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INTEGRAL EQUATIONS' APPROACH TO SCATTERING PROBLEMS

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INTEGRAL EQUATIONS' APPROACH TO SCATTERING PROBLEMS

by

VLADIMIR ROKHLIN

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DOCTOR OF PHILOSOPHY

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To the Memory of John W. Dunkin
Abstract

In the present thesis, the classical potential theory is used to derive systems of second kind integral equations corresponding to scattering of acoustic and elastic waves from both fluid and solid inclusions. These systems of integral equations are discretized by means of the Nyström algorithm. The resulting systems of linear algebraic equations are solved by means of a version of the preconditioned generalized conjugate residual algorithm. In order to obtain the time domain result, results for a sequence of frequency values are computed with subsequent application of the Fast Fourier Transformation. The computational results presented in the present thesis indicate that the resulting numerical algorithm is suitable for fairly large-scale scattering computations in two dimensions. The 3-dimensional version of the theory is also briefly discussed.
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Appendix
Introduction

The interest in solving scattering problems numerically arises in connection with several practical problems. The following list contains some of these.

1. Geophysical prospecting for oil and minerals
2. Anti-submarine warfare
3. Detection and location of earthquakes and nuclear explosions
4. Non-destructive industrial testing
5. Medical diagnostics
6. Oceanographic studies

There are two large classes of scattering problems: forward scattering and inverse scattering.

In a forward scattering problem, a certain configuration of sources, scatterers, and receivers is given, and one wants to determine numerically the signal recorded by the receivers. In an inverse scattering problem, the signal recorded by the receivers is known, and one is trying to determine some of the parameters of the physical situation that has generated this signal (such as the location of the source or the shape of the scatterer). In most applications, the inverse problem is of greatest immediate interest. However, the forward problem has to be satisfactorily solved before the inverse problem can be even approached. Unfortunately, the
forward problem is by no means a solved one from the numerical point of view.

Most practical scattering problems are 3-dimensional, but the numerical solution of 3-dimensional scattering problems tends to be prohibitively expensive. It turns out that in many cases 2-dimensional models are quite adequate since they can be interpreted as 3-dimensional problems in which the sources are infinite lines and the scatterers are infinite cylinders. Since the numerical solution of 2-dimensional scattering problems is often feasible, many numerical models used by the industry are 2-dimensional.

There are two crucial parameters determining the complexity of a forward scattering problem. They are defined by the formulae

$$\alpha = \frac{D}{\lambda}$$

$$\beta = \frac{R}{\lambda}$$

where $\lambda$ is the wavelength of the source, $D$ is the diameter of the scatterer, and $R$ is the smallest radius of curvature of the surface of the scatterer (we presume that the surface of the scatterer is smooth). Depending on the parameters $\alpha$, $\beta$ of a given problem we will classify it as belonging to one of the following groups:
1. $\alpha \ll 1$ (small scatterer). In this case the problem is usually manageable by means of multipole expansions or Born approximation (see [1], [2], [3]). Some versions of finite element or finite difference algorithms also tend to be effective. Generally, the existing computational methods are satisfactory.

2. $\alpha \gg 1$, $\beta \gg 1$ (a large scatterer with a smooth surface). In this case the ray theory or the generalized ray theory is applicable. The resulting algorithms tend to be sufficiently fast and precise.

3. $\alpha \approx 1$ (the size of the scatterer is comparable to the wavelength). In this case none of the above techniques give satisfactory results. The group of algorithms known as T-matrix method (see, for example, [9]) gives extremely good results for certain geometries but cannot be applied efficiently in general situations. Thus, there exists a wide class of problems in which the scatterers are too small for the ray theory, too large for the Born approximation or multipole expansion, and cannot be treated by means of a T-matrix method. Attempts to apply the finite difference or finite element methods to such problems tend to meet with serious difficulties. Indeed, in many scattering problems the distances between the source, the receivers, and the scatterers are very large while the scatterers themselves are reasonably small. In order to solve a scattering problem by means of either finite differences or finite elements, an area containing the receivers, the scatterers, and the sources must
be discretized as well as a substantial neighborhood of such an area. The resulting computation times tend to be prohibitively large. In such situations, it becomes attractive to represent the scattered field in the frequency domain by an integral over the surface of the scatterers by means of some version of the Stoke's theorem. A numerical implementation of such an algorithm will have to discretize only the surface of the scatterers, replacing the integrals by quadrature formulae and the integral equations by systems of linear equations. For such an algorithm to be stable, the integral equations must be of the second kind and their kernels must be continuous or, at least, weakly singular (see [11]).

4. $\alpha \gg 1, \beta \ll 1$. This case is similar to the preceding one, but the situation is even worse. It corresponds to a large scatterer with a relatively coarse surface. The author is not aware of any serious attempts to solve such problems numerically.

The purpose of this work is to develop a numerical technique that in combination with modern computers would be able to solve scattering problems of the types 3 and 4 in 2-dimensional situations. Most of this thesis deals with isolated scatterers imbedded in a homogeneous 2-dimensional space. Multiple scatterers and the 3-dimensional version of the theory are discussed briefly in Chapter 10.
I. Scattering Theory in Two Dimensions

1. Equations of Motion in Two Dimensions

   a) Notation. The coordinates in the plane will be denoted by \( x, y \). The \( x \)-component of the displacement will be denoted by \( u \), and the \( y \)-component by \( v \). Obviously,

\[
\frac{\partial u}{\partial x} \, dx + \frac{\partial u}{\partial y} \, dy ,
\]

\[
\frac{\partial v}{\partial x} \, dx + \frac{\partial v}{\partial y} \, dy ,
\]

(1.1.1)

and, following [4] we will introduce the following notation.

\[
\frac{\partial u}{\partial x} = \varepsilon_{xx}, \quad \frac{\partial v}{\partial y} = \varepsilon_{yy} ,
\]

(1.1.2)

\[
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \varepsilon_{xy} .
\]

We will be writing the generalized Hook's law in the following form:

\[
\sigma_{xx} = c_{11} \varepsilon_{xx} + c_{12} \varepsilon_{xy} + c_{13} \varepsilon_{yy} ,
\]

\[
\sigma_{xy} = c_{21} \varepsilon_{xx} + c_{22} \varepsilon_{xy} + c_{23} \varepsilon_{yy} ,
\]

\[
\sigma_{yy} = c_{31} \varepsilon_{xx} + c_{32} \varepsilon_{xy} + c_{33} \varepsilon_{yy} ,
\]

(1.1.3)

where the coefficients \( \{c_{ij}\} \), \( i,j = 1,2,3 \) are elastic constants of the material and \( \sigma \) is the stress tensor.
It can be shown that in an isotropic medium (the only kind we will be considering in this thesis),

\[ C_{13} = C_{31} = \lambda , \]
\[ C_{22} = \mu , \]
\[ C_{11} = C_{33} = \lambda + 2\mu , \]  \hspace{1cm} (1.1.4)

and equations (1.1.3) can be rewritten as

\[ \sigma_{xx} = \lambda \Delta + 2\mu \varepsilon_{xx} , \]
\[ \sigma_{yy} = \lambda \Delta + 2\mu \varepsilon_{yy} , \]  \hspace{1cm} (1.1.5)
\[ \sigma_{yx} = \mu \varepsilon_{yx} , \]
\[ \Delta = \varepsilon_{xx} + \varepsilon_{yy} . \]

The two elastic constants \( \lambda \) and \( \mu \) are known as Lame's constants, and they completely define the elastic behavior of an isotropic medium.

b) Equations of motion in time domain. The motion of the 2-dimensional elastic medium in the absence of friction is described by the equations
\[
\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y},
\]
\[
\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}.
\]

Substituting (1.1.5) into (1.1.6), we obtain
\[
\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} (\lambda \varepsilon_{yy} + (\lambda + 2\mu)\varepsilon_{xx}) + \frac{\partial}{\partial y} (\mu \varepsilon_{yx}) ,
\]
\[
\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x} \mu \varepsilon_{yx} + \frac{\partial}{\partial y} (\lambda \varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy} ).
\]

or
\[
\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial x} + \mu \varepsilon^2 u,
\]
\[
\rho \frac{\partial^2 v}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial x} + \mu \varepsilon^2 v,
\]

where \( \nabla^2 \) is a differential operator defined by the formula
\[
\nabla^2 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).
\]

In the fluid case when \( \mu = 0 \), the equations (1.1.8) assume the form
\[
\rho \frac{\partial^2 u}{\partial t^2} = \lambda \frac{\partial \Delta}{\partial x} = \lambda \frac{\partial \varepsilon_{xx}}{\partial x} + \lambda \frac{\partial \varepsilon_{yy}}{\partial x},
\]
\[
\rho \frac{\partial^2 v}{\partial t^2} = \lambda \frac{\partial \Delta}{\partial y} = \lambda \frac{\partial \varepsilon_{xx}}{\partial y} + \lambda \frac{\partial \varepsilon_{yy}}{\partial y},
\]
or, using (1.1.2),
\[ \rho \cdot \frac{\partial^2 u}{\partial t^2} = \lambda \cdot \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} \right), \]
\[ \rho \cdot \frac{\partial^2 v}{\partial t^2} = \lambda \cdot \left( \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial y^2} \right). \]  
(1.1.10)

The equations (1.1.8) are called Navier equations.

c) Equations of motion in terms of potentials. Using the

notion of dilatational and shear potentials \( \phi, \psi : R^2 \rightarrow R \), the

equations (1.1.8) can be rewritten as

\[ u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}, \]
\[ v = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}, \]
\[ \rho \cdot \frac{\partial^2 \phi}{\partial t^2} = (\lambda + 2\mu) \cdot \nabla^2 \phi, \]
\[ \rho \cdot \frac{\partial^2 \psi}{\partial t^2} = \mu \cdot \nabla^2 \psi. \]  
(1.1.11)

The form (1.1.11) for the equations of motion is valid for any
twice continuously differentiable functions \( u, v : R^2 \rightarrow R \). A detailed
proof can be found in [5], pages 348-356. In the fluid case only one
potential function is required, and (1.1.11) assumes the form
\[
\begin{align*}
    u &= \frac{\partial \phi}{\partial x}, \\
    v &= \frac{\partial \phi}{\partial y}, \\
    \rho \frac{\partial^2 \phi}{\partial t^2} &= \lambda \nabla^2 \phi. \tag{1.1.12}
\end{align*}
\]

It is well known that in terms of displacement potentials, the pressure at the point \((x,y)\) of the fluid medium is equal to \(\lambda \nabla^2 \phi\), and this fact will be used in Chapter V.

d) Frequency-domain version of the theory. The equations (1.2.11), (1.2.12) are hyperbolic linear equations with constant coefficients. One standard way to deal with such equations is to perform the Fourier transformation with respect to time and deal with the resulting system of elliptic equations. This operation presents both theoretical and computational difficulties. These difficulties are discussed in detail in Chapter VI. Following the usual practice, we will denote the coordinate dual to time by \(\omega\) and call it angular frequency. The variables dual to \(\phi, \psi, u, \text{ and } v\) will be denoted by \(\hat{\phi}, \hat{\psi}, \hat{u}, \hat{v}\). However, whenever it is clear from the context that we are working in frequency domain, the hats will be omitted. The equations dual to (1.1.11) are
\[
\begin{align*}
    u &= \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}, \\
    v &= \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}, \\
    \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + K_p^2 \phi &= 0, \\
    \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + K_s^2 \psi &= 0,
\end{align*}
\]

with \( K_p = \frac{\omega}{(\frac{\lambda + 2\mu}{\rho})^{1/2}} \), \( K_s = \frac{\omega}{(\frac{\lambda}{\rho})^{1/2}} \),

and the equations dual to (1.1.12) are

\[
\begin{align*}
    u &= \frac{\partial \phi}{\partial x}, \\
    v &= \frac{\partial \phi}{\partial y}, \\
    \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + K^2 \phi &= 0,
\end{align*}
\]

with \( K = \frac{\omega}{(\frac{\lambda}{\rho})^{1/2}} \).

2. Boundary Conditions at the Interface of Two Media

For definitiveness, we will be considering the following situation:
The body is immersed into the 2-dimensional space. Various combinations of the material for the body and the space (solid/solid, fluid/fluid, etc) will be considered. For convenience the notation is given in the following table.

\[
\begin{align*}
\phi & \quad \text{p-potential inside the body} \\
\psi & \quad \text{shear potential inside the body} \\
\phi_{\text{out}} & \quad \text{p-potential outside the body} \\
\psi_{\text{out}} & \quad \text{shear potential outside the body} \\
\sigma_{NN} & \quad \text{normal-normal stress on the inside of the boundary} \\
\sigma_{NN}_{\text{out}} & \quad \text{normal-normal stress on the outside of the boundary} \\
\sigma_{NT} & \quad \text{tangent-normal stress on the inside of the boundary} \\
\sigma_{NT}_{\text{out}} & \quad \text{tangent-normal stress on the outside of the boundary} \\
\sigma_{NN} & \quad \text{normal-normal strain on the inside of the boundary} \\
\sigma_{NN}_{\text{out}} & \quad \text{normal-normal strain on the outside of the boundary} \\
\varepsilon_{NT} & \quad \text{tangent-normal strain on the inside of the boundary} \\
\varepsilon_{NT}_{\text{out}} & \quad \text{tangent-normal strain on the outside of the boundary}
\end{align*}
\]
\( \varepsilon_{TT}^{\text{in}} \) \hspace{1cm} \text{tangent-tangent strain on the inside of the boundary}

\( \varepsilon_{TT}^{\text{out}} \) \hspace{1cm} \text{tangent-tangent strain on the outside of the boundary}

\( \lambda \) \hspace{1cm} \text{Lame's coefficients}

\( \mu \) \hspace{1cm} \text{the tangent component of displacement on the inside of the boundary}

\( u_{\text{in}} \) \hspace{1cm} \text{the tangent component of displacement on the outside of the boundary}

\( u_{\text{out}} \) \hspace{1cm} \text{the normal component of displacement on the inside of the boundary}

\( v_{\text{in}} \) \hspace{1cm} \text{the normal component of displacement on the outside of the boundary}

\( N \) \hspace{1cm} \text{the internal normal to the boundary (||N|| = 1)}

\( T \) \hspace{1cm} \text{the tangent to the boundary in the positive direction (||T|| = 1)}

When the superscript 'source' is added to a characteristic of the field (like \( u^{\text{source}} \) \( \sigma_{NN}^{\text{source}} \)), it makes it the characteristic of the field created by the source in the absence of the scatterer.

When the superscript "total" is added to a characteristic of the field, it makes it a characteristic of the total field, i.e. of the sum of the source field and the scattered field. A characteristic of the field without either of the above superscripts pertains to the
scattered field. For example, \((u^{\text{source}})^{\text{in}}\) would mean the normal
displacement of the field on the inside of the boundary created by
the sources inside the body. Ordinarily, in the scattering problems,
no sources are located inside the scatterer.

The following boundary conditions are reproduced (with a change
in notation) from [6], [7]. We will state these conditions without
derivation. Naturally, they are the same in the time and frequency
domains.

a) Free solid surface (a solid body inside an empty space):
\[
\sigma_{NN}^{\text{in}} = \sigma_{NT}^{\text{in}} = 0
\]

b) Surface of a solid body welded to a rigid body (a rigid
inclusion inside a solid space):
\[
u^{\text{out}} = u^{\text{out}} = 0
\]

c) Welded surface of two solids:
\[
\sigma_{NN}^{\text{in}} = \sigma_{NN}^{\text{out}}, \quad u^{\text{in}} = u^{\text{out}},
\sigma_{NT}^{\text{in}} = \sigma_{NT}^{\text{out}}, \quad v^{\text{in}} = v^{\text{out}}.
\]

d) Solid/fluid interface (a solid body immersed in a fluid
space):
\[
v^{\text{in}} = v^{\text{out}}, \\
\sigma_{NT}^{\text{in}} = 0, \\
\sigma_{NN}^{\text{in}} = \sigma_{NN}^{\text{out}}.
\]
e) Fluid/fluid interface:

\[ u^{\text{in}} = u^{\text{out}} , \]

\[ \sigma^{\text{in}}_{\text{NN}} = \sigma^{\text{out}}_{\text{NN}} . \]

f) Interface of a fluid and infinitely rigid body (an infinitely rigid body immersed in a fluid):

\[ v^{\text{out}} = 0 . \]

It is interesting to notice that the free fluid interface is not on the above list because such an interface is not physically possible, since something must keep the interface in place. However, the Dirichlet problem is a good approximation to the physical situation when the surface of a fluid is kept in place by the combination of gravity and surface tension (see [26], page 1469).
II. Boundary Conditions In Terms of Potentials

In Section 1.3, the boundary conditions at various types of interfaces were given. In order to apply the method of potential functions, one would like to reformulate these conditions in terms of potentials. The boundary conditions a)-f) from Section 1.3 contain the following expressions:

\[ \sigma_{NN}, \sigma_{TN}, u, v \]

Let us express the above quantities in terms of potentials.

a) Solid body

\[ \sigma_{NN} = \lambda \cdot \Delta + 2 \mu \cdot \varepsilon_{NN} = \]

\[ = \lambda \cdot (\varepsilon_{TT} + \varepsilon_{NN}) + 2 \mu \cdot \varepsilon_{NN} = \]

\[ = \lambda \cdot \varepsilon_{TT} + \lambda + 2 \mu \cdot \varepsilon_{NN} = \]

\[ = \lambda \cdot \frac{\Delta u}{\Delta t} + (\lambda + 2 \mu) \cdot \frac{\Delta v}{\Delta N} = \]

\[ = \lambda \cdot \frac{\Delta^2 \phi}{\Delta T^2} \left( \frac{\Delta \psi}{\Delta N} \right) + (\lambda + 2 \mu) \cdot \frac{\Delta^2 \phi}{\Delta N^2} \left( \frac{\Delta \psi}{\Delta T} \right) = \]

\[ = \lambda \cdot \frac{\Delta^2 \phi}{\Delta T^2} - \frac{\Delta \psi}{\Delta N} \right) + (\lambda + 2 \mu) \cdot \frac{\Delta^2 \phi}{\Delta N^2} \left( \frac{\Delta \psi}{\Delta T} \right) = \]

\[ = \lambda \cdot \frac{\Delta^2 \phi}{\Delta T^2} \left( \frac{\Delta \psi}{\Delta N^2} + 2 \mu \cdot \frac{\Delta^2 \phi}{\Delta N^2} \right) + 2 \mu \cdot \frac{\Delta \psi}{\Delta T} = \]

\[ = \lambda \cdot \frac{\Delta^2 \phi}{\Delta T^2} \left( \frac{\Delta \psi}{\Delta N^2} + 2 \mu \cdot \frac{\Delta^2 \phi}{\Delta N^2} \right) + 2 \mu \cdot \frac{\Delta \psi}{\Delta N} = \]

\[ = \left( \lambda + 2 \mu \right) \left( \frac{\Delta^2 \phi}{\Delta T^2} \right) + 2 \mu \left( \frac{\Delta \psi}{\Delta N} \right) \right). \]
Similarly,

\[
\sigma_{NT} = \mu \cdot \varepsilon_{NT} =
\]

\[
= \mu \cdot \left( \frac{\partial u}{\partial N} \cdot \left( \frac{\partial \phi}{\partial T} - \frac{\partial \psi}{\partial N} \right) + \frac{\partial \psi}{\partial T} \cdot \left( \frac{\partial \phi}{\partial N} + \frac{\partial \psi}{\partial T} \right) \right)
\]

\[
= \mu \cdot \left( \frac{\partial^2 \phi}{\partial N \partial T} - \frac{\partial^2 \psi}{\partial N^2} + \frac{\partial^2 \phi}{\partial N \partial T} + \frac{\partial^2 \psi}{\partial T^2} \right)
\]

\[= \mu \cdot \left( \frac{\partial^2 \phi}{\partial N \partial T} + \frac{\partial^2 \psi}{\partial T^2} \right)
\]

(2.1.2)

\[
= \mu \cdot \left( \frac{\partial^2 \psi}{\partial T^2} + \frac{\partial^2 \phi}{\partial N \partial T} + \frac{\partial^2 \psi}{\partial T^2} \right)
\]

Writing the above results together with the expressions for \( u \) and \( v \), we have

\[
u = \frac{\partial \psi}{\partial N} - \frac{\partial \psi}{\partial T} \]

\[
v = \frac{\partial \psi}{\partial N} + \frac{\partial \psi}{\partial T} \]

(2.1.3)

\[
\sigma_{NN} = (\lambda + 2\mu) \cdot \nabla^2 \phi + 2\mu \cdot \left( \frac{\partial^2 \phi}{\partial N \partial T} - \frac{\partial^2 \phi}{\partial T^2} \right)
\]

\[
\sigma_{NT} = -\mu \cdot \nabla^2 \psi + 2\mu \cdot \left( \frac{\partial^2 \phi}{\partial N \partial T} + \frac{\partial^2 \phi}{\partial T^2} \right)
\]

(2.1.3)

Using (1.2.13), the above equalities can be rewritten as
\[ u = \frac{\partial \phi}{\partial T} - \frac{\partial \psi}{\partial N} \]

\[ v = \frac{\partial \phi}{\partial N} + \frac{\partial \psi}{\partial T} \]

\[ \sigma_{NN} = -\rho \omega^2 \phi + 2 \mu (\frac{\partial^2 \psi}{\partial N \partial T} - \frac{\partial^2 \phi}{\partial T^2}) \]

\[ \sigma_{NT} = \rho \omega^2 \psi + 2 \mu (\frac{\partial^2 \phi}{\partial N \partial T} + \frac{\partial^2 \psi}{\partial T^2}) \]

(2.1.4)

Now, armed with the formulae (2.1.4), we can express the boundary conditions a) - e) from the Section 1.3 in terms of potentials.

a) Free solid surface:

\[ \text{Total} \]
\[ (\sigma_{NN}) = 0 \]

\[ \text{Total} \]
\[ (\sigma_{NT}) = 0 \]

Using (2.1.4), the above equations can be rewritten as

\[ \sigma_{NN}^{\text{source}} - \rho \omega^2 \phi + 2 \mu (\frac{\partial^2 \psi}{\partial N \partial T} - \frac{\partial^2 \phi}{\partial T^2}) = 0 \]

(2.1.5)

\[ \sigma_{NT}^{\text{source}} + \rho \omega^2 \psi + 2 \mu (\frac{\partial^2 \phi}{\partial N \partial T} + \frac{\partial^2 \psi}{\partial T^2}) = 0 \]

b) Surface of a solid body welded to a rigid body:

\[ (u)^{\text{Total}} = 0 \]

\[ (v)^{\text{Total}} = 0 \]
According to (2.1.4), this is equivalent to

\[ \frac{\partial \phi}{\partial t} - \frac{\partial \psi}{\partial N} = - u \text{ source} , \]
\[ \frac{\partial \phi}{\partial N} + \frac{\partial \psi}{\partial T} = - v \text{ source} . \]

(2.1.6)

c) Welded interface of two solids:

\[
\begin{align*}
\sigma_{\text{in}}^\text{Total} &= \sigma_{\text{out}}^\text{Total} , \\
\sigma_{\text{NT}}^\text{Total} &= \sigma_{\text{NT}}^\text{Total} , \\
u^\text{in} &= u^\text{out} ,
\end{align*}
\]

Using (2.1.4) these equations can be rewritten as:

\[
\begin{align*}
\left( \sigma_{\text{source}}^{\text{in}}_{\text{NN}} - \rho \omega^2 \phi + 2 \mu \left( \frac{\partial^2 \psi}{\partial N \partial T} - \frac{\partial^2 \phi}{\partial T^2} \right) \right) &= \\
\left( \sigma_{\text{source}}^{\text{out}}_{\text{NN}} - \rho \omega^2 \phi + 2 \mu \left( \frac{\partial^2 \psi}{\partial N \partial T} - \frac{\partial^2 \phi}{\partial T^2} \right) \right) , \\
\left( \sigma_{\text{source}}^{\text{in}}_{\text{NT}} + \rho \omega \psi + 2 \mu \cdot \left( \frac{\partial^2 \phi}{\partial N \partial T} + \frac{\partial^2 \psi}{\partial T^2} \right) \right) &= \\
\left( \sigma_{\text{source}}^{\text{out}}_{\text{NT}} + \rho \omega \psi + 2 \mu \cdot \left( \frac{\partial^2 \phi}{\partial N \partial T} + \frac{\partial^2 \psi}{\partial T^2} \right) \right) ,
\end{align*}
\]
\[
(\ u_{\text{source}} + (\frac{\partial \phi}{\partial t} - \frac{\partial \psi}{\partial N}) \)_{\text{in}} = (u_{\text{source}} + (\frac{\partial \phi}{\partial t} - \frac{\partial \psi}{\partial N}) )_{\text{out}},
\]
and
\[
(\ v_{\text{source}} + (\frac{\partial \phi}{\partial N} + \frac{\partial \psi}{\partial T}) \)_{\text{in}} = (v_{\text{source}} + (\frac{\partial \phi}{\partial N} + \frac{\partial \psi}{\partial T}) )_{\text{out}}.
\]

\[\text{(2.1.7b)}\]

\[d) \quad \text{Solid/liquid interface (we will assume that the body is fluid and the space is solid):}\]

\[
(v_{\text{in}})_{\text{Total}} \ = \ (v_{\text{out}})_{\text{Total}},
\]

\[
(\sigma_{\text{NT}})_{\text{in}} \ = \ (\sigma_{\text{NT}})_{\text{Total}},
\]

\[
(\sigma_{\text{NN}})_{\text{in}} \ = \ (\sigma_{\text{NN}})_{\text{out}},
\]

is equivalent to

\[
(v_{\text{source}} + \frac{\partial \phi}{\partial N})_{\text{in}} = (v_{\text{source}} + (\frac{\partial \phi}{\partial N} + \frac{\partial \psi}{\partial T}) )_{\text{out}},
\]

\[
(\omega^2 \psi + 2\mu \cdot (\frac{\partial^2 \phi}{\partial N^2} + \frac{\partial^2 \psi}{\partial T^2}) + \sigma_{\text{NT source}})_{\text{out}} = 0, \quad \text{(2.1.8)}
\]

\[
(\sigma_{\text{source}} - \rho \omega^2 \psi)_{\text{in}} = (\sigma_{\text{source}} - \rho \omega^2 \psi + 2\mu \left(\frac{\partial^2 \psi}{\partial N \partial T} - \frac{\partial^2 \phi}{\partial T^2}\right) )_{\text{out}}.
\]

\[e) \quad \text{Fluid/Fluid interface}\]

\[
(\sigma_{\text{in}})_{\text{Total}} = (\sigma_{\text{out}})_{\text{Total}}.
\]
is equivalent to

\[(\sigma_{\text{source}}^{\text{source}} - \rho \omega^2 \phi)_{\text{in}} = (\sigma_{\text{source}}^{\text{source}} - \rho \omega^2 \phi)_{\text{out}}, \]

\[(\nu_{\text{source}} + \frac{\partial \phi}{\partial N})_{\text{in}} = (\nu_{\text{source}} + \frac{\partial \phi}{\partial N})_{\text{out}}. \]

(2.1.9)

f) Infinitely rigid body imbedded in a fluid space:

\[(\nu^\text{out})_{\text{total}} = 0 \]

is equivalent to

\[(\nu_{\text{source}} + \frac{\partial \phi}{\partial N})_{\text{out}} = 0. \]

(2.1.10)

It is easy to see that (2.1.10) is the Neumann problem for the Helmholtz equation. The other five conditions, however, involve the potentials, their tangent derivatives and their normal derivatives. The following section is devoted to building a theory of such mixed problems based on integral equations.
III. Single and Double Layer Potentials

1. Definitions

For an arbitrary complex number $k$ and $x_0 \in \mathbb{R}^2$, we will consider the function $p_{k,x_0} : \{\mathbb{R}^2 \times \{x_0\}\} \to C^1$ defined by the formula

$$p_{k,x_0}(x) = H_0(k \cdot ||x - x_0||) \quad (3.1.1)$$

where $H_0$ is the first kind Hankel function of order zero and $||\cdot||$ is the Euclidean norm. The function $p_{k,x_0}$ is known to be analytic everywhere, except at $x_0$ where it is not defined, and to satisfy the Helmholtz equation

$$\nabla^2 \phi + k^2 \phi = 0. \quad (3.1.2)$$

We will call $p_{k,x_0}$ the potential of a unity charge located at the point $x_0$ with the Helmholtz coefficient $k$.

Suppose that $h \in \mathbb{R}^2$, $||h|| = 1$. We will denote by $p_{k,x_0}^h$ the function $\{\mathbb{R}^2 \times \{x_0\}\} \to C^1$ defined by the formula

$$p_{k,x_0}^h(x) = \frac{d}{dt} \bigg|_{t=0} \left( p_{k,(x_0+t\cdot h)}(x) \right); \quad (3.1.3)$$

$p_{k,x_0}^h$ will be referred to as the potential of a unity dipole located at the point $x_0$ and oriented in the direction $h$. Obviously, $p_{k,x_0}^h$ satisfies the equation (3.1.2). The reason for the terms "charge" and "dipole" is the obvious analogy with the Laplace equation.
Suppose that $\gamma : [0,1] \to \mathbb{R}^2$ is a Jordan $C^1$-curve and $\sigma : [0,1] \to C^1$ is a $C^1$-mapping. We will denote by $N(t)$ the external normal to the curve $\gamma$ at the point $t \in [0,1]$. We will define the potential $p_{k,t}^0(x)$ at the point $x \in \mathbb{R}^2 \setminus \text{Im}(\gamma)$ of the single layer on the boundary $\gamma$ with the charge density $\sigma$ by the formula

$$p_{k,\sigma}^0(x) = \frac{1}{0} \int p_{k,\gamma}(t)(x) \cdot \sigma(t) \, dt. \quad (3.1.4)$$

Similarly, the potential $p_{k,\sigma}^1(x)$ of the double layer on the boundary $\gamma$ with the dipole density $\sigma$ will be defined by the formula

$$p_{k,\sigma}^1(x) = \frac{1}{0} \int p_{k,\gamma}(t)(x) \cdot \sigma(t) \, dt. \quad (3.1.5)$$

Obviously, the potentials $p_{k,t}^0, p_{k,t}^1$ are analytical on $\mathbb{R}^2 \setminus \text{Im}(\gamma)$, and both of them satisfy the equation (3.1.2).

We will define the mapping $g_0^k : (\mathbb{R}^2 \times \mathbb{R}^2 \setminus \{x, y : x=y\}) \to C^1$ by the formula

$$g_0^k(x,y) = p_{k,x}(y)$$

and the mapping $g_1^k : \text{Im}(\gamma) \times (\mathbb{R}^1 \setminus \text{Im}(\gamma)) \to C^1$ by the formula

$$g_1^k(x,y) = p_{k,\gamma}(t)(y)$$

where $t = \gamma^{-1}(x)$. When it is clear what the Helmholtz coefficient is, we will take the liberty to write $g_0^k, g_1^k$ instead of $g_0^k, g_1^k$. 

The rest of this chapter is devoted to the behavior of \( g_0(x) \) and \( g_1(x) \) when \( x \) approaches \( \text{Im}(\gamma) \).

2. Local Properties of Single and Double Layer Potentials

The local behavior of single and double layer potentials for the Laplace equation is well known. In this section we will see that the behavior of the potential for Helmholtz equation is very similar. First of all, let us rewrite the definitions of Bessel and Hankel functions.

**Definition 3.2.1.** The J-type Bessel function is defined by the formula

\[
J(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \cdot \left( \frac{z}{2} \right)^{2k}.
\]  

(3.2.1)

**Definition 3.2.2.** The Y-type Bessel function is defined by the formula

\[
Y(z) = \frac{2}{\pi} \cdot (\log(z) + \gamma) \cdot J(z) + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(\frac{z}{2})^{2k}}{(k!)^2} \cdot \frac{(-1)^{k+1}}{k} \cdot \frac{1}{j}, \ z \neq 0.
\]  

(3.2.2)

**Definition 3.2.3.** The first kind Hankel function \( H_0(z) \) is defined by the formula

\[
H_0(z) = J(z) + i \cdot Y(z).
\]  

(3.2.3)
It is easy to see from (3.2.3) that

$$H_0(z) = A(z^2) + B(z^2) \cdot \log(z^2)$$

(3.2.4)

where $A$ and $B$ are functions analytical in the whole complex plane, defined by the formulae

$$A(z) = \frac{2i\gamma - 2}{\pi} \cdot J(\sqrt{z}) + \frac{2i}{\pi} \sum_{k=1}^{\infty} \left( \frac{z}{4} \right)^k \cdot \frac{(-1)^{k+1}}{(k!)^2} \cdot \sum_{j=1}^{k} \frac{1}{j} ,$$

(3.2.5)

$$B(z) = \frac{2}{\pi} \cdot J(\sqrt{z}).$$

(3.2.6)

We will say that the function $f : R^k \to C^1$ has a logarithmic singularity at $x_0$ if there exist such $C^1$-functions $u, v : R^1 \to C^1$ that $f(x) = u(x) + v(x) \cdot \log(||x-x_0||)$.

In the following five theorems we will presume that

$\gamma, \sigma, \sigma_0 \in C^1([0,1])$.

**Theorem 3.2.1.**

The function $p_{k,\gamma(t_o)}(x)$ has a logarithmic singularity for $x = \gamma(t_o)$. The function $p_{k,t}$ is defined and continuous on the whole $R^2$.

Let us define the function $Q_{t_o}^0 : ([0,1], \{t_0\}) \to C^1$ by the formula

$$Q_{t_o}^0(t) = p_{k,\gamma(t_0)}(\gamma(t)) \cdot \sigma(t_0).$$
It is easy to see that
\[
p_{k_0}^1(\gamma(t_0)) = \frac{1}{\gamma(t_0)} \int_{0}^{t_0} Q^0_t(t) \, dt \tag{3.2.7}
\]
and the following two theorems can be formulated.

**Theorem 3.2.2.**

The function \( Q^0_{t_0} \) can be continuously extended on \([0,1]\).

**Theorem 3.2.3.**

The function \( p_{k_0}^1 \) is defined on the whole \( \mathbb{R}^2 \). It is discontinuous on \( \text{Im}(\gamma) \) and when the point \( x \) approaches \( \gamma(t_0) \) from the inside at a non-zero angle to the curve \( \gamma \),

\[
\lim_{x \to \gamma(t_0)} p_{k_0}^1(x) - p_{k_0}^2(\gamma(t_0)) = 2i \cdot \sigma(t_0). \tag{3.2.8}
\]

When the point \( x \) approaches the point \( \gamma(t_0) \) under non-zero angle to the curve \( \gamma \) from the outside,

\[
\lim_{x \to \gamma(t_0)} p_{k_0}^1(x) - p_{k_0}^1(\gamma(t_0)) = -2i \cdot \sigma(t_0). \tag{3.2.9}
\]

The above three theorems deal with the potentials of single and double layers. The following theorems deal with the normal derivative of the single layer potential.
We will denote by $Q_{t_0}^1$ the function $([0,1] \to \mathbb{R}) \to C^1$ defined by the formula

$$Q_{t_0}^1(t) = \frac{d}{dt} \int_0^t p_{k,\gamma(t)}(\gamma(t) + \tau \cdot N(t)) \cdot \sigma(t) \, d\tau \mid_{\tau = 0}. \quad (3.2.10)$$

**Theorem 3.2.4**

The function $Q_{t_0}^1(t)$ can be continuously extended on $[0,1]$.

**Theorem 3.2.5**

The function $p_{k,\sigma}^0$ is continuously differentiable on $\mathbb{R} \setminus \operatorname{Im}(\gamma)$.

When the point $x$ approaches the point $\gamma(t)$ at a non-zero angle from the inside,

$$\lim_{x \to \gamma(t)} \frac{\partial}{\partial N(t)} p_{k,\sigma}^0(x) = \frac{1}{Q_{t_0}^1(t)} \int_0^1 Q_{t_0}^1(t) \, dt + 2i \cdot \sigma(t_0). \quad (3.2.11)$$

And when the point $x$ approaches $\gamma(t_0)$ from the outside at a non-zero angle,

$$\lim_{x \to \gamma(t_0)} \frac{\partial}{\partial N(t_0)} p_{k,\sigma}^0(x) = \frac{1}{Q_{t_0}^1(t)} \int_0^1 Q_{t_0}^1(t) \, dt - 2i \cdot \sigma(t_0). \quad (3.2.12)$$
The analogs of the Theorems 3.2.1 - 3.2.5 for the Laplace equation are well known. For example, they can be found in [8]. Expression (3.2.4) can be interpreted as the connection between potentials for the Laplace and Helmholtz equations, and we can rewrite it as

\[
H(z) = A(z) + B(z) \cdot \log(z) = 
\]

\[
A(z) + i \cdot \frac{2}{\pi} \cdot \log(z) + B_1(z) \cdot z^2 \cdot \log(z) 
\]

(3.2.13)

with the functions \( A \) and \( B_1 \) analytical on the whole complex plane. Since the expression \( A(z) + B_1(z) \cdot z^2 \cdot \log(z) \) is locally continuously differentiable, (3.2.13) effectively reduces Theorems 3.2.1 - 3.2.5 to their counterparts for Laplace's equation.

The normal derivative of the potential of a smooth double layer is also smooth, and it stays smooth when the point approaches the boundary. This follows from the regularity theorems of the following section. However, there is no simple decomposition into a local part and an integral of a continuous or weakly singular function of its limiting values on the boundary of a form similar to (3.2.11) or (3.2.13). The latter fact will become a problem in Chapter IV when we will be formulating boundary conditions at the interface of two fluids in terms of integral equations. The following theorem will be used to bypass these difficulties.
Suppose that \( k_1, k_2 \in \mathbb{C}^1 \). We will define the function

\[
Q^2_{t_0}(t) : ([0,1] \setminus \{t_0\}) \to \mathbb{C}^1
\]

by the formula

\[
Q^2_{t_0}(t) = \sigma(t) \cdot \frac{N(t_0)}{p_{k_1,\gamma}(t)(x)} - \frac{N(t_0)}{p_{k_2,\gamma}(t)(x)}.
\]

(3.2.14)

**Theorem 3.2.6**

The function \( Q^2_{t_0}(t) \) has a logarithmic singularity at \( t = t_0 \), and when \( x \) approaches \( \gamma(t_0) \) from the inside or from the outside,

\[
\lim_{x \to \gamma(t_0)} \frac{\partial}{\partial N(t_0)} (p^1_{k_1,\tau}(x) - p^1_{k_2,\tau}(x)) = \frac{1}{\int_0^1 Q^2_{t_0}(t) \, dt}.
\]

**Proof.**

Obviously, \( \log(k \cdot z) = \log(k) + \log(z) \) for any complex \( k, z \).

This means that the singular part of the double layer potential does not in fact depend on the Helmholtz coefficient \( k \). As a result, when two such potentials are subtracted the singular part disappears, which is easy to see from the formulae (3.2.1) - (3.2.6).
For \( k_1, k_2 \in C^1 \) and \( t_0 \in [0,1] \) we will define the function

\[
Q^3_{t_0} : ([0,1] \setminus \{t_0\}) \rightarrow C^1
\]

by the formula

\[
Q^3_{t_0}(t) = p_{k_1, \gamma(t_0)}(\gamma(t)) - p_{k_2, \gamma(t_0)}(\gamma(t)) \quad (3.2.15)
\]

Obviously, \( Q^3_{t_0}(t) \) is not defined at \( t = t_0 \). We will define the function \( \hat{Q}^3_{t_0} : ([0,1] \setminus \{t_0\}) \rightarrow C^1 \) by the formulae

\[
\hat{Q}^3_{t_0}(t) = Q^3_{t_0}(t) \quad \text{for} \quad t \neq t_0
\]

and

\[
\hat{Q}^3_{t_0}(t_0) = \lim_{t \to t_0} Q^3_{t_0}(t) \quad (3.2.16)
\]

**Theorem 3.2.7**

The function \( \hat{Q}^3_{t_0} \) is continuously differentiable on the interval \([0,1]\) for any \( t_0 \in [0,1] \).

The above theorem is a trivial consequence of the formulae (3.1.1), (3.2.4), and (3.2.5).
IV. Regularity

1. Regularity Theorems

Theorems 3.2.1 - 3.2.5 of the preceding chapter establish that the potentials of smooth single and double layers are continuous on the closure of the interior (or exterior) of a smooth Jordan curve. These theorems also establish the continuity of the appropriately defined normal derivative of a smooth single layer potential. It seems to be common knowledge among specialists that these theorems can be dramatically strengthened by replacing continuity with smoothness of analyticity under appropriate conditions. Let us formulate these theorems in their strengthened form.

Suppose that $f$ is a function defined on the subset $\Omega$ of $\mathbb{R}^n$, $n \geq 1$. The statement $f \in C^k(\Omega)$ will mean that $f$ is $k$ times continuously differentiable on $\Omega$ and we will say that $f$ is a $C^k$ function or that $f$ is $k$-smooth on $\Omega$. Here $k$ can be a natural number, $\infty$, or a (analyticity). The convention is that $\infty \pm 1 = \infty$, $a \pm 1 = a$.

For a Jordan curve $\gamma$, we will denote by $\Omega$ the open interior of $\gamma$.

Theorem 4.1.1

Suppose that $\gamma$, $\sigma \in C^m[0,1]$. Then $p^0_{k,\sigma} \in C^m(\overline{\Omega})$ and

$p^0_{k,\sigma} \in C^m(\mathbb{R}^2 \setminus \Omega)$. 
If \( \gamma, \sigma \in c^m [0,1] \) then, according to the Theorem 3.2.3, the function \( p^1_{k,\sigma} \) can be continuously extended from \( \Omega \) to \( \overline{\Omega} \), and we will denote the extended function by \( S^+_{k\sigma} \). Similarly, \( p^1_{k\sigma} \) can be continuously extended from \( R^2 \setminus \overline{\Omega} \) to \( R^2 \setminus \Omega \), and we will denote the extended function by \( S^-_{k\sigma} \).

**Theorem 4.1.2**

Suppose that \( \gamma \in c^m [0,1], \sigma \in c^m [0,1], m \geq 1 \). Then \( S^+_{k,\sigma} \in c^m(\overline{\Omega}) \) and \( S^-_{k\sigma} \in c^m(R^2 \setminus \Omega) \).

2. Classical Regularity

Suppose that \( L \) is an elliptic linear differential operator in \( L^2 \) with coefficients and \( f : R^2 \to C^1 \) is a \( c^m \)-function, \( k \geq 1 \). We will be considering the equation

\[
L(u) = f \tag{4.2.1}
\]

inside a Jordan curve \( \gamma : [0,1] \to R^2 \) with one of the following two boundary conditions:

\[
|\text{Im}(\gamma) = \phi \tag{4.2.2}
\]

or

\[
\frac{\partial u}{\partial N} |\text{Im}(\gamma) = \phi \tag{4.2.3}
\]
where \( \phi : \text{Im}(\gamma) \to C^1 \) is a continuous function. Obviously the boundary value problem ((4.2.1), (4.2.2)) is a Dirichlet problem, and ((4.2.1), (4.2.3)) is a Neumann problem.

**Theorem 4.2.1 (Classical Regularity)**

If \( \gamma, \phi \) are \( m \)-smooth and the internal boundary value problem (4.2.1), (4.2.2) has a unique solution \( u \), then \( u \) is \( m \)-smooth on \( \overline{\Omega} \).

**Theorem 4.2.2 (Schauder Estimates)**

Under the conditions of the preceding theorem,

\[
||u||_m \leq M \cdot (||u||_0 + ||f||_m + ||\phi||_m)
\]

(4.2.4)

Here \( ||\cdot||_m \) is the standard norm in \( C^m \) and the coefficient \( M \) depends only on the operator \( L \) and the curve \( \gamma \).

Theorem 4.2.1 is a particular case of the Theorem 17.3 from [20]. Theorem 4.2.2 has been copied with changes in notation from [12], page 335.

3. Analytical Lemmas

In order to prove the Theorems 4.1.1, 4.1.2, we will need the following simple analytical result.
Lemma 4.3.1

Suppose that the functions \( \phi, f : [0,1] \rightarrow C^1, m \geq 1 \) are such that

a) \( \phi, f \in C^m [0,1] \)

b) \( \phi(x) \geq 0 \) for any \( x \in [0,1] \)

c) There exists a real \( \varepsilon > 0 \) such that if \( \phi(x) = 0 \) then in the neighborhood of \( X \)

\[ \phi(x + h) > \varepsilon \cdot h^2. \]

Then the function \( F : [0,1] \rightarrow C^1 \) defined by the formula

\[ F(x) = \frac{1}{1} \int_{0}^{1} f(t - x) \cdot \log(\phi(t - x)) \, dt \]

is \( m \)-smooth on \([0,1] \).

Proof

The second of the following two statements \( \alpha, \beta \) is known to imply the first one (see, for example, [31]).

\( \alpha \) \( F \in C^{m-2} [0,1] \)

\( \beta \) There exist integers \( C, M \) such that in the Fourier expansion,

\[ F(x) = \sum_{m=0}^{\infty} \left[ A_m(F) \cdot \cos(2 \cdot \pi \cdot M \cdot x) + B_m(F) \cdot \sin(2 \cdot \pi \cdot m \cdot x) \right], \]
\[ |A_m(F)| < \frac{C}{m^k} , \quad (4.3.1) \]

\[ |B_m(F)| < \frac{C}{m^k} , \quad (4.3.2) \]

for any \( m > M \).

Obviously, \( \alpha \) is the statement of the lemma, and we will prove the lemma by proving \( \beta \).

Indeed,

\[ A_m(F) = \int_0^1 \cos (2 \pi \cdot m \cdot X) \cdot F(X) \, dX = \]

\[ \int_0^1 \cos (1 \pi \cdot mX) \cdot \int_0^1 f(t - X) \cdot \log(\phi(t - X)) \, dt \, dx = \]

\[ \int_0^1 \log(\phi(t - X)) \cdot \int_0^1 f(t - X) \cdot \cos (2 \pi \cdot m \cdot X) \, dx \, dt . \]

But

\[ \int_0^1 f(t - X) \cdot \cos (2 \pi \cdot m \cdot X) \, dX = b_m(g_t) , \]

where \( g_t : [0,1] \rightarrow \mathbb{C}^1 \) is defined by the formula

\[ g_t(X) = f(t - X) . \]
Obviously, $g_t$ is $m$-smooth and, consequently, there exist $C_t$, $M_t$ such that

$$| A_m (g_t) | < \frac{C_t}{M_k^k}$$

for any $m > M_t$. The existence of $C_A, M_A$ common for all $t \in [0,1]$ easily follows from the compactness of the interval $[0,1]$.

Now we can write

$$| A_m (F) | = | \int_0^1 \log (\phi(t - X)) \cdot \frac{C_t}{M_k^k} \, dt | \leq$$

$$\frac{C_A}{M_k^k} \cdot | \int_0^1 \log (\phi(t - X)) \, dX |$$

Finally, the integral

$$\int_0^1 \log (\phi(t - X)) \cdot dX$$

is limited uniformly with respect to $t \in [0,1]$ due to the condition c). Denoting this uniform limiting value by $D_a$, we obtain
\[ A_m (F) \leq \frac{C_A D_A}{m^k} \]  

(4.3.3)

for any \( m \geq M_a \), which proves (4.3.1).

The proof of (4.3.2) is identical to the proof of (4.3.1).

Combining the result of the Lemma 4.3.1 with the formulae (3.2.3), (3.1.1), we arrive at the following lemma.

**Lemma 4.3.2**

Suppose that \( \gamma : [0,1] \rightarrow \mathbb{R}^2 \) is a \( m \)-smooth Jordan curve and \( \sigma : [0,1] \rightarrow \mathbb{C} \) is also \( m \)-smooth. Suppose further that \( k \) is a complex number. Then the mapping \( S : [0,1] \rightarrow \mathbb{C} \) defined by the formula

\[ S(t) = p_{k,\sigma}^0 (\gamma(t)) \]

is \( m \)-smooth.

**Remark 4.3.1**

We will define the mapping \( L^0_\gamma : c^m[0,1] \rightarrow c^m[0,1] \) by the formula

\[ L^0_\gamma (\sigma)(X) = p_{k_o}^0 (\gamma(X)). \]  

(4.3.4)
The above lemma can be interpreted as the proof of correctness of the definition (4.3.4). It is easy to see from the proof of Lemma 4.3.2 that in fact the mapping $L^0_\gamma$ is strongly continuous in $c^k$ under the conditions of the lemma.

**Lemma 4.3.3.**

Suppose that $\gamma: [0,1] \to \mathbb{R}^2$ is a $m$-smooth Jordan curve and $\sigma: [0,1] \to C^1$ is also $m$-smooth. Suppose further that $k$ is a complex number. Then the mapping $S_1: [0,1] \to C^1$ defined by the formula

$$S_1(t) = p^1_{k\sigma}(\gamma(t))$$

is $m$-smooth.

The proof of this lemma is almost literally similar to the proof of Lemma 4.3.2.

**Remark 4.3.2**

Remark 4.3.1 can be repeated here with Lemma 4.3.2 replaced by Lemma 4.3.3.

4. Proof of Theorems 4.1.1 and 4.1.2

**Proof of Theorem 4.1.1.**
a) Inside the domain $\Omega$.

Obviously, $p_{K\sigma}^0$ is the solution of the Dirichlet boundary value problem:

$$
\nabla^2 u + k^2 u = 0 \text{ on } \Omega
$$

(4.4.1)

$$
u = p_{K\sigma}^0 \quad \text{on } \partial \Omega
$$

(4.4.2)

According to the Lemma 4.3.2, the restriction $p_{K\sigma}^0|_{\text{Im}(\gamma)} = \partial \Omega$ is $m$-smooth and, consequently, the boundary value problem ((4.4.1), (4.4.2)) satisfies the condition of the Theorem 4.2.1. Now we conclude that $p_{K\sigma}^0 \in C^m(\overline{\Omega})$.

b) Outside the domain $\Omega$.

In this case Theorem 4.2.1 is not applicable directly, and we will reduce the external case of Theorem 4.1.1 to the internal case.

Suppose that $\gamma_1 : [0,1] \rightarrow R^2$ is a circle containing some neighborhood of $\overline{\Omega}$. We will denote by $\Omega_m$ the open circle bounded by $\text{Im}(\gamma_1)$ and by $\Omega_s$ the intersection $\overline{\Omega_m} \cap (R^2 \setminus \Omega)$. Finally, we will define the mapping $\sigma_2 : [0,1] \rightarrow C^1$ to be zero everywhere, and the mapping $Q : \overline{\Omega_s} \rightarrow C^1$ will be defined by the formula

$$
Q(x) = p_{K\sigma}^0(x) + p_{K\sigma}^0(x).
$$

(4.4.4)
The proof of m-smoothness of (4.4.4) is literally identical to the proof of m-smoothness of $p_{K\sigma}^0$ inside $\Omega$ and, obviously, $Q(x) = p_{K\sigma}^0(x)$ everywhere, which concludes the proof.

Proof of Theorem 4.1.2

The proof of Theorem 4.1.2 is identical to the proof of Theorem 4.1.1.

5. Strong Regularity

Suppose that $\gamma : [0,1] \to \mathbb{R}^2$ is a m-smooth Jordan curve and $k \in C^1$.

Define the linear mapping $L_{Yk}^0 : c^m [0,1] \to c^m (\Omega)$ by the formula

$$L_{Yk}^0 (\sigma) (x) = p_{K\sigma}^0 (x)$$

and the linear mapping $L_{Yk}^1 : c^m [0,1] \to c^m (\Omega)$ - by the formula

$$L_{Yk}^1 (\sigma) (x) = S_{K\sigma}^+ (x).$$

The correctness of the above definitions follows from Theorems 4.1.1, 4.1.2. The following theorem follows immediately from the Schauder estimates (Theorem 4.2.2).
**Theorem 4.5.1** (Strong Regularity)

The mappings \( L^0_{\gamma k}, L^1_{\gamma k} : c^m \ [0,1] \to c^m(\Omega) \) are strongly continuous in the \( c^m \) - norm.
V. Integral Equations Corresponding to Boundary Conditions

1. Systems of Linear Integral Equations with Weak Singularities.

Suppose that the linear integral operator \( L : L^2[0,1] \rightarrow L^2[0,1] \) is defined by the formula:

\[
L(P)(x) = \frac{1}{\alpha} \int_0^1 P(t) \cdot K(t,x) dt \tag{5.1.1}
\]

where \( P \in L^2[0,1] \). We will say that the kernel \( K \) is weakly singular if there exists \( 0 < \alpha < 1 \) such that the function \( |x-t|^{-\alpha} \cdot K(t,x) \) is bounded. We will say that the kernel \( K \) is completely continuous if for any given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \(|x_1 - x_2| < \delta\) then

\[
\frac{1}{\alpha} \int_0^1 |K(t_1x_1) - K(t_1x_2)| dt < \varepsilon
\]

for any \( x_1, x_2 \in [0,1] \).

Finally, we will define the system of second kind Fredholm integral equations with weak singularities as the expression

\[
P_j(x) - \lambda_j \sum_{\ell=1}^n \int_0^1 K_{j\ell}(t,x) \cdot P_\ell(t) \, dt = f_j(x) \tag{5.1.2}
\]

with \( j = 1, 2, \ldots, n \) where each of \( K_{j\ell} \) is a weakly singular kernel and \( \lambda_j \in \mathbb{C} \) for all \( j = 1, 2, \ldots, n \).
Theorem 5.1.1

The Fredholm alternative holds for systems of second kind integral equations with weak singularities.

Theorem 5.1.2

Under the conditions of the previous theorem, if the kernels $K_{j1}$, $j, l = 1, 2, \ldots, n$ are completely continuous and the functions $f_i$, $i = 1, 2, \ldots, n$ are continuous, then the solutions of (5.1.2) are also continuous.

The above two theorems can be found in [11] in somewhat different notation.

Definition 5.1.1

We will say that the kernel $K : [0,1] \times [0,1] \to C^1$ is logarithmically singular if there exist two continuous kernels $K_1, K_2 : [0,1] \times [0,1] \to C^1$ such that

$$K(t,x) = K_1(t,x) \cdot \log(|t-x|) + K_2(t,x) \text{ for all } t, x \in [0,1].$$

Noticing that for any $\alpha > 0$, the function $x^\alpha \cdot \log(x)$ is bounded on the interval $[0,1]$, we obtain the following theorem:
Theorem 5.1.3

Kernels with logarithmic singularities are completely continuous.

2. Mixed Fredholm-Volterra Equations

A Volterra equation is an expression of the form

$$\lambda \phi(x) + \int_0^x \phi(t) \cdot K(t,x) \, dt = f(x) \quad (5.2.1)$$

where \( K : [0,1] \times [0,1] \rightarrow \mathbb{R}^1 \) is an \( L^2 \) function. We will say that the operator \( V : L^2[0,1] \rightarrow L^2[0,1] \) is a Volterra operator if it is defined by the formula

$$V(\phi)(x) = \int_0^x \phi(t) \cdot K(t,x) \, dt. \quad (5.2.2)$$

A Volterra operator can be looked upon as a particular form of Fredholm operator. Indeed,

$$\int_0^x \phi(t) \cdot K(t,x) \, dt = 1 \int_0^1 \phi(t) \cdot \hat{K}(t,x) \, dt \quad (5.2.2)$$

with \( K : [0,1] \times [0,1] \rightarrow \mathbb{R}^1 \) defined by the formulae

$$\hat{K}(t,x) = K(t,x) \text{ for } t \leq x,$$

$$\hat{K}(t,x) = 0 \text{ for } t > x. \quad (5.2.4)$$

Kernels of the form (5.2.4) will be referred to as Volterra kernels.
Remark 5.2.1

Theorems 5.1.1 and 5.1.2 hold if some or all of the kernels $K_{j,l}$ are weakly singular Volterra kernels. It follows from the fact that Volterra equations are a particular case of Fredholm equations.

3. Regularity

We have seen that under the conditions of Theorem 5.1.2, solutions of systems of integral equations are continuous. It turns out that these theorems can be strengthened by replacing continuity with smoothness when the kernels satisfy some simple additional conditions. We will not prove these stronger theorems here, since their proofs would almost literally repeat the proof of Theorem 5.1.2 found in pages 91-99. It turns out that the kernels of the integral equations resulting from scattering problems satisfy the conditions of these strengthened theorems.

Definition 5.3.1

We will say that the periodic kernel $K : [0,1] \times [0,1]$ is $m$-stable if for any $m$-smooth $\phi : [0,1] \to C^1$ the mapping $\psi = K(\phi)$ defined by the formula

$$\psi(x) = K(\phi)(x) = \frac{1}{\int_0^1 K(t,x) \cdot \phi(t) dt}$$

is $m$-smooth and the mapping $\hat{K} : C^m[0,1] \to C^m[0,1]$ defined by the formula
\[ \hat{K}(\phi)(x) = \int_0^1 K(t,x) \cdot \phi(t) dt \]

is strongly continuous on \( C^m [0,1] \).

**Theorem 5.3.1**

Under the conditions of Theorem 5.1.2, if the kernels \( K_j \) are m-stable and the functions \( f_j \) are m-smooth, then the solution of the problem (5.1.1) is m-smooth.

4. Contact of two fluids

We will be representing the scattered field both inside and outside the scatterer by linear combinations of single and double layer potentials with appropriate Helmholtz coefficients. Using the notation of Chapter III, the scattered field both inside and outside the scatterer will be represented by the formula

\[ \phi_{\text{scat}}(x) = \int_\Gamma \sigma_0(t) \cdot g_0(\gamma(t),x) dt \]

(5.4.1)

\[ + \int_\Gamma \sigma_1(t) \cdot g_1(\gamma(t),x) dt. \]

When the point \( x \) approaches \( \Gamma \) from the inside, the limiting form of (5.4.1) is
\[ f_{\text{scat}}^{\text{in}}(\gamma(\tau)) = \int_{\Gamma} \sigma_0^{\text{in}}(t) \cdot g_0^{\text{in}}(\gamma(t), \gamma(\tau)) \, dt + \]

\[ + 2 \cdot i \cdot \sigma_1^{\text{in}}(\tau) + \int_{\Gamma} \sigma_1^{\text{in}}(t) \cdot g_1^{\text{in}}(\gamma(t), \gamma(\tau)) \, dt \]  

(5.4.2)

where \( \tau \in [0,1] \) is such that the point \( x \) is approaching \( \gamma(\tau) \). When \( x \) approaches \( \Gamma \) from the outside, the limiting form of (5.4.1) is

\[ f_{\text{scat}}^{\text{out}}(\gamma(\tau)) = \int_{\Gamma} \sigma_0^{\text{out}}(t) \cdot g_0^{\text{out}}(\gamma(t), \gamma(\tau)) \, dt - \]

\[ - 2 \cdot i \cdot \sigma_0^{\text{out}}(\tau) + \int_{\Gamma} \sigma_0^{\text{out}}(t) \cdot g_1^{\text{out}}(\gamma(t), \gamma(\tau)) \, dt. \]  

(5.4.3)

Similarly, when \( x \) approaches \( \Gamma \) from the inside, the limit of the normal derivative of (5.4.1) is

\[ \lim_{x \to \gamma(\tau)} \frac{\partial}{\partial N} f_{\text{scat}}^{\text{in}}(x) = 2 \cdot i \cdot \sigma_0^{\text{in}}(\tau) + \]

\[ + \int_{\Gamma} \sigma_0^{\text{in}}(t) \cdot \frac{\partial}{\partial N(\tau)} \cdot (g_0^{\text{in}}(\gamma(t), \gamma(\tau))) \, dt + \]

\[ + \lim_{x \to \gamma(\tau)} \int_{\Gamma} \sigma_1^{\text{in}}(t) \cdot \frac{\partial}{\partial N(\tau)} \cdot (g_1^{\text{in}}(\gamma(t), x)) \, dt \]  

(5.4.4)

And when \( x \) approaches \( \Gamma \) from the outside,
\[ \lim_{x \to \gamma(t)} \frac{\partial}{\partial N} f^{\text{out scat}}(x) = -2 \cdot i \cdot \sigma_0^{\text{out}}(t) + \]
\[ + \int_{\Gamma} \sigma_0^{\text{out}}(t) \frac{\partial}{\partial N(t)} (g_0^{\text{out}}(\gamma(t), \gamma(t))) dt + \]
\[ + \lim_{x \to \gamma(t)} \int_{\Gamma} \sigma_1^{\text{out}}(t) \cdot \frac{\partial}{\partial N(t)} (g_2^{\text{out}}(\gamma(t), x)) dt. \]

In the above formulae, the charge densities \( \sigma_0^{\text{in}}, \sigma_0^{\text{out}}, \sigma_1^{\text{in}}, \sigma_1^{\text{out}} \) are to be found from the boundary conditions (2.1.9). Let us rewrite these conditions:

\[ \omega^2 (\rho^{\text{in}} - \rho^{\text{out}} \cdot \phi^{\text{out}}) = \rho_{\text{source}}^{\text{in}} - \rho_{\text{source}}^{\text{out}}, \quad (5.4.6a) \]

\[ \frac{\partial \phi^{\text{in}}}{\partial N} - \frac{\partial \phi^{\text{out}}}{\partial N} = u_{\text{source}}^{\text{in}} - u_{\text{source}}^{\text{out}}. \quad (5.4.6b) \]

Substituting (5.4.2), (5.4.5) into (5.4.6), we would like to obtain the charge densities \( \sigma_0^{\text{in}}, \sigma_0^{\text{out}}, \sigma_1^{\text{in}}, \sigma_1^{\text{out}} \) as the solution of the resulting system of linear integral equations. However, two apparent problems arise.

a) We need to determine four charge densities from a system of two coupled integral equations (5.4.6).
b) We would like our system of integral equations to satisfy the conditions of Theorems 5.4.1 and 5.4.2.

Theorem 5.1.3 means that the kernels in the expressions (5.4.2), (5.4.3) do not pose any problem. However, the expressions (5.4.4), (5.4.5) contain normal derivatives of a double layer potential. As we have seen in the preceding chapter, those derivatives exist and are smooth for smooth dipole density functions but they obviously do not satisfy the conditions of Theorems 5.1.1, 5.1.2.

We can bypass the difficulty by noticing that if the inside dipole density function \( \sigma_1^{in} \) is equal to the outside dipole density function \( \sigma_1^{out} \), then the difference

\[
\int_{\Gamma} \sigma_1(t) \cdot \frac{\partial}{\partial N(\tau)} g_1^{out}(\gamma(t), \gamma(\tau)) dt - \int_{\Gamma} \sigma_1(t) \cdot \frac{\partial}{\partial N(\tau)} g_1^{in}(\gamma(t), \gamma(\tau)) dt = \]

\[
+ \int_{\Gamma} \sigma_1(t) \cdot \frac{\partial}{\partial N(\tau)} (g_1^{out}(\gamma(t), \gamma(\tau)) - g_1^{in}(\gamma(t), \gamma(\tau))) dt
\]

does satisfy the conditions of the Theorems 5.1.1 and 5.1.2 due to Theorem 3.2.6. In the above formula,
\[ \sigma_1 = \sigma_1^{\text{in}} = \sigma_1^{\text{out}} \]  \hspace{1cm} (5.4.7)

is the common dipole density function. The constraint (5.4.7) also eliminates a part of the problem b) since we now have to determine three charge densities \( \sigma_1, \sigma_0^{\text{in}}, \sigma_0^{\text{out}} \) from two equations (5.4.6). Let us introduce an arbitrary additional constraint of the form

\[ a \cdot \sigma_0^{\text{in}} - b \cdot \sigma_0^{\text{out}} = 0 \]  \hspace{1cm} (5.4.8)

with at least one of the complex coefficients \( a, b \) not equal to zero. We will presume for definiteness that \( b \neq 0 \). In this case, (5.4.8) can be rewritten as

\[ \sigma_0^{\text{out}} = \frac{a}{b} \cdot \sigma_0^{\text{in}} \]  \hspace{1cm} (5.4.9)

and we have reduced the number of unknown density functions to two. Substituting the expressions (5.4.2), (5.4.5) and the equalities (5.4.7), (5.4.9) into the equations (5.4.6), we obtain the following system of integral equations with respect to the functions \( \sigma_0^{\text{in}}, \sigma_1 \)
\[4 \cdot (\rho^\text{in} + \rho^\text{out}) \cdot \sigma_1(t) +
\]
\[+ \int_\Gamma (\rho^\text{in} \cdot g^\text{in}_1(\gamma(t), \gamma(t)) - \frac{a}{b} \cdot \rho^\text{out} \cdot g^\text{out}_0(\gamma(t), \gamma(t))) \cdot \sigma_0^\text{in}(t) \, dt \]
\[= \frac{p^\text{in}}{p^\text{source}} - \frac{p^\text{out}}{\omega^2} ;
\]
\[4 \cdot i \cdot (1 + \frac{a}{b}) \cdot \sigma_0^\text{in}(t) +
\]
\[+ \int_\Gamma \frac{\partial}{\partial N(t)} (g^\text{in}_0(\gamma(t), t)) - \frac{a}{b} \cdot g^\text{out}_0(\gamma(t), \gamma(t))) \cdot \sigma_0^\text{in}(t) \, dt +
\]
\[= \lim_{x \to \gamma(t)} \int_\Gamma \frac{\partial}{\partial N(t)} (g^\text{in}_1(\gamma(t), x) - g^\text{out}_1(\gamma(t), x)) \cdot \sigma_1(t) \, dt =
\]
\[= u^\text{in}_\text{source} (\gamma(t)) - u^\text{out}_\text{source} (\gamma(t)) .
\]

Applying Theorems 3.2.1, 3.2.2 and 3.2.3 to the kernels of the equation (5.4.10), we obtain the following lemma:

**Lemma 5.4.1**

The kernel
\[(\rho^\text{in} \cdot g^\text{in}_0(\gamma(t), \gamma(t)) - \frac{a}{b} \cdot \rho^\text{out} \cdot g^\text{out}_0(\gamma(t), \gamma(t)))\]
in the equation (5.4.1) is weakly (logarithmically) singular.

The kernel

\[ \rho \cdot g^\text{in}_1(\gamma(t), \gamma(\tau)) + \rho \cdot g^\text{out}_1(\gamma(t), \gamma(\tau)) \]

in the equation (5.4.1) is continuous.

Applying Theorems 3.2.4, 3.2.6 to the equation (5.4.11), we obtain a similar lemma:

**Lemma 5.4.2**

The kernel

\[ \frac{\partial}{\partial N(\tau)} \left( g^\text{in}_0(\gamma(t), \gamma(\tau)) - \frac{a}{b} \cdot g^\text{out}_0(\gamma(t), \gamma(\tau)) \right) \]

in the equation (5.4.11) is continuous.

The kernel

\[ \lim_{x \to \gamma(\tau)} \left( \frac{\partial}{\partial N(\tau)} \left( g^\text{in}_1(\gamma(t), x) - g^\text{out}_1(\gamma(t), x) \right) \right) \]

in the equation (5.4.11) is weakly (logarithmically) singular.

Combining the above two lemmas with Theorem 5.1.1, we obtain the following theorem.
Theorem 5.4.1

Suppose that \( \frac{a}{b} + 1 \neq 0 \). Then the system of integral equations (5.4.10), (5.4.11) satisfies the conditions of first, second, third, and fourth Fredholm theorems.

So far the coefficients \( a, b \) have been chosen arbitrarily with the only conditions that \( b \neq 0, a \neq -b \). It makes sense to try to choose them in such a way that the equations would be simpler in some sense. The author is in favor of the choice

\[
a = \rho_{\text{in}}, \quad b = \rho_{\text{out}}
\]

(5.4.12)
due to the following lemma:

Lemma 5.4.3

If \( a = \rho_{\text{in}}, \quad b = \rho_{\text{out}} \), then both kernels in the equation (5.4.10) are continuous.

Proof

Let us denote \( a \cdot \sigma_{0}^{\text{in}} = b \cdot \sigma_{0}^{\text{out}} \) by \( n \). Then \( \sigma_{0}^{\text{in}} = \frac{n}{\rho_{\text{in}}}, \)

\( \sigma_{0}^{\text{out}} = \frac{n}{\rho_{\text{out}}} \) and the equation (5.4.10) assumes the form
\[ 4 \cdot i \cdot (\rho^\text{in} + \rho^\text{out}) \cdot \sigma_1(\tau) + \]

\[ + \int_{\Gamma} (g_o^{\text{in}}(\gamma(t),\gamma(\tau)) - g_o^{\text{out}}(\gamma(t),\gamma(\tau))) \cdot \eta(t) \, dt + \]

\[ = \frac{p^\text{in}_{\text{source}} - p^\text{out}_{\text{source}}}{\omega^2}. \]

(5.4.13)

Now the conclusion of the lemma follows from Theorem 3.2.7.

\textbf{Remark 5.4.1}

Substituting (5.4.12) into (5.4.11), we obtain

\[ 4 \cdot i \cdot \left( \frac{1}{\rho^\text{in}} + \frac{1}{\rho^\text{out}} \right) \cdot \eta(\tau) + \]

\[ + \int_{\Gamma} \frac{\bar{\rho}}{\rho^\text{in}} \left( g_o^{\text{in}}(\gamma(t),\gamma(\tau)) \right) \frac{\rho^\text{in}}{\rho^\text{out}} - \frac{g_o^{\text{out}}(\gamma(t),\gamma(\tau))}{\rho^\text{out}} \right) \cdot \eta(t) \, dt + \]

\[ + \lim_{x \to \gamma(\tau)} \int_{\Gamma} \frac{\bar{\rho}}{\rho^\text{in}} \left( g_1^{\text{in}}(\gamma(t),x) - g_1^{\text{out}}(\gamma(t),x) \right) \cdot \sigma_1(t) \, dt = \]

\[ = u^\text{in}_{\text{source}}(\gamma(\tau)) - u^\text{out}_{\text{source}}(\gamma(\tau)). \]

(5.4.14)
For the rest of this thesis, the boundary conditions at the interface of two fluids will be written in the form (5.4.13), (5.4.14). This choice of parameters a, b makes both kernels non-singular in the equation (5.4.3). In addition, it has some other advantages which will be discussed in Section V.7.

5. Welded Contact of Two Solids

In this case we will be representing the p and s-potentials of the scattered field inside the scatterer by the expressions:

\[
\begin{align*}
\phi_{\text{scat}}^\text{in}(x) = & \int_{\Gamma} \phi_0^\text{in}(t) \cdot g_0(\gamma(t), x) \, dt + \\
& \int_{\Gamma} \phi_1^\text{in}(t) \cdot g_1(\gamma(t), x) \, dt,
\end{align*}
\]

(5.5.1)

\[
\begin{align*}
\psi_{\text{scat}}^\text{in}(x) = & \int_{\Gamma} \chi_0^\text{in}(t) \cdot h_0^\text{in}(\gamma(t), x) \, dt + \\
& \int_{\Gamma} \chi_1^\text{in}(t) \cdot h_1^\text{in}(\gamma(t), x) \, dt.
\end{align*}
\]

The potentials outside the scatterer will be defined by the expressions:
The potentials outside the scatterer will be defined by the expressions:

\[
\sigma^\text{in}_{\text{scat}}(x) = \int_{\Gamma} \sigma_0^\text{out}(t) \cdot g_0(\gamma(t), x) dt + \int_{\Gamma} \sigma_1^\text{out}(t) \cdot g_1(\gamma(t), x) dt ,
\]

(5.5.2)

\[
\phi^\text{out}_{\text{scat}}(x) = \int_{\Gamma} \chi_0^\text{out}(t) \cdot h_0^\text{out}(\gamma(t), x) dt + \int_{\Gamma} \chi_1^\text{out}(t) \cdot h_1^\text{out}(\gamma(t), x) dt
\]

where \( \sigma^\text{in}_0, \sigma^\text{out}_0, \sigma^\text{in}_1, \sigma^\text{out}_1, \chi^\text{in}_0, \chi^\text{out}_0, \chi^\text{in}_1, \chi^\text{out}_1 \) are unknown charge densities to be found.

All the problems arising in the case of the fluid/fluid contact arise here. We have four coupled equations and eight unknown functions to be found. An attempt to substitute (5.5.51) and (5.5.2) into (2.1.7) will involve normal derivatives of double layer potentials which, as we have seen in the preceding section, will prevent us from obtaining Fredholm integral equations. In addition, tangent derivatives of \( p \)- and \( s \)-potentials are present in (2.1.7), making direct analytical treatment, similar to that for the fluid/fluid interface, extremely messy.
In order to simplify the analysis, we will notice that

$$\frac{\partial^2 \psi}{\partial N \partial T} - \frac{\partial^2 \phi}{\partial T^2} = \frac{\partial}{\partial T} \left( \frac{\partial \psi}{\partial N} - \frac{\partial \phi}{\partial T} \right),$$

and substituting (2.1.7b) into (2.1.7a), we obtain

$$\begin{align*}
(c_{\text{source}}^{\text{in}} - \rho \omega^2 \phi)^{\text{in}} N_N & - (c_{\text{source}}^{\text{out}} - \rho \omega^2 \phi)^{\text{out}} N_N \\
& = \frac{\partial}{\partial T} \left( (u_{\text{source}})^{\text{out}} - (u_{\text{source}})^{\text{in}} \right) + 2 \cdot (\mu_{\text{in}} - \mu_{\text{out}}) \cdot \frac{\partial}{\partial T} \left( \frac{\partial \psi}{\partial N} - \frac{\partial \phi}{\partial T} \right),
\end{align*}$$

(5.5.3a)

$$\begin{align*}
(c_{\text{source}}^{\text{in}} + \rho \omega^2 \psi)^{\text{in}} N_T & - (c_{\text{source}}^{\text{out}} - \rho \omega^2 \psi)^{\text{out}} N_T \\
& = \frac{\partial}{\partial T} \left( (v_{\text{source}})^{\text{out}} - (v_{\text{source}})^{\text{in}} \right) + 2 \cdot (\mu_{\text{in}} - \mu_{\text{out}}) \cdot \frac{\partial}{\partial T} \left( \frac{\partial \phi}{\partial N} - \frac{\partial \psi}{\partial T} \right).
\end{align*}$$

(5.5.4b)
Now, applying the reasoning of the preceding section to the equations (5.5.3), (5.5.4), we easily obtain the following conditions:

\[(\sigma_1')^\text{in} = -(\chi_0')^\text{in}, \quad (5.5.5a)\]

\[(x_1')^\text{in} = (\sigma_0)^\text{in}. \quad (5.5.5b)\]

Applying the same reasoning to the equations (2.1.7b), we obtain

\[\sigma_1^\text{in} = \sigma_1^\text{out}, \quad (5.5.5a)\]

\[x_1^\text{in} = x_1^\text{out}. \quad (5.5.5b)\]

In order to make the equations (5.5.5a) compatible with the rest of the theory, we will integrate them with respect to the boundary of the scatterer, getting

\[(\sigma_1(x))^\text{in} = -\int_0^x \chi_0^\text{in}(t)dt, \quad (5.5.6)\]

\[(x_1(x))^\text{in} = \int_0^x \sigma_0^\text{in}(t)dt. \]
Now we can formulate the following theorem.

**Theorem 5.5.1**

The system of integral equations consisting of (2.1.7b), (5.5.4a), (5.5.4b), 5.5.5b) and (5.5.6) is a Fredholm system with respect to the functions

\[
\sigma^\text{in}_0, \sigma^\text{out}_0, \sigma^\text{in}_1, \sigma^\text{out}_1, \chi^\text{in}_0, \chi^\text{out}_0, \chi^\text{in}_1, \chi^\text{out}_1.
\]

The proof of the above theorem is a repeat of the proof of Theorem 5.4.1.

6. The Table of Auxiliary Conditions

We have seen in the preceding two sections that deriving the integral equations for the boundary conditions of Chapter III consists mainly of finding auxiliary linear constraints on the densities of single and double layer potentials inside and outside the scatterer. In the cases other than fluid/fluid and solid/solid contact, the derivation is very similar to that in these two cases. Therefore, the results will be listed without derivation in the same order as the boundary conditions were listed in Chapter II.
a) Free solid surface:

\[ \sigma_1' = -\chi_0', \]
\[ \chi_1' = \sigma_0'. \]

b) Surface of a solid body welded to a rigid body:

\[ \sigma_1 = \chi_1 = 0. \]

c) Solid/solid interface:

\[ (\sigma_1^{\text{in}})', = -\chi_0^{\text{in}}, \]
\[ (\chi_1^{\text{in}})', = \sigma_0^{\text{in}}, \]
\[ \sigma_1^{\text{out}} = \sigma_1^{\text{in}}, \]
\[ \chi_1^{\text{out}} = \chi_1^{\text{in}}. \]

d) Solid/fluid interface:

\[ \sigma_1^{\text{in}} = \sigma_1^{\text{out}}, \]
\[ (\chi_1^{\text{out}})', = \sigma_0^{\text{out}}, \]
\[ (\sigma_1^{\text{out}})', = -\chi_0^{\text{out}}. \]
e) Fluid/fluid interface:

\[ \sigma_1^{\text{in}} = \sigma_1^{\text{out}}, \]

\[ \rho^{\text{in}} \cdot \sigma_0^{\text{in}} = \rho^{\text{out}} \cdot \sigma_0^{\text{out}}. \]

f) Infinitely rigid body imbedded in a fluid space:

\[ \sigma_1 = 0. \]

g) A cavity inside a fluid space:

\[ \sigma_0 = 0. \]

Remark 5.6.1

A discussion of acoustic scattering problems from the point of view of second kind integral equations can be found in [30]. The condition (5.4.7) is introduced there in a form somewhat different from the one we are using. However, the physical interpretation of the following section is not applicable to the integral equations from [30] due to lack of the condition (5.4.12). The author is not aware of any actual numerical treatment of the fluid/fluid scattering problems by means of second type integral equations. The present work seems to be the first successful attempt to reduce elastic scattering problems to second kind integral equations on the boundary of the scatterer.
7. Physical Considerations; Multiple Reflections

The present section is devoted to the physical interpretation of the results of the Section 5 in the case of acoustic scattering. It is partly heuristic, and most of the reasoning is only outlined since it is not directly related to the purpose of this paper.

In the preceding section we have seen that the constraint (5.4.7) is necessary if we want our equations to satisfy the conditions of the Fredholm theorems. The constraint (5.4.12), however, seems almost arbitrary. It makes the kernels of the equation (5.4.6a) continuous. Any other constraint of a similar form would make it logarithmically singular which, mathematically speaking, does not make much difference. However, (5.4.12) turns out to be necessary for a certain physical interpretation of the equations (5.4.10), (5.4.11) to hold.

For \( m=1,2,\ldots \), we will denote by \( S^m \) the pair \( (\sigma_2^m, n^m) \) obtained as a result of the \( m \)-th Neumann iteration of the system (5.4.13), (5.4.14). The potentials inside and outside the scatterer associated with \( S^m \) will be denoted by \( \phi_{in}^m \) and \( \phi_{out}^m \) respectively. The fields corresponding to \( \phi_{in}^m, \phi_{out}^m \) possess a very transparent physical meaning. They can be looked upon as the part of the field that has interacted with the surface of the scatterer no more than \( m \) times. The validity of this interpretation can be established by analyzing some simple situations.
a) Scattering from an infinite line. In this case it is possible to establish by a direct computation that \( S^m = S^1 \) for any \( m \geq 2 \), which is equivalent to saying that the whole scattered field in this case is equal to the first reflection.

b) Ray theory approximation. Since in the ray theory approximation all surfaces are replaced locally by line scatterers, it follows from a) that in this case \( S^1 \) can be interpreted as the first reflection. It is also clear that by iterating this process, we will obtain the second, third, and subsequent reflections. Hence in the ray theory approximation, the field created by \( S^m \) can be interpreted as the portion of the scattered field that has interacted with the surface of the scatterer no more than \( m \) times.

c) Scattering from a non-existent scatterer, i.e. when the physical parameters of the scatterer are the same as those of the containing space. In this case no multiple reflections should be present since all transmission coefficients are equal to 1 and all reflection coefficients are equal to zero. Here the necessity of (5.4.12) becomes obvious for this interpretation to hold. A simple analysis shows that the Neumann series for (5.4.13), (5.4.14) contains only one non-zero term and that (5.4.12) is necessary for this to be true.
Using the above interpretation, it is easy to see that the problem (5.4.13), (5.4.14) is well conditioned as long as the contrast between the physical parameters of the scatterer and that of the containing space is not too great. Indeed, at every Neumann iteration a certain percentage of the total energy is radiated out to infinity, and this percentage is roughly determined by the transmission and reflection coefficients on the boundary. This means that the Neumann series for (5.4.13), (5.4.14) converges rapidly, which in turn indicates that the system is well conditioned.
VI. Solution of Transient Problems by Laplace's Transformation

It was stated in the Introduction that scattering problems in the time domain are equivalent to families of boundary value problems in the frequency domain. This fact is widely used in physics. It is proven in [12] in a form very close to the one we need. This chapter is devoted to formulating this equivalence precisely. The theory will be given for the scattering of acoustic waves by fluid inclusions. All other cases are exactly similar.

1. The Time Domain Problem

The equation inside the scatterer is

$$\rho_{\text{in}} \cdot \frac{\partial^2 \Phi_{\text{in}}}{\partial t^2} = \lambda_{\text{in}} \cdot \nabla^2 \Phi_{\text{in}} \quad (6.1.1)$$

and the equation outside the scatterer is

$$\rho_{\text{out}} \cdot \frac{\partial^2 \Phi_{\text{out}}}{\partial t^2} = \lambda_{\text{out}} \cdot \nabla^2 \Phi_{\text{out}} \quad (6.1.2)$$

The conditions on the boundary of the scatterer are

$$\alpha_{\text{in}} \cdot \Phi_{\text{in}} - \alpha_{\text{out}} \cdot \Phi_{\text{out}} = \phi_{\text{source}} \quad (6.1.3)$$

$$\beta_{\text{in}} \cdot \frac{\partial \Phi_{\text{in}}}{\partial N} - \beta_{\text{out}} \cdot \frac{\partial \Phi_{\text{out}}}{\partial N} = \frac{\partial \phi_{\text{source}}}{\partial N} \quad (6.1.4)$$
where

\begin{equation}
\phi_{\text{source}} : (R^2 \setminus \{X_s\}) \times [0, \infty) \rightarrow C^1
\end{equation}

is a function analytical with respect to the first coordinate and at least \( C^2 \) with respect to the second coordinate. The subset \( X_s \) of \( R^2 \) will be presumed to be compact. Finally, it will be presumed that:

\begin{align*}
\phi_{\text{source}} |_{(R^2 \setminus \{X_s\}) \times \{0\}} &= 0, \\
\frac{\partial \phi_{\text{source}}}{\partial t} |_{R^2 \times \{0\}} &= 0, \\
\phi_{\text{in}} |_{\Omega \times \{0\}} &= 0, \\
\frac{\partial \phi_{\text{in}}}{\partial t} |_{\Omega \times \{0\}} &= 0, \\
\phi_{\text{out}} |_{(R^2 \setminus \overline{\Omega}) \times \{0\}} &= 0, \\
\frac{\partial \phi_{\text{out}}}{\partial t} |_{(R^2 \setminus \overline{\Omega}) \times \{0\}} &= 0.
\end{align*}

2. The Frequency Domain Problem

For a fixed \( \omega \in C^1 \), the equation inside the scatterer is

\begin{equation}
\nabla^2 \phi_{\text{in}} + (k_{\text{in}})^2 \cdot \phi_{\text{in}} = 0
\end{equation}

and the equation outside the scatterer is

\begin{equation}
\nabla^2 \phi_{\text{out}} + (k_{\text{out}})^2 \cdot \phi_{\text{out}} = 0
\end{equation}
with \( k^\text{in} = \omega/c^\text{in}, k^\text{out} = \omega/c^\text{out} \).

Here \( c^\text{in} \) and \( c^\text{out} \) are the speeds of sound inside and outside the scatterer. The function \( \phi^\text{in}: \overline{\Omega} \rightarrow \mathbb{C} \) is presumed to be continuous on \( \overline{\Omega} \) and analytic on \( \Omega \), and the function \( \phi^\text{out}: \mathbb{R}^2 \setminus \overline{\Omega} \rightarrow \mathbb{C} \) is presumed to be continuous on \( \mathbb{R}^2 \setminus \overline{\Omega} \) and analytic on \( \mathbb{R}^2 \setminus \overline{\Omega} \).

3. Existence and Uniqueness.

**Theorem 6.3.1**

If none of the coefficients \( \alpha^\text{in}, \alpha^\text{out}, \beta^\text{in}, \beta^\text{out} \) is equal to zero, then problem 1 (time domain) has a unique solution.

This theorem is proven in a considerably more general form in [12].

4. Equivalence.

**Theorem 6.4.1**

For any \( x \in \mathbb{R}^2 \setminus \overline{\Omega} \), if there exist real numbers \( A, B \) such that

\[
|\phi(x,t)| < A \cdot e^{Bt}
\]

for any \( t \in [0,\infty) \), then the function \( \phi^\chi(\omega) = \phi(x,\omega) \) is analytic for all values of \( \omega \) such that \( \text{Im}(\omega) > B \).

The converse is also true.
Theorem 6.4.2

For any \( x \in \mathbb{R}^2 \setminus \{\omega\} \), if there exists such a real number \( \beta \) that the function \( \phi_x(\omega) \) is analytic in the half-plane \( \text{Im}(\omega) > \beta \), then there exists such a real positive \( A \) that

\[
|\phi(x,t)| < A \cdot e^{B \cdot t}.
\]

Theorem 6.4.3

Under the conditions of Theorems 6.4.1, 6.4.2, the solutions of the problems I and II at the point \( x \) are connected by the equality

\[
\phi(x,t) = \sum_l \frac{\hat{g}(x,\omega)}{\omega} \cdot e^{i\omega t} d\omega,
\]

where \( L \) is an arbitrary line parallel to the real axis laying in the half-plane \( \text{Im}(\omega) > \beta \).

Theorems 6.4.1 - 6.4.3 are the basis for the use of the Laplace transformation (and also Fourier transformation) for the numerical solution of wave propagation problems. The standard technique is to evaluate the frequency domain solution for a finite number of values of \( \omega \) and to approximate the integral (6.4.1) numerically.

A very detailed proof of the one-dimensional analogues of Theorems 6.4.1 - 6.4.3 can be found in [12]. The proofs of Theorems 6.4.1 - 6.4.3 are quite similar to those found in [12]. In fact this whole area is treated in a very general form in Electrical Engineering (see, for example, [13]). Electrical Engineers usually refer to it as the theory of linear systems.
VII. Discretization of The Integral Equations

In order to solve integral equations of the Chapter V numerically, one has to discretize them by replacing integrals with quadrature formulae and integral equations with systems of linear algebraic equations.

1. Quadrature Formulae for Logarithmically Singular Functions

The kernels of the equations of the Chapter V are logarithmically singular, which necessitates the use of quadrature formulae suitable for logarithmically singular functions. In this section such formulae will be derived by means of a version of Richardson's extrapolation technique. Our approach is very similar to that of [14] and [15]. The difference is that [14] and [15] are dealing with singularities of the form $x^\alpha$ (for non-integer $\alpha$) and we are dealing with singularities of the form $x^m \cdot \log(x)$, $m = 0, 1, 2, \ldots$.

For a continuous function $f: (0,1] \rightarrow \mathbb{R}$, we will define $T_n(f)$ by the formula

$$T_n(f) = h \cdot \sum_{i=1}^{n} f(x_i),$$

(7.1.1)

where $h = \frac{1}{n}$ and $x_i = i \cdot h$ with $i = 1, 2, \ldots, n$. For functions $f$ with a logarithmic singularity at zero, we will consider $T_n(f)$ an approximation to the integral

$$\int_{0}^{1} f(x) \, dx.$$
Actual quadrature formulae will be obtained by applying Richardson-type extrapolation technique to \( T_n(f) \). For a fixed \( f \), we will be considering \( T_n(f) \) as a function of \( h \) and it will be denoted by \( T(h) \).

**Lemma 7.1.1**

If \( f \) is defined by the formula \( f(x) = x^m \cdot \log(x) \), \( m = 1, 2, \ldots \), then

\[
(h) = I(f) - T(h) = U_1(h) + V_1(h) \cdot \log(h)
\]

(7.1.2)

where \( U_1, V_1: [-1,1] \rightarrow \mathbb{R}^1 \) are functions analytical in some neighborhood of 0.

**Proof**

We will denote the sum \( U_1(h) + I(f) \) by \( U(h) \). Since \( I(f) \) does not depend on \( n \), it is sufficient to show that

\[
T(h) = U(h) + V(h) \cdot \log(h)
\]

(7.1.3)

where \( U \) and \( V \) are functions, analytical in a neighborhood of 0. The following exercise in elementary calculus does it. Indeed,

\[
T_n(x^m \cdot \log(|x|)) = h \cdot \sum_{i=1}^{n} \left( \frac{i}{n} \right)^m \cdot \log \left( \frac{i}{n} \right) =
\]

(7.1.4)

\[
= h \cdot \alpha_n(m) + h \cdot \beta_n(m)
\]
with

\[ \alpha_n(m) = \frac{1}{n^m} \sum_{i=1}^{n} i^m \cdot \log(i) \]  
\[ \beta_n(m) = \frac{1}{n^m} \sum_{i=1}^{n} i^m \cdot \log(n) \]  

(7.1.5)

Now

\[ \alpha_n(m) = \frac{1}{n^m} \cdot \log(\exp(\sum_{i=1}^{n} i^m \cdot \log(i))) = \]

\[ = \frac{1}{n^m} \log(\prod_{i=1}^{n} e^{i^m}) = \]

\[ = \frac{1}{n^m} \log(n! \cdot e^{i^m}) = \]

\[ = \frac{1}{n^m} \log(n!) + \frac{1}{n^m} \sum_{i=1}^{n} i^m = \]

\[ = h^m \cdot \log(n^{1/m}) + h^m \cdot P_m(\frac{1}{h}) , \]

where \( P_m(t) \) is the m-th order polynomial defined by the formula

\[ P_{m+1}(n) = \sum_{i=1}^{n} i^m . \]

Similarly,

\[ \beta_n(m) = \frac{\log(n)}{n^m} \cdot \sum_{i=1}^{n} i^m = \]

\[ = h^m \cdot \log(\frac{1}{h}) \cdot P_m(\frac{1}{h}) . \]
Combining formulae (7.1.3) - (7.1.6), we obtain

\[ T_n(x^m \cdot \log(x)) = h^{m+1} \cdot \log(n^{1/h} + 1) + P_m \cdot \left( \frac{1}{h} \right) \cdot (1 + \log(h)). \]

According to Stirling's formula (see [16]),

\[ r(z) = e^{-z} \cdot z^{z - \frac{1}{2}} \cdot S\left(\frac{1}{2}\right), \]

where \( S(t) \) is a function analytical on the whole complex plane.

Euler - Maclaurin formula.

**Theorem 7.1.1**

If the function \( f \) is defined by the formula

\[ f(x) = A(x) + B(x) \cdot \log(x) \]

with \( A, B: [0,1] \rightarrow \mathbb{R} \) \( m \)-smooth functions,

then

\[ I(f) - T(h) = u(h) + v(h) \cdot \log(h) \quad (7.1.10) \]

where \( u, v: [0,1] \rightarrow \mathbb{R} \) are \( m \)-smooth functions such that

\[ u(0) = v(0) = 0. \]

Using (7.1.10), Richardson-type extrapolation algorithms can be developed. An example of such an algorithm (based on doubling the number of nodes) is given by the following theorem. Its proof consists of applying Richardson's extrapolation twice.
Theorem 7.1.2

If the function $\phi: [0,1] \rightarrow \mathbb{R}$ is such that

$$\phi(x) = \phi_0 + \alpha x^k + \beta x^k \cdot \log(x) + o(x^{k+1} \cdot \log(x))$$

in some neighborhood of 0, then the equality

$$\psi_1(x) = \phi_0 + \gamma x^k + o(x^{k+1} \cdot \log(x))$$

is true for the function $\psi_1$ defined by the formula

$$\psi_1(x) = \frac{1}{2^k - 1} \cdot (2^k \cdot \phi(x) - \phi(2x))$$.

Furthermore, the equality

$$\psi_2(x) = \phi_0 + o(x^{k+1} \cdot \log(x))$$

is true for the function $\psi_2$ defined by the formula

$$\psi_2(x) = \frac{1}{2^k - 1} \cdot (2^k \cdot \psi_1(x) - \psi_1(2x))$$.

If the logarithmic singularity is at the center of the interval, then some of the extrapolation steps can be omitted due to the fact that

$$\int_{-a}^{a} x^k \cdot \log(|x|) = 0$$.
for odd $k$. Since in our case we perform the integration over the
boundary of the scatterer, all of the integrands are periodic, and
we always can move the singularity to the center of the integration
interval by a simple shift of the system of coordinates on the
boundary.

The following quadratically convergent formula was used in the
actual implementation of the algorithm:

$$
\int_{-a}^{a} f(x) dx \sim h \cdot \sum_{i=1}^{n} f_i + \frac{h}{2} \cdot \left( f_{\frac{1}{2}} - f_{-\frac{1}{2}} \right) - \frac{\gamma_n}{2 \ln(2)} \cdot \left( f_2 - f_1 - f_{-1} - f_{-2} \right),
$$

(7.111)

$$
\gamma_n = 2 \cdot \sum_{i=1}^{n} \ln(x_i).
$$

In the above formulae,

$$
x_i = h \cdot i \quad \text{for} \quad i = -n, -(n-1), \ldots, -2, -1, 1, 2, \ldots, n
$$

and

$$
f_i = f(x_i).
$$
2. Rates of Convergence

When solving integral equations numerically, the integral equations are replaced by finite integration formulae. There are several standard approaches to this discretization process. The author chose to use the so-called Nyström method (see [17]). In this approach, the integral equation

$$\lambda \cdot \phi(x) + \int_{0}^{1} \phi(t) \cdot K(t,x) \, dt = f(x) \quad (7.2.1)$$

is replaced by the expression

$$\lambda \cdot \phi_n(x_i) + \sum_{j=1}^{n} w_j \cdot \phi_n(x_j) \cdot K(x_j,x_i) = f(x_i) \quad (7.2.2)$$

with $i = 1, 2, \ldots, n$. Here $\{x_i\}, i = 1, 2, \ldots, n$, is a finite subset of the interval $[0,1]$ and $\{w_i\}, i = 1, 2, \ldots, n$ are coefficients of some (prechosen) integration formula. We will denote the integration formula corresponding to $\{x_i, w_i\}, i = 1, 2, \ldots, n$ by $P_n$. The solution of (7.2.1) will be denoted by $\phi$ and the solution of (7.2.2) by $\phi_n$. The following theorem means that under fairly broad assumptions the convergence rate of a Nyström type algorithm is the same as the convergence rate of the integration formula it is based on. It can be found in a somewhat different notation in [17], page 99.
Theorem 7.2.1

Assume that the equation (7.2.1) is non-degenerate. Assume further that

$$\left| \int_{0}^{1} K(t,x) \cdot f(t) dt - \sum_{j=1}^{n} w_{j} \cdot f(x_{j}) \cdot K(x_{j},x) \right| < \frac{\alpha}{n^{k}}$$

for any \( x \in [0,1] \) where \( \alpha \) is some positive real number independent from \( x \), and \( k \geq 1 \) is an integer number. Then

$$|\phi_{n} - \phi| < \frac{\beta}{n^{k}}$$

where \( \beta \) is some positive real number.

Theorem 7.2.1 is formulated in terms of R-convergence (see [10]). A similar result can be proven in terms of Q-convergence. In the actual implementation, a version of Nyström's algorithm was used in combination with the formula (7.1.11), resulting in a quadratically convergent algorithm.
VIII. Solution of The Linear Systems

1. Generalized Conjugate Residual Algorithm for Complex Non-Hermitian Linear Systems

In this section, we will describe a generalization of the conjugate residual algorithm for complex non-Hermitian systems to linear equations. Our approach is somewhat different from the traditional one. Its equivalence to the standard conjugate residual algorithm in the real symmetric case is proven in [9].

A complex $n \times n$ - system of linear algebraic equations

$$AX = Y$$  \hspace{1cm} (8.1.1)

will be considered. We will be seeking an approximate solution to the system (8.1.1) in the form

$$\tilde{X} = P_k(A) \cdot Y$$  \hspace{1cm} (8.1.2)

where $P_k$ is a polynomial of degree $k$ with complex coefficients. The polynomial $P_k$ will be chosen to minimize the quadratic form

$$||A \cdot P_k(A) \cdot Y - Y||_2^2$$  \hspace{1cm} (8.1.3)

among all polynomials of order $k$. As is well known (see, for example, [18], page 199), minimizing (8.1.3) is equivalent to solving the Hermitian $(k+1) \times (k+1) - $ system

$$B(P) = Z$$  \hspace{1cm} (8.1.4)
where the element $b_{ij}$ of the matrix $B$ is defined by the formula

$$
    b_{ij} = (A^{i-1}y, A^{j-1}y)
$$

and the $j$-th element of the vector $Z$ is defined by the formula

$$
    Z_j = (A^{j-1}y, y)
$$

it is easy to see that if the vectors

$$
    A^0 y, A^1 y, A^2 y, \ldots, A^k y
$$

are linearly independent, then the system (8.1.4) has a unique solution. The suggestion to solve (8.1.4) instead of (8.1.1) has a number of obvious drawbacks. Its condition number is bound to be greater, and one step of the conjugate gradient algorithm for the system (8.1.4) would require four matrix-vector multiplications. To make things worse, one does not know in advance how large $k$ must be to yield the required precision in the solution of (8.1.1). And finally, attempts to apply high degree matrix polynomials to arbitrary vectors are known to lead to unacceptable round-off errors. However, the following simple observation enables one to bypass all these difficulties. Namely, the matrix $B_{k+1}$ can be obtained from $B_k$ by addition of a column and a row, and the inverse of $B_{k+1}$ can be obtained from the inverse of $B_k$ by means of the Sherman-Morrison-Woodbury formula (see [10]). Numerical algorithms based on the Sherman-Morrison-Woodbury formula tend to be unstable. We eliminate this instability by means of Theorem 8.1.1.
Lemma 8.1.1 (Sherman-Morrison-Woodbury formula)

Suppose that $A$ is a complex non-singular $n \times n$ matrix and $u, v \in \mathbb{C}^n$. Then $A + u \cdot v^*$ is non-singular if and only if $1 + v^* \cdot A^{-1} \cdot u \neq 0$ and

$$(A + u \cdot v^*)^{-1} = A^{-1} - \frac{A^{-1}uv^*A^{-1}}{1 + v^*A^{-1}u}.$$ 

The above lemma can be found, for example, in [10]. We will use it to construct a recursive algorithm computing $B_k^{-1}$ for $k = 2, 3, \ldots$. At each step, we will represent $B_k$ as a sum of matrices $G_k$ and $H_k$, such that the inverse of $G_k$ is already known, and the rank of $H_k$ is equal to 2. Then $H_k$ is broken into a sum of two rank-one matrices, and $B_k^{-1}$ is computed by applying Lemma 8.1.1 twice.

For $k \geq 2$, the $k \times k$ matrices $H_k$, $G_k$ are defined on the figures below.

\[
H_k = \begin{bmatrix}
0 & b_{1,k} & \cdots & b_{k-1,k} \\
. & . & \ddots & . \\
. & . & . & . \\
b_{k,1} & \cdots & b_{k,k-1} & 0
\end{bmatrix} \quad (8.1.8)
\]
\[ G_k = \begin{pmatrix} B_{k-1} & 0 \\ 0 & b_{kk} \end{pmatrix} \]  \hspace{1cm} (8.1.9)

Obviously, \( B_k = G_k + H_k \). Also, it is clear that

\[ G_k^{-1} = \begin{pmatrix} B_k^{-1} & 0 \\ 0 & b_k^{-1} \end{pmatrix} \]  \hspace{1cm} (8.1.10)

**Lemma 8.1.2**

\[ H_k = p \cdot p^* - q \cdot q^*, \]

where the vectors \( p, q \in c^k \) are defined by the formulae

\[ p_i = q_i = \frac{1}{2} b_{ik} \text{ for } i \leq k-1, \]

\[ p_k = 1, q_k = -1. \]
The above lemma can be easily verified by substitution. Combining
the Lemmas 8.1.1 and 8.1.2, a recursive algorithm can be constructed to
obtain $B_k^{-1}$ from $(B_{k-1}^{-1})^{-1}$. Indeed,

$$B_k = G_k + H_k = G_k + p \cdot p^* - q \cdot q^*$$

and $B_k^{-1}$ can be obtained by applying Lemma 6.1.1 twice, provided that
$(B_{k-1}^{-1})^{-1}$ is already known. The number of operations required for the
k-th step of the recursion is of the order $k^2$.

The above described algorithm has one obvious drawback. Namely,
it represents $H_k$ as a difference of two rank-one positive semi-definite
operators which creates a possibility of substantial round-off errors
due to cancellation. This problem can be bypassed by imposing an addi-
tional constraint that these rank-one operators be orthogonal to each
other.

We will define the vector $b_k^k \in \mathbb{C}^k$ by the formula

$$b_i^k = (H_k)_{ik}.$$  

We will denote by $m$ an integer number such that $|b_m^k| \geq |b_i^k|$ for any
$1 \leq i \leq k$. The coefficient $\gamma$ will be defined by the formula

$$\gamma = \frac{\frac{1}{2} \cdot ||b_k||}{||b_m||}.$$  \hspace{1cm} (8.1.11)

The vector $q$ will be defined by the formulae
The vector \( q \) will be defined by the formulae

\[
q_m = \frac{2}{\gamma} ,
\]

\[
q_i = b_i^k \cdot q_m \cdot \frac{1}{b_m^k} \quad \text{for} \quad 1 \leq i \leq k , \tag{8.1.12}
\]

\[
q_k = (\gamma \cdot b_m^k)^* ,
\]

and the vector \( p \) will be defined by the formulae

\[
p_i = q_i + \frac{1}{\gamma \cdot b_m^k} \quad \text{for} \quad 1 \leq i \leq k , \tag{8.1.13}
\]

\[
p_k = q_k .
\]

The following theorem can be verified by substitution.

**Theorem 8.1.1**

For the above defined vectors \( p, q, \)

\[
p^* \cdot p - q \cdot q^* = H_k ,
\]

\[
(p, q) = 0 .
\]

The resulting algorithm can be viewed as a generalization of the classical conjugate residual algorithm, since for real symmetric matrices it will (in exact arithmetic) coincide with the later. However, even for such matrices, its numerical behavior is considerably more stable than that of the classical conjugate residual algorithm.
This fact is the subject of a separate investigation by the author.

When applied to the matrices of the preceding chapter, the above algorithm turns out to be extremely effective, especially when combined with the preconditioning described in the following section.

2. Preconditioning

In the solution of time-domain scattering problems, a sequence of frequency-domain problems is solved with subsequent application of the Fast Fourier Transformation (the Cooley-Tukey version of the algorithm has been used in the present work). Since a whole sequence of such problems is solved, the temptation is great to use the results of computations for preceding frequency values to precondition the linear systems for the subsequent ones. In this section we will presume that we have to solve a finite sequence of \( n \)-dimensional linear systems

\[
A_k \cdot x = y_k, \quad k = 1, 2, \ldots, m. \tag{8.2.1a}
\]

We will also presume that (8.2.1) is a discretization of a smooth family of linear systems, i.e.

\[
A_k = A(\omega_k), \quad y_k = y(\omega_k), \quad \tag{8.2.1b}
\]

where \( A : C^1 \rightarrow C^{n \times n} \), \( Y : C^1 \rightarrow C^n \) are \( \epsilon \)-smooth \((\epsilon > 1)\) mappings, and \( \{\omega_k\}, k = 1, 2, \ldots, m \) is a discretization of some smooth path \( \sigma : [0,1] \rightarrow C^1 \) in the complex plane. We will be assuming that the distances \( |\omega_k - \omega_{k-1}| \) are in some sense small.
Several approaches to designing a preconditioning algorithm based on "looking back" suggest themselves. The author has investigated two of them:

a) Broyden preconditioning. For the k-th value of $\omega$, each iteration of the generalized conjugate residual algorithm produces two pairs $(e_1, a_1), (e_2, a_2)$ such that $a_1 = A_k \cdot e_1, a_2 = A_k \cdot e_2$. These pairs can be used to create an approximation $B_k$ to $A_k^{-1}$ by means of a complex version of the Broyden update (see, for example, [10]). The identity matrix can be chosen as an initial approximation to $A^{-1}$. When $k > 2$, the already computed approximation to $A_{k-1}^{-1}$ is used as the initial approximation to $A_k^{-1}$.

The author's experiments indicate that the actual algorithm works extremely well in terms of the number of iterations required per frequency. However, it requires a substantial amount of overhead work associated with performing Broyden updates. Since the matrices involved in the solution of actual scattering problems tend to be very well conditioned (see Section V.7), a simplification of this idea has also been tried.

b) Simplified preconditioning algorithm. This algorithm only uses several preceding values of the frequency to precondition a matrix $A_k$. We will denote the number of preceding frequencies used in the preconditioning process by $p$. It will be presumed that $A(\omega), Y(\omega)$ are both $p$-smooth. The solution of the $(k-1)$-th system will be
denoted by $z_i$, and the vector $A_k z_i$ will be denoted by $w_i$.

Finally, we will denote by $W_p$ the subspace of $\mathbb{C}^n$ spanned by the vectors $w_1, w_2, \ldots, w_p$ and by $V_p$ the orthogonal complement of $W_p$ so that

$$\mathbb{C}^n = W_p + V_p.$$  \hspace{1cm} (8.2.2)

For any $y \in \mathbb{C}^n$, we will denote by $y_w$ its projection on $W_p$ and by $y_v$ its projection on $V_p$. Naturally, $y = y_w + y_v$, and we will define the linear operator $B_k$ by the formula

$$V_k(y) = A^{-1}(y_w) + y_v.$$ \hspace{1cm} (8.2.3)

Suppose that this preconditioning scheme is applied to (8.2.1). Then

$$X(\omega_k) = \sum_{i=1}^{p} \alpha_i \cdot X(\omega_{k-i}) + O(h^p)$$ \hspace{1cm} (8.2.4)

where $h$ is the maximum distance between $\omega_k$, $\omega_{k+1}$, and $\{\alpha_i\}$, $i = 1, 2, \ldots, p$ are coefficients of some extrapolation formula (see [18], page 49). Substituting (8.2.4) into (8.2.1a), we obtain

$$A_k \left( \sum_{i=1}^{p} \alpha_i \cdot X_{k-i} + O(h^p) \right) = Y_k$$ \hspace{1cm} (8.2.5) \\

or

$$\sum_{i=1}^{p} \alpha_i \cdot A_k(x_{k-i}) + O(h^p) = Y_k.$$ \hspace{1cm} (8.2.6)
Now the equation

\[ B_k \cdot A_k(x_k) = B_k(y_k) \]  \hspace{1cm} (8.2.7)

can be rewritten as

\[ B_k(\sum_{i=1}^{p} \alpha_i \cdot A_k(x_{k-i}) + O(h^p)) = B_k(y_k), \]  \hspace{1cm} (8.2.8)

or

\[ \sum_{i=1}^{p} \alpha_i \cdot B_k \cdot A_k(x_{k-i}) + O(h^p) = B_k(y_k), \]  \hspace{1cm} (8.2.9)

or

\[ B_k(y_k) - \sum_{i=1}^{p} \alpha_i \cdot B_k \cdot A_k(x_{k-i}) = O(h^p) \]  \hspace{1cm} (8.2.10)

which means that the initial error in the conjugate residual process is of the order \( h^p \), which in practical applications tends to be quite small.

In the examples the author has run, the above two preconditioning algorithms have resulted in comparable execution times. With the decrease in \( h \), the Broyden-type algorithm became appreciably faster, while the simplified algorithm tended to be somewhat faster for coarse discretizations.
IX. Implementation and Numerical Results

A program has been written dealing with three types of acoustic scattering: from empty inclusions (Dirichlet problem), from infinitely rigid inclusions (Neumann problem), and from fluid inclusions (an essential part of this work). The first two cases served as a proving ground for several techniques to be used for the third. Several purely technical problems had to be overcome. One of them had to do with Hankel functions of complex argument (see Appendix A). The storage requirements of the program also turned out to be a problem. Indeed, if the discretization of the scatterer consists of 200 points, then the matrix of the discretized system will contain $(200 \cdot 2)^2 \cdot 2 = 320,000$ real numbers and in order to store it, we would need 1.28 megabytes of core. As a result, it was decided to always store the matrices on disk, and sufficiently efficient storage - retrieval software had to be written. The block-diagram of the actual program is reproduced in the appendix B. Results of applying this program to three types of scattering (fluid/empty, fluid/rigid, fluid/fluid) are given below.

Following are the results of 15 numerical experiments performed by the author. For each experiment, the figure showing the physical situation being modeled is followed by the plot depicting the response of receivers in the preceding figure. In all plots, the time axis is vertical, the positive direction being down. Receivers
measure the pressure as a function of time, and each vertical trace represents the response of a receiver.
\[ \rho^\text{in} = 2.5 ; \quad \rho^\text{out} = 2.0 ; \quad \text{TYPICAL WAVELENGTH} = 10 \text{ FT} \]
\[ c^\text{in} = 300 ; \quad c^\text{out} = 250 ; \quad \text{NUMBER OF RECEIVERS} = 21 \]
\[ \rho_{\text{in}} = 2.0 ; \quad \rho_{\text{out}} = 2.5 ; \quad \text{TYPICAL WAVELENGTH} = 10 \text{ FT} \]
\[ c_{\text{in}} = 250 ; \quad c_{\text{out}} = 300 ; \quad \text{NUMBER OF RECEIVERS} = 21 \]
\[ \rho_{\text{in}} = 2.5 ; \quad \rho_{\text{out}} = 2.0 ; \quad \text{TYPICAL WAVELENGTH} = 10 \text{ FT} \]
\[ c_{\text{in}} = 300 ; \quad c_{\text{out}} = 250 ; \quad \text{NUMBER OF RECEIVERS} = 21 \]
\[ \rho^{in} = 2.0 ; \quad \rho^{out} = 2.5 ; \quad \text{DOMINANT WAVELENGTH} = 10 \text{ FT} \]
\[ c^{in} = 250 ; \quad c^{out} = 300 ; \quad \text{NUMBER OF RECEIVERS} = 21 \]
BODY - EMPTY; \( \rho_{\text{out}} = 2.0 \); TYPICAL WAVELENGTH = 10 FT
\( c_{\text{out}} = 250 \); NUMBER OF RECEIVERS = 21
BODY - RIGID; \( \rho_{\text{out}} = 2.0; \)  
TYPICAL WAVELENGTH = 10 FT  
\( c_{\text{out}} = 250; \)  
NUMBER OF RECEIVERS = 21
\[ \rho^{\text{in}} = 2.0 ; \quad \rho^{\text{out}} = 5.0 ; \quad \text{TYPICAL WAVELENGTH} = 10 \text{ FT} \]
\[ c^{\text{in}} = 250 ; \quad c^{\text{out}} = 500 ; \quad \text{NUMBER OF RECEIVERS} = 21 \]
\( \rho_{in} = 2.5 \); \( \rho_{out} = 2.0 \);

\( c_{in} = 300 \); \( c_{out} = 250 \);

TYPICAL WAVELENGTH = 10 FT

NUMBER OF RECEIVERS = 61
\[ \rho_{\text{in}} = 2.0 \quad \rho_{\text{out}} = 2.5 \quad \text{TYPICAL WAVELENGTH} = 10 \text{ FT} \]
\[ c_{\text{in}} = 250 \quad c_{\text{out}} = 300 \quad \text{NUMBER OF RECEIVERS} = 61 \]
\[ \rho_{\text{in}} = 2.0; \quad \rho_{\text{out}} = 2.5; \quad \text{TYPICAL WAVELENGTH} = 10 \text{ FT} \]

\[ c_{\text{in}} = 250; \quad c_{\text{out}} = 200; \quad \text{NUMBER OF RECEIVERS} = 121 \]
\[ \rho_{\text{in}} = 2.0 ; \quad \rho_{\text{out}} = 2.5 ; \quad \text{TYPICAL WAVELENGTH} = 10 \text{ FT} \]
\[ c_{\text{in}} = 250 ; \quad c_{\text{out}} = 300 ; \quad \text{NUMBER OF RECEIVERS} = 41 \]
X. Generalizations

The problems treated in this thesis are those of acoustic and elastic scattering in two dimensions. There are several directions in which the results of this present work can be generalized.

1. Two-dimensional Scatterers With Corners

From the practical point of view, this is a most interesting problem. It turns out that in the case of scattering of acoustic waves by fluid inclusions, the problem is fairly simple. It is easy to show that the equations (5.4.10), (5.4.11) do not become singular when the scatterer has corners. The charge and dipole density functions undergo a jump at the corner points, but they stay $L^2$ and the whole theory holds. The author is planning to publish a rigorous proof of this fact in the near future.

As for the 2-dimensional scattering of waves in solid media, the problem is considerably more complicated. The equations (5.5.3), (5.5.4), and (2.1.7) in this case become essentially singular, and the author has not investigated the behavior of their singularities in sufficient depth.

2. Three-Dimensional Scattering

The theory of acoustic waves in three dimensions is very similar to that in two dimensions. As far as the author can tell, the theory
of Section V.5 holds with only minor changes. A practical implementation of such an algorithm would face substantial programming difficulties due to the amount of bookkeeping involved. However, for certain practical applications, such a program could be quite useful.

An attempt to apply the techniques of the Chapter V to 3-dimensional scattering in solid media runs into certain theoretical difficulties. One ends up with integro-differential equations on the boundary of the scatterer very similar to those of Chapter V. However, the boundary is now 2-dimensional, and problems of stability of solutions of certain PDEs on 2-dimensional manifolds become involved. The author does not expect these difficulties to be insurmountable, but a substantial amount of work would be involved in producing a workable algorithm.

3. Multiple Scatterers

The case of multiple scatterers does not differ substantially from the case of one scatterer. The scattered field inside each scatterer is represented in a manner precisely similar to the one used to solve the isolated scatterer problem. The scattered field outside the scatterers is represented by the sum of the fields of single and double layer potentials on the surfaces of all scatterers. The integral equations on the boundary of one scatterer are replaced by integral equations on the union of the boundaries of all scatterers. The whole procedure is completely straightforward.
BIBLIOGRAPHY


Appendix

The author failed to find in the literature a sufficiently fast algorithm for computing Hankel functions of complex arguments. A subroutine was written taking advantage of the fact that for a given frequency the arguments of Hankel functions are located on a ray emanating from the origin. Indeed, the argument always has the form \( k \cdot R \)

where \( k \) is the Helmholtz coefficient (generally, complex) and \( R \) is the distance between two points in \( \mathbb{R}^2 \) (real). This means that for a fixed frequency the functions we are interested in are in fact complex functions of a real argument, which automatically reduces the computational effort approximately by the factor of 2. Further, we are only interested in the values of the argument located in the first quadrant, which permits the use of fairly simple asymptotic expansions for sufficiently large values of the argument. The first quadrant was divided into four areas:

1. \(|Z| < \frac{1}{2}\),
2. \(\frac{1}{2} \leq |Z| \leq 4\),
3. \(4 < |Z| \leq 8\),
4. \(|Z| > 8\).
In the first area, the standard local expansion was used (see [16], 9.1.2, 9.1.3). In the fourth area, the asymptotic expansion was used (see [16], 9.2.5 – 9.2.10). In the transition regions (areas 2 and 3), advantage was taken of the fact that the argument is, in fact, real and Chebyshev approximations can be used.

In the beginning of the processing of each frequency value, an initialization entry point of the subroutine would precompute the coefficients of interpolating polynomials corresponding to this value for all four regions. The time required by this entry point is insignificant taking into account that it is called once per frequency value. After that, the computation of each pair $H_0(k\cdot R), H_1(k\cdot R)$ takes about 1.2 times longer than computing one complex exponential.