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A SELF-CONSISTENT COMPUTER MODEL FOR THE SOLAR POWER SATELLITE-PLASMA INTERACTION

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A SELF-CONSISTENT COMPUTER MODEL
FOR THE SOLAR POWER
SATELLITE-PLASMA INTERACTION

by

David Lyttleton Cooke

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ABSTRACT

"A Self-Consistent Computer Model for the Solar Power Satellite-Plasma Interaction"

by

David Lyttleton Cooke

High-power solar arrays for satellite power systems are presently being planned with dimensions of kilometers, and with tens of kilovolts distributed over their surface. Such systems will face many plasma interaction problems, such as power leakage to the plasma, enhanced surface damage due to particle focusing, and anomalous arcing to name a few. In most cases, these effects cannot be adequately modeled without detailed knowledge of the plasma-sheath structure and space charge effects. A computer program (PANEL) has been developed to model the solar power satellite (SPS)-plasma interaction by an iterative solution of the coupled Poisson and Vlasov equations. PANEL uses the "inside-out" method and a finite difference scheme to calculate densities and potentials at selected points on either a two or three dimensional grid. PANEL was originally developed by Dr. Lee W. Parker to solve the Laplace equation for the potential distribution about a planar spacecraft and to calculate the plasma currents to the spacecraft surface. After some
improvements, this version was tested and used to model the plasma interaction of the MSFC/Rockwell design for the SPS. Those results are presented in chapter three. More recently, with the aid of Dr. Parker, charge density calculation routines have been added to PANEL to include space charge effects. These routines along with some necessary improvements have been installed, resulting in the present version of PANEL. Among these improvements are: selectable boundary conditions, stop and start capability, a grid cell division technique to improve trajectory accuracy, and a method of phase space boundary tracking that greatly increases program efficiency by avoiding the repeated tracing of most trajectories.

In this thesis, the history of the spacecraft charging problem is reviewed, the theory of the plasma screening process is discussed and extended, program theory is developed, and a series of models is presented. These models are primarily two-dimensional (2-D) for two reasons; one being that large 3-D models require more computing time than I have been able to afford, and the other being that most analytic models suitable for testing PANEL are 1-D and the 3-D capabilities were not required.

These models include PANEL's predictions for two variations on the Child-Langmuir diode problem and two models of the interaction of an infinitely long one meter wide solar array with a dense 10 eV plasma. These models are part of an ongoing effort to adapt PANEL to augment the laboratory
studies of a 1 x 10 meter solar array in a simulated low Earth orbit plasma being conducted in the Chamber A facili-
ties at the NASA/Johnson Space Center. Also included are
two 3-D test models. One is a "point potential" in a hot plasma and is compared to the Debye theory of plasma screen-
ing. The other is a flat disc in charge free space. For the Child-Langmuir diode problem, a good agreement is
obtained between PANEL results and the classical theory.
This is viewed as a confirming test of PANEL. Conversely,
in the solar array models, the agreement between the PANEL and Child-Langmuir predictions for the plasma sheath thick-
ness is presented as a numerical confirmation of the use of the Child-Langmuir diode theory to estimate plasma sheath thickness in the spacecraft charging problem.
Acknowledgements

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1: INTRODUCTION

Any object in space will interact with the local charged particle environment, i.e., the ambient plasma, resulting in a perturbation of the plasma and the acquisition by the object of an electrostatic charge. The study of the various interactions encountered by man-made satellites is known as the field of Spacecraft Charging. Until recently this field was primarily concerned with the relatively small satellites with which our exploration of space began. However, the coming space shuttle era will open the door for the exploitation of space by government, military and private industry, resulting in the construction of larger and more diverse space facilities. An example of this is the proposed Solar Power Satellite (Glaser, 1968) which will use many square kilometers of solar cells to convert sunlight to electricity. This electricity can, in turn, be converted to microwaves and beamed to earth where it can be converted back to electricity for general consumption. Another variant of this theme would be a similar sized satellite to convert sunlight to electricity for use in space by manufacturing and communication industries. These giant satellites will have their own unique plasma interactions, thus we can expect a continuing demand for theories and methods that will not only explain currently observed satellite-plasma interactions, but be capable of predicting the behavior of planned space facilities so that costly design errors can be avoided.
This chapter introduces the subject of spacecraft charging which best begins with a discussion of the equilibrium potential of a body immersed in a plasma.

The equilibrium potential of a body in a plasma is determined by the condition that all currents to the body add to zero. If the only sources of current are plasma electrons and ions with equal temperatures, a body could be expected to charge to a negative potential comparable to the voltage equivalent of the plasma temperature. This happens because the lighter electrons drive a higher current to an uncharged body, thus depressing the potential until the reduced electron current and an enhanced ion current balance. If the body is sunlit, photoelectrons will be emitted and raise the potential. In sunlight, positive potentials of a few volts are typical.

The charging phenomenon was apparently first considered by Jung (1937) who studied the equilibrium potential of interstellar grains. He concluded that the dominant processes would be photoelectron emission and electron bombardment, and that the equilibrium potential would be a few volts positive. The study of the charging phenomenon continued to be limited to grains and microscopic particles until Lehnert (1956) anticipated the charging of Earth satellites. He predicted a potential of only a few volts negative due to enhanced ion collection resulting from the satellite motion through the ionospheric plasma. An extensive review of the early history of equilibrium potential
calculations is given by Whipple (1965).

Spacecraft charging received increased attention after numerous satellites stationed in geosynchronous orbit (GEO = 6.6 Earth radii) were observed to undergo mysterious switching activity and degradation of thermal control properties (McPherson and Schoeber, 1976). The best explanation for this weirdness seems to be the charging of these satellites by hot electrons from the magnetosphere. DeForest (1973) has analyzed data from the GEO satellite ATS-5 and concluded that the satellite had at times charged to potentials as high as -10 kV. Furthermore, incidences of anomalous behavior and observations of extreme spacecraft potential have been correlated with geomagnetic substorms (Fredricks and Scarf, 1973). During a substorm the magnetosphere by high pressure solar wind causes the magnetosheet plasma to become energized and injected into the GEO region.

Variations in the overall spacecraft potential do not necessarily represent a threat to the well-being of a spacecraft, but can complicate particle measurement experiments and possibly make docking maneuvers hazardous. Differential charging is frequently a more serious problem in its effect on hardware and data collection (Grard et al., 1977). An example of this would be a dielectric surface on the shaded side of a satellite with a surface potential of a few kilovolts negative, adjacent to a conductor with a low positive potential because of its connection to a sunlit conductor. Arcs between such surface can damage the surfaces and gener-
ate electromagnetic transients which can saturate and/or damage sensitive electronics. Furthermore, electric fields arising from differential spacecraft charging can drastically influence particle measurement experiments by altering collection efficiencies and particle energies. Whipple (1976) has shown that an electrostatic barrier can form around a spacecraft, returning spacecraft generated particles (primarily photoelectrons), and shutting out ambient electrons.

One approach to the spacecraft charging problem is to engineer the spacecraft components to withstand possible high potential differences and discharges, and to alter spacecraft designs so as to minimize differential charging, e.g., avoiding the use of ultra-high resistivity materials. Accurate and effective spacecraft designs, however, require a detailed description of the spacecraft plasma interactions. This requires both an understanding of the involved plasma processes, material properties, and reliable space weather forecasting (Garrett, 1979). Most aspects of the problem have received attention to date and the serious reader is referred to the proceedings of the 1977 and 1978 Spacecraft Charging Technology Conferences (see references for Garrett (1979) and Massaro (1977)).

Another approach that has been used with some success is active control of the spacecraft potential by the use of electron guns and electric thrusters (Bartlett and Purvis, 1978). With the thruster off, operation of the plasma
bridge neutralizer (the source of electrons which neutralize the ion beam) of the cesium ion thruster onboard ATS-6 was observed to raise the spacecraft potential from -8 kV to a few volts negative but did not fully alleviate differential charging. Operation of the main thruster clamped the spacecraft potential at approximately -5 volts (the potential differences between the ion beam and spacecraft) and eliminated differential charging (Olsen and Whipple, 1978).

Although most spacecraft charging problems have been encountered in GEO, spacecraft-plasma interactions can be a concern in all space environments. Reiff et al. (1980) have predicted 3% power losses and anomalous surface-to-plasma arcing for high voltage space power systems placed in the relatively cool but dense plasma at low earth orbit (LEO). Also, planetary exploration probes can expect to encounter plasmas hot and dense enough to cause charging problems, as in the Jovian magnetosphere for example.

The next decade will probably see the deployment of large high voltage power systems in space. Not only are plans for space based solar power for earth being considered, but the communications industry and the military will surely find a need for large amounts of energy for use in space, such as for particle beam and laser weapons. It can be expected that due to the large size, unusual shapes, and high voltages of these satellites, charging problems will be encountered that will be substantially different from those that have been observed and modeled for existing spacecraft.
From these projected needs comes the continued interest in the development of powerful, flexible models capable of reliably predicting satellite-plasma behavior.
2: SPACECRAFT CHARGING

The equilibrium potential of a body or a portion of the body surface is determined by the condition of constant charge, i.e.,

\[
\frac{dV}{dt} = \frac{1}{C} \frac{dQ}{dt} = 0,
\]

where \( C \) is the capacitance, \( Q \) is charge, \( V \) is the potential, and \( t \) is time. This is equivalent to a condition of zero net current between the body or surface and anywhere else (such as the plasma);

\[
\sum_S I_S = 0,
\]

where the sum is over all sources of current. Commonly important sources of current are plasma electrons and ions, photoelectrons, and other body generated particles such as secondary electrons. Before we can address the equilibrium potential problem, we must first understand how current collection for each source depends on the surface potential and the potentials of neighboring surfaces. However, if the plasma or the geometry is too complex, explicit expressions for the currents in terms of the surface potential may not be obtainable, and we must resort to computer models.

The variables, symbols, and constants used in this and other chapters are listed in appendix A.
In kinetic theory, a complete statistical description of a collection of $M$ particles is given by the $6M + 1$ dimensional distribution function, $F(\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_M, \dot{v}_1, \dot{v}_2, \ldots, \dot{v}_M, t)$, where \(\int F d\dot{x}_1 d\dot{x}_2 \ldots d\dot{x}_M d\dot{v}_1 \ldots d\dot{v}_M = 1\). This function gives the probability of all the particles having a particular set of the $6M$ phase space coordinates, i.e., the probability of finding the system in a particular state. By considering the evolution of the distribution of systems over the states contained in a volume of the $6M$ dimensional phase space, one can derive the Liouville equation (Reiff, pp. 626) for the total time derivative of $F$,

\[
\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \sum_{i=1}^{M} \left( \frac{\partial F}{\partial \dot{x}_i} \cdot \dot{v}_i + \frac{\partial F}{\partial \dot{v}_i} \cdot \dot{a}_i \right) = 0
\]

where $\dot{a}_i$ is the total acceleration due to collisions and all external and internal forces. This equation suggests a conclusion, known as Liouville's Theorem, that the total time derivative of the $6M + 1$ dimensional distribution function is constant along a system's path through phase space. In practice, one cannot hope to calculate a function like $F$, and using it would be equally difficult. The standard approach is therefore to attempt a reduction of $F$ and the Liouville equation to something more tractable. Thus we can define the single particle distribution function,

\[
f = M \int F d\dot{x}_2 \ldots d\dot{x}_M d\dot{v}_2 \ldots d\dot{v}_M,
\]
which may be considered to be the density of particles in a six-dimensional phase space. For a particle species \( s \), \( f_s \) has the property that the density and current density at a point \( \mathbf{x}' \) are given by the 0th and 1st velocity moment of \( f_s \):

\[
N_s(\mathbf{x}) = \int f_s'(\mathbf{x}', \mathbf{\dot{v}}') \, d^3\mathbf{\dot{v}}'
\]  

\[
\mathbf{J}_s(\mathbf{x}') = q_s \int f_s'(\mathbf{x}', \mathbf{\dot{v}}') \, \mathbf{\dot{v}}' \, d^3\mathbf{\dot{v}}',
\]  

(2-1)  

(2-2)

The reduction of the Liouville equation is more difficult because of \( \mathbf{\dot{\mathbf{x}}}_i \). This factor includes two-, three-, etc., particle interactions, thus preventing the integration of the Liouville equation by each coordinate independently. This problem is more severe for a plasma than for a neutral gas because of the long range of the Coulomb force. A great deal of theoretical effort has gone into determining when these interparticle interactions can be ignored, i.e., when is a plasma collisionless (Montgomery and Tidman, 1964)? It turns out that a plasma may be considered collisionless when the number of particles in a sphere with radius equal to the Debye length is much greater than unity. Thus for a "collisionless" plasma we may integrate the Liouville equation over \( 6(M - 1) \) phase space coordinates to obtain the colli-
sionless Boltzmann or Vlasov\(^\dagger\) equation,

\[
\frac{Df_s}{Dt} = \frac{\partial f_s}{\partial t} + \dot{v} \cdot \nabla f_s + \frac{q_s}{m_s} \left( \dot{\mathbf{x}} + \dot{v} \times \mathbf{B} \right) \cdot \nabla f_s = 0. \quad (2-3)
\]

This equation states that in six-dimensional phase space, \(f_s\) is constant along a particle's path, which connects \((\dot{x}, \dot{v}, t)\) with \((\dot{x}', \dot{v}', t')\), i.e.

\[f_s(\dot{x}', \dot{v}') = f_s(\dot{x}, \dot{v}).\]

In general, this path may be traced from some initial point in phase space using the equations of motion derivable from Hamilton's equations. For the time independent case, the trajectories may be characterized by certain constants of the motion such as the total energy,

\[H_s(\dot{x}, \dot{v}) = \frac{1}{2} m_s \dot{v}^2 + q_s V(\dot{x})\]

where \(q_s V(\dot{x})\) is the potential energy of the particle at \(\dot{x}\).

The six-dimensional phase space path projected onto the usual space coordinates is just the usual trajectory of a particle prescribed by Newtonian mechanics.

\(^\dagger\)The Vlasov equation represents the zeroth order terms in a cluster expansion of the Liouville equation, with smallness parameter \(g = (n_0 \lambda_d^{-3})^{-1}\), the inverse of the number of particles in a Debye sphere. For GEO under substorm condition \(n_e = 1/\text{cc}\), and \(kT_e \approx 10\ \text{kev}, \ g \approx 10^{-9}\), so the collisionless approximation is a very good one. However, in the F region with \(n = 10^6/\text{cc}\) and \(kT_e \approx .2\ \text{eV}, \ g \approx 10\) suggesting that the transport problem there needs a more detailed treatment.
Consider the trajectory connecting \( (\dot{x}', \dot{v}') \) with \( (\dot{x}, \dot{v}) \) for a given electrostatic field, where at \( \dot{x} \), the distribution of particles of species \( s \) is known to be \( f_s(\dot{x}, \dot{v}) \). We now limit our discussion to an isotropic plasma at \( \dot{x} \) where we further assume that \( f_s \) can be written as a function of only \( H(\dot{x}, \dot{v}) \). Since \( H(\dot{x}', \dot{v}') = H(\dot{x}, \dot{v}) \) we have

\[
 f_s(H(\dot{x}, \dot{v})) = f_s(H(\dot{x}', \dot{v}')). \tag{2-4}
\]

This result of the Vlasov equation provides us with a means of finding \( f \) anywhere by following particle trajectories, and we may now, in principle, proceed to evaluate the integrals in equations (2-1) and (2-2).

Note that some different value of \( \dot{v}' \) will map to a different point \( (\dot{x}_2, \dot{v}_2) \) where we might know the distribution function to be different (or zero). Thus, in evaluating the integrals in equations (2-1) and (2-2), equation (2-4) must be used to develop a composite expression for \( f' \). For example, consider the problem of a non-emitting body immersed in a Maxwellian plasma. At infinity, the distribution function in three-dimensions, for species \( s \) is,

\[
 f_s'(\dot{x}, \dot{v}) = N_{so}(\frac{m_s}{2\pi kT_s})^{3/2} \exp\left[ -\frac{H(\dot{v})}{kT_s} \right]. \tag{2-5}
\]

where \( N_{so} \) is the undisturbed density. At some point \( \dot{x}' \) near the body, the distribution function will be,
\[ f_s'(\dot{x'}, \dot{v'}) = N_{s0} \left( \frac{m_s}{2\pi kT_s} \right)^{3/2} \exp \left[ -\left( \frac{1}{2} \frac{m_s}{kT_s} \dot{v'}^2 + q_s V(\dot{x'}) \right) \right] / kT_s \]

\[ \times G_s(\dot{x'}, \dot{v'}) \]  

(2-6)

where \( G_s \) is a function with a value of either zero or one depending on whether \((\dot{x'}, \dot{v'})\) maps to a non-source or source at infinity. In other formulations, the \( G \) function is effectively replaced by reconstructing the limits of integration in equations (2-1) and (2-2).

As a simple example of the determination of the equilibrium potential of a body from the current balance condition, we can look at the problem of a uniformly conducting spherical body in a Maxwellian plasma where the only important sources of current are the plasma electrons and ions. To use equation (2-3) for this purpose we must be able to evaluate the function \( G(\dot{x'}, \dot{v'}) \). In even the simplest of problems this can be difficult because of the dependence of \( G \) on the global distribution of potential. Potentials can be determined from the Poisson equation,

\[ \nabla^2 V(\ddot{x}) = - \rho(\ddot{x}) / \varepsilon_0 \]  

(2-7)

where the charge density \( \rho(\ddot{x}) \) can be found with equations (2-1) and (2-6). However, this again requires knowledge of \( G \), which means that to rigorously solve any spacecraft charging problem, we must first solve the space-charge problem, i.e., the simultaneous and self-consistent determi-
nation of $V$ and $\rho$ everywhere. This region of perturbed $V$ and $\rho$ is known as the plasma sheath, which always has a polarity opposite that of the body, and thus screens remote regions of the plasma from the body potential. The traditional method for avoiding the space-charge problem is to use the time-honored sharp-edged sheath approximation due to Mott-Smith and Langmuir (1929); the results of which I will present next while putting the space-charge problem off until later chapters.

**The Repelled Particles**

The density of repelled particles, $N_r$, ($q_r V > 0$) is given by the combination of equations (2-1) and (2-6),

$$N_r(x) = N_{r0}(\frac{m_r}{2\pi kT_r})^{3/2} \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \exp[-(\frac{1}{2}m_r v^2$$

$$+ q_r V/kT_r] \times G(x, \hat{v}) \quad (2-8)$$

If $G = 1$ for all $\hat{v}$, then we have after performing the integrals,

$$N_r(x) = N_{r0} \exp[-q_r V/kT_r] \quad (2-9)$$

which is simply the Boltzman equilibrium condition (Reiff, 1965). This is usually a good approximation since $G$ would be zero for only those trajectories intersecting the body. These null trajectories span a limited solid angle due to
the repulsion, and all have energies greater than the body potential; so because of the exponential in (2-8), the erroneous inclusion of the trajectories in the integral causes minimal error. This approximation is common to most theories.

The current density at a point on the surface of a body is given by the dot product of the current density with the inward normal vector for that point on the surface. If we choose a coordinate system with the \( \hat{z} \) axis aligned with the outward surface normal, the collected surface current in \( J_z = \int f v_2 d^3\hat{r} \). For temporary convenience we can choose the spherical polar coordinate system of figure 5/2 where \( v_2 = v \cos \alpha \), and \( d^3\hat{r} = v^3 \sin \alpha \, d\alpha \, d\beta \). Combining equations (2-2) and (2-6), we have,

\[
J_r(\hat{x}) = qN_{ro} \left( \frac{m_r}{2\pi k T_R} \right)^{3/2} \int_0^\infty v^3 dv \int_0^\pi d\alpha \int_0^{2\pi} d\beta \\
\times \sin \alpha \cos \alpha \exp\left[\left(-\frac{1}{2}m_r v^2 + qV/k T_R\right)\right] G(\hat{x}, \hat{r})
\]

(2-10)

Making the approximation \( G = 1 \) for \( \alpha < \pi/2 \), this integrates to,

\[
J_r(\hat{x}) = J_{ro} \exp\left[-qV/k T_R\right],
\]

(2-11)

where \( J_{ro} \) is just the undisturbed current density,

\[
J_{ro} = qN_{ro} \left( \frac{k T_R}{2\pi m_r} \right)^{1/2}
\]

(2-12)
Note that the current density in equation (2-11) is just $J_{ro}$ multiplied by the Boltzman factor.

**The Attracted Particles**

The calculation of densities and currents due to the attracted plasma particles is significantly more difficult than for the repelled species. This is because for $\mathbf{x}$ near the body, many to most trajectories originating at $\mathbf{x}$ map back to the body instead of a source region, and therefore the function $G(\mathbf{x}, \mathbf{v})$ (or alternatively, the limits of integration) must be considered carefully. Mott-Smith and Langmuir (1926) have shown that the repelled densities and currents are independent of the plasma-sheath structure, but the opposite is true for the attracted species.

The approach of Mott-Smith and Langmuir (1926), is to assume a sharp sheath edge with radius, $s$, beyond which exists an undisturbed plasma. Currents are obtained by integrating the distribution function at the top of the sheath over limits (derived from conservation of energy and angular momentum considerations) that allow particles to reach the body surface. Their result for the total current of attracted particles, $I_a$, from a Maxwellian plasma to a spherical body of radius $R$, is,

$$I_a = 4\pi s^2 J_{or} \left[ 1 - \frac{s^2 - R^2}{s^2} \exp\left[\frac{R^2 q_a V}{kT_a} (s^2 - R^2)\right]\right] \quad (2-13)$$
Since the sheath radius cannot be found without solving the space-charge problem, the two commonly used limits of equation (2-13) are: the thin sheath limit,

\[ I_a = J_\text{or} \ 4\pi s^2, \quad s = R \]  \hspace{1cm} (2-14)

and the thick sheath or orbit limited case,

\[ I_a = J_\text{or} \ 4\pi R^2 \left(1 - \frac{q_a V}{kT_a}\right) \]  \hspace{1cm} (2-15)

To continue our example problem of finding the equilibrium potential of a body in a Maxwellian plasma, I assume conditions where the thin sheath limit is appropriate and that the equilibrium potential will be negative. This means that equation (2-12) should be used for the electrons, and (2-14) should be used for the ions. Denoting the respective currents by \( I_e \) and \( I_i \), and applying the current balance condition, we have,

\[ 4\pi s^2 J_{oi} = +4\pi R^2 J_{oe} \exp\left(\frac{eV}{kT_e}\right) \]  \hspace{1cm} (2-16)

where \( e \) is the magnitude of the electric charge. Solving for \( V \) and substituting for \( J_{oe} \) and \( J_{oi} \) from equation (2-12) I get,

\[ V = -\frac{kT_e}{Ze} \ln \left[ \frac{T_e m_i R^4}{T_i m_e s^4} \right] \]  \hspace{1cm} (2-17)
If we set $s = R$, we get Chopra's (1961) equation for the potential.

For conditions where $s > R$, we must somehow find $s$ and then use equation (2-13) for the ion current. Nevertheless, the potential given by (2-17) provides a good estimate for the potential of a satellite in the dark. If the satellite is illuminated, photoelectrons will be generated, some of which will not return, therefore raising the equilibrium potential by an amount that depends on the production rate and the escape probability. The production rate depends on the emission efficiency and the illumination, and there is no simple way to determine the fraction of photoelectrons that will escape the satellite. Whipple (1976) has shown that photoelectrons can create their own space charge barrier returning lower energy photoelectrons. This possibility has been confirmed by the studies of Guernsey and Fu (1970) and Fu (1971). Finally, the consideration of secondary electrons with dependence on surface properties, magnetic fields and satellite motion makes the problem even more intractable. Whipple (1965) provides an exhaustive study of the problem of a spherical conductor in a flowing plasma and considers all of the complications just mentioned.

Because of the need to account for as many details as possible, and the geometric asymmetries of real spacecraft, most recent efforts have been towards the development of computer models. There are perhaps two broad classes of charging models: spacecraft oriented, attempting to resolve
details of spacecraft geometry; and plasma oriented, attempting a self-consistent solution to the plasma problem. Of course, a complete description of the charging phenomenon would be included in both classes, but no such comprehensive treatment exists so far to date. Parts of the following review are taken from Whipple (1977) and Parker (1976).

One example of a spacecraft oriented model is that of Massaro et al. (1977). They represented the plasma fluxes to elements of the spacecraft surface by equations identical to equations (2-11) and (2-14) (thin sheath approximation). Photoelectrons and secondaries were also included. These equations served as current sources to a detailed three-dimensional spacecraft model, resulting in an equivalent circuit which predicted time dependent surface potentials. Perhaps the most important general result of this study was that potentials were always highest in the steady state. A similar approach was used by Robinson (1977) to predict the differential charging behavior of the Pioneer-Venus orbiter, and by Inouye (1976) to model the DSCS II satellite (geosynchronous). The three-dimensional NASCAP program (Katz et al., 1978) is probably the closest to being in both categories. NASCAP neglects plasma space charge effects but includes an approximation for photoelectron space charge, and calculates fluxes to the spacecraft by tracing particle trajectories. Non-linear bulk properties of materials are included in the equivalent circuit model, as well as a model for discharges in dielectrics. NASCAP has been primarily
written to model the interaction of the SCATHA (spacecraft charging at high altitudes) satellite with GEO plasma environment. SCATHA was launched on January 1, 1979, but to my knowledge there have been no published comparisons between data and NASCAP predictions.

The spacecraft charging models just reviewed all used approximate plasma models. This approach is perhaps dictated by engineering needs of immediate estimates, and by the expense in computer time of adding detailed plasma models.

Plasma models fall roughly into two categories: particle pushers, and direct integration models. Particle pushers track typically a few thousand particles simultaneously across a grid of cells in small time increments. Densities and currents are obtained by counting the particles in each cell. This is a popular approach because of the ability to study time dependent problems. Rothwell et al. (1976) have used this method to simulate the plasma sheath surrounding a charged satellite. Their model injects weighted computer particles at a sheath edge (defined to be where \( V = kT/e \)) and follows their evolution to the satellite. They restricted their study to spherical symmetry and have obtained good agreement with orbit limited (thick sheath) Langmuir theory. In later work (Rothwell, 1977) has observed fluctuations near the electron plasma frequency that exhibit Landau damping when the high energy tail of the electron distribution function is appropriately enhanced. She cautions however, that the possibility that the oscilla-
tions are computer related had not been fully investigated. This approach has also been used by DeGroot (1977) to investigate dynamic current interruption and double layers in magnetized plasmas.

If time dependence is not needed, a great deal of computer storage and time can be saved by following complete trajectories and recording the "time" spent in each cell to obtain densities and currents. This variation has been studied by Parker (1964) and labeled the "outside-in" method because trajectories are started at the outside boundary and followed inside. The main disadvantage of this method and particle pushers in general is that it is difficult to choose trajectories that will map to some regions of interest.

In the special case where trajectories do not cross or reverse directions, the "flux tube" method of choosing trajectories may be used. Applications of this method are generally limited to objects in cold flowing plasmas. The technique is developed in detail by Parker (1964), and applications are reviewed by Whipple (1977).

A direct integration method obtains densities and currents by integrating distribution functions at the various grid points. This approach is common to many one-dimensional (1-D) kinetic theories such as the BGK theory of electrostatic waves (Bernstein et al., 1957). For a three-dimensional numerical model, the integrations are performed by tracing trajectories in reverse to either source or non-
source boundaries. This technique is due to Parker (1964) who dubbed it the "inside-out" method. Since this is the method that I have used in my research, it will be described in detail in subsequent chapters.

The inside-out method has been used by Parker and Whipple (1967, 1970) to model two-electrode probes on a satellite, but they did not self-consistently solve for the sheath potential distribution. In Parker (1970) the method was used to study the collection of ions by a rocket-borne mass spectrometer with an attractive orifice, and in Parker (1973) it was used for the problem of a small probe in the wake of large electrode. More recently Parker (1976, 1972) has calculated sheath and wake structures about disk and pill-box shaped objects in flowing plasmas. Fournier (1971) and Grabowski and Fischer (1975) have used the inside-out method to calculate the wake of an infinitely-long moving cylinder, and Liu and Hung (1968) used it to predict wave-like behavior in the far wake zone of a satellite. It was also used by Taylor (1967) for the wake of an infinitely-long cylinder of rectangular cross-section, but the calculation was not carried beyond the first iteration. So far, applications of the method have been limited to models with one and two-dimensional symmetries. Our (Parker and Cooke) program PANEL represents the first use of the method for a fully self-consistent three-dimensional model.
3: THE SOLAR POWER SATELLITE

Peter Glaser (1968) is usually credited for introducing the solar power satellite (SPS) concept. In recent years, various organizations including NASA have begun to seriously investigate the feasibility of the SPS. A number of power conversion schemes have been considered but the two leading designs, proposed by NASA/JSC/Boeing and NASA/MSFC/Rockwell International (Hanley, 1978), both call for about 75 km$^2$ of photo-voltaic cell arrays, operated at voltages in excess of 40 kV, producing $5 \times 10^9$ watts of power that is to be beamed to Earth via microwave. Both designs have reached the definition stage and are currently being reviewed.

John Freeman and Arthur Few (1979) of Rice University, under contract to Marshall Space Flight Center, have reviewed the MSFC/Rockwell design (fig. 3/1) for environmental impact. The two main areas of concern were the protection of the rectenna (microwave receiver-antenna) from lightning damage and the interaction of the SPS with the GEO environment (Reiff et al., 1980).

The SPS-GEO plasma interaction represents a somewhat different problem than the usual spacecraft charging problem. Four major differences are:

1) The large size; The Debye length at GEO under substorm conditions is on the order of a kilometer. This is much larger than any conventional satellite but only a fraction of typical SPS dimensions.
2) High voltage; To date, the highest used power system voltage has been the 100 V system on Skylab. In contrast, the MSFC/Rockwell design calls for solar cell string potential differences of 45 kV.

3) Planar geometry; Almost all charging models for conventional spacecraft have been able to assume spherical or cylindrical symmetry, and monopole behavior at reasonable distances. For the SPS, the geometry and voltages will invalidate both approximations.

4) Also, the presence of ion thrusters for attitude control will be a complication previously experienced with only the GEO satellites, ATS-5 and ATS-6.

Foremost among concerns for the SPS is the amount of power that will be lost in driving currents through the ambient plasma and the anticipated photoelectron sheath. Our study (Freeman et al., 1979) concluded that the principal parasitic load will be due to photoelectron currents to and from the reflectors which we calculated to be 0.7% of the array output power. Parker (1978a) gives 0.1% for an idealized array under similar conditions. Reiff et al. (1980) and Parker and Oran (1978b) have studied the possibility of using the satellite currents to magnetically shield the SPS from the ambient plasma. The studies considered different winding schemes, both concluded that a significant improvement in shielding could be obtained if needed. However, it is doubtful that this approach would lend much aid to the photoelectron problem, and the materials for the
windings would certainly add to the cost of any SPS project.

A related problem is that of determining the ground point for the system, i.e., how much of the solar cell string would be above and below the space potential if the string were not electrically connected to other parts of the satellite. Although it has not been proven, it is felt that if the solar cell string can be connected to the rest of the satellite such that no currents flow between the cells and the structure, power loss will be minimized. This would be especially important for the MSFC/Rockwell design which calls for reflectors of greater surface area than the solar-cell arrays. Parker (1978a) has estimated the voltage distribution of a rectangular array with a linear voltage gradient along the full length. Using the thick sheath approximation and the zero net current criterion for a plasma with equal ion and electron temperatures, he gives the relation,

\[ \lambda = \frac{L_2}{L_1} = \left( \frac{\text{me}}{\text{m}_i} \right)^{1/4} \cdot \text{RAM}^{1/2} \]  

(3-1)

where \(L_1\) = the length below the plasma potential,
\(L_2\) = the length above the plasma potential,
RAM = \exp \left( -M^2 \right) + \sqrt{\pi} \ M \ (1 + \text{erf} \ (M)),
M = the ion mach number,
and \(\text{erf} \ ()\) is the error function. If \(V_p\) is the end to end potential difference, a linear distribution of potential gives for the negative end potential,
\[ V^- = \frac{L_1 V_p}{L_1 + L_2} = \frac{V_p}{1 + \frac{\lambda}{\kappa}} \quad (3-2) \]

and for the positive end potential,

\[ V^+ = \frac{\lambda V_p}{1 + \frac{\lambda}{\kappa}}. \quad (3-3) \]

At GEO with \( \text{RAM} = 1 \) (\( M = 0 \)), he predicts for a proton plasma, \( \lambda = 0.15 \). For LEO, using a thin sheath approximation, he gives

\[ \lambda = \left( \frac{m_e}{m_i} \right)^{1/2} \cdot \text{RAM} \quad (3-4) \]

Thus for \( O^+ \) ions and \( \text{RAM} = 23 \) (\( M = 6.4 \)), \( \lambda = 0.13 \). Parker also comments that the consideration of photoelectrons might raise these ratios to near unity. Our own study of the MSFC/Rockwell design (Freeman et al., 1979) produced similar results with the following argument. Make the assumptions that: the negative area \( A^- \), collects current from the plasma ions \( J_i \), and from the loss of photoelectrons \( J_{ph^-} \); the positive area \( A^+ \), collects current from the plasma electrons \( J_e \), and from the negative end photoelectrons \( J_{ph^-} = J_{ph^+} \). We further assume that a thin sheath approximation is adequate because the nearby reflectors and the alternating voltage pattern will limit current collection from the plasma to less than that predicted by a thick sheath assumption. For a worst case, we used the plasma conditions, \( N_i = N_e = 2/\text{cc}, T_i = 10 \text{ keV}, T_e = 5 \text{ keV} \). For
these conditions, the random thermal current densities are: $J_{oi} = 1.25 \times 10^{-7} \text{ A/m}^2$, $J_{oe} = 3.79 \times 10^{-6} \text{ A/m}^2$. Arguing that both the front and back sides of the panel will collect plasma currents while only the front will emit and collect the photoelectron current, the current balance condition gives:

$$\lambda = \frac{J_{ph} + 2J_e}{2J_i} = \frac{A^-}{A^+} \quad (3-5)$$

The thin sheath approximation gives $J_e = J_{oe}$, and $J_i = J_{oi}$; and taking the photoelectron current to be $J_{ph} = 3 \times 10^{-5} \text{ A/m}^2$, we concluded $\lambda = 1.17$. With this result I modeled a section of the MSFC design using a early version of PANEL that did not include space charge effects. These results are reproduced in figures 3/2, 3/3, and 3/4. The average electron current to the positive panel was calculated to be $J_e = 7 \times 10^{-6} \text{ amp/m}^2$, and the average negative end ion current (protons) was $J_i = 2 \times 10^{-7} \text{ amp/m}^2$. Ignoring space charge effects is equivalent to an infinite sheath approximation so these computer results were viewed as supporting our limited sheath = thin sheath assumption. Photoelectrons were not included in the PANEL calculation.

Since there are presently no high voltage satellites in orbit, there can be no in situ testing of these predictions. For lack of such, there are active programs to conduct vacuum chamber tests of high voltage system models and components. The most notable is that of McCoy and Konradi
(1978) who have simulated the behavior of a large solar array in LEO using the NASA/JSC 20 meter vacuum chamber, chamber A. The simulation consists of a 1 m x 10 m panel biased at voltages to ± 3 kV in an Argon plasma \((10^3-10^6/\text{cc})\) produced by a Kaufman ion thruster. The panel is constructed of a conductive plastic (Velostat) mounted on a dielectric panel of Lexan, leaving a 10 cm exposed border along all edges. Using the thin sheath approximation, the current balance criterion for their plasma predicts \(\lambda = 0.023\), and the value observed for the floating panel was \(\lambda = 0.025 \pm 0.002\). One unexpected effect was a lens action by the plasma when the panel was operated at negative voltages, resulting in a strong focusing of ions towards the center of the panel near the ends. Over an order of magnitude variation in the ion currents across the panel face was observed.

Stevens (1980) at NASA–Lewis Research Center has also been experimentally studying various aspects of the coupling of high voltage systems and plasmas. Of particular interest is his investigation into the current collection properties of solar cell arrays. He finds that an array of quartz covered solar cells with exposed interconnects will collect an electron current proportional to the interconnect area at solar cell bias voltages less than 100 volts positive. At these voltages the quartz will maintain the usual slightly negative equilibrium potential. For positive voltages above 100 V, he observes a sudden "snap-over" effect where the interconnect fields appear to engulf the quartz cover
slides, and at higher voltages the electron collection is proportional to the total array area as if it were entirely conducting. Current collection for negative bias voltages was proportional to the interconnect area until the fields built up to a point where discharges occurred. Breakdown occurred at $-600\ \text{V}$ for $N_e = 10^4/\text{cc}$, and at $-750\ \text{V}$ for $N_e = 10^3/\text{cc}$, showing a dependence on the plasma density. McCoy and Konradi (1978) have also produced similar surface to plasma arcing for positive voltages over $+400\ \text{V}$ and negative voltages over $-1\ \text{kV}$. Also, tests at JSC using a stainless steel panel have shown drastically increased leakage currents at negative voltages upon covering more than 90% of the surface with mylar tape. This insulation also lowered the onset voltage for the arcing problem (McCoy, 1980).

This experimental approach to anticipating the plasma interaction problems of solar power systems in LEO has so far been the most fruitful. However, it is quite expensive and the extrapolation and scaling of empirical results to other conditions is difficult, thus implying a need for accurate theoretical models. On the other hand, any theoretical model will involve some degree of approximation and may neglect important interactions. Thus, as is common in science, the symbiosis of theory and experiment will produce the greatest insight into the SPS-plasma interaction problem.
4: THE PLASMA SHEATH

Perhaps the best known example of plasma screening is the Debye treatment of the plasma screening of an isolated test charge. A positive test charge, $\delta Q$, placed in a plasma of temperature $T$, will attract electrons and repel ions so as to develop a surrounding sheath with a potential distribution given by,

$$V(r) = \frac{\delta Q}{4\pi \varepsilon_0 r} \exp\left(-\frac{r}{\lambda_D}\right)$$

where $\lambda_D = (\varepsilon_0 kT/N_e e^2)^{1/2}$ is the Debye length. Implicit in the derivation of this equation (Jackson, 1962) are the assumptions that the charge has negligible cross-section, and that $V(r) \ll kT/e$ for $r > \lambda_D$. For a microscopic body of radius $R$, satisfying these assumptions, we can write

$$V(r) = \frac{V_b R}{r} \exp\left[\frac{(R-r)/\lambda_d}{\lambda_d}\right], \quad r > R \quad (4-1)$$

where $V_b$ is the surface potential of the body.

For macroscopic bodies with a high degree of symmetry, sheath structures may be calculated by capitalizing on the constants of the motion allowed by the symmetry, e.g., angular momentum (Whipple, 1977). In general though, self-consistent treatment of a macroscopic body requires computer modeling. In spite of this difficulty, a better understanding of the shielding process can be gained by studying current limiting by space charge in the 1-D planar electron
diode.

The first theoretical treatments of the electron diode were published independently by Child (1911) and Langmuir (1913). Variations on this problem have been studied by Fay et al. (1938), and the general topic of space charge effects in vacuum tubes is treated in the book by Birdsall and Bridges (1966).

Consider the three electrode system shown in Figure 4/1. At \( x = -d \), we have a cathode, with zero potential capable of emitting unlimited quantities of electrons all with zero velocity. At \( x = 0 \), we have a transparent screen at potential \( V_0 \), and at \( x = x_1 \), we have a non-emitting anode at potential \( V_1 \). The kinetic energy of an electron at \( x \) is

\[
\frac{1}{2} m_e v^2 = eV. \quad (4-2)
\]

The current density for electrons at \( x \) is

\[
J = \rho v \quad (4-3)
\]

where \( \rho \) is the charge density. Poisson's equation in one dimension is

\[
\frac{d^2 V}{dx^2} = -\frac{\rho}{\varepsilon_0}. \quad (4-4)
\]

Substituting for \( \rho \) from equation (2-3) and then for \( v \) from equation (4-1), we have
\[
\frac{d^2V}{dx^2} = - \frac{J}{\varepsilon_0} \left( \frac{me}{2eV} \right)^{1/2}.
\] (4-4)

Multiply this by \(2\frac{dV}{dx}\) and integrate from \((0, V_0)\) to \((x, V_x)\);

\[
\left( \frac{dV}{dx} \right)^2 \mid_0^x = - \frac{J}{b^2} \cdot \frac{16}{9} \left( V_x^{1/2} - V_0^{1/2} \right)
\] (4-5)

where \(b^2 = \frac{4\epsilon_0}{9} \left( \frac{2e}{me} \right)^{1/2} = 2.336 \times 10^{-6} \) (amps/volts\(^{3/2}\)).

The boundary condition that we desire at \(x = 0\), is \(E_x = -\frac{dV}{dx} \mid_0^0 = 0\), a common definition of the sheath edge in the spacecraft charging problem. Using this condition, and taking the positive square root of equation (4-5),

\[
\frac{dV}{dx} = \frac{4}{3b} \left[ J_e \left( V_x^{1/2} - V_0^{1/2} \right) \right]^{1/2}
\] (4-6)

where \(J_e = -J\). Solving for \(dx\) we integrate again from \((0, V_0)\) to \((x_1, V_1)\);

\[
x_1 = \frac{b}{J_e} V_1^{3/4} \left[ 1 + 2 \left( \frac{V_0}{V_1} \right)^{1/2} \right] \cdot \left[ 1 - \left( \frac{V_0}{V_1} \right)^{1/2} \right]^{1/2}.
\] (4-7)

This equation can be applied to region I, \(-d < x < 0\), where \(V_0 = 0\) (zero initial velocity) to recover the Child-Langmuir (C-L) result

\[
d^2 = 2.336 \times 10^{-6} \frac{V^{3/2}}{J_e},
\] (4-8)

where \(d^2\) and \(J_e\) must have the same unit of area. If \(d\) and \(V\) are fixed, equation (4-8) gives the maximum conducted
current despite an unlimited supply of electrons. If \( d, V, \) and \( J_e \) are all considered independent, the sheath edge electric field that was set to zero will become the dependent variable.

To apply equation (4-7) to a planar spacecraft surface at potential \( V \), we identify \( x_1 \) with the spacecraft surface, and \( x = 0 \) with the sheath edge where \( E = 0 \). Region I is now identified with the undisturbed plasma where \( V_o \) represents the average thermal energy of the electrons. Therefore, a transformation of voltage is needed because we want the sheath edge to be at zero potential; thus, \( V_1 + V + V_o \)

\[
\psi = \frac{V}{V_o} = \frac{V_1}{V_o} - \frac{V_o}{V_o} = \frac{V}{kT_e},
\]

and

\[
\frac{V_o}{V_1} = (\psi + 1)^{-1}.
\]  

(4-9)

Substituting this last result into equation (4-7) we get for the sheath thickness \( a \),

\[
x = a = b \left( \frac{V^{3/4}}{\sqrt{J_e}} \right) \cdot S(\psi)
\]

(4-10)

\[
S(\psi) = \{(1 + \psi^{-1})^{3/4} \cdot [(1 + 2(1 + \psi)^{-1/2})
\]

\[
\cdot (1 - (1 + \psi)^{-1/2})^{1/2}\}\}.
\]

(4-11)
Equation (4-10) is written such that $S(\psi)$ is the correction to the usual Child-Langmuir result due to non-zero initial velocities. Equation (4-11) is plotted in Figure 4/2 as a function of $\log(\psi^{-1})$. When applied to a Maxwellian plasma, $S(\psi)$ will be only qualitatively correct since there will be a distribution of initial velocities, but it should be reasonably accurate for larger values of $\psi$.

Another variation of this problem is given by Birdsall (1966). The conditions are illustrated in the lower portion of Figure 4/1, with the grids at $x = 0$ and $x_1$ both at the same positive potential $V_1$, and the separation distance $x_1$ considered fixed. The negative space charge of the electrons in the gap between zero and $x_1$ will depress the potential in the gap and give rise to current limitation if the potential drops to zero. This variant is more suited for comparison to PANEL, since the geometry is fixed and only voltages and charge densities vary. The potential distribution in the gap is determined by subdividing region II into regions A and B whose boundary at $x_m$ is the point of minimum potential where we have also the condition of zero electric field. The potential can be obtained separately in each region with exactly the same approach that led to equation (4-7) to give
\[ b[(2 - f)J]^{-1/2} \cdot (V_A^{1/2} + 2V_m^{1/2}) \cdot (V_A^{1/2} - V_m^{1/2})^{1/2} = x_m - x_A \]  
\[ (4-12) \]

\[ b(fJ)^{-1/2} \cdot (V_B^{1/2} + 2V_m^{1/2}) \cdot (V_B^{1/2} - V_m^{1/2})^{1/2} = x_B - x_m \]

The factor, \( f \), is the fraction of transmitted current; if \( V_m > 0 \), \( f = 1 \). Equations (4-12) may be solved for \( x_m \) by setting \((V_A, x_A) = (V_1, 0)\) and \((V_B, x_B) = (V_1, x_1)\). For the case \( V_m > 0 \) and \( f = 1 \), we find \( x_m = x_1/2 \), and equations (4-12) can be combined to give Birdsall's equation:

\[ (\phi^{1/2} - \phi_m^{1/2}) \left( \phi^{1/2} + 2\phi_m^{1/2} \right)^2 = \beta (\xi - 1/2)^2 \]  
\[ (4-13) \]

where \( \phi = V/V_1 \), \( \phi_m = V_m/V_0 \) and \( \xi = x/x_1 \). The dimensionless current \( \beta \), is defined by \( \beta = J_e/b^2 \cdot V_1^{3/2} \cdot x_1^{-2} \), where the normalizing current is C-L current for a diode with separation \( x_1 \), and potential drop \( V_1 \). The value of the minimum potential for a given current is found by evaluating equation (4-13) at \( x = 0 \), giving

\[ 4\phi_m^{3/2} - 3\phi_m^{1/2} + (\frac{\beta}{4} - 1) = 0. \]  
\[ (4-14) \]

Figure 4/3a (from Birdsall) gives \( \phi_m \) as a function of input current \( \beta \). The dotted portion of the curve, has been labeled the "C-overlap" by Fay et al. (1938) and has been shown to contain no stable solutions (Birdsall, 1966). Figure 4/3b (also from Birdsall) gives selected potential profiles from equation (4-13) for the range \( 0 < \beta < 8 \).
For solutions with $\beta > 8$ we set $\xi_m = 0$ in equation (4-12). Allowing now for $f < 1$ and a non-symmetric potential distribution, we can derive Birdsall's dimensionless equations:

\[
\phi_A^{3/2} = (2 - f) \beta (\xi - \xi_m)^2
\]  
\[\phi_B^{3/2} = f\beta (\xi - \xi_m)^2,\]  
\[\xi_m = \frac{[f (2 - f)]^{1/2} - f}{2(1 - f)},\]  
\[\beta = 2 \left[ \frac{[f (2 - f)]^{1/2} + 1}{f (2 - f)} \right] \]  
\[(4-15)\]  
\[(4-16)\]  
\[(4-17)\]

where $\xi_m = x_m/x_1$. Figure 4/3c (also borrowed from Birdsall), shows a few selected potential profiles from equations (4-15).

The multiplicity of solutions in the region $4 < \beta < 8$ means that there should be hysteresis in the behavior of the classical diode model. This hysteresis was observed experimentally by Gill (1925). He also observed the predicted current limiting. It should be mentioned that although the condition $V_m = 0$ does lead to mathematical solutions for the range $4 < \beta < \infty$ these solutions are a result of the enforced time-independence of the classical method and can be shown to be unstable when time dependence is considered (Birdsall, 1966). This lack of stability has been confirmed by the experiments of Salzberg and Haeff (1938).
These solutions of what I call the gap problem have been very useful in the development of PANEL. PANEL's predictions for the gap model will be presented in section 4.

Returning now to the spacecraft plasma sheath problem; for appropriate conditions, it may be possible to use the C-L sheath thickness given in equation (4-8) as an estimate of satellite sheath thickness for the sharp-edge sheath model (the accuracy of a C-L sheath model is the subject of this and other current research). These appropriate conditions are:

1. Satellite dimensions should be much larger than all estimates of sheath thickness, such that a planar approximation is justified.

2. The surface potential is much greater than the plasma temperature, so the initial velocities of particles entering the sheath can be neglected [or accurately accounted for by equation (4-10)]. Also repelled particles must not penetrate significantly into the sheath since the C-L treatment considers only attracted particles.

3. The current is assumed to be the random thermal current \( J_0 = N_0 e \sqrt{kT/2\pi m} \) of attracted particles falling on the sheath edge.

For diode geometries other than planar, Langmuir (1913) has shown that the space charge limited current will always be proportional to \( V^{3/2} \), however, the distribution of potential in space does depend on geometry. The problems of cur-
rent flow between concentric spheres and cylinders has been addressed by Langmuir and Blodgett (1924). Their solutions take the form of equation (4-8) with \(d\) replaced by various series expansions in terms of the ratios of the electrodes' radii, with the results presented in tabular form. Parker (1979) has adapted these results to estimate sheath thickness for charged spherical satellites, and provides the convenient fit to those results presented in equation (4-18). In the following equations \(a = \) sheath radius, \(r_0 = \) body radius, and \(d\) is the C-L screening distance given by equation (4-10) with \(\psi = \infty\), or by equation (4-20) below:

\[
\frac{1}{2} + \left[ \frac{1}{4} + \frac{d}{r_0} \right]^{1/2} \quad ; \quad \frac{d}{r_0} < 0.2
\]

\[
\frac{a}{r_0} = \frac{1}{2} + \left[ \frac{1}{4} + \frac{d}{r_0} \right]^{1/2} + 0.052 \frac{d}{r_0} \quad ; \quad 0.2 < \frac{d}{r_0} < 19
\]

\[
\left[ 1 + \left( \frac{d}{r_0} \right)^{7.53} \right]^{7.524} = \left( \frac{d}{r_0} \right)^{5.67} \quad ; \quad \frac{d}{r_0} > 19
\]

(4-18)

It is interesting to stop at this point and compare screening length estimates. On one hand, we have the Debye length for the microscopic case with low potentials

\[
\lambda_d = \left[ \frac{e^2}{N_0 e^2} \right]^{1/2} \text{meters} \quad (4-19)
\]

and, on the other hand, the planar C-L screening length
\[ d_{cl} = \frac{2}{3} \left( \frac{4\pi\varepsilon_0 e^2}{N_0^2 kT} \right)^{1/4} V^{3/4} = 9.34 \left( T(\text{ev}) \right)^{-1/4} \]

where \( N_0 \) is the density in \( \text{cm}^{-3} \), \( V \) is the potential, and \( T \) is the temperature in \( \text{ev} \).

\[ N_0(\text{cm}^{-3})^{-1/2} V^{3/4} \text{ meters} \quad (4-20) \]

where I have substituted for the current the thermal current \( J_0 \). Note the difference in the temperature and density dependencies; \( (T/N)^{1/2} \) as opposed to \( (N^2 T)^{-1/4} \). A brief example will help demonstrate the inappropriateness of applying the Debye model to large objects. Consider an object of radius \( r_0 = 10 \text{m} \), at a potential of 100V in a plasma with \( T_e = 1 \text{ ev} \), and \( N = 100/\text{cc} \). For these conditions, the C-L distance is by (4-20), \( d = 29.5 \text{ m} \), and from (4-18) we have \( a = 2.4 r_0 = 24 \text{m} \); whereas with a Debye length of \( 0.74 \text{ m} \), equation (4-1) predicts \( a = 13 \text{ m} \) for a sheath edge potential of one volt. The Debye model predicts screening on the order of a few Debye lengths which significantly underestimates the sheath thickness.

The planar, C-L/sharp-edge sheath model has its own shortcomings. In applying it to space conditions one assumes that the screening length will be small compared to surface dimensions so that the surface can be approximated as infinite. But this is incompatible with the assumed boundary condition of an undisturbed plasma one screening length away from the surface. Since repelled particles will be reflected at the sheath edge, the repelled particle distribution will be nearly isotropic. On the other hand, the attracted particles come to the edge from one direction.
only, resulting in only a hemispherical distribution out to a point where the surface no longer appears infinite. Such a plasma cannot even be quasi-neutral unless the potential at the "sheath edge" is significantly non-zero, which would be contrary to the original conditions of the C-L model. This difficulty is not as severe for the cylindrical and spherical extensions. This situation can be improved somewhat if we relax the definition of the sheath edge to require only that the electric field be near zero there, and let the potential deviate from zero (closer to the surface potential) so as to draw in more attracted particles and reduce the repelled particle density. This in turn requires that we invoke a presheath region to match the sheath edge to regions where the attracted and repelled distributions are identical, and the potential and electric fields vanish.

This presheath problem was recognized and accounted for by Langmuir (1929) in his analysis of a plasma discharge between plane parallel electrodes, however the resolution of that problem is too involved for presentation here, and not directly applicable to the problem of a planar satellite in a collisionless plasma. The presheath problem has also been studied by Parker (1980) for an extremely large spherical body (large compared to all estimates of sheath thickness) in a collisionless plasma, using a technique similar to the inside-out method but solving a condition of quasineutrality instead of Poisson's equation. The sheath was modeled as a potential discontinuity at the surface. Results indicate
that no matter how thin the sheath gets, the presheath thickness will always be comparable to body dimensions (as one would expect from geometrical shadowing considerations). In the presheath region the potential drop is of order $kT/e$ and the ion and electron densities are essentially equal, but reduced from ambient values. One-dimensional theory predicts that such a body will collect a current density of attracted particles equal to the random thermal density current, but a conclusion of Parker's study was that the collected current density will be increased in the presheath region by a factor dependent upon the body potential but approaching a limiting value of 1.45 for infinite body potential, independent of body shape.

Another complication is presented by secondary and photoelectron emission. The extent to which these additional sources will modify a plasma sheath is of course dependent upon the emission flux. Guernsey and Fu (1970) and Fu (1971), have studied the case characterized by $(N_u/N_e) > (T_e/T_u) > 1$ where the u subscript refers to the photo or secondary electrons, and the e for the plasma electrons. We would expect these conditions to lead to a positive surface potential of a few volts and a monotonic decrease to zero with increasing distance from the surface. Their study confirmed that solution, but also revealed the existence of another non-monotonic type with a negative "overshoot" potential minimum located about one photoelectron Debye length from the surface. This overshoot was also
accompanied by a lowering of the equilibrium surface potential by an amount roughly equal to the overshoot potential. Without a time dependent analysis one cannot decide which solution is the true steady-state, but energy considerations suggested that the non-monotonic solution is the true steady-state when \( \left( \frac{N_e}{N_0} \right) \left( \frac{H_u}{4\pi kT} \right) < 1 \), where \( H_u \) is the peak in the photoelectron kinetic energy spectrum.
5: THE INSIDE-OUT METHOD

The classical theory of electrodynamics states that the scalar electrostatic potential \( V(\mathbf{x}) \) and the charge density \( \rho(V(\mathbf{x})) \) will satisfy Poisson's equation

\[
\nabla^2 V(\mathbf{x}) = -\rho(V(\mathbf{x}))/\varepsilon_0. \tag{5-1}
\]

In problems where the charge density does not depend upon the potential, equation (5-1) becomes an inhomogeneous linear elliptic partial differential equation. For such equations, the theory of partial differential equations (Jackson, 1962), will guarantee a unique solution interior to a closed boundary \( S \), on which is specified either (but not both) the potential \( V(x_s) \) (Dirichlet boundary conditions), or the normal derivative \( \partial V(x_s)/\partial n_s \) (Neuman boundary conditions). Unique solutions may also be obtained for problems with mixed boundary conditions with Dirichlet conditions on part of the boundary, and Neuman for the rest. For the general non-linear problem where the charge density depends on the distribution of potential, there are no uniqueness or existence guarantees for solutions to equation (5-1). Experience, however, leads us to believe that the physically real problems that we encounter in the study of plasma screening do have at least one self-consistent solution for \( V \) and \( \rho \). It is this experience that leads us to pursue solutions to such problems.
The inside-out method adopts an iterative approach to solving plasma sheath problems. The best estimate for $\rho(\hat{x})$ is used in equation (5-1) to obtain a new estimate for $V(\hat{x})$. Next, new estimates for $\rho(\hat{x})$ are obtained via the Vlasov equation and the latest values for $V(\hat{x})$ and the process is repeated. The calculation of $\rho(\hat{x})$ has been labeled the "Vlasov problem" and the problem of finding solutions to (5-1) is called the "Poisson problem".

PANEL has the feature of being able to operate in both a two-dimensional and a three-dimensional mode. The three-dimensional version is presented first, and the conversions to two-dimensional operation are found in Appendix C.

**The Poisson Problem**

With $\rho(\hat{x})$ temporarily considered known and independent of $V(\hat{x})$, equation (5-1) becomes linear, thus a well posed boundary value problem will have a unique solution. PANEL uses a standard finite-difference method to solve Poisson's equation (Collatz, 1960). The approach is to discretize the space to be modeled by constructing a three-dimensional grid of points $P_{i,j,k}$. An x-y plane at constant z is illustrated in figure 5/1. The standard approach is to let the x, y, and z spacings all be a constant h so that there is a cube of volume $h^3$ associated with each interior point. But, in modeling many objects it is convenient to use variable spacing to achieve greater economy by allowing a higher density of points where a need is anticipated. This means
that each interval must be calculated, but the symbol $h$ will still be used to represent a typical interval. With variable spacing, the volume associated with a point $P$ becomes a rectangular parallelepiped with faces located at the midpoints between $P$ and its neighbors. The shaded area in figure 5/1 represents the $x$-$y$ projection of this volume. Also indicated in the figure is the sense of the directions represented by the notation, $N$, $S$, $E$, $W$, $U$, $D$ for north, south, east, west, up, and down, respectively.

On this grid, we now develop a difference equation to approximate (5-1). We start with the central difference operator, $\delta$, which is defined as

$$\delta f(x) = f(x + h/2) - f(x - h/2).$$

Applying this operator twice we get,

$$\delta^2 f(x) = \delta [f(x + h/2) - f(x - h/2)]$$
$$= [f(x + h) - f(x)] - [f(x) - f(x - h)] \tag{5-2}$$
$$= f(x + h) - 2f(x) + f(x - h).$$

The connection between this last expression and the second derivative can be observed by first writing the Taylor series for $f(x \pm h)$,

$$f(x \pm h) = f(x) \pm h \frac{df}{dx}(x) + \frac{1}{2!} h^2 \frac{d^2f}{dx^2}(x) \pm \frac{1}{3!} h^3 \frac{d^3f}{dx^3}(x) \pm \cdots \tag{5-3}$$
substituting these series into (5-2), we find

\[
\delta^2 f(x) = f(x + h) - 2f(x) + f(x - h) \\
= 2\left[h^2 \frac{1}{2!} \frac{\partial^2 f}{\partial x^2}(x) + \frac{1}{4!} h^4 \frac{\partial^4 f}{\partial x^4}(x)\right] + O(h^6),
\]

(5-4)

and solving for \( \frac{\partial^2 f}{\partial x^2} \),

\[
\frac{\partial^2 f}{\partial x^2} = \frac{\delta^2 f(x)}{h^2} - \frac{h^2}{12} \frac{\partial^4 f}{\partial x^4} + O(h^6).
\]

(5-5)

The simplest approximation that we can make for the second partial derivative is thus,

\[
\frac{\partial^2 f}{\partial x^2}(x, y, z) = \frac{\partial^2 f}{\partial x^2} = \frac{\delta^2 f}{h^2}
\]

\[
= \frac{f(x + h, y, z) - 2f(x, y, z) + f(x - h, y, z)}{h^2},
\]

(5-6)

with an error of order \( h^2 \). Higher order approximations can be obtained by a similar Taylor series analysis (Collatz, 1960), but the resulting formulas complicate the consideration of boundary conditions enough to discourage their use in favor of just reducing \( h \) as much as possible. This of course increases the number of points required to model a given object; so if a machine size limit is reached and greater accuracy is still required, the use of higher order approximations would be an option.

To investigate the effect of using variable spacing we can let \( h + h_+, h_- \) where both are positive numbers. Starting again at (5-3) with this change, we see that the odd
order contributions to (5-4) no longer cancel, thus (5-5) becomes

\[
\frac{d^2 f}{dx^2} = \frac{2 \delta^2 f}{(h_+^2 + h_-^2)} - \frac{2(h_+ - h_-)}{(h_+^2 + h_-^2)} \frac{df}{dx} (x) - \cdots \cdots \quad (5-7)
\]

so, if \( h_+ \) and \( h_- \) are not nearly equal, accuracy will be reduced.

We could now formally construct a differenced form of Poisson's equation from (5-6), however it is more honest to present PANEL's Poisson algorithm as originally developed by Parker (1977b). We first throw Poisson's equation into partially dimensionless form by dividing by \( kT/e \), so with \( \phi = Ve/kT \) and \( \lambda_D^2 = \varepsilon_0 kT/N_0 e^2 \) we get

\[
\nabla^2 \phi (\vec{x}) = \lambda_D^{-2} (n_e - n_i) = R, \quad (5-8)
\]

where \( n_e \) and \( n_i \) are the electron and ion densities in units of the ambient density \( N_0 \). Integrate now (5-8) over the cell volume associated with point \( P \), and apply the divergence theorem to the left hand side;

\[
\iiint \nabla^2 \phi \, d^3x = \oint_S \frac{\partial \phi}{\partial n} \, ds = \iiint R \, d^3x = Q, \quad (5-9)
\]

where \( \frac{\partial \phi}{\partial n} \) is the outward normal derivative at the surface of the cell. \( Q \) can be identified as the net charge within the cell, however, this identification is not implicit in the formal development. We next approximate the surface
integral in (5-9) by the sum:

\[ \sum_F A_F \frac{\partial \phi}{\partial n} = Q \quad (5-10) \]

where \( F = N, S, E, W, U, D \), and \( A_F \) is the area on each of these faces. These areas are given by,

\[ A_N = A_S = \frac{1}{4} (x_{i+1} - x_{i-1}) ((z_{k+1} - z_{k-1}) \]

\[ A_E = A_W = \frac{1}{4} (y_{j+1} - y_{j-1}) (z_{k+1} - z_{k-1}) \quad (5-11) \]

\[ A_U = A_D = \frac{1}{4} (x_{i+1} - x_{i-1}) (y_{j+1} - y_{j-1}) \].

The partials \( \frac{\partial \phi}{\partial n} \) are approximated by the difference quotients:

\[ \frac{\partial \phi}{\partial n} = \frac{\phi_N - \phi}{y_{j+1} - y_j}, \quad \frac{\partial \phi}{\partial n} = \frac{\phi_S - \phi}{y_j - y_{j-1}} \quad (5-12) \]

and similarly for the \( E, W, U, \) and \( D \) directions, where \( \phi \) is the potential at the point \( P \), and \( \phi_N, \phi_S, \) etc. are the neighboring potentials. Thus substituting equations (5-11) and (5-12) into (5-10) we obtain the algebraic expression,

\[ C_N \phi_N + C_S \phi_S + C_E \phi_E + C_W \phi_W + C_U \phi_U + C_D \phi_D - C \phi = Q \quad (5-13) \]

where

\[ C_N = \frac{(x_{i+1} - x_{i-1})(z_{k+1} - z_{k-1})}{4(y_{i+1} - y_i)} \].

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and likewise for $C_g$ through $C_D$; $C = \sum_F C_F$.

Equation (5-13) can be applied to each interior point in the model, but exterior or boundary points require a modified treatment so as to include the required boundary conditions. The types of boundary conditions (B.C.) used in PANEL are:

1. Floating, where the outward normal derivative on the cell and model boundary is linearly related to the potential on the boundary$^\dagger$.

2. Neuman, where the inward normal component of the electric field is specified.

3. Dirichlet, where the boundary potential is specified.

$^\dagger$In the theory of boundary value problems, independent specification of the normal derivative and potential is an over specification of the boundary conditions and there will be no solution unless the solution was already known and used to specify the B.C. The floating B.C. is an approximation based on the following argument. Green's second identity may be generalized to the electrostatic boundary value problem as,

$$\phi(\hat{x}) = \int_V \phi(\hat{x}') G(\hat{x}, \hat{x}') \, dx' + \frac{1}{4\pi} \int_S \left[ G(\hat{x}, \hat{x}') \frac{\partial \phi}{\partial n'} - \phi(\hat{x}') \frac{\partial G(\hat{x}, \hat{x}')}{\partial n'} \right] \, da'. \quad (F-1)$$

Here, $\phi$ is the potential at $\hat{x}$, $G(x, \hat{x}')$ is the Green function (not to be confused with the previously defined $G(\hat{x}, \hat{v})$), $\rho$ is charge density, and $V$ is a volume interior to a surface $S$ with normal $n$. For either Dirichlet or Neuman B.C., the surface term breaks into a part for the panel surface, and a part for an outer bounding surface. If the outer surface is moved out to infinity, $G$ can be chosen such that the outer surface term goes to zero. This leaves only the panel.
4. Extended Dirichlet, where a boundary potential of zero is assumed to exist one interval beyond the usual model boundary.

5. Reflection, where like condition 4, an extended boundary is assumed, but with a potential equal to the nearest interior neighbor.

\[
\phi(x) = \int_{v<T} \rho(x') G(x,x') \, dx'^3 + \int_{s_1} \, [ ] \, da' + \int_{T-s_1} \, [ ] \, da'
\]

For our model boundary to have no effect on the problem we need,

\[
\int_{T-s_1} \, [ ] \, da' = \int_{v>T} \rho(x') G(x,x') \, dx'^3 \quad (F-2)
\]

If our boundary placement was sufficiently removed to exclude a negligible \( \rho \) in \( v > T \), then we may approximate both sides of (F-2) to zero. A null integral does not insure a zero integrand, but since its average value will be zero we can make the further approximation that it is everywhere identically zero if we argue that for a wise choice of boundary location, the induced error is minimal for interior points. Thus, for certain conditions, we can write,

\[
\frac{\partial \phi}{\partial n} G = \phi \frac{\partial G}{\partial n}
\]

on \( T \). For the case where the boundary is far enough away from the "object" for the object to look like a point charge or at least a uniformly charged sphere, we can assume a Green function of \( 1/r \), so we have the relations:

\[
\frac{\partial \phi}{\partial n} = \hat{n} \cdot \vec{\nabla} \phi = \frac{\hat{n} \cdot \vec{r}}{r^2} \phi.
\]

For a closer boundary, the possibility exists for finding the appropriate Green function, but this has not been pursued far enough to produce a useful algorithm.
When a boundary is assumed to represent "infinity", i.e. a source of undisturbed plasma at zero potential, the boundary should be far enough away from the "object" that all boundary conditions give the same results. It is frequently not possible to make grids that large so it becomes necessary to choose the B.C. which best approximates "infinity" on a limited grid. Parker and Sullivan (1969) addressed this problem, and concluded that for the spherical diode problem, the floating B.C. (1) produced the best "infinity" approximation with the least computing time. The zero gradient B.C. (2) produced an effective infinity at a distance comparable to the floating B.C., but required about twice the computing time as B.C. (1). The zero potential B.C. (3) required a more distant boundary to produce similar results, and the required computing time was between that required for B.C. (1) and B.C. (2).

All of these boundary conditions are effected by treating a boundary point as an interior point, and by adding the appropriate "off-grid" potential. Equation (5-12) and (5-13) also require modification at exterior points since if the maximum (minimum) value of x in the model is $X_{II}$ ($X_I$) the point $X_{II+1}$ ($X_0$) does not exist. The required modifications for $A_N$, $A_S$, $(\partial \phi / \partial n)_E$ and $(\partial \phi / \partial n)_W$ are respectively:
\[ A_N = \frac{1}{4} (x_{i+1} - x_{i-1})(z_{k+1} - z_{k-1}) \]
\[ + \frac{1}{2} (x_{II} - x_{II-1})(z_{k+1} - z_{k-1}), \text{ for } i = II; \]
\[ A_S = \frac{1}{2}(x_2 - x_1)(z_{k+1} - z_{k-1}), \text{ for } i = 1; \]
\[
\left( \frac{\partial \phi}{\partial n} \right)_E = \frac{\phi_E - \phi}{x_{i+1} - x_i} \Rightarrow \frac{\phi_E^* - \phi}{-(x_{II-1} - x_{II})},
\]
and
\[
\left( \frac{\partial \phi}{\partial n} \right)_W = \frac{\phi_W - \phi}{x_i - x_{i-1}} = \frac{\phi_W^* - \phi}{(x_2 - x_1)}
\]

where the asterisk indicates that \( \phi^* \) is chosen subject to the boundary condition.

With the appropriate consideration of exterior points, we can now apply equation (5-13) to all grid points giving a system of linear equations that is solved by the method of over-relaxation (O.R.) (Stiefel, 1963). Faster and more sophisticated methods are discussed by Hockney (1965), but O.R. has been chosen for its programming simplicity and versatility. To derive the O.R. formula used in Panel's relaxation algorithm, we first cast the system of equations produced by equation (5-13) into the form

\[
\sum_{m=1}^{M} C_{pm} \phi_m - Q_p = 0, \quad p = 1, 2, \ldots \ldots M \quad (5-14)
\]

where \( M \) is the total number of grid points. The solution of the \( p \) equation with respect to the central unknown \( \phi_p \) yields.
\[ \phi_p = \frac{1}{C_{pp}} [Q_p - \sum_{m \neq p} C_{pm} \phi_m] \]  

(5-15)

This equality will not be satisfied until the problem has relaxed or converged to the final result. When (5-15) is not satisfied, we assume the righthand side to be the better value for \( \phi_p \), so by denoting the various approximations by the index \( u \) we step through the index \( p \), replacing \( \phi_p^u \) with \( \phi_p^{u+1} \) to arrive at the iteration algorithm,

\[ \phi_p^{u+1} = \frac{W}{C_{pp}} [Q_p - \sum_{m=1}^{p-1} C_{pm} \phi_m^{u+1} - \sum_{m=p+1}^{M} C_{pm} \phi_m^u] \]

We have jumped ahead and added the over-relaxation factor \( W \). In the ordinary single step method, \( W = 1 \), and for \( W > 2 \) the method will diverge. The matrix \( C_{pm} \) obtained from the discretization of a boundary value problem is of the banded symmetric-definite type, and for such, convergence is insured for \( W < 2 \); Panel uses \( W = 1.9 \) with no divergence problems.

In the program, the subroutine FIELD controls the Poisson calculation. The calculation of the interior coefficients is delegated to the subroutines CNS, CEW, and CUD. The boundary conditions and B.C. influenced coefficients are effected in FIELD and the subroutine RELAX performs the relaxation operation.
The Vlasov Problem

In kinetic theory, the density and current density at a point $\mathbf{x}'$ are given by the 0th and 1st velocity moment of the single particle distribution function:

$$N_x(\mathbf{x}') = \int f_s'(\mathbf{x}', \mathbf{v}') \, d^3\mathbf{v}'$$  \hspace{1cm} (5-16)

$$J_s(\mathbf{x}') = q_s \int f_s'(\mathbf{x}', \mathbf{v}') \, \mathbf{v}' \cdot \hat{n} \, d^3\mathbf{v}' ,$$  \hspace{1cm} (5-17)

where $\hat{n}$ is the unit vector in the direction of $\mathbf{j}_s'$. In chapter 2, I showed that for a Maxwellian plasma, the Vlasov equation allows us to write for the distribution function at $\mathbf{x}'$:

$$f_s'(\mathbf{x}', \mathbf{v}') = N_s \left( \frac{m_s}{2\pi k T_s} \right)^{3/2} \exp \left[ -\frac{1}{2} \frac{m_s \mathbf{v}'^2}{k T_s} + q_s V(\mathbf{x}')] / k T_s \right]$$

$$\times G_s(\mathbf{x}', \mathbf{v}') ,$$  \hspace{1cm} (5-18)

In practice, the integrals in (5-16) and (5-17) are approximated by summations over a discrete set of velocities where each value of $\mathbf{v}'$ represents a trajectory that must be followed to evaluate $G(\mathbf{x}', \mathbf{v}')$. We now have the choice of either starting trajectories at "infinity" and following them in; or because of the assumed time-independence, we could start at $\mathbf{x}'$ and follow trajectories backwards in time to "infinity". The first technique has been dubbed the "Outside-in Method" by Parker (1964) and has the advantage
of having all trajectories successfully connecting to a source and of supplying useful trajectory information to all points along the trajectory. Its chief disadvantage lies in the difficulty of getting adequate trajectory probing of some regions of the problem. The Inside-out Method adopts the other approach of following trajectories backwards in time. This allows one to evaluate \( G(\dot{x}', \dot{v}') \) at all points with equal accuracy, but can lead to large numbers of trajectories to be retraced with each iteration. This last difficulty has been recently overcome by recording the fate of each trajectory so that in subsequent iterations, that information can be used to trace only those trajectories that lie on the velocity space boundary between null and escaping trajectories. This "boundary tracking" innovation can greatly increase storage requirements, but the reduction in time requirements make it essential.

In the following paragraphs I shall described how PANEL performs the integrals (5-16) and (5-17). Parts of this description has been taken directly from Parker (1977).

It is convenient to transform (5-16) and (5-17) to energy and angle variables in velocity space. Since we will be primarily interested in Maxwellian energy distributions, we may adopt the following units in terms of which dimensionless variables may be defined:

\[
kT = \text{unit of energy, where } T \text{ is the temperature of the Maxwellian distribution};
\]
\[
\sqrt{2kT/m} = \text{unit of velocity, namely, the most probably}
\]
thermal velocity;
\[ N_0 = \text{unit of particle density, the unperturbed density}; \]
\[ J_0 = qN_0 \sqrt{\frac{kT}{2\pi m}} = \text{unit of current, the undisturbed thermal current}. \]

The energy and angle variables are:
\[ H = \text{energy in multiples of } kT; \]
\[ \alpha = \text{polar angle with respect to } z\text{-axis}; \]
\[ \beta = \text{azimuthal angle with respect to the plane containing the } z\text{-axis and the point } \hat{x}. \]

These angles which define the orientation of the velocity vector \( \hat{v} \), are illustrated in figure 5/2. Note that the potential energy \( \phi \) is also in units of \( kT \), so with the new unit of velocity we can write: \( H = v^2 + \phi \).

The density and current integrals (5-16) and (5-17) may be written

\[ N_S = \iiint f'v^2 dv \sin \alpha \, d\alpha \, d\beta \, G \]
\[ J_S = \iiint f'v^3 dv \cos \alpha \sin \alpha \, d\alpha \, d\beta \, G \]

where \( J_S \) is assumed to be the current to a surface perpendicular to the \( z \)-axis. Introducing now, the Maxwell distribution (without drift), we have

\[ n = \frac{N_S}{N_0} = \frac{1}{2\pi^{3/2}} \int_{\max(0,\phi)}^{0} e^{-H} \sqrt{H - \psi} \, dH \int_{0}^{\pi} \sin \alpha \, d\alpha \int_{0}^{2\pi} G \, d\beta \]

(5-19)
\[ j = \frac{J_s}{J_0} = \frac{1}{\pi} \int_0^{\infty} e^{-H} (H - \Phi) \, dH \int_0^{\pi/2} \cos \alpha \sin \alpha \, d\alpha \int_0^{2\pi} G \, d\beta \max(0, \Phi) \]

(5-20)

The lower limit on the energy integral is chosen to be zero for an attractive potential ($\Phi < 0$), and to be $\Phi$ for a repulsive potential ($\Phi > 0$). This ensures that we never consider particles with negative kinetic energy.

The integrals in (5-19) and (5-20) are evaluated by the method of Gaussian quadrature (Jennings, 1964). It is not feasible to derive it here, but the method can be illustrated by stating that in the formula

\[ \int_a^b f(x) \, dx = \sum_{i=0}^n A_i f(x_i) + R \]

it is possible to choose $A_i$ and $x_i$ such that $R = 0$ for $f(x)$ any polynomial of degree $\leq 2n + 1$. When attempting to integrate a function of undetermined degree, it is desirable to make $n$ as large as possible. This poses two problems: 1) the formulas for $x_i$ and $A_i$ get increasingly complicated as $n$ increases and 2) the step function nature of $G(\vec{x}, \vec{v})$ implies a polynomial of infinite degree. Both problems are partially overcome by dividing the integration interval uniformly. as for ordinary trapezoidal integration, and then applying a Gaussian quadrature of order 2 to each subinterval. Thus, in preparation for this, we transform the ranges of inte-
gratation into intervals between -1 and +1 by the transformations:

\[ H(c) = \frac{1 + \frac{c}{\phi}}{1 - \frac{c}{\phi}} + \max (0, \phi), \quad -1 < c < +1 \quad (5-21) \]

\[ \alpha = \cos^{-1} \sqrt{(1 - a)/2} \quad \text{for current} \quad -1 < a < +1 \quad (5-22) \]

\[ \alpha = \cos^{-1} (-a) \quad \text{for density} \]

\[ \beta = \pi (1 + b), \quad -1 < b < +1 \quad (5-23) \]

The transformed current and density integrals then become,

\[
n = \frac{1}{\sqrt{\pi}} \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} e^{-H(c)} \sqrt{H(c)} - \phi \frac{G \, da \, db \, dc}{(1 - c)^2} \quad (5-24)\]

\[
j = \frac{1}{2} \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} e^{-H(c)} (H(c) - \phi) \frac{G \, da \, db \, dc}{(1 - c)^2} \quad (5-25)\]

We now have the integrals in a form suitable for Gaussian quadratures. We now divide the c-range into \( M_c \) sub-intervals, and apply a Gaussian quadrature of order 2 to each. Similarly, we divide the a-range and b-range into \( M_a \) and \( M_b \) sub-intervals, with Gaussian quadratures of order 2 applied to each sub-interval. Now, both equations (5-24) and (5-25) can be put in the form

\[
I = \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} W(H) \cdot G(H, \alpha, \beta) \cdot da \, db \, dc \quad (5-26)\]

which may be approximated by the sum:
\[
I = \frac{1}{M_a M_b M_c} \sum_{K_c=1}^{M_c} \sum_{K_a=1}^{M_a} \sum_{K_b=1}^{M_b} [W(H^-) \cdot G(H^-, \alpha^-, \beta^-) + W(H^+) \cdot G(H^+, \alpha^+, \beta^+)] 
\] (5-27)

where \( W \) is the energy weight function defined by,

\[
W(H) = \frac{e^{-H(c)} [H(c) - \phi]}{2(1 - c)^2} \quad \text{for current} \\
W(H) = \frac{e^{-H(c)} \sqrt{H(c) - \phi}}{\sqrt{\pi} (1 - c)^2} \quad \text{for density}
\] (5-28)

\( H^- = H(c^-), \ H^+ = H(c^+) \)

with \( \alpha^- = \alpha(a^-), \ \alpha^+ = \alpha(a^+) \)

\( \beta^- = \beta(b^-), \ \beta^+ = \beta(b^+) \).

Gaussian quadrature of order 2 applied to the interval \([-1, 1]\) yields the abscissas \( \pm (3)^{-1/2} \) with a weight coefficient of unity. Applying this simple formula to each sub-interval gives the formula,

\[
i^\pm = \frac{1}{M_i} \left[ \pm \frac{1}{\sqrt{3}} + 2K_i - 1 + M_i \right] \quad \text{for} \ i = a, b, c. \] (5-29)

so, with these formulae for \( a, b, \) and \( c \), we may use equations (5-21) through (5-23) to choose sets of trajectories to be followed backwards in time to either source or non-source regions and thus approximate the integrals for density and current.
In the first iteration only a small set of the total $8M_cM_aM_b$ trajectories are used, and in following iterations the number is increased until the maximum number of trajectories is being followed. This is an economizing move that doesn't affect the accuracy of the calculation since in the beginning iterations, densities are only approximate. As each trajectory is followed to its end-point, its fate (escape = true, absorbed = false) is recorded in the four-dimensional logical matrix ($N \times 2M_c \times 2M_a \times 2M_b$) called TRYE for electrons and TRYI for ions, where $N$ identifies the point. When each trajectory has been used at least once, the TRY matrices are used in subsequent iterations to trace only those that lie on a velocity space boundary. This is accomplished by simply comparing the last recorded fate for the trajectory in question to that for each of its six energy and angle neighbors in the TRY matrix. If all seven fates are identical, the trajectory is not followed and its fate is merely read from TRY. If they are not all identical, the trajectory is followed and any change of fate is recorded in TRY. If all the fates for a point and particle become the same, a shut-out would occur and no trajectories would be traced. To prevent this, all of the highest energy trajectories are exempted from the boundary searching process, and traced each iteration.

In PANEL, trajectories are initiated and analyzed in the subroutine DEN, and the energy coefficients and Gaussian quadrature formulae are calculated in the subroutine COABS.
A flow chart and subroutine linkage chart can be found in appendix B. Because of the length of the complete program listing, it is not practical to reproduce it in this document. Copies of PANEL may be obtained by writing to David L. Cooke, c/o the Dept. of Space Physics & Astronomy, Rice University, Houston, TX 77001.

**Trajectories**

The equation of motion of a particle in an electrostatic field with no magnetic field is,

$$\frac{d^2}{dt^2} \ddot{x} = \frac{q}{m} \ddot{E}(\ddot{x}).$$

Where $\ddot{E}$ is the electric field, and $t$ is time or since $\ddot{E}$ is time independent here, $t$ can be considered a parametric path coordinate. This equation must be integrated twice to find a particle's position $\ddot{x}(t)$. In a finite difference scheme, $E(\ddot{x})$ is calculated from potentials specified only at the grid points. It would be possible to define a suitable cell and solve the interior boundary value problem for continuous potentials and electric field. However, this approach would be time consuming, so PANEL makes the assumption that $E(\ddot{x})$ is constant within the cells or subcells described below; thus the equation of motion within a cell or subcell integrates to,

$$\ddot{x} = \ddot{x}_0 + \ddot{v}_0 t + \frac{1}{2} \frac{q}{m} \ddot{E} t.$$
This equation can be put into the units common to PANEL with the transformation \( t + t' \) (2kT/m). Thus we have for the individual components,

\[
x = x_0 + v_{ox} t' + \frac{1}{2} \left( - \frac{1}{2} \frac{\partial \phi}{\partial x} \right) t'^2
\]

\[
y = y_0 + v_{oy} t' + \frac{1}{2} \left( - \frac{1}{2} \frac{\partial \phi}{\partial y} \right) t'^2
\]

\[
z = z_0 + v_{oz} t' + \frac{1}{2} \left( - \frac{1}{2} \frac{\partial \phi}{\partial z} \right) t'^2
\]  \hspace{1cm} (5-30)

where as before, \( v \) is in units of the most probable thermal velocity, and \( \phi \) is the dimensionless potential.

PANEL traces particle trajectories on the same grid that is used for the Poisson calculation. At each interior grid point, the six neighboring intermediate points each define a face of a cell enclosing the grid point. The velocities in equations (5-30) are always given as a result of a previous step, or as initial conditions as a trajectory starts. The partial derivatives of the potential in (5-30) are approximated by divided differences calculated in the following manner. At the point \( P(x_i, y_j, z_k) \), the west and east potential differences, are:

\[
\Delta \phi_W = \phi(i, j, k) - \phi(i-1, j, k)
\]

\[
\Delta \phi_E = \phi(i+1, j, k) - \phi(i, j, k)
\]

and similarly for the \( y \) and \( z \) directions. Next, the absolute value \( |(\Delta \phi_W - \Delta \phi_E)| \) is compared to the particle's total energy, \( H \), multiplied by an input resolution factor.
RES. If the variation in potential differences is less than $H \cdot \text{RES}$, PANEL uses

$$\frac{\Delta \phi}{\Delta x} = \frac{\Delta \phi_W + \Delta \phi_E}{x_{i+1} - x_{i-1}}$$

for the entire cell, and if the variation is too great, the cell is halved and for the west and east halves we use,

$$\left(\frac{\Delta \phi}{\Delta x}\right)_W = \frac{\Delta \phi_W}{x_i - x_{i-1}}$$

and

$$\left(\frac{\Delta \phi}{\Delta x}\right)_E = \frac{\Delta \phi_E}{x_{i+1} - x_i}.$$ 

Thus, a cell can conceivably be divided into eight sub-cells. This subdivision is always performed at the start of a trajectory when the particle will probably never enter most of the sub-cells.

Once the cell or sub-cell has been defined and the "electric field" calculated, equations (5-30) are solved independently for the times required to cross the cell, and the shortest positive time is chosen. Using this time, the particle is stepped to another face of the cell and the process begins again in the next cell until an outer boundary is reached. At a boundary, a particle can escape, be absorbed on a surface, or be reflected (for a reflection boundary condition).
Stability and Convergence

Parker and Sullivan (1970) have analyzed the stability and convergence of the inside-out method, applied to a uniformly charged sphere in a uniform plasma. Although a three-dimensional method like PANEL could be expected to differ from a simple one-dimension method in its stability properties, tests have shown that the results of their study are applicable to PANEL. That analysis will be briefly outlined.

If we imagine that the Poisson solving process can be represented by the operator $L(\phi)$ and that the Vlasov process can be represented by $\dot{\Phi}(\phi)$, then the state of a system would be prescribed by

$$L(\phi) = F(\phi), \quad (5-31)$$

and the iterative procedure previously described would follow the Picard iteration rule,

$$L(\phi_{n+1}) = F(\phi_n), \quad (5-32)$$

where $n$ is the iteration index. This iteration scheme, however, was found to diverge when the distance between their sphere and the model boundary exceeded the Debye length. An effective cure for this, is to replace rule (5-32) with

$$L(\phi_{n+1}) = F(\alpha\phi_n + (1 - \alpha) \phi^M_{n-1}), \quad 0 < \alpha < 1. \quad (5-33)$$
This technique is called mixing, and the superscript $M$ indicates previously mixed potentials, i.e.,

$$
\phi_{n-1}^M = \alpha \phi_{n-1} + (1 - \alpha) \phi_{n-2}^M.
$$

Their analysis of rule (5-33) predicts monotone convergence for $\alpha < 2/(2 + y)$, oscillatory convergence for $2/(2 + y) < \alpha < 2/(1 + y)$, and divergence for $\alpha > 2/(1 + y)$, with an optimum value, $\alpha_{\text{opt}} = 2/(1 + y)$. The parameter $y$ is given by

$$
y = 2d^2 / \pi^2 \lambda_D^2,
$$

where $d$ is the boundary-object separation distance, and $\lambda_D$ is the Debye length for the plasma. These predictions have been proven to be accurate for the one-dimensional sphere model.

One could insure convergence by choosing $\alpha$ very small, but then a large number of iterations would be required. Therefore it is desirable to optimize $\alpha$. With PANEL, I have found this prediction for $\alpha_{\text{opt}}$ to be a good first approximation; but to really obtain optimal convergence, calculations must be stopped every three or four iterations to adjust $\alpha$. 
The results presented here are of two distinct types: "calibration" models designed to test PANEL against problems for which analytic answers are available, and two production models to investigate plasma sheath structure. Thus far, the emphasis has been on the former. Just as with an instrument, results are worthless without calibration. This emphasis has been rewarded, as many subtle errors (both with PANEL and my use of PANEL) have been detected and corrected. For this reason, the runs presented here represent only a fraction of those that have been made.

The models called Gap 06, Gap 07, and Gap 08 are calibration models of the problems described in equations (4-12) through (4-17) in chapter 4. Pan 21 is a model of a planar electron diode, and can be compared to the Child-Langmuir law, equation (4-20). Finally, Pan 29 and Pan 36 are production models of charged panel in a plasma similar to that encountered in the Chamber A experiments at the Johnson Space Center (McCoy and Konradi, 1978). All of these are two-dimensional models. Three-dimensional tests have also been made, but limited computing time has prevented the running of physically meaningful three-dimensional models.

In the gap problem, electrons are accelerated from a cold cathode \( T = 0 \) to the potential \( V_1 \) (see figure 4/1) at \( x = 0 \), to produce a beam current \( J \). PANEL models this experiment by assuming that there is an undisturbed Maxwell-
ian plasma at $x_1 < -d$, so that the current $J$ is the random thermal current ($J_0 = N_o e \sqrt{kT/2\pi m}$) crossing the grid at $x = -d$. As described in chapter 4, this current is normalized by the relevent C-L current, thus we have the current ratio,

$$\beta = J/J_{CL}.$$  

The results of Gap 06, Gap 07, and Gap 08 are plotted in figures 6/1 and 6/2. In these plots, the transmitted electrons travel from right to left across a gap of one meter. This is modeled by 24 grid points; 12 z and 2 x coordinates. At $z = 0$ electrons are absorbed; at $z = 11$ (not shown), they are generated; and they are reflected at both x boundaries. (Since this is a one-dimensional problem, PANEL could have been fitted with a one-dimension option, but unlike the two-dimension option, a one-dimension option would have only limited applications.) In all three plots, the potentials predicted by the classical theory are labeled as curve A, the potentials calculated by PANEL are labeled P, and the densities are labeled D (although the plasma has density $N_o$, the electrons crossing the grid have $N = N_o/2$). The features of these models are:

Gap 06: $\beta = 10$, $J = 2.373 \times 10^{-2}$ A/m$^2$, $T_e = 10$ eV, $V_1 = 100$ V, $N_o = 2.8 \times 10^5$ cm$^{-3}$, $M_e = 4$, $M_a = 32$;
Gap 07: $\beta = 10, J = 2.373 \times 10^{-2} \text{ A/m}^2, T_e = 1 \text{ eV}, V_1 = 100 \text{ V}, N_0 = 8.9 \times 10^5 \text{ cm}^{-3}, M_e = 4, M_a = 32;\nGap 08: \beta = 4, J = 9.49 \times 10^{-3} \text{ A/m}^2, T_e = 1 \text{ eV}, V_1 = 100 \text{ V}, N_0 = 3.5 \times 10^5 \text{ cm}^{-3}, M_e = 4, M_a = 32.\n
Gap 06 and 08 are both well converged, but Gap 07 has an uncertainty indicated by the error bar on the plot. I consider these models to be a positive test of PANEL, in spite of the deviations from the classical predictions. The classical theory considers a source of electrons with no thermal spread. By comparing Gap 06 with Gap 07 we can see that as the source plasma cools from a temperature of 10 eV to 1 eV, the results get closer to the classical prediction. In Gap 08 where $\beta = 4$, the predicted minimum potential is 75.0 volts while PANEL gives $75.1 \pm 0.7$ (the error indicates the degree of convergence). This again indicates that the disagreement with the classical theory in Gap 06 and Gap 07 are due to non-zero temperatures since one would expect this effect to be most pronounced with low minimum potentials, and least pronounced with higher minimum potentials.

From Gap 08 we can also learn something about the number of trajectories that must be traced to give accurate densities. In figure 6/2, the lower curves labeled D and C are densities for PANEL and classical theories respectively. Briefly, the classical densities are derived by eliminating $v$ from
\[ J = N e^v \]  \hspace{1cm} (6-1) \\
\text{and} \quad \frac{1}{2} \, mv^2 = eV \hspace{1cm} (6-2) \\
to get \quad N = J \left( \frac{m_e}{2e^3 V} \right)^{1/2}. \hspace{1cm} (6-3) \\

For Gap 08 the total zenith angle range of 2\pi is covered by 64 trajectories to give a trajectory separation of .098 rad. or 5.63°. This separation was further reduced by one half by noting that due to symmetry, positive and negative angles of equal magnitude lead to equivalent trajectories; thus all trajectories were shifted by half of the separation angle. The result is that although the voltages were obtained with good accuracy, the densities still lack resolution.

Pan 21 represents a simple but important test of PANEL. This is a comparison of PANEL with the Child-Langmuir law shown in fig. 6/3. Due to the close agreement, a curve has been drawn only through the PANEL points. At selected points, PANEL and C-L potentials are given for comparison. The C-L potentials are given in parentheses and C-L densities are plotted with crosses. Here 32 points (2 x 16) were used to model a diode with a 16.51 meter plate separation, and a 100 volt potential difference. The model parameters are:

Pan 21; \( T_e = 1 \text{ eV}, \ N = 3.2 \times 10^2 \text{ cm}^{-3}, \ \lambda_d = 0.4m, \ M_e = 4, \) 
\[ M_a = 32, \ J = 8.58 \times 10^{-6} \text{ A/m}^2. \]
The greatest disagreement between PAN 21 and the C-L theory occurs at \( z = 14 \), where the PANEL prediction is 22\% high, with improved agreement at lower \( z \) values. At \( z = 8 \), the disagreement is only 1\%. The larger deviations should be expected in the low voltage region near the cathode due to the non-zero injection velocity of the electrons. For this reason, it would be desirable to compare PANEL predictions with the modified C-L law, equation (4-10), but unfortunately, this comparison has not yet been made.

Pan 29 and Pan 36 are two-dimensional models of a cross-section of an infinitely long, one meter wide panel held at a potential of 100 volts in a hydrogen plasma with equal ion and electron temperatures of 10 eV. The chosen plasma temperature of 10 eV is higher than the usual temperatures encountered in LEO or in the JSC Chamber A experiments which are frequently less than 1 eV. Models with a panel potential of 100 V and a temperature of 1 eV have been considered, but under these conditions PANEL is significantly less stable. To achieve stability thus requires a smaller mixing parameter, more iterations and more computing time; so for these first models, a higher but not unreasonable temperature was chosen.

Due to the symmetry of the problem, it was possible to model the entire cross-section by calculating potentials and densities in one quadrant only by using the reflection boundary condition on the DOWN and WEST boundaries. For PAN 29 the UP B.C. is \( V = 0 \), and for the EAST boundary the
B.C. is the zero normal gradient boundary condition \([B.C. (2), \partial V/\partial n = 0]\). Model parameters for PAN 29 and PAN 36 are:

\[
T_i = 10 \text{ eV}, \quad T_e = 10 \text{ eV}, \quad N_0 = 1.9 \times 10^4 \text{ cm}^{-3}, \quad \lambda_0 = 0.17 \text{ m},
\]

\[
J_{oe} = 1.61 \times 10^{-3} \text{ A/m}^2, \quad ME = 4, \quad MA = 32.
\]

PAN 29 potential contours are displayed in Figures 6/4, and the PAN 36 results are displayed in Figures 6/5, 6/6 and 6/7. The potential contours shown in Figures 6/4 and 6/5 were produced by the well-known technique of eyeball interpolation. For PAN 36, potentials (labeled P) and charge densities (labeled C) along the DOWN and WEST boundaries are presented in Figures 6/6a and b respectively, and the electron and proton densities along these same boundaries are shown in Figure 6/7. In addition to the change in B.C.'s the other differences between PAN 29 and PAN 36 are location of the boundaries and the number of grid points. PAN 29 has an UP boundary distance of 1.85 meters with a 8 x 8 grid, while PAN 36 has an UP boundary distance of 2.4 meters with a 9 x 10 grid.

Several interesting features of the general problem can be observed by comparing the contour plots 6/4 and 6/5. First, by assuming that PAN 36 is the better model of the two, we can see that B.C. (2) on the EAST boundary of PAN 29 (Figure 6/4) produced potentials along the DOWN boundary that are very close to those of PAN 36; and therefore sup-
posedly better than what could have been obtained with a $V = 0$ condition. The drawback is that B.C. (2) severely destabilizes the problem. One model, PAN 35 (nearly identical to PAN 36, but with B.C. (2) on both the UP and EAST boundaries) diverged severely with a mixing factor of 0.08, while PAN 36 was stable with mixing factors up to 0.5. This suggests the possibility of speeding convergence on smaller grids by using varying mixing factors with small values near B.C. (2) boundaries and increasing to larger values near fixed potentials.

It can also be seen that the increased grid point density near the panel in PAN 36 has caused the 70 V, 50 V, and 30 V contours to move closer to the panel with smoother contours near the edge of the panel. The electron currents collected from above the panel are indicated by the arrows below the panel in figures 6/4 and 6/5, and have been normalized by the random thermal current, $J_0$. Near the center of the panel, both models give the same current, but with the increased point density in PAN 36 we begin to see a slight reduction in current collection near the edge. A further increase in point density would probably show more current focusing. However, the strong central focusing (greater than an order of magnitude difference between central and edge currents) observed by McCoy (1980) in the solar panel tests at JSC is not indicated in these models. This focusing could be dependent on the correct choice of panel voltage and plasma parameters, but is most likely due
to a band of dielectric along the edges of that test panel. This possibility will be tested in future models.

For both PAN 29 and PAN 36, the C-L screening distance is $D_{CL} = 1.2 \text{ m}$, and the corrected screening distance is $D_s = 1.73 \text{ m}$. These points are indicated in Figures 6/4, 6/5, 6/6, and 6/7, and the uncorrected C-L contour is marked with crosses in Figure 6/6. Figures 6/5 and 6/7 show that potentials have been reduced to less than $kT/e (= 10 \text{ V})$ within either estimate. There is some "compression" of the contours caused by the closeness of the $V = 0$ boundaries, as is evidenced by the most distant points in Figure 6/7 where the electron density unrealistically drops below the proton density due to the artificially high electric field between the outermost two points. However, comparison of the EAST boundaries of Figures 6/4 and 6/5 suggests that this compression is not too severe. Although all of the PAN 29 and PAN 36 boundaries are too close to allow an undisturbed plasma region to develop, Figure 6/7b shows a definite presheath region beyond the $D_{CL}$ point with electron and proton densities nearly equal but reduced from the ambient values.

It should also be noticed that a simple C-L/sharp edged sheath model would predict a surface current density equal to the ambient thermal current multiplied by the ratio of the surface area of a cylinder with radius, $R \approx D_{CL}$, to the surface area of the PANEL. Thus the C-L/SES model would predict a normalized surface current density, $j = \pi r^2/2w$ where $w$ is the panel width. From figure 6/5 we can infer a
range in $R$ of $1.2$ m to $1.73$ m. The $R$ suggested by the PAN 36 result, $j = 3.3$, is $R = 1.45$ which fits nicely that range.

**Three-Dimensional Results**

As previously mentioned, I have not had sufficient computer time to produce any full scale three-dimensional models. There has, however, been some three-dimensional testing of PANEL and since both two-dimensional and three-dimensional models are obtained with the same code, the two-dimensional calibration models can serve in part to validate three-dimensional models. The two three-dimensional tests of PANEL presented here are a model of a charged disc in charge-free space, and a comparison with the Debye model for a point at a fixed voltage.

Consider a disc of radius $a$, at a potential $V_0$, lying in the $x, y$ plane, centered at the origin. Using the cylindrical coordinates $z, r, \theta$, Jackson (1962) gives the following relations for the potential $V$ directly above the disc, and in the plane of the disc:

$$V(0, z) = \frac{2V_0}{\pi} \arctan \left( \frac{a}{z} \right) \quad (6-4)$$

$$V(r, \theta) = \frac{2V_0}{\pi} \left( \frac{a}{r} \right) \quad (6-5)$$

The grids and panels used for the models PAN 06, PAN 08, and PAN 11 are illustrated in Figure 6/8. The are Laplacian
models that were run before the space charge calculation, reflection boundary conditions, or the two-dimensional option were implemented in PANEL. The floating boundary conditions B.C. (1) was used on all boundaries except at z = 0 where the boundary condition was a reflection condition (presently optional on all boundaries).

A Laplacian solution is the first step in all PANEL space charge calculations.

Since a square is not a disc, equation (6-4) cannot be compared to these models until a disc radius is chosen. One choice might be the radius of a disc with an area equal to the panels in these models; call this radius $a_a$. However, rather than comparing tables of potentials for various choices of $a$, it is more instructive to substitute resulting model potentials into equation (6-4) and solve for the radius $a_p$ to use for comparison. An added advantage of this method of comparison is that if the result $a_p$ varies between points, we know to suspect some error in the calculation or in the model parameters, such as the boundary locations. In Figure 6/8, the radii $a_a$ and $a_p$ (from Equation (6-4) are given in terms of the dimension $L$ for the three models. For all three models, the variation in $a_p$ between points was always less than 2%. From the table in Figure 6/8, it is apparent that the "disc" resolution of PAN 06 and PAN 08 is inadequate; the radii $a_p$ both being larger than even that of a circle that would circumscribe the squares ($\sqrt{2} L$). With PAN 11, though, the resolution has improved significantly.
My other three-dimensional test of PANEL is a space charge model of a single grid point held at a potential of 10 V in a 50 eV plasma (PAN 30). The notion of a point potential needs some discussion. Point charges can be imagined, but point potentials cannot be defined. We must rather speak of the potential of a small charged sphere, and if the charge is held constant, the surface potential will vary inversely with the radius of the sphere. There are two reasons for this model: one is that it was about all that I could afford, and the other was to further investigate PANEL's resolution.

The grid for PAN 30 is 5 x 5 x 5 with grid points spaced identically along each axis at 0, 2, 4, 8, and 12 meters. The WEST, DOWN, and SOUTH boundaries are reflecting, so that only one quadrant is actually modeled. The other three boundaries are each different being either $V = 0$, $E = 0$, or floating. The 50 eV plasma has a density of 100/cc and an electron Debye length of 5.26 meters.

Figure 6/9 compares PANEL's first iteration Laplacian solution for PAN 30 with the usual Coulomb potential expression for a sphere. Again, rather than compare potentials, I give an inferred sphere radius, $R$, obtained by substituting each PANEL potential into the Coulomb expression. Potentials and radii are paired in Table 1 of Figure 6/9, where poor consistency in the inferred radii is evident. This could be due to the closeness of the boundaries, and it can be seen that the different boundary conditions do have dif-
fering effects. However, the correct interpretation is more likely that the discrepancies are due to the limitations of the finite difference method. Equation (5-5) shows that in constructing a differential form of the Laplacian operator, we can expect error terms like $\frac{h^2}{12} \frac{\partial^4 V(r)}{\partial x^4}$. The collection of the fourth order partial derivatives of $1/r^2$ for all three space variables would be very difficult to follow through the system of Equations (5-14). However, this collection will contain terms proportional to $h^2/r^5$ and lesser inverse powers of $r$ and since these all come to life as $r \to h$, the errors experienced in PAN 30 are not surprising.

Table 2 of Figure 6/9 gives PANEL's Poisson potentials obtained after eleven iterations. Here, the comparison with the Debye model is made with three different profiles of $V_D(r)$; each for different values of $R$. This method of comparison is chosen because of the difficulty in solving for $a$ in the Debye expression for $V_D(r)$. From the PANEL results in Table 2 we can note that the central potential is contained with a Debye length or so, but none of the choices for $R$ give an accurate match between $V_D$ and the PANEL results. In the Poisson results, we see that all three boundary conditions give nearly identical results indicating that the boundaries were sufficiently removed.

The higher order difference schemes mentioned on page 45 would surely improve resolution, but the lesson of this type of problem would still be to not expect accuracy at the limit of grid resolution.
7: CONCLUSION

The Models PAN 29 and PAN 36 are clearly not a complete study of the solar panel-plasma interaction problem, but the results that have been presented should demonstrate that the Child-Langmuir/sharp-edged sheath model (C-L/SES) is a significant improvement over a Debye model for the space charge sheath of a "large" satellite. This conclusion is not new, but these results represent the most rigorous numerical confirmation of that conclusion. This conclusion is also indicated by the recent experimental results of McCoy and Konradi (1978). I do not claim that a C-L/SES model will be applicable to all environments. I anticipate that future PANEL results will confirm the prediction that the C-L/sharp edge sheath model will not adequately describe the sheath of very large planar satellites, due to the presheath considerations discussed in chapter 4. It may be possible to develop (with the help of numerical modeling) a presheath model to complement a C-L/SES model, but only rigorous numerical calculations will be able to include such complications as photoelectrons or overlapping sheaths due to alternating satellite surface potentials, or give detailed predictions of surface current distributions for even simple configurations.

Accounting for increased detail in a rigorous manner with PANEL will not be easy and at some point must become impossible due to computation time limitations. The two-dimensional model PAN 29 required about twelve minutes of
processing on a ITEL AS/6 (system 370), PAN 36 required about thirty minutes, and comparable three-dimensional models are expected to require many hours of processing time. Although these are not extreme requirements, the inclusion of greater detail or additional plasma sources will further increase the required time. There are, however, many ways in which PANEL can be optimized, such as variable mixing parameters, alternating "calibrated approximations" with rigorous density calculations, or the use of a higher order Poisson algorithm to reduce grid point density requirements. The conversion to a finite-element method is also being considered. It may be possible to base an approximate but faster numerical scheme on a C-L/SES model (with a proper presheath treatment). If so, PANEL might become an important source of standard models. Finally, it is evident from the discharge phenomena discussed in chapter 3 that ultimately, time-dependent modeling must be undertaken. Even for this, though, good detailed steady state models may be helpful in spotting unstable sheath configurations.
REFERENCES


Lehnert, Tellus, 8, 408, 1956.


Appendix A: Symbols and Constants

The symbols and constants used throughout this report are reviewed in this appendix. With only a few noted exceptions, the MKS system of units has been adhered to.

T; temperature in degrees Kelvin.

k; Boltzman's constant = 8.62 x 10^{-5} eV/°K.

e; electron charge = 1.602 x 10^{-19} Coulomb.

m_e; electron mass = 9.11 x 10^{-31} kilograms.

m_p; proton mass = 1.673 x 10^{-27} grams.

ε_o; free space permittivity = 8.85 x 10^{-12} farad/meter.

V; potential in Volts.

ϕ; dimensionless potential normalized by kT/e.

v; velocity in meters/second, or normalized by \sqrt{2kT/m}.

E; dimensionless electric field.

H; total energy in Joules, or dimensionless total energy normalized by kT/e.

N; density in m^{-3}.

N_o; ambient density in m^{-3}.

n; dimensionless density normalized by N_o.

J; current in Amperes/meters² = Am^{-2}.

J_o; random thermal current in Am^{-2}.

j; dimensionless current normalized by J_o.

ρ; charge density in Coulomb m^{-3}.

a; sheath thickness.

λ_D; Debye length.

q; particle charge.
Appendix B: Subroutine linkage.

- CNS: \( \pm Y \) field coefficients
- CEW: \( \pm X \) field coefficients
- CUD: \( \pm Z \) field coefficients

FIELD:
Control of CNS, CEW, CUD, and boundary conditions

RELAX:
Solve \( \nabla^2 \phi = -\rho/\varepsilon_0 \) by over-relaxation

ARRAY:
Storage and printing of geometric coefficients

LISTB:
Listing of potential and density arrays

LIST:
Listing of XYZ coordinates

FIND:
Find XYZ

PANEL
Input/Output Calculation control Potential mixing

Input

COABS: Calculation of energies, angles, and coefficients

SPCHG: Control of DEN for density calculation

POWER: Control of DEN for current calculation

DEN: Trajectory initialization and analysis

ORBIT3: Individual trajectory increments

INTERP: Locate particles after trajectory steps, calculate E fields

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PANEL Flow Chart

Start

Input: List of X,Y,Z coordinates, panel and other fixed potentials, boundary conditions, output options

Calculate and list geometric coefficients

Yes

Is this the zeroth iteration?

No

FT08, information from previous iterations

Calculate Laplacian potentials, \(\rho = 0\) (RELAX)

Density, Temperatures, Mixing parameter, Trajectory resolution factor

Read FT08

Read Plasma information

No

Another iteration?

Choose specie

Yes

Select point for density or current integration (SPCHG)

3

2

Choose trajectory

1

Look at 6 (4 for 2D) neighboring trajectories from previous iteration, are all the same as this one? (DEN)

NO

Find grid cell and calculate E field (INTERP)

NO

Trace trajectory across cell (ORBIT3)

Has the trajectory reached a boundary? (DEN)

Yes

Yes

Assume and record fate (DEN)

Determine and record fate (DEN)

Next specie or point

No

Assume and record fate (DEN)

All points considered

Yes

Mix potentials for next iteration

Solve Poisson equation (RELAX)

Calculate density or current from stored fates (DEN or POWER)

Write FT09

Stop

Yes

Print charge density and potential arrays (LISTB)
Appendix C: The 2-D Option

When a problem exhibits sufficient symmetry, it is sometimes possible to reduce the number of integrals in equations (5-16) and (5-17) that must be performed numerically; so, in a sense, PANEL's 2-D option still produces 3-D results.

Using the Maxwellian distribution given in (5-18), and writing (5-16) and (5-17) in terms of Cartesian velocity coordinates we have,

\[
N(x') = N_0 \left( \frac{m}{2\pi kT} \right)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d^3v}{dxdydz} G(x', v') \tag{C-1}
\]

\[\times \exp \left[ -\frac{m}{2kT} \left( v_x'^2 + v_y'^2 + v_z'^2 \right) - \frac{qV(x')}{kT} \right]\]

\[
\vec{J}(x') = q N_0 \left( \frac{m}{2\pi kT} \right)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d^3v}{dxdydz} G(x', v') \tag{C-2}
\]

\[\times (\vec{v} \cdot \hat{n}) \exp \left[ -\frac{m}{2kT} \left( v_x'^2 + v_y'^2 + v_z'^2 \right) - \frac{qV(x')}{kT} \right]\]

If \(G(x', v')\) is not a function of \(v_y\), and if \(\hat{n}\) in (C-2) lies entirely in the \(x-z\) plane, the \(v_y\) integrations can be performed immediately leaving

\[
N(x') = N_0 \left( \frac{m}{2\pi kT} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d^2v}{dxdz} G(x', v') \tag{C-3}
\]

\[\times \exp \left[ -\frac{m}{2kT} \left( v_x'^2 + v_z'^2 \right) - \frac{qV(x')}{kT} \right]\]

and
\[ J_z(x') = N_o \left( \frac{m}{2\pi kT} \right) \int_0^\infty dv_x' \int_0^\infty dv_z' G(x',z') V_z \quad (C-4) \]

\[
\exp \left[ -\frac{m}{2kT} (v_x'^2 + v_z'^2) - \frac{qV(x')}{kT} \right]
\]

where we have set \( \hat{n} = \hat{z} \). Transforming now to cylindrical polar coordinates, and as in Section 3, denoting the exponential argument by \( H \); we have

\[ N(x') = N_o \left( \frac{m}{2\pi kT} \right) \int_0^\infty v' dv' \int_0^{+\pi} d\theta' G(x',z') \exp(-H) \quad (C-5) \]

and

\[ J_z = N_o \left( \frac{m}{2\pi kT} \right) \int_0^\infty v'^2 dv' \int_{-\pi/2}^{+\pi/2} \cos \theta' d\theta' G(x',z') \exp(-H) \quad (C-6) \]

Again as in Section 3, we transform to dimensionless \( v, H, \) and \( \theta, \) and normalize by \( N_{SO} \) and \( J_{SO} \) to get the two-dimensional versions of (5-19) and (5-20);

\[ n = \frac{1}{2\pi} \int_0^\infty e^{-H} dH \int_{-\pi}^{+\pi} G(x',z') \quad (C-7) \]

and

\[ j_z = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-H} \sqrt{H - \phi} dH \int_{-\pi/2}^{+\pi/2} \cos \theta d\theta G(x',z') \quad (C-8) \]

In preparation for Gaussian quadrature, we make the
following transformations:

\[ E = \frac{1 + c}{1 - c} + \max(0, \phi), \quad -1 < c < +1 \quad (C-9) \]

\[ \theta = \pi a \quad \text{, for density} \quad -1 < a < +1 \quad (C-10) \]

\[ \theta = \sin^{-1}(a) \quad \text{, for current} \]

Including these we have,

\[ n = \int_{-1}^{+1} \int_{-1}^{+1} e^{-H(c)} \frac{dc}{(1 - c)^2} \frac{d\alpha}{(1 - c)^2} G(\hat{x}', \hat{v}') \quad (C-11) \]

and

\[ j_z = \frac{2}{\sqrt{\pi}} \int_{-1}^{+1} \int_{-1}^{+1} e^{-H(c)} \sqrt{H - \phi} \frac{dc}{(1 - c)^2} \frac{d\alpha}{(1 - c)^2} G(\hat{x}', \hat{v}') \]

And, as in chapter 5, we introduce the sub-interval Gaussian quadrature approximations:

\[ n \text{ or } j_z = \frac{1}{M_c M_a} \sum_{K_c = 1}^{M_c} \sum_{K_a = 1}^{M_a} \left[ W_2(H^-) \cdot G_2(H^-, \theta^-) \right. \]

\[ + W_2(H^+) \cdot G_2(H^+, \theta^+) \quad (C-12) \]

where the two-dimensional energy weight function is

\[ W_2 = \frac{e^{-H(c)}}{(1 - c)^2} \times \left\{ \frac{1}{2} \sqrt{\frac{\phi}{\pi}} \right\}, \text{for density} \]

\[ \frac{2}{\sqrt{\pi}} \sqrt{H - \phi}, \text{for current} \quad (C-13) \]
and \[ H^\pm = H(c^\pm), \quad \theta^\pm = \theta(a^\pm). \]

The Gaussian abscissa formula (5-29) and the transformations (C-9) and (C-10) are used to initiate trajectories.
FIG. 3/1: This figure illustrates the basic configuration and solar cell layout for the MSFC January 25, 1978 baseline design. Located in the center of the satellite is the rotating microwave antenna. The voltage distribution on each panel has not been fixed; for modeling purposes, we have assumed the configuration illustrated in figure 3/2. The ultimate voltage distribution scheme will depend upon the results of continued SPS-plasma interaction studies.
FIG. 3/2

Figure 3/2 illustrates the 3 dimensional grid used to model 2 interior panels of the MSFC/Rockwell SPS design. Not shown are grid points at the intersection of all integer X and Y values and even values of Z. One unit of grid spacing corresponds to 85.0 meters, giving panel dimensions of 765 x 425 m. Fixed voltages are indicated in the figure. The plasma conditions are:

\[ N_i = N_e = 2/\text{cc}, \quad T_i = 10 \text{ keV}, \quad T_e = 5 \text{ keV} \]

For these conditions, the random thermal current densities are:

\[ J_{0i} = 1.25 \times 10^{-7} \text{ Amp/m}^2, \quad J_{0e} = 3.79 \times 10^{-6} \text{ Amp/m}^2 \]

The dimensionless numbers at selected grid points are ratios of local electron current densities to the random thermal current. For each grid point PANEL traced 864 trajectories to calculate the currents. For the two panels modeled the total collected current and power loss are: \( 6.64 \times 10^{-2} \text{ Amp} \) and \( 5.66 \times 10^2 \text{ Watts} \) for protons, and \( 2.25 \text{ Amp} \) and \( 2.74 \times 10^4 \text{ Watts} \) for electrons.

FIGURES 3/3 and 3/4

Figures 3/3 and 3/4 are equipotential contour slices at the points indicated in figure 3/2.
FIG. 4/1:

Schematic representation of a planar electron diode, showing the effect of the electron space charge on the interelectrode potential for two different arrangements of the electrode potentials.
FIG. 4/2. The Child-Langmuir correction function $S(\psi)$ plotted against $\log(\psi^{-1})$, where $\psi = eV/kT$. 
Fig. 4/3a. Minimum potential $\phi_m$ as a function of input current $\beta$. The dotted portion is called the "C-overlap" solution and is excluded by stability considerations.

Fig. 4/3b. Potential profiles in the interelectrode space for the classical, positive potential-minimum solutions with input currents between 0 and 8.

Fig. 4/3c. Potential profiles in the interelectrode space for the classical virtual-cathode solutions ($\phi_m = 0$) with input currents between 4 and $\infty$. 
FIG. 5/1. A sample grid illustrating a projection onto the X-Y plane of the volume associated with the point P, and the directions indicated by N, S, E, W, U, and D.

FIG. 5/2. An illustration of the polar-spherical angles $\alpha$ and $\beta$ used in PANEL.
FIG. 6/1: In both graphs, curve P is PANEL's prediction for the inter-electrode potentials, curve A is the classical prediction, and curve D is PANEL's prediction for the electron density. The Z distance unit is 0.1 meter. $\beta$ is the normalized current density.
FIG. 6/2: The upper curve is PANEL's prediction for the inter-electrode potentials, and the classical potentials are the unconnected crosses. Of the lower curves, D is the PANEL result for the densities, and C gives the classical densities. The Z unit of distance is 0.1 meter, and $\beta$ is the normalized current density.
FIG.6/3: PANEL's predictions for the Child-Langmuir electron diode are plotted with the connected dots, and selected values of potential and density are presented. The classical C-L density predictions are plotted by the unconnected crosses, and for comparison, selected values of the C-L potential and density are given in parenthesis. For this model, $N_o = 320/\text{cc}$, $J_e = 8.6 \times 10^{-6} \text{ A/m}^2$, and the total diode separation is 16.51 meter.
FIG. 6/4, Model Pan29: Equipotential contours about a 1 m wide, infinitely long panel, uniformly charged to 100 V; modeled with 64 grid points (not shown). The plasma temperature and density are $kT = 10$ eV, $N_0 = 1.9 \times 10^4$/cc. The thermal currents are $J_{oe} = 1.61 \times 10^{-3}$ A/m$^2$ for electrons, and $J_{op} = 3.76 \times 10^{-5}$ A/m$^2$ for protons. The electron currents collected by the panel are indicated by the arrows below the panel and are normalized to $J_{oe}$; the average proton current to the panel is $1.5 \times 10^{-9}$ A/m$^2$. The corrected and uncorrected Child-Langmuir screening distances are indicated above and are respectively: $D_S = 1.73$ m, $D_{CL} = 1.2$ m. The X and Z unit of distance is 1.0 meter.
FIG.6/5, Model Pan36: The same panel and plasma of model Pan29 are modeled here with 90 grid points (not shown), and with a $V = 0$ boundary condition on both the UP and EAST boundaries. The distance unit is 1.0 meter.
FIG. 6/6: Potential (curve P) and charge density (curve C) profiles along the DOWN (6/6a) and WEST (6/6b) boundaries for the model Pan36. The density scale on the right is normalized to the ambient density, $N_0 = 1.9 \times 10^4$/cc. In 6/6b, the Child-Langmuir potential profile is indicated by the unconnected crosses, and the initial velocity corrected C-L screening distance, $D_s$, is indicated by the arrow. The distance unit is 1.0 meter.
FIG. 6/7: Potential (curve P), and electron and proton density profiles along the DOWN (6/7a) and WEST (6/7b) boundaries for the model Pan36. The corrected and uncorrected Child-Langmuir screening length estimates $D_S$ and $D_{CL}$ are indicated in 6/7b. The distance unit is 1.0 meter.
FIG. 6/8: illustrated above are three models used to approximate a charged disc. The Z unit of spacing is the same as that shown for X and Y. The radius $a_a$ is that of a disc of area equal to the square or polygon, $a_p$ is the radius deduced from substituting PANEL potentials into the analytic expression relating radius and potential (p64).
FIG. 6/9

\[ V_L(r) = \frac{V_O R}{r} \quad \text{(Laplacian, Coulomb)} \]

\[ V_D(r) = \frac{V_O R}{r} \exp\left[\frac{(R - r)}{\lambda_D}\right] \quad \text{(Debye)} \]

\[ V_P(r) = \text{PANEL result} \]

<table>
<thead>
<tr>
<th>Radial Distance ( r )</th>
<th>Potential, and inferred body radius ( R ) for ( \begin{pmatrix} X \ Y \ Z \end{pmatrix} )</th>
<th>( V_P(x) )</th>
<th>( V_P(y) )</th>
<th>( V_P(z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 m ( 10 \text{V} )</td>
<td>( 10 \text{V} )</td>
<td>( 10 \text{V} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 m ( 3.23 \text{V}, 0.65 \text{m} )</td>
<td>( 3.22 \text{V}, 0.64 \text{m} )</td>
<td>( 3.23 \text{V}, 0.66 \text{m} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 m ( 1.45 \text{V}, 0.58 \text{m} )</td>
<td>( 1.42 \text{V}, 0.57 \text{m} )</td>
<td>( 1.43 \text{V}, 0.57 \text{m} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 m ( 0.69 \text{V}, 0.55 \text{m} )</td>
<td>( 0.58 \text{V}, 0.47 \text{m} )</td>
<td>( 0.61 \text{V}, 0.52 \text{m} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12 m ( 0.50 \text{V}, 0.59 \text{m} )</td>
<td>( 0.24 \text{V}, 0.29 \text{m} )</td>
<td>( 0.38 \text{V}, 0.46 \text{m} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B.C.</td>
<td>( E = 0 )</td>
<td>( V = 0 )</td>
<td>( E = -\frac{V}{r} )</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: PANEL potentials (Laplacian) surrounding a point at a potential of 10V, and the inferred radius for the point assuming a Coulomb potential.

| Radial Distance \( r \) | \( V_P(x) \) | \( V_P(y) \) | \( V_P(z) \) | \( V_D(r) \) for:
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( R = 0.66 \text{m} )</td>
<td>( R = 0.55 \text{m} )</td>
<td>( R = 0.46 \text{m} )</td>
<td></td>
</tr>
<tr>
<td>0 m ( 10.0 \text{V} )</td>
<td>( 10.0 \text{V} )</td>
<td>( 10.0 \text{V} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 m ( 2.45 \text{V} )</td>
<td>( 2.45 \text{V} )</td>
<td>( 2.45 \text{V} )</td>
<td>( 2.57 \text{V} )</td>
<td>( 2.07 \text{V} )</td>
</tr>
<tr>
<td>4 m ( 0.67 \text{V} )</td>
<td>( 0.67 \text{V} )</td>
<td>( 0.67 \text{V} )</td>
<td>( 0.88 \text{V} )</td>
<td>( 0.71 \text{V} )</td>
</tr>
<tr>
<td>8 m ( 0.13 \text{V} )</td>
<td>( 0.13 \text{V} )</td>
<td>( 0.13 \text{V} )</td>
<td>( 0.21 \text{V} )</td>
<td>( 0.17 \text{V} )</td>
</tr>
<tr>
<td>12 m ( 0.04 \text{V} )</td>
<td>( 0.03 \text{V} )</td>
<td>( 0.04 \text{V} )</td>
<td>( 0.06 \text{V} )</td>
<td>( 0.05 \text{V} )</td>
</tr>
</tbody>
</table>

Table 2: PANEL potentials (Poisson) surrounding a point at a potential of 10V, compared to Debye model potentials for spheres of differing radii.