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Optimization of Interplanetary Trajectories
to Mars via Electrical Propulsion

by

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Abstract

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Although chemical rocket propulsion is widely used in space transportation, large amounts of propellant mass limit designs for spacecraft missions to Mars. Electrical propulsion, which requires a smaller propellant load, is an alternative propulsion system that can be used for interplanetary flight. After the recent successes of the NASA Deep Space 1 spacecraft and the ESA SMART 1 spacecraft, which incorporate an electrical propulsion system, there is a strong need for trajectory tools to support these systems.

This thesis describes the optimization of interplanetary trajectories from Earth to Mars for spacecraft utilizing low-thrust electrical propulsion systems. It is assumed that the controls are the thrust direction and the thrust setting. Specifically, the minimum time and minimum propellant problems are studied and solutions are computed with the sequential gradient-restoration algorithm (SGRA).

The results indicate that, when the thrust direction and thrust setting are simultaneously optimized, the minimum time and minimum propellant solutions are not identical. For minimum time, it is found that the thrust setting must be at the maximum value; also, the thrust direction has a normal component with a switch at midcourse from upward to downward. This changes the curvature of the trajectory, has a beneficial effect
on time, but a detrimental effect on propellant mass; indeed, the propellant mass ratio of
the minimum time solution is about twice that of the Hohmann transfer solution. Thus,
the minimum time solution yields a rather inefficient trajectory. For minimum propellant
consumption, it is found that the best thrust setting is bang-zero-bang (maximum thrust,
followed by coasting, followed by maximum thrust) and that the best thrust direction is
tangent to the trajectory. This is a rather efficient trajectory; to three significant digits,
the associated mass ratio is the same as that of the Hohmann transfer solution, even for
thrust-to-weight ratios of order $10^{-4}$.

For a robotic spacecraft, it is clear that the minimum propellant mass solution is to
be preferred. For a manned spacecraft, the transfer time and propellant mass functionals
have comparable importance; they are in conflict with one another for the following
reason: any attempt at reducing the former increases the latter and viceversa. This
suggests the construction of a compromise functional, which is the linear combination of
the previous two functionals, suitably scaled. The compromise functional depends on a
parameter $C$ (compromise factor) in the range $0 \leq C \leq 1$ and is such that it reduces to the
transfer time functional for $C = 0$ and to the propellant mass functional for $C = 1$.

We study the solutions minimizing the compromise functional in the range
$0 < C < 1$ and we find that the thrust profile includes three subarcs. The first subarc is
characterized by maximum thrust setting in conjunction with positive (upward) thrust
direction; the second subarc is characterized by coasting; the third subarc is characterized
by maximum thrust setting in conjunction with negative (downward) thrust direction.
We investigate systematically the effect of the compromise factor on the solutions
minimizing the compromise functional.
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1. Introduction

The national and international space community have acknowledged the fact that, in order to explore deep space or conduct preliminary studies for manned missions to nearby planets, efficient power sources and cost-effective vehicle designs must be developed. Design considerations for these space vehicles include the requirement of minimal onboard propulsion weight to reduce the overall mass.

In addition to the challenge of the space vehicle hardware, trajectory design is just as crucial. Optimally designed trajectories can reduce the amount of propellant required while containing the mission time. In this regard, the motivation for this research is based on recent progress in rocket propulsion systems and the need for trajectory tools to support them. Specifically, this thesis focuses on recent advances in electrical propulsion systems and the development of optimal interplanetary trajectories for missions to Mars. In the following sections, we present general background information about the trajectory tools and the requirements for optimal interplanetary trajectories; but first we give a brief history of rocket propulsion systems and their origin.

1.1. Origins of Rocket Propulsion. Since the mid 19th century, the interest of man in deep space and the outer galaxies has inspired countless theories and inventions for space travel. The limitations of the Earth natural resources as well as a general scientific curiosity have encouraged man to consider options for alternative planetary habitats. The visionaries include Konstantin Tsiolkovsky (1857-1935), Hermann Oberth (1894-1989), Robert Goddard (1882-1945), and Wernher von Braun (1912-1977), who
introduced revolutionary concepts that have become the foundation of modern space travel (Refs. 1-2). Russian born Tsiolkovsky was one of the first scientists to publish mathematical proofs and theories of multistage rockets. Often quoted for the statement “The Earth is the cradle of the mind, but we cannot live forever in a cradle”, Tsiolkovsky’s ideas on human space travel and rocket propulsion did not become recognized until the works of Hermann Oberth were published. Although Oberth did not work directly with Tsiolkovsky, he theorized also that rocket staging could be effective for achieving weight reduction. Working as Oberth’s assistant, Wernher von Braun helped design multistage rocket vehicles.

It was during this time that Robert Goddard (an American) launched the first liquid fueled rocket on a field near Worcester, Massachusetts in 1926. Known as the father of modern rocketry, Goddard continued to develop chemical engines until World War I when he focused on the design and testing of military rockets. In the 1940s, von Braun and a team of German scientists produced chemical propulsion systems for the V2 rocket during World War II. Soon after the war, von Braun and his team came to the United States and directed the chemical propulsion technology for the Saturn V rocket and Apollo missions to the Moon. In 1947, von Braun introduced ideas from his mentor (Oberth) on using electric propulsion for space missions. Ernst Stuhlinger, a colleague of von Braun, proposed solar-powered or nuclear-powered electric propulsion for travel beyond the Moon (Ref. 3). Oberth, von Braun, Stuhlinger, and others agreed that major design progresses must be made in order to develop a space vehicle with less mass (mostly chemical fuel), while accommodating onboard passengers.
Since the launch of the Saturn V during the Apollo Moon missions in the 1960s, chemical propulsion has been used primarily in manned and unmanned space missions. In general, chemical engines are characterized by high thrust, low specific impulse produced by the use of liquid or solid propellants. The high thrust is needed and fundamental to overcome the gravitational pull of any planet. Electrical or ion propulsion, a new propulsion technology developed and test flown within the past 10 years, is characterized by low thrust and high specific impulse; it is complementary to chemical propulsion and shows great promise for interplanetary flight.

In addition to the challenges of the space vehicle power source, flight trajectories depend critically on tools that optimize space travel by reducing the fuel mass while containing the transfer time. This is why trajectory optimization must become just as reliable and efficient as the space vehicle itself.

1.2. Current Trajectory Tools. Several international space agencies have identified the challenges of trajectory design using ion propulsion and have made significant gains in developing trajectory models for unmanned interplanetary flight. In 2002, NASA-JPL (National Aeronautics and Space Administration-Jet Propulsion Laboratory) successfully launched and tested the Deep Space 1 (DS1) spacecraft, an unmanned vehicle utilizing electrical propulsion powered by solar energy (Ref. 4). The primary objective of this mission was to test ion propulsion technology while achieving a rendezvous with the comet Borelli. This was the very first time that electrical propulsion was used for the interplanetary phase of a space mission and proved to be a benchmark for future space propulsion system designs. In 2003, ESA (European Space Agency)
launched the SMART 1 (Small Missions for Advanced Research in Technology 1) spacecraft to test its electric propulsion system and conduct studies of the Moon surface (Refs. 5-8). As of late 2004, SMART1 completed its flight from the Earth to the Moon and is currently orbiting the Moon.

With these two electric propulsion technologies, advanced trajectory design methods are being developed for flight trajectory optimization. Thus far, chemical propulsion has dominated the satellite industry; therefore, reliable trajectory tools exist for these high-thrust, low specific impulse systems. However, the interest in deep interplanetary space travel has encouraged the use of electrical engines, thereby creating the need for more low-thrust high specific impulse optimization programs. Optimization packages for chemical and electrical propulsion systems vary in that the instantaneous mass and specific impulse values are constantly changing and require continuous updating of the trajectory information. On the other hand, chemical propulsion is used for only short periods of time and does not require the constant updating of trajectory data.

Current tools developed for low-thrust trajectory optimization are now reviewed (Ref. 9). CHEBYTOP (Chebyshev Trajectory Optimization Program) is a low-thrust, two-body program requiring only few input variables for preliminary studies. SEPSOT (Solar Electric Propulsion System Program for Optimization of Trajectories) is a minimum time optimization program for vehicles using electrical propulsion. VARITOP (Variational Trajectory Optimization Program) is yet another feasibility tool and is based on ballistic patched conic theory that has been used for over 30 years. VARITOP utilizes an optimization program based on a two-body, Sun-centered coordinate system and has
been used for preliminary studies for several NASA low-thrust missions, including travel to Mars, Callisto (Jupiter icy Moon), and other outer planetary bodies. A more precise version of VARITOP is SEPTOP (Solar Electric Propulsion Trajectory Optimization Program). Deep Space 1 implemented SEPTOP to determine its optimal trajectory. The upcoming Dawn mission will use SEPTOP while traveling to the Vesta and Ceres asteroids for solar system material and make-up analysis.

The SMART 1 mission of the European Space Agency utilizes a patched conics program, with trajectory optimization based on the Pontryagin maximum principle and shooting methods for low-thrust systems. The European Space Agency has conducted several studies on trajectory paths for the SMART 1 spacecraft and has used gradient-type algorithms to generate optimal trajectories. Other low-thrust missions to Mars for a sample return are planned; the international community recognizes the need for a more robust low-thrust optimal trajectory tool that can handle both continuous and discontinuous thrusting.

1.3. Sequential Gradient-Restoration Algorithm. Reflecting the emerging technology in propulsion systems, computational trajectory tools are under development as well. A trajectory tool for the interplanetary travel of space systems using electrical propulsion is the algorithm employed in this thesis, the sequential gradient-restoration algorithm (SGRA). Developed in its basic form by Miele and his collaborators within the Aero-Astronautics Group in the period 1968-1986, SGRA is a proven tool for solving both mathematical programming and optimal control problems (Refs. 10-18). SGRA is a cyclical algorithm consisting of alternating restoration phases (reduction of the constraint
error) and gradient phase (reduction of the augmented functional). Several variations of
SGRA have been produced and the most widely used is the SGRA version with complete
restoration.

In 1992, McDonnell Douglas Technical Service Company reprogrammed SGRA
at the request of NASA under the acronym SEGRAM (SEquential Gradient Restoration
Algorithm, Ref. 19). SEGRAM has been used at NASA-JSC in various optimization
studies. Note that, up until recently, most of the problems solved with either SGRA or
SEGRAM have involved the use of chemical propulsion.

This thesis presents SGRA as a potential optimization tool in connection with
electrical propulsion systems. Baseline feasibility studies are performed for
interplanetary travel for an unmanned mission to Mars.

1.4. Content. The organization of this thesis is as follows. Section 2 deals with
the Bolza-Pontryagin optimal control problem and the basic structure of the sequential
gradient-restoration algorithm (SGRA). Because each iteration of SGRA requires the
solution of a linear two-point boundary-value problem (LTPBVP), Section 3 describes
the method of particular solutions for such task. Section 4 deals with the system
equations, boundary conditions and inequality constraints plus formulation of the
optimization problems of interplanetary transfer from Earth to Mars via electrical
propulsion: minimum time solution, minimum propellant mass solution, and compromise
solutions. Section 5 describes the numerical results obtained via SGRA. Finally, the
conclusions are given in Section 6.
2. Algorithm Structure

To solve optimal control problems, two types of methods are employed: first-order methods, also known as gradient methods; second-order methods, also known as quasilinearization methods. Common to these methods is the presence of a linear two-point boundary-value problem to be solved at each iteration.

2.1. Problem Formulation. For the continuous case, it is convenient to normalize the interval of integration to unity. If $\rho$ is the running time and $\tau$ is the final time, the transformation $t = \rho/\tau$, with $0 \leq t \leq 1$, allows one to convert a problem with variable interval of integration into a problem with fixed interval of integration. Then, the final time $\tau$ (if free) is treated as a component of a vector parameter $\pi$ being optimized. With this understanding, we formulate the following Bolza-Pontryagin problem of optimal control: Minimize the functional\footnote{The Bolza-Pontryagin problem reduces to the Lagrange problem for $h \equiv 0$, $g \equiv 0$ and to the Mayer problem for $f \equiv 0.$}

$$I = \int_{0}^{1} f(x, u, \pi, t) dt + [h(x, \pi)]_{0} + [g(x, \pi)]_{1},$$

wrt the state $x(t)$, control $u(t)$, and parameter $\pi$ which satisfy the following constraints:

$$\dot{x} = \phi(x, u, \pi, t),$$

$$x_{0} = \text{given},$$

$$[\psi(x, \pi)]_{1} = 0.$$
$n$-vector function, and $\psi$ is a $q$-vector function. The dependent variables are the $n$-vector state $x(t)$, the $m$-vector control $u(t)$, and the $p$-vector parameter $\pi$. A trajectory satisfying the constraints (2) is called a feasible trajectory.

2.2. First-Order Conditions. From calculus of variations, it is known that the previous problem is one of the Bolza-Pontryagin type. It can be reformulated as

$$
\min J = \int_0^1 \left[ f + \lambda^T (\dot{x} - \phi) \right] dt + (g + \mu^T \psi) \bigg|_1
$$

$$
= \int_0^1 \left( f - \lambda^T \phi - \dot{\lambda} x \right) dt - \left( \lambda^T x \right) \bigg|_0 + (g + \mu^T \psi + \lambda^T x) \bigg|_1,
$$

(3)

s.t. \hspace{1cm} \dot{x} = \phi,

(4a)

$x_0$ = given,

(4b)

$\psi_1 = 0$.

(4c)

Here, $J$ is the augmented functional, the $n$-vector $\lambda(t)$ is a variable Lagrange multiplier, and the $q$-vector $\mu$ is a constant Lagrange multiplier.

The first variation of the augmented functional (3) can be written as

$$
\delta J = \int_0^1 \left( f_x - \phi_x \lambda - \dot{\lambda} \right)^T \Delta x dt + \int_0^1 \left( f_u - \phi_u \lambda \right)^T \Delta u dt
$$

$$
+ \left[ \int_0^1 (f_x - \phi_x \lambda) dt + (g_x + \psi_x \mu) \right]^T \Delta x + \left[ (g_x + \psi_x \mu + \lambda)^T \Delta x \right]_1.
$$

(5)

The optimal trajectory must satisfy not only the constraint equations (4) but also the first-order optimality conditions

$$
\dot{\lambda} = f_x - \phi_x \lambda,
$$

(6a)

\footnote{The second form of Eq. (3) is obtained via integration by parts.}
\[ f_x - \phi_x \lambda = 0 , \quad (6b) \]
\[ \int_0^1 (f_x - \phi_x \lambda) dt + (g_x + \psi_x \mu)_1 = 0 , \quad (6c) \]
\[ (g_x + \psi_x \mu + \lambda)_1 = 0 . \quad (6d) \]

Indeed, satisfaction of (6) ensures the vanishing of the first variation (5) for any system of variations. A trajectory satisfying (4), (6) is called an extremal trajectory.

### 2.3. Approximate Solutions

In general, the system (4), (6) is nonlinear and sometimes strongly nonlinear. Therefore, approximate methods must be used to seek a solution iteratively.

Let the norm squared of a vector be defined as

\[ N(v) = v^T v. \quad (7) \]

Then, under the assumption that the initial condition (4b) is satisfied at every stage of SGRA, the functionals

\[ P = \int_0^1 N(\dot{x} - \phi) dt + N(\psi)_1 , \quad (8) \]

\[ Q = \int_0^1 N(\dot{\lambda} - f_x + \phi_x \lambda) dt + \int_0^1 N(f_u - \phi_u \lambda) dt \]

\[ + N \left[ \int_0^1 (f_x - \phi_x \lambda) dt + (g_x + \psi_x \mu)_1 \right] + N(g_x + \psi_x \mu + \lambda)_1 \quad (9) \]

represent the constraint error and the optimality condition error, respectively. For an exact optimal solution,

\[ P = 0 , \quad (10a) \]
\[ Q = 0. \] \hspace{1cm} (10b)

For an approximation to the optimal solution,

\[ P \leq \varepsilon_1, \] \hspace{1cm} (11a)

\[ Q \leq \varepsilon_2, \] \hspace{1cm} (11b)

where \( \varepsilon_1 \) and \( \varepsilon_2 \) are small preselected numbers.

2.4. Sequential Gradient-Restoration Algorithm. As explained, SGRA has a cyclical structure, each cycle including a gradient phase and a restoration phase. The gradient phase is started whenever Ineq. (11a) is satisfied and involves a single iteration. In this gradient iteration, the objective is to reduce the augmented functional \( J \), while the constraints are satisfied to first order. The restoration phase is started whenever Ineq. (11a) is violated and involves one or several iterations. In each restorative iteration, the objective is to reduce the functional \( P \), while the constraints are satisfied to first order and the norm squared of the variations of the control and the parameter is minimized. The restoration phase is terminated whenever Ineq. (11a) is satisfied. SGRA terminates when Ineqs. (11) are both satisfied.

2.5. System of Variations. For any iteration of the gradient phase or the restoration phase, let \( x(t) \), \( u(t) \), \( \pi \) denote the nominal functions; let \( \Delta x(t) \), \( \Delta u(t) \), \( \Delta \pi \) denote the variations of the nominal functions; let \( A(t) \), \( B(t) \), \( C \) denote the variations for unit stepsize \( \alpha \). Therefore,

\[ \bar{x}(t) = x(t) + \Delta x(t) = x(t) + \alpha A(t), \] \hspace{1cm} (12a)
\[ \tilde{u}(t) = u(t) + \Delta u(t) = u(t) + \alpha B(t), \]

\[ \tilde{\pi} = \pi + \Delta \pi = \pi + \alpha C. \]

Concerning the functionals (3), (8), the terminology is as follows: \( J, P \) denote the nominal functionals; \( \tilde{J}, \tilde{P} \) denote the varied functionals; \( \Delta J, \Delta P \) denote the total variations of these functionals. Therefore,

\[ \tilde{J} = J + \Delta J, \quad \tilde{P} = P + \Delta P. \]

The variations \( \Delta x(t), \Delta u(t), \Delta \pi \) must be chosen so that, at each iteration,

either \( \Delta J < 0 \) or \( \Delta P < 0. \)

To enforce (14), SGRA is constructed so that, at each iteration,

either \( \delta J < 0 \) or \( \delta P < 0, \)

where \( \delta J \) is given by (5), rewritten here for convenience as follows:

\[
\delta J = \alpha \int_0^1 (f_x - \phi_x \lambda - \dot{\lambda})^T A dt + \alpha \int_0^1 (f_u - \phi_u \lambda)^T B dt \\
+ \alpha \left[ \int_0^1 (f_x - \phi_x \lambda) dt + (g_x + \psi_x \mu) \right]^T C + \alpha \left[ (g_x + \psi_x \mu + \lambda)^T A \right]_1
\]

and where \( \delta P \) is given by

\[ \delta P = 2 \int_0^1 (\dot{x} - \phi)^T (\Delta \dot{x} - \phi_x \Delta x - \phi_u \Delta u - \phi_x \Delta \pi) dt + 2[\psi^T (\psi_x^T \Delta x + \psi_x^T \Delta \pi)]_1, \]

rewritten here for convenience as follows:

\[ \delta P = 2\alpha \int_0^1 (\dot{x} - \phi)^T (\dot{A} - \phi_x^T A - \phi_u^T B - \phi_x^T C) dt + 2\alpha [\psi^T (\psi_x^T A + \psi_x^T C)]_1. \]
2.6. **Gradient Phase.** As explained, the gradient phase consists of a single iteration designed to decrease the augmented functional (3).

Suppose that the nominal functions \(x(t), u(t), \pi\) satisfy the feasibility conditions (4). To first order, the perturbations per unit stepsize \(A(t), B(t), C\) must satisfy the linearized constraint equations

\[
\dot{A} - \dot{\phi}_x^T A - \dot{\phi}_u^T B - \dot{\phi}_n^T C = 0, \quad (19a)
\]

\[
(A)_0 = 0, \quad (19b)
\]

\[
(\psi_x^T A + \psi_n^T C)_1 = 0. \quad (19c)
\]

By inspection of (16), we see that \(\delta J\) can be made negative through the following choice of the variations per unit stepsize:

\[
B = -(f_x - \phi_x \lambda), \quad (20a)
\]

\[
C = \left[ \int_0^1 (f_x - \phi_x \lambda) dt + (g_x + \psi_x \mu)_1 \right], \quad (20b)
\]

with \(A(t), B(t), C\) consistent with (19) and multipliers \(\lambda(t), \mu\) consistent with

\[
\dot{\lambda} - f_x + \phi_x \lambda = 0, \quad (20c)
\]

\[
(g_x + \psi_x \mu + \lambda)_1 = 0. \quad (20d)
\]

In light of (19)-(20), the first variations of the functionals under consideration become

\[
\delta J = -\alpha Q, \quad \delta P = 0, \quad (21)
\]

with optimality condition error given by

\[
Q = \int_0^1 B^T B dt + C^T C. \quad (22)
\]

Since \(Q > 0\), the first of Eqs. (21) shows that \(\delta J < 0\). Hence, for \(\alpha\) sufficiently small, the decrease in the augmented functional \(J\) is guaranteed.
The system (19)-(20) is linear and nonhomogenous in the unknowns $A(t)$, $B(t)$, $C$ and $\lambda(t)$, $\mu$ and is independent of the stepsize $\alpha$. The latter is to be determined a posteriori in such a way that the descent requirement $\Delta J < 0$ is enforced.

The LTPBVP (19)-(20) can be solved via the method of particular solutions (MPS). The number of integrations required is $n + p + 1$ if a forward integration scheme is used and $q + 1$ if a backward-forward integration scheme is used (Ref. 11).

Once $A(t)$, $B(t)$, $C$ and $\lambda(t)$, $\mu$ are known, one can form the one-parameter family of solutions (12) for which the augmented functional (3) and the constraint error (8) are functions of the form

$$\tilde{J} = \tilde{J}(\alpha), \quad \tilde{P} = \tilde{P}(\alpha).$$

(23)

Then, some one-dimensional search scheme must be employed and $\alpha$ must be selected in such a way that the following inequalities are satisfied:

$$\tilde{J}(\alpha) < \tilde{J}(0),$$

(24a)

$$\tilde{P}(\alpha) < P^*. $$

(24b)

Concerning (24a), its satisfaction is guaranteed by the descent property of the gradient phase. Concerning (24b), $P^*$ is a preselected number, not necessarily small, which limits the constraint violation at the end of the gradient phase.

In summary, the gradient stepsize must be chosen so that Ineqs. (24) are satisfied. Should any violation occur, then a smaller value of $\alpha$ must be employed and can be obtained with a bisection process, starting from a suitably chosen reference stepsize $\alpha_0$. In turn, $\alpha_0$ is obtained via a scanning process followed by cubic interpolation.
2.7. Restoration Phase. At the end of a gradient phase, the varied constraint error $\tilde{P}$ can be computed with (8). If Ineq. (11a) is satisfied, then a new gradient phase is executed; otherwise, a restoration phase must be executed. In general, the restoration phase consists of several restorative iterations, each decreasing the constraint error until Ineq. (11a) is satisfied. When this occurs, the restoration phase ends and a new gradient phase is started.

Let $x(t)$, $u(t)$, $\pi$ denote nominal functions violating Ineq. (11a). By inspection of (18), we see that $\delta P$ can be made negative via variations per unit stepsize satisfying the linearized constraint equations

\begin{align*}
\dot{A} - \phi_x^T A - \phi_u^T B - \phi_x^T C + (\dot{x} - \phi) &= 0, \\
A_0 &= 0, \\
(\psi_x^T A + \psi_x^T C + \psi)_t &= 0.
\end{align*}

Any solution $A(t)$, $B(t)$, $C$ of Eqs. (25) is such that the first variation of the constraint error becomes

$$\delta P = -2\alpha P.$$  \hspace{1cm} (26)

Since $P > 0$, Eq. (26) shows that $\delta P < 0$. Hence, for $\alpha$ sufficiently small, the decrease in the constraint error is guaranteed.

Note that Eqs. (25) admit an infinite number of solutions. We render the solution unique by determining the perturbations per unit stepsize $A(t)$, $B(t)$, $C$ so that the following functional is minimized:

$$K = (1/2) \left[ \int_0^1 B^T B \, dt + C^T C \right],$$  \hspace{1cm} (27)
subject to the linearized constraints (25). The solution of the ensuing linear-quadratic optimal control problem is characterized by the following variations per unit stepsize:

\[ B = \phi_x \lambda, \]  
(28a)

\[ C = \int_0^1 \phi_x \lambda dt - (\psi_x \mu)_t, \]  
(28b)

with \( A(t), B(t), C \) consistent with (25) and multipliers \( \lambda(t), \mu \) consistent with

\[ \dot{\lambda} + \phi_x \lambda = 0, \]  
(28c)

\[ (\psi_x \mu + \lambda)_t = 0. \]  
(28d)

The system (25), (28) is linear and nonhomogeneous in the unknowns \( A(t), B(t), C \) and \( \lambda(t), \mu \) and is independent of the stepsize \( \alpha \). The latter is to be determined a posteriori in such a way that the descent requirement \( \Delta P < 0 \) is enforced.

The LTPBVP (25), (28) can be solved via the method of particular solutions (MPS). The number if integrations required is the same as for a gradient iteration, namely \( n + p + 1 \) if a forward integration scheme is used and \( q + 1 \) if a backward-forward integration scheme is used.

Once \( A(t), B(t), C \) and \( \lambda(t), \mu \) are known, one can form the one-parameter family of solutions (12) for which the constraint error (8) is a function of the form

\[ \tilde{P} = \tilde{P}(\alpha). \]  
(29)

Then, some one-dimensional search scheme must be employed and \( \alpha \) must be selected in such a way that the following inequality is satisfied:

\[ \tilde{P}(\alpha) < \tilde{P}(0). \]  
(30)

Should a violation occur, then a smaller value of \( \alpha \) must be employed and can be obtained with a bisection process starting form \( \alpha = 1 \).
2.8. Generalized LTPBVP. The linear two-point boundary-value problems of the gradient phase and the restoration phase can be embedded into a family of LTP-BVPs, depending on two constants \( G \) and \( R \), such that their sum is

\[
G + R = 1. \tag{31}
\]

The generalized LTPBVP is given below:

\[
\dot{A} - \Phi^T A - \phi^T A B - \phi^T C + R(\dot{x} - \phi) = 0, \tag{32a}
\]

\[
A_0 = 0, \tag{32b}
\]

\[
(\psi^T A + \psi^T C + R\psi)_1 = 0, \tag{32c}
\]

and

\[
B = -(Gf_u - \phi_x \lambda), \tag{33a}
\]

\[
C = -\left[ \int_0^1 (Gf_x - \phi_x \lambda) dt + (Gg_x + \psi_x \mu)_1 \right], \tag{33b}
\]

\[
\dot{\lambda} - Gf_x + \phi_x \lambda = 0, \tag{33c}
\]

\[
(Gg_x + \psi_x \mu + \lambda)_1 = 0. \tag{33d}
\]

In Eqs. (32)-(33), the values of the constants are

\[
G = 1, \quad R = 0, \quad \text{gradient phase}, \quad \tag{34a}
\]

\[
G = 0, \quad R = 1, \quad \text{restoration phase}. \quad \tag{34b}
\]

2.9. Gradient-Restoration Cycle. Let \( I_1 \) denote the value of the functional (1) at the beginning of the gradient phase; let \( I_2 \) denote the value of (1) at the end of the gradient phase; let \( I_3 \) denote the value of (1) at the end of the restoration phase. Note that \( I_1 \) and \( I_2 \) are not comparable, since the constraints are not satisfied to the same accuracy. On the other hand, \( I_1 \) and \( I_3 \) are comparable, since the constraints are satisfied to the same
accuracy. For SGRA to be stable on a digital computer, one must require satisfaction of the inequality

\[ I_3 < I_1. \]  

(35)

If Ineq. (35) is satisfied, the gradient-restoration cycle is complete and the next cycle is started. If Ineq. (35) is violated, one must return to the previous gradient phase and reduce the gradient stepsize until, after restoration, Ineq. (35) is satisfied. This property is guaranteed in the SGRA.

**Remark.** Consider a particular iteration (gradient or restorative) characterized by the stepsize \( \alpha \). At the end of the iteration, the updating of the state is done with (12a). Concerning the time derivative of the state, the updating must be done via the relation

\[
\dddot{x} = \dot{x} + \alpha \dddot{A}
\]

\[
= \dot{x} + \alpha [\phi_x^T \phi_x + \phi_u^T B + \phi_n^T C - R(\dot{x} - \phi)].
\]  

(36)

All quantities present on the rhs of (36) are computed based on information available at the beginning of the iteration under consideration.

**Remark.** The decision-making process of the sequential gradient-restoration algorithm (SGRA) is based on the values of two scalar parameters: the constraint error \( P \) [see (8)] and the optimality condition error \( Q \) [see (9)]. Let \( \varepsilon_1 \) and \( \varepsilon_2 \) denote preselected tolerances for \( P \) and \( Q \). Three cases can occur:

(C1) If \( P > \varepsilon_1 \), SGRA executes a restorative iteration.

(C2) If \( P \leq \varepsilon_1 \) but \( Q > \varepsilon_2 \), SGRA executes a gradient iteration.

(C3) If \( P \leq \varepsilon_1 \) and \( Q \leq \varepsilon_2 \), convergence is declared and the algorithm stops.
Note that the computation of $P$ requires only the functions $x(t), u(t), \pi$. On the other hand, the computation of $Q$ requires both the functions $x(t), u(t), \pi$ and the multipliers $\lambda(t), \mu$.

**2.10. Summary of the Algorithm.** For the continuous case, SGRA can be summarized via Steps 1 to 4 below.

Step 1. Assume nominal functions $x(t), u(t), \pi$ consistent with the initial conditions (4b). Compute the constraint error via (8). If $P$ satisfies Ineq. (11a), go to Step 2. If $P$ violates Ineq. (11a) go to Step 3.

Step 2. Gradient Phase. This phase consists of a single iteration described by Steps 2a to 2e below.

Step 2a. Compute the vectors and matrices $(f_x, f_u, f_\pi), (\phi_x, \phi_u, \phi_\pi), (g_x, g_\pi)_1$,

$(\psi, \psi_x, \psi_\pi)_1$.

Step 2b. For $G = 1$ and $R = 0$, solve the linear two-point boundary-value problem (32)-(33) with the method of particular solutions. Obtain the functions $A(t), B(t), C$ and multipliers $\lambda(t), \mu$; see Section 3.

Step 2c. Compute the optimality condition error via (22). If $Q$ satisfies Ineq. (11b), stop; a solution has been found and SGRA terminates. If $Q$ violates Ineq. (11b), go to Step 2d.

Step 2d. Compute the gradient stepsize by a one-dimensional search on the augmented functional $\tilde{J}(\alpha)$. First, determine the reference stepsize $\alpha_0$ via a scanning process followed by a cubic interpolation process. Then, perform a bisection process on $\alpha$, starting from $\alpha_0$, until Ineqs. (24) are both satisfied.
Step 2e. Once the gradient stepsize is known, compute the varied functions, $\bar{x}(t)$, $\bar{u}(t)$, $\bar{\pi}$ via (12). Return to Step 1 by setting the new nominal functions $x(t)$, $u(t)$, $\pi$ equal to the varied functions just computed.

Step 3. Restoration Phase. This phase consists of several restorative iterations, each described by Steps 3a to 3d below.

Step 3a. Compute the vectors and matrices $\delta_x$, $(\phi_x, \phi_u, \phi_\pi), (\psi_x, \psi_u, \psi_\pi)$.  

Step 3b. For $G = 0$ and $R = 1$, solve the linear two-point boundary-value problem (32)-(33) with the method of particular solutions. Obtain the functions $A(t)$, $B(t)$, $C$ and multipliers $\lambda(t)$, $\mu$; see Section 3.

Step 3c. Compute the restoration stepsize by a one-dimensional search on the constraint error $\bar{P}(\alpha)$. Perform a bisection process on $\alpha$, starting from $\alpha = 1$, until Ineq. (30) is satisfied.

Step 3d. Once the restoration stepsize is known, compute the varied functions $\bar{x}(t)$, $\bar{u}(t)$, $\bar{\pi}$ via (12). Compute the constraint error via (8). If $P$ violates Ineq. (11a), return to Step 3a by setting the new nominal functions $x(t)$, $u(t)$, $\pi$ equal to the varied functions just computed. If $P$ satisfies Ineq. (11a), stop; the restoration phase ends. Go to Step 2 if the restoration phase just ended is part of an incomplete gradient-restoration cycle (this can only occur at the beginning of SGRA); otherwise, go to Step 4.

Step 4. Complete Gradient-Restoration Cycle. For each complete gradient-restoration cycle, verify whether Ineq. (35) is satisfied, that is, whether the value of the functional $I$ at the end of the cycle is smaller than that at the beginning of the
cycle. If Ineq. (35) is satisfied, go to Step 2. If Ineq. (35) is violated, return to Step 2d of the previous gradient iteration and bisect the gradient stepsize as many times as needed until, after restoration, Ineq. (35) is satisfied. Then, go to Step 2.
3. **LTPBVP Solver**

The method of particular solutions (MPS) is a general technique for solving nonhomogeneous linear two-point boundary-value problem (LTPBVP). It differs from the method of complementary functions in that it makes use of only one differential system, namely the nonhomogeneous differential system at hand, in the particular case the system (32)-(33). See Ref. 20.

MPS is based on the decomposition of the LTPBVP (32)-(33) into two subsystems: a relatively large primary system and a relatively small secondary system. First, a proper number of particular solutions of the primary system must be generated. Then, the particular solutions are combined linearly via an equal number of undetermined constants so as to form the general solution of the primary system. Finally, the constants are obtained by forcing the general solution of the primary system to satisfy the secondary system plus a scalar normalization condition (sum of the constants equal to 1).

In the following sections, we describe MPS for two cases: (i) forward integration scheme and (ii) backward/forward integration scheme.

3.1. **Forward Integration Scheme.** With reference to the LTP-BVP (32)-(33), let the overall system be decomposed into two subsystems:

- primary system [Eqs. (32a), (32b), (33a), (33c)],

\[
\text{(37a)}
\]

- secondary system [Eqs. (32c), (33b), (33d)].

(37b)

Let \( y \) denote the \((n + p)\)-vector

\[
y = \begin{bmatrix} \lambda(0) \\ C \end{bmatrix},
\]

(38)
where \( n \) is the dimension of the state vector and \( p \) is the dimension of the parameter vector. Observe that, for a particular choice of \( y \), the primary system (37a) can be integrated forward so as to generate the functions

\[
A(t), B(t), C, \lambda(t).
\]  

(39)

Observe that, if \( y \) is arbitrary, the particular solution (39) does not satisfy the secondary system.

Let \( M \) denote the \((n + p) \times (n + p + 1)\) matrix

\[
M = [I_{n+p}, 0_{n+p}]
\]

made up of the identity matrix of order \( n + p \) and the zero vector of order \( n + p \). Let \( n + p + 1 \) particular solutions of the primary system (37a) be generalized by setting \( y \) equal to each of the columns of the matrix (40). In this way, we obtain the matrices

\[
\tilde{\lambda}(0) = [\lambda_1(0), \lambda_2(0), \ldots, \lambda_{n+p}(0)]
\]

(41a)

\[
\tilde{C} = [C_1, C_2, \ldots, C_{n+p}]
\]

(41b)

\[
\tilde{y} = [y_1, y_2, \ldots, y_{n+p}] = M
\]

(41c)

\[
\tilde{A}(t) = [A_1(t), A_2(t), \ldots, A_{n+p}(t)]
\]

(41d)

\[
\tilde{B}(t) = [B_1(t), B_2(t), \ldots, B_{n+p}(t)]
\]

(41e)

\[
\tilde{\lambda}(t) = [\lambda_1(t), \lambda_2(t), \ldots, \lambda_{n+p}(t)]
\]

(41f)

Assume now that the functions (39) solving the LTPBVP are linear combinations of the particular solutions in (41), that is, they have the form

\[
A(t) = \tilde{A}(t)k, \quad B(t) = \tilde{B}(t)k, \quad C = \tilde{C}k, \quad \lambda(t) = \tilde{\lambda}(t)k,
\]

(42)

where \( k \) denotes an \((n + p + 1)\)-vector of undetermined constants,
\[ k = [k_1, k_2, \ldots, k_{n+p}, k_{n+p+1}] \quad . \quad (43) \]

With this understanding, it is easy to show that the functions (42) constitute the general solutions of the primary system (37a) providing the following normalization condition is satisfied:

\[ U^T k = 1 \quad , \quad (44) \]

where \( U \) is the \((n + p + 1)\)-vector with components all equal to 1.

The next step is to force the satisfaction of the secondary system (37b). This is precisely the case if the following relations are satisfied:

\[ (\psi_x^T \widetilde{A} + \psi_{\pi}^T \widetilde{C})_1 k + R(\psi)_1 = 0 \quad , \quad (45a) \]

\[ \left( \widetilde{C} - \int_0^1 \phi_x \hat{\lambda} dt \right) k + (\psi_\pi)_1 \mu + G \left[ (g_\pi)_1 - \int_0^1 f_\pi dt \right] = 0 \quad , \quad (45b) \]

\[ (\bar{\lambda})_1 k + (\psi_x)_1 \mu + G(g_x)_1 = 0 \quad , \quad (45c) \]

together with the normalization condition (44). Note that (44)-(45) constitute a system of \(1 + q + p + n\) equations in which the unknowns are the \(n + p + 1\) components of the vector \(k\) and the \(q\) components of the vector \(\mu\).

**Memory Reduction.** Once the constants of the particular solutions are known, the solution of the LTPBVP (32)-(33) can be obtained via (42). This requires the memorization of the particular solutions at every point of the interval of integration, thus taxing the computer memory. To avoid this situation, it is convenient to trade memory for CPU time by generating one additional particular solution: this is done by setting the vectors \(\lambda(0), C, y\) at the following levels:

\[ \lambda(0) = [k_1, k_2, \ldots, k_n]^T \quad , \quad (46a) \]
\[ C = [k_{n+1}, k_{n+2}, \ldots, k_{n+p}]^T, \quad (46b) \]

\[ y = [k_1, k_2, \ldots, k_{n+p}]^T, \quad (46c) \]

and then integrating forward once more the primary system (37a). The solution obtained satisfies automatically the secondary system (37b), except for roundoff errors.

**Baricentric Property.** The method of particular solution has an elegant baricentric property as can be seen by rewriting the solutions (42) in the form

\[ A(t) = \tilde{A}(t)k/\mathbf{U}^T k, \quad B(t) = \tilde{B}(t)k/\mathbf{U}^T k, \quad C(t) = \tilde{C}k/\mathbf{U}^T k, \quad (47a) \]

\[ \lambda(t) = \tilde{\lambda}(t)k/\mathbf{U}^T k. \quad (47b) \]

The meaning of (47) is that the solution of the LTPBVP (32)-(33) behaves as the trajectory of the center of gravity of a system of \( n + p + 1 \) particles having individual masses equal to the constants of the particular solutions and unit total mass.

The above mechanical analogy must be taken with a grain of salt for the following reason. In the mechanical world, the masses are all positive. In the MPS world, the masses (namely, the constants of the particular solutions) can be either positive or negative. However, they cannot be all negative in view of the normalization condition (44).

### 3.2. Backward-Forward Integration Scheme.

With reference to the LTPBVP (32)-(33), let the overall system be decomposed into two subsystems:

- primary system [Eqs. (32a), (32b), (33a), (33b), (33c), (33d)],
- secondary system [Eq. (32c)].

Observe that, for a particular choice of the \( q \)-vector multiplier \( \mu \), \( \lambda(1) \) is known via (33d) and then the function \( \lambda(t) \) is obtained by backward integration of Eq. (33c), leading to the
determination of $B(t)$ and $C$ via (33a) and (33b). Subsequently, the function $A(t)$ is obtained by forward integration of the linearized equation (32a) utilizing the initial condition (32b). Observe that, if $\mu$ is arbitrary, the particular solution

$$A(t), B(t), C, \lambda(t), \mu$$

does not satisfy the final condition (32c).

Let $M$ denote the $q \times (q + 1)$ matrix

$$M = [I_q, 0_q]$$

made up of the identity matrix of order $q$ and the zero vector of order $q$. Let $q + 1$ particular solutions of the primary system (48a) be generated by setting $\mu$ equal to each of the columns of the matrix (50). In this way, we obtain the matrices

$$\tilde{A}(t) = [A_1(t), A_2(t), \ldots, A_q(t), A_{q+1}(t)],$$

$$\tilde{B}(t) = [B_1(t), B_2(t), \ldots, B_q(t), B_{q+1}(t)],$$

$$\tilde{C} = [C_1, C_2, \ldots, C_q, C_{q+1}],$$

$$\tilde{\lambda}(t) = [\lambda_1(t), \lambda_2(t), \ldots, \lambda_q(t), \lambda_{q+1}(t)],$$

$$\tilde{\mu} = [\mu_1, \mu_2, \ldots, \mu_q, \mu_{q+1}] = M.$$  

Assume now that the functions (49) solving the LTPBVP are linear combinations of the particular solutions in (51), that is, the functions (49) have the form

$$A(t) = \tilde{A}(t)k, \quad B(t) = \tilde{B}(t)k, \quad C = \tilde{C}k, \quad \lambda(t) = \tilde{\lambda}(t)k, \quad \mu = \tilde{\mu}k,$$

where $k$ denotes a $(q + 1)$-vector of undetermined constants,

$$k = [k_1, k_2, \ldots, k_q, k_{q+1}].$$
With this understanding, it is easy to show that the functions (49) constitute the general solutions of the primary system (48a) providing the following normalization condition is satisfied:

$$U^T k = 1,$$

where $U$ is the $(q + 1)$-vector with components all equal to 1.

The next step is to force the satisfaction of the secondary system (48b). This is precisely the case if the following relation is satisfied:

$$\left( \bar{\lambda} + \psi_x \bar{\mu} \right) \lambda \big[ k + G(x) \big] = 0,$$

(55)

together with the normalization condition (54). Note that (54)-(55) constitute a system of $1 + q$ equations in which the unknowns are the $q + 1$ components of the vector $k$.

**Memory Reduction.** Once the constants of the particular solutions are known, the solution of the LTPBVP (32)-(33) can be obtained via (52); this requires the memorization of the particular solutions at every point of the interval of integration, thus taxing the computer memory. To avoid this situation, it is convenient to trade memory for CPU time by generating one additional particular solution: this is done by setting the vector $\mu$ at the following level:

$$\mu = [k_1, k_2, \ldots, k_q]^T,$$

(56)

and then performing one more backward-forward integration of the primary system (48a). The solution obtained satisfies automatically the secondary system (48b) except for roundoff errors.

**Baricentric Property.** For the backward/forward integration scheme, Eqs. (47a) are still valid, albeit with different dimensions of the vectors and matrices in (47).
Therefore, the baricentric property noted in connection with the forward integration scheme is still valid in connection with the backward-forward integration scheme.

**Remark.** While the forward integration scheme of Section 3.1 requires $n + p + 1$ integrations of the system (32)-(33), the backward-forward integration scheme requires $q + 1$ integrations of the same system. Since $q \leq n + p$, it is clear that the backward-forward integration scheme is computationally more efficient. Nevertheless, for stiff systems characterized by large positive eigenvalues, the forward integration system might be preferable from the point of view of containing the error propagation due to the integration process. This issue has been explored in Ref. 21.
4. System Description and Problem Formulation

The problems presented in this thesis relate to low-thrust propulsion systems exemplified by the DS1 and SMART1 spacecraft (Refs. 22-24). The objective of this study is to illustrate the application of the sequential gradient-restoration algorithm to low-thrust propulsion systems. It is assumed that the spacecraft is controlled via the thrust direction $\alpha$ and thrust setting $\beta$. Specifically, $\alpha$ is the thrust direction, the angle between the thrust vector and a reference direction; $\beta$ is the ratio of instantaneous thrust to maximum thrust, $0 \leq \beta \leq 1$; clearly, $\beta = 0$ denotes engine shut-off, while $\beta = 1$ denotes engine operating at full thrust. Note that the reference direction, hence the definition of $\alpha$, depends on the coordinate system.

4.1. Assumptions and Reference Data. The maneuver under study is the transfer of a spacecraft from a low Sun orbit (LSO) to a high Sun orbit (HSO). In particular, LSO can be the Earth orbit around the Sun and HSO can be the Mars orbit around the Sun.

The basic assumptions are as following:

(A1) The Sun is fixed in space.

(A2) LSO and HSO are circular and coplanar orbits.

(A3) The spacecraft is subject to the gravitational attraction of the Sun along the entire trajectory.

(A4) The spacecraft is controlled via the thrust direction and the thrust setting.

(A5) Circularization of the motion is assumed at both the departure and arrival.
Reference data and physical constants associated with the problem at hand include the following:

(B1) The Sun gravitational constant is
\[ \mu_S = 0.1327 \text{ E12} \text{ km}^3/\text{s}^2. \]  \hfill (57a)

(B2) The average radii of the terminal orbits are the same as the average radii of the orbits of Earth (E) and Mars (M) around the Sun,
\[ r_{\text{LSO}} = r_{\text{E}} = 149.6 \text{ E06 km}, \]  \hfill (57b)
\[ r_{\text{HSO}} = r_{\text{M}} = 227.9 \text{ E06 km}. \]  \hfill (57c)

(B3) The initial and final velocities of the spacecraft are the circular velocities at LSO and HSO,
\[ V_{\text{LSO}} = V_{\text{E}} = \sqrt{\mu_S / r_{\text{LSO}}} = \sqrt{\mu_S / r_{\text{E}}} = 29.78 \text{ km/s}, \]  \hfill (57d)
\[ V_{\text{HSO}} = V_{\text{M}} = \sqrt{\mu_S / r_{\text{HSO}}} = \sqrt{\mu_S / r_{\text{M}}} = 24.13 \text{ km/s}. \]  \hfill (57e)

(B4) The reference acceleration of gravity is the acceleration of gravity at sea level on Earth,
\[ g_{\text{SL}} = 9.81 \text{ m/s}^2. \]  \hfill (57f)

Remark. Let the reference weight \( W \) be defined as the product of the spacecraft instantaneous mass \( M \) and a reference acceleration, the acceleration of gravity \( g_{\text{SL}} \) at sea level on Earth,
\[ W = M g_{\text{SL}}. \]  \hfill (58)

The reference weight can be regarded as synonym for mass, since (58) constitutes a one-to-one relation between mass and reference weight.
4.2. Coordinate Systems. The computations presented in this paper can be done using any of three coordinate systems: (i) Cartesian coordinate system, (ii) polar coordinate system, and (iii) hybrid coordinate system; the latter is akin to the wind coordinate system of flight mechanics (Ref. 10).

**Cartesian Coordinate System.** The Cartesian coordinate system $S_{xy}$ is centered in the Sun, with the axes $x$, $y$ pointing to fixed directions in space. The spacecraft position is determined by the pair $(x, y)$ and the spacecraft velocity is determined by the pair $(u, w)$, with $u, w$ the velocity components on the Cartesian axes.

Let $M$ denote the instantaneous mass, $M_0$ the departure mass, and $m = M/M_0$ the normalized instantaneous mass (for brevity, mass). Let $\beta T$ denote the actual thrust, with $T$ the maximum thrust and $\beta$ the thrust setting, a dimensionless number between 0 and 1. Let $g_{\text{SL}}$ denote a reference acceleration (acceleration of gravity at sea level on Earth), let $M_0 g_{\text{SL}}$ denote the reference weight at departure, and let $\sigma = T/M_0 g_{\text{SL}}$ denote the ratio of the maximum thrust to the reference weight at departure (for brevity, thrust-to-weight ratio). Let $I_p$ denote the specific impulse of the electrical engine.

Let $\rho$ denote the running time, $0 \leq \rho \leq \tau$, with $\rho = 0$ the initial time and $\rho = \tau$ the final time. Let $t = \rho/\tau$ denote the normalized time, $0 \leq t \leq 1$, with $t = 0$ the initial normalized time and $t = 1$ the final normalized time. Let a dot denote derivative wrt the normalized time. With this understanding, the system equations are

$$\dot{x} = \tau u, \quad (59a)$$

$$\dot{y} = \tau w, \quad (59b)$$

$$\dot{u} = \tau \left[- \left(\mu_S / r^3\right) x + (g_{\text{SL}} \sigma \beta / m) \cos \alpha\right], \quad (59c)$$
\begin{align}
\dot{w} &= \tau \left[ -\left( \mu_s / r^2 \right) \dot{y} + (g_{SL} \sigma \beta / m) \sin \alpha \right], \\
\dot{m} &= \tau \left[ -\left( \sigma / I_{sp} \right) \beta \right],
\end{align} 
\tag{59d, 59e}

where
\begin{equation}
r = \sqrt{x^2 + y^2}
\tag{60}
\end{equation}
is the radial distance from the Sun, \( \mu \) is the Sun gravitational constant, and \( \alpha \) is the thrust direction, the angle between the thrust vector and the \( x \)-axis.

In the above equations, the state variables are \( x(t), y(t), u(t), w(t), m(t) \), the control variables are \( \alpha(t), \beta(t) \), and \( \tau \) is a parameter. While the thrust direction \( \alpha(t) \) is unconstrained, the thrust setting \( \beta(t) \) is subject to the inequality
\begin{equation}
0 \leq \beta \leq 1,
\tag{61}
\end{equation}
with \( \beta = 0 \) denoting zero thrust and \( \beta = 1 \) denoting maximum thrust.

The initial conditions are
\begin{align}
x(0) &= r_{LSO}, \\
y(0) &= 0, \\
u(0) &= 0, \\
w(0) &= \sqrt{\mu_s / r_{LSO}}, \\
m(0) &= 1,
\end{align} 
\tag{62a, 62b, 62c, 62d, 62e}

and the final conditions are
\begin{align}
x^2(1) + y^2(1) &= r_{HSO}^2, \\
u^2(1) + w^2(1) &= \mu_s / r_{HSO}, \\
x(1) u(1) + y(1) w(1) &= 0.
\end{align} 
\tag{63a, 63b, 63c}

The meaning of Eqs. (62) is that the \( x \)-axis is directed toward the spacecraft initial position and the \( y \)-axis is directed as the spacecraft initial velocity; also, the departure
from LSO takes place with radius equal to the LSO radius, tangentially, and with circular velocity. The meaning of Eqs. (63) is that the arrival to HSO takes place with radius equal to the HSO radius, tangentially, and with circular velocity. In particular, note that (63c) is an orthogonality condition between the radius vector and the velocity vector.

**Polar Coordinate System.** The polar coordinate system \(sr\theta\) is centered in the Sun. The spacecraft position is determined by the pair \((r, \theta)\) and the spacecraft velocity is determined by the pair \((V_r, V_\theta)\), with \(V_r, V_\theta\) the velocity components on the radial and transversal directions.

Let \(t = \rho/\tau\) denote the normalized time, \(0 \leq t \leq 1\), with \(t = 0\) the initial normalized time and \(t = 1\) the final normalized time. Let a dot denote derivative wrt the normalized time. Accordingly, the system equations (38) are replaced by

\[
\dot{r} = \tau V_r ,
\]

\[
\dot{\theta} = \tau \left( V_\theta / r \right) ,
\]

\[
\dot{V}_r = \tau \left[ -\mu_s / r^2 + V_\theta^2 / r + \left( g_{SL} \sigma \beta / m \right) \sin \alpha \right] ,
\]

\[
\dot{V}_\theta = \tau \left[ - V_r V_\theta / r + \left( g_{SL} \sigma \beta / m \right) \cos \alpha \right] ,
\]

\[
\dot{m} = \tau \left[ - \left( \sigma / I_{sp} \right) \beta \right] ,
\]

where \(\alpha\) is the thrust direction, the angle between the thrust vector and the transversal direction, which is the same as the local horizon.

In the above equations, the state variables are \(r(t), \theta(t), V_r(t), V_\theta(t), m(t)\), the control variables are \(\alpha(t), \beta(t), \) and \(\tau\) is a parameter. While the thrust direction \(\alpha(t)\) is unconstrained, the thrust setting \(\beta(t)\) is subject to the inequality (61).

The initial conditions are
\[ r(0) = r_{LSO}, \]  
\[ \theta(0) = 0, \]  
\[ V_r(0) = 0, \]  
\[ V_\theta(0) = \sqrt{\left( \mu_\infty / r_{LSO} \right)}, \]  
\[ m(0) = 1, \]

and the final conditions are

\[ r(1) = r_{HSO}, \]  
\[ V_r(1) = 0, \]  
\[ V_\theta(1) = \sqrt{\left( \mu_\infty / r_{HSO} \right)}. \]

**Hybrid Coordinate System.** The hybrid coordinate system is akin to the wind coordinate system of flight mechanics. The spacecraft position is determined by the pair \((r, \theta)\) and the spacecraft velocity is determined by the pair \((V, \gamma)\), where \(V\) is the velocity modulus and \(\gamma\) is the angle between the velocity vector and the local horizon.

Let \(t = \rho/\tau\) denote the normalized time, \(0 \leq t \leq 1\), with \(t = 0\) the initial normalized time and \(t = 1\) the final normalized time. Let a dot denote derivative wrt the normalized time. Accordingly, the system equations (62) are replaced by

\[ \dot{r} = \tau(V \sin \gamma), \]  
\[ \dot{\theta} = \tau \left[ (V/r) \cos \gamma \right], \]  
\[ \dot{V} = \tau \left[ -\left( \mu_\infty / r^2 \right) \sin \gamma + (g_{SL} \sigma \beta / m) \cos \alpha \right], \]  
\[ \dot{\gamma} = \tau \left[ -\left( \mu_\infty / r^2 V \right) \cos \gamma + (V/r) \cos \gamma + (g_{SL} \sigma \beta / mV) \sin \alpha \right], \]  
\[ \dot{m} = \tau \left[ -\left( \sigma / I_{sp} \right) \beta \right], \]
where $\alpha$ is the thrust direction, the angle between the thrust vector and the velocity vector.

In the above equations, the state variables are $r(t)$, $\theta(t)$, $V(t)$, $\gamma(t)$, $m(t)$, the control variables are $\alpha(t)$, $\beta(t)$, and $\tau$ is a parameter. While the thrust direction $\alpha(t)$ is unconstrained, the thrust setting $\beta(t)$ is subject to the inequality (61).

The initial conditions are

$$r(0) = r_{LSO}, \quad (68a)$$
$$\theta(0) = 0, \quad (68b)$$
$$V(0) = \sqrt{\frac{\mu_S}{r_{LSO}}}, \quad (68c)$$
$$\gamma(0) = 0, \quad (68d)$$
$$m(0) = 1, \quad (68e)$$

and the final conditions are

$$r(1) = r_{HSO}, \quad (69a)$$
$$V(1) = \sqrt{\frac{\mu_S}{r_{HSO}}}, \quad (69b)$$
$$\gamma(1) = 0. \quad (69c)$$

**Remark.** The computations presented in this paper are done using the hybrid coordinate system formulation (67)-(69). To verify the accuracy of the results, some particular problems have also been solved using the Cartesian coordinate system formulation (59), (62), (63) and/or the polar coordinate system formulation (64)-(66). In all cases, the thrust setting inequality constraint (61) has been considered.
4.3. **Engine Relations.** The normalized mass flow of propellant is the negative of the right-hand side of any of Eqs. (59e), (64e), or (67e); hence,

$$ |\dot{m}| = (\sigma / I_{sp}) \beta . $$  \hspace{1cm} (70)

In turn, the average velocity of the jet exiting the nozzle of the electrical engine is given by

$$ V_{exit} = g_{SL} I_{sp} . $$  \hspace{1cm} (71)

Finally, the maximum thrust and the maximum power satisfy the relation

$$ T = 2\eta P / V_{exit} , $$  \hspace{1cm} (72)

where $P$ is the power developed by the powerplant supplying energy to the electrical engine and $\eta$ is the efficiency of the conversion of powerplant power into mechanical power, a dimensionless number between 0 and 1. In this thesis, it is assumed that the power conversion efficiency $\eta$ and the specific impulse $I_{sp}$ are constants characteristic of the electrical engine.

4.4. **Performance Indexes.** In this thesis, we are interested in three performance indexes. The first performance index is the transfer time from LSO to HSO,

$$ I = \tau , $$  \hspace{1cm} (73)

where $\tau$ is a parameter of the problem.

The second performance index is the normalized propellant mass consumed while transferring the spacecraft from LSO to HSO,

$$ I = m_p = m(0) - m(1) = 1 - m(1) , $$  \hspace{1cm} (74a)

with

$$ m_p = M_p / M_0 , \quad \sigma = T / M_0 g_{SL} , $$  \hspace{1cm} (74b)

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where $\sigma$ is the thrust-to-weight ratio at departure.

The third performance index is the weighted linear combination of the scaled values of the transfer time and the propellant mass,

$$ I = (1 - C) \frac{\tau}{\tau_R} + C \frac{m_p}{m_{pR}} , \quad (75a) $$

with

$$ 0 \leq C \leq 1, \quad (75b) $$

where $\tau_R$ and $m_{pR}$ are suitably chosen reference values for the transfer time and the propellant mass. Note that (75) is a one-parameter family of performance indexes, which reduces to (73) for $C = 0$ and to (74) for $C = 1$. For intermediate values of $C$, the performance index (75) is intended to achieve a compromise between the transfer time and the propellant mass.

4.5. Hohmann Transfer. The reference values employed in (75) are those pertaining to the Hohmann transfer trajectory from LSO to HSO. Under the extremely idealized assumption that the propellant burns occur instantaneously at the endpoints of the trajectory, the geometry of the Hohmann transfer trajectory is an elliptical arc bitangent to the terminal orbits. The associated transfer time and normalized propellant mass are given by

$$ \tau_R = \pi \sqrt{\frac{r_{AVE}^3}{\mu_s}} , \quad (76a) $$

$$ r_{AVE} = (1/2) (r_{LSO} + r_{HSO}) , \quad (76b) $$

and

$$ m_{pR} = 1 - \exp(-\Delta V / V_{exit}) , \quad V_{exit} = g_{SL} I_{sp} , \quad (77a) $$

$$ \Delta V = \Delta V_{LSO} + \Delta V_{HSO} , \quad (77b) $$
\[
\Delta V_{LSO} = V_{LSO} \left[ -1 + \sqrt{\frac{r_{HSO}}{r_{AVE}}} \right], \tag{77c}
\]
\[
\Delta V_{HSO} = V_{HSO} \left[ +1 - \sqrt{\frac{r_{LSO}}{r_{AVE}}} \right], \tag{77d}
\]
and their values can be found in Table 1 for several values of the specific impulse. In the table, the first line refers to a chemical engine \((I_{sp} = 450s)\), while the remaining three lines refer to an electrical engine. The third line refers to the electrical engine of the DS1 spacecraft \((I_{sp} = 3000s)\). The fourth line refers to the electrical engine of Project Prometheus \((I_{sp} = 6000s)\), tested at NASA-Glenn Research Center.

The reference values (76)-(77) are important for two reasons: (i) they allow us to properly scale the components of the compromise performance index (75); (ii) they constitute a yardstick against which the goodness of the solutions obtained can be measured.

4.6. **Optimization Problems.** This thesis considers the following optimization problems of the Mayer type.

**Problem Q1. Minimum Time.** The minimization of the functional (73) is to be performed wrt the values of the state vector \([ r(t), \theta(t), V(t), \gamma(t), m(t) ]\), control vector \([ \alpha(t), \beta(t) ]\), and parameter \(\tau\) which are consistent with the differential constraints (67), boundary conditions (68)-(69), and thrust setting inequality constraints (61). The thrust direction \(\alpha(t)\) is unconstrained.

**Problem Q2. Minimum Propellant Mass.** The statement of this problem is the same as that of Problem Q1, except that the time functional (73) is now replaced by the propellant mass functional (74).
Problem Q3. Compromise Solutions. The statement of this problem is the same as that of Problem Q1, except that the time functional (73) is now replaced by the compromise functional (75).
5. Numerical Experiments

For the problems described in the previous section, computations relative to the LSO to HSO transfer were performed via an HP Pavilion PC with a Pentium 3 Processor using a version of SGRA written in the C language. The integrations were performed with a 4th order Adams-Bashforth-Moulton predictor-corrector scheme. Double precision arithmetic was used throughout SGRA.

For Problems Q1-Q3, the system equations (67) were employed together with the boundary conditions (68)-(69); the thrust setting $\beta(t)$ was subject to inequality (61). In all cases, the dimension of the state vector is $n = 5$, the dimension of the control vector is $m = 2$, and dimension of the scalar parameter is $p = 1$. The functional being minimized is represented by Eq. (73) for the minimum time solution (Problem Q1), by Eq. (74) for the minimum propellant mass solution (Problem Q2), and by Eq. (75) for the compromise solutions (Problem Q3). The unknowns of the above Mayer problems are the state $r(t)$, $\theta(t)$, $V(t)$, $\gamma(t)$, $m(t)$, control $\alpha(t)$, $\beta(t)$, and parameter $\tau$.

5.1. Minimum Time Solution. Problem Q1 is a two-control problem in which the thrust direction $\alpha(t)$ and the thrust setting $\beta(t)$ are optimized together with the transfer time $\tau$ from the point of view of the performance index (73), rewritten here for convenience as follows:

$$I = \tau.$$  \hspace{1cm} (78)

For the ensuing Mayer problem, computer runs were made for several combinations of thrust-to-weight ratio $\sigma$ and engine specific impulse $I_{sp}$. In all cases, it was found that the best thrust setting is $\beta = 1$, meaning that maximum thrust must be
employed along the entire transfer. Concerning the thrust direction, the main conclusion is that it is not aligned with the velocity direction. Indeed, the thrust vector is transversal to the velocity vector (α ≠ 0), meaning that the thrust has two components: a tangential component, mostly accelerating, and a normal component, directed upward in the first half of the trajectory and downward in the second half, with a switch at midcourse.

Summary results for the transfer time and the propellant mass are shown in Tables 2 and 3 for various combinations of thrust-to-weight ratio and engine specific impulse. Specifically, Table 2 refers to $I_{sp} = 3000s$ and shows that, as the thrust-to-weight ratio increases, the transfer time decreases, while the propellant mass increases. Table 3 refers to $\sigma = 0.5 \times 10^{-4}$ and shows that, as the specific impulse increases, the transfer time increases, while the propellant mass decreases.

For a particular pair ($\sigma$, $I_{sp}$), namely $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000s$, Tables 2-3 show that the transfer time is 237.9 days and the normalized propellant mass is 0.3425. For the Hohmann transfer (Table 1), the corresponding figures are 258.8 days for the transfer time and 0.1730 for the normalized propellant mass. Therefore, vis-à-vis the Hohmann transfer trajectory, the minimum time trajectory saves 8 percent in transfer time albeit at the expense of a nearly 100 percent increase in propellant mass.

To sum up, the results obtained via SGRA point toward the relative inefficiency of a transfer done in minimum time. Indeed, not only the thrust is everywhere transversal to the velocity (not aligned with the velocity), but its normal component is substantial. Energetically speaking, this means that the advantage due to the higher specific impulse of the electrical engine is largely dissipated due to the everywhere transversal application of the thrust.
For the same values $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000$ s, Figure 1 shows the geometry of the minimum time trajectory in the $xy$-plane. In the figure, the thrust profile is superimposed on the trajectory geometry. For the same values $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000$ s, Figure 2 (in seven parts) shows the time history of the state variables (Figs. 2A to 2E) and control variables (Figs. 2F and 2G).

5.2. Minimum Propellant Mass Solution. Problem Q2 is a two-control problem in which the thrust direction $\alpha(t)$ and the thrust setting $\beta(t)$ are optimized together with the transfer time $\tau$ from the point of the performance index (74), rewritten here for convenience as follows:

$$I = m_p = m(0) - m(1) = 1 - m(1). \tag{79}$$

For the ensuing Mayer problem, computer runs were made for several combinations of thrust-to-weight ratio $\sigma$ and engine specific impulse $I_{sp}$. In all cases, it was found that the optimal thrust setting is bang-zero-bang, meaning that we are in the presence of a three-subarc solution in which maximum thrust ($\beta = 1$) is followed by coasting ($\beta = 0$), which in turn is followed by maximum thrust ($\beta = 1$). Concerning the thrust direction, it was found that the optimal thrust direction is $\alpha = 0$ meaning that, for the powered subarcs, the thrust vector is aligned with the velocity vector.

Summary results for the transfer time and the propellant mass are shown in Tables 4-5 for various combinations of thrust-to-weight ratio and engine specific impulse. Table 4 refers to $I_{sp} = 3000$ s and shows that, as the thrust-to-weight ratio increases, the transfer time decreases while the propellant mass stays nearly the same to three significant digits. Table 5 refers to $\sigma = 0.5 \times 10^{-4}$ and shows that, as the specific impulse increases, the
transfer time increases and the propellant mass decreases.

For a particular pair \((\sigma, I_{sp})\), namely \(\sigma = 0.5 \times 10^{-4}\) and \(I_{sp} = 3000\)s, Tables 4-5 show that the transfer time is 320.2 days and the normalized propellant mass is 0.1733. For the Hohmann transfer (Table 1), the corresponding figures are 258.8 days for the transfer time and 0.1730 for the normalized propellant mass. Therefore, to three significant digits, the propellant mass of the minimum propellant mass trajectory is the same as that of the Hohmann transfer trajectory, albeit at the expense of a 24 percent increase in transfer time. Energetically speaking, this means that the advantage due to the higher specific impulse of the electrical engine is fully realized due to the everywhere tangential application of the thrust in conjunction with the bang-zero-bang thrust setting sequence.

For \(I_{sp} = 3000\)s, Table 6 extends the range of values of the thrust-to-weight ratio considered in Table 4 and includes the results pertaining to the Hohmann transfer \((\sigma = \infty, \text{ impulsive thrust})\) as a limiting case. Note that the distributed thrust for \(\sigma\) finite becomes an impulsive thrust for \(\sigma\) infinite. Also note that, to three significant digits, the asymptotic value \(m_p = 0.1730\) characterizing the Hohmann transfer is reached by the solution of Problem Q2 for values of the thrust-to-weight ratio of \(O(10^{-4})\).

For \(\sigma = 0.5 \times 10^{-4}\) and \(I_{sp} = 3000\)s, Figure 3 shows the geometry of the minimum propellant mass trajectory in the xy-plane. In the figure, the thrust setting profile is superimposed on the trajectory geometry. Clearly, we are in the presence of a three-subarc situation, with the central subarc longer timewise than the other two, taken alone or in combination.

For \(\sigma = 0.5 \times 10^{-4}\) and \(I_{sp} = 3000\)s, Figure 4 (in seven parts) shows the time history
of the state variables (Figs. 4A to 4E) and control variables (Figs. 4F and 4G). In particular, Fig. 4G shows the time history of the thrust setting. Clearly, the control is bang-zero-bang: the initial subarc with maximum thrust ($\beta = 1$) is followed by a central coasting subarc ($\beta = 0$), which is followed by a final subarc with maximum thrust ($\beta = 1$).

5.3. Compromise Solutions. Problem Q3 is a two-control problem in which the thrust direction $\alpha(t)$ and the thrust setting $\beta(t)$ are optimized together with the transfer time $\tau$ from the point of view of the compromise performance index (75), rewritten here for convenience as follows:

$$I = (1 - C) \tau / \tau_R + C m_p / m_{pR}.$$  

(80)

Because time and propellant mass are not compatible dimensionally, each term of the sum (80) is scaled by means of an appropriate reference quantity; in the particular case, the reference quantities are the transfer time $\tau_R$ and propellant mass $m_{pR}$ of the Hohmann transfer trajectory corresponding to the same specific impulse. The parameter $C$, called the compromise factor, varies in the range

$$0 \leq C \leq 1.$$  

(81)

Note that, for $C = 0$, Problem Q3 reduces to Problem Q1 (minimum time); for $C = 1$, Problem Q3 reduces to Problem Q2 (minimum propellant mass). For intermediate values of the parameter $C$, the minimization of the functional (80) produces compromise solutions having properties intermediate between those of the minimum time solution and those of the minimum propellant mass solution.

Also note that, for a robotic spacecraft, the most important performance index is the propellant mass (79). For a manned spacecraft, the transfer time and propellant mass
functionals are both important and are in conflict with one another for the following reason: any attempt at reducing the former causes an increase in the latter and vice versa. This is why it is natural to consider the compromise performance index (80) involving the scaled values of the propellant mass and transfer time weighted respectively by the compromise factor $C$ and its complement $1 - C$.

For the ensuing Mayer problem, computer runs were made for several values of the compromise factors specifically,

$$C = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0,$$

while keeping the thrust-to-weight ratio at the level $\sigma = 0.5 \times 10^{-4}$ and the specific impulse of the level $I_{sp} = 3000$s. As $C$ increases, the optimization problem gradually changes from minimizing the transfer time ($C = 0$) to minimizing the propellant mass ($C = 1$). The compromise solutions correspond to the range $0 < C < 1$ and are such that, as $C$ increases, the relative importance of the propellant mass wrt the transfer time increases.

Thrust Profile. Generally speaking, the thrust profile includes three subarcs. The first subarc is characterized by maximum thrust setting ($\beta = 1$) in conjunction with positive (upward) thrust direction ($\alpha > 0$); the second subarc is characterized by coasting ($\beta = 0$); the third subarc is characterized by maximum thrust setting ($\beta = 1$) in conjunction with negative (downward) thrust direction ($\alpha < 0$).

Effect of the Compromise Factor. As the compromise factor $C$ increases, the transfer time $\tau$ and the phase angle travel $\Delta \theta$ increase, while the propellant mass $m_p$ and the maximum path inclination $\gamma_{max}$ decrease. Correspondingly, as $C$ increases: (i) the normalized time lengths $a/\tau$ and $c/\tau$ of the first and third subarcs (powered phases) decrease slightly, meaning that thrust application occurs for shorter duration; (ii) the
normalized time length of the second subarc \( b/\tau \) increases considerably, resulting in an increase in transfer time; (iii) the average value of the thrust direction modulus in the first and third subarcs decreases, implying higher efficiency of the thrust application wrt the spacecraft energy level; (iv) as a result, as \( C \) increases, the propellant mass decreases.

First Limiting Case. For the limiting case \( C = 0 \), the compromise trajectory becomes a minimum time trajectory. The normalized time length of the second subarc (coasting) shrinks to zero and the three-subarc solution degenerate into a two-subarc solution. Maximum thrust is applied at all times, the thrust pointing upward in the first subarc and downward in the last subarc, with a switch just beyond midway.

Second Limiting Case. For the other limiting case \( C = 1 \), the compromise trajectory becomes a minimum propellant mass trajectory. Maximum thrust is applied in the first and third subarcs, the thrust direction being aligned with the velocity direction. No thrust is applied in the second subarc, with the spacecraft coasting.

Summary Results. For \( \sigma = 0.5 \times 10^{-4} \) and \( I_{sp} = 3000 \)s, summary results are shown in Table 7 and Figures 5-6. Table 7 shows the transfer time \( \tau \), propellant mass \( m_p \), phase angle travel \( \Delta \theta \), and maximum path inclination \( \gamma_{max} \) versus the compromise factor \( C \). Clearly, as the compromise factor increases, the transfer time and phase angle travel increase, while the propellant mass and maximum path inclination decrease.

Figure 5 (in nine parts) shows the following quantities as functions of the compromise factor \( C \): transfer time \( \tau \) (Fig. 5A), propellant mass \( m_p \) (Fig. 5B), phase angle travel \( \Delta \theta \) (Fig. 5C), maximum path inclination \( \gamma_{max} \) (Fig. 5D); time averaged thrust direction \( \alpha_{AVE} \) for each of the powered subarcs (Fig. 5E and 5F); thrust setting \( \beta \) for the three subarcs (Figs. 5G to 5I). Finally, Figure 6 (in seven parts) shows the time history of
the state variables (Figs. 6A to 6E) and control variables (Figs. 6F and 6G) for a particular value of the compromise factor, namely $C = 0.4$. 
6. Discussion and Conclusions

For several years, the Aero-Astronautics Group of Rice University has been studying optimal Earth-to-Mars trajectories using chemical engines and more recently hybrid engines, namely, the combination of high-thrust chemical engines for planetary flight and low-thrust electrical engines for interplanetary flight. The major tool for the study has been the sequential gradient-restoration algorithm (SGRA) for optimal control problems.

In this thesis, the focus is on deep-space interplanetary flight using low-thrust engines and on trajectory optimization via SGRA in single-subarc form. The principal optimization criteria are the transfer time (Problem Q1) and the propellant mass (Problem Q2); these criteria are in conflict with one another for the following reasons: any attempt at reducing the former increases the latter and vice versa. This suggests the construction of a compromise criterion, the convex combination of the previous two criteria, suitably scaled (Problem Q3). The associated compromise functional depends on a parameter $C$ (compromise factor) in the range $0 \leq C \leq 1$ and is such that it reduces to the transfer time functional for $C = 0$ and to the propellant mass functional for $C = 1$.

Major conclusions are as follows:

(Q1) The minimum time solution requires the use of maximum thrust setting along the entire trajectory. The thrust direction is everywhere transversal to the velocity (not aligned with the velocity) and is characterized by a midcourse switch from a large positive value to a large negative value. This has a beneficial effect on time, but a detrimental effect on propellant mass. Assuming a specific impulse of 3000s, the results
show that: (i) for an initial thrust-to-weight ratio $0.5 \times 10^{-4}$, the propellant mass ratio is 0.3425, nearly twice that of a Hohmann transfer; (ii) for an initial thrust-to-weight ratio $1.0 \times 10^{-4}$, the propellant mass ratio is 0.4779, nearly 2.76 times that of a Hohmann transfer. Energetically speaking, these results point toward the relative inefficiency of this type of transfer. Timewise however there are benefits: in case (i) the transfer time reduces from 258.84 days to 237.86 days (8 percent less); in case (ii), the transfer time reduces from 258.84 days to 165.96 days (36 percent less).

(Q2) The minimum propellant mass solution requires the thrust setting to be bang-zero-bang (maximum thrust, coasting, maximum thrust) and the thrust direction to be aligned with the velocity direction. This solution is the best for a robotic spacecraft. Assuming a specific impulse of 3000s, consider once more cases (i) and (ii) as above. For both cases, the propellant mass ratio is 0.173, which is identical to three significant digits with that of a Hohmann transfer. Energetically speaking, the results point toward the high efficiency of this type of transfer. Timewise however there is a penalty to be paid: in case (i), the transfer time increases from 258.84 days to 320.23 days (24 percent more); in case (ii), the transfer time increases from 258.84 to 289.27 days (12 percent more).

(Q3) For a manned spacecraft, the transfer time and propellant mass functionals have comparable importance; they are in conflict with one another for the following reason: any attempt at reducing the former increases the latter and vice versa. This suggests the construction of a compromise functional, which is the convex combination of the previous two functionals, suitably scaled. The compromise functional depends on
a parameter $C$ (compromise factor) in the range $0 \leq C \leq 1$ and is such that it reduces to the transfer time functional for $C = 0$ and to the propellant mass functional for $C = 1$.

We study the solutions minimizing the compromise functional in the range $0 < C < 1$ and we find that the thrust profile includes three subarcs. The first subarc is characterized by maximum thrust setting in conjunction with positive (upward) thrust direction; the second subarc is characterized by coasting; the third subarc is characterized by maximum thrust setting in conjunction with negative (downward) thrust direction. We investigate systematically the effect of the compromise factor on the solutions minimizing the compromise functional.
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Fig. 4E. Problem Q2. Mass vs time for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000s$.

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Table 1. Hohmann transfer trajectory. Transfer time and propellant mass.

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<td>$1.0 \times 10^{-4}$</td>
<td>3000</td>
<td>165.96</td>
<td>0.4779</td>
</tr>
</tbody>
</table>

Table 3. Problem Q1. Minimum time solution. Effect of the specific impulse on transfer time and propellant mass.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$I_{sp}$ [s]</th>
<th>$\tau$ [days]</th>
<th>$m_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.5 \times 10^{-4}$</td>
<td>1500</td>
<td>214.48</td>
<td>0.6177</td>
</tr>
<tr>
<td>$0.5 \times 10^{-4}$</td>
<td>3000</td>
<td>237.86</td>
<td>0.3425</td>
</tr>
<tr>
<td>$0.5 \times 10^{-4}$</td>
<td>4500</td>
<td>245.40</td>
<td>0.2356</td>
</tr>
<tr>
<td>$0.5 \times 10^{-4}$</td>
<td>6000</td>
<td>249.14</td>
<td>0.1794</td>
</tr>
</tbody>
</table>
Table 4. Problem Q2. Minimum propellant mass solution. Effect of the thrust-to-weight ratio on transfer time and propellant mass.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$I_{sp}$ [s]</th>
<th>$\tau$ [days]</th>
<th>$m_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.3 \times 10^{-4}$</td>
<td>3000</td>
<td>362.73</td>
<td>0.1736</td>
</tr>
<tr>
<td>$0.5 \times 10^{-4}$</td>
<td>3000</td>
<td>320.23</td>
<td>0.1733</td>
</tr>
<tr>
<td>$0.7 \times 10^{-4}$</td>
<td>3000</td>
<td>302.48</td>
<td>0.1732</td>
</tr>
<tr>
<td>$1.0 \times 10^{-4}$</td>
<td>3000</td>
<td>289.27</td>
<td>0.1732</td>
</tr>
</tbody>
</table>

Table 5. Problem Q2. Minimum propellant mass solution. Effect of the specific impulse on transfer time and propellant mass.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$I_{sp}$ [s]</th>
<th>$\tau$ [days]</th>
<th>$m_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.5 \times 10^{-4}$</td>
<td>1500</td>
<td>315.19</td>
<td>0.3165</td>
</tr>
<tr>
<td>$0.5 \times 10^{-4}$</td>
<td>3000</td>
<td>320.23</td>
<td>0.1733</td>
</tr>
<tr>
<td>$0.5 \times 10^{-4}$</td>
<td>4500</td>
<td>322.07</td>
<td>0.1191</td>
</tr>
<tr>
<td>$0.5 \times 10^{-4}$</td>
<td>6000</td>
<td>323.01</td>
<td>0.0907</td>
</tr>
</tbody>
</table>

Table 6. Problem Q2. Comparison of the minimum propellant mass solution with the Hohmann transfer.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$I_{sp}$ [s]</th>
<th>$\tau$ [days]</th>
<th>$m_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.2 \times 10^{-4}$</td>
<td>3000</td>
<td>420.03</td>
<td>0.1741</td>
</tr>
<tr>
<td>$0.3 \times 10^{-4}$</td>
<td>3000</td>
<td>362.73</td>
<td>0.1736</td>
</tr>
<tr>
<td>$0.5 \times 10^{-4}$</td>
<td>3000</td>
<td>320.23</td>
<td>0.1733</td>
</tr>
<tr>
<td>$1.0 \times 10^{-4}$</td>
<td>3000</td>
<td>289.27</td>
<td>0.1732</td>
</tr>
<tr>
<td>$2.0 \times 10^{-4}$</td>
<td>3000</td>
<td>274.07</td>
<td>0.1731</td>
</tr>
<tr>
<td>$\infty$</td>
<td>3000</td>
<td>258.84</td>
<td>0.1730</td>
</tr>
</tbody>
</table>
Table 7. Problem Q3. Effect of the compromise factor on the properties of the optimal trajectories.

<table>
<thead>
<tr>
<th>$C$</th>
<th>$\tau$ [days]</th>
<th>$m_p$</th>
<th>$\Delta\theta$ [deg]</th>
<th>$\gamma_{\text{max}}$ [deg]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>237.86</td>
<td>0.3425</td>
<td>176.26</td>
<td>16.46</td>
</tr>
<tr>
<td>0.2</td>
<td>259.74</td>
<td>0.2337</td>
<td>189.99</td>
<td>13.17</td>
</tr>
<tr>
<td>0.4</td>
<td>286.43</td>
<td>0.1877</td>
<td>206.93</td>
<td>11.95</td>
</tr>
<tr>
<td>0.5</td>
<td>294.88</td>
<td>0.1809</td>
<td>212.16</td>
<td>11.77</td>
</tr>
<tr>
<td>0.6</td>
<td>301.67</td>
<td>0.1771</td>
<td>216.33</td>
<td>11.67</td>
</tr>
<tr>
<td>0.8</td>
<td>312.44</td>
<td>0.1739</td>
<td>222.79</td>
<td>11.54</td>
</tr>
<tr>
<td>1.0</td>
<td>327.48</td>
<td>0.1733</td>
<td>232.19</td>
<td>11.39</td>
</tr>
</tbody>
</table>
Fig. 1. Problem Q1. Minimum time trajectory from LSO to HSO for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000\text{s}$.
Fig. 2A. Problem Q1. Radial distance vs time for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000$ s.

Fig. 2B. Problem Q1. Phase angle vs time for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000$ s.
Fig. 2C. Problem Q1. Velocity vs time for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000s$.

Fig. 2D. Problem Q1. Path inclination vs time for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000s$. 
Fig. 2E. Problem Q1. Mass vs time for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000$ s.

Fig. 2F. Problem Q1. Thrust direction vs time for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000$ s.
Fig. 2G. Problem Q1. Thrust setting vs time for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000$ s.
Fig. 3. Problem Q2. Minimum propellant mass trajectory from LSO to HSO for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000s$. 
Fig. 4A. Problem Q2. Radial distance vs time for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000s$.

Fig. 4B. Problem Q2. Phase angle vs time for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000s$. 
Fig. 4C. Problem Q2. Velocity vs time for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000s$.

Fig. 4D. Problem Q2. Path inclination vs time for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000s$. 
Fig. 4E. Problem Q2. Mass vs time for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000s$.

Fig. 4F. Problem Q2. Thrust direction vs time for $\sigma = 0.5 \times 10^{-4}$ and $I_{sp} = 3000s$. 
Fig. 4G. Problem Q2. Thrust setting vs time for $\sigma = 0.5 \times 10^{-4}$ and $J_{sp} = 3000s$. 
Fig. 5A. Problem Q3. Transfer time vs compromise factor.

Fig. 5B. Problem Q3. Propellant mass vs compromise factor.
Fig. 5C. Problem Q3. Phase angle travel vs compromise factor.

Fig. 5D. Problem Q3. Maximum path inclination vs compromise factor.
Fig. 5E. Problem Q3. Subarc 1. Time averaged thrust direction vs compromise factor.

Fig. 5F. Problem Q3. Subarc 3. Time averaged thrust direction vs compromise factor.
Fig. 5G. Problem Q3. Subarc 1. Thrust setting vs compromise factor.

Fig. 5H. Problem Q3. Subarc 2. Thrust setting vs compromise factor.

Fig. 5I. Problem Q3. Subarc 3. Thrust setting vs compromise factor.
Fig. 6A. Problem Q3. Radial distance vs time, compromise factor $C = 0.4$.

Fig. 6B. Problem Q3. Phase angle vs time, compromise factor $C = 0.4$. 
Fig. 6C. Problem Q3. Velocity vs time, compromise factor $C = 0.4$.

Fig. 6D. Problem Q3. Path inclination vs time, compromise factor $C = 0.4$. 
Fig. 6E. Problem Q3. Mass vs time, compromise factor $C = 0.4$.

Fig. 6F. Problem Q3. Thrust direction vs time, compromise factor $C = 0.4$. 
Fig. 6G. Problem Q3. Thrust setting vs time, compromise factor $C = 0.4$. 