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ALGEBRAIC TOPOLOGY OF FREDHOLM MAPS.

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Algebraic Topology of Fredholm Maps

by

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INTRODUCTION

This thesis can be approached from two directions: that of algebraic topology, and that of nonlinear functional analysis. The theorems are all generalizations of "classical" algebraic topology (although "classical" is stretched a bit; in some degree they are new even in the classical case), but one hopes they will be of use in the qualitative theory of nonlinear elliptic partial differential equations. The theorems are quite powerful from a topological point of view; from an analytical point of view, the situation is unclear; for there exist (to my knowledge) no nontrivial, interesting problems of analysis to which these theorems are applicable.

Perhaps I may be allowed a defensive paragraph. These theorems are interesting in that they derive results from a certain measure of the nonlinearity of an equation, and most problems so far encountered in physics or calculus of variations are simply not nonlinear enough. As the theorems now stand, they yield only negative results on quasilinear equations—yet quasilinear equations make up the vast bulk of problems currently studied. This is not to say that they will not be studied; but this thesis seems to be a case of art exceeding science. I plead first that the ground is fallow but unexplored,
and secondly that as topology this is interesting in its own right.

The idea of the thesis is simple: to enlarge the now-classical Leray-Schauder degree theory along lines indicated by Smale [20]. The classical degree theory suffered from several weaknesses. First, to a certain (ver restricted) class of operators an integer, the degree, was associated, and it was shown that if the degree were nonzero the equation possessed a solution. It was also clear that the size of the degree had something to do with the multiplicity of solutions, but the obvious relation resisted proof.

Smale's work clarified much of this. He indicated that the degree was equal to the multiplicity if one was willing to speak of generic (i.e. nonexceptional) solutions, and greatly enlarged the class of operators to which the theory was applicable. Thus the Leray-Schauder theory applied to operators of the form identity + compact perturbation. Smale's work extended the validity of the technique to proper elliptic operators (that is, elliptic operators with a certain qualitative type of a priori bound on the stability of solutions) and with positive index. Smale's work, however, was essentially modulo two. Ellworthy and Tromba (independently)
removed this restriction and developed many other aspects of the theory of Fredholm maps. In this thesis both conditions of properness and positivity are removed; the degree theory can be applied to a completely general elliptic operator between Banach manifolds; moreover, the "degree" of Smale is strengthened to show, in some sense, not only "how many solutions" there are, but also "where they lie" topologically.

The idea of Smale is as follows: Suppose we are given an abstract elliptic operator $\alpha: X \to Y$ (the proper general notion is that of a Fredholm map, which is defined below). We pick a "generic" point $y \in Y$ and consider the set $\alpha^{-1}(y)$. This set will be a manifold; the dimension of $\alpha^{-1}(y)$ is an invariant of $\alpha$, called the index and written $\varepsilon(I\alpha)$ (see below for the reason for this notation). The case $\varepsilon(I\alpha) = 0$ is a strong generalization of the Leray-Schauder case; in this case, the set $\alpha^{-1}(y)$ is discrete, and the number of points in it is the degree. (Since $\alpha$ is assumed proper, this number is finite.)

This thesis is much more ambitious. We map a whole compact smooth manifold "generically" into $Y$, and examine its preimage in $X$ under $\alpha$. This enables us to get information on singularities and bifurcation
theory as well as milking the maximum possible information from Smale's technique. The first main theorem may be stated:

**Theorem:** Let $\alpha: X \to Y$ be a proper oriented Fredholm map between Banach manifolds. Then there is an induced homomorphism of $\Omega^k_*(pt) - \text{modules } (k = \epsilon(I\alpha))$

$$\alpha^! : \Omega^*_{s_k}(Y) \longrightarrow \Omega^*_{s_k+k}(X; \mathcal{E}_b)$$

$\subseteq \Omega^*_{s_k}(X)$ indicates the Thom bordism groups of the space $X$, and $\Omega^*_{s_k}(X; \mathcal{E}_b)$ are the bordism groups with bounded supports, defined below. These appear to be new. If $1 \in \Omega^0(Y)$ is the identity, then $\alpha^!(1)$ is a generalization of Smale's invariant.

Now when $X$ and $Y$ are finite-dimensional manifolds, the homomorphism in the theorem is familiar in algebraic topology; at least a construction can be given easily from published literature. The construction we give works equally well in infinite dimensions and is more conceptual.

Since $\alpha^!$ is such a strong invariant of Fredholm maps, two questions occur: what are its properties, and how can it be computed. We give an answer to both. The answer to the first question is essentially: anything which is true of the finite-dimensional analogue
is true of the infinite-dimensional version. That is, we give an essentially canonical technique for reducing questions about $\alpha^!$ to the finite-dimensional use. As examples of this technique we prove two "Riemann-Roch type" theorems about $\alpha^!$ (Props. 6.9 and 6.11; the motivation for the names "Riemann-Roch type" is slightly metaphysical and will not be explicated.) Two comments on these propositions are in order. The first proposition is concerned with the behavior of $\alpha^!$ under certain "cohomology" operations. For this we restrict ourselves to unitary cobordism, for the cases of unitary cobordism and unoriented cobordism are the only ones so far in which the structure of these operations is known at all. The unoriented case follows the unitary case trivially and is omitted. The second proposition concerns the relation of $\alpha^!$ to a similar object: in fact one may construct an analogue of $\alpha^!$ in ordinary homology and ask if they are compatible.

Unfortunately the construction of the homology analogue of $\alpha^!$ depends on extremely different techniques (techniques from algebraic geometry and homological algebra) and the published proofs and unpublished notes available for use in the proof are very sketchy, and
do not allow one to get as strong a theorem as is probably possible. For this reason I have axiomatized the properties of the homological analogue needed and shown how Prop. 6.11 follows. I assert the existence of this analogue only in a limited case which is easily accessible from the published literature; but when a full account of Verdier's work is available I plan to give a much stronger theorem.

In any case the grasp on the properties of $\alpha'$ given by Props. 6.9 and 6.11 is strong enough to make $\alpha'$ essentially computable, thus answering the second question raised above. We show that the characteristic numbers, (the standard way of computing bordism classes) are directly computable from knowledge of the index bundle and the homology analogue of $\alpha'$, and write down the expected formulae.

Finally, we construct a machine to enable us to talk about singularities of differential operators and prove Prop. 7.11, which measures the bifurcation of an operator at a "generic" singular point. All of Section Seven is perhaps a bit clumsy; I plan to write a much longer analysis of singularities in general and apply it to the problems begun there, in hopes of more conceptual theorems.
The thesis is organized in seven sections and an appendix. The first three expose the differential topology of infinite-dimensional manifolds that will be needed later; much of this comes from Segal's thesis \[28\] and Abraham-Robbins \[3\]. Section Four contains the algebraic and homotopy-theoretic machinery that will be used later; the only claim to originality here are the supported bordism groups. Section Five outlines a technique for "orienting" Fredholm maps and is perhaps idiosyncratic: Ellworthy and Tromba take a completely different approach, but I think the technique of Section Five is conveniently functorial. Section Six contains the finite-dimensional reduction technique (Prop. 6.5) and its consequences, as well as the construction of \(\omega!\); this is the heart of the thesis. Section Seven introduces certain characteristic classes of singularities and applies them to the problem of measuring bifurcation, but stronger theorems are probably available. The appendix is an uneasy compromise. This thesis probably contains quite enough expository material without it, but it seemed advisable to pay lip service to some shibboleths of global analysis. It is intended for those who balk (perhaps rightly) at believing that Banach manifolds have anything
to do with analysis. It is intended to indicate how one gets from an elliptic equation to a Fredholm map; but there is an excellent expose in Palais \( \int_{24}^7 \) which the interested reader would be better advised to read.

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SECTION 1

Indexes, Fredholm Complexes

Notation: All manifolds considered will be either Banach manifolds or finite-dimensional; but a Banach manifold needn't be infinite-dimensional. All Banach manifolds will be assumed sufficiently differentiable, and will be denoted by roman capitals from the bottom of the alphabet. A smooth map whose domain is a Banach manifold will be written as a Gothic capital, and a map whose domain is definitely finite-dimensional will be denoted by a small roman letter. The tangent vector bundle of a manifold $X$ will be denoted by $\mathcal{T}_X$ regardless of dimensionality; the fiber at $x \in X$ will be written $\mathcal{T}_x$. The derivative of $\alpha: X \to Y$ will be written $d\alpha: \mathcal{T}_X \to \mathcal{T}_Y$. Except for the applications, I will be sloppy about naming model spaces for Banach manifolds. Vector bundles, Banach spaces, etc. will be real unless otherwise noted.

Defn. 1.1: If $E, F$ are fixed Banach spaces, write $L(E,F)$ for the space of bounded linear operators from $E$ to $F$ with the uniform topology. An operator $T: E \to F$ is Fredholm if both ker $T$ and coker $T$ have finite algebraic dimension. Write $\mathcal{F}(E,F) \subseteq L(E,F)$ for the subspace of Fredholm operators with the induced topology. Recall Palais [2, VII th. 4.7].
Prop 1.2: \( \mathcal{T}(E,F) \) is open in \( L(E,F) \); the function \( \text{ind}: \mathcal{T}(E,F) \rightarrow \mathbb{Z} \) defined by \( \text{ind}(T) = \dim \ker T - \dim \text{coker} T \) is continuous. //

Defn. 1.3: Let \( X, Y \) be Banach manifolds, \( \alpha: X \rightarrow Y \) a smooth map. We have a diagram of vector bundles:

\[
\begin{array}{ccc}
\mathcal{T}_X & \xrightarrow{d\alpha} & \mathcal{T}_Y \\
\pi_X & \xrightarrow{\alpha^*\pi_Y} & \downarrow \\
X & \xrightarrow{\alpha} & Y
\end{array}
\]

where \( \alpha^*\pi_Y \rightarrow \pi_X \) is the induced map of the pullback bundle \( \alpha^*\mathcal{T}_Y \) of \( \mathcal{T}_Y \) (See Lang, 187 for definitions and properties of pullbacks.). By the universality properties of pullbacks, there is then a well-defined map of vector bundles over \( X \), written \( d\alpha: \mathcal{T}_X \rightarrow \alpha^*\mathcal{T}_Y \).

ACKNOWLEDGEMENT: Much of this section is plagiarized from Graeme Segal's thesis \( \gamma^27 \).

Defn. 1.4: A Fredholm complex over a topological space \( X \) is a sequence of bundles \( E_i \) of locally convex complete Hausdorff linear topological vector spaces admitting a bounded neighborhood of \( 0 \), and a sequence \( d_i: E_i \rightarrow E_{i+1} \) of continuous bundle homomorphisms, such that \( d_{i+1}d_i = 0 \) and the \( d_i \) are chain homotopic to compact bundle.
homomorphisms; that is, there are $h_i : E_i \to E_{i-1}$ (continuous bundle homomorphisms) such that $d_{i-1} h_i + h_{i+1} d_i = 1_{E_i} - k_i$, where $k_i : E_i \to E_i$ is a bundle homomorphism and is a compact map of vector spaces on each fiber.

Defn. 1.5: A smooth map $\alpha : X \to Y$ is a Fredholm map (or $\mathcal{F}$-map in short) if the sequence

$$0 \to \mathcal{T}_X \xrightarrow{d\alpha} \alpha^* \mathcal{T}_Y \to 0$$

is a Fredholm complex. It is clear that this is equivalent to: $d\alpha_x \in \mathcal{O}_F(\mathcal{T}_X, x; \mathcal{T}_Y, \alpha x)$ for any $x \in X$; that is, $d\alpha$ is a Fredholm operator on the fibers.

Remark: Write $\mathcal{F}(X)$ for the category of Fredholm complexes defined above (with continuous homomorphisms of vector bundle complexes for morphisms). The derived category of $\mathcal{F}(X)$, written $D\mathcal{F}(X)$, is just the above category with certain morphisms adjoined. Every complex $(E_\cdot, d_\cdot) \in \mathcal{F}(X)$ has pointwise homology groups $H_i(E_\cdot, d_\cdot)_x = \ker d_x^i / \text{im } d_x^{i-1}$ for each $x \in X$.

These are finite-dimensional vector spaces (by the existence of the chain homotopy). A map $\alpha : (E_\cdot, d_\cdot) \to (F_\cdot, d'_\cdot)$ in $\mathcal{F}(X)$ is a quasi-isomorphism if it induces isomorphisms of the pointwise homology groups. Then
$\mathcal{D}(X)$ is obtained by adjoining to $\mathcal{E}(X)$ a formal inverse to each quasi-isomorphism.

Remark 1: $\mathcal{D}(X)$ is the smallest category containing $\mathcal{E}(X)$ in which every acyclic complex splits. See Verdier [33]. We could go to the other extreme and require that the "boundary operator" have split kernel and image on each fiber; we would then get "strong" Fredholm complexes, which however are apparently useless.

Defn. 1.7: Let $K_+(X)$ be the Grothendieck group of the additive category $\mathcal{D}(X)$; let $K(X)$ be the "represented K-theory" $\bigwedge X, \mathcal{F}(\mathcal{H}, \mathcal{H})$. (Where $\mathcal{H}$ is a Hilbert space; see Atiyah [1, Appendix] or Jänich [14].) We should write $K_0(X)$, but see below about reality. We then have

THEOREM: (G.Segal) If $X$ is paracompact and locally compact, then $K_+(X)$ and $K(X)$ are naturally isomorphic. (Proof: See Segal's thesis.)

Defn. 1.8: Let $\alpha: X \to Y$ be a $\mathcal{E}$-map of Banach manifolds. Then $\alpha_+$ is defined to be the image of the Fredholm complex $0 \to \tau_X \xrightarrow{d\alpha} \alpha_+ \tau_Y \to 0$ in $K_+(X)$. 
The augmentation \( \varepsilon(I\alpha) = \text{ind } \alpha \in K(\text{point}) = K(\text{point}) = \mathbb{Z} \) is clearly the value of \( \text{ind } (d\alpha \text{point}) \). (See Prop. 1.1, Def. 1.4). By continuity, it depends only of the component of \( X \). The element \( I\alpha \) will be called the "index bundle of \( \alpha \)" though it is no bundle at all, in general.

**Remark:** Define \( K_{vb}(X) \) to be the Grothendieck group of the additive category of vector bundles over \( X \). We have a sequence

\[
K_{vb}(X) \xrightarrow{r} K(X) \xrightarrow{s} K_{\Phi}(X)
\]

in which \( s \) is an isomorphism if \( X \) is (paracompact and) locally compact, and \( r \) is an isomorphism if \( X \) is compact.

**Remark 2:** If \( \alpha : X \rightarrow Y \) is a smooth map of finite-dimensional manifolds, then \( I\alpha \) is defined; and in fact

\[
I\alpha = r(T_X - \alpha^* T_Y)
\]

is the class representing the inverse to the "stable normal bundle" of \( \alpha \).

**SECTION 2:**

**Transversality, Fiber Products.**

**Defn. 2.1:** Let \( \alpha : X \rightarrow Z, \beta : Y \rightarrow Z \) be continuous maps of topological spaces. Let
\[ X \pi Y \equiv Z \pi Y \equiv \{ (x,y) \in X \times Y \mid \alpha x = \eta y \} \subseteq X \times Y \]

and give this set the relative topology. There are natural maps \( \overline{\pi} : X \pi Y \rightarrow X \), \( \overline{\alpha} : X \pi Y \rightarrow Y \) induced by the projections of \( X \times Y \) onto \( X, Y \) respectively. The diagram

\[ \begin{array}{ccc}
X \pi Y & \xrightarrow{\overline{\alpha}} & Y \\
\downarrow{\alpha \pi \eta} & & \downarrow{\eta} \\
X & \xrightarrow{\alpha} & Z \\
\end{array} \]

\[ (\alpha \pi \eta = \eta \overline{\alpha} = \alpha \overline{\pi} ) \]

is commutative.

**Lemma 2.2:** Let \( \rho : Y \rightarrow X \), \( \eta : Y \rightarrow Z \), \( \alpha : X \rightarrow Z \) be continuous maps of topological spaces. Then in the diagram

\[ \begin{array}{ccc}
V \pi Y & \xrightarrow{\overline{\rho}} & Z \pi Y \\
\downarrow{\alpha \rho \eta} & & \downarrow{\eta \overline{\alpha}} \\
X \pi Y & \xrightarrow{\overline{\alpha \rho \eta}} & Y \\
\downarrow{\alpha} & & \downarrow{\eta} \\
X & \xrightarrow{\alpha} & Z \\
\end{array} \]

there is a natural homeomorphism \( L \).
This is a trivial verification. Define
\[ \forall \pi Z \times \pi \pi Y = \{ (v, x, y) \in V \times X \times Y \mid \pi (v) = x, \sigma (x) = \pi (y) \} \]
\[ \forall \pi Y = \{ (v, y) \in V \times Y \mid \sigma \pi (v) = \pi (y) \} \]
\[ \iota (v, x, y) = (v, y); \quad \iota' (v, y) = (v, \sigma \pi (v), y) \]

Then \( \iota, \iota' \) are inverses and continuous. //

**Defn: 2.3:** Let \( \alpha : X \to Z, \pi : Y \to Z \) be smooth maps of Banach manifolds; we have a commutative diagram over \( I \),

\[
\begin{array}{ccc}
\pi X \times \pi Y & \xrightarrow{d(\alpha \pi \tau)} & \pi Y \\
\downarrow \alpha \pi & & \downarrow \pi \\
\pi X & \xrightarrow{\alpha \pi} & \pi Y \\
\downarrow \alpha & & \downarrow \pi \\
X & \xrightarrow{\pi \pi Y} & Z \\
\end{array}
\]

in which \( \pi X \times \pi Y = \{ (\xi_x, \eta_y) \in g^* T_{X \times Y} \mid d\alpha X (\xi_x) = d\pi Y (\eta_y) \} \) and the vertical maps are induced by fiber projections.

(\( g \) is the inclusion \( g : \pi X \times \pi Y \to X \times Y \)). There is a natural map \( d(\alpha \pi \tau) : g^* T_{X \times Y} \to T Z \) defined by
\[
d(\alpha \pi \tau)(\xi_x, \eta_y) = d\alpha X (\xi_x) + d\pi Y (\eta_y),
\]
and we may describe \( \pi X \times \pi Y \) as \( \ker d(\alpha \pi \tau) \).
Note that \( T_x \Pi_{t_{xy}} \) is defined as a fiber product of topological spaces by Defn. 2.1, but there is no ambiguity, because the two definitions coincide. Note also that the fiber of \( T_x \Pi_{t_{xy}} \) over a point \( p \in X_{t_{xy}} \) is a linear subspace of the fiber of \( g^* T_{x \times y} \) over \( p \).

We say that \( \alpha \) is transversal to \( \mathcal{M} \) (symbolically, \( \alpha \parallel \mathcal{M} \)) if

1) \( \delta(\alpha \cap \mathcal{M}) : g^* T_{x \times y} \to T_z \) is onto over the image of \( X_{t_{xy}} \to Z \).

2) \( T_x \Pi_{t_{xy}} \) is a split vector subbundle of \( g^* T_{x \times y} \).

This last condition is simply that the fiber product exist as a vector bundle.

If \( A \subseteq X \) is a subspace of \( X \), then \( A \Pi_{t_{xy}} \) is a subspace of \( X_{t_{xy}} \); and we say that \( \alpha \) is transversal to \( \mathcal{M} \) over \( A \) (symbolically, \( \alpha \parallel \mathcal{M} \)), if \( T_x \Pi_{t_{xy}} \mid A \Pi_{t_{xy}} \) is a split vector subbundle of \( g^* T_{x \times y} \mid A \Pi_{t_{xy}} \), and \( \delta(\alpha \cap \mathcal{M}) \mid A \Pi_{t_{xy}} \) is a surjection of vector bundles.

Note that if \( U \) is open in \( X \) and \( d\alpha \) restricted to \( U \) is a surjection of vector bundles, and \( d\alpha \) has a split kernel, then \( \alpha \parallel \mathcal{M} \).
Finally, we remark that $\alpha \not\sim \gamma$ if there is a split exact sequence of vector bundles:

$$0 \to T_{XZ} \to T_{XZ} \xrightarrow{(\eta, 1)} T_{XZ} \xrightarrow{\alpha \circ \eta} \gamma \to 0$$

over $X$.

Prop. 2.4: Let $\alpha : X \to Z$, $\gamma : Y \to Z$ be as in 1).

Then we also have a diagram:

$$\begin{array}{c}
X \times Y \pi Z \\
\uparrow \tilde{\alpha} \\
X \times Y \\
\downarrow \tilde{\alpha} \\
X \times Z \xrightarrow{\Delta} Z \\
\Delta (z) = (z, z)
\end{array}$$

inducing a commutative diagram

$$\begin{array}{c}
T_{X \times Y \pi Z} \xrightarrow{q} T_{X \times \pi Z} \\
\downarrow \\
T_{X \times Z} \xrightarrow{q} T_{Z}
\end{array}$$

with $q$ a homomorphism and $\overline{q}$ an isomorphism.

----Proof:-------

Both maps are trivial projections. We have,

$X \times Y \pi Z = \{(x, y, z) : (\alpha x, \gamma y) = (z, z)\}$

$X \times \pi Y = \{(x, y) : \alpha x = \gamma y\}$.

where $q(x, y, z) = (x, y)$. That $q$ has an inverse follows just as in Lemma 2.1. Now by definition

$$T_{X \times Y \pi Z} = \left\{(\xi_x, \gamma_y, \xi_z) \in G^* T_{X \times Y \times Z} \mid d(\alpha x y) (\xi_x, \gamma_y) = d(\xi_z) \right\}$$

$$= \left\{(\xi_x, \gamma_y, \xi_z) \in G^* T_{X \times Y \times Z} \mid \sigma_x (\xi_x) = \xi_z, d\gamma_y (\gamma_y) = \xi_z \right\}.$$
while \( T_x \pi T_y = \{(\xi_x, \eta_y) \in g^* T_{x \times y} | d\sigma_x(\xi_x) = d\gamma_y(\eta_y)\} \); so if
\[
\overline{q}(\xi_x, \eta_y, z) = (\xi_x, \eta_y)
\]
then \( \overline{q} \) is invertible just as \( q \) is, and is furthermore linear. //

\[\text{Corollary 2.5:} \quad \text{With } \alpha, \gamma \text{ as above, } \alpha \circ \gamma \text{ iff } \alpha \circ \gamma \circ \Delta.\]

\[\text{-----:Proof:------:-----}
\]

First let \( q^{-1} \) be the inverse to \( q \); then \( (q^{-1})^* g^* T_{x \times y \times z} = g^* T_{x \times y} \Theta (\pi \Delta)^* T_z \). Hence \( T_x \pi T_y \) is a split subbundle of \( g^* T_{x \times y} \) if \( T_{x \times y} \pi T_z \) is a split subbundle of \( g^* T_{x \times y \times z} \) (for \( (\pi \Delta)^* T_z \) always splits off, and the two fiber products are isomorphic). It remains to see that \( \delta(\alpha \gamma \Delta) \) is onto iff \( d(\alpha \gamma \pi \Delta) \) is. But
\[
d(\alpha \gamma \pi \Delta)(\xi_x, \eta_y, z) = (d\alpha \gamma(\xi_x) + \xi_z, d\alpha \gamma(\eta_y) + \eta_z),
\]
and
\[
d(\alpha \gamma \Delta)(\xi_x, \eta_y) = d\sigma_x(\xi_x) + d\gamma_y(\eta_y).
\]
so if the latter map spans \( T_z \), then we can choose \( z \) in the former map so that \( z \) hits anything in and vice versa. //

\[\text{Prop. 2.6:} \quad \text{Let } \alpha : X \rightarrow Y \text{ be a smooth map of Banach manifolds, } \iota : Z \subset Y \text{ a submanifold inclusion with } \iota \pi \alpha \text{.}
\]

Then
\[
1) \; \overline{\alpha}'(Z) \quad \text{is a smooth submanifold of } X.
\]
\[ T(x \cdot \pi_1 (z)) = (\omega \cdot d \omega)^{1} \text{ image of} \]

\[
\text{---Proof:---}
\]

This is just Prop. 17.2 of [3.7].

**Prop. 2.7:** Let \( \omega, \eta \) be as in I), with \( \omega \neq \eta \).
Then \( \chi_{X, Z} \) is a manifold with \( T_{x, Z} = T_{x, \pi Z} \).

\[
\text{---Proof:---}
\]

In view of Corr. 2.5 and Prop. 2.6, \( \omega \neq \eta \) implies that \( \omega \times \eta \neq \Delta \); for \( \Delta \) is a submanifold inclusion, and \( \chi_{X, Z} \) is homeomorphic to \( (\omega \times \eta) \Delta (z) = X \times \gamma \pi Z \).
Thus the only thing to check is that \( T_{x, Y \cdot \pi Z} \cong d(\omega \times \eta) \text{ im } \Delta \).
But this is obvious from the definition.

**Corr. 2.8:** If \( X \) has boundary and \( Y \) does not, and if they are transversal over \( Z \), then

\[ \partial (\chi_{X, Z}) = \partial \chi_{X, Z} \]

**Lemma 2.9:** Let \( \alpha, \beta, \omega \) be as in II), such that \( \omega \neq \eta \), \( \alpha \neq \beta \) \( \eta \).

\[
\text{---Proof:---}
\]

If \( \omega \neq \eta \), then \( \chi_{X, \pi Z} \), which is a priori only a topological space, is a vector bundle and isomorphic to \( T_{X, \pi Z} \).
Hence \( T_{V \cdot \pi Z} \) is also a vector bundle (since \( \phi \neq \eta \) ); but by lemma 2.2, this is just
\[ \tau \pi \overline{\tau} \]  
Hence \( \sigma \pi \overline{\tau} \) if the map \( \delta(\alpha \pi \overline{\pi}) \overline{\delta} \)  
\[ = \delta \circ \delta(\alpha \pi \overline{\tau}) \]  
is onto. But suppose \( \gamma \in \tau \) may be written as \( \delta \circ \delta(\alpha \pi \overline{\tau}) \). Then \( \xi \) may be written as \( \delta \circ \delta(\alpha \pi \overline{\tau}) + d \gamma \), with \( \xi \in \tau \), \( \gamma \in \tau \). Then \( \xi \) may be written as \( \delta \circ \delta(\alpha \pi \overline{\tau}) + d \gamma \), for we have  
\[ \xi = d \gamma + d \gamma \circ \delta \]  
\[ \xi = d \gamma + d \gamma \circ \delta \]  
but \( \delta \circ \delta(\alpha \pi \overline{\tau}) \), so finally \( \xi \) can be written as the sum of an element from \( \tau \) and one from \( \tau \); so \( \sigma \pi \overline{\tau} \) if.

**Defn 2.10:** Let \( M \) be a smooth compact manifold, \( \alpha : X \to Y \) a smooth \( \mathbb{R} \)-map, and \( f : M \to Y \) smooth, with \( \alpha \circ f \). Then we have

\[
\begin{array}{c}
X \\
\xleftarrow{\alpha} Y \\
\xrightarrow{f} M \\
\end{array}
\]

and an acyclic (in fact split) Fredholm complex

\[ 0 \to \tau_X \to \tau_X \otimes M \to (\alpha \pi f)^* \tau_Y \to 0 \]

We moreover have \( \tau_X \otimes M \cong \pi^* \tau_M \oplus f^* \tau_X \).
Prop. 2.11: We have a relation
\[ [\tau_\pi \pi M] = [\overline{\tau} \pi \pi M] + [\overline{\tau} \pi \pi \alpha] \]
in \[K_{\Phi}(X_{\pi M}).\]

Consider the map of Fredholm complexes
\[ E: 0 \to \tau_{\pi M} \to \pi_{\pi M} \to \pi_{\pi M} \to 0 \]
\[ F: 0 \to \tau_{\pi M} \to \tau_{\pi M} \to \pi_{\pi M} \to 0 \]
where the bottom line is the acyclic complex arising from
Def. 2.9, and the top line is a complex arising from the
sum \[\tau_{\pi M} - \overline{\tau} \pi \pi M - \overline{\tau} \pi \pi \alpha\]. Hence it suffices to
show that \[E, F\] represent the same element in
\[K_{\Phi}(M_{\pi X})\]; but \[L\] differs from the identity by
a compact map, and the statement is true if it is true
that compact perturbations of a Fredholm complex leave
the element it defines in \[K_{\Phi}(\cdot)\] undisturbed.
This is clear for a two term complex, and follows for a
general complex by induction (Segal \(\Sigma^{28}, \S 4.5.7\)).

SECTION 3:

**Density theorems**

This section sketches the proof of the density theorems
for maps of finite-dimensional manifolds which will be needed later. The published proof of Smale [29] is a little sketchy and it will be convenient to have a stronger form. The techniques used here are inspired by Abraham [1] and I am indebted to Frank Quinn for showing me how to use them.

**Def. 3.1:** Let $M$ be a compact smooth manifold, perhaps with boundary. Write $C^r(M,Y)$ for the manifold of $C^r$ maps of $M$ to $Y$ ($Y$ an exponential Banach manifold) $ev: M \times C^r(M,Y) \to Y$ the evaluation map given by $ev(m,f) = f(m)$. Let $\alpha: X \to Y$ be a Fredholm map; consider the diagram:

\[
\begin{array}{ccc}
P & \xrightarrow{ev} & X \\
\downarrow \alpha & & \downarrow \alpha \\
M \times C^r(M,Y) & \xrightarrow{ev} & Y \\
\downarrow \text{pr}_2 & & \\
C^r(M,Y)
\end{array}
\]

in which $P$ is the fiber product $M \times C^r(M,Y) \times X$ and $\text{pr}_2$ is the projection on the second factor.

**Prop. 3.2:** $ev \cong \alpha$

**Proof:** Note that $d(ev)$ is surjective; for this amounts only
to the fact that given \( m \in M, y \in Y, v \in T_y Y, \) we can choose a smooth map \( g : M \to Y \) with \( g(m) = y, v \in \text{im } dg_m \), and since \( d \alpha \) has finite dimensional cokernel, the proposition follows from Def. 2.3.\

Remark: We must consider \( C^r(M, Y) \) for \( r > 1 \) because we will need the evaluation map to be highly differentiable; it is fairly easy to pass to the \( C^\infty \) case, but we avoid Frechet spaces when possible. The requirement that \( Y \) be an exponential manifold can be weakened but it is fulfilled whenever \( Y \) is a manifold of sections, which is the main case of interest.

Prop. 3.3: \( \alpha \) is a Fredholm map.

Proof: \( P \) is, by Prop. 3.2, a manifold; and the "differential complex" of \( \alpha \) (see Def. 1.5) is just the pullback along \( \alpha^* \) of that of \( \alpha \); hence \( \alpha \) is Fredholm.\

Prop. 3.4: The regular values of \( pr_{20} \alpha \) (which is a Fredholm map) are just those \( g \in C^r(M, Y) \) such that \( g \circ \alpha \).

Proof: That is, Fredholm is just the assertion that the composition of Fredholm maps is Fredholm. Now suppose
\[(m,g,x) \in M \times C'(H,Y) \times X, \quad (v,X,w) \in T_H \times T^g Y \times T_X\] as an element of the tangent bundle over \((m,g,x)\). If \(g(m) = o_2(0)\), then \(\ker \delta(\text{ev} \circ \tau g^0)_{(m,g,x)} = \{ (v,X,w) | dg(v) + d\tau g(x) + X(w) = 0 \}\) and \(\delta(\text{pr}_2 \circ \tau g)_{(v,X,w)} = X\). Hence \((m,g,x)\) is a regular point iff \(\{ (v,w) | dg(v) + d\tau g(w) = 0 \}\) spans \(T_{M \times X(f,m,x)}\). The splitting comes easily from finite dimensionality///.

**Theorem 3.5**: Let \(\tau : X \to Y\) be a \(C^r\) Fredholm map of Banach manifolds, \(r > \max(0, \text{index } \tau)\); assume \(\tau\) is \(\Sigma\)-proper. Then the regular values of \(\tau\) are residual.

----Proof: This is the Smale-Sard density theorem; it is proved in Smale \(29\) and Abraham-Robbins \(37\). A set is residual if it is a dense \(G_\delta\) (that is, a countable intersection of open dense sets). A map is \(\Sigma\)-proper if the preimage of a compact set is an \(F_\infty\)///.

**Prop. 3.6**: Let \(\tau : X \to Y\) be a \(C^r\) \(\Sigma\) map as above, \(M\) a smooth compact \(n\)-manifold. Write \(R(H;\tau) = \{ g \in C(H,Y) | g \circ \tau = 0 \}\). Then \(R(H;\tau)\) is residual in \(C^r(H,Y)\) if \(r > \max(0, n + \text{index } \tau)\).

----Proof:---

By Prop. 3.4, the set \(R(H;\tau)\) is just the set of regular values of \(\text{pr}_2 \circ \tau\); but by theorem 3.5 that set is residual///.
Def. 3.7: The fine topology on the space of maps between two paracompact manifolds $M, N$ is generated by sets of the form
$$\mathcal{B}(f_0, \delta) = \{ f \in C^0(M, N) \mid \delta(f(x), f(y)) < \delta(x) \}$$
as a base, where $\delta$ is a metric on $N$, $\delta : N \to \mathbb{R}^+$ is a continuous positive function. The $C^r$-topology is analogous.

We will state the strongest density theorem we will need, and indicate its proof:

Prop. 3.8: Let $\alpha : X \to Y$ be a $C^r$-map as above, $M$ a smooth paracompact $n$-manifold, perhaps with boundary. We consider $C^r_p(M, Y)$, the space of $C^r$-proper maps from $M$ to $Y$ in the $C^r$ fine topology, $\tau$ as above. Then $R(M; \alpha) = \{ g \in C^r_p(M, Y) \mid g \circ \alpha \}$ is residual. Further, if $A$ is closed in $M$, and $C^r(M, A; Y) = \{ g \in C^r_p(M, Y) \mid g \not\in A \}$, then $R(M, \alpha) \cap C^r_p(M, A; Y)$ is residual in $C^r_p(M, A; Y)$.

----Proof:----

For a proof, see Frank Quinn's thesis [27]; it follows essentially from a local theorem: given $x_0 \in X$,

\[
\begin{array}{c}
m_0 \in M \xrightarrow{g} Y \\
\downarrow \alpha \\
\end{array}
\]

these exist open sets $U \ni x_0 \subset X$, $\forall m_0 \in M$ such that $\{ g \in C^r(M, Y) \mid g \not\in A \}$ is open dense in $C^r(M, Y)$. /
SECTION 4

Bordism and Cobordism

Defn. 4.1: Let $X$ be a topological space. A family of supports on $X$ is a family of closed subsets of $X$ such that

1) A closed subset of a member of $\mathcal{F}$ is in $\mathcal{F}$
2) $\mathcal{F}$ is closed under finite unions

$\mathcal{F}$ is said to be a paracompactifying family of supports if in addition

3) each element of $\mathcal{F}$ is paracompact
4) each element of $\mathcal{F}$ has a closed neighborhood in $\mathcal{F}$.

Defn. 4.2: A functorial family of supports for a category of topological spaces is a function which assigns a family of supports to each space in $\mathcal{S}$, such that the morphisms of $\mathcal{S}$ preserve supports.

Examples:
1) Let $\mathcal{S}$ = all topological spaces and all morphisms, $\mathcal{E}(X) = \{ \text{closed subsets of } X \}$. Then $\mathcal{E}$ is a functorial family of supports.

2) Let $\mathcal{S}_p$ = all topological spaces and all proper morphisms, $\mathcal{S}_{lc}$ the subcategory whose objects are all locally compact. Then $\mathcal{E}_{lc}(X) = \{ \text{compact subsets of } X \}$. On $\mathcal{S}_p$, $\mathcal{E}$ is a functorial family of supports; on $\mathcal{S}_{lc}$ it is in addition paracompactifying.
3) Let $\mathcal{M} = \text{all metric spaces and bounded maps},$

\[ \mathcal{M}_b(X) = \{ \text{closed bounded subsets of } X \} \]

Then $\mathcal{M}_b$ is functorial and paracompactifying.

4) $\mathcal{T} = \text{all topological spaces and all } \Sigma \text{-proper morphisms;}$

\[ \mathcal{T}(X) = \{ A \subset X \mid A \text{ is an } \bar{F}_0 \} \]

Then $\mathcal{T}$ is functorial.

5) Let $\mathcal{S}_F$ be the category of topological spaces with a flow (that is, an action of the real line) with equivariant morphisms. Let $\mathcal{S}_F(X) = \{ \text{closed subsets of } X \times \mathbb{R} \text{ bounded in } \mathbb{R} \}$. This doesn't quite fall under the above definitions but is useful in its own right.

**Defn. 4.3:** Let $O = \lim_{\to} O(n)$ be the infinite orthogonal group with the direct limit topology, $\subseteq 21 \subseteq 24 \subseteq 7$. Let $G \subset O$ be a subgroup with the induced topology. If we write $G(n) = O(n) \cap G$, then $G = \lim_{\to} G(n)$. Finally let $BG(n)$ be a classifying space for $G(n)$.

**Examples:** All the (nonexceptional) Lie groups such as $U$, $\text{Spin}$, $SU$, $SO$ define subgroups of $O$, and the induced topology defined above agrees with the usual topology. We should also mention the stable trivial group $\{ 0 \}$ as an example of an interesting subgroup of $O$. 
**Defn. 4.4:** Let $\mathcal{M}$ be a paracompact smooth $n$-manifold. $\mathcal{M}$ has a naturally defined tangent bundle $\mathcal{T}_\mathcal{M}$; since $\mathcal{M}$ is paracompact, $\mathcal{T}_\mathcal{M}$ is the pullback of the universal bundle on $BO(n)$. A $G$-structure on $\mathcal{M}$ is a lifting

$$
\begin{array}{ccc}
\mathcal{T}_\mathcal{M} & \xrightarrow{\sim} & BG(n) \\
\downarrow{\mathcal{L}_h} & & \\
\mathcal{M} & \xrightarrow{\mathcal{T}_\mathcal{M}} & BO(n)
\end{array}
$$

where $\mathcal{L}_h$ is induced by the natural inclusion $G(n) \hookrightarrow O(n)$. The pair $[\mathcal{M}, \mathcal{T}_\mathcal{M}]$ will be called a $G$-$n$-manifold.

**Defn. 4.5:** Let $X$ be a topological space, $\mathcal{F}$ a family of supports on $X$. Define $\Omega_n^{\mathcal{G}}(X; \mathcal{F})$, the group of bordism classes of singular $G$-manifolds in $X$ with local supports in $\mathcal{F}$, as follows:

Consider the set of (isomorphism classes of) triples $[\mathcal{M}, \mathcal{T}_\mathcal{M}, \mathcal{F}]$ where $[\mathcal{M}, \mathcal{T}_\mathcal{M}]$ is an $n$-dimensional $G$-manifold, as above, and $\mathcal{F}: \mathcal{M} \to X$ is a continuous map with the property that if $A \subseteq X, A \in \mathcal{F}$, then $\mathcal{F}(A)$ is compact. Such a map will be called $\mathcal{F}$-proper. The set $\Omega_n$ is an additive semigroup with identity, under the operation of disjoint union. We introduce an equiva-
lence relation: \( \Sigma M, \overline{\Sigma}M, \not\sim \sim \sim 0 \) if there exists an \((n+1)\)-manifold \( \Sigma W, \overline{\Sigma}W \) and a \( \varphi \)-proper map \( F: W \rightarrow X \) such that

1) \( \partial W = M \) as \( G \)-manifolds

2) \( F|\partial W = \varphi \).

It then follows as in the classical case that \( \Omega / \sim \) is an abelian group \( \Sigma 7, \S 3 \).7

Remarks: 1) We can relativize the above by considering a pair \((X, A)\), and a family of supports \( \varphi \) on \( X \) such that \( \varphi \cap A = \{ B \cap A | B \in \varphi \} \) is a family of supports on \( A \).

One then proceeds with relative bordism groups \( \Omega^G_n(X, A; \varphi) \) as in \( \Sigma 7, \S 4 \).

2) We write \( \Omega^G_n(X, A) \) for \( \Omega^G_n(X, A; \Xi cl(X)) \) for if \( \Sigma M, \overline{\Sigma}M, \not\sim \) \( \in \Omega^G_n(X, \Xi cl(X)) \), then \( X \in \Xi cl(X) \), so \( M = \varphi^{-1}(X) \) is compact and a \( \varphi \)-proper map is just a continuous map: The groups reduce to the ordinary bordism groups \( \Sigma 7, \S 3 \).

3) It is reasonable to expect the sequence

\[ 0 \rightarrow \lim_\rightarrow \lim_\leftarrow \Omega^G_{n+1}(X, X-A) \rightarrow \Omega^G_n(X, \varphi) \rightarrow \lim_\leftarrow \Omega^G_n(X, X-A) \rightarrow 0 \]

of exactness; but I don't know of a proof. See Remark \( \S 22 \).

Definition \( \S 25 \)

Let \( \Xi \) be a functorial family of supports on some category \( \mathcal{S} \) of topological spaces; we write \( \Omega^G_n(X, \Xi) = \Omega^G_n(X, \Xi cl(X)) \) and

and
speak of bordism with \( \Xi \) - supports. The groups 
\[ \Omega^G_*(X, \Xi) \]
and their induced homomorphisms form a functor from \( \mathcal{G} \) to the category of graded abelian groups and their homomorphisms.

**Defn. 4.6:** We introduce cobordism groups of topological spaces following Atiyah \( \Sigma \) and Whitehead \( \Gamma_3 \); for we will have no need of supports for cobordism.

Recall then that the universal bundle \( X(n) \rightarrow BO(n) \)
induces a pullback \( i^n_*X(n) \rightarrow BG(n) \). The Thom space \( MG(n) \) is defined to be \( B(i^n_*X)/S(i^n_*X) \) (the associated ball bundle with the associated sphere bundle collapsed to a point.) There are natural maps \( \varepsilon_n: S' \times MG(n) \rightarrow MG(n+1) \) induced by the inclusion \( G(n) \rightarrow G(n+1) \) (where \( S' \times \cdot \) represents suspension) and we obtain a spectrum \( MG = \{ \varepsilon_n\}_n MG(n) \}. Then
\[ \Omega^G_{*,k}(X, A) = \lim_{\rightarrow} [S^{n+k} \times X/A, MG(n)] \]
where square brackets represent the set of homotopy classes of maps.

**Props. 4.7:** \( \Omega^G_{*,*} \) is a cohomology theory in the sense of Whitehead; that is, it satisfies all the axioms of Eilenberg-Steenrod except the dimension axiom.
Proof: see \( \text{\cite{7, \S 13}} \).

**Defn. 4.8:** Let \((M, \mathcal{A})\) be a (paracompact) \(\mathcal{G}\)-n-manifold with boundary. Two submanifolds \((N_0, \mathcal{A}N_0), (N_1, \mathcal{A}N_1)\), with \(\mathcal{A}N_0, \mathcal{A}N_1 \subset \mathcal{A}M\) and \(\mathcal{G}\)-structures on their normal bundles are said to be \(L\)-equivalent if there is a submanifold \(W \subset M \times I, I = [0,1]\) with a \(\mathcal{G}\)-structure on its normal bundle, \(\mathcal{A}W \subset \mathcal{A}(M \times I)\) and such that \((W, \mathcal{A}W) \cap M \times \{i\} = (N_i, \mathcal{A}N_i), i = 0, 1\) the intersections being transversal.

Write \(L^G_k (M, \mathcal{A}M)\) for the set of \(L\)-equivalence classes of \(k\)-dimensional \(\mathcal{G}\)-submanifolds of \((M, \mathcal{A}M)\); we wish to stabilize this by taking cartesian products with Euclidean spaces. Hence let \(L^G_{k,q} (M, \mathcal{A}M)\) be the set of \(L\)-equivalence classes in \((M, \mathcal{A}M) \times \mathbb{R}^q\) which are bounded in the \(\mathbb{R}^q\)-coordinate (so they won't slip off to infinity.) The limit of these groups as \(q\) goes to infinity will be written \(L^G_k (M, \mathcal{A}M)\); it carries a natural abelian group structure, which is, according to Thom \(\text{\cite{30}}\), a homotopy-theoretic artifact:

**Theorem 4.9 (Thom):** There exists a natural isomorphism \(\tau: L^G_k (M, \mathcal{A}M) \rightarrow \Omega^\infty \mathcal{G} (M, \mathcal{A}M)\).

(in his original paper Thom was concerned only with mani-
folds without boundary. Taking them into account yields

**Theorem 4.9'**: There exists a natural isomorphism

$$T': L^G_k(M, \partial M) \sim \Omega^{n-k}_G(M-\partial M).$$

Now there is a natural homomorphism

$$\iota: \Omega^G_k(M-\partial M; \mathbb{E}) \rightarrow L^G_k(M-\partial M)$$

defined using the Whitney embedding theorem: given a singular $\mathbb{E}$-manifold $V \xrightarrow{f} M-\partial M$, with $f$ proper, there is a $q$ large enough such that

$$V \xrightarrow{f} M-\partial M \xrightarrow{1 \times o} (M-\partial M) \times \mathbb{R}^q$$

is properly homotopic to an embedding (bounded in $\mathbb{R}^q$), with any two such embeddings homotopic through such embeddings. Furthermore, $\iota'$ admits an inverse: given $W$ embedded in $(M-\partial M) \times \mathbb{R}^q$, we use the map $W \xrightarrow{r_2} (M-\partial M) \times \mathbb{R}^q$ as definition of the element in $\Omega^G_k(M-\partial M; \mathbb{E})$. It is proper since the $\mathbb{R}^q$ coordinate is bounded. The same procedure defines a homomorphism

$$\iota': \Omega^G_k(M, \partial M; \mathbb{E}) \rightarrow L^G_k(M, \partial M)$$

which admits an inverse just as $\iota$ does. Hence

**(Prop. 4.10)**: If $M$ is a paracompact $\mathbb{E}$-n-manifold, there
are natural homomorphisms

\[ A^{-1} : \Omega^G_k(M, \partial M; \mathcal{E}) \xrightarrow{\sim} \Omega^{n-k}_G(M, \partial M) \]
\[ K^{-1} : \Omega^G_k(M, \partial M; \mathcal{E}) \xrightarrow{\sim} \Omega^{n-k}_G(M, \partial M). \]

Defn 4.11: We assume from now on that \( G \) is a stable Lie group such as \( \text{Spin}, \, \mathcal{O}, \, \mathcal{U}, \, SU \) or the like; for we will need multiplicative properties of the bordism and cobordism groups, and a general subgroup of \( \mathcal{O} \) will not yield a well-defined multiplication. However, assuming \( G \) to be a stable Lie group we have well-defined maps,

\[ \pi_{r,s} : G(r) \times G(s) \longrightarrow G(r+s) \]

inducing \( B\pi_{r,s} : BG(r) \times BG(s) \longrightarrow BG(r+s) \) and finally

\[ ^{\wedge} \pi_{r,s} : MG(r) \wedge MG(s) \longrightarrow MG(r+s). \]

( \( \wedge \) means the smash product of pointed spaces: \( X \wedge Y = X \times Y / (x \times y_0 = x \times y) \). The latter map determines the external cup product.

\[ \Omega^*_G(X) \otimes \Omega^*_G(Y) \rightarrow \Omega^{p+q}_G(X \times Y) \]
the groups \( \Omega^*_G(X) \) become \( \Omega^*_G(\mathcal{P}^*) \)-modules, and the diagonal map makes \( \Omega^*_G(X) \) into a graded algebra over \( \Omega^*_G(\mathcal{P}^*) \).

Remark: In the classical case, \( \Omega^*_G(\cdot) \) is defined by a spectrum, and possesses a module structure over \( \Omega^*_G(\cdot) \) by general homotopy theoretic-nonsense.
For reasonable families of supports, this module structure still exists on \( \Omega^G_\bullet (\cdot, \varphi) \); however, the construction is more geometric.

**Defn. 4.12:** Let \( X \) be a topological space, \( \Phi \) a functorial family of supports. Define the cap-product pairing

\[ \eta: \Omega^G_n(X; \Phi) \otimes \Omega^k_c(X) \to \Omega^G_{n-k}(X; \Phi) \]

by the following diagram: given \( \eta = [M, \bar{\tau}_M, f] \in \Omega^G_n(X; \Phi) \), write

\[ \Omega^k_c(X) \xrightarrow{f^*} \Omega^k_c(M) \]

\[ \Omega^G_{n-k}(X; \Phi) \xleftarrow{f_*^\Phi} \Omega^G_{n-k}(M; \Phi_c) \]

where \( f_*^\Phi \) is defined the obvious way: if \( \Sigma V, \bar{\tau}_V, p I \in \Omega^G_{n-k}(M; \Phi_c) \) then \( \Sigma V, \bar{\tau}_V, f p I \in \Omega^G_{n-k}(X; \Phi) \); for all we need to verify is that \( (f p)^{-1} \) takes elements of \( \Phi(X) \) into compact sets; but \( f \) does, and \( p \) is proper, so \( f p \) does.

It remains to see that this is independent of the representation of \( \eta \). That \( \eta \eta \cdot \) is bilinear is more or less trivial, so it suffices to see that if \( \eta \) bounds, then the composition above is zero. But if \( M \) is a \( \mathcal{G} \)-
boundary, there exists \( W \) with \( \partial W = M \) and \( F : W \to X \), \( F \) proper, extending \( f \). Hence if \( i : M \to W \) is the inclusion of the boundary, then \( Fi = f \), and
\[
\kappa \cap a = F_* \circ i_* \circ A_M^{-1} \circ i^* \circ F^* \circ a. \quad (a \in \Omega^k_G(X))
\]
Hence it will suffice to show that the map
\[
i_* \circ A_M^{-1} \circ i^* : \Omega^*_G(M) \to \Omega^*_H(M; \mathbb{Z})
\]
is zero. (Note that \( i_* \circ A_M^{-1} \circ i^* \circ z = \kappa_M \cap z \), where \( \kappa_M \in \Omega^n_G(M; \mathbb{Z}) \) is the class \( \langle M, \mathbb{Z}, M, 1 \rangle \), which is in fact bordant to zero. But if \( \tau : W - M \to M \) is the other inclusion, then
\[
A_M^{-1} \circ i^* = \tau A_W^{-1} \circ \tau^* \quad \text{and} \quad i_* \circ A_M^{-1} \circ i^* = i_* \circ \tau A_W^{-1} \circ \tau^* = 0
\]
by exactness. /\.

Prop. 4.13: The pairing defined above is bilinear and satisfies
\[
\kappa \cap (a \cup b) = (\kappa \cap a) \cap b \quad (a \in \Omega^*_G(X; \mathbb{Z}),
\]
\[
a, b \in \Omega^*_G(X).
\]

Proof:---------

First note that if \( M \) is a \( G \)-manifold without boundary, the orientation class \( \kappa_M \in \Omega^n_G(M; \mathbb{Z}) \) is defined as \( \langle M, \tau_M, 1 \rangle \), and \( A_M \times = \kappa \cap x \cap \kappa_M \times \in \Omega^*_G(M) \).

Hence it suffices to show that \( (A_M \times) \cap y = A_M \cup \kappa_M \cap y \), which is the "universal example" of the above. But this is clear from the definition. /\.

Remark: In the same manner we can introduce homology
operations in bordism groups with supports: by appealing to characteristic class theory and duality. The co-
bordism operations have been introduced by Landweber \cite{17} and Novikov \cite{23}; the modifications necessary for
the covariant case are more or less trivial, but we develop them here anyway.

**Defn. 4.13:** A G-cobordism characteristic class \( \gamma \), of
degree \( i \), is a mapping from the category of G-vector
bundles over a space \( X \) to the group \( \Omega^i_G(X) \), natural
with respect to G-bundle maps, which is stable: \( \gamma(\xi) = \gamma(\xi^i) \). These operations form a graded algebra
\( C^*(G) \) under pointwise addition and multiplication; this algebra is naturally isomorphic to \( \Omega^*_G(BG) \).

Let \( \Phi \) be a functorial family of supports. \( G - \Phi \)
bordism operation of degree \( i \) is a natural homomorphism
\( \gamma: \Omega^*_G(\cdot \Phi) \to \Omega^*_{G-i}(\cdot \Phi) \) commuting
with suspension. The set of such operations is a graded
algebra under pointwise addition and composition, which
will be denoted \( A^*(G; \Phi) \). We define a natural map
\( \wedge: C^*(G) \to A^*(G; \Phi) \) of additive groups
much as the multiplication above was defined. In case
\( \Phi = \Phi C \) is the family of all closed subsets,
this map is an additive isomorphism (Landweber \cite{17} and
it seems likely that it will be in general.

The function \( \wedge \) is defined as follows: given
\[ x \in \Sigma_n^G(X, f) \text{ represented by } [M, \overline{\tau}_M, f, J], \text{ and } \gamma \in C^\ast(G), \]
we assign \( \overline{\gamma}[M, \overline{\tau}_M, f] = \overline{\gamma} A_n \gamma(\overline{\tau}_M). \)

**Proposition 4.14:** \( \wedge : C^\ast(G) \to A^\ast(G; \infty) \) is a well-defined additive homomorphism.

**Proof:**

We need to check that \( \overline{\gamma}[M, \overline{\tau}_M, f] \) is

1) independent of the bordism class of \( [M, \overline{\tau}_M, f, J] \);

2) additive in \( \gamma \);

3) additive in \( x \);

4) invariant under suspension.

Now 3) is trivial and 2) is only slightly less so. 1) follows, then, if \( \overline{\gamma} \) vanishes on classes bordant to zero.

Let \( [W, \overline{\tau}_W, f] \) be that bordism then; and \( i : M \to W \) the inclusion. Hence
\[ \overline{\gamma} A_n \gamma(\overline{\tau}_M) = \overline{\gamma} A_n i \gamma(\overline{\tau}_W) = 0 \quad (\overline{\tau}_W = \overline{\tau}_M, \gamma \text{ is stable.}) \]
But the same stability implies that \( \overline{\gamma} \) commutes with suspension/.

Cobordism (and hence bordism) operations have been computed by Novikov and Landweber. We will use the \( V \)-bordism operations as an example, so their relevant properties are listed below.

**Recollection 4.15:** The ring \( C^\ast(V) \leq \Omega^\ast V \text{ where} \)
\[ \Omega^\ast V = \Omega^\ast (\text{point}) \text{ and } M^\ast \text{ is isomorphic to the} \]
ring of symmetric functions (on infinitely many indeterminates) properly graded. The generators of $M$ corresponding to the elementary symmetric functions $\sigma_i$, are written $c_{\alpha}^i$ and called Conner-Floyd Chern classes. If $(\alpha_1, \ldots, \alpha_n)$ is a multiindex $\in (\mathbb{Z}^+)^n$, we write $\sigma_{\alpha} = \sum t_{\alpha}^{\alpha_1} \cdots t_{\alpha_n}$ for the sum over all permutations of the $t_i$ and write $c_{\alpha}^{\alpha}$ for the associated "chern class." $\hat{\otimes}$ denotes the completed tensor product, for $M^*$ is positively graded.\

Recollection 4.16: The ring $\hat{\otimes}^*(U)$ (as a subring of $A^*(U; \Xi)$) is a hopf algebra and is isomorphic to $\Omega^*_U \otimes S$, where $S$ is the image of $M^*$ under $\psi$. $S$ is a cocommutative hopf algebra over $\mathbb{Z}$.

We define $c_{\alpha}^i = s_{\alpha}^i$ and write $s_{\alpha}^i$ for the dual to $s_i$ in $S^* = \text{Hom}^*(S, \mathbb{Z})$ (the graded dual.) Write $s = \sum s^i \epsilon S$, $s' = \sum s_{\alpha}^i \epsilon S^*$. Then

1) $S^*$ is a polynomial algebra over $\mathbb{Z}$ with generators $s_{\alpha}^i$.

11) $\Delta s' = \sum_{i=0}^{\infty} (s')^{i+1} \otimes s_{\alpha}^i$

This completely defines the Hopf Algebra structure on $S$.

Note that $s_{\alpha}^*(x \cup y) = \sum_{\alpha + \beta = \alpha} s_{\alpha}^*(x) \cup s_{\beta}^*(y)$; that is, the Cartan formulae hold, and that if $\xi \rightarrow X$ is a complex vector bundle, $\phi: \Omega^*_U(X) \rightarrow \Omega^*_{U+1}(X)$ the Thom isomorphism, then $\phi^1 s_{\alpha}^* \phi(1) = c_{\alpha}^*(\xi)$.
SECTION 5

Orientations:

Defn. 5.1: Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{B} = \mathcal{B}(\mathcal{H})$, the Banach algebra of bounded operators on $\mathcal{H}$ in the uniform topology, $\mathcal{C}$ the closed ideal of compact operators in $\mathcal{B}$. Then $\mathcal{B}/\mathcal{C}$ is a Banach algebra (in fact a $C^*$-algebra) which will be denoted $\mathcal{A}$.

If $\mathcal{A}$ is an algebra, we write $\mathcal{G}(\mathcal{A})$ for the group of invertible elements in $\mathcal{A}$. Kuiper's theorem \cite{15} states that $\mathcal{G}(\mathcal{B})$ is contractible; a corollary is that $\mathcal{G}(\mathcal{A})$ has the homotopy type of $\mathbb{Z} \times \mathcal{BO}$ (see Atiyah, \textit{French} or Janich \cite{14}). We have a commutative diagram of fiber bundles

$$
\begin{array}{ccc}
\mathcal{O} & \longrightarrow & \mathcal{B} \\
\downarrow e & & \downarrow e \\
\mathcal{G}(\mathcal{A}) & \longrightarrow & \mathcal{A}
\end{array}
$$

where $\mathcal{O}$ is the space of Fredholm operators on $\mathcal{H}$.

Now since $\pi_1(\mathbb{Z} \times \mathcal{BO}) = \pi_1(\mathcal{O}) = \pi_1(\mathcal{G}(\mathcal{A})) = \mathbb{Z}$, the group $\mathcal{G}(\mathcal{A})$ has a universal cover $\tilde{\mathcal{G}}$ which must have the homotopy type of $\mathbb{Z} \times \mathcal{BSO}$. If we let $\mathbb{Z}_2$ act on $\mathcal{G}$ as the group of covering transformations and on $\mathbb{R}$ as $e^{\pi i t}$, then $\mathbb{C} \times \mathbb{R} / \mathbb{Z}_2$ defines a line bundle $\gamma$ over $\mathcal{G}(\mathcal{A})$.

Prop. 5.2: Let $\xi \rightarrow X$ be a (stable) vector bundle over a compact space $X$. Then $\xi$ is induced by a map $\xi \rightarrow \mathcal{G}(\mathcal{A})$, and a reduction of the structure group of
from $\mathbb{Z} \times \text{BO}$ to $\text{BSO}$ is equivalent to giving a nonzero section of $\gamma$ over $X$,

\[ \xymatrix{ \cdots \ar[r] & \gamma \ar[d] \ar@{.>}[r] & \mathbb{Z} \times \text{BO} \ar[r] \ar[d] & X \ar[r] \ar@{=>}[d] & \gamma \ar[d] \ar@{.>}[r] & \text{BSO} } \]

Proof: This is standard; see for example Steenrod [30].

Remark: It would be very nice to have a direct functional-analytic construction of $\mathbb{Z} \times \text{BSO}$ as, for example, the group of units in a Banach algebra. It would be even nicer to have a direct construction for the bundle $\gamma$. This contains various technical difficulties; a direct construction for $\gamma$ amounts to a definition of determinant for a compact operator. Many dense subalgebras of $\mathcal{C}$ admit such determinants (e.g. Hilbert-Schmidt operators) but $\mathcal{C}$ itself does not (to the best of my knowledge.) For further data on such things, see Dixmier [9, I.6.11].

Defn. 5.3: Let $\bar{\eta} \to \mathcal{F}$ be the pullback $p^*\eta$, where $p: \mathcal{F} \to \text{G(\mathbb{Z})}$ is the projection. Suppose $\alpha: X \to Y$ to be a Fredholm map of Hilbert manifolds, $I\alpha: X \to \mathcal{F}$ the induced map defining the index. An orientation for the map $\alpha$ is a map $I\alpha$, 

\[ \xymatrix{ \cdots \ar[r] & \mathcal{F} \ar[r] \ar[d] & X \ar[r] \ar[d] & Y \ar[r] \ar[d] & \mathcal{F} \ar[r] \ar[d] & \cdots } \]
such that \( \overline{I\alpha} \) is never zero. Such a map always exists, for example if \( \omega_1(I\alpha) = 0 \) (the first Stiefel-Whitney class); so it certainly exists if \( X \) is simply connected.

Perhaps the preceding is motivation enough for the next definition:

**Defn. 5.4:** Let \( \alpha: X \to Y \) be a Fredholm map of Hilbert manifolds, \( I\alpha: X \to OF \) the induced map defining the index. Let \( G \) be a stable lie group; there is a map \( \xi' \), natural up to homotopy, defined by inclusion of \( G \) into \( O: \overline{I\alpha} \to \mathbb{Z} \times BG \)

A \( G \)-structure on the \( \overline{E} \)-map is a lifting \( \overline{I\alpha} \) of \( I\alpha \) such that the above diagram homotopy-commutes.

**Remark 1:** As defined here, a \( G \)-structure is homotopy-theoretic and thus very coarse. In case \( G = SO \), there is a slightly sharper definition at hand; in case \( G = U \),
we have the very nice model \( \mathcal{F}_C \) of Fredholm operators on complex Hilbert space. The problem of constructing \( \mathcal{F} \)-structures on Fredholm maps in general is very interesting.

Ellworthy and Tromba \( \text{\cite{12}} \) have developed theories of reduction of the structural group of a Banach manifold to the group of operators of the form identity plus a compact operator. We have avoided this here, for certain difficulties arise from the fact that there may exist of a orientation-reversing maps Banach manifold into itself which are homotopic to the identity. In some sense an orientation on the map, as defined above, is the bare minimum needed for a workable pullback homomorphism; but there are other occasions which the existence of orientation the manifolds themselves are more useful.

**Remark 2:** Atiyah \( \text{\cite{5}} \), Appendix 3 \( \text{\cite{7}} \) has shown that KR-theory is the natural place to look for invariants of real linear elliptic operators. Inspired by this, Landweber \( \text{\cite{18}} \) has developed a cobordism theory of "real" manifolds. There are immediate generalizations of the concepts I have used to the study of real nonlinear operator one can construct a real index in KR \((X)\) and so forth. I do not carry this out because the doubling of indexes it involves is combersome and unenlightening.
Defn 6.1: A map $\alpha: X \to Y$ of topological spaces is called proper if for any compact $K \subseteq Y$, the set $\alpha^{-1}(K)$ is compact. If $X$ is a metric space, we say that $\alpha$ is preproper if given any closed bounded set $B \subseteq X$, $\alpha^{-1}(B)$ is proper. (The importance of this concept was first noticed by Ellworthy.)

Assumptions: In this section we assume that $\alpha: X \to Y$ is a $C^\infty$ map with $X$, Lindelof (X,Y) Banach manifolds), that $\alpha$ is either proper or preproper, and that $\alpha$ has a $G$-structure. This requires that $\mathcal{I}\alpha \in K_\mathcal{E}(X)$ be defined by a map $X \to \mathcal{F}$. This will be the case if $X$ is modeled on a Kuiper space (by Kuiper's theorem \cite{15.7}) or if $\alpha$ is a differential operator on sections of a fiber bundle. The first is trivial: $\mathcal{I}\alpha$ is the Fredholm complex $0 \to \mathcal{T}_X \xrightarrow{\partial_X} \alpha^* \mathcal{T}_Y \to 0$; but $\mathcal{T}_X$, $\mathcal{T}_Y$ are Kuiper bundles, hence parallelizable, so we may pull back to $K(X)$. The second case is proven in P lais.

Without loss of generality we may assume that $X$ is connected, and that index $\mathcal{I}\alpha = k$ is constant.

Theorem 6.2: Let $\alpha: X \to Y$ be as above. If $\alpha$ is proper, there is a group homomorphism.
\[ \alpha! : \Omega^g_*(Y) \to \Omega^g_*(X) \]

If \( X \) is metric and \( \alpha \) is preproper, there is a group homomorphism
\[ \alpha! : \Omega^g_*(Y) \to \Omega^g_*(X; \mathcal{F}_b) \]

-----:Proof:--------

We first note that any class \( \alpha \in \Omega^g_*(Y) \) may be represented by a triple \( \alpha = [M, \iota_1, f] \) with \( F_1 \alpha \), and further that any two such representatives are bordant through \( [W_1, \iota_2, F_2] \) in which \( F_2 \alpha \).

Lemma 6.3: If \( M \xrightarrow{f} Y \) are both transversal to \( \alpha \), then \( \# \alpha : M \cup N \to Y \) (disjoint union) is transversal to \( \alpha \).

-----:Proof:--------

Note first that \( (M \cup N) \pi X = M \pi X \cup N \pi X \), and then that \( \delta (\# \alpha) \pi \alpha = \delta (f \pi \alpha) \cup \delta (g \pi \alpha) \).

Hence it suffices to prove that i) give \( M \xrightarrow{f} Y \), there is an arbitrarily small homotopy \( f \sim f' \) such that \( f' \alpha \), and ii) that for any \( f' : M \to Y, f' \alpha \), bordant to zero, there exists a bordism \( F : W \to Y \), with \( F \pi \alpha \) \( (\partial W = M, F/M = f') \). But these bordisms follow immediately from the homotopies guaranteed by the Smale-Sard theorem.

Lemma 6.4: If \( \overline{\alpha} : M \pi X \xrightarrow{f} Y \) is a fiber product with \( M \)
compact, and \( \alpha \) proper, then \( M\pi X \) is compact. If \( \alpha \) is preproper, then \( \overline{f} \) is \( \Xi_{b} \)-proper.

--------:Proof:--------

In case \( \alpha \) is proper, we note that \( M\pi X \) is a closed subset of \( Mx\alpha^{t}FM \), which is compact, so \( M\pi X \) is itself compact. In case \( \alpha \) is preproper, let \( i: B \subset X \) be the inclusion of a closed bounded subset in \( X \); then

\[
M\pi X = M\pi B \ . \text{ But } \alpha i \text{ is proper, so this set is compact; but since } i \text{ is an inclusion,}
\]

\[
M\pi X = (i)^{t}(B), \text{ so } \overline{f} \text{ is } \Xi_{b} \text{-proper.} \]

Hence we define

\[
\alpha! : [M, \overline{t}_{H}] = [X\pi M, \alpha^{t}\overline{t}_{H} \Theta e^{*} \overline{t}_{\alpha}, \overline{f}] \text{ in the above theorem} \quad \overline{f} \text{ is supposed transversal to } \alpha .
\]

This is well-defined, for it is additive by lemma 1; hence it suffices to check that if \( [M, \overline{t}_{M}] \overline{f} \) is bordered to zero, then the right-hand side is also. But we can choose \( [W, \overline{t}_{W}] F \) with \( F \not\alpha \overline{f} \), the triple \( [X\pi W, \alpha^{t}\overline{t}_{W} \Theta e^{*} \overline{t}_{\alpha}, F] \) defines the bordism of the right-hand side. \( \overline{f} \) Recall that \( \partial(X\pi W) = XT \omega W \) from Corollary 2.87.

Note that \( \overline{t}_{W} \) is a \( G \)-structure on \( \overline{t}_{W}, \overline{t}_{\alpha} \) is a \( G \)-structure on \( \overline{t}_{\alpha} \), and thus that \( \alpha^{t}\overline{t}_{W} \Theta e^{*} \overline{t}_{\alpha} \) is a \( G \)-structure on \( \alpha^{t}\overline{t}_{W} \Theta e^{*} \overline{t}_{\alpha} \); so \( X\pi M \) is in a natural way a \( G \)-manifold.
Prop. 6.5: A fiber product of manifolds

\[
\begin{array}{c}
\Omega^\infty(X; \Phi) \\
\sigma_! \downarrow \\
\Omega^\infty(Y; \Phi) \\
\end{array}
\]

with \(M\) compact, \(\sigma_\sharp\) as above, and \(f_\sharp\sigma\) induces a commutative diagram.

\[
\begin{array}{c}
\Omega^\infty_k(M; \Phi) \\
\sigma_! \downarrow \\
\Omega^\infty_k(Y; \Phi) \\
\end{array}
\]

in which the upper left-hand corner is the ordinary finite-dimensional gysin homomorphism; \(\overline{\Phi} = \Phi\) if \(\sigma\) is proper, \(\overline{\Phi} = \overline{\Phi}_b\) if \(\sigma\) is preproper.

**Proof:** Let \(\alpha \in \Omega^\infty_k(M)\). By the reasoning above, we can take a representative \([V, \overline{\nu}, \nu, g]\) for \(\alpha\), such that in the diagram

\[
\begin{array}{c}
V \rightarrow M \rightarrow X \\
\sigma_\sharp \downarrow \sigma_\sharp \downarrow \\
\overline{\alpha} \rightarrow \overline{\alpha} \\
\end{array}
\]

\(g\) is transversal to \(\overline{\alpha}\). But by lemma 2.9, \(g \delta \overline{\alpha}\) and \(f_\sharp \sigma\) implies that \(f_\sharp \sigma \alpha\). Hence \(V_\sharp M \pi X = V_\sharp X\). But
\[ \bar{\overline{f}}_* \overline{\alpha}_{\overline{2}!} [V, \bar{\nu}_V, \bar{g}] = \bar{\overline{f}}_* [V_{\bar{\nu}} M_G X, \bar{\nu}_{\bar{\nu} M_G X} \bar{g}] \\
= [V_{\bar{\nu}} M_G X, \bar{\nu}_{\bar{\nu} M_G X} \bar{g}] \\
= [V_{\bar{\nu}} M_G X, \bar{\nu}_{\bar{\nu} M_G X} \bar{g}] \\
= \alpha_1 [V, \bar{\nu}_V, \bar{g}] \\
= \alpha_1 \bar{\overline{f}}_* [V, \bar{\nu}_V, \bar{g}] . \]

Hence

\[ \bar{\overline{f}}_* \overline{\alpha}_{\overline{2}!} = \overline{\alpha}_{\overline{2}!} \bar{\overline{f}}_* . \]

Remark: Since every smooth map of finite-dimensional manifolds is a \( \Phi \)-map, the above construction defines a Gysin homomorphism in the finite-dimensional case. The classical construction of this homomorphism uses Atiyah-Thom duality, which has no appropriate analogue for Fredholm maps. (See however Wells-McAlpine \( \sim 10.7 \)). We will need to see that the "classical" Gysin map and the Gysin map above agree; that is, that

Prop 6.6: There is a commutative diagram

\[
\begin{array}{ccc}
\Omega^*_G (\mathcal{M}) & \xrightarrow{\pi^*} & \Omega^*_G (\mathcal{V}) \\
\uparrow & & \uparrow \\
\Omega^*_G (\mathcal{M}) & \xrightarrow{\pi_1} & \Omega^*_G \oplus p-n (V, \Phi_c)
\end{array}
\]

where \( \mathcal{M}, \mathcal{V} \) are \( G \)-manifolds (without boundary), \( \mathcal{M} \) is compact, and \( \pi: \mathcal{V} \longrightarrow \mathcal{M} \) is smooth.

Proof: First rewrite the diagram as
Now \( \overline{\pi} \) is defined exactly as \( \pi \) is; only we are dealing with embedded submanifolds rather than singular manifolds mapped in, and we need to use the density of transversal maps for embeddings rather than arbitrary smooth maps. This is a classical geometric argument (see Smale [20]) and predates the density theorem for arbitrary mappings. The bottom square commutes and it suffices to see that the top one does.

In the following we can replace \( M \) with \( M \times \mathbb{R}^n \), etc., so without loss of generality we may deal with \( \mathcal{L}_{n-\ast}(M) \);

hence we get a diagram

\[
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{\pi} & \mathcal{N} \\
\downarrow{i} & \quad & \downarrow{i} \\
\mathcal{V} & \xrightarrow{i} & \mathcal{M}
\end{array}
\]

in which \( i \) is an embedding, \( i \circ \pi \); hence \( \overline{i} \) is also. We also know that if \( \mathcal{V}_\mathcal{N} \) is a small enough normal bundle
of $N$ in $M$, then $\pi_1^*(\nu_N)$ is a normal bundle for $\nu_{\pi N}$ in $\nu$. In terms of classifying maps we have a commutative diagram

$$
\begin{array}{ccc}
\nu_{\pi N} & \xrightarrow{\nu} & BG(n-\ast) \\
\downarrow \pi & & \downarrow \\
\nu_N & \xrightarrow{\pi} & \ast
\end{array}
$$

But this implies that we have

$$
\begin{array}{ccc}
\nu & \xrightarrow{\nu} & \nu_{\pi N} \\
\downarrow \pi & & \downarrow \pi \\
M/\nu_N & \xrightarrow{\sim} & T(\nu_N)
\end{array}
$$

where $T(\cdot)$ is the Thom space functor of vector bundles; but the compositions along the top and bottom are respectively $\ell(\nu_{\pi N})$ and $\ell(N)$, and we have shown that $\ell(\pi_1^*(N)) = \pi_* \ell(N)$.

**Remark:** It is a general principle that anything that can be done with the finite-dimensional Gysin homomorphism can be done with the infinite-dimensional one. In fact if $\mathcal{M}_1(\tau_H)^f \tau$ is a class in $\Omega_n(Y)$, we may write it as $\mathcal{M}_1(\tau_H)^f \tau = \pi_* \mathcal{M}_1$, where $\mathcal{M}_1$ is the
orientation class of $M$. Then $\alpha! \gamma M, \gamma H, \gamma = \alpha! \gamma H = \gamma^* \alpha M$ by prop. 6.5. Hence $\alpha M$ can always be decomposed into a functorial $\gamma^*$ and a finite-dimensional Gysin homomorphism, the index of $\alpha$ turns into the normal bundle of $\alpha^r$, and most properties of the Gysin homomorphism become exercises in translation. For example,

Prop. 6.8: (Projection formula) With $\alpha: X \to Y$ as above, we have $\alpha! (x \wedge y) = \alpha! x \wedge \alpha^* y$

$(x \in \Omega^X, y \in \Omega^Y)$. \\
----------Proof-----------

$\alpha! (x \wedge y) = \alpha! (\gamma^* A_M \gamma^* y) = \gamma^* \alpha M \gamma^* y$

$= \gamma^* A_M \gamma^* \alpha \gamma^* y = \gamma^* A_M \gamma^* \gamma^* \gamma^* y$

$= \alpha^r! x \wedge \alpha^* y$.

Similarly for the case of $\alpha$ only preproper.  \\

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Corollary 6.8: $\alpha!$ is a homomorphism of $\Omega^*(*)$-modules.

Remark: The next proposition establishes a formula of Riemann-Roch-Nu type for unitary bordism. The restriction to $\Omega^U$-bordism is unnecessary; a similar formula is true in general, but one has especially good hold on $\Omega^U$-operations.

Prop. 6.9: Let $\alpha: X \to Y$ be a smooth $\gamma^*$-map with a $V$-structure (on the index); let $s: \Omega^\nu(*) \to \Omega^\nu(*)$
be the total Novikov operation \( \sum_{i=0}^{\infty} s_i \) (see recollection 4.16); we have a commutative diagram.

\[
\begin{array}{ccc}
\Omega^U_\ast(Y) & \xrightarrow{\alpha_1} & \Omega^U_{\ast+k}(X) \\
\downarrow s & & \downarrow s(\cdot) \cap cf(-\overline{\Theta}) \\
\Omega^U_\ast(Y) & \xrightarrow{\alpha_2} & \Omega^U_{\ast+k}(X)
\end{array}
\]

where the map in the right-hand column is defined by

\[ x \mapsto \sum_{k=0}^{\infty} \sum_{p+q=k} s^p(x) \cap cf^q(-\overline{\Theta}) \] and we have assumed \( \alpha \) to be proper. When \( \alpha \) is proper we replace the right-hand groups by \( \Omega^U_{\ast+k}(X; \Phi_B) \).

**Proof:**

We need to show that \( \alpha_1 S^k(x) = \sum_{p+q=k} s^p(x) \cap cf^q(-\overline{\Theta}) \). Given \( x \in \Omega^U_{\ast}(Y) \), choose a transversal representative 

\[ [M, \overline{\Theta}_M, \mathcal{T}] \]. Then \( \alpha_1(x) = [M_{\overline{\Theta}_1}, \overline{\Theta}_{M_{\overline{\Theta}_1}}, \mathcal{T}] \) and

\[ s^p \alpha_1(x) = \mathcal{T}^* A_{H_{M_{\overline{\Theta}_1}}} cf^p(\overline{\Theta}_{M_{\overline{\Theta}_1}}) \]

Now \( \alpha_1 S^k(x) = \alpha_1 \mathcal{T}^* s^k(x_M) = \mathcal{T}^* \alpha_1 A_M cf^k(\overline{\Theta}_M) = \mathcal{T}^* A_{H_{M_{\overline{\Theta}_1}}} \overline{\Theta}^* cf^k(\overline{\Theta}_M) \).

But \( \overline{\Theta}_{M_{\overline{\Theta}_1}} = \overline{\Theta}_M \Theta \Theta^* \), so

\[
\sum_{p+q=k} s^p \alpha_1(x) \cap cf^q(-\overline{\Theta}) = \sum_{p+q=k} \mathcal{T}^* A_{H_{M_{\overline{\Theta}_1}}} cf^p(\overline{\Theta}_M \Theta \Theta^* \overline{\Theta}) \cap cf^q(-\overline{\Theta})
\]

\[
= \sum_{p+q=k} \mathcal{T}^* A_{H_{M_{\overline{\Theta}_1}}} [cf^p(\overline{\Theta}_M \Theta \Theta^* \overline{\Theta}) \cap cf^q(-\overline{\Theta})] \]

\[
= \mathcal{T}^* A_{H_{M_{\overline{\Theta}_1}}} cf^k(\overline{\Theta}_M \Theta) = \alpha_1 S^k(x) \]
Defn 6.10: Fix a principal ideal domain $\mathcal{L}$ as coefficient ring. Let $M^p, V^q$ be finite-dimensional manifolds, $f: M \rightarrow V$ a continuous map, $\omega_M(\mathcal{L}), \omega_V(\mathcal{L})$ the appropriate orientation sheaves. We say that $f$ is $\mathcal{L}$-orientable if the sheaf $\omega_M \otimes f^*\omega_V = \text{Hom}(\omega_M, f^*\omega_V)$ is trivial; an orientation for $f$ is a section of that sheaf. The pair consisting of $f$ and an orientation for $f$ will be called an $\mathcal{L}$-oriented map.

Let $\mathcal{D}_f(L)$ be the category whose objects are finite-dimensional manifolds and whose morphisms are smooth proper $\mathcal{L}$-oriented maps. We will want to consider categories $\mathcal{M}_L \supset \mathcal{D}_f(L)$ whose objects are Banach manifolds and whose morphisms are proper $\mathcal{L}$-oriented Fredholm maps. Note that we haven't defined $\mathcal{L}$-orientation for Fredholm maps: we are axiomatizing an argument which goes through whenever one can construct an extension of orientation to the Banach case. Thus all we assume about $\mathcal{M}_L$ is that we are given a definition of orientation for Fredholm maps which reduces to the usual one when spaces are finite-dimensional. We require that $\mathcal{M}_L$ be closed under fiber products with elements of $\mathcal{D}_f(L)$. Thus if $\alpha: X \rightarrow Y$ is in $\mathcal{M}_L, M$ is finite-dimensional, and $g: M \rightarrow Y$ is transversal
to $\omega$, then we have

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{cccc}
\pi_X \longrightarrow & M\pi X & \longrightarrow & X
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{cccc}
\pi_X \longrightarrow & M\pi X & \longrightarrow & X
\end{array}
\end{array}
\end{array}
\]

$M\pi X \in M_L$

and $\bar{g}: M\pi X \longrightarrow X$ is a morphism of $M_L$.

A **homology gysin homomorphism** is a functor which assigns to each morphism $\omega: X \longrightarrow Y$ in $M_L$ a homomorphism $\omega^H$, (All homology and cohomology are taken with coefficients in $L$) $\omega^H: H_*(Y) \longrightarrow H_{*+k}(X)$, where $k = \text{index } \omega$.

A **natural** homology gysin homomorphism is such a functor which is natural on fiber products: the above diagram *

induces a commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{cccc}
\omega^H & \longrightarrow & \bar{g}^H & \longrightarrow \\
\omega^H & \longrightarrow & \bar{g}^H & \longrightarrow
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{cccc}
\omega^H & \longrightarrow & \bar{g}^H & \longrightarrow \\
\omega^H & \longrightarrow & \bar{g}^H & \longrightarrow
\end{array}
\end{array}
\end{array}
\]

Now there is a natural functor defining $\omega^H$ when $\omega \in \mathcal{O}_L$. we will require that a natural gysin homomorphism agree with this on finite-dimensional objects.
Prop. 6.11: The natural transformation \( \mu_L : \Omega^k_\ast (\cdot) \to H^\ast (\cdot) \)
induces a commutative diagram

\[
\begin{array}{ccc}
\Omega^k_\ast (Y) & \xrightarrow{\alpha_Y} & \Omega^k_{\ast+k} (X) \\
\downarrow \mu_L & & \downarrow \mu_L \\
H^\ast (Y) & \xrightarrow{\alpha_Y^H} & H^\ast +k (X)
\end{array}
\]

for any natural homology gysin homomorphism.

--------:Proof:--------

Recall the definition of \( \mu_L : [H, \bar{T}, t] \in \Omega^k_\ast (Y) \),
then there exists an orientation \([H] \in H_\ast (M)\) and
\( \mu_L [H, \bar{T}, t] = \bar{T} * [H] \).
Now it is trivial that
the above diagram commutes when \( X, Y \) are finite-dimensional.
But by the finite-dimensional reduction technique,
we have that
\[
\mu_L \alpha_Y (x) = \mu_L \alpha_Y \bar{T} \cdot x_H = \mu_L \bar{T} \cdot \alpha_Y x_H \\
= \bar{T} \cdot \alpha_Y H \mu_L x_H = \alpha_Y H \bar{T} \cdot \mu_L (x_H) \\
= \alpha_Y H \mu_L (x)
\]

Remark: If \( \Phi \) is a functorial family of supports, we
can introduce cohomology groups \( H^\ast_{\Phi \ast} (X) \) as derived
functors of appropriate section functors; and using, say,
Borel-Moore homology we may construct \( H^\ast_{\Phi \ast} (X) \). Then
we can carry through the above argument for larger classes
of morphisms in \( \mathcal{M}_L \); for example, \( \oplus b \)-supports allow us to construct \( \alpha \mathcal{H}^H \) for preproper maps. Note that \( H_{\mathcal{E}^C}^\#(\cdot) = H_{\mathcal{E}^L}^\#(\cdot) \) are just the homology groups on infinite locally finite chains.

**Remark 2:** Note that we haven't asserted the existence of homology gysin homomorphisms. If we are satisfied with the restriction that \( L \) be a field, the problem is equivalent to defining a natural cohomology gysin homomorphism; for then

\[
\text{Hom}(H^*(X); L)
\]

The way to construct such an object in cohomology has been sketched by Verdier [24]. The techniques involved are quite different from those used here: derived categories, sheaf theory, Hyperexts and the like. Since the construction for Fredholm maps involves technicalities, it seems better to wait for the final version of Verdier's work before writing down the construction. For the interim we record a provisional theorem, without proof:

**Defn 6.12:** Let \( \alpha : X \to Y \) be a continuous map of topological spaces; write \( \dim \alpha \) for

\[
\max \{ k \mid H_{\mathcal{E}^C}^k(\alpha(X)) \neq 0 \}
\]

Clearly if \( \alpha \) is a Fredholm map, \( \dim \alpha \) is always finite for any \( X \in X \).

**Proposition:** Let \( \mathcal{M}_L^1 \) be the category whose objects are
Banach manifolds and whose morphisms are proper Fredholm maps with \( \dim \mathcal{F} : X \rightarrow \mathcal{Z} \) uniformly bounded. Then there exists a natural cohomology Gysin map for \( \mathcal{M}_L^1 \); similarly for preproper maps.

**Remark:** The global bound on \( \dim \mathcal{F} \) can probably be removed, but this depends on technicalities of the (unpublished) proof.

**Defn. 6.13:** We recall the definition of characteristic numbers of bordism classes in \( \Omega^*_\mathcal{F}(X) \).

Define \( \tilde{\mu}_L : \Omega^*_\mathcal{F}(X) \rightarrow \text{Hom}^\ast (H^*_\mathcal{F}(X) \otimes H^*_\mathcal{G}(\mathcal{B}G) ; \mathcal{L}) \)

\[
\tilde{\mu}_L([\mathcal{M}, \mathcal{F}, \mathcal{C}]) (x \otimes c) = [\mathcal{F} \circ \mathcal{C} \otimes \overline{\mathcal{L}}]_\mathcal{M}(\mathcal{C})[\mathcal{M}]
\]

where the \([\mathcal{M}]\) on the right-hand side means that the term preceding it is evaluated on the fundamental class of \( \mathcal{M} \). It is known that \( \tilde{\mu}_L \) is injective when \( \mathcal{G} = \emptyset \) and \( X \) has no torsion, or \( \mathcal{G} = \mathcal{S} \mathcal{D} \) and \( X \) has no odd torsion, and so forth.

The correspondence between \( H^*_\mathcal{F}(\mathcal{B}U; \mathcal{Z}) \) and the Novikov ring \( \mathcal{S} \) gives an especially nice way of computing characteristic numbers in \( \Omega^*_U \) — bordism.

**Lemma 6.14:** Let \( \varepsilon : H^*_\mathcal{F}(X) \rightarrow \mathcal{L} \) be the augmentation.

Then \( \varepsilon(\mu_L S^\mathcal{G}x \cap x) = \tilde{\mu}_L(x)(x \otimes c x) \).
Proof:

The left-hand side equals $\varepsilon[t^* D_M^c a(T_H) \cap x]$

$$\varepsilon! (t^* x \cup c^a(T_H))$$

$$= (t^* x \cup c^a(T_H))[M] / l.$$ 

The next proposition is included to show that the Riemann-Roch formula (Prop. 6.9) and Prop. 6.11 can be combined to make $\alpha!$ computable in terms of $\text{I}_0 a$ and $\alpha ! H^i$. In fact this proposition could be proved directly; but in some sense prop 6.9 carries the most information available and it seems best to derive the formula from that proposition.

Prop. 6.15: Define $\tilde{\alpha}: H^o(Y) \otimes H^o(BG) \rightarrow H^o(Y) \otimes H^o(BG)$

by $\tilde{\alpha}(x \otimes c) = \Sigma \alpha_{-1} (x \cup c \cup (-\text{I}_0 c)) \otimes c_i^c$, where

$\Delta c = \Sigma c_i^c \otimes c_i$. ( $H^*(BG)$ is a hopf algebra).

Then the diagram

$$\begin{array}{ccc}
\Omega^G_*(Y) & \xrightarrow{\alpha !} & \Omega^G_*(X) \\
\downarrow \mu_L & & \downarrow \mu_L \\
\text{Hom}^e (H^*(Y) \otimes H^*(BG); L) & \xrightarrow{\text{Hom}(\alpha_{-1})} & \text{Hom}^e (H^*(X) \otimes H^*(BG); L)
\end{array}$$

commutes.

Proof:-----------------

We carry $\tilde{\alpha}$ one better: define $\tilde{\alpha}': \Omega^G_*(Y) \otimes S \rightarrow \Omega^G_*(X) \otimes S$

by
\[ \omega'(x \otimes s) = \sum_{\beta + \gamma = \alpha} \omega_1(x \otimes \chi(-\overline{\tau})) \otimes s^\beta \]

\[ \mu \otimes \omega' = \omega \mu \]

where \( \pi: \Omega^*_\mathbb{C}(\cdot) \rightarrow \Omega^*_\mathbb{C}(\cdot) \otimes S \) acts by sending \( \nu \mapsto \nu \otimes 1 \). Then \( \pi \) is an \( S \)-module homomorphism (by the Riemann-Roch theorem) and the theorem is proved.

Remark: Again a similar statement is true for nonproper maps if appropriate supports are used. The above proof works because \( H^*(BU) \) is a free \( S \)-module on one generator, and will work for any characteristic class in the image of \( \Omega^*_G(BG) \rightarrow H^*(BG) \).

Corr. 6.16: Assume \( \omega: X \rightarrow Y \) proper of index \( k \); assume that \( \omega \) carries a \( \mathbb{G} \)-structure. The Smale index \( \eta_\omega = \omega_1(1) \) has characteristic numbers given by \( \omega_1^*(c^*(-\overline{\tau})) \otimes L \) for \( c \in H^*(BG) \).
tions: may coalesce at exceptional points. Bifurcation theory asks about the behavior of solutions at such exceptional points.

The problems of bifurcation (or singularities) have been studied deeply by Thom, Whitney, Levine and others in the finite-dimensional case. In particular, Thom \([32]\) and Whitney \([36]\) have introduced canonical stratifications of the sets of singularities, given canonical forms for them, and so forth. Their work has made it possible to speak of various types of singularities, to measure "how big" a singularity is -- in terms of a homology class it carries. The transversality theory required to put singularities in their generic form, however, has so far resisted attempts to extend it to the infinite-dimensional case. This fails even for hilbert spaces; however, Frank Quinn has shown it true for manifolds modelled on \(\mathbb{C}^\infty\).

Because of this we use a finite-dimensional singularity theory and bootstrap back to the infinite-dimensional case by the trick of Prop. 6.5.

The idea is quite simple. We suppose we are given a Fredholm map \(\mathbf{a}: X \rightarrow Y\); we choose a "test function" \(g: M \rightarrow Y\). We form the fiber product
Then the singularities of the map $\overline{\sigma}$ are in some sense those of $\sigma$ restricted to $g(M)$. Thus we imagine the map $\sigma$ to display certain singularities, which we measure by intersecting transversally with finite-dimensional sub manifolds. Thus if $M$ is a single point, we can get nothing but generic data about $\sigma$; but if $M$ is a 2-manifold, we can inquire about generic singularity of codimension 2, and similarly for any finite $k$.

Thom has shown \[132\] that the subset of $k^{th}$ singularities carries a canonical cohomology class when the singularity is generic; we could then use this class as a measure of the size. Since we are in a bordism context, it seems more sensible to notice that the $k^{th}$ singularities carry a bordism class (which reduces to the cycle dual to the cohomology class of Thom under the natural map between bordism and cobordism.) These bordism classes have never been computed, even in the finite case, so we first compute them and then apply the finite-dimensional case. As above, we restrict attention to the almost-complex case. The oriented case is more interesting
in applications but its theory involves many more technicalities, and (as above again) the "real" manifold theory of Landweber probably gives best theoretical insight. The basic reference for this section is Thom-Levine \[16\] and Porteous \[26\].
Preliminaries:

We will work exclusively with almost-complex manifolds and complex vector bundles. The dimension of a manifold will be the real dimension of the underlying space, and dimension of vector bundles will be the complex dimension of the fibers.

**Def. 7.1:** Given a vector bundle \( p : E \rightarrow X \) we can form various grassmannian bundles; write \( G_k(E) \) for the bundle of \( k \)-grassmannian of \( E \). Thus if \( x \in X \) and \( \pi_{k,E} : G_k(E) \rightarrow X \) is the bundle projection, then \( \pi_{k,E}^*(x) = G_k(E_x) \) is the set of complex \( k \)-planes in \( E_x \). Note that \( G_k(E) \) carries a canonical \( k \)-plane bundle \( K_E \) whose fiber over \( (x, p) \) is simply \( p \), and that the complement of \( P \) in \( \pi_{k,E}^* \) defines an \( n-k \)-plane bundle \( L_E \) \( (n = \dim E) \). The dimension of \( G_k(E_x) \) is of course \( 2k(n-k) \). If \( X \) is a point and \( E \) merely \( \mathbb{C}^n \) we write \( G_k(E) = G_k, n \).

The spaces \( G_k, n \) have torsion-free integral homology; the spectral sequence for \( \Omega^*_U(G_k, n) \) collapses, and more generally we may use Dold's theorem to compute \( \Omega^*_U(G_k(E)) \).

**Prop. 7.2:** Consider \( \Omega^*_U(G_k(E)) \) as an \( \Omega^*_U(X) \)-module through the map
$\pi^*_k : \Omega^p_U(X) \rightarrow \Omega^p_U(G_k(E))$. Then

$\Omega^p_U(G_k(E)) = \Omega^p_U(X) \left[ u_1, \ldots, u_k; v_1, \ldots, v_{n-k} \right] / R$

where $R$ is the module of relations generated by

$\sum u_{pq} v_q = c_f^j(E)$. Further $c_f^p(K_E) = u_p$, $c_f^p(L_E) = v_p$.

(The proof generalizes the case $k = 1$ which is given in Conner-Floyd, \$8, \{397\}).

Remark on Chern classes: Let $\alpha$ be a multiindex, $E \rightarrow X$ a vector bundle. The Chern class $c_f^\alpha(E) = c_f^{\alpha_1}(E) \cup c_f^{\alpha_2}(E) \cup \cdots \cup c_f^{\alpha_n}(E)$, while the Chern class $c_f^\alpha(E)$ is given by

$P_\alpha (c_f^1, \ldots, c_f^n)$, where $P_\alpha (\sigma_1, \ldots, \sigma_n) = \sum t_1^{\sigma_1} \cdots t_n^{\sigma_n}$

(The $\sigma_i$ are the elementary symmetric functions on the $t_i$, and the summation extends over all permutations of the $t_i$).

Singularities of Maps:

We repeat the definitions of Porteous \$26.7$:

Defn. 7.3: Let $f : \mathbb{H}^{2n} \rightarrow \mathbb{V}^{2p}$ be a smooth map of manifolds. Consider the diagram

$$
\begin{array}{ccc}
K_M & \xrightarrow{i} & T_M \\
\downarrow & & \downarrow \\
G_K(T_M) & \xrightarrow{\pi_K} & M \\
\end{array}
\quad
\begin{array}{ccc}
K_M & \xrightarrow{i} & T_M \\
\downarrow & & \downarrow \\
G_K(T_M) & \xrightarrow{\pi_K} & M \\
\end{array}
$$

where $i$ is the inclusion of a $k$-plane in $T_M$. This
diagram defines a section \( \Phi_k \) of the vector bundle

\[
\text{Hom}(K_M, \pi_k^* \tau^* \tau_V) \xrightarrow{\Phi_k} G_k(T_M)
\]

In general let \( E \xrightarrow{s} X \) be a section of a smooth vector bundle over a smooth manifold; write \( \text{O}_X \) for the 0-section considered as a submanifold of \( E \). Then \( s(X) \subset E \) is another copy of \( X \); we say that \( s \) is transversal to the 0-section if \( s(X) \) intersects \( \text{O}_X \) transversally. Sections transversal to zero are generic in the appropriate topology on the space of sections. See Abraham-Robbins. \[3-7\]

**Defn. 7.4:** We say that \( f: M^{2n} \xrightarrow{} V^{2p} \) is \( k \)-transversal to zero (when \( n \geq p \)) if \( f_k \) is transversal to the zero-section of the bundle \( \text{Hom}(K_M, \pi_k^* f^* \tau_V) \). (This only makes sense for positive \( k \); but we agree that a map as above is always \( k \)-transversal to zero for negative \( k \).)

Let \( Z_k(f) = \{ f_k \} \) be the preimage of the zero-section. When \( f \) is \( k \)-transversal to zero, \( Z_k(f) \) is a smooth compact submanifold of \( G_k(T_M) \) and we may consider the bordism class

\[
[Z_k(f), \pi_k^* \tau_k] \in \Omega^U_2(n+k(n-p-k)) (M)
\]

where \( Z_k(f) \xrightarrow{\Phi_k} G_k(T_M) \xrightarrow{\pi_k} M \). Let \( Z_k(f) = 0 \) for \( k < 0 \); when \( k = 0 \) it is consistent to say that \( Z_0(f) = M \).
In case $p \geq n$ we make an analogous construction
with cotangent bundles. Roughly, the idea is that a map
displays a $k$-singularity when its dual displays a $-k$-singularity:

**Def. n 7.5**: Consider the diagram $(p \geq n)$

\[
\begin{array}{ccc}
K \cdot \Omega V & \xrightarrow{\pi} & \Omega V \\
\downarrow & & \downarrow \\
G_k \cdot \Omega V & \xrightarrow{\pi_k} & M
\end{array}
\]

(where $\pi$ is $\text{Hom}(E; \Omega)$, the dual of $E$).

Consider the section

\[\tilde{f}_k: G_k \cdot \Omega V \rightarrow \text{Hom}(K \cdot \Omega V, \tilde{\Omega} M)\]

Then $\tilde{f}$ is $k$-transversal to zero ($k < 0$) if $\tilde{f}_k$ is
transversal to the 0-section of $\text{Hom}(K \cdot \Omega V, \tilde{\Omega} M)$. Then

\[Z_k(\tilde{f}) = (\tilde{f}_k)^{-1}(0_{G_k \cdot \Omega V})\]

is a closed submanifold as above and we have another bordism class

\[\left[ Z_k(\tilde{f}), \pi_k i_k \right] \in \Omega^U_{2(n+k(n-p-k))}(M)\]

In this case ($p \geq n$) we agree that $\tilde{f}$ is always $k$-transversal to zero for $k > 0$ and that $Z_k(\tilde{f})$ is always empty.

By Atiyah-Thom duality there is a dual cobordism
class $\tilde{Z}_k(F) \in \Omega^{2k(n-p-k)}(M)$ $(k \in \mathbb{Z})$

$\tilde{Z}_k(F) = A^{-1}_H [Z_k(F), \pi_k i_k]$. 

To compute $\tilde{Z}_k(F)$ it is convenient to introduce some characteristic classes $Z_k(E, F)$:

**Defn. 7.5**: Let $E(F)$ be a $p(q)$-plane bundle over a manifold $M$. Recall that $\pi_{k,E} : G_k(E) \rightarrow M$ has an associated Gysin homomorphism in cobordism,

$$\pi_{k,E}^! : \Omega^*_U(G_k(E)) \longrightarrow \Omega^{*-2k(p-k)}(M)$$

Then for $k > 0$, define

$$Z_k(E, F) = \pi_{k,E}^! c^{kq} (K_E \otimes \pi_{k,E}^* F);$$

for $k < 0$, set $Z_k(E, F) = \bar{Z}_k(F, E)$. Somewhat surprisingly, $Z_k(E, F)$ is a stable characteristic class:

**Prop. 7.6**: $Z_k(E \oplus P, F \oplus P) = Z_k(E, F)$ where $P$ is any $L$-plane bundle on $M$.

**Proof**: It suffices to assume $k > 0$. Consider the commutative diagram

$$
\begin{array}{ccc}
G_k(E) & \xrightarrow{j} & G_k(E \oplus P) \\
\pi_E \downarrow & & \pi_{E \oplus P} \downarrow \\
M & \xrightarrow{\pi_E} & M
\end{array}
$$

The normal bundle $V$ of $j$ is easily computed. The
tangent bundle of \( G_k(E) \) along the fibers is \( \tilde{K}_E \otimes L_E \); it is immediate that \( \mathcal{V} = \tilde{K}_E \otimes \pi_{EOP}^* P \).

Now \( \mathcal{V} \) is the restriction of \( \text{Hom}(K_{EOP}, \pi_{EOP}^* P) \) to \( G_k(E) \). There is a canonical section \( s: G_k(E) \to \text{Hom}(K_{EOP}, \pi_{EOP}^* P) \) induced by the composition \( K_{EOP} \xrightarrow{i} \pi_{EOP}(EOP) \xrightarrow{pr_2} \pi_{EOP}^* P \).

Clearly \( s|_{j^*G_k(E)} \) is zero; in fact \( s \) is transversal to the zero section of the bundle and \( j^*G_k(E) \) is the transversal intersection of the two sections.

Hence \( j^!(1) = cf^{k_2}(\tilde{K}_{EOP} \otimes \pi_{EOP}^* P) \).

Returning to \( Z_k(E,F) \), we have

\[
Z_k(E,F) = \pi_{EOP}^! \cdot j^! j^* cf^{k_4}(\tilde{K}_{EOP} \otimes \pi_{EOP}^* F) \\
= \pi_{EOP}^! \cdot cf^{k_4}(\tilde{K}_{EOP} \otimes \pi_{EOP}^* F) \cup cf^{k_2}(\tilde{K}_{EOP} \otimes \pi_{EOP}^* P) \\
(\text{By the computation } j^!(1) \text{ and the projection formula for the Gysin homomorphism.})
\]

\[
= \pi_{EOP}^! \cdot cf^{k_4+k_2}(\tilde{K}_{EOP} \otimes \pi_{EOP}^* (F \oplus P)) \\
= Z_k(E \oplus P, F \oplus P)
\]
Now $M$ is a compact space, so the vector bundle $E$ admits a stable inverse $\bar{E}$ such that $E \otimes \bar{E} = \mathbb{C}P^2$.

Hence

$$Z_k(E,F) = Z_k(\mathbb{C}P^2, F \otimes \bar{E}) = \pi_1^{k+1}(k \otimes \pi^k(F \otimes \bar{E})).$$

The quantity on the right depends only on the stable class of $F \otimes \bar{E} = F - E + p + q$, so we can write

$$Z_k(E,F) = \pi_1^{k+1}(k \otimes \pi^k(F - E + p + q))$$

($\pi: G_k \times M \times M \to M$ is the trivial projection.) It can be checked that this quantity depends only on $E - F$, $k$, and $p - q$.

To compute the tensor product involved, we have to appeal to weights and the splitting principle. That the splitting principle carries through in the case of unitary bordism is trivial; see Conner-Floyd $\xi^2$, § 27. The real difficulty occurs in taking tensor products of line bundles. It is shown in Novikov $\xi^2$, Appendix 7 that if $\xi, \eta$ are complex line bundles, then
\[ c_f'(\xi \otimes \eta) = \sum_{i,j \geq 0} \chi_{i,j} \left[ c_f'(\xi) \right]^i \left[ c_f'(\eta) \right]^j \]

\[ \chi_{i,j} \in \Omega_{\mathfrak{U}}^{-2(i+j-1)} \] (pf.)

and the \( \chi_{i,j} \) are quite nontrivial. However, Grothendieck has a formalism for Chern classes which eliminates this difficulty.

**Prop. 7.7:** Consider the ring of "formal power series"

\[ \Omega^*_{\mathfrak{U}}(X) \left[ [t] \right] \]

whose elements are sequences

\[ \sum_{i=0}^{\infty} A_i t^i, \quad A_i \in \Omega^{2i}_{\mathfrak{U}}(X). \]

This is a module over \( \Omega^*_{\mathfrak{U}}(\mathfrak{U}) \left[ [t] \right] \); let \( \Omega^*_{\mathfrak{U}}(X) \) be the group of invertible elements in the ring \( \Omega^*_{\mathfrak{U}}(X) \left[ [t] \right] \).

Then \( \Omega^*_{\mathfrak{U}}(X) \) is an abelian group, and may be endowed with a multiplication making it into a commutative ring with identity such that

\[ c_f : K(X) \to \Omega^*_{\mathfrak{U}}(X) \]

(where

\[ c_f(\xi) = \sum_{i=0}^{\infty} c_f'(\xi) t^i \]

) is a ring homomorphism.

(This is just the theorem of Grothendieck, for \( \Omega^*_{\mathfrak{U}}(\cdot) \) rather than \( H^*(\cdot) \). The difference of course is that the universal polynomials defining \( * \) are infinitely more complicated here.)

In terms of this multiplication, we have

**Prop. 7.8:** \( \mathbb{Z}_k(\mathcal{E}, F) = c_f(\pi^*_1 K) * c_f(\pi^*_2 (F-E)) \)

\[ \frac{1}{\pi^*_1(1)} \]
where $\pi_i: C_k \times X \rightarrow \mathbb{C}_k^p X$ is the projection on the $i$th factor, $\beta_q$ the operation which assigns to $\sum_{i=0}^{a_i} A_i t^i$ the element $A_q$, and is the slant product (Whitehead $\{35, 7\}$).

Finally we state Porteous' result in the form we will use:

**Prop. 7.9:** Given $\gamma: \mathbb{M}^{2n} \rightarrow \mathbb{V}^{2p}$, then

$$Z_k(\gamma) = Z_k(\tau M, f^* \tau V)$$

when $k, n-p$ have the same sign, and zero otherwise.

**Proof:**

This is essentially the first step of Porteous' prop 1.3; he uses no properties of his cohomology functor except for the existence of Poincare duality to establish this.

**Remark:** from Prop 7.6 it is clear that $Z_k(\gamma)$ is dependent only on $f^* \tau V - \tau M = I f$ and the integer $n-p$; that is, there is a universal polynomial $P_{k, n-p} (c_{f1}, \ldots, c_{f2}, \ldots)$ such that

$$Z_k(\gamma) = P_{k, n-p} (c_{f1}(I f))$$

is essentially given by Prop. 7.8.

We turn to the problem of measuring singularities of a fixed Fredholm map $\omega: X \rightarrow Y$ between Banach mani-
ifolds. Since a general transversality theory of sufficient strength is lacking, we define "relative" singularity classes.

Defn. 7.10: Let \( \alpha: X \rightarrow Y \) be a smooth proper \( \Phi \)-map, \( M \) a paracompact \( n \)-dimensional \( \mathcal{U} \)-manifold; (we assume \( \Phi \) carries a \( \mathcal{U} \)-structure); let \( g: M \rightarrow Y \) be a smooth map transversal to \( \alpha \). Then the \( r \)th singularities of \( \alpha \) over \( g \) are defined to be the \( r \)th singularities of \( \overline{\alpha}: X \pi M \rightarrow M \) in the diagram

\[
\begin{array}{ccc}
X \pi M & \xrightarrow{\overline{\alpha}} & M \\
\downarrow g & & \downarrow g \\
Y & \xrightarrow{\alpha} & X
\end{array}
\]

It is easy to see that by an arbitrarily small perturbation of \( g \), we may assure that \( \overline{\alpha} \) has only generic singularities.

The set \( Z_r(\overline{\alpha}) \) will be denoted \( Z_r(\alpha/g) \). The invariant we are interested in is the bordism class

\[
[Z_r(\alpha/g), g \pi_k \iota_k] \in \Omega^U_{k}(X).
\]

The operation which sends the triple \( [M, \overline{\mathcal{E}}_M, g] \) to the class of \( [Z_r(\alpha/g), g \pi_k \iota_k] \) depends only on the bordism class \( [M, \overline{\mathcal{E}}_M, g] \) and thus defines a homomorphism

\[
\alpha^{(r)}: \Omega^U_{2n}(Y) \rightarrow \Omega^{U}_{2n+k-r}(k-r)) (X).
\]
Prop. 7.11: Let $\omega \in \Omega^u_{2n}(Y)$. Then

$$\omega^{(r)}(\omega) = Pr_k(\overline{\omega}) \cap \omega_1(\omega)$$

---------Proof:---------

Clearly $\omega^{(r)}(\omega) = \left[ Z_r(\omega/k), \bar{g} \pi_k i_k \right]$

$$= \bar{g} \ast A_{\pi_k X} Z_r(\omega)$$

But $\hat{Z}_r(\omega) = Pr_k(\overline{\omega}) = \bar{g} \ast Pr_k(\overline{\omega})$, so

$$\omega^{(r)}(\omega) = \bar{g} \ast A_{\pi_k X} \bar{g} \ast Pr_k(\overline{\omega})$$

$$= Pr_k(\overline{\omega}) \cap [H \pi X, \bar{g}]$$

$$= Pr_k(\overline{\omega}) \cap \omega_1(\omega).$$

Remark: Where enough transversality is available—say $X$ modelled on $C^0$ or hilbert space—the cobordism class $Pr_k(\overline{\omega})$ can be interpreted as a submanifold of codimension $2r(k-r)$ in $X$; thus although the singularities may not actually be stratifiable, in some sense they are stratifiable up to homotopy.
APPENDIX: ON GLOBAL ANALYSIS

Introduction

The preceding has developed a theory of the Gysin homomorphism, and, thus, the Smale invariant, for a smooth, preproper Fredholm map between Banach manifolds. It is the aim of this section to show how elliptic boundary-value and eigenvalue problems give rise to such maps. There is absolutely nothing original in this; this section will be almost entirely a homage to Palais. However, it seemed worthwhile to collect the definitions and theorems needed to pass from nonlinear elliptic differential operators to smooth preproper Fredholm maps somewhere convenient for reference. Besides, in the preceding work strong use has been made of the index bundle $\text{Ln}$ of a Fredholm map $\mathcal{r}$. This "bundle" is in fact computable from the symbol of an elliptic operator (nonlinear, of course) in a way much like that of the index of a linear elliptic operator. The computation grows naturally out of Palais' formalism for nonlinear operators, and this is a convenient place to indicate it.

The organization of the section is quite simple. We first introduce certain section functors on vector bundles which globalize into nice functors on the category of smooth fiber bundles to the category of Banach manifolds. We then indicate the "derived" functors, which
take chains of Banach manifolds as values. Using these, we show how nonlinear differential operators give rise to smooth maps of Banach manifolds. The next set of ideas are those of linearization and symbol; these arise from the definition of nonlinear operator and yield, under an obvious definition of ellipticity, that the maps of interest are Fredholm. Since our maps are actually maps of chains of manifolds, we can attach natural Finsler structures and apply a theorem of Ellworthy to see that our maps are proper.

Finally, we return to the symbol and its associated invariant in K-theory and compute the Chern character of \( Z \) by appealing to the parameterized Atiyah-Singer index theorem.

**Defn. A1:** A **Palais Section Functor** \( \Pi \) is a functor from the category of Riemannian vector bundles over a smooth compact manifold to Banach spaces satisfying the following two axioms:

**P1)** If \( M, V \) are compact \( n \)-manifolds, \( \varphi: M \to V \) is a diffeomorphism of \( M \) into \( V \), and \( \mathcal{E} \) is a vector bundle over \( V \) then \( s \mapsto s \circ \varphi \) is a continuous linear map of \( \Gamma(\mathcal{E}) \) into \( \Gamma(\varphi^* \mathcal{E}) \).

**P2)** There is a natural continuous inclusion map.
\( \Gamma(\xi) \rightarrow C^0(\xi) \). Moreover if \( \eta \) is a second vector bundle over \( \mathcal{V} \) and \( g : \xi \rightarrow \eta \) is a smooth fiber preserving map, then

\[
\begin{array}{ccc}
C^0(\xi) & \xrightarrow{g^o} & C^0(\eta) \\
\uparrow & & \uparrow \\
\Gamma(\xi) & \xrightarrow{g^r} & \Gamma(\eta)
\end{array}
\]

where \( g^r = g^o \mid C^0(\xi) \).

**Examples:** The section functors are essentially defined by their properties on coordinate patches, so we may restrict ourselves to open sets in \( \mathbb{R}^n \) and Banach spaces of functions on them. Then some canonical examples of Palais section functors are given locally by

1) \( C^0 \) (i.e. continuous functions in the supnorm.

2) \( \Lambda^\alpha, 0 < \alpha < 1 \) (Hölder continuous functions which are \( o(1) \) rather than \( O(1) \)).

III) \( C^\alpha, 0 < \alpha < 1 \) (ordinary Hölder continuous functions.)

The main reason for studying Palais section functors is that they behave well under globalization:

**THM (Palais)** Every Palais section functor has a unique canonical extension to a functor from the category of smooth fiber bundles over a compact manifold to the cathe-
gory of smooth Banach manifolds. (The morphisms in the category of smooth fiber bundles are taken to be smooth fiberwise maps.)

If \( E \xrightarrow{F} B \) is such a fiber bundle, then the tangent space of \( PE \) at a section \( s \in PE \) is \( \Gamma(s^*TB) \), where \( TB \) is the tangent bundle of \( E \) along the fibers, \( s^*TB \) is the pullback over \( B \), and \( \Gamma(s^*TB) \) is the space of sections over \( B \).

For proof of this, see Palais \( \text{§} \), \( \text{§}13\_7 \). The second section is standard in the theory of manifolds of sections.\( \text{II} \).

Note that \( \Gamma \) is a functor from Riemannian vector bundles to Banach spaces; thus a Riemannian structure on \( s^*TB \) induces a natural finsler structure on \( \Gamma PE \).

But if \( g : TB \rightarrow \mathbb{R}^+ \) is a Riemannian metric on the tangent bundle of \( E \) along the fibers, \( s^*TB \xrightarrow{g} \mathbb{R}^+ \) is a Riemannian metric on \( s^*TB \). Write \( \hat{g} : \Gamma PE \rightarrow \mathbb{R}^+ \) for the induced finsler structure; it is locally uniformly continuous in the sense of Abraham, \( \text{§}2, \text{§}5\_7 \).

\textbf{Defn. A2:} A chain \( \{ A, \tau_k, \pi_k \} \) of section functors is a directed family of functors, consisting of a directed index set \( A \), a section functor \( \pi_k \) defined for each
\[ \alpha \in A, \ \text{and a natural transformation } \pi_{\alpha'}^\alpha : \Gamma_{\alpha'} \rightarrow \Gamma_\alpha \]
of functions defined whenever \( \alpha' \geq \alpha \) such that

1) \( \pi_{\alpha}^\alpha \Gamma_{\alpha} = \Gamma_{\alpha} \) and ii) if \( \alpha'' \geq \alpha' \geq \alpha \),
then \( \pi_{\alpha''}^\alpha \pi_{\alpha'}^\alpha = \pi_{\alpha''}^{\alpha''} \). (The \( \pi_{\alpha}^\alpha \) are bounded linear transformations.)

Given a section functor \( \Pi \), there is a canonical way of constructing a chain \( \{ \Pi^k, \pi^k \} \) with \( \Pi^0 = \Gamma \)
called the derived chain of \( \Gamma \). Let \( C^k(\xi) \) be the
Banach space of \( C^k \)-sections of \( \xi \) with the sup-norm
on derivatives of order \( \leq k \), \( (\Pi_k)(\xi) = \{ s \in C^k(\xi) \mid j^k_s \in \Gamma^k(\xi) \} \),
where \( J^k \xi \) is the \( k \)-jet bundle of \( \xi \) and \( j^k \)
is the \( k \)-jet extension map. (For properties of \( J^k \xi \) when

\( \xi \) is a vector bundle, see Palais (1, 54); The general
case, when \( \xi \) is a fiber bundle, is treated in Palais (24),
Abraham (2), and Ehresmann. . Now \( (\Pi_k)(\xi) \) is
a subspace of \( C^k(\xi) \); let \( \Pi_k(\xi) \) be its closure, with
the induced norm. The \( \pi^k : \Pi_k(\xi) \rightarrow \Pi_k(\xi) \) are (the
extensions to the closures of ) the maps which forget
the last \( l-k \) derivatives of \( s \in \Pi_k(\xi) \). Palais shows
that this is a functor from section functors to chains
of them. He also proves there that if \( \Pi \) satisfies
P1,P2, then \( \Pi_k \) does for all \( k \in \mathbb{Z}^+ \). There are,
however, section functors which become Palais only when
sufficiently derived.
Proposition A3: Let \( \Gamma(\xi) = L^p(\xi) \) be the space of (Lebesgue) \( p \)-integrable sections of \( \xi \). Then \( \Gamma_k(\xi) = L^p_k(\xi) \) is a Palais section functor when \( k > \frac{n}{p} \) \( (n=\dim \xi) \).

Proof: See Palais, \( \xi \, 24 \), \( \xi \, 5 \). (This follows from the Sobolev embedding theorems.) There is an obvious definition for a chain of Banach (or Finsler) manifolds. The globalization theorem extends in the obvious way for arbitrary chains of Palais section functors. However, when the chain is the derived functor of some \( \Gamma \), the globalization has an extra naturality:

Prop A4: Let \( E \rightarrow \Xi \) be a smooth fiber bundle, \( J^kE \) the \( k \)-jet bundle of \( E \). If \( (\Gamma_k)(E) = \{ \text{sections } c \in \mathcal{C}^k(E) | j^k c \in \Gamma_j^k E \} \) then the closure of \( (\Gamma_k)(E) \) in \( \mathcal{C}^k(E) \) is naturally isomorphic to \( \Gamma_k(E) \).

Proof: From construction. See Palais \( \xi \, 24 \), th. 15.2.

Thus \( \Gamma_k E \) has a natural embedding in \( \Pi^k_j E \); we use this to define nonlinear differential operators.

Defn. A5: Let \( E_1, E_2 \) be smooth fiber bundles over \( M \). A \( k^{th} \) order partial differential operator is defined by a smooth fiberwise map \( D : J^k E_1 \rightarrow E_2 \).

This induces a smooth map
\[
\Gamma_k E_1 \xrightarrow{\pi_k} \Gamma^k j^k E_1 \xrightarrow{\eta D} \Gamma E_2
\]
defining \( D = \pi D \circ j^k \). Since \( (\Gamma_k)_c = (\Gamma^k_c) \), we in fact get a
family of operators $D: \pi_{k+\ell} E_1 \to \pi_\ell E_2$ for any $\ell \in \mathbb{Z}^+$, which commutes with the $\pi$'s: thus $D$ induces a map of order $-k$ from the chain $\pi_{-k} E_1$ to that of $E_2$.

Note that each of the spaces involved is smooth, and that since $D$ is $C^\infty$, the associated map $D$ is a smooth map of Banach manifolds. This would not be generally true without the use of Sobolev spaces; $D$ does not define a smooth map between $\pi_\ell E_1$ and $\pi_\ell E_2$, only between $\pi_{k+k} E_1$ and $\pi_\ell E_2$.

For bifurcation theory, it is convenient to be able to use a parameter space:

**Defn. A5:** Let $E_1, E_2$ be smooth fiber bundles over $M$ another smooth manifold. A family of differential operators parameterized by $P$ is defined by a map $D': P \times \mathbb{N} E_1 \to E_2$ fiberwise over $M$. Thus there is an induced map $D': P \times \pi_{k+\ell} E_1 \to \pi_\ell E_2$.

Finally, note that boundary-value problems are easily included in this formalism. If $B$ is a boundary-value operator, $B: \pi_{q+\ell} E_1 \to \pi_\ell (E_3|\partial M)$ we get a map $D \times B: \pi_\ell E_1 \to \pi_{-k} E_2 \times \pi_{-q} (E_3|\partial M)$. 
We will be interested in Fredholm maps which arise from nonlinear differential operators. Thus \( D : \Pi_{k \times \ell} E_1 \rightarrow \Pi_{k \times \ell} E_2 \) will be a Fredholm map if the derivative \( dD_s : T_{\Pi_{k \times \ell} E_1} s \rightarrow T_{\Pi_{k \times \ell} E_2} d\) is a Fredholm operator for all \( s \). From the definition of the tangent space of \( \Pi E \), however, it is easy to identify \( dD_s \) with a linear differential operator \( \Lambda(D)_s : \Pi_{k \times \ell} (s^* T_{E_1}^h) \rightarrow \Pi_{k \times \ell} (s^* T_{E_1}^h) \). This will be a Fredholm map when \( \Lambda(D)_s \) is a Fredholm operator; and \( \Lambda(D)_s \) will be a Fredholm operator when its symbol is elliptic.

Hence the symbol \( g_k(\Lambda(D)_s) \) becomes a most interesting invariant of \( D \). It has been studied in detail by Palais, who has shown how the "index bundle" \( ID \) may be computed from it. First recall the "derivative along the fibers" \( \frac{d}{dt} D : T_t \Pi_{k \times \ell} E_1, j_s \rightarrow T_{(s \times T_{E_1})} d\). Since over \( j_s \in j_k E_1, j_s = j_k (s^* T_{E_1}) \), we can identify \( \frac{d}{dt} D \) with a map \( SD_{j_s} : j_k (s^* T_{E_1}) \rightarrow (s^* T_{E_1}) d\). Palais calls this the vertical derivative of \( D \); it defines a linear differential operator, and the top-order symbol of \( SD \) is the required symbol of \( \Lambda(D)_s \).

We can piece these symbols together into a "global linearized symbol" of \( D \) as follows: let \( T_{k \times \ell} (E_1) = j_k E_1 \prod_m T_{(E_1)}^m \) be the fiber product of the
the cotangent bundle of $\mathcal{M}$ and the $k$-jet bundle of $E_1$ over $\mathcal{M}$; let $\mathcal{T}: T_k(E_1) \to E_1$, $D: T_k(E_1) \to E_2$ be defined by $\mathcal{T}(\mathcal{J}(s), v, x) = s(x)$, $D(\mathcal{J}(s), v, x) = D(s(x))$; then the symbols of $\Lambda(D)$, for different $s$ define a section $\sigma_k(D) \in \Gamma\text{Hom}(\mathcal{T}^*T_{E_1}, \overline{D^*T_{E_2}})$.

**Defn. A6:** Consider the vector bundle homomorphism $\sigma_k(D)$ restricted to $S(T_k(E_1)) = J^k(E_1) \pi^*S(T_{E_1})$, the “sphere bundle” of $T_k(E_1)$. The operator $D$ is said to be elliptic if $\sigma_k(D) \vert S(T_k(E_1))$ is an isomorphism. This is equivalent to requiring that $\Lambda(D)$ be an elliptic operator for every $s \in \Gamma\mathcal{E}_1$.

**Remark:** Since $\Lambda(D)$ is elliptic, it defines a Fredholm operator from the tangent spaces $\Gamma_{k+l}(\mathcal{T}^*T_{E_1})$ to $\Gamma_{l}(\mathcal{T}^*T_{E_2})$ for all $l \in \mathbb{Z}^+$. Hence $D$ becomes not only a Fredholm map of manifolds, but a Fredholm map of chains of Banach manifolds. The added power of the chains becomes clear from the following theorem of Ellsworth [7], th4.27:

**Theorem A7:** Let $(P_0, P_1)$, $(Q_0, Q_1)$ be pairs of Banach spaces, such that $P_0 \subset P_1$, $Q_0 \subset Q_1$ are dense subspaces and the inclusion maps are compact. Let $\iota: (P_0, P_1) \to (Q_0, Q_1)$ be a $C^p$-Fredholm map of such pairs; that is, there is a commutative diagram
Then \( f_1 \) is proper on each closed bounded subset of \( P_1 \).

This theorem is local; we may globalize it directly from the above form or by replacing Banach space at each step in Ellworthy's proof by Banach manifolds. The only subtlety is, perhaps, that we must require that the Banach space carry a Finsler structure to define the bounded sets. Fortunately, Banach manifolds of sections carry the natural Finsler structures introduced above. We then have

**Theorem A7.** Let \( \{ X_\alpha, \pi^\alpha_\beta \} \) and \( \{ Y_\alpha, \pi^\alpha_\beta \} \) be two chains of Banach manifolds indexed by the same set \( A \); we suppose \( A \) has a smallest element \( a \). Then if the chains satisfy the Rellich condition (that is, the \( \pi^\alpha_\beta \) are compact inclusions for \( \alpha > \beta \) and \( \pi^\alpha_\beta(X_\alpha) \) is dense in \( X_\beta \)) and

\[
D: \{ X_\alpha, \pi^\alpha_\beta \} \rightarrow \{ Y_\alpha, \pi^\alpha_\beta \}
\]

is a map of chains, then \( D_\alpha: X_\alpha \rightarrow Y_\alpha \) is preproper for any \( \alpha > a, a \in A \).

**Proof:**

It suffices to consider the case \( A = \{ 0,1,2 \} \). Then any \( x \in X \) has a neighborhood, restricted to which \( D_1 \) is preproper. Thus \( X_1 \) possesses a cover
of such sets. Since $\Pi_2$ is a compact inclusion, if $B$ is closed and bounded in $X_1$, then $\Pi_2^1(B)$ is compact in $X_2$, and is covered with a finite set of $\Pi_2^1(U, \lambda)$. Hence $D|B$ is proper. //

Hence a nonlinear elliptic operator $D$ will be proper whenever we apply it to a Rellich chain; in particular, whenever the section functor $\gamma$ satisfies the Rellich condition. But it is known from functional analysis that the section spaces $C_0^1 L_p^p_k$ (in particular) satisfy this condition.

Although we do not go into it here, it should be clear that the same procedure works for boundary-value problems; thus there is an obvious meaning for a coercive nonlinear elliptic boundary-value problem: that is, one whose linearization at each point is coercive. The problem of parameterized families of nonlinear operators is amusing:

**Prop. A8:** Let $M^n$ be a finite-dimensional connected paracompact manifold without boundary, $X$ and $Y$ Banach manifolds. If $\alpha: M \times X \to Y$ is such that for every $m \in M$, $\alpha(m, \cdot)$ is a Fredholm map from $X$ to $Y$ of index $k$, then $\alpha_z$ is a Fredholm map of index $n + k$. If $M$ is compact and each $\alpha_z(m, \cdot)$ is proper, then $\alpha_z$ is proper; otherwise, if each $\alpha_z(m, \cdot)$ is preproper and $M$ is given any Riemannian
metric, then \( \alpha \) is preproper.

-----Proof is trivial:-----

Hence the parameterized problems define Fredholm operators of positive index, and fall under the same analysis of Fredholm maps as boundary-value problems.

Finally, as promised in the introduction, we return to the symbol and its \( K \)-theory. Recall that the nonlinear operator \( D \) has a symbol \( s_k(D) \in \text{Hom}(\pi^*T^*_E, D^*T^*_E) \). When \( D \) is elliptic, \( s_k(D) \) restricted to the "sphere-bundle" of \( T_k(E_i) \) is an isomorphism. Hence we may use the difference-construction in \( K \)-theory to define an element

\[
[s_k(D)] \in KO(\overline{J^k(E_i)}_M \otimes B(c^*_H), \overline{J^k(E_i)}_M \otimes S(c^*_H))
\]

Where \( B(c^*_H) \) is the ball-bundle of \( T^*_M \). Now \( \overline{J^k(E_i)} \) has the same homotopy type as \( E_i \), so we may consider \( [s_k(D)] \) as an element of \( KO(E_i \pi B(-), ...) \) without reference to its degree. This element is explicitly computable from the knowledge of \( D \), and Palais has shown how the chern characters of \( ID \) may be computed from it. His procedure may be sketched as follows:

Let \( j: \overline{E_i \times T^*_M} \to \overline{E_i \pi T^*_M} \) be defined by the map

\[
j((s, v, x)) = (sc(w), v, x)
\]

Then \( [s_k(D)] \)
pulls back under \( j \) to an element of \( \text{KO}(\mathcal{P}E_i \wedge T(\mathcal{C}_M)) \)
(where \( T(\mathcal{C}_M) \) is the Thom space of the cotangent bundle of \( \mathcal{M} \), and \( \wedge \) is the smash product.) This element defines a family of differential operators parameterized by the set \( \mathcal{P}E_i \); the index homomorphism of Atiyah-Singer defines a map \( \text{ind} : \text{KO}(\mathcal{P}E_i \wedge T(\mathcal{C}_M)) \to \text{KO}(\mathcal{P}E_i) \).

From the definition of \( \text{ind} \) it follows that \( \text{ind} \circ j^* [\mathcal{O}_k(D)] = \text{ID} \). Since the parameterization of \( [\mathcal{O}_k(D)] \) is given by a product rather than by a fibration, \( \text{ind} \) is essentially an exterior slant product and may be computed as such in K-theory. Alternatively, we may apply the parameterized Atiyah-Singer index theorem and compute
\[
\chi_h(\text{ID}) = \left\{ \varphi^j_{\mathcal{C}_M} \cdot \text{ch}(j^* \mathcal{O}_k(D)) \cdot \varphi(t(\mathcal{M})) \right\} / [\mathcal{M}]
\]
where \( \varphi^j_{\mathcal{C}_M} \) is the Thom isomorphism in cohomology for the bundle \( T_{\mathcal{C}_M} \) pulled back over \( \mathcal{P}E_i, T\mathcal{M} \) is the total Todd genus, and \( [\mathcal{M}] \) is the fundamental class of \( \mathcal{M} \).

This may be displayed in a diagram as
\[
\begin{array}{ccc}
K(\mathcal{P}E_i \wedge T(\mathcal{C}_M)) & \xrightarrow{\text{ind}} & K(\mathcal{P}E_i) \\
\downarrow \varphi & & \downarrow \varphi \\
H^*(\mathcal{P}E_i \wedge T(\mathcal{C}_M); \mathbb{Q}) & \xrightarrow{\varphi^j} & H^*(\mathcal{P}E_i; \mathbb{Q}) \\
\downarrow \varphi' & & \downarrow [\mathcal{M}] \\
H^*(\mathcal{P}E_i \times \mathcal{M}; \mathbb{Q}) & \xrightarrow{\partial(N)} & H^*(\mathcal{P}E_i \times \mathcal{M}; \mathbb{Q}).
\end{array}
\]
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