{ \frac{1}{4} \left( \frac{a}{H} \right)^2 + \sum_{n=1}^{\alpha} \frac{2}{\alpha_n^2} \left[ (-1)^{n+1} - \frac{2}{\alpha_n} \right] \frac{I_2(\alpha_n \frac{a}{H})}{I_1'(\alpha_n \frac{a}{H})} \right\} m_l H^2 \]  

(B.21)
where $I_2$ is the modified Bessel function of the second order.

**Convective Component:** The velocity potential function, $\phi_c$, that satisfies equation 5.1 along with the first and third boundary conditions specified in Eq. B.2 is given by

$$
\phi_c = \sum_{j=1}^{\infty} \int_0^t F_j(t') dt' J_1(\lambda_j \frac{r}{\alpha}) \cosh(\lambda_j \frac{z}{\alpha}) \cos \theta
$$

(B.22)

in which $F_j(t)$ is a time function that must be determined by making use of the second boundary condition in Eq. B.2; $J_1$ is the Bessel function of the first kind; $\lambda_j$ is the $j$th zero of $J_1''$.

The convective component of the hydrodynamic pressure, $p_c = p_c(r,\theta,z,t)$, is determined from Eq. 5.3 to be

$$
p_c = \rho \frac{\partial \phi_c}{\partial \theta} = \sum_{j=1}^{\infty} F_j(t) J_1(\lambda_j \frac{r}{\alpha}) \cosh(\lambda_j \frac{z}{\alpha})
$$

(B.23)

The second equation in Eq. B.2 can also be expressed as

at $z=H$

$$
\frac{\partial^2 p_c}{\partial t^2} + \rho \frac{\partial p_c}{\partial z} = -\rho \frac{\partial^2 p_c}{\partial z^2}
$$

(B.24)

which is determined by taking the partial derivative of it and multiplying it by $\rho \chi$.

The function $F_j(t)$ is determined by substituting Eqs. B.12, B.23 into B.24, and by making use of eigenfunction expansion technique. The result is

$$
F_j(t) = \frac{\omega_j}{\lambda_j J_1(\lambda_j)} \frac{1}{\tanh(\lambda_j \frac{H}{\alpha})} \frac{1}{\cosh(\lambda_j \frac{\alpha}{\alpha})} \frac{2}{\lambda_j^2 - 1} \left\{ \lambda_j \tanh(\lambda_j \frac{H}{\alpha}) \right\}
$$

$$
\frac{-\lambda_j^2}{H} \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha_i} \frac{1}{\lambda_j^2 + (\alpha_i \frac{H}{\alpha})^2 + \frac{a}{H}} \right\} A_j(t)
$$

(B.25)

in which $\omega_j$ is the $j$th circular natural frequency of the liquid in sloshing, given by
\[
\omega_j^z = \frac{\lambda_j q}{a} \tanh \left( \frac{\lambda_j H}{a} \right)
\]
\[(B.26)\]

and
\[
A_j(t) = \omega_j \int_0^t H \tilde{\sigma}_j(\tau) \sin(\omega_j(t-\tau)) d\tau
\]
\[(B.27)\]
is a pseudoacceleration function.

The convective component of the hydrodynamic pressure can be expressed in a more compact form as
\[
P_c^r = \int_0^a \cos \theta \sum_{j=1}^\infty \mathcal{C}_j^r(r, z) A_j(t)
\]
\[(B.28)\]
where
\[
\mathcal{C}_j^r(r, z) = \left\{ \lambda_j \tanh \left( \frac{\lambda_j H}{a} \right) - \frac{\lambda_j a}{H} \sum_{n=1}^\infty \frac{4}{\alpha_n} \frac{1}{\lambda_j^2 + \left( \frac{\alpha_n a}{H} \right)^2} \right\}
\]
\[+ \frac{a}{H} \left( \frac{\lambda_j}{\lambda_j^2 - 1} \right) \frac{1}{\lambda_j \tanh \left( \frac{\lambda_j H}{a} \right)} \frac{J_0(\lambda_j \frac{r}{a}) \cosh \left( \frac{\lambda_j z}{a} \right)}{J_0(\lambda_j \frac{H}{a}) \cosh \left( \frac{\lambda_j H}{a} \right)}
\]
\[(B.29)\]
and the corresponding wall pressure is obtained by evaluating \(\mathcal{C}_j^r(r, z)\) at \(r=a\). The latter function is denoted by \(\mathcal{C}_j^r(z)\) in Eq. 5.8.

The convective component of the base shear, \(Q_c^r(t)\), is determined from Eq. B.13 by replacing for \(p\) the expression defined by Eq. B.28. On making this substitution and integrating, one obtains
\[
Q_c^r(t) = \sum_{j=1}^\infty m_j^r A_j(t)
\]
\[(B.30)\]
in which
\[ m_j^r = \left\{ \frac{H}{a} \lambda_j \tanh(\lambda_j \frac{H}{a}) - \lambda_j^2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\alpha_n}{\lambda_n + (\alpha_n \frac{a}{H})^2} \right\} \lambda_j \]  
\[ + 1 \]  
\[ + \frac{1}{\lambda_j^2} \left( \frac{\alpha}{H} \right)^2 \right\} m_j \]  
\[ \text{(B.31)} \]

The convective components of the base moments induced by the wall pressure and base pressure, \( M_c^r(t) \) and \( \Delta M_c^r(t) \), are computed similarly from Eqs. B.16 and B.17, respectively. The results are

\[ M_c^r(t) = \sum_{j=1}^{\infty} m_j^r h_j A_j(t) \]  
\[ \text{(B.32)} \]

in which

\[ h_j = \left[ 1 - \frac{1}{\lambda_j} \frac{a}{H} \tanh \left( \frac{\lambda_j}{2} \frac{H}{a} \right) \right] H \]  
\[ \text{(B.33)} \]

and

\[ \Delta M_c^r(t) = \sum_{n=1}^{\infty} m_j^r \Delta h_j A_j(t) \]  
\[ \text{(B.34)} \]

in which

\[ \Delta h_j = \left\{ \frac{1}{\lambda_j} \frac{\alpha}{\sinh \left( \frac{H}{a} \lambda_j \right)} \frac{a}{H} \right\} H \]  
\[ \text{(B.35)} \]

The vertical surface displacement of the fluid, \( \text{dir.} \theta, t \), may finally be obtained from the following equation (Ref. (43))

\[ \bar{\rho} \frac{\partial \phi}{\partial t} \bigg|_{z=H} = \int_{\Gamma} \rho \mathcal{G}(r, \theta, t) \]  
\[ \text{(B.36)} \]

Since the impulsive component of the hydrodynamic pressure is zero at \( z = H \), the left-hand member of this equation must represent the convective pressure component at that level; the latter may be determined from Eq. B.25 by letting \( z = H \). On making this substitution, one obtains
\[ d(r, \theta, t) = a \cos \theta \sum_{j=1}^{\infty} C_j^r (r, H) \left( \frac{A_j(t)}{\delta} \right) \]  

(B.37)
APPENDIX C

MATHEMATICAL DETAILS FOR VERTICAL VIBRATION OF RING FOUNDATIONS

Computation of \( d_{ij} \). The influence coefficient, \( d_{ij} \), is given by the following specialized form of Eq. 7.4:

\[
d_{ij} = \frac{1}{g} \int_{0}^{\infty} \frac{\phi(\xi)}{F(\xi)} M(\xi) J_{0}(\bar{r}_{i}\xi) \, d\xi
\]  

(C.1)

The integration of this expression for the static and dynamic conditions of loading is considered separately.

Static Loading. The functions \( \phi(\xi) \) and \( F(\xi) \) in Eq. C.1 are the only terms that depend on the exciting frequency, \( \omega \). For a statically loaded foundation for which \( \omega = 0 \), each of these terms tends to zero, and their ratio must be evaluated by application of L'Hopital's rule. On differentiating the individual terms twice with respect to \( \omega \) and letting \( \omega \to 0 \), one obtains

\[
\frac{\phi(\xi)}{F(\xi)} \bigg|_{\omega \to 0} = -\frac{1 - \nu}{\xi}
\]  

(C.2)

Finally, on substituting this result into Eq. C.1 and making use of Eq. 7.5c, one obtains:

\[
d_{ij} = \frac{1 - \nu}{g} \int_{0}^{\infty} \frac{1}{\xi} \left[ r_{j} J_{1}(r_{j}\xi) - r_{j-1} J_{1}(r_{j-1}\xi) \right] J_{0}(\bar{r}_{i}\xi) \, d\xi
\]  

(C.3)

The evaluation of Eq. C.3 may further be simplified by making use of the identity

\[
\int_{0}^{\infty} \frac{1}{\xi} J_{1}(r_{j}\xi) J_{0}(\bar{r}_{i}\xi) \, d\xi = r_{j} \int_{0}^{\infty} J_{0}(r_{j}\xi) J_{0}(\bar{r}_{i}\xi) \, d\xi - \bar{r}_{j} \int_{0}^{\infty} J_{1}(r_{j}\xi) J_{1}(\bar{r}_{i}\xi) \, d\xi
\]  

(C.4)

and expressing the integrals on the right member of the latter expression in terms of...
elliptic integrals as follows (39):

\[ \int_0^\infty J_0(r_j \xi) J_0(\tilde{r}_i \xi) \, d\xi = \begin{cases} \frac{2}{\pi r_j} K(\tilde{r}_i / r_j) & \text{for } r_j > \tilde{r}_i \\ \frac{2}{\pi \tilde{r}_i} K(r_i / r_j) & \text{for } r_j < \tilde{r}_i \end{cases} \] (C.5)

and

\[ \int_0^\infty J_1(r_j \xi) J_1(\tilde{r}_i \xi) \, d\xi = \frac{1}{\pi (\tilde{r}_i + r_j)} \left[ \frac{\tilde{r}_i}{r_j} + \frac{r_j}{\tilde{r}_i} \right] K\left( \frac{2 \sqrt{\tilde{r}_i r_j}}{\tilde{r}_i + r_j} \right) \\
- \frac{1}{\pi} \left[ \frac{1}{\tilde{r}_i} + \frac{1}{r_j} \right] E\left( \frac{2 \sqrt{\tilde{r}_i r_j}}{\tilde{r}_i + r_j} \right) \] (C.6)

in which \( K(x) \) and \( E(x) \) are complete elliptic integrals of modulus \( x \) of the first and second kind, respectively. Eq. C.4 is valid only when \( r_j \) and \( r_i \) are different from each other; for \( r_j = r_i \), its value is \( 2/\pi \).

In addition to being needed in the static analysis of the system, the static values of \( \sigma_{ij} \) may be used to accelerate the convergence of their corresponding dynamic values. This matter is considered further in the following section.

Harmonic Loading.-- The integrand of Eq. C.1 has branch points at \( \xi = h_c \) and \( \xi = h_s \) and a Rayleigh pole at \( \xi = \xi_R \) (the value of \( \xi \) for which \( f(\xi) = 0 \)). Eq. C.1 was integrated by a combination of analytical and numerical techniques along the real axis of the complex \( \xi \)-plane, as shown in Fig. 7.13. The values of the integral for the semicircles around the branch points are zero, whereas the value around the pole is given by

\[ -i \pi \frac{\delta(\xi)}{F'(\xi)} M(\xi) J_0(\tilde{r}_i \xi) \bigg|_{\xi = \xi_R} \]

in which a prime superscript denotes differentiation with respect to \( \xi \).

The values of the integral along the straight line segments of the integration path
were computed by Gaussian quadrature. Integration for the segment between \( \xi = 0 \) and \( \xi = h_s \) is straightforward, but care is required to evaluate the contribution of the segment to the right of \( \xi = h_s \). The principal value of the integral for \( \xi \) between \( h_s \) and \( 2\xi_R - h_s \) was computed by folding technique \((28)\); and for the larger values, use was made of the limiting asymptotic behavior of the factor \( \Phi(\xi)/F(\xi) \) as indicated below.

If the quantity \( \xi^2 \) in the expressions for \( \Phi(\xi) \) and \( F(\xi) \) is factored out of the terms involving the square root signs, and each of these terms is then represented by a polynomial expansion, it can be shown that, for large values of \( \xi \),

\[
\frac{\phi(\xi)}{F(\xi)} = - \frac{1 - \nu}{\xi} \quad (C.7)
\]

Considering that this is the same limit as that defined by Eq. C.2, it is desirable to rewrite the part of the integral in Eq. C.1 corresponding for \( \xi > 2\xi_R - h_s \) as follows:

\[
\int_{2\xi_R - h_s}^{\xi} \frac{\phi(\xi)}{F(\xi)} \mathcal{J}_0(\rho \zeta) \, d\xi = \int_{2\xi_R - h_s}^{\xi} \left[ \frac{\phi(\xi)}{F(\xi)} + \frac{1 - \nu}{\xi} \right] \mathcal{J}_0(\rho \zeta) \, d\xi \\
- \int_{0}^{\xi} \frac{1 - \nu}{\xi} \mathcal{J}_0(\rho \zeta) \, d\xi + \int_{0}^{2\xi_R - h_s} \frac{1 - \nu}{\xi} \mathcal{J}_0(\rho \zeta) \, d\xi \quad (C.8)
\]

The first term on the right-hand side of this equation can now be shown to converge rapidly; the second term represents the static value of \( d_{ij} \); which, as previously noted, can be computed effectively from Eqs. C.3 through C.6 in terms of elliptic integrals; and the third term offers no difficulty. The integration for values of \( \xi \geq 2\xi_R - h_s \) was carried out incrementally considering successive zeroes of \( \mathcal{J}_0 \) as suggested by Longman \((29)\), and it was terminated when the difference in the contributions of the last two increments was less than a prescribed tolerance.

Analysis of Model.-- The motion of the inner plate in the model is governed by the differential equation
\[ G \left( \frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} \right) = \rho \frac{\partial^2 (z+w)}{\partial t^2} + c_m \frac{\partial z}{\partial t} \]  

(C.9)

in which \( z \) = the displacement of a point on the plate relative to its moving boundary, and \( w \) = the displacement of the boundary. On specializing this equation to harmonic vibrations, solving it subject to the conditions of zero slope at the center and \( z = w \) at \( r = R_1 \), and recalling that the impedance \( K_{mi} \) is the ratio of the total vertical shear at \( r = R_1 \) to the foundation displacement \( w \), the following expression is obtained:

\[ K_{mi} = 2\pi G \frac{J_1(\lambda a_o R_1/R_0)}{\lambda J_0(\lambda a_o R_1/R_0)} a_o \frac{R_1}{R_0} \]  

(C.10)

in which

\[ \lambda = \sqrt{1 - i2\zeta_m} \]  

(C.11)

and \( \zeta_m \) is defined by Eq. 7.28. Eq. C.10 can be recast in the form of Eq. 7.29.

The impedance for the outer element of the plate model, \( K_{mo} \), is given by (35)

\[ K_{mo} = i 2\pi G \frac{K_1(ia_o'/a_o)}{K_0(ia_o'/a_o)} a_o \]  

(C.12)

in which the quantities \( K_0 \) and \( K_1 \) represent second kind of modified Bessel functions of order zero and one, respectively. Eq. C.12 may be similarly be cast in the form of Eq. 7.29.
APPENDIX D

DERIVATION OF $d_{ij}$ IN CHAPTER 9

The displacement components of the halfspace in the radial, circumferential and vertical directions, $u$, $v$, and $w$, are related to the potential functions, $\psi$, $\eta$, and $\chi$ by

$$u = \frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial \eta}{\partial \theta} + \frac{\partial^2 \chi}{\partial r \partial z}$$  \hspace{1cm} (D.1a)

$$v = \frac{1}{r} \frac{\partial \psi}{\partial \theta} - \frac{\partial \eta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2}$$  \hspace{1cm} (D.1b)

$$w = \frac{\partial \psi}{\partial z} + \frac{\partial^2 \chi}{\partial z^2} + h_s^2 \chi$$  \hspace{1cm} (D.1c)

The potential functions are solutions of the following Helmholtz equations

$$\nabla^2 \psi + h_s^2 \psi = 0$$  \hspace{1cm} (D.2a)

$$\nabla^2 \eta + h_s^2 \eta = 0$$  \hspace{1cm} (D.2b)

$$\nabla^2 \chi + h_s^2 \chi = 0$$  \hspace{1cm} (D.2c)

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

The stress-displacement relationships are given by

$$\sigma_{rz} = G \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)$$

$$\sigma_{r\theta} = G \left( \frac{1}{r} \frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z} \right)$$

$$\sigma_{rz} = \frac{2G(1-\nu)}{1-2\nu} \left( \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial v}{\partial \theta} \right) + \frac{2G(1-\nu)}{1-2\nu} \frac{\partial w}{\partial z}$$  \hspace{1cm} (D.3)

and the boundary conditions are taken as:

$$\sigma_{rz} \bigg|_{z=0} = 0$$  \hspace{1cm} (D.4a)

$$\sigma_{z\theta} \bigg|_{z=0} = 0$$  \hspace{1cm} (D.4b)

and
\[
C_{z_0} = \begin{cases} 
-\frac{G}{\mu} \frac{r}{R_0} \cos \theta & \text{if } r_{j-1} \leq r \leq r_j \\
0 & \text{otherwise}
\end{cases}
\] (D.4c)

The three potential functions are taken as
\[
\begin{align*}
\psi(r, \theta, z) &= \psi(r, z) \cos \theta \\
\eta(r, \theta, z) &= \eta(r, z) \sin \theta \\
\chi(r, \theta, z) &= \chi(r, z) \cos \theta
\end{align*}
\] (D.5)
in which \(\psi(r, z), \eta(r, z)\) and \(\chi(r, z)\) are determined from the solutions of Eqs. D.2a through D.2c to be
\[
\begin{align*}
\psi(r, z) &= \int_0^\infty \xi A(\xi) \exp\left[\left(-\sqrt{\xi^2 - h_c^2}\right)z\right] J_1(\xi r) d\xi \\
\eta(r, z) &= \int_0^\infty \xi B(\xi) \exp\left[\left(-\sqrt{\xi^2 - h_c^2}\right)z\right] J_0(\xi r) d\xi \\
\chi(r, z) &= \int_0^\infty \xi C(\xi) \exp\left[\left(-\sqrt{\xi^2 - h_c^2}\right)z\right] J_0(\xi r) d\xi
\end{align*}
\] (D.6)
in which \(A(\xi), B(\xi)\) and \(C(\xi)\) are the unknown functions of \(\xi\) which have to be determined from the boundary conditions. The boundary conditions Eqs. D.4a and D.4b state that there is no shear stresses on the surface; therefore, the potential function, \(\eta\), corresponding to SH waves should be equal to zero, i.e., \(B(\xi) = 0\). The unknown functions \(A(\xi)\) and \(C(\xi)\) can then be determined from either one of the equations Eq. D.4a or Eq. D.4b and Eq. D.4c. By doing so, one obtains
\[
(2\xi^2 - h_s^2)A - 2\xi \sqrt{\xi^2 - h_c^2} C = M(\xi) \frac{G}{\mu} \frac{C}{R_0} \\
-2\xi \sqrt{\xi^2 - h_c^2} A + (2\xi^2 - h_s^2)C = 0
\] (D.7)

The vertical displacement at the surface of the halfspace is obtained from Eq. D.1c by letting \(z = 0\); the result is Eq. 9.5.

Computation of \(d_{ij}\) -- The influence coefficient, \(d_{ij}\), is given by the following special-
ized form of Eq. 9.5

\[ d_{ij} = \frac{1}{GR_0} \int_0^K \frac{\bar{\Phi}(\xi)}{F(\xi)} M(\xi) J_i(\bar{T}_z \xi) d\xi \]  

(D.8)

The integration of this expression for the static loading can be obtained by doing the same procedure as that in Appendix C, and the result is

\[ d_{ij} = -\frac{(1-\nu)}{GR_0} \int_0^K \frac{1}{\xi} M(\xi) J_i(\bar{T}_z \xi) d\xi \]  

(D.9)

The elliptical integrals given in the Appendix C are used to simplify the computation of Eq. D.9.

As for the dynamic loading, Eq. D.8 is computed by making use of exactly the same procedure as that used to compute the \( d_{ij} \) for vertically loaded case. The residual value of the integral at Rayleigh pole is given by

\[ -i \pi \frac{\bar{\Phi}(\xi)}{F'(\xi)} M(\xi) J_i(\bar{T}_z \xi) \bigg|_{\xi = \xi_R} \]  

(D.10)
Table 2.1 Basic Parameters for Rigidly Supported, Clamped-Base Tanks

<table>
<thead>
<tr>
<th>H/a</th>
<th>$\delta$</th>
<th>$A_w$</th>
<th>$A_\xi$</th>
<th>$\beta$</th>
<th>$\Gamma$</th>
<th>$C_A$</th>
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<tbody>
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<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
<td>(6)</td>
<td>(7)</td>
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</tbody>
</table>

(a) Steel tanks: h/a = 0.001, $\nu = 1/3$, $\sigma_\xi/\sigma = 0.127$

| 0.3  | 17.25    | 0.397  | 0.254   | 0.418   | 0.327   | 0.781 |
| 0.5  | 28.74    | 0.438  | 0.319   | 0.451   | 0.357   | 0.793 |
| 0.75 | 43.11    | 0.459  | 0.392   | 0.467   | 0.373   | 0.798 |
| 1.0  | 57.48    | 0.469  | 0.470   | 0.475   | 0.381   | 0.801 |
| 1.5  | 86.23    | 0.479  | 0.644   | 0.484   | 0.389   | 0.804 |
| 2.0  | 114.97   | 0.485  | 0.830   | 0.488   | 0.393   | 0.806 |
| 3.0  | 172.46   | 0.490  | 1.216   | 0.492   | 0.397   | 0.807 |
| 5.0  | 287.43   | 0.494  | 2.011   | 0.495   | 0.400   | 0.809 |

(b) Concrete tanks: h/a = 0.01, $\nu = 0.17$, $\sigma_\xi/\sigma = 0.4$

| 0.3  | 5.54     | 0.181  | 0.0991  | 0.248   | 0.183   | 0.736 |
| 0.5  | 9.24     | 0.309  | 0.199   | 0.347   | 0.264   | 0.760 |
| 0.75 | 13.86    | 0.372  | 0.291   | 0.398   | 0.308   | 0.775 |
| 1.0  | 18.48    | 0.404  | 0.378   | 0.423   | 0.332   | 0.783 |
| 1.5  | 27.71    | 0.436  | 0.558   | 0.449   | 0.356   | 0.792 |
| 2.0  | 36.95    | 0.452  | 0.746   | 0.462   | 0.368   | 0.796 |
| 3.0  | 55.43    | 0.468  | 1.133   | 0.474   | 0.380   | 0.801 |
| 5.0  | 92.38    | 0.481  | 1.928   | 0.485   | 0.390   | 0.805 |
Table 2.2 Fundamental Natural Frequencies and Wall Pressure Coefficients for Rigid Supported, Clamped-Base Tanks

<table>
<thead>
<tr>
<th>Value of $\omega/\omega_0$</th>
<th>Exact</th>
<th>Present Solution</th>
<th>Membrane Solution</th>
<th>Base Pressure Coefficient $C_{AX}(0)$</th>
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<td>Approx.</td>
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### Table 2.3 Fundamental Natural Frequencies and Wall Pressure Coefficients for Rigid Supported Hinged-Base Tanks

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### Table 2.4 Coefficient $C_p$ in Expression for Equivalent Static Radial Force, $P_w$, and Effective Liquid Mass, $m_1$

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(a) Steel Tanks

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TABLE 4.1
VALUES OF $\alpha_i$ AND $\beta_i$ FOR UNIFORM CANTILEVER BEAM

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For $i > 5$ $\alpha_i=1.0$ $\beta_i=(2i-1)\pi/2$
TABLE 4.2
CONVERGENCE TABLE OF NATURAL FREQUENCIES
(FULL STEEL TANKS WITH h/a=0.001, ρ_L/ρ=0.127 AND ν=0.3)

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TABLE 4.9
SENSITIVITY OF RESULTS TO h/a
(STEEL TANKS WITH h/a=0.001 and ρf/ρ=0.127)

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VALUES OF m01h01/mfH

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VALUES OF m01Δh01/mfH

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VALUES OF m01/m5

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VALUES OF m01L1/m5H

†CONCRETE TANKS WITH ρf/ρ=0.4 AND ν=0.17
TABLE 5.1
COEFFICIENT FOR WALL MOMENT AND BASE MOMENT

<table>
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<tr>
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<th>ΔI_o^r/(m_e H^2)</th>
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TABLE 5.2
COORDINATES OF SURFACE DISPLACEMENT AT WALL

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### TABLE 6.1
C$_o$(z), COEFFICIENT OF PRESSURE ALONG THE WALL

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### Table 6.2

**Effective Masses in Expression for Impulsive Component of Base Shear, \( Q_1 \).**

(Steel tanks with \( h/R = 0.001 \), \( \rho_f/\rho = 0.127 \) and \( \nu = 0.3 \))

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<th>( \dot{m}_{02}/\dot{m}_f )</th>
<th>( \dot{m}_{S1}/m_s )</th>
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**TABLE 6.3**

COEFFICIENTS IN EXPRESSION FOR IMPULSIVE COMPONENT OF BASE MOMENT, \( M_\chi \).

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### TABLE 6.5
COEFFICIENTS IN EXPRESSION FOR IMPULSIVE COMPONENT OF FOUNDATION MOMENT, $\Delta M_f^i$.
(STEEL TANKS WITH $h/R=0.001$, $\rho_e/\rho=0.127$ and $\nu=0.3$)

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Table 7.3 Stiffness and Damping Coefficients Corresponding to a Uniform Pressure at Foundation-Soil Interface

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<tr>
<td>$\Delta R/R_0$</td>
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<td>$\Delta R/R_0 = 0.5$</td>
<td>$\Delta R/R_0 = 1.0$</td>
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<tr>
<td>$a_0$</td>
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<td>$\nu = 0.5$</td>
<td>$\nu = 0$</td>
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<td>0.820</td>
<td>0.980</td>
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<td>0.592</td>
<td>0.823</td>
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<td>0.640</td>
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<td>0.622</td>
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Table 7.5  Convergence of Stiffness and Damping Coefficients for Foundations on a Halfspace with $\nu = 1/3$

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<th>$a_0 = 3$</th>
<th>$a_0 = 6$</th>
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<td>$\alpha$</td>
<td>$\beta$</td>
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<td>-----------</td>
<td>-----------</td>
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<tr>
<td>7a</td>
<td>0.816</td>
<td>1.329</td>
<td>0.175</td>
</tr>
<tr>
<td>7b</td>
<td>0.818</td>
<td>1.333</td>
<td>0.177</td>
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<tr>
<td>9</td>
<td>0.819</td>
<td>1.337</td>
<td>0.178</td>
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<td>11</td>
<td>0.820</td>
<td>1.338</td>
<td>0.178</td>
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</table>

(a) For $\Delta R/R_0 = 0.1$

| 5  | 0.969     | 1.843     | 0.976     | 0.520     | 0.721     |
| 7  | 0.975     | 1.816     | 0.996     | 0.533     | 0.734     |
| 9  | 0.977     | 1.806     | 1.002     | 0.536     | 0.739     |
| 11 | 0.979     | 1.795     | 1.005     | 0.551     | 0.747     |
| 13 | 0.980     | 1.790     | 1.008     | 0.552     | 0.750     |
| 15 | 0.980     | 1.787     | 1.010     | 0.554     | 0.751     |

(b) For $\Delta R/R_0 = 0.5$

| 5  | 0.991     | 0.557     | 0.991     | 0.665     | 1.060     |
| 7a | 0.991     | 0.564     | 0.993     | 0.662     | 1.047     |
| 7b | 0.998     | 0.563     | 1.006     | 0.687     | 1.074     |
| 10 | 0.998     | 0.563     | 1.006     | 0.688     | 1.074     |
| 12 | 0.998     | 0.570     | 1.008     | 0.672     | 1.061     |
| 15a| 0.9995    | 0.571     | 1.009     | 0.676     | 1.064     |
| 15b| 0.998     | 0.571     | 1.011     | 0.694     | 1.059     |
| 20 | 0.998     | 0.571     | 1.009     | 0.672     | 1.060     |
| 23 | 0.996     | 0.572     | 1.012     | 0.674     | 1.063     |
| Shah's Solution (38) | (0.573) | (1.013) | (0.679) | (1.064) |
Table 7.6 Definition of Ring Elements Considered in Convergence Study

<table>
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<tr>
<th>n</th>
<th>Values of $r_i/R_0$ considered</th>
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<td>(a) For $\Delta R/R_0 = 0.1$</td>
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<tr>
<td>3</td>
<td>0.9 0.92 0.98 1.0</td>
</tr>
<tr>
<td>5</td>
<td>0.9 0.91 0.92 0.98 0.99 1.0</td>
</tr>
<tr>
<td>7a</td>
<td>0.9 0.91 0.92 0.94 0.96 0.98 0.99 1.0</td>
</tr>
<tr>
<td>7b</td>
<td>0.9 0.905 0.91 0.92 0.98 0.99 0.995 1.0</td>
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<tr>
<td>9</td>
<td>Same as 7b except for addition of 0.902 and 0.998</td>
</tr>
<tr>
<td>11</td>
<td>Same as 9 except for addition of 0.94 and 0.96</td>
</tr>
<tr>
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<td>(b) For $\Delta R/R_0 = 0.5$</td>
</tr>
<tr>
<td>5</td>
<td>0.5 0.55 0.6 0.9 0.95 1.0</td>
</tr>
<tr>
<td>7</td>
<td>0.5 0.52 0.55 0.6 0.9 0.95 0.98 1.0</td>
</tr>
<tr>
<td>9</td>
<td>0.5 0.51 0.52 0.55 0.6 0.9 0.95 0.98 0.99 1.0</td>
</tr>
<tr>
<td>11</td>
<td>0.5 0.51 0.52 0.55 0.6 0.8 0.95 0.98 0.99 1.0</td>
</tr>
<tr>
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<td>0.7 0.9</td>
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<tr>
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<td>0.5 0.505 0.52 0.55 0.6 0.8 0.95 0.98 0.99 1.0</td>
</tr>
<tr>
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<td>0.51 0.7 0.9</td>
</tr>
<tr>
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<td>0.995</td>
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<td>15</td>
<td>Same as 13 except for addition of 0.502 and 0.998</td>
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<td>(c) For $\Delta R/R_0 = 1.0$</td>
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<td>0 0.8 0.9 0.95 1.0</td>
</tr>
<tr>
<td>7a</td>
<td>0 0.3 0.5 0.7 0.8 0.9 0.95 1.0</td>
</tr>
<tr>
<td>7b</td>
<td>0 0.5 0.8 0.9 0.95 0.97 0.99 1.0</td>
</tr>
<tr>
<td>10</td>
<td>0 0.5 0.8 0.9 0.95 0.97 0.99 1.0</td>
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<tr>
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<td>0.93 0.96 0.98</td>
</tr>
<tr>
<td>12</td>
<td>0 0.3 0.5 0.7 Rest same as for n = 10</td>
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<tr>
<td>15a</td>
<td>Same as 12 except for addition of 0.985, 0.995 and 0.998</td>
</tr>
<tr>
<td>15b</td>
<td>Same as 12 except for addition of 0.15, 0.4 and 0.6</td>
</tr>
<tr>
<td>20</td>
<td>0 0.225 0.45 0.675 0.825 0.9 0.95 0.97 0.99 1.0</td>
</tr>
<tr>
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<td>0.075 0.3 0.525 0.7 0.93 0.96 0.98</td>
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<tr>
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<td>0.15 0.375 0.6 0.75</td>
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<td>Same as n = 20 except for addition of 0.985, 995 and 0.998</td>
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Table 7.7 Comparison of Solutions for a Disk Foundation on a Halfspace with $\nu=1/3$

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<tr>
<th>$a_o$</th>
<th>Lysmer</th>
<th>Luco and Westmann*</th>
<th>Shah</th>
<th>Present Study n=15a</th>
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<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
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<td>0.0164</td>
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(a) Values of $f$

(b) Values of $-g$

*As reported in (48)
Table 8.1 Values of Effective Damping Factor, $\bar{\xi}$, for Ring Foundations

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<th>$\Delta R/R_D$</th>
<th>$\nu$</th>
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<td>1/3</td>
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<td>0.320</td>
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<tr>
<td>From Eq.8.12</td>
<td>(0.601)</td>
<td>(0.425)</td>
<td>(0.301)</td>
<td>(0.245)</td>
<td>(0.190)</td>
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</tr>
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<tr>
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<td>$\Delta R/R_0$=0.05</td>
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<td>$a$</td>
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## TABLE 9.4
CONVERGENCE OF STIFFNESS AND DAMPING COEFFICIENTS FOR FOUNDATIONS ON HALFSPACE WITH $\nu=1/3$

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Fig. 2.1 System considered
Fig. 2.2 Distributions of Assumed Displacement Function for Clamped-Base Tanks
Fig. 2.3 Elastically Supported Liquid Mass, $m_1$, for Clamped-Base Tanks
Fig. 2.4 Models for Rigidly and Elastically Supported Tanks
Fig. 2.5 Frequency Response Curves for Elastically Supported Clamped-Base Tanks
Fig. 2.6 Comparison of Exact and Approximate Frequency Response Curves for Elastically Supported Clamped-Base Steel Tanks

--- Exact Solution for SDF Oscillator

- H/a = 0.5 (Right Scale)
- H/a = 1 (Left Scale)
- H/a = 3 (Right Scale)

(a) V'/V = 10
(b) V'/V = 30
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Fig. 2.8 Effective Frequency and Damping of Elastically Supported Clamped-Base Tanks
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Fig. 3.2 The First Mode of Equivalent Static Wall Pressure
Fig. 3.3 Models for Elastically Supported Tanks
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Fig. 3.5 Frequency Response Curves for Elastically supported tanks with $H/a=0.5$ and $v/v_s=40$. 
Fig. 3.6 Frequency Response Curves for Elastically supported tanks with H/a=1.0 and ν/νg=10.

- One Mode
- Two Modes
- Three Modes
Fig. 3.7 Frequency Response Curves for Elastically supported tanks with $H/a=1.0$ and $v/v_s=40$. 
Fig. 3.8 Frequency Response Curves for Elastically supported tanks with H/a=3.0 and v/v_s=10.
Fig. 3.9 Frequency Response Curves for Elastically supported tanks with $H/a=3.0$ and $v/v_s=40$. 
Fig. 4.1 System Considered
Fig. 4.2 Frequency Coefficient $C_1$ for Steel Tanks with $\nu=0.3$, $\rho_f/\rho=0.127$
\[ \omega_j = \frac{C_j \sqrt{E}}{H \sqrt{\rho}} \]

Fig. 4.3 Frequency Coefficient \( C_j \) for Steel Tanks with 
\( \nu = 0.3, \frac{\rho_f}{\rho} = 0.127 \)
Fig. 4.4 Frequency Coefficient $C_j$ for Concrete Tanks with $\nu=0.17$, $\rho_i/\rho=0.4$
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Fig. 4.6 Distributions of Fundamental Modal Hydrodynamic Pressure and Radial Displacement on Concrete Tanks with $\nu=0.17$, $\rho_f/\rho=0.4$
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Fig. 4.6 Base Shear Coefficients for Liquid-Filled Steel Tanks with $\nu=0.3$, $\rho_f/\rho=0.127$ and $h/a=0.001$
Fig. 4.9 Wall Moment Coefficients for Liquid-Filled Steel Tanks with $\nu=0.3$, $\rho_f/\rho=0.127$ and $h/a=0.001$
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Fig. 5.4 Distributions of Components of Impulsive Pressure on the Wall
Fig. 5.5 Distributions of Convective Pressure on the Wall
Fig. 5.6 Moment Coefficients $I_o^r$ and $\Delta I_o^r$
Fig. 5.7 Distributions of Impulsive Pressure on the Base
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Fig. 6.2 Base Shear Coefficients for Liquid-Filled Steel Tanks with $\nu=0.3$, $\rho_f/\rho=0.127$ and $h/a=0.001$

Fig. 6.3 Wall Moment Coefficients for Liquid-Filled Steel Tanks with $\nu=0.3$, $\rho_f/\rho=0.127$ and $h/a=0.001$
Fig. 6.4 Pressure Distributions of Instantaneous Base Moment on the Base
Fig. 6.5 Instantaneous Base Moment
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Fig. 7.3 Distributions of Normal Contact Pressure for Statically Loaded Foundations
Fig. 7.4 Stiffness and Damping Coefficients $\alpha$ and $\beta$; $\nu=1/3$
Fig. 7.5 Amplitude of Foundation Impedance; $\nu=1/3$
Fig. 7.6 Flexibility Coefficients $f$ and $-g$; $\nu=1/3$
Fig. 7.7 Radial Distributions of Vertical Surface Displacements; $\nu=1/3$
Fig. 7.8 Model for Ring Foundation-Soil System

Fig. 7.9 Stiffness Coefficient for Model
fig 7.10 Amplitude of Relative Displacement $|Z|$
Fig. 7.11 Distributions of Normal Contact Pressure for Dynamically Excited Foundations
Fig. 7.12 Effect of Poisson's Ratio $\nu$ on Amplitude of Foundation Impedance
Fig. 7.13 Integration Path Considered
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Fig. 8.2 Maximum Harmonic Response of Ring Foundations; \( \nu=1/3 \)
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Fig. 9.2 Distributions of Normal Contact Pressure for Statically Loaded Foundations
Fig. 9.3 Stiffness and Damping Coefficients $\alpha$ and $\beta$; $\nu=1/3$
Fig. 9.4 Amplitude of Foundation Impedance; $\nu = 1/3$
Fig. 9.5 Flexibility Coefficients $f$ and $-g$: $\nu=1/3$