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The Computation of Optimal Rendezvous Trajectories Using the Sequential Gradient-Restoration Algorithm

by

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Abstract

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In recent years, there has been a growing demand for the autonomous rendezvous and docking capability of a spacecraft. Current guidance methods in existence are based on the human control of the chaser spacecraft and are not suitable nor sufficient for an autonomous vehicle.

The optimal solution of the rendezvous problem investigated in this thesis consists of finding an allowable finite control distribution which minimizes some prescribed performance index (i.e. time, fuel, etc) and brings a chaser vehicle into coincidence with a target vehicle. This thesis first derives the well-known Clohessy-Wiltshire (CW) differential equations (Ref. 1) and focuses on the optimal solution of a linearized three-dimensional rendezvous with bounded thrust and limited fuel. To accomplish this, the sequential gradient-restoration algorithm is utilized to optimize several rendezvous trajectories for the case of a target spacecraft in a circular orbit at the International Space Station (ISS) altitude and a chaser spacecraft with typical initial conditions during the terminal phase of the rendezvous with bounded thrust and bounded ΔV.

First, the time-optimal rendezvous is investigated followed by the fuel-optimal rendezvous for three values of the max thrust acceleration via the sequential gradient-
restoration algorithm (SGRA). Then, the time-optimal rendezvous for given fuel and the fuel-optimal rendezvous for given time are investigated. There are three controls, two of which determine the thrust direction in space and one which determines the thrust magnitude.

The main conclusion is that the optimal control distribution can result in two, three, or four subarcs depending on the performance index and the constraints. The time-optimal case results in a two-subarc solution with max thrust. The fuel-optimal case results in a four subarc solution consisting of an initial coasting period, followed by a maximum thrust phase, followed by another coasting period, followed by another maximum thrust phase. Regardless of the number of resulting subarcs, the optimal thrust distribution requires the thrust magnitude to be either at the maximum value or at zero. The coasting periods are finite in duration and their length increases as the time to rendezvous increases and/or as the max allowable thrust increases. Another finding is that, for the fuel-optimal rendezvous with the time unconstrained, the minimum fuel required is nearly constant and independent of the max available thrust.

Based on the above observations, the final portion of this thesis applies the multiple-subarc version of SGRA to solve the guidance problem based on the implementation of constant-control finite-thrust functions during each subarc. Here, the intent is to replace the optimal trajectory with a simplified finite-thrust guidance trajectory which can be executed in real time, while still closely approximating the results of the optimal continuous control trajectory. For all of the problems studied, the performance indexes of the pieced guidance trajectories using the multiple-subarc SGRA are in excellent agreement (within a fraction of one percent) with the performance indexes of the continuous optimal trajectories obtained via
the single-subarc SGRA. Thus, the methodology covered in this thesis could be the basis for an autonomous real-time rendezvous guidance algorithm.

**Key Words.** Rendezvous, target spacecraft, chaser spacecraft, optimal rendezvous trajectories, Clohessy-Wiltshire (CW) equations, calculus of variations, constant thrust guidance, mathematical programming, optimal control.
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1. Introduction

Although the first rendezvous and docking of two spacecraft occurred nearly 30 years ago (it involved the Gemini 6 and Gemini 7 spacecraft), all subsequent rendezvous between spacecraft have continued to rely on humans to control the approaching spacecraft, that is, astronauts onboard or ground controllers. In the past few years, there has been an overwhelming demand for the development of automated rendezvous and docking capability which would enable a chaser spacecraft to autonomously rendezvous and dock with a target spacecraft without human intervention (Refs. 3 - 6). As a result, government agencies and commercial aerospace corporations have begun to spend substantial resources in efforts to develop and prove this technology on programs like the DARPA (Defense Advanced Research Projects Agency) DART Program, the DARPA Orbital Express Program (Ref. 7), and the Air Force Research Laboratory XSS-11 Program (Ref. 8). Likewise, with the end of the space shuttle approaching, NASA has expressed a similar interest in this capability. For example, NASA has imposed the requirement of automated rendezvous and docking capability on the new Crew Exploration Vehicle (CEV) currently in acquisition phase (Ref. 9) and has issued a solicitation to the aerospace industry in January 2006 for the development of a robotic spacecraft capable of autonomously delivering cargo and payload to and from the International Space Station (Ref. 10). In addition, the commercial space industry has also expressed an interest in the automated rendezvous and docking in support of new emerging markets such as satellite refueling and servicing in order to extend the life of current assets in space (Refs. 11 – 14). Despite this widespread interest in the automated rendezvous,
capability of an autonomous spacecraft being able to successfully rendezvous and dock remains yet to be demonstrated (Ref. 4).

The rendezvous maneuvers performed by NASA in the past as well as the present involve a series of ground-computed steps which are then uplinked to a spacecraft and executed by the crew onboard (Refs. 15 – 19). This series of maneuvers is called the rendezvous profile; for a typical shuttle mission, it consists of ten steps as seen in Figure 1a. Once the chaser spacecraft comes within view of the target spacecraft, the crew takes manual control of the vehicle and flies a so-called glideslope approach. During the glideslope approach, the chaser vehicle moves along a quasi-straight line toward its target as seen in Figure 1b. Controlled by a human operator, this rendezvous sequence has been used successfully many times, from the Apollo era to present shuttle/ISS missions. Besides proven heritage, one advantage of the glideslope approach is a continuous line-of-sight with the target spacecraft so that the human controller can monitor continuously the closing rate and status of the approach. In addition, the glideslope approach requires relatively light computational requirements. It is well known from the literature (Refs. 20 - 30) that the Clohessy-Wiltshire equations of relative motion, governing the rendezvous of two spacecraft, are not solvable in general form for the case of forced motion. In light of this, the adopted approach is to assume that the thrusting maneuvers required to guide the target spacecraft are performed instantaneously, thus allowing the use of an analytical solution to the Clohessy-Wiltshire equations. For this reason, the glideslope method is broken up into a sequence of multiple small thrusting maneuvers such that the approximation of instantaneous thrusting maneuvers results in little error. However, this comes at the expense of obvious fuel
penalties, since a straight line approach requires continual actuation against the force of gravity and results in the rendezvous requiring many multiple maneuvers during the course of the rendezvous. In the past, since this method was convenient for the humans controlling the chaser spacecraft, the benefits have justified the fuel costs. In this dissertation however, we seek to investigate the generation of optimal rendezvous trajectories using finite thrust durations to be ultimately utilized into a robust guidance formulation implementable autonomously onboard a spacecraft.

Although an extensive literature search proves that many optimization studies have been performed previously on this problem from authors such as Prussing, Carter, Guzman, and Shen (Refs. 31 – 42), they have all assumed impulsive guidance schemes or employed primer vector theory to apply instantaneous velocity changes. Although, as in the glideslope approach, this method is justified in some cases and results in very little error when the on-periods are small in time with respect to the total transfer time, that is not always the case as will be shown in the subsequent investigation of the time-optimal rendezvous. Another example of the insufficiency of instantaneous velocity changes is with regards to collision avoidance. A spacecraft capable of autonomously rendezvous and docking must also have the ability to perform rapid collision avoidance maneuvers, which would again require that the on-periods be large in time with respect to the total transfer period. Therefore, the contribution of this investigation comes from the fact that the sequential gradient-restoration algorithm (SGRA) computes an optimal control distribution as a function of time allowing for optimal finite duration maneuvers, as opposed to the current method approximations made by assuming instantaneous velocity changes.
The organization of this thesis is as follows. Section 2 deals with the Bolza-Pontryagin optimal control problem and the basic structure of SGRA. Section 3 deals with the system equations, boundary conditions, and inequality constraints plus formulation of the optimal rendezvous problems. Section 4 describes the numerical results obtained via the single-subarc version of SGRA. Section 5 describes the application of the multiple-subarc version of SGRA to the guidance problem via the implementation of constant thrust functions during each subarc. Finally, Section 6 contains the conclusion.
2. Algorithm Structure

Developed in its basic form by Miele and collaborators within the Aero-Astronautics Group in the period 1968-1986, SGRA is a proven tool for solving both mathematical programming and optimal control problems, as well as trajectory problems of atmospheric and space flight. Applications and extensions of this algorithm have been reported in the US, Japan, Germany, Spain, and other countries around the world; in particular, a version of this algorithm is used at NASA-JSC under the code name SEGRAM, developed by McDonnell Douglas Technical Service Company. SGRA is an iterative technique which includes a sequence of two-phase cycles, each composed of a gradient phase and a restoration phase. This technique is designed to achieve a decrease in the functional being minimized at the end of each cycle, while the constraints are satisfied to a predetermined accuracy.

2.1. Formulation of the Problem. In order to implement the algorithm in a more suitable way, it is convenient to normalize the interval of integration from 0 to 1. If $\rho$ is the running time and $\tau$ is the final time, the transformation $t = \rho/\tau$, with $0 \leq t \leq 1$, converts a problem with variable interval of integration into one with a fixed interval of integration. In light of this time normalization, the time length $\tau$ (if free) can be treated as a component of a vector parameter $\pi$ to be optimized.

The basic optimization problem considered here is of a Bolza-Pontryagin type and involves the minimization of a functional consisting of a line integral and a function of the boundary conditions. In detail, the problem is to minimize the following functional
\[ I = \int_0^1 f(x,u,\pi,t)dt + [g(x,\pi)]_1 , \]  
with respect to the state vector \( x(t) \), control vector \( u(t) \), and parameter vector \( \pi \) satisfying the differential constraint:

\[ \dot{x} = \phi(x,u,\pi,t) , \]  
with initial condition:

\[ x_o = \text{given}, \]  
and the final boundary constraint:

\[ [\psi(x,\pi)]_1 = 0 . \]  

The above relations, the independent variable is the normalized time \( t \), \( 0 \leq t \leq 1 \); the dot superscript denotes derivative with respect to the normalized time; \( f \) and \( g \) are scalar functions, \( \phi \) is a \( n \)-vector function, and \( \psi \) is a \( q \)-vector function. The dependent variables are the \( n \)-vector state \( x(t) \), the \( m \)-vector control \( u(t) \), and the \( p \)-vector parameter \( \pi \). Any trajectory satisfying the constraints of (2) is called a feasible trajectory; among the infinite number of feasible trajectories, we seek the special trajectory which minimizes the functional (1).

2.2. First-Order Conditions. We know that from the calculus of variations, that the previous problem is one of the Bolza-Pontryagin type. It can be reformulated as a Mayer problem

\[
\begin{align*}
\min J &= \int_0^1 f + \lambda^T (\dot{x} - \phi)dt + (g + \mu^T \psi)_1 \\
&= \int_0^1 (f - \lambda^T \phi - \dot{\lambda}^T x)dt - (\lambda^T x)_0 + (g + \mu^T \psi + \lambda^T x)_1 ,
\end{align*}
\]  

(3)
which is subject to the same set of initial conditions and constraints as

\[ \dot{x} = \phi, \]  
\[ x_0 = \text{given}, \]  
\[ \psi_1 = 0. \]  

(4a)  
(4b)  
(4c)

Here, \( J \) is the augmented functional, the \( n \)-vector \( \lambda(t) \) is a variable Lagrange multiplier, and the \( q \)-vector \( \mu \) is a constant Lagrange multiplier.

In this case, the first variation of the augmented functional (3) is

\[
\delta J = \int_0^1 (f_x - \phi_x \lambda - \dot{\lambda})^T \Delta x \, dt + \int_0^1 (f_u - \phi_u \lambda)^T \Delta u \, dt
\]

\[
+ \left[ \int_0^1 (f_x - \phi_x \lambda) \Delta t + (g_x + \psi_x \mu) \right]^T \Delta \pi \left[ (g_x + \psi_x \mu + \lambda)^T \Delta x \right].
\]  

(5)

In light of that, the optimal trajectory must satisfy the constraint equations (4) as well as satisfying the first order optimality conditions below

\[ \dot{\lambda} = f_x - \phi_x \lambda, \]  
\[ f_u - \phi_u \lambda = 0, \]  
\[ \int_0^1 (f_x - \phi_x \lambda) \Delta t + (g_x + \psi_x \mu) \Delta t = 0, \]  
\[ (g_x + \psi_x \mu + \lambda) = 0. \]  

(6a)  
(6b)  
(6c)  
(6d)

We call a trajectory satisfying (4) and (6) an extremal trajectory. Indeed, satisfaction of (6) ensures the vanishing of the first variation (5) for any system of variations.
2.3. Approximate Solutions. Because the system of (4) and (6) is frequently nonlinear, approximate methods must be used to seek a solution iteratively.

Let the norm squared of a vector be

\[ N(v) = v^T v. \] (7)

So that under the assumption that the initial condition (4b) is satisfied at every stage of SGRA, the functionals,

\[ P = \int_0^1 N(\dot{x} - \phi) dt + N(\nu), \] (8)

\[ Q = \int_0^1 N(\dot{\lambda} - f_x + \phi_x \lambda) dt + \int_0^1 N(f_u - \phi_u \lambda) dt \]

+ \[ N\left( \int_0^1 (f_x - \phi_x \lambda) dt + (g_x + \psi_x \mu) \right) + N(g_x + \psi_x \mu + \lambda), \] (9)

represent respectively the constraint error and the optimality condition error.

In the case of an exact optimal solution,

\[ P = 0, \] (10a)

\[ Q = 0. \] (10b)

Whereas an approximation to the optimal solution results in,

\[ P \leq \zeta, \] (11a)

\[ Q \leq \zeta, \] (11b)

with \( \zeta \) and \( \zeta \) being small preselected constants.

2.4. Sequential Gradient-Restoration Algorithm. The characteristic of SGRA is that has a cyclical structure, each cycle including a gradient phase and a restoration phase.
First, the gradient phase is started whenever Ineq. (11a) is satisfied and involves a single iteration. The target of the gradient iteration is to reduce the value of the augmented functional $J$, while keeping the constraints satisfied to first order. Conversely, the restoration phase is started whenever Ineq. (11a) is violated and might involve several iterations. The restoration phase is terminated whenever Ineq. (11a) is satisfied. Finally, SGRA terminates when Ineqs. (11) are both satisfied.

2.5. System of Variations. For all iterations, let $x(t)$, $u(t)$, $\pi$ denote the nominal functions; let $\Delta x(t)$, $\Delta u(t)$, $\Delta \pi$ denote variations of the nominal functions; let $A(t)$, $B(t)$, $C$ denote the variations for unit stepsize $\alpha$. Therefore, with this understanding

\begin{align*}
\tilde{x}(t) &= x(t) + \Delta x(t) = x(t) + \alpha A(t), \\
\tilde{u}(t) &= u(t) + \Delta u(t) = u(t) + \alpha B(t), \\
\bar{\pi} &= \pi + \Delta \pi = \pi + \alpha C.
\end{align*}

Next, let $J$, $P$ denote the nominal functionals; $\bar{J}$, $\bar{P}$ denote the varied functionals; $\Delta J$, $\Delta P$ denote the total variations of these functionals. Therefore,

\begin{equation}
\bar{J} = J + \Delta J, \quad \bar{P} = P + \Delta P.
\end{equation}

At each iteration, the perturbations per unit stepsize, $A(t)$, $B(t)$, $C$ must be chosen so that

\begin{equation}
\text{either } \Delta J < 0 \text{ or } \Delta P < 0.
\end{equation}

In order to enforce (14), SGRA is constructed so that for each iteration,

\begin{equation}
\text{either } \delta J < 0 \text{ or } \delta P < 0,
\end{equation}

where $\delta J$ is given by (5) and $\delta P$ is given by
\[ \delta P = 2\alpha \left[ (\dot{x} - \phi) \left( \dot{A} - \phi^T x A - \phi^T u B - \phi^T \pi C \right) dt + 2\alpha \left( \psi^T (A^T \pi A + \psi^T C) \right) \right]. \] \hspace{1cm} (16)

### 2.6. Gradient Phase

As mentioned previously, the gradient phase consists of a single iteration designed to decrease the augmented functional (3). Suppose that the nominal functions \( x(t), u(t), \pi \) satisfy the feasibility conditions (4). The perturbations per unit stepsize \( A(t), B(t), C \) must satisfy the linearized constraint equations to first order

\[ \dot{A} - \phi_x^T A - \phi_u^T B - \phi_\pi^T C = 0, \hspace{1cm} (17a) \]

\[ (A)_0 = 0, \hspace{1cm} (17b) \]

\[ \left( \psi_x^T A + \psi_\pi^T C \right)_1 = 0. \hspace{1cm} (17c) \]

It is clear that \( \delta J \) can be made negative through the following choice of the variations per unit stepsize:

\[ B = -(f_u - \phi_u \lambda), \hspace{1cm} (18a) \]

\[ C = \left[ \int_0^1 \left( (f_x - \phi_x \lambda) dt + (g_x + \psi_x \mu) \right) \right], \hspace{1cm} (18b) \]

where \( A(t), B(t), C \) are consistent with (17), the and multipliers \( \lambda(t), \mu \) are consistent with

\[ \dot{\lambda} - f_x + \phi_x \lambda = 0, \hspace{1cm} (18c) \]

\[ (g_x + \psi_x \mu + \lambda)_1 = 0. \hspace{1cm} (18d) \]

As a consequence, the first variations of the functionals under consideration become

\[ \delta J = -\alpha Q, \hspace{1cm} \delta P = 0, \hspace{1cm} (19) \]

and the optimality condition error is given by
\[ Q = \left\{ \begin{array}{c}
B^T B \, dt + C^T C \\
0
\end{array} \right. \]  \hspace{1cm} (20)

Since \( Q > 0 \), the first of Eqs. (19) shows that \( \delta J < 0 \). Hence, for \( \alpha \) sufficiently small, the decrease in the augmented functional \( J \) is guaranteed.

The system (17)-(18) is linear and nonhomogenous in the unknowns \( A(t), B(t), C \) and \( \lambda(t), \mu \) and is independent of the stepsize \( \alpha \). The latter is to be determined a posteriori in such a way that the descent requirement \( \Delta J < 0 \) is enforced.

The linear two-point boundary value problem (LTPBVP) of Eqs. (17)-(18) can be solved via the method of particular solutions (MPS). The number of integrations required is \( n + p + 1 \) if a forward integration scheme is used and \( q + 1 \) if a backward-forward integration scheme is used.

Once \( A(t), B(t), C \) and \( \lambda(t), \mu \) are known, one can form the one-parameter family of solutions (12) for which the augmented functional (3) and the constraint error (8) are functions of the form

\[ \bar{J} = \bar{J}(\alpha), \hspace{1cm} \bar{P} = \bar{P}(\alpha). \]  \hspace{1cm} (21)

Then, some one-dimensional search scheme must be employed and \( \alpha \) must be selected in such a way that the following inequalities are satisfied:

\[ \bar{J}(\alpha) < \bar{J}(0), \]  \hspace{1cm} (22a)

\[ \bar{P}(\alpha) < \bar{P}. \]  \hspace{1cm} (22b)

Concerning (22a), its satisfaction is guaranteed by the descent property of the gradient phase. Concerning (22b), \( \bar{P} \) is a preselected number, not necessarily small, which limits the constraint violation at the end of the gradient phase.
In summary, the gradient stepsize must be chosen so that Ineqs. (22) are satisfied. Should any violation occur, then a smaller value of $\alpha$ must be employed and can be obtained with a bisection process, starting from a suitably chosen reference stepsize $\alpha_o$. In turn, $\alpha_o$ is obtained via a scanning process followed by cubic interpolation.

2.7. Restoration Phase. At the end of a gradient phase, the varied constraint error $\tilde{P}$ can be computed with (8). If Ineq. (11a) is satisfied, then a new gradient phase is executed; otherwise, a restoration phase must be executed. In general, the restoration phase consists of several restorative iterations, each decreasing the constraint error until Ineq. (11a) is satisfied. When this occurs, the restoration phase ends and a new gradient phase is started.

Let $x(t)$, $u(t)$, $\pi$ denote nominal functions violating Ineq. (11a). By inspection of (16), we see that $\delta P$ can be made negative via variations per unit stepsize satisfying the linearized constraint equations

$$A - \phi_x^T A - \phi_u^T B - \phi_\pi^T C + (\dot{x} - \phi) = 0,$$

$$A_0 = 0,$$

$$\left(\psi_x^T A + \psi_u^T B + \psi_\pi^T C + \psi\right) = 0.$$

Any solution $A(t)$, $B(t)$, $C$ of Eqs. (23) is such that the first variation of the constraint error becomes

$$\delta P = -2\alpha P.$$

Since $P > 0$, Eq. (24) shows that $\delta P < 0$. Hence, for $\alpha$ sufficiently small, the decrease in the constraint error is guaranteed.
Note that Eqs. (23) admit an infinite number of solutions. We render the solution unique by determining the perturbations per unit stepsize $A(t), B(t), C$ so that the following functional is minimized:

$$ K = \left( \frac{1}{2} \right) \left[ \int_0^1 B^T B \, dt + C^T C \right]. $$

(25)

subject to the linearized constraints (23). The solution of the ensuing linear-quadratic optimal control problem is characterized by the following variations per unit stepsize:

$$ B = \phi_x \lambda, $$

(26a)

$$ C = \int_0^1 \phi_x \lambda \, dt - (\psi_x \mu), $$

(26b)

with $A(t), B(t), C$ consistent with (23) and multipliers $\lambda(t), \mu$ consistent with

$$ \dot{\lambda} + \phi_x \lambda = 0, $$

(26c)

$$ (\psi_x \mu + \lambda)_1 = 0. $$

(26d)

The system (23), (26) is linear and nonhomogeneous in the unknowns $A(t), B(t), C$ and $\lambda(t), \mu$ and is independent of the stepsize $\alpha$. The latter is to be determined a posteriori in such a way that the descent requirement $\Delta P < 0$ is enforced.

The LTPBVP (23), (26) can be solved via the method of particular solutions (MPS). The number of integrations required is the same as for a gradient iteration, namely $n + p + 1$ if a forward integration scheme is used and $q + 1$ if a backward-forward integration scheme is used.

Once $A(t), B(t), C$ and $\lambda(t), \mu$ are known, one can form the one-parameter family of solutions (12) for which the constraint error (8) is a function of the form
\[ \tilde{P} = \tilde{P}(\alpha). \]  

Then, some one-dimensional search scheme must be employed and \( \alpha \) must be selected in such a way that the following inequality is satisfied:

\[ \tilde{P}(\alpha) < \tilde{P}(0). \]  

Should a violation occur, then a smaller value of \( \alpha \) must be employed and can be obtained with a bisection process starting form \( \alpha = 1 \).

2.8. Gradient-Restoration Cycle. Let \( I_e \) denote the value of the functional (1) at the beginning of the gradient phase; let \( I_s \) denote the value of (1) at the end of the gradient phase; let \( I_r \) denote the value of (1) at the end of the restoration phase. Note that \( I_e \) and \( I_s \) are not comparable, since the constraints are not satisfied to the same accuracy. On the other hand, \( I_s \) and \( I_r \) are comparable, since the constraints are satisfied to the same accuracy. For SGRA to be stable on a digital computer, one must require satisfaction of the inequality

\[ I_r < I_s. \]  

If Ineq. (29) is satisfied, the gradient-restoration cycle is complete and the next cycle is started. If Ineq. (29) is violated, one must return to the previous gradient phase and reduce the gradient stepsize until, after restoration, Ineq. (29) is satisfied. This property is guaranteed in the SGRA.

Remark. The decision-making process of the sequential gradient-restoration algorithm (SGRA) is based on the values of two scalar parameters: the constraint error \( P \) [see (8)] and
the optimality condition error $Q$ [see (9)]. Let $\varepsilon_1$ and $\varepsilon_2$ denote preselected tolerances for $P$ and $Q$. Three cases can result:

(C1) If $P > \varepsilon_1$, SGRA executes a restorative iteration.

(C2) If $P \leq \varepsilon_1$ but $Q > \varepsilon_2$, SGRA executes a gradient iteration.

(C3) If $P \leq \varepsilon_1$ and $Q \leq \varepsilon_2$, convergence is declared and the algorithm stops.

It is important to see that the computation of $P$ requires only the functions $x(t), u(t), \pi$. On the other hand, the computation of $Q$ requires both the functions $x(t), u(t), \pi$ and the multipliers $\lambda(t), \mu$.

2.9. Structure of the Algorithm. For the continuous case as we are employing here, SGRA can be summarized via Steps 1 to 4 below.

Step 1. Choose nominal functions $x(t), u(t), \pi$ consistent with the initial conditions (4b).

Then compute the constraint error via (8). In the case $P$ satisfies Ineq. (11a), go to Step 2. Whereas if $P$ violates Ineq. (11a) go to Step 3.

Step 2. Gradient Phase. It consists of a single iteration described by Steps 2a to 2e below.

Step 2a. Compute the vectors and matrices $(f_x, f_u, f_\pi), (\phi_x, \phi_u, \phi_\pi), (g_x, g_\pi)_1, (\psi, \psi_x, \psi_\pi)_1$.

Step 2b. Find the solution of the linear two-point boundary-value problem (17) - (18) with the method of particular solutions. Obtain the functions $A(t), B(t), C$ and multipliers $\lambda(t), \mu$.

Step 2c. Compute the optimality condition error via (20). In the case that $Q$ satisfies Ineq. (11b), stop; the solution has been found and SGRA terminates. In the case $Q$ violates Ineq. (11b), go to Step 2d.
Step 2d. Calculate the gradient stepsize by a one-dimensional search on the augmented functional $\tilde{J}(\alpha)$. First, determine the reference stepsize $\alpha_0$ via a scanning process followed by a cubic interpolation process. Then, perform a bisection process on $\alpha$, starting from $\alpha_0$, until Ineqs. (22) are both satisfied.

Step 2e. Since the gradient stepsize is known, compute the varied functions $\tilde{x}(t), \tilde{u}(t), \tilde{\pi}$ via (12). Return to Step 1 by setting the new nominal functions equal to the varied functions just computed.

Step 3. Restoration Phase. This phase consists of several restorative iterations, each described by Steps 3a to 3d below.

Step 3a. Compute the vectors and matrices $\dot{x} - \phi, (\phi_s, \phi_u, \phi_\pi), (\psi, \psi_s, \psi_\pi)$.

Step 3b. Solve the linear two-point boundary-value problem (23) - (26) with the method of particular solutions. Obtain the functions $A(t), B(t), C$ and multipliers $\lambda(t), \mu$.

Step 3c. Compute the restoration stepsize by a one-dimensional search on the constraint error $\tilde{P}(\alpha)$. Perform a bisection process on $\alpha$, starting from $\alpha = 1$, until Ineq. (28) is satisfied.

Step 3d. Once the restoration stepsize is known, compute the varied functions $\tilde{x}(t), \tilde{u}(t), \tilde{\pi}$ via (12). Compute the constraint error via (8). If $P$ violates Ineq. (11a), return to Step 3a by setting the new nominal functions $x(t), u(t), \pi$ equal to the varied functions just computed. If $P$ satisfies Ineq. (11a), stop; the restoration phase ends. Go to Step 2 if the restoration phase just ended is part of an incomplete gradient-restoration cycle (this can only occur at the beginning of SGRA); otherwise, go to Step 4.
Step 4. Complete Gradient-Restoration Cycle. For each complete gradient-restoration cycle, verify whether Ineq. (29) is satisfied, that is, whether the value of the functional $I$ at the end of the cycle is smaller than that at the beginning of the cycle. If Ineq. (29) is satisfied, go to Step 2. If Ineq. (29) is violated, return to Step 2d of the previous gradient iteration and bisect the gradient stepsize as many times as needed until, after restoration, Ineq. (29) is satisfied. Then, go to Step 2.
3. System Description

3.1 Differential System. The relative motion of a chaser spacecraft near a target spacecraft in a circular orbit can be represented by the Clohessy-Wiltshire (CW) differential equations. To derive the CW equations, the local-vertical local-horizontal (LVLH) coordinate system is used for this analysis and can be seen graphically in Figure 2a. The convention used here assumes that the LVLH coordinate frame is centered at the center of mass of the target spacecraft with the $x$-axis aligned with the velocity vector (positive opposite the direction of motion), the $y$-axis aligned with the orbital radius vector (positive toward zenith), and the $z$-axis orthogonal to the $xy$-plane and completing the right-hand rule. The angular velocity of the target spacecraft $\omega_{\text{target}}$ is assumed constant because of the circular orbit.

Given this convention, the total acceleration of the chaser spacecraft moving relative to the rotating reference frame of the target spacecraft can be written as the sum of several acceleration components as seen in Eq. (30) below:

$$\ddot{u} + \ddot{g} = \ddot{a}_A = \ddot{a}_R + \ddot{a}_T + \ddot{a}_C$$  \hspace{1cm} (30)

These are the transport acceleration $\ddot{a}_T$, due the translation and rotation of the target frame with respect to an inertial reference frame, the Coriolis acceleration $\ddot{a}_C$, the relative acceleration of the chaser spacecraft relative to target frame $\ddot{a}_R$, the absolute acceleration of the target spacecraft frame itself with respect to an inertial frame $\ddot{a}_A$, the acceleration due to the engine thrusting $\ddot{u}$, and finally the acceleration due to gravity $\ddot{g}$. 
We can write expressions for each of the individual acceleration components above in vector form.

The Coriolis acceleration term is

\[ \ddot{a}_c = 2 \omega_{\text{target}} \times \ddot{V}_A = 2 \omega_{\text{target}} \tilde{k} \times (\dot{x}\tilde{i} + \dot{y}\tilde{j} + \dot{z}\tilde{k}) \]

\[ = 2 \omega_{\text{target}} (\dot{x}\tilde{j} - \dot{y}\tilde{i}) \]

\[ = -2 \omega_{\text{target}} \dot{y}\tilde{i} + 2 \omega_{\text{target}} \dot{x}\tilde{j}. \] (31)

The transport acceleration term is

\[ \ddot{a}_r = \ddot{a}_{\text{target}} + \ddot{\omega}_{\text{target}} \times (\dot{\omega}_{\text{target}} \times \delta\tilde{r}) \]

\[ = \ddot{g}_{\text{target}} + \ddot{\omega}_{\text{target}} \times (\dot{\omega}_{\text{target}} \times \delta\tilde{r}), \]

\[ \ddot{\omega}_{\text{target}} \times \delta\tilde{r} = \omega_{\text{target}} \tilde{k} \times (\dot{x}\tilde{i} + \dot{y}\tilde{j} + \dot{z}\tilde{k}) = -\omega_{\text{target}} \dot{y}\tilde{i} + \omega_{\text{target}} \dot{x}\tilde{j}. \]

\[ = \omega_{\text{target}} \dot{x}\tilde{j} - y\dot{j} \]

\[ = -\omega_{\text{target}}^2 \dot{x}\tilde{i} - \omega_{\text{target}}^2 \dot{y}\tilde{j}, \]

\[ \ddot{a}_r = \ddot{a}_{\text{target}} - \omega_{\text{target}}^2 \dot{x}\tilde{i} - \omega_{\text{target}}^2 \dot{y}\tilde{j}. \] (32)

The translational acceleration term is

\[ \ddot{a}_r = \ddot{x}\tilde{i} + \ddot{y}\tilde{j} + \ddot{z}\tilde{k}. \] (33)

For the acceleration of the chaser spacecraft due to gravity, simplifying assumptions must be made. To begin with, recall Newton’s law of gravitation, which gives the force of the Earth’s gravity acting on a satellite:
\[ F = -\frac{GMm}{r^2} = -\frac{\mu m}{r^2} \quad (34) \]

From Newton’s law of gravitation we can then write the equation of motion for each satellite in orbit independently. The two-body equation of motion for the target spacecraft due to gravity is

\[ \ddot{a}_{\text{target}} = \ddot{g}_{\text{target}} = -\mu \frac{\vec{r}_{\text{target}}}{r_{\text{target}}^3} = -\omega_{\text{target}}^2 \vec{r}_{\text{target}} \cdot \vec{j} = -\omega_{\text{target}}^2 r_{\text{target}} \vec{j} \quad (35) \]

For a circular orbit, \( \omega_{\text{target}}^2 \) is constant and the following equality holds:

\[ \omega_{\text{target}}^2 = \frac{\mu}{r_{\text{target}}^3} \quad (36) \]

Therefore,

\[ \ddot{g}_{\text{target}} = -\mu \frac{\vec{r}_{\text{target}}}{r_{\text{target}}^3} = -\omega_{\text{target}}^2 \vec{r}_{\text{target}} \cdot \vec{j} = -\omega_{\text{target}}^2 r_{\text{target}} \vec{j} \quad (37) \]

The acceleration of gravity on the chaser spacecraft is given by

\[ \ddot{g}_{\text{chaser}} = -\mu \frac{\vec{r}_{\text{chaser}}}{r_{\text{chaser}}^3} = -\frac{\mu}{r_{\text{target}}^3} \frac{r_{\text{target}}^3}{r_{\text{chaser}}^3} \vec{r}_{\text{chaser}} \quad (38) \]

with

\[ \vec{r}_{\text{chaser}} = \vec{r}_{\text{target}} \vec{j} + x \vec{i} + y \vec{j} + z \vec{k} \quad (39) \]

Factoring the above expression and replacing \( \frac{\mu}{r_{\text{target}}^3} \) with \( \omega_{\text{target}}^2 \) yields a term for \( \ddot{g}_{\text{chaser}} \) which can be approximated by a Taylor series expansion as seen in Eq. (40).

\[ \ddot{g}_{\text{chaser}} \approx -\omega_{\text{target}}^2 \left( \frac{1}{1 + \frac{y}{r_{\text{target}}}} \right)^3 \vec{r}_{\text{chaser}} = -\omega_{\text{target}}^2 \left( 1 - 3 \frac{y}{r_{\text{target}}} \right) \vec{r}_{\text{chaser}} \quad (40) \]
Substituting the expression for $\vec{r}_{\text{chaser}}$ yields

$$
\vec{g}_{\text{chaser}} = -\omega_{\text{target}}^2 \left( 1 - \frac{3}{r_{\text{target}}} \right) \left( r_{\text{target}} \vec{i} + x\vec{j} + y\vec{j} + z\vec{k} \right).
$$

(41)

Upon inspection, higher-order terms can be dropped resulting in

$$
\vec{g}_{\text{chaser}} = -\omega_{\text{target}}^2 \left( x\vec{i} - 2y\vec{j} + z\vec{k} \right) - \omega_{\text{target}}^2 r_{\text{target}} \vec{j}.
$$

(42)

As a result of this approximation, the difference in gravitational acceleration between the target and the chaser, written in the target reference frame is given by

$$
\vec{g}_{\text{target}} - \vec{g}_{\text{chaser}} = \omega_{\text{target}}^2 \left( x\vec{i} - 2y\vec{j} + z\vec{k} \right).
$$

(43)

Now putting all the terms of Eq. (30) together, we obtain the following expression for the total acceleration of the chaser spacecraft,

$$
\vec{u} = \omega_{\text{target}}^2 \left( x\vec{i} - 2y\vec{j} + z\vec{k} \right) + \left( \omega_{\text{target}}^2 x\vec{i} - \omega_{\text{target}}^2 y\vec{j} \right)
$$

$$
+ \left( -2\omega_{\text{target}} y\vec{i} + 2\omega_{\text{target}} x\vec{j} \right) + \ddot{x}\vec{i} + \dot{y}\vec{j} + \ddot{z}\vec{k},
$$

(44)

that is,

$$
\vec{u} = \left( \ddot{x} - 2\omega_{\text{target}} \dot{y} \right)\vec{i} + \left( \ddot{y} - 3\omega_{\text{target}}^2 y + 2\omega_{\text{target}} \dot{x} \right)\vec{j} + \left( \ddot{z} + \omega_{\text{target}}^2 z \right)\vec{k}.
$$

(45)

And results in the well-known CW differential equations.

$$
u_1 = \ddot{x} - 2\omega_{\text{target}} \dot{y},
$$

(46a)

$$
u_2 = \ddot{y} - 3\omega_{\text{target}}^2 y + 2\omega_{\text{target}} \dot{x},
$$

(46b)

$$
u_3 = \ddot{z} + \omega_{\text{target}}^2 z.
$$

(46c)
After replacing the variables \( x, y, z \) with \( x_1, x_2, x_3 \), we rewrite the CW equations as shown below:

\[
\ddot{x}_1 - 2\omega \dot{x}_2 = \frac{F_1}{m} = u_1, \tag{47a}
\]

\[
\ddot{x}_2 + 2\omega \dot{x}_1 - 3\omega^2 x_2 = \frac{F_2}{m} = u_2, \tag{47b}
\]

\[
\ddot{x}_3 + \omega^2 x_3 = \frac{F_3}{m} = u_3. \tag{47c}
\]

In these equations, the dot superscript denotes derivative with respect to the actual time \( \rho \); the symbols \( F_1, F_2, F_3 \) denote the thrust components along the directions \( x_1, x_2, x_3 \); the symbols \( u_1, u_2, u_3 \) denote the thrust components per unit mass, hence the thrust acceleration components. In Eqs.(47), the \( \omega \) terms are due to the Coriolis acceleration, which affects the in-plane motion, while leaving the out-of-plane motion undisturbed. The \( \omega^2 \) terms are due in part to the transport acceleration and in part to the fact that the gravitational attractions on the chaser and target spacecraft are different in magnitude and direction.

In first order form, Eqs. (47) can be rewritten as follows:
\[ \dot{x}_1 = x_4 , \] 
\[ \dot{x}_2 = x_3 , \] 
\[ \dot{x}_3 = x_6 , \] 
\[ \dot{x}_4 = 2\omega x_5 + u_1 , \] 
\[ \dot{x}_5 = -2\omega x_4 + 3\omega^2 x_2 + u_2 , \] 
\[ \dot{x}_6 = -\omega^2 x_3 + u_3 , \] 

where \( x_1, x_2, x_3 \) are the separation coordinates in the downrange, radial, and out-of-plane directions, and where \( x_4, x_5, x_6 \) are the separation velocities in the downrange, radial, and out-of-plane directions.

### 3.2 Time Normalization

To bring the system of Eqs. (48a-48f) into the format required by SGRA, it is necessary to normalize the interval of integration to unity. If \( \rho \) is the running time and \( \tau \) is the final time to rendezvous, then the transformation \( t = \rho/\tau \), with \( 0 \leq t \leq 1 \), allows one to convert a rendezvous problem with variable interval of integration into a rendezvous problem with fixed interval of integration. Then, the final time \( \tau \) is treated as a component of the vector parameter \( \pi \) being optimized. Upon normalization of the interval of integration, the system of Eqs. (48) is rewritten as
\[ x'_1 = \tau x_4, \quad (49a) \]
\[ x'_2 = \tau x_3, \quad (49b) \]
\[ x'_3 = \tau x_6, \quad (49c) \]
\[ x'_4 = \tau (2 \omega x_5 + u_1), \quad (49d) \]
\[ x'_5 = \tau (-2 \omega x_4 + 3 \omega^2 x_2 + u_2), \quad (49e) \]
\[ x'_6 = \tau (-\omega^2 x_3 + u_3). \quad (49f) \]

3.3 Control Transformation. In light of Eqs. (49), it is evident that the relative position of the chaser with respect to the target can be optimized and driven by proper selection of the optimal controls as functions of time. For this approach, the controls \( u_1, u_2, u_3 \) were selected as the components of the thrust acceleration in the downrange, radial, and out-of-plane directions; they are themselves dependent on the auxiliary controls \( \alpha, \theta, \phi \). While \( \alpha \) controls the thrust magnitude (see Eqs. (50-51)), \( \theta \) and \( \phi \) control the thrust direction (see Figure 2b and Eqs. (50-51)).

Let \( F, 0 \leq F \leq F_{\text{max}} \), denote the total thrust magnitude and let \( \sigma = \frac{F}{m}, 0 \leq \sigma \leq \sigma_{\text{max}}, \) denote the corresponding thrust acceleration. This inequality can be converted into an equality via the following trigonometric transformation where \( \alpha \) denotes an auxiliary control:
\[ \sigma = \frac{\sigma_{\text{max}}}{2} (1 + \sin \alpha). \] (50)

Also, let \( \theta \) denote the angle of inclination of the thrust vector with respect to the \( x_1x_2 \)-plane and let \( \phi \) denote the angle which the projection of the thrust vector on the \( x_1x_2 \)-plane forms with the \( x_1 \)-axis. With this understanding, the original controls \( u_1, u_2, u_3 \) can be rewritten in terms of the auxiliary controls \( \alpha, \theta, \phi \) as follows:

\[ u_1 = \sigma \cos \theta \cos \phi = \frac{\sigma_{\text{max}}}{2} (1 + \sin \alpha) \cos \theta \cos \phi, \] (51a)

\[ u_2 = \sigma \cos \theta \sin \phi = \frac{\sigma_{\text{max}}}{2} (1 + \sin \alpha) \cos \theta \sin \phi, \] (51b)

\[ u_3 = \sigma \sin \theta = \frac{\sigma_{\text{max}}}{2} (1 + \sin \alpha) \sin \theta. \] (51c)

It should be noted that, in this investigation, the change in the mass of the chaser spacecraft due to fuel consumption is neglected. However, if desired, the change in the mass of the chaser spacecraft can be easily accounted for in SGRA via the introduction of an additional state variable.

3.4 Optimal Control Problems. For the optimal rendezvous investigation at hand, there arise four problems which could be investigated:

(P1) minimum time rendezvous, fuel free,

(P2) minimum fuel rendezvous, time free,

(P3) minimum time rendezvous, fuel given,

(P4) minimum fuel rendezvous, time given.
For the minimum fuel problems, there is need for some direct or indirect measure of the fuel quantity. This results in the necessity of an additional state variable \( x_7 \) as shown in Eq. (52) below. Simply stated, this new state variable is the time integral of the thrust acceleration,

\[
x_7(t) = x_7(0) + \tau \int_0^t \sigma \, dt \quad \rightarrow \quad x'_7 = \tau \left[ \frac{\sigma_{\max}}{2} (1 + \sin \alpha) \right].
\]  

Therefore, in light of the new state variable \( x_7 \), the four problems can be rewritten as:

(P1) minimum time rendezvous, \( I = \tau \), \( x_7(1) \) free,

(P2) minimum fuel rendezvous, \( I = x_7(1) \), \( \tau \) free,

(P3) minimum time rendezvous, fuel given, \( I = \tau \), \( x_7(1) \) given,

(P4) minimum fuel rendezvous, time given, \( I = x_7(1) \), \( \tau \) given.

Of these, Problems (P1) and (P2) constitute the two extreme cases of minimum time rendezvous and minimum fuel rendezvous respectively. Depending on the given value of fuel or time, Problems (P3) and (P4) constitute an infinite number of possible in-between problems offering a compromise between fuel-optimal and time-optimal solutions.

As it turns out, there exists a reciprocity between Problems (P3) and (P4). This is a well-known general property of the Calculus of Variations: the reciprocity of isoperimetric integrals. It is stated as follows: “The solution extremizing a functional \( J \) for given value of the functional \( K \) is the same as the solution extremizing the functional \( K \) for given value of the functional \( J \).” For this investigation, it was decided to explore (P3), the time-optimal rendezvous problem for given \( \Delta V \). One reason why (P3) was
chosen is due to its simple problem formulation, since for this case the only change required for the bounded $\Delta V$ case is that there is now a fixed endpoint condition for the state variable $x_7(1)$.

At this point then, all of the problems to be considered [(P1), (P2), (P3)] have been reduced to Mayer type optimal control problems. Figures 3-5 summarize the problem formulation as implemented in SGRA.
4. Results

4.1 Minimum Time, Fuel Free. For the problem in Figure 3, SGRA was run for three different values of the max thrust acceleration: 0.1524, 0.3048, and 0.6096 m/s². The results show that the minimum time to rendezvous for a max thrust acceleration of 0.1524 m/s² is 735 s, the minimum time to rendezvous for a max thrust acceleration of 0.3048 m/s² is 553 s, and the minimum time to rendezvous for a max thrust acceleration of 0.6096 m/s² is 412 s. As the max thrust acceleration increases, the minimum time to rendezvous decreases, while the expended ΔV increases. Therefore, with additional thrust, one can rendezvous in a shorter time, but there is an increased ΔV cost for doing so. This fact can be seen in Figure 6.

To further look into the time-optimal case, it is beneficial to plot the behavior of the state and control variables over the course of the rendezvous. Here, we refer to a max thrust acceleration of 0.3048 m/s². Figure 6 shows the separation distance components \( x_1, x_2, x_3 \) during the rendezvous, while Figure 7 separation velocity components \( x_4, x_5, x_6 \). Figure 8 shows the behavior of the controls \( u_1, u_2, u_3 \) during the rendezvous while Figure 9 shows a plot of the total thrust acceleration. Finally, Figure 10 demonstrates the in-plane projection of the thrust vector during the rendezvous maneuver and

For the time-optimal case, a first observation is evident from Figures 8 - 9. The time-optimal rendezvous requires a two-subarc solution for the optimal thrust distribution with constant max thrust magnitude over the duration of the rendezvous. In the first-subarc, the thrust action is accelerating; in the second subarc, the thrust action is braking. Furthermore, when comparing the time-optimal results for the various values of
the max thrust acceleration, the max thrust magnitude result is the same regardless of the available thrust acceleration. Another finding regarding the application of thrust can be seen in Figure 8. The first portion of the thrusting period at the beginning of the rendezvous maneuver seems to establish the chaser and target spacecraft on a collision course. Also at the start of the maneuver, a small component of the total acceleration is applied to begin removing the out-of-plane initial separation distance. However, approximately midway through the rendezvous, the thrust direction is switched and the remainder of the thrusting period is utilized to null the velocities as the time of closure is approached. This same two-subarc trend was observed for all values of the maximum thrust acceleration.

4.2 Minimum Fuel, Time Free. For the problem in Figure 4, SGRA was run again for the same three values of the max thrust acceleration and a much different trend was discovered. The fuel-optimal rendezvous results in a four-subarc solution for the optimal thrust distribution. It was observed that the minimum ΔV required to rendezvous is nearly constant and independent of the max thrust acceleration (Table 2). One interesting attribute is that, as the max thrust acceleration is increased, although the minimum fuel usage approaches asymptotically a minimum value of 44.80 m/s, the time to rendezvous decreases with increased thrust (just as in the time-optimal case, albeit at a weaker rate). The behavior of the state and control variables is shown in Figures 11-15.

Compared to the minimum time case, a different trend was observed for the minimum fuel rendezvous. The optimal thrust distribution is now off-on-off-on in character with thrust magnitude either at the maximum value or zero as seen in Figure
14. To explain the necessity of the first coasting subarc for the fuel-optimal case, it is helpful to consider the natural unforced relative motion of the chaser spacecraft (Figure 16). For the initial conditions of the problem, it is seen that the radial separation velocity at the initial point in negative. This means that, not only the chaser spacecraft is below the target spacecraft, but is moving further away from the target spacecraft. However, as Figure 16 shows, the radial separation velocity vanishes at approximately 230 s, and then the chaser begins to move toward the target spacecraft. Indeed, the first thrusting subarc begins at this point.

Unlike the time-optimal case, there is a significant coasting period between the thrusting periods. Furthermore, the coasting periods are each finite in duration and again their lengths increase as the max allowable thrust increases. For the minimum fuel problem, the thrust acceleration components follow the same trend at the beginning and end of the rendezvous as for the minimum time problem. The first thrusting period is applied near the beginning of the rendezvous and seems to establish the chaser and target spacecraft on a collision course. Also during the first thrusting period, a small component of the total thrust acceleration is applied to begin removing the out-of-plane separation distance. Then, after a long period of coasting, a second thrusting period is required to null the velocities as the time of closure is approached.

4.3 Minimum Time, Fuel Given. For the problem in Figure 5, SGRA was run again for the same three values of the max thrust acceleration. The available ΔV for each rendezvous was given and the time-optimal rendezvous was determined with SGRA.
Summary results can be seen in Table 3 and Figure 17. Clearly, for a fixed amount of propellant, an increase in the max available thrust results in a shorter time to rendezvous.

For a max thrust acceleration of 0.3048 m/s$^2$, Figure 18 shows the time history of the fuel expended $\Delta V(t)$ for various fixed amounts of the final fuel expended $\Delta V(1)$. As the fixed amount $\Delta V(1)$ is increased from the minimum fuel value to the value associated with the time-optimal case, the coasting time decreases tending to zero for the time-optimal case.

Once again, the state and control variables were plotted for the various cases of Problem (P3) that were run with SGRA. Since there is not adequate room in this paper to present the plots for each case, Figures 19-23 show the results for a max thrust acceleration of 0.3048 m/sec$^2$ and a given $\Delta V(1)$ of 90 m/s for which a three-subarc solution was obtained. The optimal thrust distribution is on-off-on with thrust magnitude either at the maximum value or zero as seen in Figure 22. In the initial thrusting period, thrust was applied in a direction to send the chaser vehicle toward the target such that, at the time of rendezvous, they might collide. Then, in the final thrusting period, the thrust components were switched in the opposite direction to arrest the motion and null the relative velocities between the target and chaser. Regarding the coasting period, it can be seen that its duration is in between that of the fuel-optimal problem and that of the time-optimal problem.
5. Constant-Control Guidance Implementation

Up to this point, we considered the three-dimensional rendezvous between a target spacecraft in a circular orbit and a chaser spacecraft with an initial separation distance and separation velocity. We have employed the Clohessy-Wiltshire equations describing the relative motion of the chaser spacecraft vis-à-vis the target spacecraft. We assumed that the trajectory of the chaser spacecraft is governed by three controls, one determining the thrust magnitude and two determining the thrust direction. Within the confines of the above model, we employed the single-subarc version of the sequential gradient-restoration algorithm to produce continuous optimum trajectories for four problems: (P1) minimum time, fuel free; (P2) minimum fuel, time free; (P3) minimum time, fuel given; (P4) minimum fuel, time given. The achieved continuous solutions are characterized by two, three, or four subarcs depending on the performance index (time, fuel) and the constraints.

At this point, we employ the multiple-subarc version of SGRA to produce pieced guidance trajectories implementable in real time via constant control components. To be specific, we now force the controls to behave as parameters in each subarc. We consider again Problems P1 to P4 in light of the previous results and under the following scenario.

Problem P1. We study a two-subarc solution, each subarc with constant controls: a max thrust accelerating subarc followed by a max thrust braking subarc.

Problem P2. We study a four-subarc solution, each subarc with constant controls: an initial coasting subarc, followed by a max thrust accelerating subarc, followed by another coasting subarc, followed by a max thrust braking subarc.
Problems P3 and P4. We study two-subarc solutions, three-subarc solutions, and four-subarc solutions, depending on the performance index and the constraints, albeit with constant controls in each subarc.

As the subsequent analysis shows, for Problems P1-P4 the pieced guidance trajectories of this multiple-subarc SGRA approach approximate well the continuous optimal trajectories presented previously.

5.1 Multiple-Subarc SGRA Algorithm. The multiple-subarc SGRA is an important extension of the single-subarc SGRA. While the single-subarc SGRA deals with the optimization of a single system with initial and final boundary conditions, the multiple-subarc SGRA deals with the optimization of multiple systems with initial, final, and inner boundary conditions. In the multiple-subarc SGRA (see Refs. 9-10), a large and complicated overall system is decomposed into several subsystems along the time domain: each subsystem, having relatively simple properties, corresponds to a subarc; the connection between consecutive subsystems takes place via the inner boundary conditions.

In the application of the multiple-subarc SGRA, the actual time is denoted by $\rho$, the total number of subarcs is denoted by $s$, and the actual time length at each subarc is denoted by $\tau_i$, $i = 1, 2, \ldots, s$. Then, the end times of all the subarcs are given by the recurrence relation

$$\rho_i = \rho_{i-1} + \tau_i, \quad i = 1, 2, \ldots, s,$$

(53a)

in which we set

$$\rho_0 = 0.$$

(53b)
We denote by $t$ the dimensionless virtual time, which is such that the time length of each subarc is normalized to 1. For a specified subarc $i$, the direct transformation from actual time to virtual time can be written as

$$t = (\rho - \rho_{i-1}) / \tau_i, \quad i = 1, 2, \ldots, s; \quad (54a)$$

the corresponding inverse transformation from virtual time to actual time can be written as

$$\rho = \rho_{i-1} + \tau_i t, \quad i = 1, 2, \ldots, s. \quad (54b)$$

Then, with reference to the virtual time domain, the Bolza-Pontryagin optimization problem is formulated as follows: minimize the functional

$$I = \sum_{i=1}^{s} \int_{0}^{1} f(x, u, \pi, t, i) \, dt + g(y, \pi), \quad (55)$$

with respect to the $n$-dimensional state vectors $x(t, i)$, the $m$-dimensional control vectors $u(t, i)$, and the $p$-dimensional parameter vector $\pi$ which satisfy the $n$-dimensional differential constraints

$$x' = \phi(x, u, \pi, t, i), \quad 0 \leq t \leq 1, \quad i = 1, 2, \ldots, s, \quad (56a)$$

plus a general $q$-dimensional boundary condition

$$\psi(y, \pi) = 0, \quad q \leq 2ns, \quad (56b)$$

incorporating the initial conditions, final conditions, and inner boundary conditions. In the above relations, $s$ is total number of subarcs, $i$ is the index identifying a particular subarc, and the $2ns$-vector $y$ is given by

$$y = [x^T(0,i), x^T(1,i), \ldots, i = 1, 2, \ldots, s]^T. \quad (56c)$$

Clearly, the vector $y$ includes the initial and final state vectors of all the subarcs. The time scaling factors $\tau_i$, characterizing the mapping from the virtual time to the actual
time, can be either fixed or free, depending on the optimization problem to be solved. In the latter case, the time scaling factors $\tau_i$ can be treated as elements of the unknown parameter vector $\pi$.

In the problem (55)-(56), the independent variable is the normalized time $t, 0 \leq t \leq 1$; the prime denotes derivative with respect to the normalized time; $f, g$ are scalar functions, $\phi$ is an $n$-vector function, and $\psi$ is a $q$-vector function. The dependent variables are the $n$-vector state $x(t, i)$, the $m$-vector control $u(t, i)$, and the $p$-vector parameter $\pi$. Any trajectory satisfying the constraints (56) is called a feasible trajectory; among the infinite number of feasible trajectories, we seek the special trajectory which minimizes the functional (55).

With the sequential gradient-restoration algorithm, a series of linear multipoint boundary-value problems is solved iteratively using the method of particular solutions (Ref. 8) to obtain feasible solutions and ultimately a locally optimal one. The SGRA algorithm involves a cyclical scheme whereby first the constraints (56) are satisfied to a prescribed accuracy (restoration phase); then, using a first-order gradient method, a step is taken in the optimal direction to improve the performance index (gradient phase). Note that, in this implementation, Eqs. (56) are satisfied at the end of each restoration phase. The convergence tolerances for each phase are expressed as

$$P \leq \zeta_1$$  \hspace{1cm} (57a)

for the restoration phase and
\[ Q \leq \zeta_2 \] \quad (57b)

for the gradient phase, where \( P \) is the norm squared of the error in the constraints, \( Q \) is the norm squared of the error in the optimality conditions, and \( \zeta_1, \zeta_2 \) are small preselected constants.

The term “sequential” is most appropriate for this algorithm due to the successive application of either a gradient iteration or a restorative iteration depending on the values of the scalar quantities \( P \) and \( Q \). As before, three cases can occur:

\begin{itemize}
  \item [(C1)] If \( P > \zeta_1 \), SGRA executes a restorative iteration leading to the decrease of the constraint error.

  \item [(C2)] If \( P \leq \zeta_1 \) but \( Q > \zeta_2 \), SGRA executes a gradient iteration leading to the decrease of the so-called augmented functional (original functional augmented by the constraints weighted via appropriate Lagrange multipliers).

  \item [(C3)] If \( P \leq \zeta_1 \) and \( Q \leq \zeta_2 \), convergence is declared and the algorithm stops.
\end{itemize}

\textbf{5.2 Transformed System.} The intent of this section is to generate guidance trajectories implementable in real time. This is done by replacing the single-subarc variable-control trajectories considered previously with multiple-subarc constant-control trajectories. This means that, in each subarc, the auxiliary controls \( \alpha, \theta, \phi \) are kept constant, so that the original controls \( u_1, u_2, u_3 \) are also kept constant.
Let $s$ denote the total number of subarcs and let $i$ denote the index identifying a particular subarc. Let $\tau_i$, with $i = 1, \ldots, s$, denote the actual time length of each subarc. With this understanding, use of the transformation (54), normalizing the time length of each subarc to 1, allows us to rewrite the system in the following form:

\[
x'_1 = \tau_i x_4, \quad i = 1, \ldots, s, \tag{58a}
\]
\[
x'_2 = \tau_i x_5, \quad i = 1, \ldots, s, \tag{58b}
\]
\[
x'_3 = \tau_i x_6, \quad i = 1, \ldots, s, \tag{58c}
\]
\[
x'_4 = \tau_i (2\omega x_5 + u_1), \quad i = 1, \ldots, s, \tag{58d}
\]
\[
x'_5 = \tau_i (-2\omega x_4 + 3\omega^2 x_2 + u_2), \quad i = 1, \ldots, s, \tag{58e}
\]
\[
x'_6 = \tau_i (-\omega^2 x_3 + u_3), \quad i = 1, \ldots, s, \tag{58f}
\]
\[
x'_7 = \tau_i \sigma, \quad i = 1, \ldots, s. \tag{58g}
\]

In Eqs. (58), the prime superscript denotes derivative wrt the normalized time $t$, $0 \leq t \leq 1$.

We recall that the thrust acceleration $\sigma$ and its components $u_1, u_2, u_3$ are given by

\[
\sigma = (1/2) \sigma_{\max} (1 + \sin \alpha), \tag{59a}
\]
\[
u_1 = \sigma \cos \theta \cos \phi, \tag{59b}
\]
\[
u_2 = \sigma \cos \theta \sin \phi, \tag{59c}
\]
\[
u_3 = \sigma \sin \theta. \tag{59d}
\]
We seek pieced solution of the system (58) made up of the combination of coasting
subarcs and max thrust subarc. A coasting subarc is obtained by setting $\alpha = -\pi/2$ in
Eqs. (59), which become

\begin{align}
\sigma &= 0, \tag{60a} \\
u_1 &= 0, \tag{60b} \\
u_2 &= 0, \tag{60c} \\
u_3 &= 0. \tag{60d}
\end{align}

A max thrust subarc is obtained by setting $\alpha = \pi/2$ in Eqs. (59), which become

\begin{align}
\sigma &= \sigma_{\text{max}}, \tag{61a} \\
u_1 &= \sigma_{\text{max}} \cos \theta \cos \phi, \tag{61b} \\
u_2 &= \sigma_{\text{max}} \cos \theta \sin \phi, \tag{61c} \\
u_3 &= \sigma_{\text{max}} \sin \theta. \tag{61d}
\end{align}

**5.3. Boundary Conditions.** We again consider the rendezvous between a target
spacecraft and a chaser spacecraft under the following scenario. The orbit of the target
spacecraft is circular and is located at the Space Station altitude ($h = 390$ km, $r = 6768$
km); its eccentricity is 0.00 and its orbital inclination is 0.00 deg. The orbit of the chaser
spacecraft is elliptic ($373 \times 500$ km); its eccentricity is 0.0093 and its orbital inclination is
0.0789 deg.
For the chaser spacecraft, the initial conditions are

\[ x_1(0,1) = 30,480 \text{ m (downrange separation distance)}, \]  
\[ x_2(0,1) = -15,240 \text{ m (radial separation distance)}, \]  
\[ x_3(0,1) = 7,620 \text{ m (out-of-plane separation distance)}, \]  
\[ x_4(0,1) = -60.96 \text{ m/s (downrange separation velocity)}, \]  
\[ x_5(0,1) = -15.24 \text{ m/s (radial separation velocity)}, \]  
\[ x_6(0,1) = -6.10 \text{ m/s (out-of-plane separation velocity)}, \]  
\[ x_7(0,1) = 0.00 \text{ m/s (\Delta V expended)}. \]

Clearly, vis-à-vis the target spacecraft, the chaser spacecraft is initially behind [see (62a)], below [see (62b)], and to the left [see (62c)].

Also for the chaser spacecraft, the desired final conditions are

\[ x_1(1, s) = 0 \text{ m}, \]  
\[ x_2(1, s) = 0 \text{ m}, \]  
\[ x_3(1, s) = 0 \text{ m}, \]  
\[ x_4(1, s) = 0 \text{ m/s}, \]  
\[ x_5(1, s) = 0 \text{ m/s}, \]  
\[ x_6(1, s) = 0 \text{ m/s}, \]  
\[ x_7(1, s) = \text{free or given (m/s)}, \]

with \( s = 2,3,4 \) depending on the particular problem being solved.

Finally, at the interface between any two contiguous subarcs, the following continuity conditions must be satisfied:
\[ x_1(i, i) = x_1(0, i+1), \quad i = 1, \ldots, s-1, \quad (64a) \]
\[ x_2(i, i) = x_2(0, i+1), \quad i = 1, \ldots, s-1, \quad (64b) \]
\[ x_3(i, i) = x_3(0, i+1), \quad i = 1, \ldots, s-1, \quad (64c) \]
\[ x_4(i, i) = x_4(0, i+1), \quad i = 1, \ldots, s-1, \quad (64d) \]
\[ x_5(i, i) = x_5(0, i+1), \quad i = 1, \ldots, s-1, \quad (64e) \]
\[ x_6(i, i) = x_6(0, i+1), \quad i = 1, \ldots, s-1, \quad (64f) \]
\[ x_7(i, i) = x_7(0, i+1), \quad i = 1, \ldots, s-1. \quad (64g) \]

5.4. Optimization Problems. We intend to generate implementable, pieced, constant-control guidance trajectories by solving four basic optimization problems:

(P1) Minimum Time, Fuel Free. The performance index is
\[ I = \tau = \tau_1 + \tau_2 + \ldots + \tau_s, \quad (65a) \]
where \( \tau \) is the final time and \( \tau_1, \tau_2, \ldots, \tau_s \) are the actual time lengths of the component subarcs. In this problem, the cumulative velocity change is
\[ x_7(1, s) = \Delta V(1, s) = \text{free}. \quad (65b) \]

(P2) Minimum Fuel, Time Free. The performance index is
\[ I = x_7(1, s) = \Delta V(1, s). \quad (66a) \]
In this problem, the final time is
\[ \tau = \tau_1 + \tau_2 + \ldots + \tau_s = \text{free}. \quad (66b) \]

(P3) Minimum Time, Fuel Given. The performance index is
\[ I = \tau = \tau_1 + \tau_2 + \ldots + \tau_s. \quad (67a) \]
In this problem, the cumulative velocity change is
\[ x_7(1, s) = \Delta V(1, s) = \text{given} \] \hspace{1cm} (67b)

(P4) Minimum Fuel, Time Given. The performance index is

\[ I = x_7(1, s) = \Delta V(1, s) \] \hspace{1cm} (68a)

In this problem the final time is

\[ \tau = \tau_1 + \tau_2 + \ldots + \tau_s = \text{given} \] \hspace{1cm} (68b)

Problems (P1)-(P4) are of the Mayer type, since the line integral in Eq. (55) of the Bolza-Pontryagin problem is missing. Because both the original controls \( u_1, u_2, u_3 \) and the auxiliary controls are kept constant along each subarc, Problems (P1)-(P4) involve only state variables and parameters (the time intervals \( \tau_i \) plus the parameters \( \alpha_i, \theta_i, \phi_i \) determining the magnitude and direction of the thrust).

If the above parameters are chosen so that the solutions of the differential equations (58) are consistent with the boundary conditions (62)-(64), this means that the resulting trajectory is feasible. Among the infinite number of feasible trajectories, we seek the particular feasible trajectory which minimizes the performance index \( I \) in one of the forms (65)-(67).
5.4.1. Minimum Time, Fuel Free. The results from the single-subarc SGRA indicate that the continuous solution of Problem P1 can be approximated by a two-subarc, pieced, constant-control solution: a max thrust accelerating subarc 1 and a max thrust braking subarc 2. Since

\[ \alpha_1 = \pi/2, \quad \sigma_1 = \sigma_{\text{max}}, \quad \text{subarc 1}, \]  
(69a)

\[ \alpha_2 = \pi/2, \quad \sigma_2 = \sigma_{\text{max}}, \quad \text{subarc 2}, \]  
(69b)

we are left with the following free parameters:

\[ \tau_1, \theta_1, \phi_1, \quad \text{subarc 1}, \]  
(70a)

\[ \tau_2, \theta_2, \phi_2, \quad \text{subarc 2}. \]  
(70b)

Clearly, the dimension of the parameter vector \( \pi \) is \( p = 6 \) while the dimension of the vector of final conditions is \( q = 6 \). Not only the optimal control problem degenerates into a mathematical programming problem, but the number of degrees of freedom is \( p - q = 0 \). Thus, Problem P1 is no longer an optimization problem but a feasibility problem. This means that a very precise solution can be obtained at the end of the first restoration phase of the multiple-subarc sequential gradient-restoration algorithm (SGRA).

The results obtained for a max thrust acceleration \( \sigma_{\text{max}} = 0.3048 \text{ m/s}^2 \) can be seen in Tables 4–5 and Figure 24-27. Table 4 shows the values of the parameters and constant controls of the problem. Table 5 presents the major quantities of the problem, namely: transfer angle, time to rendezvous, propellant mass ratio, and cumulative velocity change.
Also, Table 5 presents a comparison of the discretized solutions obtained from the multiple-subarc SGRA and the continuous solutions presented previously. The agreement is indeed excellent.

Figures 24-27 refer to a max thrust acceleration $\sigma_{\text{max}} = 0.3048 \text{ m/s}^2$. Figure 24 shows the separation coordinate $x_1, x_2, x_3$ during the rendezvous, while Figure 25 shows the separation velocities $x_4, x_5, x_6$. Figure 26 shows the constant controls $u_1, u_2, u_3$ during the rendezvous, while Figure 27 shows a plot of the total thrust acceleration $\sigma$. Even though the continuous controls have been replaced by discretized controls, the qualitative agreement between the separation coordinates and separation velocities with their continuous counterparts is excellent.

**Remarks.** In Tables 5, 7, 9 the term “transfer angle” refers to the angular travel of the target spacecraft during the rendezvous maneuver of the chaser spacecraft. Therefore, it equals the product of the angular velocity of the target spacecraft $\omega$ and the rendezvous time $\tau$. The term “propellant mass ratio” refers to the chaser spacecraft and is computed with the formula

$$m_p / m_0 = 1 - \exp[-\Delta V(1, s)/g_{\text{ref}} I_{sp}], \quad (71a)$$

in which $m_p$ is the propellant mass consumed, $m_0$ is the initial mass of the chaser spacecraft, $g_{\text{ref}}$ is a reference acceleration of gravity (acceleration of gravity at sea level on Earth), and $I_{sp}$ is the specific impulse of the rocket engine. In the computation, the following values are assumed:
\[ g_{\text{ref}} = 9.81 \text{ m/s}^2, \quad I_{sp} = 300 \text{ s}. \] (71b)

5.4.2. Minimum Fuel, Time Free. The results of the single-subarc SGRA indicate that the continuous solution of Problem P2 can be approximated by a four-subarc, pieced, constant-control solution: a coasting subarc 1, followed by a max thrust accelerating subarc 2, followed by another coasting subarc 3, followed by a max thrust braking subarc 4. Since

\[ \alpha_1 = -\pi/2, \quad \sigma_1 = 0, \quad \text{subarc 1}, \] (72a)

\[ \alpha_2 = \pi/2, \quad \sigma_2 = \sigma_{\text{max}}, \quad \text{subarc 2}. \] (72b)

\[ \alpha_3 = -\pi/2, \quad \sigma_3 = 0, \quad \text{subarc 3}, \] (72c)

\[ \alpha_4 = \pi/2, \quad \sigma_4 = \sigma_{\text{max}}, \quad \text{subarc 4}, \] (72d)

we are left with the following free parameters:

\[ \tau_1, \quad \text{subarc 1}, \] (73a)

\[ \tau_2, \theta_2, \phi_2, \quad \text{subarc 2}, \] (73b)

\[ \tau_3, \quad \text{subarc 3}, \] (73c)

\[ \tau_4, \theta_4, \phi_4, \quad \text{subarc 4}. \] (73d)

Since \( \sigma_1 = 0 \) and \( \sigma_3 = 0 \), the original controls take the values \( u_{11} = u_{21} = u_{31} = 0 \) and \( u_{13} = u_{23} = u_{33} = 0 \) regardless of the values assigned to \( \theta_1, \phi_1, \theta_3, \) and \( \phi_3 \). Clearly, the dimension of
the parameter vector \( \pi \) is \( p = 8 \), while the dimension of the vector of final conditions is \( q = 6 \). Thus, the optimal control problem degenerates into a mathematical programming problem with a number of degrees of freedom \( p - q = 2 \).

The results obtained for a max thrust acceleration \( \sigma_{\text{max}} = 0.3048 \text{ m/s}^2 \) can be seen in Tables 6-7 and Figures 28-31. Table 6 shows the values of the parameters and constant controls of the problem. Table 7 presents the major quantities of the problem, namely: transfer angle, time to rendezvous, propellant mass ratio, and cumulative velocity change. Also, Table 7 presents a comparison of the discretized solutions of this paper and the continuous solutions of the single-subarc SGRA. The agreement is indeed excellent.

Figures 28-31 refer to a max thrust acceleration \( \sigma_{\text{max}} = 0.3048 \text{ m/s}^2 \). Figure 28 shows the separation coordinate \( x_1, x_2, x_3 \) during the rendezvous, while Figure 29 shows the separation velocities \( x_4, x_5, x_6 \). Figure 8 shows the constant controls \( u_1, u_2, u_3 \) during the rendezvous, while Figure 31 shows a plot of the total thrust acceleration \( \sigma \). Even tough the continuous controls of the single-subarc SGRA have been replaced by discretized controls, the qualitative agreement between the separation coordinates and separation velocities with their multiple-subarc SGRA counterparts is excellent.
5.4.3. Minimum Time, Fuel Given. The results of the single-subarc SGRA indicate that the continuous solutions of Problem P3 can be approximated by either a two-subarc solution [on-on thrust], or a three-subarc solution [on-off-on thrust], or a four-subarc solution [off-on-off-on thrust] depending on the assumed value of the max thrust acceleration and the fuel expended. To fix the ideas, let us assume that the max thrust acceleration is $\sigma_{\text{max}} = 0.3048 \text{ m/s}^2$ and that the fuel expended is $\Delta V(1, s) = 90 \text{ m/s}$. For this particular combination, the previous results indicate that the continuous solution of Problem P3 can be approximated by a three-subarc pieced constant-control solution: a max thrust accelerating subarc 1, followed by a coasting subarc 2, followed by a max thrust braking subarc 3.

Since

$$\alpha_1 = \pi/2, \quad \sigma_1 = \sigma_{\text{max}}, \quad \text{subarc 1,} \quad (74a)$$

$$\alpha_2 = -\pi/2, \quad \sigma_2 = 0, \quad \text{subarc 2,} \quad (74b)$$

$$\alpha_3 = \pi/2, \quad \sigma_3 = \sigma_{\text{max}}, \quad \text{subarc 3,} \quad (74c)$$

we are left with the following free parameters:

$$\tau_1, \theta_1, \phi_1, \quad \text{subarc 1,} \quad (75a)$$

$$\tau_2, \quad \text{subarc 2,} \quad (75b)$$

$$\tau_3, \theta_3, \phi_3, \quad \text{subarc 3,} \quad (75c)$$
Since $\sigma_2 = 0$, the original controls take the values $u_{12} = u_{22} = u_{32} = 0$, regardless of the values assigned to $\theta_2$ and $\phi_2$. Clearly, the dimension of the parameter vector $\pi$ is $p = 7$ and the dimension of the vector of final conditions is $q = 7$. Not only the optimal control problem degenerates into a mathematical programming problem, but the numbers of degree of freedom is $p - q = 0$. This means that a very precise solution can be obtained at the end of the first restoration phase of the multiple-subarc sequential gradient-restoration algorithm (SGRA).

The results obtained for a max thrust acceleration $\sigma_{\text{max}} = 0.3048 \text{ m/s}^2$ and a cumulative velocity change $\Delta V(1, s) = 90 \text{ m/s}$ can be seen in Tables 8-9 and Figures 32-35. Table 8 shows the values of the parameters and constant controls of the problem. Table 9 presents the major quantities of the problem, namely: transfer angle, time to rendezvous, propellant mass ratio, and cumulative velocity change. Also, Table 9 presents a comparison of the discretized solutions of this paper and the continuous solutions presented previously. The agreement is indeed excellent.

Figures 32-35 refer to a max thrust acceleration $\sigma_{\text{max}} = 0.3048 \text{ m/s}^2$ and cumulative velocity change $\Delta V(1, s) = 90 \text{ m/s}$. Figure 32 shows the separation coordinate $x_1$, $x_2$, $x_3$ during the rendezvous, while Figure 33 shows the separation velocities $x_4$, $x_5$, $x_6$. Figure 34 shows the constant controls $u_1$, $u_2$, $u_3$ during the rendezvous, while Figure 35 shows a plot of the total thrust acceleration $\sigma$. Even tough the continuous controls of the single-subarc SGRA have been replaced by discontinuous controls, the qualitative agreement between the separation coordinates and separation velocities with their previous counterparts is excellent.
6. Conclusions

This thesis has discussed the successful utilization of the single-subarc and multiple-subarc SGRA algorithm to solve the time-optimal and fuel-optimal rendezvous problems under the assumptions of a target spacecraft in a circular ISS orbit and a chaser spacecraft with typical initial conditions during the terminal phase of a rendezvous with bounded thrust and bounded ΔV.

The results of the optimization investigation using the single-subarc SGRA and continuously variable controls are as follows:

(i) For the time-optimal rendezvous, fuel free, a two-subarc bang-bang solution is obtained: a max thrust accelerating subarc is followed by a max thrust braking subarc. During each subarc, regardless of the initial conditions chosen, the control magnitude is constant although the control components continuously change.

(ii) For the fuel-optimal rendezvous, time free, a four-subarc zero-bang-zero-bang solution is obtained for the typical initial conditions chosen: an initial coasting subarc is followed by a max thrust accelerating subarc, which is followed by another coasting subarc, which is followed by a max thrust braking subarc. Again, during each powered subarc, the control magnitude is constant although the control components continuously change.

(iii) For the time-optimal rendezvous, fuel given, the solution is characterized by a two-subarc bang-bang solution, or a three-subarc bang-zero-bang solution, or a four-subarc zero-bang-zero-bang solution, depending on the max thrust acceleration and the
assumed value of the fuel expended. Again, during each powered subarc the control magnitude is constant although the control components continuously change.

The results of the guidance investigation using the multiple-subarc SGRA and constant control components during each subarc are as follows:

(iv) For the time-optimal rendezvous, fuel free, a two-subarc bang-bang solution is obtained: a max thrust accelerating subarc is followed by a max thrust braking subarc, with control components constant in each subarc.

(v) For the fuel-optimal rendezvous, time free, a four-subarc zero-bang-zero-bang solution is obtained: an initial coasting subarc is followed by a max thrust accelerating subarc, which is followed by another coasting subarc, which is followed by a max thrust braking subarc, with control components constant in each subarc.

(vi) For the time-optimal rendezvous, fuel given, the solution is characterized by a two-subarc bang-bang solution, or a three-subarc bang-zero-bang solution, or a four-subarc zero-bang-zero-bang solution, depending on the assumed values of the max thrust acceleration and the fuel expended. In all cases, the powered subarcs are characterized by control components constant in each subarc.

While it is difficult to implement in real-time continuously variable controls, it is simple to implement in real-time controls which are constant in each subarc. Also, comparison of the performance indexes obtained with continuously variable controls with the performance indexes obtained with controls which are constant in each subarc shows that the relative differences are within a fraction of one percent for all cases studied. For this reason, the author believes that the results represented by items (iv) to
(vi) are eminently suitable for flight implementation in an autonomous guidance algorithm.

To sum up, the methodology presented in this thesis offers an obvious advantage as the basis for an automated rendezvous and docking guidance algorithm over the current glideslope approach for two reasons: (a) the reduction in the total number of maneuvers required to rendezvous and (b) the associated reduction in the fuel required.
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Figure 28. Problem (P2). Separation distance vs. time for fuel-optimal rendezvous.
Figure 29. Problem (P2). Separation velocity vs. time for fuel-optimal rendezvous.

Figure 30. Problem (P2). Thrust acceleration components vs. time for fuel-optimal rendezvous.

Figure 31. Problem (P2). Total thrust acceleration vs. time for fuel-optimal rendezvous.

Figure 32. Problem (P3). Separation distance vs. time for time-optimal rendezvous.

Figure 33. Problem (P3). Separation velocity vs. time for time-optimal rendezvous.

Figure 34. Problem (P3). Thrust acceleration components vs. time for time-optimal rendezvous.

Figure 35. Problem (P3). Total thrust acceleration vs. time for time-optimal rendezvous.
Table 1. Minimum time rendezvous, fuel free.

<table>
<thead>
<tr>
<th>$\sigma_{\text{max}}$ [m/s²]</th>
<th>Transfer angle [rad]</th>
<th>Time to rendezvous [s]</th>
<th>Propellant mass ratio</th>
<th>$\Delta V(1)$ [m/s]</th>
<th>Number of subarcs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1524</td>
<td>0.83</td>
<td>735.0</td>
<td>0.0373</td>
<td>112.0</td>
<td>2</td>
</tr>
<tr>
<td>0.3048</td>
<td>0.63</td>
<td>553.3</td>
<td>0.0557</td>
<td>168.6</td>
<td>2</td>
</tr>
<tr>
<td>0.6096</td>
<td>0.47</td>
<td>412.3</td>
<td>0.0819</td>
<td>251.3</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 2. Minimum fuel rendezvous, time free.

<table>
<thead>
<tr>
<th>$\sigma_{\text{max}}$ [m/s²]</th>
<th>Transfer angle [rad]</th>
<th>Time to rendezvous [s]</th>
<th>Propellant mass ratio</th>
<th>$\Delta V(1)$ [m/s]</th>
<th>Number of subarcs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1524</td>
<td>2.19</td>
<td>1932</td>
<td>0.0152</td>
<td>44.9</td>
<td>4</td>
</tr>
<tr>
<td>0.3048</td>
<td>2.18</td>
<td>1920</td>
<td>0.0151</td>
<td>44.8</td>
<td>4</td>
</tr>
<tr>
<td>0.6096</td>
<td>2.17</td>
<td>1914</td>
<td>0.0151</td>
<td>44.8</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 3a. Minimum time rendezvous, fuel given, $\sigma_{\text{max}} = 0.1524$ m/s².

<table>
<thead>
<tr>
<th>Transfer angle [rad]</th>
<th>Time to rendezvous [s]</th>
<th>Propellant mass ratio</th>
<th>$\Delta V(1)$ [m/s]</th>
<th>Number of subarcs</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.83</td>
<td>735.0</td>
<td>0.0373</td>
<td>112.0</td>
<td>2</td>
<td>Absolute min time</td>
</tr>
<tr>
<td>0.84</td>
<td>742.9</td>
<td>0.0334</td>
<td>100.0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>0.86</td>
<td>755.0</td>
<td>0.0301</td>
<td>90.0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>0.93</td>
<td>823.2</td>
<td>0.0235</td>
<td>70.0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1.06</td>
<td>935.8</td>
<td>0.0202</td>
<td>60.0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1.21</td>
<td>1066</td>
<td>0.0185</td>
<td>55.0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1.50</td>
<td>1322</td>
<td>0.0168</td>
<td>50.0</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2.19</td>
<td>1932</td>
<td>0.0152</td>
<td>44.9</td>
<td>4</td>
<td>Absolute min fuel</td>
</tr>
</tbody>
</table>

The propellant mass ratios in Tables 1-3 refer to a chemical engine with a specific impulse of 300 s.
Table 3b. Minimum time rendezvous, fuel given, $\sigma_{\max} = 0.3048 \, \text{m/s}^2$.

<table>
<thead>
<tr>
<th>Transfer angle [rad]</th>
<th>Time to rendezvous [s]</th>
<th>Propellant mass ratio</th>
<th>$\Delta V(1) , \text{[m/s]}$</th>
<th>Number of subarcs</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.63</td>
<td>553.3</td>
<td>0.0557</td>
<td>168.6</td>
<td>2</td>
<td>Absolute min time</td>
</tr>
<tr>
<td>0.63</td>
<td>558.7</td>
<td>0.0497</td>
<td>150.0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>0.65</td>
<td>569.4</td>
<td>0.0432</td>
<td>130.0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>0.72</td>
<td>637.3</td>
<td>0.0301</td>
<td>90.0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>0.97</td>
<td>854.8</td>
<td>0.0202</td>
<td>60.0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1.17</td>
<td>1030</td>
<td>0.0185</td>
<td>55.0</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>1.49</td>
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<td>0.0168</td>
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<td>4</td>
<td></td>
</tr>
<tr>
<td>2.18</td>
<td>1920</td>
<td>0.0151</td>
<td>44.8</td>
<td>4</td>
<td>Absolute min fuel</td>
</tr>
</tbody>
</table>

Table 3c. Minimum time rendezvous, fuel given, $\sigma_{\max} = 0.6096 \, \text{m/s}^2$.

<table>
<thead>
<tr>
<th>Transfer angle [rad]</th>
<th>Time to rendezvous [s]</th>
<th>Propellant mass ratio</th>
<th>$\Delta V(1) , \text{[m/s]}$</th>
<th>Number of subarcs</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.47</td>
<td>412.3</td>
<td>0.0819</td>
<td>251.3</td>
<td>2</td>
<td>Absolute min time</td>
</tr>
<tr>
<td>0.47</td>
<td>415.0</td>
<td>0.0752</td>
<td>230.0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>0.47</td>
<td>418.6</td>
<td>0.0689</td>
<td>210.0</td>
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<td></td>
</tr>
<tr>
<td>0.50</td>
<td>439.4</td>
<td>0.0561</td>
<td>170.0</td>
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</tr>
<tr>
<td>0.55</td>
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</tr>
<tr>
<td>0.65</td>
<td>577.3</td>
<td>0.0301</td>
<td>90.0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>0.93</td>
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<td>0.0202</td>
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<td></td>
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<tr>
<td>1.14</td>
<td>1005</td>
<td>0.0185</td>
<td>55.0</td>
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<tr>
<td>1.48</td>
<td>1308</td>
<td>0.0168</td>
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<td></td>
</tr>
<tr>
<td>2.17</td>
<td>1914</td>
<td>0.0151</td>
<td>44.8</td>
<td>4</td>
<td>Absolute min fuel</td>
</tr>
</tbody>
</table>

The propellant mass ratios in Tables 1-3 refer to a chemical engine with a specific impulse of 300 s.
Table 4. Problem (P1). Minimum time, fuel free, $\sigma_{\text{max}} = 0.3048$ m/s$^2$. Parameters and controls.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_i$ [rad]</th>
<th>$\theta_i$ [rad]</th>
<th>$\phi_i$ [rad]</th>
<th>$\tau_i$ [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\pi/2$</td>
<td>3.422</td>
<td>-0.884</td>
<td>202.9</td>
</tr>
<tr>
<td>2</td>
<td>$\pi/2$</td>
<td>2.902</td>
<td>2.229</td>
<td>353.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\sigma_i$ [m/s$^2$]</th>
<th>$u_{1i}$ [m/s$^2$]</th>
<th>$u_{2i}$ [m/s$^2$]</th>
<th>$u_{3i}$ [m/s$^2$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3048</td>
<td>-0.1857</td>
<td>0.2265</td>
<td>-0.0845</td>
</tr>
<tr>
<td>2</td>
<td>0.3048</td>
<td>0.1811</td>
<td>-0.2342</td>
<td>0.0723</td>
</tr>
</tbody>
</table>

Table 5. Problem (P1). Minimum time, fuel free, $\sigma_{\text{max}} = 0.3048$ m/s$^2$. Major quantities.

<table>
<thead>
<tr>
<th>Transfer angle [rad]</th>
<th>Time to rendezvous [s]</th>
<th>Propellant mass ratio</th>
<th>$\Delta V(1)$ [m/s]</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.631</td>
<td>556.6</td>
<td>0.05599</td>
<td>169.6</td>
<td>Multiple subarc</td>
</tr>
<tr>
<td>0.627</td>
<td>553.3</td>
<td>0.05569</td>
<td>168.6</td>
<td>Single Subarc</td>
</tr>
</tbody>
</table>

Table 6. Problem (P2). Minimum fuel, time free, $\sigma_{\text{max}} = 0.3048$ m/s$^2$. Parameters and controls.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_i$ [rad]</th>
<th>$\theta_i$ [rad]</th>
<th>$\phi_i$ [rad]</th>
<th>$\tau_i$ [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\pi/2$</td>
<td>0</td>
<td>0</td>
<td>232.3</td>
</tr>
<tr>
<td>2</td>
<td>$\pi/2$</td>
<td>0.2944</td>
<td>0.1504</td>
<td>115.8</td>
</tr>
<tr>
<td>3</td>
<td>$-\pi/2$</td>
<td>0</td>
<td>0</td>
<td>1544</td>
</tr>
<tr>
<td>4</td>
<td>$\pi/2$</td>
<td>2.4028</td>
<td>0.1250</td>
<td>31.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\sigma_i$ [m/s$^2$]</th>
<th>$u_{1i}$ [m/s$^2$]</th>
<th>$u_{2i}$ [m/s$^2$]</th>
<th>$u_{3i}$ [m/s$^2$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.3048</td>
<td>0.2884</td>
<td>0.0437</td>
<td>0.0884</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0.3048</td>
<td>-0.2236</td>
<td>-0.0281</td>
<td>0.2052</td>
</tr>
</tbody>
</table>
Table 7. Problem (P2). Minimum fuel, time free, $\sigma_{\text{max}} = 0.3048 \text{ m/s}^2$.
Major quantities.

<table>
<thead>
<tr>
<th>Transfer angle [rad]</th>
<th>Time to rendezvous [s]</th>
<th>Propellant mass ratio</th>
<th>$\Delta V(1)$ [m/s]</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.18</td>
<td>1924</td>
<td>0.0151</td>
<td>44.85</td>
<td>Multiple subarc</td>
</tr>
<tr>
<td>2.17</td>
<td>1920</td>
<td>0.0151</td>
<td>44.84</td>
<td>Single subarc</td>
</tr>
</tbody>
</table>

Table 8. Problem (P3). Minimum time, fuel given, $\sigma_{\text{max}} = 0.3048 \text{ m/s}^2$, $\Delta V(1, s) = 90$ m/s.
Parameters and controls.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_i$ [rad]</th>
<th>$\theta_i$ [rad]</th>
<th>$\phi_i$ [rad]</th>
<th>$\tau_i$ [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\pi/2$</td>
<td>5.9883</td>
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<td>78.2</td>
</tr>
<tr>
<td>2</td>
<td>$-\pi/2$</td>
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<td>0.0000</td>
<td>344.4</td>
</tr>
<tr>
<td>3</td>
<td>$\pi/2$</td>
<td>0.2416</td>
<td>-1.0002</td>
<td>217.1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\sigma_i$ [m/s$^2$]</th>
<th>$u_{11}$ [m/s$^2$]</th>
<th>$u_{2i}$ [m/s$^2$]</th>
<th>$u_{3i}$ [m/s$^2$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3048</td>
<td>-0.1061</td>
<td>0.2717</td>
<td>-0.0886</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>3</td>
<td>0.3048</td>
<td>0.1598</td>
<td>-0.2491</td>
<td>0.0729</td>
</tr>
</tbody>
</table>

Table 9. Problem (P3). Minimum time, fuel given, $\sigma_{\text{max}} = 0.3048 \text{ m/s}^2$, $\Delta V(1, s) = 90$ m/s.
Major quantities.

<table>
<thead>
<tr>
<th>Transfer angle [rad]</th>
<th>Time to rendezvous [s]</th>
<th>Propellant mass ratio</th>
<th>$\Delta V(1)$ [m/s]</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.725</td>
<td>639.6</td>
<td>0.0301</td>
<td>90</td>
<td>Multiple subarc</td>
</tr>
<tr>
<td>0.722</td>
<td>637.3</td>
<td>0.0301</td>
<td>90</td>
<td>Single subarc</td>
</tr>
</tbody>
</table>
Figure 1a. Shuttle rendezvous trajectory profile

Figure 1b. Glideslope approach of the Space Shuttle during the terminal phase of rendezvous
Figure 2a. LVLH coordinate frame (two-dimensional representation).

Figure 2b. Components of thrust acceleration.
(P1) Minimize $I = \tau$

subject to the following constraints:

$x'_1 = \tau x_4$

$x'_2 = \tau x_5$

$x'_3 = \tau x_6$

$x'_4 = \tau \left( 2 \omega x_5 + u_1 \right)$

$x'_5 = \tau \left( -2 \omega x_4 + 3 \omega^2 x_2 + u_2 \right)$

$x'_6 = \tau \left( -\omega^2 x_3 + u_3 \right)$

$x'_7 = \tau \left[ \frac{\sigma_{\text{max}}}{2} (1 + \sin \alpha) \right]$

where

$u_1 = \frac{\sigma_{\text{max}}}{2} (1 + \sin \alpha) \cos \theta \cos \phi$

$u_2 = \frac{\sigma_{\text{max}}}{2} (1 + \sin \alpha) \cos \theta \sin \phi$

$u_3 = \frac{\sigma_{\text{max}}}{2} (1 + \sin \alpha) \sin \theta$

Initial Conditions

$x_1(0) = 30,480 \text{ m} = \text{separation distance in the downrange direction}$

$x_2(0) = -15,240 \text{ m} = \text{separation distance in the radial direction}$

$x_3(0) = 7,620 \text{ m} = \text{separation distance in the out-of-plane direction}$

$x_4(0) = -60.96 \text{ m/s} = \text{separation velocity in the downrange direction}$

$x_5(0) = -15.24 \text{ m/s} = \text{separation velocity in the radial direction}$

$x_6(0) = -6.10 \text{ m/s} = \text{separation velocity in the out-of-plane direction}$

$x_7(0) = 0.00 \text{ m/s} = \Delta V \text{ expended}$

Final Conditions

$x_1(1) = x_2(1) = x_3(1) = 0 \text{ m}, \quad x_4(1) = x_5(1) = x_6(1) = 0 \text{ m/s}, \quad x_7(1) = \text{free}$

Figure 3. Minimum time rendezvous problem formulation, fuel free.
(P2) Minimize $I = x_7(1)$

subject to the following constraints:

$x'_1 = \tau\ x_4$
$x'_2 = \tau\ x_5$
$x'_3 = \tau\ x_6$
$x'_4 = \tau\ (2\ \omega\ x_3 + u_1)$
$x'_5 = \tau\ (-2\ \omega\ x_4 + 3\ \omega^2 x_2 + u_2)$
$x'_6 = \tau\ (-\omega^2 x_3 + u_3)$
$x'_7 = \tau\ \left[ \frac{\sigma_{\max}}{2} (1 + \sin \alpha) \right]

where

$u_1 = \frac{\sigma_{\max}}{2} (1 + \sin \alpha) \cos \theta \cos \phi$
$u_2 = \frac{\sigma_{\max}}{2} (1 + \sin \alpha) \cos \theta \sin \phi$
$u_3 = \frac{\sigma_{\max}}{2} (1 + \sin \alpha) \sin \theta$

**Initial Conditions**

$x_1(0) = 30,480\ \text{m} = \text{separation distance in the downrange direction}$
$x_2(0) = -15,240\ \text{m} = \text{separation distance in the radial direction}$
$x_3(0) = 7,620\ \text{m} = \text{separation distance in the out-of-plane direction}$
$x_4(0) = -60.96\ \text{m/s} = \text{separation velocity in the downrange direction}$
$x_5(0) = -15.24\ \text{m/s} = \text{separation velocity in the radial direction}$
$x_6(0) = -6.10\ \text{m/s} = \text{separation velocity in the out-of-plane direction}$
$x_7(0) = 0.00\ \text{m/s} = \Delta V\ \text{expended}$

**Final Conditions**

$x_1(1) = x_2(1) = x_3(1) = 0\ \text{m}$, \quad $x_4(1) = x_5(1) = x_6(1) = 0\ \text{m/s}$, \quad $\tau = \text{free}$

Figure 4. Minimum fuel rendezvous problem formulation, time free.
(P3) Minimize \( I = \tau \)

subject to the following constraints:
\[
\begin{align*}
x_1' &= \tau x_4 \\
x_2' &= \tau x_5 \\
x_3' &= \tau x_6 \\
x_4' &= \tau (2 \omega x_5 + u_1) \\
x_5' &= \tau (-2 \omega x_4 + 3 \omega^2 x_2 + u_2) \\
x_6' &= \tau (-\omega^2 x_3 + u_3) \\
x_i' &= \tau \left[ \frac{\sigma_{\text{max}}}{2} (1 + \sin \alpha) \right]
\end{align*}
\]

where
\[
\begin{align*}
u_1 &= \frac{\sigma_{\text{max}}}{2} (1 + \sin \alpha) \cos \theta \cos \phi \\
u_2 &= \frac{\sigma_{\text{max}}}{2} (1 + \sin \alpha) \cos \theta \sin \phi \\
u_3 &= \frac{\sigma_{\text{max}}}{2} (1 + \sin \alpha) \sin \theta
\end{align*}
\]

**Initial Conditions**
- \( x_1(0) = 30,480 \) km = separation distance in the downrange direction
- \( x_2(0) = -15,240 \) km = separation distance in the radial direction
- \( x_3(0) = 7,620 \) km = separation distance in the out-of-plane direction
- \( x_4(0) = -60.96 \) m/s = separation velocity in the downrange direction
- \( x_5(0) = -15.24 \) m/s = separation velocity in the radial direction
- \( x_6(0) = -6.10 \) m/s = separation velocity in the out-of-plane direction
- \( x_7(0) = 0.00 \) m/s = \( \Delta V \) expended

**Final Conditions**
- \( x_1(1) = x_2(1) = x_3(1) = 0 \) m, \( x_4(1) = x_5(1) = x_6(1) = 0 \) m/s, \( x_7(1) \) given

Figure 5. Minimum time rendezvous problem formulation, fuel given.
Figure 6. Problem (P1). Separation distance vs. time for time-optimal rendezvous.

Figure 7. Problem (P1). Separation velocity vs. time for time-optimal rendezvous.
Figure 8. Problem (P1). Thrust acceleration components vs. time for time-optimal rendezvous.

Figure 9. Problem (P1). Total thrust acceleration vs. time for time-optimal rendezvous.
Figure 10. Problem (P1). In-plane projection of thrust vector for time-optimal rendezvous.

Figure 11. Problem (P2). Separation distance vs. time for fuel-optimal rendezvous.
Figure 12. Problem (P2). Separation velocity vs. time for fuel-optimal rendezvous.

Figure 13. Problem (P2). Thrust acceleration components vs. time for fuel-optimal rendezvous.
Figure 14. Problem (P2). Thrust acceleration components vs. time for fuel-optimal rendezvous.

Figure 15. Problem (P2). In-plane projection of thrust vector for fuel-optimal rendezvous.
Figure 16. In-plane projection of unforced relative motion of the chaser spacecraft.

Figure 17. Problem (P3). Expended $\Delta V$ vs. time to rendezvous, time-optimal rendezvous, fuel given.
Figure 18. Problem (P3). Expended $\Delta V$ vs. time for time-optimal rendezvous.

Figure 19. Problem (P3). Separation distance vs. time for time-optimal rendezvous, fuel given.
Figure 20. Problem (P3). Separation velocity vs. time for time-optimal rendezvous, fuel given.

Figure 21. Problem (P3). Thrust acceleration components vs. time for time-optimal rendezvous, fuel given.
Figure 22. Problem (P3). Total thrust acceleration vs. time for time-optimal rendezvous, fuel given.

\[
\sigma_{\text{max}} = 0.3048 \text{ m/s}^2, \quad \Delta V = 90 \text{ m/s}
\]

Figure 23. Problem (P3). In-plane projection of thrust vector for time-optimal rendezvous, fuel given.

\[
\sigma_{\text{max}} = 0.3048 \text{ m/s}^2, \quad \Delta V = 90 \text{ m/s}
\]
Figure 24. Problem (P1). Separation distance vs. time for time-optimal rendezvous.

Figure 25. Problem (P1). Separation velocity vs. time for time-optimal rendezvous.
Figure 26. Problem (P1). Thrust acceleration components vs. time for time-optimal rendezvous.

Figure 27. Problem (P1). Total thrust acceleration vs. time for time-optimal rendezvous.
Figure 28. Problem (P2). Separation distance vs. time for fuel-optimal rendezvous.

Figure 29. Problem (P2). Separation velocity vs. time for fuel-optimal rendezvous.
Figure 30. Problem (P2). Thrust acceleration components vs. time for fuel-optimal rendezvous.

Figure 31. Problem (P2). Total thrust acceleration vs. time for fuel-optimal rendezvous.
Figure 32. Problem (P3). Separation distance vs. time for time-optimal rendezvous.

Figure 33. Problem (P3). Separation velocity vs. time for time-optimal rendezvous.
Figure 34. Problem (P3). Thrust acceleration components vs. time for time-optimal rendezvous.

Figure 35. Problem (P3). Total thrust acceleration vs. time for time-optimal rendezvous.