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DYNAMIC RESPONSE OF GROUND-EXCITED BUILDING FRAMES

by

J. L. ROEHL

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

Doctor of Philosophy

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1. Background

Inasmuch as the design philosophy for building frames which are likely to be subjected to high-intensity transient base excitations relies on the ability of the structure to deform without distress beyond the linearly elastic range, it is important that the response of such structures in the nonlinear and inelastic ranges of behavior be properly understood.

The evaluation of the dynamic behavior of inelastic systems may be carried out with the aid of high speed computers in a more or less straightforward manner by procedures that are well established. The interpretation of the results, however, is complicated by the large number of parameters that are involved and by the high sensitivity of the response to variations in several of these parameters.

Considering the complexity of the problem, a complete nonlinear analysis of a structure does not appear to be possible for design purposes at the present time. On the other hand, there is a need for basic information and concepts which may be used to estimate the significant aspects of the inelastic response of a structure based on a simple characterization of the system, or, perhaps, based on the dynamic response of the system in the linear range.

Of the numerous studies of the inelastic response of ground-excited systems that have been made in recent years, by far the larger group has been concerned with relatively simple structures behaving as single-degree-of-freedom systems.
While several aspects of the response of such systems require further study, the effect and relative importance of the various factors which influence the absolute maximum deformation of these systems are reasonably well understood, and simple approximate rules have been developed for relating the deformation spectra for certain classes of nonlinear systems to the corresponding spectra of similarly excited linear systems (1, 2). The latter spectra can, in turn, be related to certain gross characteristics of the acceleration, velocity and displacement of the exciting function.

With the exception of some recent studies (3), past investigations of the inelastic behavior of multi-degree-of-freedom (MDF) systems have been concerned almost exclusively with fairly complex models of multistory building frames subjected to earthquake excitations. Systems of the flexible girder type have also been considered (4, 5, 6, 7, 8, 9) but the response has been evaluated for only a few combinations of the many parameters involved. Although they have provided valuable insight into the response of the specific systems analyzed, on the whole these studies have not been sufficiently detailed and comprehensive to permit an adequate assessment to be made of the effect of the various parameters and to arrive at conclusions of wide applicability. It is generally recognized that this problem requires further study.

Studies of simple frames of the shear-beam type conducted recently at Rice University (3) have shown that the significant aspects of the response of a MDF nonlinear system can be estimated to a reasonable degree of accuracy from the results of a linear analysis. The concepts developed appear to be capable of ex-
tension to systems of greater complexity than those considered and it would be desirable to explore this possibility further. It would be particularly desirable to investigate the response of systems with flexible girders for which yielding may be concentrated in either the girders or the columns.

2. **Object**

   The objective of this investigation is to make a thorough evaluation of the nonlinear dynamic behavior of relatively simple multistory building frames which have flexible girders and are subjected to transient ground excitations. It is planned to assess the influence of the most important factors affecting the response of such systems, and to develop information and concepts whereby the significant aspects of the nonlinear response of multistory building frames may be estimated by simplified analyses.

3. **Scope**

   The present investigation was limited to uniform, five-story frames having from one to several bays and subjected to a pulse-like ground motion. In each case, the behavior of the system was first investigated under linear conditions of response, then its yield resistance was lowered to specified fraction of the value required for linear behavior, and the response of the nonlinear system was analyzed to determine the regions of inelastic action and the magnitude of such action for both the individual members and for the system as a whole. In this preliminary study, only the girders were allowed to yield, and the P-Δ effect was neglected. At later stages, the effect of inelastic action in the columns will
also be considered, as will also the P-Δ effect.

The results were analyzed with reference to the response of the associated linear systems, existing information on the response of appropriately chosen single-degree-of-freedom inelastic systems, and the concepts developed in the studies of multi-degree-of-freedom systems of the shear-beam type referred previously.

In addition to the dynamic response, the static behavior of the systems was studied under monotonically increasing lateral loads to determine the load-deformation characteristics and the magnitude of the nonlinear action; different distributions of lateral loads were considered to represent approximately the distributions of lateral inertia forces expected in selected frequency ranges of the spectrum and at different intensities of excitation.

The effect of the following principal parameters was investigated:

a) the duration of the exciting function as compared to the low-amplitude fundamental natural period of the structure

b) the value of relative stiffness of girders and columns

c) the yield resistance of the individual members, and

d) the magnitude of the direct lateral load.

The governing equations of motions were integrated numerically by step by step integration procedure which approaches the nonlinear problem as a sequence of linear solutions. Whenever the characteristics of the system change as a result of yielding in a member, the necessary modifications in the stiffness matrix of the system are made considering the properties of the structure to
remain constant within each integration step.

The complete solution involves two major parts: (1) the reduction of the general stiffness matrix of the frame into the lateral stiffness matrix, which involves only the horizontal forces associated with lateral displacements of the floor levels, and (2) the integration of the resulting equations of motion by a numerical procedure, using an iterative scheme and integration formulas which assume the acceleration of the system to vary linearly with respect to time within each integration interval.

The elastic response of the system was evaluated by the modal superposition method to determine the contributions of the various natural modes and to investigate the extent to which the generalized concept of modal analysis used to interpret the nonlinear response of systems of the shear-beam type are also applicable to the more complex systems considered in this study.
II. METHOD OF ANALYSIS

1. General

The methods used in this study are for the analysis of the following structural systems:

a. A plane building frame with flexible members, subjected to a transient ground motion, which is represented by a piecewise linear acceleration diagram. The response in both the elastic and inelastic ranges of deformation is investigated.

b. The same frame subjected to a monotonically increasing static lateral load, the spatial distribution of which is fixed.

The methods used are well known in structural analysis and generally need not be described in detail; however, the following special features of the method deserve attention, and will be considered in detail:

a. the procedure used in the dynamic analysis to account for the slope discontinuities in either the acceleration diagram of the ground motion or the load-deformation relationship of the system;

b. the procedure used to evaluate the change in the stiffness of the system due to the formation or removal of plastic hinges.

2. System Considered

The rectangular plane frames considered in this study were assumed to have the following characteristics:

- rigid joints and fixed column bases;
prismatic members, with inelastic action assumed to be localized in plastic hinges at the ends; the moment-angle relationship for the plastic hinges is assumed to be of the rigid-elastic type with hysteretic action;

- axial and shear deformations are negligible compared to those due to bending;

- the effect of the axial force in a member on the rotational stiffness of the member is small in comparison with the effect of end moments;

- masses are concentrated at the joints:

- the inertia forces associated with the rotations of the joints can be disregarded since they are associated with frequencies of vibration which are much higher than those associated with the lateral motion;

- damping is of the viscous type.

3. **Mathematical Model**

Under the assumptions referred to above, the frame may be represented by the model shown in Fig. 1. Each member is considered to consist of a prismatic elastic element and two rigid-elastic hinges at the end. If $\theta_e$ represents the rotation at an end of the elastic element of a member, and $\alpha$ the plastic rotation or kink of the hinge at that end, then the rotation $\theta$ at the junction of the member with an adjoining member (the joint rotation) is

$$\theta = \theta_e + \alpha.$$  

The moment-kink relationship for the hinge is considered to be of the rigid-elastic type, as shown in Fig. 3. The yield levels in the two directions of deformation need not necessarily be the same, and unloading and reloading from regions of
inelastic action are assumed to take place along vertical lines as indicated in the figure.

It is desirable to comment on the relationship between the plastic angle \( \alpha \) and the curvature \( \phi \) of the end of the member in the original structure. Let the moment-curvature relationship for the end of the original member, \( M-\phi \), be represented by a bilinear diagram as shown in Fig. 4a, and \( M_y \) and \( \phi_y \) be the limiting elastic values of \( M \) and \( \phi \). Also, let the slope of the \( M-\alpha \) diagram be represented as the product of a dimensionless coefficient \( d \) and the Hardy Cross stiffness for the member, \( 4EI/L \), as shown in Fig. 4b. It can be shown (see Appendix I) that the plastic angle, \( \alpha \), is related to \( \phi \) by the equation

\[
\alpha = (\phi - \frac{M}{M_y} \phi_y) \ell_y
\]  

(1a)

in which \( \ell_y \) is the so-called effective length over which yielding in the original member is assumed to be spread. The value of \( \phi \) along \( \ell_y \) is assumed to be constant in this interpretation. For the special case of a elastoplastic \( M-\phi \) diagram (corresponding to a rigid-plastic \( M-\alpha \) relationship), Eq. 1a reduces to

\[
\alpha = (\phi - \phi_y) \ell_y
\]  

(1b)

It should be noted that \( \ell_y \) is assumed to be constant in the procedure used herein. In actuality, \( \ell_y \) varies as a function of the magnitude and distribution of yielding in the system. The use of a variable \( \ell_y \) would imply a variable second slope coefficient, \( d \), in the \( M-\alpha \) diagram with a corresponding increase in the computation effort required. This refinement is considered unnecessary at the present stage of knowledge.
4. Stiffness Matrices

Consider a typical member supported at the ends, and let \( u \) be the relative displacement of the supports, \( \theta_L \) and \( \theta_R \) be the rotations at the left and right supports, respectively, and let \( Q_L \) and \( Q_R \) be the associated force and moments, as shown in Fig. 2. Also, let \( \Delta u \), \( \Delta \theta_L \), \( \Delta \theta_R \), \( \Delta Q_L \), \( \Delta M_L \), and \( \Delta M_R \) be the changes in these quantities during a small time interval \( \Delta t \). The changes in the force and moments are related to the displacement changes by the equation

\[
\begin{bmatrix}
\Delta Q \\
\Delta Q_L \\
\Delta Q_R \\
\Delta M_L \\
\Delta M_R
\end{bmatrix} = [k] \begin{bmatrix}
\Delta u \\
\Delta \theta_L \\
\Delta \theta_R
\end{bmatrix}
\]

(2)

in which \([k]\) is the instantaneous stiffness matrix of the member and may be written as

\[
[k] = \begin{bmatrix}
k_{11} & k_{12} & k_{13} \\
k_{12} & k_{22} & k_{23} \\
k_{13} & k_{23} & k_{33}
\end{bmatrix}
\]

(3)

and each of these elements of \([k]\) can be expressed (5) in terms of the Hardy Cross stiffness, \( 4EI/L \), and the second slope coefficient of the M-\( \alpha \) diagram, \( d \), as shown in Table 1.

| Table 1. Expressions for Element \( k_{ij} \) of Stiffness Matrix \([k]\), in terms of \( 4EI/L \). |
|---------------------------------|----------------|----------------|----------------|----------------|
| \( k_{22} \)                    | 1              | \( 0.75+d \)   | \( d \)         | \( d(1+\frac{4}{3}d) \) |
|                                |                | \( \frac{1+d}{1+d} \) | \( \frac{1}{1+d} \) | \( 1+\frac{4}{3}(2d+d^2) \) |
| \( k_{33} \)                    | 1              | \( \frac{d}{1+d} \) | \( \frac{0.75+d}{1+d} \) | \( \frac{d(1+\frac{4}{3}d)}{1+\frac{4}{3}(2d+d^2)} \) |
|                                |                | \( \frac{1}{1+d} \) | \( \frac{1}{1+d} \) | \( \frac{1+\frac{4}{3}(2d+d^2)}{1+\frac{4}{3}(2d+d^2)} \) |
| \( k_{23} \)                    | 0.5            | 0.5d            | 0.5d            | \( \frac{2d^2}{1+\frac{4}{3}(2d+d^2)} \) |
and
\[ k_{13} = \frac{k_{23} + k_{33}}{L} \]
\[ k_{12} = \frac{k_{22} + k_{32}}{L} \]
\[ k_{11} = \frac{k_{12} + k_{13}}{L} \]

Now let \( F \) be the vector of the instantaneous values of the total horizontal forces on the floors of the frame, and \( \Delta F \) be the vector of the changes in these forces during a time interval \( \Delta t \). Also, let \( M \) be the vector of the unbalanced bending moments at the joints, and \( \Delta M \) be vector of the changes in these moments. Similarly, let \( x \) and \( \theta \) be the vectors of the floor displacements relative to ground and of the joint rotations, respectively, and \( \Delta x \) and \( \Delta \theta \) be the vectors of the changes in these displacements. The positive directions of the forces and displacements, and the designations of the floors and joints are shown in Fig. 1. The elements of \( x \) and \( F \), identified with the appropriate subscripts, are arranged in the order \( 1, 2, 3, \ldots S \), and the elements of \( \theta \) and \( M \) are arranged in the order \( 11, 12, \ldots, 1C; 21, 22, \ldots, 2C; 31, 32, \ldots, 3C; \ldots S1, S2, \ldots, SC \). The symbol \( C \) denotes the number of columns, and \( S \) denotes the number of stories. The relationship between the force changes, \( \Delta F \) and \( \Delta M \), and the displacement changes, \( \Delta x \) and \( \Delta \theta \), may be stated as

\[
\begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta \theta
\end{bmatrix} =
\begin{bmatrix}
\Delta F \\
\Delta M
\end{bmatrix}
\]  

(4)

where

\[ [K_{11}] = \text{a tri-diagonal submatrix of size } S \times S, \text{ representing the horizontal forces associated with horizontal displacements of the floors and no rotations of the joints;} \]
\[ [K_{22}] = \text{a banded submatrix of size } J \times J \text{ representing the joint moments associated with joint rotations and no horizontal displacement,} \]
where \( J = C \times S \) = the number of joints in the frame. This matrix has \( C \) nonzero upper off-diagonals.
\[ [K_{12}] = [K_{21}]^T \text{ = a submatrix of size } S \times J \text{, representing the horizontal forces associated with joint rotations and no horizontal displacements.} \]

These stiffnesses are generated by combining in a suitable manner the member stiffness matrices \([k]\) by use of standard procedures of structural matrix analysis.

The ordering of the displacements and rotations in Eq. 4 and the form of the associated matrices differ from those used in previous studies (4, 5, 6, 7, 8, 9). The particular arrangement used herein was deemed particularly well suited to the procedure used to account for the modification in the stiffness due to yielding and to the procedure used to integrate the equations. These matters are discussed further in a later section.

The lateral stiffness matrix of the system, representing the horizontal forces associated with horizontal displacements with no constraint on the rotation of the joints, may be obtained from the stiffness matrices in Eq. 4 as follows. Rewriting Eq. 4 in the form,

\[ [K_{11}] [\Delta x] + [K_{12}] [\Delta \theta] = [\Delta F] \quad \text{(5a)} \]
\[ [K_{21}] [\Delta x] + [K_{22}] [\Delta \theta] = [\Delta M] = [0] \quad \text{(5b)} \]
and solving for \( [\Delta \theta] \) from Eq. 5b, one obtains
\[
[\Delta \theta] = [T] [\Delta x]
\] (6)

in which
\[
[T] = - [K_{22}]^{-1} [K_{21}]
\] (7)

Substitution of Eq. 6 into Eq. 5a yields
\[
[K'] [\Delta x] = [\Delta F]
\] (8)

in which
\[
[K'] = [K_{11}] + [K_{12}] [T]
\] (9)

is a full matrix of size \( S \times S \), referred to as the lateral stiffness matrix.

For a linear system, the relation between the forces \( F \) and the displacements \( x \) is the same as that given in Eq. 8 for the incremental values.

5. Dynamic Analysis

5.1. General. Inasmuch as both the linear and nonlinear behavior of the frame were to be studied, and inasmuch as the expansion of the linear solution in terms of the modal contributions is helpful in understanding both the linear and the nonlinear behavior, an independent linear solution by modal analysis was also developed. Of course, the method used in the analysis of the nonlinear case may also be employed to obtain the linear response.

5.2. Equations of Motion. The equations of motion for the idealized system considered may be written in an incremental form as follows
\[
[m] \{\Delta \ddot{x}\} + [c] \{\Delta \dot{x}\} + [K'] \{\Delta x\} = - \ddot{y} (t) [m] \{f\}
\] (10)

where
\[
[m] = \text{diagonal mass matrix of size } S \times S
\]
\([c] = \text{damping matrix of size } S \times S\)

\(\dot{y}(t) = \text{input ground acceleration function}\)

In the following, the damping matrix \([c]\) will be taken in the form

\[
[c] = \alpha [K'] + \gamma [m]
\]

where \([K']\) must be interpreted as the stiffness of the system in the linear range of deformation. In other words, \([c]\) is considered to be constant. Equation 11 makes possible the uncoupling of the equations for linear response and offers no difficulty in the method used for nonlinear or inelastic response.

5.3. Dynamic Linear Analysis

5.3.1. Uncoupling of the Equations of Motion. Equations 10 were uncoupled using the orthogonality properties of the natural modes as shown in Appendix II. Specifically, the displacement vector \([x]\) was expressed in the form

\[
[x] = [\phi] [\eta]
\]

where \([\phi]\) is square matrix of the natural modes \(\phi_1, \phi_2, \ldots, \phi_i, \ldots, \phi_S\). It can then be shown that the \(r\)th element of \([\eta]\) is given by the solution of

\[
\ddot{\eta}_r + 4\pi \zeta_r f_r \dot{\eta}_r + 4\pi^2 f_r^2 \eta_r = - C_r \dot{y}(t)
\]

in which \(f_r\) is the \(r\)th natural frequency of the system in c.p.s.; \(\zeta_r\) is the damping factor for the \(r\)th mode; and \(C_r\) is the participation factor for the \(r\)th mode, given by the equation

\[
C_r = \frac{[\phi_r]^T [m] \{1\}}{[\phi_r]^T [m] \{\phi_r\}}
\]

If one expresses \(\eta_r\) in the form \(\eta_r = C_r y_o (I.A.F.)_r\) in which \((I.A.F.)_r\) is the instantaneous amplification factor for the deformation of a single-degree-
of-freedom system having a natural frequency \( f_r \); then any effect \( E(t) \) in the structure can be expressed as

\[
E(t) = \sum_{r=1}^{S} E_r \text{(I.A.F.)}_r
\]

where \( E_r \) is the static component of the effect considered in the \( r \)th mode. Expressed differently, \( E_r \) represents the static effect induced in the structure by the displacement \( C_r y_0 \{\phi_r\} \).

5.3.2. Integration Techniques. The solution of Eqs. 12 was obtained numerically by assuming the accelerations of the masses to vary linearly during a small time interval. The details of the procedure are given in Appendix III. The criterion for stability of this procedure has been discussed in Ref. 10.

5.3.3. Integration Step. The time interval of integration used, \( \Delta t \), was smaller than, or at most equal to, the following two values:

(a) \( \Delta t = T_1/50 \), where \( T_1 \) is the fundamental or longest natural period of the system in its elastic range of response, and

(b) \( \Delta t = T_S/r \), where \( T_S \) is the shortest natural period, and \( r = 20, 15, 10 \) for \( f_1 < 0.35 \text{ c.p.s.} \), \( 0.35 \leq f_1 \leq 1.5 \) and \( f_1 > 1.5 \text{ c.p.s.} \), respectively. All frequencies and periods refer to the elastic system.

The first criterion was used to insure accurate resolution of the response of the system in the fundamental mode, whereas the second criterion was used to insure stability of the solution and adequate representation of the contribution of the higher modes. Some exceptions were made in the case of high frequency systems for which the contributions of higher modes were expected to be small.
The integration step $\Delta t$ was further subdivided to account for all points of discontinuity in the slope of the input accelerogram. The computations were generally carried out beyond the pulse for a period equal to $1.5T_1$.

5.4. Dynamic Nonlinear Analysis

5.4.1. Form of Equations Used. For this analysis, the equations of motion were used in the form

$$[m] [\Delta \ddot{x}] + (\alpha [K'] + \gamma [m]) [\Delta \dot{x}] + [K'] [\Delta x] = -\ddot{y}(t) [m] [1] \quad (15)$$

5.4.2. Integration Techniques. The integration of these equations was performed using the formulas referred to in Section 5.3.2, along with an iterative scheme in which the values of the acceleration at the end of the integration step are assumed and used to compute the associated velocities and displacements. These displacement and velocity values are then substituted in the equations of motion and new values are obtained for the acceleration and compared with the assumed values. With an adequate integration interval, this procedure converges in 2 or 3 cycles (see Appendix III.). This iterative scheme was used in preference to the direct technique used in previous studies (4, 5, 6, 7, 8, 9) in order to avoid the necessity of inverting matrices every time that the integration interval had to be changed to account for the slope discontinuities either in the acceleration diagram of the ground motion or in the load-deformation relationship of the system.

Whenever the yield state of the frame changed, the appropriate stiffness modifications were introduced taking advantage of the fact that only a few columns of the matrix $[K_{22}]$ which must be inverted to obtain the new value of $[K']$ are modified at a time. The details of the procedure used to account for these changes
are given in Appendix IV. The modifications in \([K']\) were introduced practically at the exact moment they occurred, using a linear interpolation to subdivide the integration step for this purpose. The details of the interpolation scheme are discussed in the next chapter.

With this approach, the computation time required to vary the integration step and to account for the stiffness changes was greatly reduced. The iterative scheme used becomes inefficient for high-frequency systems, beyond those considered in the present study, for which the time interval of integration required to insure convergence of the procedure becomes very small. This is particularly true in the very-high-frequency subregion of the spectrum for which the contribution of the higher modes is practically negligible, and one might employ larger integration steps if a non-iterative procedure is used.

5.4.3. Integration Step. In this case, the integration interval size had also to satisfy the requirements related to the convergence of the iteration and to the adequacy of the linear interpolation scheme to determine the exact instants of occurrence and disappearance of hinges. As shown in Appendix III., the integration interval must be less than \(0.39T_s\) for the procedure to be convergent. These requirements were satisfied using the same criteria as those for the linear problem described in Section 5.3.3.

The duration of the integration used for the nonlinear solutions was also the same as for the linear solutions.

5.4.4. Dead Weight Effect. This refers to the static effect produced by the dead load on the structure. This effect was considered on the assumption
that the load on the girders is uniformly distributed and that the intensity of it is below that required to cause yielding.

The key steps in the procedure used to consider this effect may be summarized as follows. Let \( \{ M_F \} \) be the vector of the unbalanced fixed-end moments due to the dead weight. Then

\[
\begin{align*}
[K_{11}] \{x\} + [K_{12}] \{\theta\} &= \{0\} \\
[K_{21}] \{x\} + [K_{22}] \{\theta\} &= \{ M_F \}
\end{align*}
\] (16) (17)

From Eqs. 17 and 7

\[
\{\theta\} = [K_{22}]^{-1} \{ M_F \} - [K_{22}]^{-1} [K_{21}] \{x\} = [K_{22}]^{-1} \{ M_F \} + [T] \{x\}
\]

Substituting this expression for \( \{\theta\} \) in Eq. 16, and noting that \( [K_{12}] = [K_{21}]^T \) and that \( [K_{22}] \) is symmetric, one obtains

\[
([K_{11}] + [K_{12}][T]) \{x\} = -[K_{12}] [K_{22}]^{-1} \{ M_F \} = [T]^T \{ M_F \}
\]

Finally, making use of Eq. 9, one obtains

\[
\{x\} = [K']^{-1} [T]^T \{ M_F \}
\]

\[
\{\theta\} = [K_{22}]^{-1} \{ M_F - [K_{21}] \{x\} \}.
\]

Note that the above expressions involve the same matrices as those needed in the dynamic analysis.

With \( \{x\} \) and \( \{\theta\} \) determined, the forces at the end of the members are evaluated by standard procedures.

5.4.5. P-\( \Delta \) Effect. The analysis described so far neglected the effect of the axial forces in the members on the stiffnesses of these members. This effect was investigated approximately by considering only the so-called P-\( \Delta \) effect which refers to the effect of the moment produced by the axial force on a member,
P, acting through the relative displacement of the ends of the member, u. More specifically, consideration of this effect changes the element \( k_{11} \) of the member stiffness in Eq. 3 to

\[
k_{11}^* = k_{11} + \frac{P}{L}
\]

This change, in turn, causes corresponding changes in the submatrix \([K_{11}]\) of Eq. 4 and in the lateral stiffness matrix, \([K']\), in Eq. 8.

It is a simple matter to show that the change in the lateral stiffness matrix, \([\Delta K']\), is given by

\[
\begin{bmatrix}
\frac{P_1 + P_2}{h_1} & -\frac{P_1}{h_2} \\
\frac{P_2}{h_2} & -\frac{P_2}{h_3} \\
\end{bmatrix}
\]

in which \(P_i\) is the instantaneous value of the sum of the axial forces in the columns of the ith story. This value is, in turn, equal to the total dead load above the ith story.

Note that the matrix \([\Delta K']\) is time-invariant and that its inclusion does not complicate the analysis.

For a specified distribution of dead load, the smallest value of the load for
which the stiffness matrix \([K' + \Delta K']\) becomes singular is designated as the "approximate sidesway buckling load". This load is evaluated by solving the characteristic value problem \([K'] [x] = -\nu [\Delta K'] [x]\) in which \(\nu\) is an appropriate load factor.

In investigating the P-\(\Delta\) effect, it is desirable to express the magnitude of the total dead load on the structure as a percent of this critical load.

6. **Static Analysis**

6.1 **General.** Knowledge of the behavior of the frame under a monotonically increasing static lateral load is helpful in understanding the dynamic response of the system in the inelastic range. For the solution of this problem, it was found preferable to write the force-displacement relationship in the form recommended in Ref. 4 rather than in the form used for the dynamic analysis referred to in the preceding sections. In this approach, the story displacements and joint rotations are grouped in such a way that the stiffness matrix is banded with 2\(C + 1\) nonzero upper off-diagonals.

6.2 **Equilibrium Equations.** The equilibrium equations for elastic response are written in the form:

\[
\begin{bmatrix}
K_s
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_s
\end{bmatrix} =
\begin{bmatrix}
Q_1 \\
Q_2 \\
Q_s
\end{bmatrix}
\]

in which \(q_i\) is the displacement vector \((x_{i1}, \theta_{i1}, \theta_{i2}, \ldots, \theta_{iC})\) and \(Q_i\) is the force vector \((F_i, M_{i1}, M_{i2}, \ldots, M_{iC})\); if only lateral loads are considered, then \(M_{i1} = M_{i2} = \ldots = M_{iC} = 0\). The stiffness matrix \([K_s]\) is obtained from the
member stiffness defined in Section 4. by standard procedures.

6.3 Solution for Increasing Lateral Loads with Prescribed Distribution.

This solution is obtained by considering successively the effects of a series of static load increments, with the magnitude of each load increment selected to as to produce a new hinge in the structure.

The solution is started by forming the stiffness matrix $[K_s]$ for the linear system, and by inverting it to compute the associated flexibility matrix $[F_s]$. This was done by use of a standard IBM Subroutine which takes advantage of the banded nature of the matrix. The response of the system is next evaluated for a lateral load of unit intensity, and the load intensity associated with the formation of the first hinge is computed, along with the associated forces and stresses on the structure.

Then, the flexibility matrix $[F_s]$ is modified considering the changes which are introduced in the stiffness matrix $[K_s]$ due to the formation of the hinge. These changes are concentrated only in a few columns of the original stiffness matrix. This fact enables one to use with advantage the same procedure as that used in modifying the matrix $[K_{22}]^{-1}$ in the dynamic nonlinear analysis. The details of the procedure are described in Appendix IV.

The modified structure with one plastic hinge is then analyzed for a new load increment, and the intensity of the load needed to produce the second hinge is determined. The procedure is repeated until the rate of increase of displacements with respect to load exceeds a prescribed value, or until some desired value of load or displacement is reached.
III. COMPUTER PROGRAMS

1. General

Three independent computer programs were developed for a B-5500 Burroughs Computer using XALGOL. Two of these programs are for the analysis of the dynamic response of the frames to a prescribed ground excitation; the first considers the linear case using modal superposition, and the second is concerned with the nonlinear behavior. The third program analyzes the static behavior of the frame under a monotonically increasing lateral load.

2. Program for Dynamic Linear Response

2.1. Scope and Flow Chart. This program considers a frame 5 stories high and B bays wide subjected to a ground motion specified by an arbitrary, piecewise linear accelerogram. The program computes the maximum values of the response quantities for the total solution as well as for the partial solutions obtained by considering only the first, first two, and first three modes. The effect of the vertical loads on the girders was not incorporated in this program, as this effect may be evaluated independently and superposed on the effect of the ground excitation.

A general flow chart for this program is given on the next page.

2.2. Details of Program. The program organization follows standard procedures which do not require detailed discussion; it may be of interest, however, to comment on the following features of the program.

To find the lateral stiffness matrix, \([K']\), it is necessary to invert the sub-
Program for Dynamic Linear Analysis

Flow Chart

1. Begin
2. Read General Data and Process Input
3. Set up Stiffness Matrix
4. Find Lateral Stiffness Matrix
5. Compute Natural Frequencies and Mode Shapes
6. Evaluate Modal Contributions
7. Uncouple and Prepare Equations
8. Read $f_1$, Time Step and Duration of Integration
9. Set Damping and Stiffness Coefficients
10. Execute Integration
11. Output Results
12. End
matrix $[K_{\infty}]$ in Eq. 4. The matrix $[K_{\infty}]$ is positive definite, symmetric and banded, with C upper off-diagonals. The procedure used to perform the inversion takes advantage of these features to reduce the number of operations required, as this is the most important and time-consuming single step in the solution.

The integration was carried out with a constant time increment, using the same value for each modal contribution, so that the component effects may be superimposed. The integration step was further subdivided to account for all changes in the slope of the input acceleration.

The response quantities evaluated include the story displacements, the shear per story, the overturning moment for each story, the bending moments in the girders and columns, and the axial forces in the columns.

3. Program for Dynamic Nonlinear Response

3.1. Scope and Flow Chart. This program considers the same frame as before, but assumes that the end sections of the members follow a bilinear, rigid-elastic moment-plastic angle relationship. Since the modal superposition method is not valid in this case, the equations of motion are integrated in their original form following an iterative scheme. This program includes options to consider or neglect the effect of uniformly distributed vertical loads on the girders, and to consider or neglect the P-\(\Delta\) effect.

A general flow chart for this program is given on the next page.

3.2. Details of Program. The integration is generally carried out using a constant time increment, but the latter is subdivided, as needed, to include the times corresponding to:
Program for Dynamic Nonlinear Analysis

Flow Chart

Begin

Read General Data and Process Input

Set up Stiffness Matrix

Find Lateral Stiffness Matrix

Compute Natural Frequencies and Mode Shapes

Read $f_1$ and other Particular Data

Set Yield Moments

Compute Vertical Load Effects, If so

Include P-$\Delta$ Effect, If so

Set Damping and Stiffness Matrices According to $f_1$

Execute Integration

Output Results

End
1. a change in the slope of the ground acceleration; and

2. the formation or disappearance of a plastic hinge.

The first item has already been discussed; the second is discussed under the following two headings:

**Placement and Removal of Hinges.** A hinge must be placed when the bending moment at the end of a member exceeds the yield moment. The time corresponding to this event was determined by assuming that all moments vary linearly with respect to time during the small integration interval, \( \Delta t \), and a linear interpolation was used to subdivide \( \Delta t \) into \( \Delta t' \) and \( \Delta t'' \) as shown in Fig. 5a. However, as this procedure may admit some overpassing of the yield level, it may lead to some difficulty later on when a hinge must be removed. To avoid this potential difficulty, an additional condition was imposed. A hinge was placed when

\[
|M| > |M_y|
\]

and, in addition, \( M \) and \( \dot{M} \) had the same sign.

When a hinge is to be removed the problem is different because it is not possible to determine accurately the instants of unloading or reloading by just examining the values of the moments or of the plastic angles \( \alpha \). It is also necessary to examine the first derivatives of the moments or plastic angles, to determine the extremum values of these quantities. If the second slope factor of the \( M - \alpha \) diagram, \( d \), is different from zero, then one may work with the \( M - t \) function; however, for \( d = 0 \) the first derivative of \( M \) with respect to the time is zero during yielding, and it is necessary to work with the \( \alpha - t \) function. The procedure used is as follows:

1. After yielding starts at some member end, the program examines the values of \( \dot{\alpha} \), or of \( \dot{M} \) if \( d \neq 0 \).
2. When a point is reached such that the sign of $\dot{\alpha}$, or $\dot{M}$, changes during integration interval, linear interpolation is used to approximate the instant that this change took place. Thus, the integration interval is subdivided into two subintervals.

3. The integration is carried out for the first subinterval and the necessary modification in the properties of the system are made.

4. The integration process is then continued by considering the second subinterval with the modified properties of the system.

When the integration subinterval which follows the removal of a hinge is too small, two difficulties to the continuation of the integration may arise. One occurs when the value of $\Delta M$ corresponding to this small subinterval is smaller than the amount by which the yield level had been overpassed. Since the bending moment at the location under consideration remained beyond the yield moment, at the end of the subinterval the program would tend to replace the hinge which was just removed. This difficulty was avoided by comparing the signs of $M$ and $\dot{M}$, as previously indicated. A second difficulty may arise in the case shown in Fig. 6 where, after the first subinterval has been taken and the hinge removed, one may find that the newly computed value of $\dot{M}$ or $\dot{\alpha}$ is still of the same sign as at the beginning of the interval. Now, if the next subinterval is very small, it may turn out that at the end of this subinterval $|M|$ is still greater than $|M_y|$ and $M$ and $\dot{M}$ are still of the same sign. The program in this case would tend to replace the hinge that was just removed, even though this does not represent the true behavior. To avoid this difficulty, the value of $|\dot{M}|$ or $|\dot{\alpha}|$ was checked to in-
sure that it was less than a prescribed tolerance before the hinge was removed. If not, the time interval was subdivided further.

**Possibility of Overlooking Yielding.** Fig. 5b illustrates a case in which the moment computed at the end of a member for both the beginning and the end of the integration interval is less than the yield value moment, but the maximum moment actually exceeds the yield moment. The dotted portion of the curve indicates the variation of the moment obtained on the assumption that the system behaves linearly. The fact that the structure does not, in reality, behave linearly within the integration interval and that the member yields could not be recognized by merely examining the values of the moment. By examining both \( M \) and \( \dot{M} \) it is possible to recognize this fact and by properly subdividing the integration interval to account for the true plastic action.

A few additional comments are in order before completing this discussion of the computer program. First, it should be noted that the linear interpolation used to locate the extreme points of the \( M - t \) and \( \alpha - t \) functions may be expected to be quite accurate, since these functions are reasonably well approximated by quadratic functions and their first derivatives by straight lines.

The second comment relates to the expressions used to compute the quantities \( \dot{M} \) and \( \dot{\alpha} \). For the typical member shown in Fig. 2 the incremental end moments are related to the incremental end rotations and relative end displacements by Eq. 2. Dividing both members of this equation by the integration interval, \( \Delta t \), and considering \( \Delta t \) to be small one obtains

\[
\dot{M}_l = k_{12} \ddot{t} + k_{22} \dot{\theta}_l + k_{33} \dot{\theta}_r
\]
\[ \dot{M}_r = k_{13} \dot{u} + k_{23} \dot{\theta}_f + k_{33} \dot{\theta}_r \]

The following expressions for \( \dot{\alpha} \) are obtained in a similar manner by first expressing the \( \alpha \)'s in terms of the end moments, and then expressing the moments in terms of the end displacements.

\[
\dot{\alpha}_f = \left( 0.5 - \frac{2k_{12} - k_{13}}{3} \right) \dot{u} + \left( 1 - \frac{2k_{22} - k_{23}}{3} \right) \dot{\theta}_f - \left( \frac{2k_{23} - k_{33}}{3} \right) \dot{\theta}_r 
\]

\[ \dot{\alpha}_r = \left( 0.5 - \frac{2k_{13} - k_{12}}{3} \right) \dot{u} - \left( \frac{2k_{23} - k_{22}}{3} \right) \dot{\theta}_f + \left( 1 - \frac{2k_{23} - k_{33}}{3} \right) \dot{\theta}_r \]

By similar reasoning one may consider Eq. 6 and write

\[ \{ \ddot{\theta} \} = [T] \{ \ddot{x} \} \]

This equation shows that \{ \ddot{\theta} \} is discontinuous at the instants of formation or removal of a hinge because the matrix [T] changes at those points. Consequently, \{ \ddot{M} \} and \{ \ddot{\alpha} \} are also discontinuous, and it is necessary to recompute these quantities following each addition or removal of a hinge.

It should finally be noted that several "singular points" may occur in different locations in the structure during the one integration interval. The program determines which of these points is reached first and considers the smallest subinterval in each case.

4. Program for Static Analysis

4.1. Scope and Flow Chart. This program may be used to analyze the response of a frame S stories high and B bays wide to monotonically increasing lateral loads at the floor levels and uniformly distributed constant vertical loads on the floor girders. A general flow chart for this program is given on the next page.
Program for Static Analysis

Flow Chart

Begin

Read Data and Process Input

Set Lateral Load Distribution

Set Stiffness Matrix

Find Flexibility Matrix

Compute Vertical Load Effects, If so

Determine Load increment to Form Next Hinge

If Max Load is Overpassed then Stop at Max Load

Determine Displacements & Stresses at Hinge Form.

Modify Flexibility Matrix and Check for Singularity

Output Results

End
4.2. **Details of Program.** The program may consider any distribution of lateral loads applied at the floor levels. The special cases of a uniform distribution and of a triangular distribution with the maximum either at the top or at the bottom may be specified by single parameters. The moment-plastic angle relationship for the end of the members was assumed to be of the rigid-plastic type in this program.

The formation of the hinges under increasing lateral load is determined incrementally according to the method of analysis described in Chapter II, Section 6.3. When a hinge is formed the flexibility matrix of the system is modified by the simplified procedure described in Appendix IV.

The output quantities include the load intensities, joint displacements, member forces, and the distribution and magnitude of inelastic action in the joints.

5. **Verification of Accuracy**

In order to verify the accuracy of the programs for dynamic analysis, a comparison was made of results obtained with these programs and corresponding data obtained with the programs used in the studies reported in Refs. 12 and 13. The first of the latter programs (12) gives for all practical purposes an exact solution for SDF systems. The second program (13), applicable to multi-degree-of-freedom (MDF) inelastic systems of the shear-beam type, uses an interactive procedure to converge to the exact instants of extremum deformation and of formation and removal of plastic hinges. By contrast, the new programs determine these instants approximately by linear interpolation.
Tables 1 and 2 summarize representative results for SDF systems and MDF uniform systems. The ground motions considered include the simple pulse-type excitation shown in Fig. 8, and the first 6.29 seconds of the NS component of the El Centro earthquake of 1940, as digitized in the studies reported in Refs. 12 and 13. The symbol C in the tables denotes the ratio of the yield point deformation of the system to the absolute maximum deformation for the associated linear system.

It can be seen that the two sets of results in Tables 1 and 2 are generally in good agreement. Note further that the results obtained by the simplified version of the inelastic program are inferior to those obtained by the complete program, and that the difference in the two results increases for the low values of the yield level. Finally, comparison of the results obtained by the linear and inelastic programs shows that the first, even without any provisions to improve the solution in the neighborhood of the maxima of the response quantities, presents very good accuracy. This is attributed to the fact that this program uses uncoupled equations and a direct integration scheme.
### TABLE 1. COMPARISON OF RESPONSE DATA FOR SDF SYSTEMS

Systems considered to be undamped for simple pulse and to have two percent of critical damping for earthquake input.

(a) For Simple Pulse

<table>
<thead>
<tr>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>( C )</th>
<th>( u/y_0 )</th>
<th>( \Delta t )</th>
<th>( u/y_0 )</th>
<th>( \Delta t )</th>
<th>( u/y_0 )</th>
<th>( \Delta t )</th>
<th>( u/y_0 )</th>
<th>( \Delta t )</th>
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<td>1</td>
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<td>0.981</td>
<td>250</td>
<td>0.959</td>
<td>0.980</td>
<td>250</td>
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(b) For Earthquake Excitation

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</table>

*In this version of the program the integration interval, \( \Delta t \), is subdivided only as needed to include the maximum and minimum points of the accelerogram.*
TABLE 2. COMPARISON ON RESPONSE DATA FOR UNIFORM MDF SYSTEMS OF SHEAR BEAM TYPE

All masses, spring stiffnesses and yield deformation are considered to be the same. Systems for simple input are considered to be undamped, whereas those for earthquake input are considered to have 2 percent of critical damping.

(a) For Simple Pulse (5DF)

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(b) For Earthquake Excitation (3DF)

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IV. PRESENTATION AND DISCUSSION OF RESULTS

1. General

The present study considered special five-story frames having 1, 3 and 5 bays as shown in Fig. 7, and subjected to a pulse-like ground motion. The system was first studied under linear conditions of response, then the yield resistance of the girders was lowered to specified fraction of the value required for linear behavior, and the response of the nonlinear system was analyzed. In addition, the static behavior of the system under monotonically increasing lateral loads, as well as the behavior under prescribed distribution of lateral deflections, were studied as an aid in understanding the dynamic behavior at particular regions of the response.

This investigation was limited to these relatively simple frames and excitation to make possible consideration of a wide variation of the parameters which govern the response, and the establishment of concepts and ideas which it is hoped are also capable of extension to more complex situations.

2. Ground Motion

The ground motion considered was the half-cycle displacement pulse shown in Fig. 8. This simple excitation was used since it has been shown (3) that there is a close relationship between the response of systems subjected to pulse-type inputs and to other transient ground motions, such as earthquakes, and that considerable insight into the seismic response of structures may be gained from studies of the effects of simple excitations.
3. **Studies of Linear Response**

3.1. **General**

3.1.1. **Object.** The broad objective of these studies was the understanding of the linear behavior of the system to provide a basis for interpreting the nonlinear response. In particular, it was desired to investigate the behavior of the system for the limiting conditions of low and high frequencies and to evaluate the relative contribution of natural modes.

3.1.2. **Systems Analyzed.** The five-story structures considered were uniform frames with the width of the bays equal to twice the height of the stories. A uniform frame was defined as having equal columns, girders, story masses, story heights and girders widths. The stiffness of the columns, $EI/h$, and the floor mass, $m$, were used as reference quantities in presenting the results. The system was considered without damping and no vertical loads were considered acting on the girders.

The frames were idealized in the manner described in Chapter II, and two parameters were used to describe its characteristics:

- Frequency Parameter, $f_1 t_1$
- Stiffness Ratio, $\rho$

The first of these parameters has been widely used before (1, 2, 3), and measures the pulse duration in terms of the natural period of the system or, gives the frequency of the system in terms of a pseudo-frequency of the excitation. This parameter was varied from the extremely-low to the extremely-high-frequency sub-regions of the spectrum (1). Although the limiting frequency values considered
may not be representative of actual building frames, knowledge of the behavior of a system for these limiting values is helpful in understanding and even predicting its response for more realistic values of the frequency parameter.

The Stiffness Ratio, \( \rho \), used by Blume as Joint Rotation Index (11) is defined as the ratio of the sum of the stiffnesses of all the girders at the mid-height story of the frame to the summation of the stiffnesses of all the columns at the same story. It is expressed as follows

\[
\rho = \frac{\sum_{i=1}^{C-1} \frac{EI_g}{L_g}}{\sum_{j=1}^{C} \frac{EI_c}{L_c}}
\]

in the case of the uniform frames considered in this study this parameter has the expression

\[
\rho = \frac{C-1}{2C} \frac{L_g}{L_c} = \frac{C-1}{2C} \frac{I_g}{I_c}
\]

This parameter is a measure of the relative girder-to-column stiffness and tells how much the system may be expected to behave as a frame. For low values of \( \rho \) the system behaves as a beam imposing no constraints to the rotations of the joints; for high values of \( \rho \) the system approaches a shear-beam for which the joints do not rotate; and, for intermediate values of \( \rho \) the structure behaves as a frame. In terms of dynamic characteristics, this parameter controls the mode shapes.

In Table 3 are shown the values of the undamped natural frequencies, \( f_n \), of five-story one-bay uniform frame and in Fig. 9 the ratio of \( f_n / f_1 \) as a function of \( \rho \). The frequencies \( f_n \) are expressed in cycles per unit of time.
### TABLE 3 - FREQUENCY COEFFICIENTS $w_n$ IN EXPRESSION $f_n = w_n \sqrt{\frac{\text{CEI}}{m h^3}}$

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<th>$\rho$</th>
<th>$w_1$</th>
<th>$w_2/w_1$</th>
<th>$w_3/w_1$</th>
<th>$w_4/w_1$</th>
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In the present study, six values of $\rho$ were considered: $\rho = 0$ and $\rho = \infty$ which characterize a beam and a shear-beam, respectively, and values of 0.05, 0.125, 0.05 and 2 which are representative of actual building frames and for which the frame effects are expected to be important.

3.2. Overall Effects. The maximum deflection of the top story, $X_T$, the maximum value of the total shear at the base, $V$, and the maximum value of the overturning moment at the base, $M$, are selected as the quantities to be considered in the discussion of the overall behavior of the system. In Figs. 10, 11 and 12 is shown the variation of these quantities with $\rho$ and $f_1 t_1$. The deflection $X_T$ is normalized with respect to the maximum ground displacement, $y_o$; $V$ is given in terms of the effective weight of the system, $W^*$, defined in the next section, and the ratio of the maximum ground acceleration, $\dot{y}_o$, to the gravitational acceleration, $g$; finally, $M$ is expressed in terms of $M^* = W^* h^*$ where $h^*$ is the height to the center of the effective weight as defined in the next section.

Effective Weight and Center. The instantaneous value of the total base shear is expressed in the form of Eq. 14 as

$$V(t) = \sum_{i=1}^{S} v_i (L.A.F.)_{d,i}$$

in which $(L.A.F.)_{d,i}$ is the instantaneous amplification factor for displacements of the $i$th mode, and $v_i$ is the static component of the $i$th mode to the base shear. The latter quantity may be expressed as

$$v_i = C_i [1]^T [K'] [\phi_i] y_o$$

where the participation factor $C_i$ is defined by Eq. 13. Equation 19 may also be
written as

\[ v_i = p_i^2 C_1 \{1\}^T [m] \{ \phi_i \} y_o \] (20)

Now, let \( m_i^* \) be the effective mass of the \( i \)th mode, defined as

\[ m_i^* = C_1 \{1\}^T [m] \{ \phi_i \} = C_1 \{ \phi_i \}^T [m] \{1\} \]

or

\[ m_i^* = \left( \frac{\{ \phi_i \}^T [m] \{1\}}{\{ \phi_i \}^T [m] \{ \phi_i \}} \right)^2 \]

and let \( W_i^* = m_i^* g \) be the associated effective weight. Equation 20 may then be written as

\[ v_i = p_i^2 y_o \frac{W_i}{g} \]

and Eq. 18 as

\[ V(t) = \sum_{i=1}^{S} W_i^* \frac{\ddot{y}_o}{g} \text{(I. A. F.)}_{a,i} \]

where \((\text{I. A. F.})_{a,i}\) is the instantaneous amplification factor for accelerations of the \( i \)th mode, and it is related to \((\text{I. A. F.})_{d,i}\) as follows:

\[ (\text{I. A. F.})_{a,i} = \frac{p_i^2 y_o}{\ddot{y}_o} \frac{1}{(\text{I. A. F.})_{d,i}} \]

It is known from previous studies of simpler structures (14), and it will be shown again for the structures under study, that the first mode solution is a very good approximation at the high-frequency region of the spectrum. For that reason one may then write

\[ V(t) \simeq W_1^* \frac{\ddot{y}_o}{g} (\text{I. A. F.})_{a,1} \]

and it is instructive to normalize \( V(t) \) in the form presented in Fig. 11.
In a similar way, the instantaneous value of the overturning moment at the base may be written as

\[ M(t) = \sum_{i=1}^{S} W_i^* h_i^* \frac{\ddot{y}_0}{g} \text{ (I. A. F.)}_{a, i} \]

where \( h_i^* \) is the distance from the base to the center of the effective weight of the \( i \)th mode. Again, for the high-frequency portion of the spectrum one may write

\[ M(t) \approx M_1^* \frac{\ddot{y}_0}{g} \text{ (I. A. F.)}_{a, 1} \]

in which \( M_1^* = W_1^* h_1^* \); and it becomes convenient to normalize \( M(t) \) in the form used in Fig. 12.

Because of the importance of the fundamental mode, the quantities \( W_1^* \) and \( h_1^* \) will be referred to in the following as the "effective weight of the structure", and will be denoted simply as \( W^* \) and \( h^* \). In a similar manner, the subscript \( 1 \) will be deleted from \( M_1^* \).

Table 4 gives the values of \( W^* \) and \( h^* \) for different values of \( \rho \) for the 5-story, single-bay frame under consideration; \( W^* \) is given in terms of the individual floor weight, \( mg \), and \( h^* \) in terms of the basic height, \( h \).

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<th>0.125</th>
<th>0.5</th>
<th>2</th>
<th>( \infty )</th>
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<td>( W^*/mg )</td>
<td>3.394</td>
<td>3.821</td>
<td>3.982</td>
<td>4.175</td>
<td>4.307</td>
<td>4.398</td>
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<tr>
<td>( h^*/h )</td>
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<td>3.785</td>
<td>3.709</td>
<td>3.619</td>
<td>3.557</td>
<td>3.513</td>
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**Top Story Deflection** - It is desired to investigate \( X_T \) not only because the maximum deflection of the frame is of interest in itself, but also because this quantity is an index of the behavior of the entire frame since the effects of the
ground excitation have to travel throughout the system before they reach the top story.

In Fig. 10, one observes that $X_T/y_0$ does not vary appreciably with $\rho$; this means that $X_T$ is not sensitive to variations in the relative distribution of stiffness between girders and columns. For low-frequency systems, the maximum deflection is equal to the maximum ground displacement, indicating that the top story practically does not move and all the ground motion is absorbed by the frame. For extremely-low-frequency systems, the ground moves below the structure without transmitting any appreciable lateral displacements to the floor levels. In this case, the behavior may be approximated by a set of static lateral forces distributed in such a way as to produce a deformation $y_0$ in the first story and no deformation in the upper stories.

At the right end of the spectrum, the pseudo acceleration of the top story, $A_T = p_T^2 X_T$, approaches $\ddot{y}_0$ for large values of $f_1 t$. This fact has been discussed previously (2), where it was noted that the system in this case behaves essentially as a rigid body, with each mass experiencing a maximum acceleration equal to the maximum ground acceleration. The maximum response of the system in this case is the same as that due to a set of static lateral forces acting at the floor levels, the magnitude of the force at each floor being equal to the product of the mass at that floor and the maximum ground acceleration.

**Maximum Shear and Overturning Moment at Base.** These two quantities are the basis of the design provisions in building codes which use a simplified static analysis to approach the dynamic problem; they are also the quantities needed in
the analysis of the foundation system.

Referring to Fig. 11 and 12, one may first observe that the curves have the general appearance of an acceleration spectrum. What is represented is indeed the variation of the acceleration of the masses in terms of the maximum ground acceleration. In this type of plot, note that the individual curves tend to zero at the left end, as does the acceleration spectrum, and that for high values of \( f_1 t_1 \) the curves approach a value of one.

From Figs. 11 and 12, one may further note that \( V \) and \( M \) vary significantly with \( \rho \) at the left half of the figure, whereas at the right half they are practically independent of \( \rho \). Note also that the variation of \( M \) with \( \rho \) is not as great as is that of \( V \). A decrease in \( \rho \) is associated with a reduction in girder stiffness, and a corresponding increase in column stiffness if the natural frequency of the system remains constant. Since at the left end of the spectrum the system behaves as if all floors were subjected to a static deflection \( y_o \), one may expect that the frames with the lower values of \( \rho \) (i.e. stiffer columns) will experience the larger shears and overturning moments. Furthermore, since the axial forces in the columns decrease when the girder stiffness decreases, and since the overturning moment is related directly to the axial forces in the columns, one may understand why \( M \) is not as sensitive to changes in \( \rho \) as is \( V \).

As the frequency parameter increases and one moves to the right end of the spectrum all the curves coincide; this means that the acceleration of the masses are the same for all values of \( \rho \). Since the normalization was made considering the quantities \( W^* \) and \( h^* \) which are related with the first natural mode it is valid
to conclude that only the first mode is important in this region. For a better
understanding it would be interesting to examine the relative contributions of the
higher modes; this will be done in the next section.

In Table 5 the values of the overall effects for the two extreme regions of
the spectrum obtained by the dynamic analysis are compared with the values from
a static analysis considering the spacial distribution of the lateral loads at the
floor levels to be (a) such as to produce equal floor displacements, and (b) the
same as that of the mass of the system. The magnitudes of the loads for case (a)
are such that the lateral displacements are equal to \( y_o \), and for case (b), they
are equal to the product of the floor masses times the maximum ground accelera-
tion. For simplicity in writing, case (a) will be referred to as the "static approach
for soft systems" and will be designated as SASS, whereas case (b) will be re-
ferred to as "static approach for rigid systems" and will be designated as SARS.
This comparison shows that these static approaches give a good approximation
to the dynamic response of the system at the extreme regions of the spectrum.

### TABLE 5. SUMMARY OF OVERALL EFFECTS DETERMINED
BY DYNAMIC AND STATIC APPROACHES.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>( \rho = 0 )</th>
<th>( \rho = 0,125 )</th>
<th>( \rho = \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_T/y_o )</td>
<td>( f_1 t_1 = 0,01 )</td>
<td>( f_1 t_1 = 0,01 )</td>
<td>( f_1 t_1 = 0,01 )</td>
</tr>
<tr>
<td>( V* g/W y_o )</td>
<td>( 0,090 )</td>
<td>( 1,707 )</td>
<td>( 0,006 )</td>
</tr>
<tr>
<td>( M g/M* y_o )</td>
<td>( 0,012 )</td>
<td>( 1,374 )</td>
<td>( 0,001 )</td>
</tr>
</tbody>
</table>
TABLE 5. SUMMARY OF OVERALL EFFECTS DETERMINED BY DYNAMIC AND STATIC APPROACHES (cont'd.)

<table>
<thead>
<tr>
<th>Quantity</th>
<th>$\rho = 0$</th>
<th>$\rho = 0.125$</th>
<th>$\rho = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b) Static Approach</td>
<td>SASS</td>
<td>SARS</td>
<td>SASS</td>
</tr>
<tr>
<td>$X_T / y_o$</td>
<td>1.000</td>
<td>0.009</td>
<td>1.000</td>
</tr>
<tr>
<td>$V / W^* y_o$</td>
<td>0.072</td>
<td>1.473</td>
<td>0.006</td>
</tr>
<tr>
<td>$M / M^* y_o$</td>
<td>0.011</td>
<td>1.115</td>
<td>0.001</td>
</tr>
</tbody>
</table>

Relative Contribution of Modes. In Figs. 13, 14 and 15 are shown the relative contributions of the natural modes to the overall effects for three different values of $\rho$, including the two limiting cases of $\rho = 0$ and $\rho = \infty$, and a value of $\rho = 0.125$.

Analysis of the information presented in these figures justifies the following conclusions:

- Along the medium-frequency and high-frequency regions (for a definition of these regions, refer to Ref. 2), the first mode controls the response, and solutions which neglect the contribution of the higher modes give a good approximation. The boundary of the two regions is identified by the symbol $c$, and the low-frequency end of the medium frequency region is identified by $b$.
- In the low frequency region, to the left of line $b$, the contribution of the higher modes is important and it increases as the value of $\rho$ decreases.
- The relative contribution of the higher modes is substantially greater
for total shear and overturning moment at the base than for top story
deflections, and compared to the overturning moment, the shear is
more affected by the higher modes.

These conclusions are in perfect agreement with the previous studies which con-
sidered simpler systems (14).

The influence of \( \rho \) on the contribution of the higher modes can be under-
stood considering Eq. 14, the values of the natural frequencies in Table 3., and
the deformation spectrum for SDF systems given in Fig. 16. As \( \rho \) decreases,
the ratios of the higher frequencies to the fundamental frequency increases, and
the higher frequencies are differently positioned in the spectrum. Depending on
the value of \( f_1 \), the amplification factors associated with the higher frequencies
may vary substantially, as it can be verified by reference to Fig. 16. However,
as the fundamental frequency decreases, all amplification factors vary within the
relatively narrow range of 1 and 1.7, and as the system becomes still more
flexible, all the amplification factors tend to unity. One concludes that the large
variation in the contribution of the higher modes just observed cannot be attributed
to the variation of the amplification factors and, by exclusion, it must be associated
with the second factor in Eq. 14, the quantity \( E_1 \), which reflects the influence of the
mode shapes. This conclusion is supported by the data in Table 6 which show the
variation with \( \rho \) of the modal contribution factors, \( E_i \), for the overall effects under
consideration.

It is interesting to notice that in certain cases of extremely-low-frequency
systems consideration of only a few modes may overestimate the result. This
occurs with $X_T$ for all the values of $\rho$ and with $M$ for moderately low values of $\rho$, and it is due to the fact that for the extremely-soft systems the maxima occur early in the pulse when the contributions of certain modes are of opposite signs to that of the fundamental mode. This is also evidenced by the analysis of the signs of the modal contributions in Table 6.

Now, referring to Figs. 13, 14 and 15 let us determine the frequency value for which the contribution of each mode starts to be significant as one moves from the right to the left end of the spectrum. Study of the data presented herein leads to the conclusion that the $i$th mode need be considered only if the value of $f_i t_{1i}a$ for that mode is equal or less than 0.6. The symbol $t_{1i}a$ denotes the average duration of the dominant pulse in the input acceleration diagram, and the value of 0.6 corresponds to the frequency value for which the I.A.F. in Eq. 14 is proportional to $1/f^k$, where $k$ is a number greater than two (1).

For the input function considered $t_{1i}a$ may be taken as $2t_1/3$. The condition $f_i t_{1i}a = 0.6$ then leads to $f_i t_1 = 0.9$. Now, using the values of $f_i/f_1$ given in Table 3, one finds that for a system with $\rho = 0.125$ the values of $f_i t_1\zeta_1$ beyond which the various modes must be considered are given by the points $n, o, p, q$ in Fig. 13(b). Specifically, point $n$ represents the frequency value beyond which only the second mode is important, point $o$ represents the frequency value beyond which the third mode is important, etc. It may be verified that these results are in good agreement with the results of the exact solutions.

These considerations suggest the following procedure for superimposing the different modal contributions for systems in the low-frequency region of the
spectrum. Suppose it is desired to compute $X_T$, $V$ and $M$ for $f_1 t_1 = 0.15$ and $\rho = 0.5$. Then, in the deformation spectrum for SDF systems, Fig. 16, the amplification factors for the different natural frequency of the system are determined. Since the values of $f_1 t_1$ corresponding to the 4th and 5th modes can be shown to be greater than the critical value referred to above, their contributions to the response will be neglected.

The combination of the modal contribution factors given in Table 6 with the amplification factor from Fig. 16 give the solution:

$$X_T = [1.269 \times 0.90 + 0.402 \times 1.66 + 0.197 \times 0.93] y_o = 1.992 y_o$$

$$V = [0.811 \times 0.90 + 0.975 \times 1.66 + 1.195 \times 0.93] \frac{4EI}{h^2} y_o = 0.448 W \frac{\ddot{y}_o}{g}$$

$$M = [2.936 \times 0.90 + 0.562 \times 1.66 + 1.016 \times 0.93] \frac{4EI}{h^2} y_o = 0.161 W h \frac{\ddot{y}_o}{g}$$

The "exact" values from the dynamic analysis considering all the modes are

$$X_T = 1.778 y_o; \quad V = 0.383 W \frac{\ddot{y}_o}{g}; \quad M = 0.143 W h \frac{\ddot{y}_o}{g}$$

and they show that the values computed by the approximate procedure constitutes a good upper bound to the exact values. In the application of the procedure to extremely-low-frequency systems one has to be careful because, in certain cases, the summation of the absolute values of the modal contributions may overestimate too much the response. It might be recalled, however, that these extremely-low-frequency systems are, in general, out of the range of interest for all practical purposes and, if necessary, the response of them can be very well estimated by the static approach described before.
TABLE 6 - MODAL CONTRIBUTION FACTORS

<table>
<thead>
<tr>
<th>Quantity</th>
<th>i</th>
<th>(\rho=0)</th>
<th>(\rho=0.125)</th>
<th>(\rho=0.5)</th>
<th>(\rho=\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{X_T}{y_o})</td>
<td>1</td>
<td>1.384</td>
<td>1.297</td>
<td>1.269</td>
<td>1.252</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.528</td>
<td>-0.441</td>
<td>-0.402</td>
<td>-0.362</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.204</td>
<td>0.211</td>
<td>0.197</td>
<td>0.159</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>-0.080</td>
<td>-0.089</td>
<td>-0.087</td>
<td>-0.063</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.020</td>
<td>0.023</td>
<td>0.022</td>
<td>0.015</td>
</tr>
</tbody>
</table>

(a) Quantity \(E_i\) in Eq. 14

| \(\frac{V h^3}{4E I y_o}\) | 1  | 0.023       | 0.313          | 0.811       | 2.137            |
|                            | 2  | 0.289       | 0.537          | 0.975       | 1.805            |
|                            | 3  | 0.787       | 0.967          | 1.195       | 1.246            |
|                            | 4  | 1.381       | 1.337          | 1.164       | 0.638            |
|                            | 5  | 1.109       | 0.859          | 0.541       | 0.173            |

| \(\frac{M h^2}{4E I y_o}\) | 1  | -0.092      | -1.160         | -2.936      | -7.509           |
|                            | 2  | -0.329      | 0.041          | 0.562       | 2.172            |
|                            | 3  | -0.551      | -0.794         | -1.016      | -0.952           |
|                            | 4  | -0.723      | -0.424         | -0.047      | 0.379            |
|                            | 5  | -0.501      | -0.408         | -0.271      | -0.090           |

(b) Quantity \(E_i\) in Eq. 14 with (I. A. F.)\(_d\) replaced by (I. A. F.)\(_a\)

| \(\frac{V}{W^* y_o}\) | 1  | 1.000       | 1.000          | 1.000       | 1.000            |
|                       | 2  | 0.303       | 0.148          | 0.123       | 0.098            |
|                       | 3  | 0.103       | 0.065          | 0.048       | 0.028            |
|                       | 4  | 0.048       | 0.032          | 0.022       | 0.009            |
|                       | 5  | 0.018       | 0.011          | 0.006       | 0.002            |

| \(\frac{M}{W^* h y_o}\) | 1  | -1.000      | -1.000         | -1.000      | -1.000           |
|                       | 2  | -0.087      | -0.003         | 0.020       | 0.034            |
|                       | 3  | -0.019      | -0.014         | -0.011      | -0.006           |
|                       | 4  | -0.006      | -0.003         | -0.0002     | 0.001            |
|                       | 5  | -0.003      | -0.001         | -0.0009     | -0.0003          |
3.3. Local Effects. Under this heading are considered the resultant stresses in the individual members, namely the bending moments in the columns and girders and the axial forces in the vertical members. The variation of the maxima of these quantities with respect to $f_{1}$ is shown in Figs. 17, 18 and 19 for different values of $\rho$ using a displacement type of plot. Furthermore, in Figs. 20 and 21 are shown similar plots for the maximum bending moments in the individual girders and columns. Finally, the relative contribution of the natural modes is considered in Figs. 22, 23 and 24.

In the first set of figures one observes the variation of the maximum local effects with $\rho$. The general trends of these curves are similar. As $\rho$ decreases, the local effects also decrease, and as $\rho$ tends to zero the bending moments in the girders and the axial forces in the columns tend to zero, whereas the maximum bending moments in the columns tend to the values corresponding to a cantilever beam. One may be surprised with the fact that the bending moments decrease as $\rho$ decreases since one may expect that with the decreasing of the stiffness of the girders larger moments would be concentrated on the columns. This apparent inconsistency is due to the fact that the column moments are expressed in terms of the rigidity of the columns which, for a fixed value of the natural frequency $f_{1}$, increases when $\rho$ decreases.

The behavior of the systems at the extreme left and the extreme right ends of the spectrum can adequately be represented by the two static loading conditions, SASS and SARS, considered previously. The results obtained by these approaches are compared with the exact values in Table 7.
TABLE 7. SUMMARY OF LOCAL EFFECT DETERMINED BY DYNAMIC AND STATIC APPROACHES

<table>
<thead>
<tr>
<th>Quantity</th>
<th>$\rho = 0$</th>
<th>$\rho = 0.125$</th>
<th>$\rho = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$f_1 t_1 = 0.01$</td>
<td>$f_1 t_1 = 5$</td>
<td>$f_1 t_1 = 0.01$</td>
</tr>
<tr>
<td>$\frac{M_{ch}^3}{EIy_0}$</td>
<td>4.844</td>
<td>0.002</td>
<td>4.448</td>
</tr>
<tr>
<td>$\frac{M_{gh}^3}{EIy_0}$</td>
<td>-</td>
<td>-</td>
<td>0.997</td>
</tr>
<tr>
<td>$\frac{P_{ch}^3}{EIy_0}$</td>
<td>-</td>
<td>-</td>
<td>0.835</td>
</tr>
</tbody>
</table>

(b) Static Approach

<table>
<thead>
<tr>
<th></th>
<th>SASS</th>
<th>SARS</th>
<th>SASS</th>
<th>SARS</th>
<th>SASS</th>
<th>SARS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{M_{ch}^3}{EIy_0}$</td>
<td>4.392</td>
<td>0.002</td>
<td>4.675</td>
<td>0.006</td>
<td>6</td>
<td>0.021</td>
</tr>
<tr>
<td>$\frac{M_{gh}^3}{EIy_0}$</td>
<td>-</td>
<td>-</td>
<td>0.994</td>
<td>0.004</td>
<td>6</td>
<td>0.038</td>
</tr>
<tr>
<td>$\frac{P_{ch}^3}{EIy_0}$</td>
<td>-</td>
<td>-</td>
<td>0.816</td>
<td>0.014</td>
<td>6</td>
<td>0.106</td>
</tr>
</tbody>
</table>

A question that may be raised concerning these local effects is where do the reported maxima occur and what is the magnitude of the maximum effect in the other members. In Fig. 20 the distributions of the maximum bending moments in the girders is considered for two values of $\rho$. The general trend apparent in this figure are representative of those applicable to other. The same is also true of the corresponding plots for bending moments in the columns, Fig. 21, and axial forces in the columns.

Along the medium and high-frequency regions of the spectra the maximum
effects occur in the first and second stories. Their magnitudes decrease as 
one moves up to the upper floors, and the rate of reduction is greater for the 
larger values of $\rho$. For the low-frequency region one has to consider two cases: 
in the extremely-low-frequency sub-region the maximum local effects are con- 
centrated at the bottom story while the effects in the other stories are small; 
and, in the moderately-low-frequency sub-region, there is a tendency for the 
distribution of maximum effects to be nearly uniform. These observations can 
all be explained with the concepts developed previously concerning the response 
of the system at the extreme ends of the spectrum and the relative contribution 
of the higher modes.

Considering now the relative contribution of the natural modes, one may 
verify in Figs. 22, 23 and 24 that the observations made in the discussion of the 
overall effects are also valid for these localized effects. Note in particular the 
reduction in the relative contribution of the higher modes with increasing $\rho$, the 
region in which the contribution of each natural mode is important, and the fact 
that, for extremely-low-frequency systems, the response computed by considering 
only a few modes may be in excess of the actual maximum value. The procedure 
presented for superimposing the different modal contributions for systems in the 
low-frequency region also works conveniently in this case. This can be seen from 
the following data which refer to the example considered previously, i.e. a 

 system with $\rho = 0.5$ and $f_{1}t_{1} = 0.15$ for which only the first three modes had to 
be superimposed:
Three Mode Approximation  "Exact" Solution

Absolute Maximum  \( M_c = 4.054 \frac{EIy_0}{h^2} \)  \( M_c = 3.452 \frac{EIy_0}{h^2} \)
Bending in Columns

Absolute Maximum  \( M = 4.075 \frac{EIy_0}{h^2} \)  \( M = 3.815 \frac{EIy_0}{h^2} \)
Bending in Girders

Absolute Maximum  \( P_c = 8.956 \frac{EIy_0}{h^3} \)  \( P_c = 8.322 \frac{EIy_0}{h^3} \)
Axial Force in Columns

3.4. Influence of Number of Bays. The effect of the number of bays was studied considering three and five-bay frames and comparing their dynamic characteristics and some response quantities to the values computed previously for the single-bay frame.

It turns out that the parameter \( \rho \) is very suitable for this purpose, since it makes possible to establish a direct correlation between the dynamic characteristics and maximum response quantities of the multi-bay frames and those of the single-bay case.

Dynamic Characteristics. In this section it is desired to examine mainly the variation of the natural frequencies and mode shapes with the number of bays. The comparison of the frequency coefficients, \( w_n \), in Table 8 shows that the fundamental frequencies practically do not change with the number of bays. As one could predict at this point, the natural modes also do not experience significant changes. This is shown in Table 9, in which the results corresponding to different values of \( \rho \) are presented.

Now, since the natural frequencies and modes remain practically unchanged, one may reach the important conclusion that the dynamic response of a system having a specified fundamental frequency and value of \( \rho \) is practically independent.
of the number of bays involved.

Overall and Local Effects: Since the values of (I.A.F.) in Eq. 14 do not vary significantly with the number of bays, the only source of variation in the response quantities is to be found in the values of the modal contribution factors, $E_i$. In so far as the overall effects are concerned, one cannot expect them to be sensitive to the number of bays, since the modal contribution factors are mainly a function of the natural modes and of the distribution of the total mass which are the same, or practically the same, in all cases. This preliminary conclusion is completely verified by the results shown in Table 10 which refer to systems with a constant value of $\rho$ and several values of $t_1 t_{\perp}$.

If one now considers the local effects, the situation is a little different because the modal contributions factors may vary appreciably as the number of bays is varied. However, with a suitable change in the way in which these effects are normalized, it was possible to interrelate them reasonably well.

A simple way to understand the situation is as follows. Consider two uniform multi-bay frames of the type considered herein and having the same stiffness ratio $\rho$ and equal column stiffness $Eh/h$. Let $C$ be the number of columns in the first frame and $C'$ the corresponding number in the other. Also, let $I_g$ and $I'_g$ be the moments of inertia of the girders in the first and second frame, respectively. The expression for $\rho$ applied to the case under consideration may be written as

$$\rho = \frac{C-1}{2C} \frac{I}{I_g} = \frac{C'-1}{2C'} \frac{I'}{I_g}$$

whence

$$\frac{I_g}{I'} = \frac{C}{C-1} \frac{C'-1}{C'}$$ (21)
TABLE 8 - FREQUENCY COEFFICIENTS $w_n$ IN EXPRESSION $f_n = w_n \sqrt{\frac{CEI}{mh^3}}$

Results are for the 5-story frames shown in Fig. 7

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$w_1$</th>
<th>$w_2/w_1$</th>
<th>$w_3/w_1$</th>
<th>$w_4/w_1$</th>
<th>$w_5/w_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.04567</td>
<td>3.7275</td>
<td>8.4355</td>
<td>15.0172</td>
<td>21.4821</td>
</tr>
<tr>
<td>0.10</td>
<td>0.05863</td>
<td>3.4761</td>
<td>7.2385</td>
<td>12.2515</td>
<td>17.0332</td>
</tr>
<tr>
<td>0.125</td>
<td>0.06277</td>
<td>3.4123</td>
<td>6.9253</td>
<td>11.5015</td>
<td>15.8085</td>
</tr>
<tr>
<td>0.15</td>
<td>0.06685</td>
<td>3.3652</td>
<td>6.6953</td>
<td>10.9398</td>
<td>14.8867</td>
</tr>
<tr>
<td>0.20</td>
<td>0.07373</td>
<td>3.2984</td>
<td>6.3662</td>
<td>10.1384</td>
<td>13.5664</td>
</tr>
<tr>
<td>0.30</td>
<td>0.08415</td>
<td>3.2182</td>
<td>5.9768</td>
<td>9.1777</td>
<td>11.9803</td>
</tr>
<tr>
<td>0.40</td>
<td>0.09186</td>
<td>3.1697</td>
<td>5.7448</td>
<td>8.6054</td>
<td>11.0378</td>
</tr>
<tr>
<td>0.50</td>
<td>0.09789</td>
<td>3.1365</td>
<td>5.5872</td>
<td>8.2186</td>
<td>10.4033</td>
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<tr>
<td>0.75</td>
<td>0.10863</td>
<td>3.0850</td>
<td>5.3456</td>
<td>7.6317</td>
<td>9.4477</td>
</tr>
<tr>
<td>1.00</td>
<td>0.11582</td>
<td>3.0547</td>
<td>5.2055</td>
<td>7.2959</td>
<td>8.9062</td>
</tr>
<tr>
<td>2.00</td>
<td>0.13070</td>
<td>2.9998</td>
<td>4.9556</td>
<td>6.7082</td>
<td>7.9707</td>
</tr>
<tr>
<td>5.00</td>
<td>0.14402</td>
<td>2.9571</td>
<td>4.7657</td>
<td>6.2741</td>
<td>7.2935</td>
</tr>
</tbody>
</table>

(a) For three-bay frames

(b) For five-bay frames
<table>
<thead>
<tr>
<th>No. of Bays</th>
<th>$\rho$</th>
<th>Mode</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.05</td>
<td>1</td>
<td>0.1115</td>
<td>0.3440</td>
<td>0.5996</td>
<td>0.8250</td>
<td>1.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>5.3903</td>
<td>-6.1154</td>
<td>5.5074</td>
<td>-3.3940</td>
<td>1.0000</td>
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### TABLE 10 - MAXIMUM LINEAR RESPONSE FOR UNIFORM FIVE-STORY FRAMES

Value of $\rho = 0.125$; For Dimensions See Fig. 7

<table>
<thead>
<tr>
<th></th>
<th>Single-Bay</th>
<th></th>
<th>Three-Bay</th>
<th></th>
<th>Five-Bay</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_T \over y_o$</td>
<td>1.000</td>
<td>1.448</td>
<td>2.074</td>
<td>0.030</td>
<td>1.000</td>
</tr>
<tr>
<td>$V \over W^* \frac{g}{y_o}$</td>
<td>0.0057</td>
<td>0.311</td>
<td>1.862</td>
<td>1.194</td>
<td>0.0057</td>
</tr>
<tr>
<td>$M \over M^* \frac{g}{y_o}$</td>
<td>0.0011</td>
<td>0.068</td>
<td>1.861</td>
<td>1.016</td>
<td>0.0011</td>
</tr>
<tr>
<td>$M h^2 \over E I y_o$</td>
<td>4.488</td>
<td>2.803</td>
<td>0.922</td>
<td>0.016</td>
<td>4.565</td>
</tr>
<tr>
<td>$M g^2 \over E I y_o C^{-1}$</td>
<td>0.498</td>
<td>0.742</td>
<td>0.376</td>
<td>0.0056</td>
<td>0.511</td>
</tr>
<tr>
<td>$P c h^3 \over E I y_o C^{-1}$</td>
<td>0.418</td>
<td>1.470</td>
<td>1.394</td>
<td>0.020</td>
<td>0.419</td>
</tr>
</tbody>
</table>

56
As has already been noted, these two frames will experience practically the same displacements. If this is so, the bending moments in the columns will be practically the same in both cases since the stiffnesses of the members are the same for both frames. However, the bending moments in girders and axial forces in columns depend directly on the stiffnesses of the girders, and the value for these response quantities will be different unless a normalization involving the ratio given in Eq. 21 is used. The results shown in Table 10 were obtained with such a modified normalization. It may be verified that when expressed in this manner the maximum response values for the single-bay and the multi-bay frames are very well interrelated.

In concluding this section it should be noted that this interrelationship exists only for the absolute maximum values of the response quantities. For example, for direct forces in the columns this refers to the outer columns only. It is generally not possible to estimate the maximum forces in the interior columns in a multi-bay frame from knowledge of the behavior of a single-bay structure.

4. Studies of Nonlinear Response

4.1. Object and Scope. The main objectives of the studies reported herein were:

a. to determine the amount and distribution of the nonlinear action developed in the structure; and

b. to evaluate the effects of this nonlinear action on the different response quantities.

In addition, the understanding of the behavior of the systems was considered to
be a complimentary objective, intended to provide a logical basis for evaluating the generality of the observations made throughout the study.

The systems analyzed were the same as those considered in the studies of linear response. To obtain the desired nonlinear effects, the yield moments of the girders were taken equal to specified fractions of the absolute maximum bending moment computed in the girders for linear response. Thus, besides the parameters $f_t t_1$ and $\rho$, one additional parameter had to be considered, namely the Girder Yield Factor, $C_g$, which is defined as the ratio of the yield moment of the girders to the absolute maximum value of the bending moments induced in all girders under linear conditions of response. The effect of $C_g$ upon the response was analyzed for a system having a constant value of $\rho$. Additional solutions were obtained for systems having different values of $\rho$ and a value of $C_g = 0.25$, which is representative of those encountered in design practice.

The effect of the vertical load on the girders was investigated on the assumption that the intensity of the load on the individual girders was the same.

4.2. Ductility Ratios. The nonlinear or plastic action in the frame was expressed by two different ductility ratios: the overall ductility ratio, $\mu_o$, and the local ductility ratio, $\mu_x$, defined in the following.

The overall ductility ratio, $\mu_o$, is defined as

$$\mu_o = \frac{X_T}{C_g \langle X_T \rangle_o}$$  \hspace{1cm} (22)$$

where $X_T$ = the maximum value of the deflection relative to ground of the top story; $C_g$ = the girder yield factor, as previously defined; and $\langle X_T \rangle_o$ the value
of $X_T$ computed on the assumption that the system responds linearly. This definition of $\mu_0$ was suggested by the observation that the top-story deflection is a good indicator of the behavior of the entire frame.

The local ductility factor, used to express the inelastic action at the end of a member, is defined as

$$\mu_g = 1 + \frac{\alpha}{\theta}$$

(23)

where $\theta = \frac{M_L}{EI}$ is the end rotation for which yielding would start if the ends of the member were subjected to two equal moments acting in the same direction.

4.3 Overall Effects. The quantities investigated under this section are the top-story deflection, and the total shear and overturning moment at the base of the structure.

Top-Story Deflection. In Fig. 25 are shown plots of the maximum top-story deflection for two values of the stiffness ratio, $\rho$, and different values of the girder yield factor, $C_g$.

It may be observed that the general trends of these plots are the same as those for the SDF elastoplastic systems considered in Ref. 3. In the extremely-low-frequency subregion of the spectrum, $X_T = (X_T^*o) = y_o$; in the moderately-low-frequency subregion $X_T$ is less than $(X_T^*o$); in the medium-frequency region, $X_T$ is approximately equal to $(X_T^*o$) ; and, in the high-frequency region $X_T$ is generally greater than $(X_T^*o$).

This observation is considered to be very important because it permits a direct correlation between the behavior of the building frames under study and the behavior of SDF systems which has been widely investigated before. This
observation indicates further that of the concepts of frequency shift and mode-shape change discussed in Ref. 3, only the first is important in this case. By contrast, both of these concepts are needed to explain the observed trends for the other response quantities.

**Maximum Shear and Overturning Moment at Base.** The influence of girder yielding on the maximum base shear is shown in Fig. 26 for two different values of \( \rho \). As a first general observation, one may state that yielding reduces the total shear all over the spectrum, except possibly in the very-high-frequency subregion. Note further that the reduction is more significant for the higher values of \( \rho \), and that for a given value of \( \rho \), it is greater for medium-frequency and for moderately-low-frequency systems.

Considering now the region of the spectrum to the right of the frequency at which the elastic spectrum attains its maximum value, it may be concluded that the change in the vibrational mode of the system due to yielding has a significant influence on the response. This follows from the fact that the response in this case is governed almost entirely by the contribution of the fundamental mode. Since a reduction in the apparent frequency of this mode is associated with an increase in response in this region of the spectrum, the observed reduction in response must be a consequence of the change in the apparent mode of vibration. That this is indeed the case is verified by examining the values of the static modal contributions for the basic frame, and for several softer frames, obtained from the basic one by the addition of hinges in the girders. Such data are presented in Table 11 for a frame with \( \rho = 0.125 \). The response quantities include the
shear and overturning moment at the base. Note that the relative importance of the fundamental mode generally decreases with increasing number of hinges.

Understanding of the behavior of the system in the lower-frequency region of the spectrum requires consideration of the effects of both the frequency shift and the mode change discussed in Ref. 3.

The variation of the overturning moment at the base is shown in Fig. 27. The general trends in this case are the same as for the base shear, but it is worth noting that, for the higher values of \( \rho \), the difference between the linear and nonlinear response values is more significant than in the case of the base shear. It may be recalled at this point that in the studies of the linear response it was found that the higher modes affect less the overturning moment than the shear and that their effect decreases as \( \rho \) increases. This explains why the overturning moment is more significantly affected by nonlinear action for higher values of \( \rho \). In effect, the contributions of the higher-frequency modes are not increased with yielding as is the case for base shear.

In concluding this discussion of the overall effects, it should be noted that it is possible to show that, as in the case of linear response, the two static approaches, SASS and SARS, referred to previously also provide good approximations to the behavior of the nonlinear systems at the two ends of the spectrum. In the SARS approach, the behavior of the frame is examined under a monotonically increasing set of lateral forces proportional to the masses, and in the SASS approach, the behavior is examined under a monotonically increasing deformation in the first story. The desired response quantities in the first case are those
TABLE II. EFFECT OF HINGE FORMATION ON NATURAL FREQUENCIES AND RESPONSE QUANTITIES OF A LINEAR STRUCTURE -- Typical five-story frame with $\rho = 0.125$

<table>
<thead>
<tr>
<th>$f_i/f_1$</th>
<th>$i = 1$</th>
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<th>$i = 3$</th>
<th>$i = 4$</th>
<th>$i = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{i_{st}}^{o} h^3 / 4 EI y_0$</td>
<td>0.313</td>
<td>0.537</td>
<td>0.967</td>
<td>1.337</td>
<td>0.859</td>
</tr>
<tr>
<td>$(M_{i_{st}}^{o} h^2) / 4 EI y_0$</td>
<td>-1.160</td>
<td>0.041</td>
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<tbody>
<tr>
<td>$(V_{i_{st}}^{o} / V_{i_{st}}^{o})$</td>
<td>0.859</td>
<td>0.803</td>
<td>0.979</td>
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<tr>
<td>$(M_{i_{st}}^{o} / M_{i_{st}}^{o})$</td>
<td>0.878</td>
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<tr>
<td>$(V_{i_{st}}^{o} / V_{i_{st}}^{o})$</td>
<td>0.452</td>
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<td>$(M_{i_{st}}^{o} / M_{i_{st}}^{o})$</td>
<td>0.491</td>
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<tr>
<td>$(V_{i_{st}}^{o} / V_{i_{st}}^{o})$</td>
<td>0.197</td>
<td>1.080</td>
<td>0.915</td>
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<td>1.145</td>
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<tr>
<td>$(M_{i_{st}}^{o} / M_{i_{st}}^{o})$</td>
<td>0.220</td>
<td>-22.807</td>
<td>0.763</td>
<td>1.273</td>
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TABLE II. EFFECT OF HINGE FORMATION ON NATURAL FREQUENCIES AND RESPONSE QUANTITIES OF A LINEAR STRUCTURE -- Typical five-story frame with $\rho = 0.125$ (cont'd.)

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<td>$f_i / f_1$</td>
<td>0.0228*</td>
<td>5.874</td>
<td>15.354</td>
<td>28.562</td>
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<tr>
<td>$(V_{i_{st}} / (V_{i_{st}})^O)$</td>
<td>0.104</td>
<td>0.763</td>
<td>1.130</td>
<td>1.103</td>
<td>1.143</td>
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<tr>
<td>$(M_{i_{st}} / (M_{i_{st}})^O)$</td>
<td>0.114</td>
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<td>$f_i / f_1$</td>
<td>0.0186*</td>
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<tr>
<td>$(V_{i_{st}} / (V_{i_{st}})^O)$</td>
<td>0.074</td>
<td>0.537</td>
<td>0.813</td>
<td>1.033</td>
<td>1.291</td>
</tr>
<tr>
<td>$(M_{i_{st}} / (M_{i_{st}})^O)$</td>
<td>0.080</td>
<td>-7.984</td>
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<tbody>
<tr>
<td>$f_i / f_1$</td>
<td>0.0518*</td>
<td>4.132</td>
<td>8.116</td>
<td>13.611</td>
<td>19.182</td>
</tr>
<tr>
<td>$(V_{i_{st}} / (V_{i_{st}})^O)$</td>
<td>0.672</td>
<td>0.851</td>
<td>1.257</td>
<td>0.928</td>
<td>1.013</td>
</tr>
<tr>
<td>$(M_{i_{st}} / (M_{i_{st}})^O)$</td>
<td>0.671</td>
<td>3.447</td>
<td>1.546</td>
<td>1.392</td>
<td>0.988</td>
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<tbody>
<tr>
<td>$f_i / f_1$</td>
<td>0.0432*</td>
<td>4.648</td>
<td>9.507</td>
<td>16.314</td>
<td>22.918</td>
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<tr>
<td>$(V_{i_{st}} / (V_{i_{st}})^O)$</td>
<td>0.488</td>
<td>0.596</td>
<td>0.948</td>
<td>0.908</td>
<td>1.150</td>
</tr>
<tr>
<td>$(M_{i_{st}} / (M_{i_{st}})^O)$</td>
<td>0.476</td>
<td>7.060</td>
<td>1.112</td>
<td>1.336</td>
<td>1.264</td>
</tr>
</tbody>
</table>
TABLE II. EFFECT OF HINGE FORMATION ON NATURAL FREQUENCIES AND RESPONSE QUANTITIES OF A LINEAR STRUCTURE -- Typical five-story frame with $\rho = 0.125$ (cont'd.)

<table>
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<tbody>
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<td>$f_i/f_1$</td>
<td>0.0346*</td>
<td>5.368</td>
<td>11.488</td>
<td>19.927</td>
<td>28.600</td>
</tr>
<tr>
<td>$(V_i)<em>{st}/(V_i)^O</em>{st}$</td>
<td>0.299</td>
<td>0.747</td>
<td>0.790</td>
<td>1.018</td>
<td>1.128</td>
</tr>
<tr>
<td>$(M_i)<em>{st}/(M_i)^O</em>{st}$</td>
<td>0.299</td>
<td>1.710</td>
<td>1.251</td>
<td>1.058</td>
<td>1.155</td>
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</tbody>
</table>

<table>
<thead>
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<th>$i = 4$</th>
<th>$i = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_i/f_1$</td>
<td>0.0288*</td>
<td>5.373</td>
<td>13.237</td>
<td>23.888</td>
<td>34.126</td>
</tr>
<tr>
<td>$(V_i)<em>{st}/(V_i)^O</em>{st}$</td>
<td>0.198</td>
<td>0.651</td>
<td>0.872</td>
<td>1.005</td>
<td>1.152</td>
</tr>
<tr>
<td>$(M_i)<em>{st}/(M_i)^O</em>{st}$</td>
<td>0.203</td>
<td>-3.269</td>
<td>1.072</td>
<td>0.967</td>
<td>1.396</td>
</tr>
</tbody>
</table>

Notes.
The quantities $(V_i)_{st}$ and $(M_i)_{st}$ correspond to the quantities $E_i$ in Eq. 14. The first defines the base shear and the second the overturning moment at the base. The superscript $O$ refers to the basic frame, without any hinges.

* This value gives the fundamental frequency, $f_i$, in terms of $\sqrt{\frac{2EI}{m h^3}}$. 
obtained when the magnitude of the forces is \( m \{1\} \ddot{y}_0 \), and in the second case when the magnitude of the deformation is \( y_0 \).

4.4. Local Effects. Under this heading are considered the bending moments and axial forces in the columns. The bending moments in the girders are not considered since, for the rigid-plastic \( M-\alpha \) relationship investigated, their values are equal to the specified yield moments. The behavior of the girders is discussed in a later section.

**Bending Moments in Columns.** The curves in Fig. 28 show how yielding affects the maximum bending moments in columns. The general trends may be described as follows. In the high-frequency region the values of the maximum bending moments in the columns of the nonlinear frame are higher than those of the associated linear structure; in the medium- and low-frequency regions the nonlinear response values are smaller than the linear ones, and this difference decreases in the extremely low-frequency subregion becoming constant at the extreme left end of the spectrum.

These general trends may again be understood in terms of the concepts of frequency shift and mode-shape change discussed in Ref. 3. In the high-frequency region, the effect of the shift is the dominant factor. As one approaches the medium-frequency region, the importance of the frequency shift decreases, and the effect of the mode change which leads to a smaller participation for the fundamental mode begins to become important. In the moderately-low-frequency subregion, the observed behavior is attributed to the increased contribution of the higher modes of vibration. Finally, in the extremely-low-frequency subregion, the frequency shift effect is insignificant, and the behavior of the system is
adequately described by the static approach SASS referred to in a previous section.

**Axial Forces in the Columns.** The effect of yielding on the absolute
maximum dynamic force in the columns is shown in Fig. 29. It can be seen that
the effect is to reduce the values of the axial forces over the entire spectrum.
This result could have been expected since the axial forces in the columns depend
directly on the bending moments in the girders and these moments are decreased
as the girder yield factor, \( C_g \), is decreased.

Concerning the magnitude of this reduction, it can be said that the ratio of
the maximum effect for the nonlinear and linear systems may be as low as the
yield factor, \( C_g \), since what happens in the columns is directly related to the
bending moments in the girders. However, the actual reduction in a given case
may be significantly smaller since the factor \( C_g \) is applicable only to the girders
that yield. For example note that the reduction is fairly small in the extremely-
low-frequency subregion of the spectrum. This is consistent with the fact that
yielding in this case is concentrated only in the lower floor levels.

4. 5. **Ductility Requirements.** This section is devoted to a discussion
of the amount and distribution of the inelastic action developed in the frames.
The discussion concerns mainly the variation of the two ductility ratios defined
in Section 4. 2.. In Figs. 30, 31 and 32 the quantities \( \mu_o \) and \( \mu_\ell \) are plotted
as a function of the frequency parameter \( f_1 \). The first of these figures
refers to systems with \( \rho = 0.125 \) and different values of the girder yield factor,
\( C_g \), whereas the other two refer to system with other values of \( \rho \).
Overall-Ductility Ratio. It was shown in Section 4.3. that the maximum
top-story deflection varies in the same way as the maximum deformation of SDF
systems. Thus, all the rules developed previously for estimating the ductility
ratio of single-degree-of-freedom systems (1, 2, 3) can also be used for the
overall ductility of the multi-degree-of-freedom systems considered herein. The
general trends can be summarized with reference to Fig. 35 as follows:

. in the extremely-low-frequency subregion, the overall ductility ratio is
equal to $1/C_g$;

. in the moderately-low-frequency subregion, $\mu_o$ is generally smaller than
$1/C_g$;

. in the medium-frequency region, $\mu_o$ is practically equal to $1/C_g$; and

. in the high-frequency region, $\mu_o$ increases rapidly toward some finite high
value.

Considering now the two extremes of the spectrum, one may show that the
SASS and SARS approaches described in Section 4.3. give excellent approxi-
mations to the behavior of the system in the left and right end, respectively.
The ductility ratios computed by these approaches are shown in Fig. 30 by
horizontal lines.

The same trends can also be observed in Fig. 31 which refers to systems
having different values of $\rho$. These plots show further that in the moderately-
high-frequency subregion the variation of $\mu_o$ can be approximated by a straight
line with a slope equal to 3, i.e. $\mu_o$ in this region is proportional to $(f_1 t_1)^3$.
For design purposes, the variation of $\mu_o$ with frequency may be approximated
by the diagram shown in Fig. 35, where point c represents the boundary between
the medium- and high-frequency regions of the spectrum. For a detailed discussion
of these regions the reader is referred to Refs. 2 and 3.

Local-Ductility Ratio. Examination of the information presented in Fig. 30
leads to the following conclusions:

- the values of the local-ductility ratio, \( \mu_L \), are always greater than those
  of the overall-ductility ratio, \( \mu_o \);
- in the extremely-low-frequency subregion and in the high-frequency region
  the curves representing the variation of \( \mu_o \) and \( \mu_L \) with frequency are
  practically parallel to one another;
- in the moderately-low-frequency subregion, the values of \( \mu_L \) vary signifi-
  cantly due to the influence of the higher-frequency components;
- at the extreme left and right ends of the spectrum, \( \mu_L \) tends to the values
  determined by the SASS and SARS approach, respectively;
- the yield factor, \( C_y \), does not affect substantially the general trends
  referred to above.

The effect of \( \rho \) on the local-ductility ratio is shown in Fig. 32. Note that
\( \mu_L \) increases substantially with \( \rho \). This trend can be understood by noting that
the change in the rotational stiffness of a joint resulting from the formation of
a plastic hinge at that joint is greater for the higher values of \( \rho \). In Fig. 32 note
further that the dips of the curves in the moderately-low-frequency subregion of
the spectrum are larger for the larger values of \( \rho \). This is attributed to the fact
that the higher natural frequencies of the system are more closely spaced for the
higher values of $\rho$. As a consequence, the amplification factors for the higher "modes" are generally greater in these cases.

For design purposes, the variation of $\mu^f$ with the frequency parameter may be approximated by the solid curve in Fig. 35. Note that the left hand portion of the diagram is approximated by the ratio $\frac{\mu^f}{\mu^o} = 1 + \rho$. Pending the results of further studies, it is recommended that the use of this empirical relationship be limited to values of $\rho \leq 2$.

It is desirable to investigate at this stage the distribution of the local ductility ratio over the frame. Representative information on this matter is given in Figs. 33 and 34, in which $\mu_{f,j}$ represents the local ductility ratio for the girder of the jth floor. It is worth noting the following trends:

. at the extreme left end of the spectrum, practically all the inelastic action is concentrated in the bottom story and the other stories remain practically linear;

. in the high-frequency region, all the girders experience significant amounts of yielding;

. the low- and medium-frequency region represent a transition between the two conditions referred to above, with the effect of $\rho$ being significant in this region; for low values of $\rho$, the nonlinear action is concentrated primarily at the top stories but as $\rho$ increases the nonlinear action becomes more important at the bottom stories.

4.6. **Influence of Number of Bays**. Based on the studies of linear response reported in Section 3.4, one may anticipate that the concepts used to relate the maximum response of a multi-bay frame to that of a single-bay frame may also
be applicable in the nonlinear case.

In order to explore the validity of this concept, the response of five-story frames with three bays was evaluated for a value of $\rho = 0.125$ and for different values of the frequency parameter, $f_1 t_1$, and of the girder yield factor, $C_g$. The results showed that the same relations do indeed hold true in the inelastic case.

The maximum values of the overall- and local-ductility ratios for the multi-bay frames were found to be for all practical purposes the same as those in a single-bay frame having the same value of $\rho$. Also, the distribution of the local ductility over the frame was practically the same in both cases.

4.7. Influence of Dead Weight. The main effect of the dead load of the girders is to modify the characteristics of the moment-kink diagram for the ends of these members. More specifically, consideration of this factor leads to a shift of the origin of the $M-\alpha$ diagram, and results to different values of yield moment in the two directions of deformation. Furthermore, for symmetric structures, the response will no longer be antisymmetric since hinges at symmetric locations will be forming at different instants.

The results of the studies made to investigate the effect of this factor are summarized in Tables 12 and 13. The intensity of the lateral load on the girders is expressed by the parameter $\delta$, which defines the ratio of the absolute maximum bending moment in the girders induced by the lateral load under static conditions to the yield moment of the girders. The latter quantity is assumed to be the same for all girders. In the first of these tables are presented the maximum values of the response quantities for a five-story, one-bay frame with $\rho = 0.125$ and
\(C_g = 0.25\), for different values of \(f_1 t_1\) and \(\rho\). In Table 13 are shown the maximum values of the response quantities for the same frame but for a constant value of \(f_1 t_1\) and different values of \(C_g\) and \(\delta\).

It can be seen from these tables that consideration of the "dead load effect" does not influence materially the maximum base shear, but as would be expected, it increases significantly the maximum axial forces in the columns. The maximum base shear remains practically constant because the magnitude and distribution of the mass of the structure which controls the lateral inertia forces are the same in the two cases. As far as the ductility ratios are concerned, it can be seen from the tables that the dead load effect does not alter the values of these quantities in the extremely-low-frequency subregion of the spectrum. For moderately low and medium-frequency systems the dead load effect reduces the ductilities, whereas for high-frequency systems it increases them significantly.

The behavior of the extremely-low and extremely-high-frequency systems could have been predicted in terms of the static approaches referred to previously and the distribution of the local ductility discussed in Section 4.5.

It has been shown that the behavior of extremely-low-frequency systems can be approximated by a static solution in which the floors of the system are held against deflection and the base is gradually displaced until the deformation in the first story is equal to the maximum displacement of the prescribed ground motion, \(y_0\). It has further been shown that the maximum values of local ductility are concentrated in the bottom story and that the nonlinear effects in the rest of the frame are very small. The fact that the plastic hinges at the first floor level will be formed earlier if the dead load effect is considered cannot be
expected to be very significant in this case.

For high-frequency systems, the situation is different. The static approach for this case involves a set of lateral forces proportional to the masses of the floor levels, and inelastic action is generally spread throughout the height of the structure. Premature yielding in such a case may have a significant effect on the values of the resulting ductility ratios.
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V. SUMMARY AND CONCLUSIONS

A study of the response of five-story, single- and multi-bay, uniform building frames subjected to a pulse-type excitation of the ground has been presented. The response in both the linear and the inelastic ranges of deformation has been investigated. In the studies of inelastic systems, the columns of the frames were assumed to remain elastic, and inelastic action was assumed to be concentrated in rigid-plastic hinges at the ends of the girders. Also the static behavior of the nonlinear structures was investigated for monotonically increasing lateral loads, and the results were used to interpret the dynamic response.

The analysis of the linear systems was carried out by use of the modal superposition method, whereas the nonlinear systems were analyzed by direct integration of the governing equations of motion using an iterative technique. The equations were written in a form that offered several advantages over previous formulations for the particular integration scheme adopted. A simple, efficient procedure was used to modify the stiffness of the frame at the instants of formation or disappearance of a plastic hinge.

Three separate computer programs were developed, one for the dynamic analysis of linear frames, one for the dynamic analysis of the frames in the nonlinear and inelastic ranges of behavior, and one for the static analysis of the frames in the nonlinear range.

The main parameters investigated were the frequency parameter, \( f_1 t_1 \), the relative column-to-girder stiffness ratio, \( \rho \), and the yield level of the girders, \( C_g \).
The results were analyzed with reference to available information on the response of single-degree-of-freedom systems and of multi-degree-of-freedom systems of the shear-beam type. The results of nonlinear action were expressed in terms of the corresponding response of an associated linear system.

The principal conclusions of the studies of linear response may be summarized as follows:

a. With the relative girder stiffness ratio, \( \rho \), used, the maximum responses of a single-bay and a multi-bay frame can be interrelated simply.

b. The concepts developed in Ref. 3 for estimating the maximum response of multi-degree-of-freedom systems of the shear-beam type can also be extended to the systems with flexible girders considered herein. In particular, it is possible to determine in advance the relative importance of the various natural modes and to evaluate with reasonable accuracy the maximum response of multi-story frames by considering the contribution of only a few modes.

c. The behavior of extremely-low and of very-high-frequency systems can be well determined by the static approaches, SASS and SARS, described in the body of the dissertation. These approaches are also applicable to nonlinear systems, and may be used to estimate the low and high frequency limits of the response spectra for such systems.

d. The results of these studies indicate that the formulas used in most building codes to compute the maximum shear and overturning moment at the base are inappropriate for low-frequency systems. This is due to the fact that these formulas do not consider the contribution of the
higher modes of vibration, which is important for low-frequency systems. In addition, it has been shown that the girder stiffness ratio $\rho$, is an important factor in the response of the system in the low-frequency region of the spectrum.

The principal conclusions of the nonlinear studies were as follows:

a. The girder stiffness ratio, $\rho$, is also suitable in this case for relating the response of a multi-bay frame to the response of a single-bay frame.

b. The behavior of the entire frame is well represented by the overall ductility ratio, $\mu_o$, defined by Eq. 22. The response spectrum for this ductility ratio presents the same general trends as the response spectrum for deformation of single-degree-of-freedom systems of the elastoplastic type, which have been investigated extensively.

c. The magnitude of inelastic action at the ends of the members has been expressed by the local ductility ratio, $\mu_l$, which is defined in Eq. 23. This ductility ratio is always greater than the overall ductility ratio, the difference being quite significant in some regions of the spectrum and for the high values of the relative girder stiffness ratio, $\rho$.

d. Based on the information presented herein, simple approximate rules have been presented for estimating the maximum values of the overall and local ductility ratios.
Fig. 1. - MODEL OF FRAME
Fig. 2. - MODEL OF TYPICAL MEMBER

Fig. 3. - MOMENT-PLASTIC ANGLE DIAGRAM
Fig. 4. - RESISTANCE-DEFORMATION RELATIONSHIPS

a) Moment-Curvature Diagram

b) Moment-Plastic Angle Diagram
Fig. 5. - INTERPOLATION IN BENDING MOMENTS

Fig. 6. - INTERPOLATION IN FIRST DERIVATIVES
All Columns Identic; All Girders Identic; All Floor Masses Same;

Fig. 7. - FRAMES CONSIDERED
Fig. 8. - GROUND MOTION CONSIDERED - SIMPLE PULSE
Fig. 9 - NATURAL FREQUENCY RATIOS FOR SINGLE-BAY FRAME
Fig. 10. - EFFECT OF $\rho$ ON MAXIMUM DEFLECTION OF TOP STORY RELATIVE TO GROUND
(Single-Bay Linear Frame Subjected to Simple Pulse)
Fig. 11 - EFFECT OF $\rho$ ON MAXIMUM BASE SHEAR
(Single-Bay Linear Frame Subjected to Simple Pulse)
Fig. 12. - EFFECT OF $\rho$ ON MAXIMUM OVERTURNING BASE MOMENT
(Single-Bay Linear Frame Subjected to Simple Pulse)
Fig. 13 - MODAL CONTRIBUTIONS TO MAXIMUM DEFLECTION OF TOP STORY RELATIVE TO GROUND
(Single-Bay Linear Frame Subjected to Simple Pulse)
Fig. 14. - MODAL CONTRIBUTIONS TO MAXIMUM BASE SHEAR
(Single-Bay Linear Frame Subjected to Simple Pulse)
Fig. 15 - MODAL CONTRIBUTIONS TO MAXIMUM OVERTURNING BASE MOMENT
(Single-Bay Linear Frame Subjected to Simple Pulse)
Fig. 16. - DEFORMATION SPECTRUM FOR LINEAR RESPONSE OF SDF SYSTEMS SUBJECTED TO SIMPLE PULSE
Fig. 17. - EFFECT OF $\rho$ ON MAXIMUM BENDING MOMENT IN COLUMNS
(Single-Bay Linear Frame Subjected to Simple Pulse)
Fig. 18. - EFFECT OF $\rho$ ON MAXIMUM BENDING MOMENT IN GIRDER
(Single-Bay Linear Frame Subjected to Simple Pulse)
Fig. 19. - EFFECT OF $\rho$ ON MAXIMUM AXIAL FORCE IN COLUMNS
(Single-Bay Linear Frame Subjected to Simple Pulse)
Fig. 20. - MAXIMUM BENDING MOMENT IN INDIVIDUAL GIRDER
(Single-Bay Linear Frame Subjected to Simple Pulse)

For Girder in Floor $i=5$

(a) For $\rho = 0.125$

(b) For $\rho = 2$

$\frac{M_0_i h^2}{E_1 \gamma_0}$

Frequency Parameter, $f_i$
Fig. 21. - MAXIMUM BENDING MOMENT IN INDIVIDUAL COLUMNS
(Single-Bay Linear Frame Subjected to Simple Pulse)
Fig. 22. - MODAL CONTRIBUTIONS TO MAXIMUM BENDING MOMENT IN COLUMNS
(Single-Bay Linear Frame Subjected to Simple Pulse)
Fig. 23 - Modal Contributions to Maximum Bending Moment in Girders (Single-Bay Linear Frame Subjected to Simple Pulse)
Fig. 24. - MODAL CONTRIBUTIONS TO MAXIMUM AXIAL FORCE IN COLUMNS
(Single-Bay Linear Frame Subjected to Simple Pulse)
Fig. 25. - EFFECT OF GIRDER YIELDING ON MAXIMUM DEFLECTION OF TOP STORY RELATIVE TO GROUND
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(Single-Bay Frame Subjected to Simple Pulse)
For Systems with $C_g = 0.25$

Stiffness Ratio, $\rho = 2$

Overall Ductility Factor, $\eta_0$

Frequency Parameter, $f_{11}$

Fig. 31. - EFFECT OF $\rho$ ON RESPONSE SPECTRA FOR OVERALL DUCTILITY (Single-Bay Frame Subjected to Simple Pulse)
Fig. 32. - EFFECT OF $\rho$ ON RESPONSE SPECTRA FOR LOCAL DUCTILITY (Single-Bay Frame Subjected to Simple Pulse)
For Systems with $p = 0.125$
and $C_g = 0.25$

Girder in Floor $j = 5$

Fig. 33. - RESPONSE SPECTRA FOR LOCAL DUCTILITY IN INDIVIDUAL GIR德RS
(Single-Bay Frame Subjected to Simple Pulse)
For Systems with $\rho = 2$
and $C_g = 0.25$

Girder in Floor $j = 1$

Fig. 34. - RESPONSE SPECTRA FOR LOCAL DUCTILITY IN INDIVIDUAL GIRDERS
(Single-Bay Frame Subjected to Simple Pulse)
Fig. 35. - DESIGN APPROACH FOR RESPONSE SPECTRA FOR OVERALL AND LOCAL DUCTILITIES
APPENDIX I

EXPRESSION RELATING
PLASTIC ANGLE AND CURVATURE AT END OF MEMBER

Consider the idealized member in Fig. 2 and the M-\(\phi\) diagram in Fig. 4a corresponding to an end section of the member. Suppose that the bending moment, \(M\), at this section is greater than the yield moment, \(M_y\), and let \(\phi_e\) be the value of curvature corresponding to \(M\) on the assumption that the member behaves linearly. Then \(\phi\) is related to \(M\) by the equation

\[
\phi_e = \frac{M}{EI}
\]  \hspace{1cm} (A.1)

Noting that \(EI = M_y/\phi_y\), one may also write Eq. A.1 in the form

\[
\phi_e = \frac{M}{M_y} \phi_y
\]  \hspace{1cm} (A.2)

The plastic angle, \(\alpha\), at the end of the member can be expressed as

\[
\alpha = (\phi - \phi_e) l_y
\]  \hspace{1cm} (A.3)

in which \((\phi - \phi_e)\) is the curvature in excess of the elastic value, \(\phi_e\), and \(l_y\) is the so-called effective length over which yielding in the original member is assumed to be spread.

Substituting Eq. A.3 into Eq. A.3, one obtains

\[
\alpha = (\phi - \frac{M}{M_y} \phi_y) l_y
\]  \hspace{1cm} (A.4)
APPENDIX II

DETAILS OF METHOD OF ANALYSIS FOR LINEAR RESPONSE

This appendix describes the details of the method used to analyze the response of the linear systems and the manner of specifying the amount of damping in the systems.

The equations of motion, presented in Chapter II, are:

\[
\begin{bmatrix}
  [m] \\ [c] \\ [k']
\end{bmatrix}
\begin{bmatrix}
  \ddot{x} \\ \dot{x} \\ x
\end{bmatrix}
+ \begin{bmatrix}
  [k'] \\ [c] \\ [m]
\end{bmatrix}
\begin{bmatrix}
  \dot{x} \\ x
\end{bmatrix}
= -\ddot{y}(t)
\begin{bmatrix}
  [m] \\ [c] \\ [k']
\end{bmatrix}
\begin{bmatrix}
  \ddot{x} \\ \dot{x} \\ x
\end{bmatrix}
\] (A.5)

where

\[
\begin{bmatrix}
  [m]
\end{bmatrix} = \text{the diagonal mass matrix}
\]

\[
\begin{bmatrix}
  [k']
\end{bmatrix} = \text{the lateral stiffness matrix}
\]

\[
\begin{bmatrix}
  [c]
\end{bmatrix} = \alpha \begin{bmatrix}
  [k']
\end{bmatrix} + \gamma \begin{bmatrix}
  [m]
\end{bmatrix}
\] is the damping matrix

\[
\begin{bmatrix}
  x
\end{bmatrix} = \text{the vector of the horizontal displacements of the floor levels relative to ground}
\]

\[
\ddot{y}(t) = \text{the input ground acceleration}
\]

The solution of the characteristic value problem

\[
\begin{bmatrix}
  [m]^{-1}
\end{bmatrix}
\begin{bmatrix}
  [k']
\end{bmatrix}
\begin{bmatrix}
  x
\end{bmatrix}
= \lambda \begin{bmatrix}
  x
\end{bmatrix}
\]

gives the values of the undamped circular natural frequencies of the system

\[
\rho_i = \sqrt{\lambda}
\]

or

\[
\rho_i = \frac{\rho_i}{2\pi}
\]

and the corresponding mode shapes, \(\phi_i\). The modal matrix is defined as

\[
\begin{bmatrix}
  \Phi
\end{bmatrix} = \begin{bmatrix}
  \phi_1 & \phi_2 & \ldots & \phi_s
\end{bmatrix}
\]

The equations of motion are uncoupled by use of the coordinate transforma-
\[ \{ \dot{x} \} = [\Phi] \{ \dot{\eta} \} \quad (A.6) \]

Substitution of Eq. A.6 into Eq. A.5 leads to
\[
\begin{bmatrix} m \end{bmatrix} [\Phi] [\dot{\eta}] + \begin{bmatrix} c \end{bmatrix} [\Phi] [\dot{\eta}] + [\kappa'] [\Phi] [\eta] = -\ddot{y}(t) \begin{bmatrix} m \end{bmatrix} \{ 1 \}
\]
\[
[\Phi] \begin{bmatrix} \ddot{m} \end{bmatrix} [\Phi] [\ddot{\eta}] + [\Phi] \begin{bmatrix} \ddot{c} \end{bmatrix} [\Phi] [\dot{\eta}] + [\Phi] \begin{bmatrix} \ddot{\kappa} \end{bmatrix} [\Phi] [\eta] = -\ddot{y}(t) [\Phi] \begin{bmatrix} \ddot{m} \end{bmatrix} \{ 1 \} \quad (A.7)
\]

Making use of the orthogonality of the natural modes and the form assumed for \([c]\), the quadratic forms in the above equations become
\[
[\Phi] [m] [\Phi] = [\ddot{m}_\epsilon]
\]
\[
[\Phi] [\kappa'] [\Phi] = [\ddot{m}_\epsilon] [\ddot{m}_\epsilon]
\]
\[
[\Phi] [c] [\Phi] = (\alpha [\ddot{m}_\epsilon] + \gamma \{ 1 \}) [\ddot{m}_\epsilon]
\]
\[
\text{in which } [\ddot{m}_\epsilon] \text{ is a diagonal matrix with the elements } \ddot{m}_r \text{ given by}
\]
\[
\ddot{m}_r = \begin{bmatrix} \phi_r \end{bmatrix} \begin{bmatrix} m \end{bmatrix} \begin{bmatrix} \phi_r \end{bmatrix}
\]
\[
\text{(A.9)}
\]

Substitution of the values given by Eq. A.8 into Eq. A.7 leads to the system of uncoupled equations
\[
\ddot{m}_1 (\ddot{\eta}_1 + (\alpha \ddot{p}_1 + \gamma) \dot{\eta}_1 + \ddot{p}_1 \eta_1) = -[\ddot{\phi}_1] \begin{bmatrix} m \end{bmatrix} \{ 1 \} \ddot{y}(t)
\]
\[
\ddot{m}_2 (\ddot{\eta}_2 + (\alpha \ddot{p}_2 + \gamma) \dot{\eta}_2 + \ddot{p}_2 \eta_2) = -[\ddot{\phi}_2] \begin{bmatrix} m \end{bmatrix} \{ 1 \} \ddot{y}(t) \quad (A.10)
\]
\[
\vdots
\]
\[
\ddot{m}_s (\ddot{\eta}_s + (\alpha \ddot{p}_s + \gamma) \dot{\eta}_s + \ddot{p}_s \eta_3) = -[\ddot{\phi}_s] \begin{bmatrix} m \end{bmatrix} \{ 1 \} \ddot{y}(t)
\]

The \(r\)th equation in this system may be written as
\[
\ddot{\eta}_r + 4 \pi \zeta_r \ddot{p}_r \dot{\eta}_r + 4 \pi^2 \ddot{p}_r^2 \eta_r = -C_r \ddot{y}(t) \quad (A.11)
\]
in which \( C_r \), the so-called participation factor of the \( r \)th mode is given by

\[
C_r = \frac{[\varphi_r^T] [m] \{1\}}{m_r} = \frac{\{\varphi_r^T\} [m] \{1\}}{[\varphi_r^T] [m] \{\varphi_r\}} \tag{A.12}
\]

and \( \zeta_r \), the damping factor for the \( r \)th mode is given by

\[
\zeta_r = \frac{4}{2 f_r^2} \left( a \frac{b}{f_r^2} + \gamma \right) = \frac{1}{4 \pi f_r^2} \left( 4 \pi^2 a \frac{b}{f_r^2} + \gamma \right) \tag{A.13}
\]

The method of analysis for linear response uses the equations of motion in the form of Eq. A.11. The amount of damping is specified in terms of the percent of critical damping value for the fundamental mode and the ratio \( \alpha/\gamma \), as follows:

a. In the general case for which both \( \alpha \) and \( \gamma \) are different from zero, the ratio \( \alpha/\gamma \) is determined from Eq. A.13 as

\[
\frac{\zeta_2}{\zeta_1} = \frac{4 \pi^2 a \frac{b}{f_r^2} + \gamma}{4 \pi^2 a \frac{b}{f_1^2} + \gamma} \frac{f_1}{f_r} \tag{A.14a}
\]

b. In the particular case of mass-proportional damping, \( \alpha = 0 \) and Eq. A.14a becomes

\[
\frac{\zeta_2}{\zeta_1} = \frac{f_1}{f_r} \tag{A.14b}
\]

c. In the particular case of stiffness-proportional damping, \( \gamma = 0 \) and

Eq. A.14a becomes

\[
\frac{\zeta_2}{\zeta_1} = \frac{f_r}{f_1} \tag{A.14c}
\]
APPENDIX III

INTEGRATION TECHNIQUES

1. Integration Formulas

The integration formulas used assume that the acceleration of the masses during the time interval, $\Delta t$, vary linearly. The pertinent formulas are obtained from Newmark's $k$-Formulas (10) by taking $k = 1/6$.

Considering $x(t)$ to be the function under study, one has

$$
\ddot{x}_{t+\Delta t} = \ddot{x} + \frac{\ddot{x}_{t+\Delta t}}{2} \Delta t
$$

(A. 15)

$$
\dddot{x}_{t+\Delta t} = \dot{x}_t + \ddot{x}_t \Delta t + \dddot{x}_t \frac{\Delta t^2}{3} + \dddot{x}_{t+\Delta t} \frac{\Delta t^2}{6}
$$

For the types of problems considered, it is convenient to write these formulas in incremental form as follows:

$$
\Delta \dddot{x} = \dddot{x}_t \Delta t + \Delta \dddot{x} \frac{\Delta t}{2}
$$

(A. 16)

$$
\Delta \dddot{x} = \dot{x}_t \Delta t + \dddot{x}_t \frac{\Delta t^2}{2} + \Delta \dddot{x} \frac{\Delta t^2}{6}
$$

where $\Delta$ denotes the incremental change in the quantity under consideration.

2. Dynamic Linear Analysis

The incremental form of Eq. A.11 is

$$
\Delta \dddot{\eta}_x + 4 \pi \xi f_1 \Delta \dddot{\eta}_x + 4 \pi^2 f_1^2 \Delta \dddot{\eta}_h = -C_n \Delta \dddot{y}
$$

(A. 17)

Application of the integration formulas in Eq. A.17 leads to

$$
\Delta \dddot{\eta}_x + 4 \pi \xi \Delta t \eta_{x,t} \Delta t + \frac{\Delta t^2}{2} \Delta \dddot{\eta}_x + 4 \pi^2 \Delta t \frac{\Delta t^2}{2} + \frac{\Delta t^2}{6} \Delta \dddot{\eta}_x
$$

$$
1 + 2 \pi \xi \frac{f_1}{\Delta t} + \frac{2}{3} \pi^2 (f_1 \Delta t)^2 \Delta \dddot{\eta}_x = - C_n \Delta \dddot{y} + 4 \pi \xi \Delta t \frac{\Delta t^2}{2} + 4 \pi^2 \frac{\Delta t^2}{6}
$$

$$
\left[1 + 2 \pi \xi \left(\frac{f_1}{\Delta t} + \frac{2}{3} \pi^2 (f_1 \Delta t)^2\right)\right] \Delta \dddot{\eta}_x = - C_n \Delta \dddot{y} + 4 \pi \xi \Delta t \frac{\Delta t^2}{2} + 4 \pi^2 \frac{\Delta t^2}{6}
$$
whence

\[ \Delta \tilde{\dot{\eta}}_n = -\frac{C_n \Delta \tilde{\dot{y}} + 4 \Delta^2 F_n^2 \Delta t \tilde{\eta}_n \dot{t} + 2 \Delta t \int_{t_n} \Delta t \left( 2 \tilde{\eta}_n + \int_{t_n} \Delta t \right) \tilde{\eta}_n \dot{t}}{1 + 2 \Delta t \left( \tilde{\eta}_n + \frac{1}{2} \int_{t_n} \Delta t \right)} \quad (A. 18) \]

Once the values of \( \Delta \tilde{\dot{\eta}}_n \) are known, one may compute the values of \( \Delta \tilde{\dot{\eta}}_n \) and \( \Delta \eta_n \) by application of Eq. A.16, and finally determine the values of \( \eta_n, \tilde{\eta}_n \) and \( \dot{\eta}_n \) at the end of the time interval from

\[
\eta_n, t+\Delta t = \eta_n, t + \Delta \eta_n \\
\tilde{\eta}_n, t+\Delta t = \tilde{\eta}_n, t + \Delta \tilde{\eta}_n \\
\dot{\eta}_n, t+\Delta t = \dot{\eta}_n, t + \Delta \dot{\eta}_n \\
\]

Since this is a self-starting procedure it is only necessary to set \( \eta_n, t = 0 \), \( \dot{\eta}_n, t = 0 \) and \( \ddot{\eta}_n, t = -C_n \tilde{\dot{y}}(0) \) to start the integration algorithm.

3. **Dynamic Nonlinear Analysis**

In this case, the equations of motion are written in an incremental form, Eq. (15). The incremental accelerations are then expressed as

\[
\{ \Delta \ddot{x} \} = -\left[ \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial \dot{x}} \right) \right] \{ \ddot{x} \} = \left[ \frac{\partial}{\partial \ddot{x}} \right] \{ \Delta \ddot{x} \} = -\Delta \ddot{y} \quad (A. 20)
\]

Considering known the values of \( \{ x \}_t, \{ \dot{x} \}_t, \) and \( \{ \ddot{x} \}_t \) at the beginning, \( t \), of the time step, \( \Delta t \), the values of \( \{ x \}_{t+\Delta t}, \{ \dot{x} \}_{t+\Delta t} \) and \( \{ \ddot{x} \}_{t+\Delta t} \) are found by the following algorithm

a. Assume a vector \( \{ \Delta \ddot{x}' \} = \bar{E} \{ 1 \} \), where \( \bar{E} \) is a scalar;
b. Form \( \{ \Delta \ddot{x} \} = \{ \Delta \ddot{x}' \} ; \{ \dot{x} \}_{t+\Delta t} = \{ \ddot{x} \}_t + \{ \Delta \ddot{x} \} ; \)
c. Evaluate \( \{ \Delta x \} = \frac{\Delta t}{2} \left[ \{ \dot{x} \}_t + \{ \dot{x} \}_{t+\Delta t} \right] \) ; and

\( \{ \Delta x \} = \Delta t \{ \dot{x} \}_t + \frac{\Delta t^2}{2} \{ \ddot{x} \}_t + \frac{\Delta t^2}{6} \{ \ddot{x} \}_{t+\Delta t} ; \)
d. From Eq. A. 24 compute a new value for

\[ \{ \Delta \ddot{x}' \} = f (\{ \Delta \dot{x}' \}, \{ \Delta x \}, \Delta \ddot{y}) ; \]

e. For each element of \( \{ \Delta \ddot{x}' \} \) compute the relative error (or discrepancy)

\[ E_i = \left| \frac{\Delta \ddot{x}'_i - \Delta \ddot{x}_i}{\Delta \ddot{x}_i} \right| ; \]

f. If \( E_i \leq \bar{E} \), for \( i = 1, 2, \ldots, S \), then proceed with step g; otherwise, check the number of cycles to insure that the process is convergent, and go to step b;

\[ g. \text{ Compute } \{ \dot{x} \}_{t+\Delta t} = \{ \dot{x} \}_t + \{ \Delta \dot{x} \} ; \]
\[ \{ \ddot{x} \}_{t+\Delta t} = \{ \ddot{x} \}_t + \{ \Delta \ddot{x} \} ; \]
\[ \{ \dddot{x} \}_{t+\Delta t} = \{ \dddot{x} \}_t + \{ \Delta \dddot{x} \} . \]

To insure stability and convergence of this iteration algorithm the time interval \( \Delta t \) must be less than the following value (10)

\[ \Delta t = 0.389 T_s \]

where \( T_s \) = the shortest natural period of the system.

In order to minimize the number of iteration cycles needed, it is advisable to use a time interval equal to \( 1/3 \) or \( 1/4 \) of the above limit, i.e.

\[ \Delta t \leq (0.10 \text{ to } 0.15) T_s \]

With this time interval, convergence may be attained in 4 to 5 cycles if \( \bar{E} \) is taken as \( \bar{E} = 0.001 \). The magnitude of relative error at the end of the \( r \)th cycle of this iterative procedure is given by

\[ E = \left( \frac{\Delta t}{0.389 T_s} \right)^{2(n-1)} \]

where \( \Delta t \) = the integration interval used.
APPENDIX IV

STIFFNESS MODIFICATIONS

Whenever the state of yielding at the end of a member changes, the stiffness and flexibility matrices of the frame are modified. Since these modifications are concentrated in a few columns of the original stiffness matrix it is possible to use, with advantage, the procedure described below to find the new flexibility without having to invert again the stiffness matrix. Considering that the evaluation of the modified flexibility is a major step in the nonlinear analysis, one may well appreciate the advantage of this procedure.

Suppose one knows \([K]\) of size \(N \times N\) and its inverse \([F]\). Suppose also that one modifies the ith column of \([K]\) by the values defined by the vector \(\{a\}\), and wishes to determine the new inverse \([F^*]\) without making a formal inversion of the modified matrix \([K^*]\).

Express the modified version of \([K]\) as

\[
[K^*] = [K] + [A]
\]  \hspace{1cm} (A. 21)

and let \(\{e_i\}\) be a vector of size \(N \times 1\) for which all elements are zero except for the ith element which is equal to 1. Then, the matrix \([A]\) may be written as

\[
[A] = \{a\} \{e\}^T
\]

Now, perform the following matrix operations

\[
[A][F][A] = \{a\}\{e_i\}^T[F]\{a\}\{e_i\}^T = \{a\}\{\{e_i\}^T[F]\{a\}\} \{e_i\}^T
\]

\[
= \{\{e_i\}^T[F]\{a\}\} [A]
\]  \hspace{1cm} (A. 22)
and

\[ [K^*[F][A][F] = [K+A][F][A][F] = [A][F] + [A][F][A][F] \]  \tag{A.23} \]

substituting the product \([A][F][A] \) from Eq. A.22 into the latter expression of Eq. A.23, one obtains

\[ [K^*[F][A][F] = (1 + \{e_i\}^T[F]\{a\})[A][F] \]
\[ = (1 + \{e_i\}^T[F]\{a\})((A)[F] + [K][F] - [I]) \]  \tag{A.24} \]

where the unit matrix \([I] = [K][F] \) By virtue of Eq. A.21, Eq. A.24 may also be written as

\[ [K^*[F][A][F] = (1 + \{e_i\}^T[F]\{a\})([K^*[F] - [I]) \]  \tag{A.25} \]

Premultiplying both sides of Eq. A.25 by \([F^*] \), one finds that

\[ [F][A][F] = (1 + \{e_i\}^T[F]\{a\})([F] - [F^*]) \]

and solving for \([F^*] \) one finally obtains

\[ [F^*] = [F] - \left( \frac{1}{1 + \{e_i\}^T[F]\{a\}} \right) [F][A][F] \]

or

\[ [F^*] = [F] - \left( \frac{1}{1 + \{e_i\}^T[F]\{a\}} \right) [F]\{a\}\{e_i\}^T[F] \]

This equation gives the new matrix \([F^*] \) in terms of the original matrix \([F] \) and the vector of the changes in the ith column of \([K]\)  

The elements of \([F^*] \) may be written in expanded form as follows

\[ f^*_m = f_m - \sum_{k=1}^{N} \frac{f_{km} a_m f_{i\ell}}{1 + \sum_{r=1}^{N} f_{ir} a_r} \]

where \(N\) = the order of the matrix.

When more than one column of \([K]\) is modified, the procedure just described has to be applied successively as many times as the number of columns changed.
In the present study, the procedure was conveniently specialized to the special case where \([K]\) is symmetric and the changes are concentrated in only two columns of the matrix and only two elements per column. The elements under consideration are the two diagonal elements \(k_{ii}\) and \(k_{jj}\) and the two off-diagonal elements \(k_{ij}\) and \(k_{ji}\).

In the dynamic nonlinear analysis it is necessary to modify \([K_{22}]^{-1}\) due to changes in \([K_{22}]\). When a hinge is placed or removed two columns of \([K_{22}]\) are changed, two elements per column, and the procedure specialized for such a case is used with advantage.

In the static analysis, when the change in \([K]\) results from yielding in a girder, only two columns of the matrix \([K_s]\) are changed, and the modified \([F_s^*]\) is found in the same way as for the dynamic case. However, when the modification results from yielding in a column, four columns of \([K_s]\) are modified and the new \([F_s^*]\) is obtained by applying the original procedure four times in succession.
APPENDIX V

REFERENCES


APPENDIX VI

NOTATION

\[ A_T = \text{pseudo acceleration of top story;} \]
\[ C = \text{number of lines of columns;} \]
\[ C_r = \text{participation factor for the } r\text{th mode;} \]
\[ C_g = \text{girder yield factor;} \]
\[ E = \text{Young's modulus of elasticity;} \]
\[ E(t) = \text{transient effect in structure;} \]
\[ E_r = \text{modal contribution factor for } E(t) \text{ in } r\text{th mode; see Eq. 14;} \]
\[ I = \text{second moment of cross-sectional area with reference to centroidal axis;} \]
\[ (I.A.F.)_{d,i} = \text{instantaneous amplification factor for displacements of } r\text{th mode; same as } (I.A.F.)_i; \]
\[ (I.A.F.)_{a,i} = \text{instantaneous amplification factor for accelerations of } r\text{th mode;} \]
\[ L = \text{span of member;} \]
\[ M = \text{maximum value of overturning moment at base of frame;} \]
\[ M^* = W^* h^*; \]
\[ M_g = \text{maximum value of bending moment in girders;} \]
\[ M_{g,j} = \text{maximum value of bending moment for girder in } j\text{th floor;} \]
\[ M_c = \text{maximum value of bending moment in columns;} \]
\[ M_{c,j} = \text{maximum value of bending moment for column in } j\text{th floor;} \]
\[ M_{l,r} = \text{bending moment at left and right end of member, respectively;} \]
\[ (M_i)_{st} = \text{modal contribution factor for overturning moment at base in the } i\text{th mode, in Eq. 14;} \]
\[ M_y = \text{yield moment at end of member}; \]
\[ \text{MDF} = \text{multi-degree-of-freedom}; \]
\[ P = \text{axial force in typical member}; \]
\[ P_c = \text{maximum value of axial load in columns}; \]
\[ P_i = \text{instantaneous value of sum of axial forces in columns of ith floor}; \]
\[ Q = \text{shear force at end of member}; \]
\[ S = \text{number of stories}; \]
\[ \text{SDF} = \text{single-degree-of-freedom}; \]
\[ \text{SASS} = \text{static approach for soft systems}; \]
\[ \text{SARS} = \text{static approach for rigid systems}; \]
\[ T_r = \text{rth natural period of system}; \]
\[ U = \text{maximum value of deformation for SDF system}; \]
\[ V = \text{maximum value of total shear at base}; \]
\[ (V_{i_{\text{st}}}) = \text{modal contribution factor for total shear at base in ith mode}; \]
\[ W_i^* = \text{effective weight associated with ith mode}; \]
\[ W^* = \text{effective weight of structure, same as } W_i^*; \]
\[ X_T = \text{maximum value of top story deflection relative to ground}; \]
\[ (X_{T_{\text{to}}}) = \text{same as } X_T \text{ for linear response}; \]
\[ d = \text{second slope coefficient in } M - \alpha \text{ diagram}; \]
\[ e = \text{second slope coefficient in } M - \phi \text{ diagram}; \]
\[ f_{1t_1} = \text{frequency parameter}; \]
\[ f_r = \text{rth natural frequency of the system in c.p.s.}; \]
\[ g = \text{gravitational acceleration}; \]
\( h_i \) = height of \( i \)th floor;

\( h \) = floor height in uniform frame;

\( h^* \) = height to center of effective weight;

\( \ell_y \) = effective length of yielding;

\( m^*_i \) = effective mass associated with \( i \)th mode;

\( p_r \) = \( r \)th circular natural frequency of system;

\( t \) = time;

\( 2t_1 \) = duration of pulse;

\( t_{1,a} \) = average duration of dominant pulse in input acceleration diagram;

\( u \) = relative displacement of ends of member;

\( \ddot{y}(t) \) = input ground acceleration function;

\( y_0 \) = maximum ground displacement;

\( \dot{y}_0 \) = maximum ground velocity;

\( \ddot{y}_0 \) = maximum ground acceleration;

\( \alpha \) = plastic rotation of hinge at end of member;

\( \sigma \) = intensity of dead load;

\( \zeta_r \) = damping factor for \( r \)th mode;

\( \theta \) = rotation at support of member;

\( \theta_e \) = rotation of elastic element of member;

\( \mu_\ell \) = local ductility ratio;

\( \mu_o \) = overall ductility ratio;

\( \nu \) = load factor;

\( \rho \) = girder-to-column stiffness ratio;
\[ \phi = \text{curvature of axis of member}; \]
\[ \phi_y = \text{curvature at which yielding starts}; \]
\[ \Delta = \text{incremental operator}; \]
\[ \{M_F\} = \text{vector of unbalanced moments due to dead load}; \]
\[ \{x\} = \text{vector of horizontal displacements of floor levels relative to ground}; \]
\[ \{\theta\} = \text{vector of joint rotations}; \]
\[ \{\eta\} = \text{vector of generalized displacements}; \]
\[ \{\phi_i\} = \text{ith natural mode} \]
\[ [K_s] = \text{stiffness matrix of frame used in static analysis}; \]
\[ [F_s] = [K_s]^{-1} \]
\[ [K_{11}] = \text{stiffness matrix representing horizontal forces associated with horizontal displacements of floors and no rotations of joints}; \]
\[ [K_{22}] = \text{stiffness matrix representing the joint moments associated with joint rotations and no horizontal displacements}; \]
\[ [K_{12}] = [K_{21}]^T = \text{stiffness matrix representing the horizontal forces associated with joint rotations and no horizontal displacements}; \]
\[ [K'] = \text{lateral stiffness matrix}; \]
\[ [T] = \text{matrix in equation } \{\Delta \theta\} = [T]\{\Delta x\}; \]
\[ [\phi] = \text{square matrix of natural modes } \phi_1, \phi_2, \ldots, \phi_s \]
\[ \{\} = \text{column vector}; \]
\[ [\ ] = \text{rectangular matrix}; \]
\[ \text{dots } = \text{derivative with respect to time}; \]

subscripts
\[ \ell = \text{left end of member}; \]
\[ r = \text{right end of member}. \]