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Impurity Effects in Interacting Quantum Many-Body Systems

by

Jun Sun

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APPROVED, THESIS COMMITTEE:

Qimiao Si, Chair
Professor of Physics and Astronomy

Peter J. A. Nordlander
Professor of Physics and Astronomy

Junichiro Kono
Professor of Electrical and Computer Engineering

Houston, Texas
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ABSTRACT

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Impurities have a wide range of effects in interacting quantum many-body systems. They can interplay with interactions and lead to new electronic states of matter. They can also serve as a probe of an “intrinsic” many-body system. In this thesis, we consider the effects of impurities in three quantum many-body systems. First, we study the transport properties of a two-dimensional interacting electronic system with dilute quenched disorder. We find that the ground state is in fact a metallic state and in-plane magnetic-field can drive it to an insulating one. Second, we address the orthogonality catastrophe in Bose-Einstein condensate with a local impurity at its center. It is shown that the orthogonality effect in a Bose system has a stretched-exponential form, stronger than the algebraic orthogonality of a Fermi counterpart. The corresponding absorption spectrum is also determined. Finally, we analyze the effects of a spin resonance mode on the scattering tunneling microscopy(STM) spectra.
of a d-wave superconductor near a potential scattering center. We identify a counterintuitive two-unit-cell spatial modulation, at \( \omega \approx \pm (\Delta_0 + \Omega_0)/\hbar \), where \( \Delta_0 \) is the energy gap and \( \Omega_0 \) is the resonance mode energy. This effect can be tested by the Fourier-transformed STM technique.
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Contents

Abstract ii
Acknowledgments iv
Preface viii
List of Figures xi

I Transport Equation for Disordered Interacting Electrons in 2D and Magnetic Metal 1

1 Introduction and Motivation 2

2 Transport Equation 7
   2.1 Green’s functions, irreducible interaction, and vertex equations . . . . 7
       2.1.1 Single particle Green’s function .......................... 7
       2.1.2 Product of two Green’s functions .......................... 7
       2.1.3 Irreducible interactions .................................. 9
       2.1.4 Vertex equations ...................................... 10
   2.2 Ward identities and limiting forms of correlation functions ........ 11
       2.2.1 Ward identities ........................................ 11
       2.2.2 Correlation functions ................................. 14
3 Perturbative Results ........................................ 18

4 Magnetic Metal and Magneto-resistance .................. 22

5 Summary .......................................................... 26

II Orthogonality Catastrophe in Bose-Einstein Conden-
sates .............................................................. 27

6 Introduction and Motivation ................................... 28

7 Bosons in a Uniform Background ............................. 30
   7.1 Ideal bosons in a uniform background ................... 30
   7.2 Weakly interacting bosons in a uniform background ... 35

8 Bosons in a Harmonic Confining Potential ................ 37
   8.1 Ideal bosons in a harmonic confining potential ......... 37
   8.2 Weakly-interacting bosons in a harmonic confining potential . 38

9 Change of Absorption Spectrum due to Orthogonality .... 43

10 Experimental Implications .................................... 48

III Effects of a Collective Spin Resonance Mode on the
STM Spectra of D-Wave Superconductors 51

11 Introduction and Motivation 52

12 General Formulation 54

13 Numerical Treatment 57

14 Discussion and Summary 63

References 65

A Details of Calculation 71

A.1 Ward identities .......................... 71

A.2 Ward identity due to nonanalytic interactions .... 76

A.3 Correction to conductivity .................. 77

B Figures 83
Preface

Many-body physics is one of the frontier topics in modern physics. The major reason it attracts many brilliant minds lies in that the strong correlation between particles makes it really hard to determine the collective behavior of quantum many-body systems. In this thesis, three different many-body physics problems are addressed. They all involve the effects of impurities that are added to those of the interactions between particles. The impurities either serve as a probe of the many-body system or, even more dramatically, lead to new electronic states of matter.

First, we develop a transport formalism for interacting electrons in the presence of quenched disorder. Quantum effects on transport, due both to quantum interference and interaction effects, are incorporated through non-analytic terms in the irreducible interactions and appropriate contributions to the electron self-energy. Perturbatively, our approach recovers the standard results on quantum corrections to the Drude conductivity. We argue that the strong coupling fixed point is a magnetic metal beyond perturbation theory. Extensions of the theory into the case with an in-plane magnetic field applied to the system are outlined.

The second problem concerns orthogonality catastrophe in Bose systems. The orthogonality catastrophe in fermionic systems is well known: in the thermodynamic limit, the overlap between the ground state wavefunctions with and without a single
local scattering potential approaches zero algebraically as a function of the particle number $N$. Here we examine the analogous problem for bosonic systems. In the homogeneous case, we find that ideal bosons display an orthogonality stronger than algebraic: the wavefunction overlap behaves as $\exp[-\lambda N^{1/3}]$ in three dimensions and as $\exp[-\lambda N/\ln^2 N]$ in two dimensions. With interactions, the overlap becomes finite but is still (stretched-)exponentially small for weak interactions. We also consider the cases with a harmonic trap, reaching similar (though not identical) conclusions. Finally, we discuss the experimental implications of our results.

In the third and final problem, we consider the effects of a collective spin resonance mode on the STM spectra of D-Wave Superconductors. A high-energy spin resonance mode is known to exist in many high-temperature superconductors. Motivated by recent scanning tunneling microscopy (STM) experiments in superconducting $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$, we study the effects of this resonance mode on the local density of states (LDOS). The coupling between the electrons in a $d$-wave superconductor and the resonance mode produces high-energy peaks in the LDOS, which displays a two-unit-cell periodic modulation around a nonmagnetic impurity. This suggests a new means to not only detect the dynamical spin collective mode but also study its coupling to electronic excitations.

While impurities serve as an overarching theme of this thesis, the three problems are distinct. We will therefore divide the thesis into three separate parts. Each part
has its own introduction, technical development and conclusions. It is hoped that the format will facilitate readers to read the thesis in the order that suits their own interest.
List of Figures


B.3 The product of two Green’s functions. The upper and lower propagators are for one particle and one hole, respectively. .......... 85

B.4 Self-energy of single particle to the first order in interaction and disorder. Only the contributions from the exchange channel are displayed. 86

B.5 Transport equation in general. Here, $\gamma$ is the interaction vertex, $\Lambda$ the field vertex, and $S_{\mu\nu}$ the current-current correlation function. ........ 86

B.6 Isolation of the nonanalyticity in vertex and transport equation. .... 87

B.7 Irreducible interaction vertices to the first order in interaction and disorder. Only those that have a logarithmic singularity are kept. ... 88

B.8 Vector vertices which are nonzero at either the static or the dynamic limit. The first two diagrams are accompanied by those obtained by exchanging the particle and hole lines. Both the second and third diagrams belong to $\Lambda_{34}$. ....................... 89
B.9 Diagrams for scalar vertices to the first order in interaction and disorder. Diagrams linear or higher order in $\epsilon$ are neglected.

B.10 Schematics of the resistivity as a function of parallel field at different temperatures. The system is metallic at small fields, but becomes insulating at high fields. The dashed lines represent interpolations between the two limits.

B.11 Phase shift as a function of the system size in three dimensions, with $\tilde{\nu} = 0.1$. Inset: the phase shift vs. $a/R$.

B.12 The analogous plot for two dimensions, with $\tilde{\nu} = 1$. Inset: phase shift vs. $1/\ln(R/a)$.

B.13 Absorption spectrum as a function of frequency in 3 dimensions, where $\lambda_{sp} = 0.25$.

B.14 Absorption Spectrum as a function of frequency in 2 dimensions, which is a Lorentzian function. $\alpha_{sp} = 0.10$.

B.15 a) Diagrams for $G(p, p + q; \omega)$. The thick solid line represents the full conduction electron Green's function in the coupling to an impurity, specified by a cross. The wavy line denotes the propagator of the collective mode; b) Self-energy diagrams for the conduction electrons in the absence of impurity, $\tilde{\Sigma}$. The thin solid line is the bare (BCS) conduction electron Green's function, $G_0$.  


B.16 Spectral function $A_k(E)$ at the wavevector $k = (\pi, 0)$ for $g/\Delta_0 = 3/\sqrt{2}$ (a). The density of states, $\sum_k A_k(E)$, is shown in (b), for various values of $g/\Delta_0 = 0, 1/\sqrt{2}, 2/\sqrt{2}$, and $3/\sqrt{2}$ (from lower to upper); here, for easier viewing, the consecutive curves are shifted by 0.2 along the vertical axis. ................................. 94

B.17 Local density of states at the impurity site (a) and at its nearest neighbor (b). The coupling constant $g/\Delta_0 = 3/\sqrt{2}$. ......................... 95

B.18 The spatial variation of the LDOS around the impurity with the mode coupling $g/\Delta_0 = 3/\sqrt{2}$ at $E = -E_1$ (a), and its corresponding Fourier spectrum $\rho(q, -E_1)$ (b). The density of states is measured with respect to its spatial average value. The image window of $25 \times 25$ plaquettes is taken from a system of size $1000 \times 1000$. Here the impurity-induced Friedel oscillation along the diagonals have been filtered away; see main text for details. ................................. 95
Part I

Transport Equation for Disordered Interacting Electrons in 2D and Magnetic Metal
Chapter 1
Introduction and Motivation

The nature of disordered interacting electrons in two dimensions is a long-standing problem. A succession of theoretical treatment has appeared over the past four decades, with increasing levels of sophistication [1, 2, 3, 4, 5]. Yet, the problem remains open. The Hamiltonian for such a system is composed of three parts,

\[ H = H_0 + H_T + H_e \]

\[ H_0 = \sum_{k,\sigma} \epsilon_k c_{k,\sigma}^+ c_{k,\sigma} \]

\[ H_T = \sum_{k,q,\sigma} u(q) c_{k+q,\sigma}^+ c_{k,\sigma} \]

\[ H_e = \frac{1}{2} \sum_{k,k',q,\sigma,\sigma'} v(q) c_{k+q,\sigma}^+ c_{k',-q,\sigma'}^+ c_{k',\sigma'} c_{k,\sigma} \quad (1.1) \]

The first part \( H_0 \) is the kinetic energy of free electrons in 2D. The second part \( H_T \) is the energy due to scattering by impurities. The last part \( H_e \) is the electron-electron interaction. If only the impurity effect and the kinetic energy are taken into account, then an Anderson insulator will be obtained[2]. In the absence of disorder, on the other hand, the system is a normal Fermi Liquid in the weak interaction limit and becomes a Wigner crystal in the strong interaction limit[6]; the Wigner crystal is pinned by an infinitesimal disorder and is an insulator as well. It is very natural to ask what the phase is, when both interaction and impurity effects are in presence. It
is believed up to 1990’s that it is an insulating phase as well.

In 1994, Kravchenko et. al. found that the temperature dependence of resistivity for high mobility Si MOSFET is metallic-like at certain range of low electron densities [9, 10], as shown in Fig.B.1. The 2D system is located on the depletion or inversion layer in Si MOSFET, with the layer width much less than the mean free path. Such metallic behavior was confirmed by other groups as well, using different materials and devices [11, 12, 13, 14, 15, 18]. Another interesting phenomenon observed in such experiments is that the magnetoresistivity is dramatically increased in the presence of parallel magnetic field [11], as shown in Fig.B.2. The parallel magnetic field couples predominantly to spin instead of to the orbital motion as in Quantum Hall Effect [22]. The unexpected metal-insulator transition (MIT) and its sensitivity to magnetic field are very exciting and have motivated considerable current research.

Our work builds on preceding theories in this field. The following is a brief review of what has been done.

Fermi-liquid theory for interacting electrons weakly scattering off dilute impurities was derived by Betbeder-Matibet and Nozières (B-MN hereafter) [1]. This theory is valid only when the scattering rate

\[ \frac{1}{\tau} = 2\pi n_i |T|^2 N_F << E_f \]  

(1.2)

or when the mean-free path \( \ell \) satisfies \( k_f \ell >> 1 \). Here \( n_i \) is the density of impurities, \( |T|^2 \) is the average of the squared scattering matrix for the impurities, and \( N_F \) is the
density of states at the Fermi-level. To this order, the essential difference between
the problem with and without impurities is that, in the former, the Fermi-surface is
not defined in momentum space. But since the energy $\omega$ is conserved in the collisions,
the Fermi energy at $\omega = \mu$ continues to be well defined and the electron self-energy
has a discontinuity at $\omega = \mu$[cf Eq.(2.2)]. This introduces some technical differences
in the derivation of Fermi-liquid theory compared to the pure case.

Altshuler and co-workers have discovered that to order $1/k_f \ell$, logarithmic singu-
larities appear in the scattering amplitude at low energies/momenta of impure
interacting electrons in two dimensions [3]. The reason for the singularity is that
the density correlation function for impure electrons is diffusive. The energy of the
fluctuations $\epsilon$ is therefore related to their momentum $q$ as $\epsilon \sim q^2$. Recall that in the
density correlation function for non-interacting electrons in the pure limit $\epsilon$ scales as
$q$ [7]. For any interaction in the pure limit logarithmic singularities then arise only
for one dimension. For the diffusive propagator the logarithmic singularity appears
perturbatively in the interaction for $d = 2$. In two dimensions, to the same order in
$1/k_f \ell$, non-interacting electrons also have the so-called weak-localization singularities
which too are logarithmic[2].

The most complete work for this problem is due to Finkelstein[4], who carried out a
perturbative scaling analysis. Here, interaction effects are considered to infinite orders
and disorder effects to first order. As the temperature is lowered or, equivalently, the
spatial scale is increased, the triplet interaction is found to scale towards infinity, the conductivity to a finite value, and the electron density of states towards a pseudo-gap form. The run-away flow of the triplet amplitudes implies that the scaling analysis eventually fails at sufficiently low temperatures. The nature of the strong coupling fixed point has been left as an open problem.

In this thesis, we generalize the usual transport equation approach for impure interacting electrons to incorporate the singular effects associated with both diffusion and quantum interference. As with B-MN, we work with the standard vector vertex, which describes the coupling of a physical electron to an external vector field. The advantage of working with such a quantity is that, we do not need to know the elementary excitations of the system. In addition, there are Ward identities associated with these vector vertices that can be taken advantage of. Unlike B-MN, however, we find that the irreducible interactions have to be taken to be non-analytic in order to incorporate the diffusive and quantum interference effects.

Formally, this problem has its novel aspects. Singularities arise from irreducible vertices rather than from \((GG)_{\text{coherent}}\) as they do in Landau-Pomeranchuk instabilities as well as SDW, CDW or superconductivity. The method developed here may be useful in such problems.

Physically, our chief motivation is to address the nature of the strong coupling fixed point hinted by the weak-coupling scaling approach of Finkelstein. In this paper, we
argue that the fixed point is a magnetic metal through applying the transport equation we developed. In addition, we extend our theory to the case where a parallel magnetic field is present, which only couples to the spin of the system.

The remainder of this part of the thesis is organized as follows. In Chapter 2, we constructed the general transport equation for disordered interacting electrons in 2D. In Chapter 3, we apply the theory to the case where density of states (DOS) can be assumed as a constant, and test the validity of the transport equation perturbatively. In chapter 4, we address the ground state of such a 2D system by using our transport equation in the non-perturbative regime with a pseudo-gap in the DOS. We summarize our conclusions in Chapter 5. In Appendix A, the calculational details for Ward Identity and conductivity are given and all the diagrams for this thesis are shown in Appendix B.
Chapter 2
Transport Equation

2.1 Green's functions, irreducible interaction, and vertex equations

2.1.1 Single particle Green's function

The disorder-averaged Green's function $G(k, \omega)$ may be expressed in terms of the non-interacting eigenvalues of the pure system $\epsilon(k)$ and the self-energy $M(k, \omega)$:

$$G(k, \omega) = \frac{1}{\omega - \epsilon(k) - M(k, \omega)}. \quad (2.1)$$

The chemical potential is $\mu$. In $d = 2$ and for finite disorder, $M(k, \omega)$ has singularities for $\omega \to \mu$, so that $G(k, \omega)$ has branch-cuts rather than poles. Nevertheless, causality requires that

$$\text{Im}M(k, \omega) \sim \text{sgn}(\omega - \mu). \quad (2.2)$$

We make extensive use of this property. It is standard to separate $G$ into a 'coherent' part and an 'incoherent' part for $\omega$ close to $\mu$. In the presence of singular diffusive corrections, we will no longer make such a separation.

2.1.2 Product of two Green's functions

Just as in the derivation of Fermi-liquid theory, it is important to separate the singularities in the products of the Green's function $G(p + \omega/2)G(p - \omega/2)$. Here
and subsequently the following notation is used

\[ p = (\mathbf{k}, \omega); \]

\[ \varpi = (\mathbf{q}, \epsilon), \]

so that in the particle-hole Green's function, see Fig.B.3, the total incoming momentum-energy is \( \varpi \). The product can be separated as

\[ G(p + \varpi/2)G(p - \varpi/2) = \Phi_{\text{inc}}(p, \varpi) + Q(p, \varpi)y(\omega, \epsilon), \]

where

\[ y(\omega, \epsilon) = \begin{cases} 1, & \text{if } |\omega - \mu| < \epsilon/2; \\ 0, & \text{if } |\omega - \mu| > \epsilon/2. \end{cases} \]

Then the second part in Eq.(2.4) is non-zero only if the incoming particle and hole have energies on opposite sides of the chemical potential \( \mu \). For \( \epsilon \to 0 \), we can set \( y(\omega, \epsilon) = \epsilon \delta(\omega) \). For later purposes we also define \( \bar{y}(\omega, \epsilon) \), which is the complement of \( y(\omega, \epsilon) \); i.e. it is non-zero only if the incoming energies are on the same side of \( \mu \).

Obviously,

\[ \bar{y}(\omega, \epsilon) + y(\omega, \epsilon) = 1 \]

\[ \bar{y}(\omega, \epsilon)y(\omega, \epsilon) = 0. \]

In the decomposition Eq.(2.4)

\[ \Phi_{\text{inc}}(p, \varpi) = \left( \frac{1}{-\epsilon_+ + \omega_+ - \text{Re}M_+ + i\text{Im}M} \right) \left( \frac{1}{-\epsilon_- + \omega_- - \text{Re}M_- + i\text{Im}M} \right) \]
\[ Q(p, \omega) = \left( \frac{1}{-\epsilon_+ + \omega + ReM_+ - iImM} \right) \left( \frac{1}{-\epsilon_- + \omega - ReM_- + iImM} \right) \]

\[ - \left( \frac{1}{-\epsilon_+ + \omega + MeM_+ + iImM} \right) \left( \frac{1}{-\epsilon_- + \omega - MeM_- + iImM} \right) \]  

(2.7)

(2.8)

As BMN pointed out, due to the sign change of the imaginary part of the self-energy at the Fermi energy, the first term of Q has two poles in the complex \( \epsilon_k \) plane, located on opposite sides of the real-axis. Once the diffusion correction is included, the second term of Q as well as \( \phi_{mc} \) also have two poles on the opposite sides of the real-axis in the complex \( \epsilon_k \) plane.

### 2.1.3 Irreducible interactions

In the Fermi liquid case, the irreducible interactions \( I(p, p', \omega) \) are analytic. Once the diffusive and interference singularities are included, this is no longer the case: the value of \( I(p, p', \omega) \) in the limit \( \omega_i \to \mu \) depends on whether \( \omega_i \) approaches \( \mu \) from the same or different sides of \( \mu \). We find it useful to separate the irreducible interactions \( I(p, p', \omega) \) into three parts,

\[ I(p, p', \omega) = I_1(p, p', \omega) + I_2(p, p', \omega)\delta(\omega - \omega') \]

(2.9)

as perturbatively illustrated in Fig.B.7. \( I_1 \) is the part of the irreducible interaction which contains interaction lines that connect the particle and the hole-lines directly. \( I_2 \) is the part in which no interaction line connects the particle and the hole-lines.
directly, so that the energy of the particle and the hole lines is separately conserved. Therefore, $I_2$ has a δ function attached to it. And $I_1$ and $I_2$ contain non-analytic contributions, which is the essential cause of the finite conductivity in the strong-coupling fixed point.

To treat the non-analyticities of $I_1$ and $I_2$, we further decompose them into

$$I_1(p, p', \omega) = I_{11}(p, p', \omega) y(\omega, \epsilon) y(\omega', \epsilon) + I_{12}(p, p', \omega) y(\omega, \epsilon) \bar{y}(\omega', \epsilon)$$

$$+ I_{13}(p, p', \omega) \bar{y}(\omega, \epsilon) y(\omega', \epsilon) + I_{14}(p, p', \omega) \bar{y}(\omega, \epsilon) \bar{y}(\omega', \epsilon) \quad (2.10)$$

$$I_2(p, p', \omega) = I_{21}(p, p', \omega) y(\omega, \epsilon) + I_{22}(p, p', \omega) \bar{y}(\omega, \epsilon) \quad (2.11)$$

The $y$ and $\bar{y}$ factors take care of how $\omega_i$ approach $\mu$ and accounts for the non-analyticity. The remaining factors, $I_{11}$, $I_{12}$, $I_{13}$, $I_{14}$, $I_{21}$, and $I_{22}$, are all analytic functions.

2.1.4 Vertex equations

The vertex for scattering is given in terms of the irreducible interactions $I$ and the single-particle Green's functions $G$ by, see Fig. B.5

$$\gamma(p, p', \omega) = I(p, p', \omega) + \sum_{p''} I(p, p'', \omega) G(p'' - \omega/2) G(p'' + \omega/2) \gamma(p, p'', \omega) \quad (2.12)$$

Given an external perturbation $\lambda(\mu)$, where $K = (k, \sigma)$, the vertex to the external perturbation $\Lambda(P, \omega)$ is given by

$$\Lambda_{\mu} = \lambda_{\mu} + \lambda_{\mu} GG\gamma \quad (2.13)$$
which can be rewritten using Eq.(2.12) for $\gamma$ as

$$
\Lambda_\mu = \lambda_\mu + \Lambda_\mu G G I.
$$

(2.14)

The correlation functions $S_{\mu\nu}(\pi)$, see Fig.B.5, are given by

$$
S_{\mu\nu} = \lambda_\mu G G \Lambda_\nu.
$$

(2.15)

The vertex equations so far are valid in general. What is unique in the current case with disorder, is the irreducible interaction has a strong singularity $\delta(\omega - \omega')$, and the vertex as the product of Green functions has non-analyticity in the vicinity of Fermi surface as well. To treat such non-analyticity, we make the decomposition.

$$
\Lambda_\mu = \Lambda_{\mu,1} + \Lambda_{\mu,2} y(\omega, \epsilon)
$$

(2.16)

2.2 Ward identities and limiting forms of correlation functions

2.2.1 Ward identities

The limiting forms of the vertex functions, $\Lambda_\mu(p, r)$, where $r = \frac{q}{\epsilon}$, as well as the correlation functions $S_{\mu\nu}(r)$ for $r = 0$ and $r = \infty$ will now be considered. The discussion of BMN about the Ward identities connecting the limiting forms of the Vertex functions to the derivatives of the self-energy carries through, even for the cases that the irreducible vertices have singularities provided that these singularities at $\epsilon = 0$ are functions of $(\omega - \omega')$, and at $q = 0$ are functions of $(k - k')$. As in BMN,
not all the Ward-identities for the pure system are preserved for the problem with disorder.

The scalar vertex $\Lambda_4(p,0)$ for $q = 0$ in the limit $\epsilon \to 0$, coupling to density fluctuations follows

$$\Lambda_4(p,0) = 1 - \frac{\partial M(p)}{\partial \omega}. \quad (2.17)$$

The vector vertex in the opposite limit, for $\epsilon = 0$ and the limit $q_\alpha \to 0$, coupling to velocity in the $\alpha$-th direction follows

$$\Lambda_\alpha(p, \infty) = \frac{k_\alpha}{m} + \frac{\partial M(p)}{\partial k_\alpha} \equiv \hat{\epsilon}(p) \frac{k_\alpha}{m} \quad (2.18)$$

where $\hat{\epsilon} = 1 + \frac{\partial M(p)}{\partial k_\alpha}$ is the ratio of the bare density of states to the renormalised density of states $\nu_0/\nu$. Therefore

$$\Lambda_\alpha(p, \infty) = \frac{k_\alpha}{m} (\nu_0/\nu) \quad (2.19)$$

Just as we mentioned in the previous section, in the case of a non-Fermi liquid, the derivative of self-energy will be singular, so does the limiting form of the vertices. Moreover, at different limits, the scalar vertex is one order more singular than the vector vertex.

The current is not conserved due to the impurities. Therefore no Ward identity exists for the current vertex, $\Lambda_4(p,0)$, i.e. the vertex for $q = 0$ in the limit $\epsilon \to 0$. 
However, it is easy to see from the general equation for the vertex in the following that, if the irreducible vertex does not provide a singularity stronger than $\epsilon^{-1}$, then

$$\Lambda_\alpha(p, \infty) = \Lambda_\alpha(p, 0).$$

(2.20)

In the pure system Galilean invariance enforces that $\Lambda_\alpha(p, \infty) \sim p_\alpha/m$ while $\Lambda_\alpha(p, 0) \sim p_\alpha/m^*$. In the impure system the two as proved as follows are equal.

We start with the general Ward identity, which is required by gauge invariance[8].

$$
\epsilon \cdot \Lambda_4(k, \omega; q, \epsilon) - q_\alpha \cdot \Lambda_\alpha(k, \omega; q, \epsilon) = \epsilon - q_\alpha \cdot k_\alpha - (M(k + q, \omega + \epsilon) - M(k, \omega))
$$

(2.21)

We can obtain the two aforementioned Ward identities from this by choosing either the static limit, i.e., $\epsilon = 0, q \to 0$, or the dynamic limit, i.e. $q = 0, \epsilon \to 0$. In order to arrive at Eq.(2.18), one needs to assume $\Lambda_4$ is no more singular than $\epsilon^{-1}$ in the static limit. Similarly, to deduce Eq.(2.17), it is assumed that $\Lambda_\alpha$ has a singularity softer that $q^{-1}$. The two assumptions are easily seen to be satisfied through analyzing the singularity of the irreducible interaction vertices.

Now, if we take derivative with respect to $q_\beta$ of Eq.(2.21) without approaching any limit, and then let $q = 0$, we reach

$$
\Lambda_1(k, \omega; q = 0, \epsilon \to 0) = (1 + \frac{\partial M}{\partial k}) \Lambda^0
$$

(2.22)

$$
\Lambda_2(k, \omega; q = 0, \epsilon)y(\omega, \epsilon) = \frac{\partial \Lambda_4(k, \omega; q, \epsilon)}{\partial q}|_{q=0}
$$

(2.23)
In fact, there is one more term in the derivative, $q_\alpha \partial \Lambda_\alpha / \partial q_\beta$, which is negligible as long as the $\partial \Lambda_\alpha / \partial q_\beta$ has no singularity in $q$ stronger than $q^{-1}$. In the end, we let $\epsilon \to 0$, and we arrive at the conclusion that the vector vertex is invariant in different limiting cases provided that $\Lambda_4$ is no more singular than $\epsilon^{-1}$, which we used in deducing the first Ward identity.

In the perturbation approach, where both the interaction and disorder($1/g$) are taken to the first order, Ward identities, i.e. Eq.(2.17), Eq.(2.18), and Eq.(2.23) are proved explicitly by computing both sides to the leading singularity, the details are shown in the appendix.

2.2.2 Correlation functions
Density-density correlation function

Given that the Ward identity Eq.(2.17) holds, the procedure in [7] (p.277) may be used to prove that the limiting form of the correlation function $S_{44}(r = 0)$ is always zero,

$$S_{44}(0) = 0. \quad (2.24)$$

This follows directly from the Gauge invariance or equivalently the continuity equation.
Current-current correlation functions

For pure Fermi-liquids as well as for those in which the singular effects of the impurities are neglected, the dynamical vertices are calculated by first deriving an equation for the static vertices. This procedure is not possible when the irreducible interactions are singular. To proceed, we first define auxiliary vertices, \( \tilde{\Lambda} \) and Inter-
actions \( \tilde{\gamma} \) which satisfy

\[
\tilde{\Lambda}(k, \omega; q, \epsilon) = \lambda(q, \epsilon) + \int dk' d\omega' I(k, \omega, k', \omega'; q, \epsilon) \Phi_{inc}(k', \omega') \tilde{\Lambda}(k, \omega, ; q)(2.25)
\]

\[
\tilde{\gamma} = I + \int I \Phi_{inc} \tilde{\gamma}
\]  (2.26)

which leads to

\[
\tilde{\Lambda} = \lambda + \lambda \Phi_{inc} \tilde{\gamma}
\]  (2.27)

The actual vertex satisfies,

\[
\Lambda = \tilde{\Lambda} + \tilde{\gamma} Q_y(\omega', \epsilon) \Lambda
\]  (2.28)

as shown in Fig.B.6.

We also define auxiliary Correlation functions

\[
\tilde{S}_{\alpha \beta} = \lambda_\alpha \Phi_{inc} \tilde{\Lambda}_{\beta}
\]  (2.29)

in terms of which the actual correlation function is given by, as shown in Fig.B.6,

\[
S_{\alpha \beta} = \tilde{S}_{\alpha \beta} + \tilde{\Lambda}_\alpha Q_y(\omega, \epsilon) \Lambda_{\beta}
\]  (2.30)
Now we make the decomposition as we do with the actual vertex,

\[ \tilde{\Lambda} = \tilde{\Lambda}_1 + \tilde{\Lambda}_2 y(\omega, \epsilon) \quad (2.31) \]

The auxiliary reducible interaction vertices share the same non-analytic form as the original irreducible interaction vertices. Now, we plug them into Eq. (2.28) and note step function in it, we may rewrite the equation as

\[
\Lambda_1 + \Lambda_2 y(\omega, \epsilon) = \tilde{\Lambda}_1 + \tilde{\Lambda}_2 y(\omega, \epsilon) \\
+ \int dk' d\omega' [\tilde{\gamma}_1 + \tilde{\gamma}_{11} y(\omega, \epsilon) + \tilde{\gamma}_{13} \bar{y}(\omega, \epsilon)] Q(k', \omega'; q, \epsilon) y(\omega', \epsilon) (\Lambda_1 + \Lambda_2) \\
+ \int dk' \tilde{\gamma}_{21} y(\omega, \epsilon) Q(k', \omega'; q, \epsilon) (\Lambda_1 + \Lambda_2) \quad (2.32)
\]

Then, we find the relations between the actual vertices and the auxiliary vertices.

\[
\Lambda_1 = \tilde{\Lambda}_1 + \int dk' (\tilde{\gamma}_1 + \tilde{\gamma}_{13}) Q(\Lambda_1 + \Lambda_2) \epsilon \\
\Lambda_2 = \tilde{\Lambda}_2 + \int dk' d\omega' (\tilde{\gamma}_{11} - \tilde{\gamma}_{13}) Q y(\omega', \epsilon) (\Lambda_1 + \Lambda_2) \\
+ \int dk' \tilde{\gamma}_{21} Q(\Lambda_1 + \Lambda_2) \\
= \tilde{\Lambda}_2 + \int dk' \tilde{\gamma}_{21} Q(\Lambda_1 + \Lambda_2) + O(\epsilon) \quad (2.34)
\]

**DC Conductivity**

The DC conductivity is related to current-current correlation function by the Kubo formula,
\[ \sigma = \lim_{\epsilon \to 0} \frac{i}{\epsilon} (S_{\mu\nu}(q = 0, \epsilon) + \frac{n_0}{m} \delta_{\mu\nu}) \quad (2.35) \]

where, \( n_0 \) is electron density and the second term come from the difference between the canonical momentum and real momentum and plays trivial role here.

Inserting the expression for auxiliary vertices into the current-current correlation function, or Eq.(2.30), and using the aforementioned Kubo formula, we obtain the DC-conductivity at zero temperature:

\[ \sigma = \sigma_1 + \sigma_2 \quad (2.36) \]

where

\[
\sigma_1 = \text{Re} \lim_{\epsilon \to 0} \lim_{q \to 0} \int dk d\omega L^0 \frac{\partial}{\partial \epsilon} [\Phi_{\text{inc}} \tilde{A}_1] \\
+ \text{Re} \lim_{\epsilon \to 0} \lim_{q \to 0} \int dk L^0 \Phi_{\text{inc}} \tilde{A}_2 \quad (2.37)
\]

\[
\sigma_2 = \text{Re} \lim_{\epsilon \to 0} \lim_{q \to 0} \int dk (1 - Q \tilde{\gamma}_{21}) Q (\Lambda_1 + \Lambda_2)^2 \quad (2.38)
\]

where, \( \Lambda^0 \) is the bare vector vertex. Eq. (2.37,2.38) are the key results of our approach. It will be used to address the nature of the strong coupling limit hinted at the weak-coupling scaling analysis of Finkelstein in the last section. In the next section, however, we will show that our formalism recovers the standard results when disorder and interactions are treated to the leading orders of the perturbation theory.
Chapter 3
Perturbative Results

We now apply this formalism to the perturbative regime, where both the interaction and disorder \((1/g)\) are taken to the first order. We will consider the case of short-range interactions. The self-energy in this case and in the vicinity of \(\omega = 0\) is shown in Fig.B.4 or

\[
M(k, \omega) = -\frac{i}{2\tau} \text{sgn}(\omega) - \frac{V_1}{2g\tau^2} \ln\left|\frac{1}{|\omega \tau|}\right| G_0(k, \omega; -\text{sgn}(\omega)) + \frac{iV_1 k^2 G_0^3}{2g k_F^2 \tau^3} \omega \ln\left|\frac{1}{|\omega \tau|}\right|
\]

\[
+ \frac{iV_1}{2g\tau} \ln\left|\frac{1}{|\omega \tau|}\right| \text{sgn}(\omega) + \frac{3V_1}{2g} \omega \ln\left|\frac{1}{|\omega \tau|}\right|
\]

\[
= -\frac{i}{2\tau} \text{sgn}(\omega) - \frac{V_1}{2g\tau^2} \ln\left|\frac{1}{|\omega \tau|}\right| G_0(k, \omega; -\text{sgn}(\omega))
\]

\[
+ \frac{iV_1}{2g\tau} \ln\left|\frac{1}{|\omega \tau|}\right| \text{sgn}(\omega) + O(\omega)
\]

(3.1)

where, \(V_1 = V_0 N_F\), \(g = 4\pi^2 \sigma_0\), and \(G_0\) is the non-interacting Green function in the presence of disorder (and with disorder treated in Born approximation). The existence of \(G'_0\) in self-energy actually gives extra pole in \(\epsilon_k\) space. This is an important distinction diffusive corrections bring in.

We also need to know what the irreducible interaction vertices look like perturbatively, in order to compute the conductivity. To the first order in interaction and \(1/g\), we give all the diagrams of irreducible interaction in Fig.B.7. Where, the diffuson, \(I_{21}^d\), and the cooperon, \(I_{21}^c\), are well known particle-hole propagators[5]. Here, we only give the expression in the regime \(\theta(-\omega(\omega+\epsilon)) = 1\). Hereafter, the definition of \(y(\omega, \epsilon)\)
would be changed to \( y(\omega, \epsilon) = \theta(-\omega_0 + \epsilon) \) for simplicity.

\[
I_{21}^d(p, p', \omega) = \frac{1}{2\pi N_F r^2(-i[\epsilon] + Dq^2)} \\
I_{21}^s((p, p', \omega) = \frac{1}{2g r^2} \ln[\frac{1}{\epsilon r}] \delta(K + K' + q)
\]  

(3.2) (3.3)

Among all the diffusion-corrected interactions shown in Fig.B.7, at the perturbative level, we only consider the most singular ones i.e. \( \sim \frac{1}{g r^2} \ln[\frac{1}{g r^2}] \ln \). We will compute the vector vertices, therefore only the p-wave components, denoted by \( l = 1 \), of the irreducible interactions are to our concern. They are

\[
I_{22,l=1}^{--}(k, k'; \omega, q, \epsilon) = \frac{(G_{0,0}^{0}(k, \omega)G_{0,0}^{0}(k', \omega)(\frac{k_s + q_s}{m})(\frac{k'_s + q'_s}{m})}{2\pi N_F r^4} \cdot \frac{V_1}{8gD} \ln[\frac{1}{-\omega + \epsilon}]
\]

\[
I_{14,l=1}^{--}(k, k'; \omega, q, \epsilon) = \frac{G_{0,0}^{0}(k, \omega)G_{0,0}^{0}(k, \omega)(\frac{k_s + q_s}{m})(\frac{k'_s + q'_s}{m})}{2\pi N_F r^4} \cdot \frac{V_1}{8gD} \ln[\frac{1}{-\omega + \epsilon}] \delta(\omega + \epsilon)
\]

\[
+ \frac{V_1}{2g r^2} \ln[\frac{1}{\omega r}] \delta(\omega' + \omega) \delta(k - k)^{l=1}
\]

(3.4)

The two terms in \( I_{14} \) corresponds to the two diagrams in Fig.B.7. The RHS of \( I_{22} \) and the second term in \( I_{14} \), in fact need to be doubled due to interchanging the particle and hole line in the diagram of the vertices. The \( \delta \) function regarding \( \omega \) and \( \omega' \) in the above expression, means that the logarithmic corrections come from scattering, where the energy change is small but does make the particle propagating on the opposite side of fermi surface.

We will first find out the relationship between \( \tilde{\Lambda}_\alpha \) and \( \Lambda_\alpha \).
\[ \tilde{\Lambda}_\alpha = \Lambda_\alpha - \int dk' d\omega' [I_{11} y(\omega, \epsilon) + I_{13} \bar{y}(\omega, \epsilon) + I_{21} \delta(\omega - \omega')]_{l=1} \mathcal{Q}(0) y(\omega', \epsilon) \Lambda_\mu^{(0)} + O(V^2), \] (3.5)

where (1) means both interaction and disorder are taken to first order. Since we are dealing with vector vertices, only the p-wave component of the interaction vertex contributes. Furthermore, we are interested only in the first order of interaction, \(1/g\), and logarithmic singularities. Taking all of these into account, we find that

\[ \tilde{\Lambda}_\alpha = \Lambda_\alpha - y(\omega, \epsilon) \int dk' I_{21,l=1}^{(0)} Q^{(0)} \Lambda_\alpha^{(0)} + O(V^2) \] (3.6)

where, \( I_{21,l=1}^{(0)} \) represents the cooperon in \( I_{21} \), which makes a contribution in the order of \(1/g\) ln.

We then solve for \( \Lambda_\alpha \) to first order of \( V_1 \) and \(1/g\) ln:

\[ \Lambda_\alpha = \Lambda_\alpha^{(0)} + I_{l=1}^{(1)} GG \Lambda_\alpha^{(0)} \]
\[ = \Lambda_\alpha^{(0)} + \bar{y}(\omega, \epsilon) \Lambda_{\alpha 14} + \bar{y}(\omega, \epsilon) \Lambda_{\alpha 22} + y(\omega, \epsilon) \Lambda_{\alpha 21}^{\epsilon} \] (3.7)

where,

\[ \Lambda_{\alpha,14} = \int dk' d\omega' I_{14} \bar{y} G_0 G_0 \Lambda_\alpha^{(0)} \] (3.8)
\[ \Lambda_{\alpha,22} = \int dk' I_{22} \bar{y} G_0 G_0 \Lambda_\alpha^{(0)} \] (3.9)
\[ \Lambda_{\alpha,21}^{\epsilon} = \int dk' I_{21}^{\epsilon} G_0 G_0 \Lambda_\alpha^{(0)} \] (3.10)
The diagrams for $\Lambda_{\alpha,14}, \Lambda_{\alpha,22}$ and $\Lambda_{\alpha,21}^c$ are shown in Fig.B.8, and calculational details are given in appendix. We, then, have the explicit perturbative expression for $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$.

\[
\tilde{\Lambda}_1 = \Lambda_1 = \Lambda_0^{(0)} + \Lambda_{\alpha,14} + \Lambda_{\alpha,22}
\]
\[
\tilde{\Lambda}_2 = \Lambda_2 - \int d\mathbf{k}' I_{21}^c Q^{(0)} \Lambda_\alpha^{(0)} = \Lambda_{\alpha,21}^c(\Phi_{inc}) - \Lambda_{\alpha,14} - \Lambda_{\alpha,22} \tag{3.11}
\]

where, $\Lambda_{\alpha,21}^c(\Phi_{inc})$ is defined as in Eq.(3.10), only that now the two Green’s function involved are of the same sign. Bearing in mind that, perturbatively, $\gamma_{21}$ is just $I_{21}^c$, and inserting these results into Eq. (2.37,2.38), we find that

\[
\sigma_1 = \frac{V_1}{g} \sigma_0 \ln\left[\frac{1}{\epsilon_T}\right]
\]
\[
\sigma_2 = \sigma_0 - \frac{2V_1}{g} \sigma_0 \ln\left[\frac{1}{\epsilon_T}\right] - \frac{1}{g} \sigma_0 \ln\left[\frac{1}{\epsilon_T}\right]. \tag{3.12}
\]

Details of the calculation are given in Appendix A.3.

This result is in agreement with the known work of Altshuler and Aronov [3, 33]. To first order in $1/g \ln$ and interaction, the cooperon correction and the diffusion-corrected interaction are both taken into account with no magnetic field or impurity in presence. The gauge invariance is still satisfied, as long as vertices have both diffusion-corrected interaction and cooperon at the same time are screened out. In other words, we are satisfying the Ward identity by keeping track of corrections to both vertices and self-energy in a consistent fashion.
Chapter 4
Magnetic Metal and Magneto-resistance

While considerable progress has been made in the understanding of the metal-insulator transition in two dimensions [10, 11, 12, 13, 14, 15, 16, 17, 18], the subject has remained controversial [20, 21]. A very basic issue concerns whether or not electron interactions play an important role. In this context, transport in a magnetic field applied parallel to the plane has emerged as a key measurement. There are two significant features in the phenomenology. First, a strong parallel field suppresses the metallic behavior[11, 14, 19, 23, 24, 25, 26, 27]. Second, for a small parallel field the metallic behavior appears to persist[24, 19, 23]. In this paper, we show that both features can be understood when interaction effects are taken into account. Our results also provide further support for a picture of the metal-insulator transition proposed earlier[28].

Theoretically, the nature of the strong coupling fixed point in interacting disordered electrons in two dimensions has been a long-standing open problem. It is mentioned earlier, the results of the one-loop renormalization group (RG) analysis[4, 29] include a) the triplet interaction amplitude is scaled to infinity; b) the single-particle density of states develops a pseudo-gap, going to zero at the Fermi energy; and c) the conductivity is scaled to a finite value. The divergence of the triplet amplitude
occurs at a finite length scale, which has been interpreted as signaling the formation of local moments at such length scales. Since local moments are ordinarily in favor of localization, a natural question is whether the strong coupling fixed point corresponds to a metal or an insulator.

The crucial observation we make here are two-folds. First, a pseudo-gap in the density of states implies that the electron self-energy is singular. This singularity, in turn, dictates that certain effective interactions between the electrons as well as the coupling of the electrons to an external electromagnetic field are also singular. To address the nature of the strong coupling fixed point, then, requires a systematic treatment of these singularities. In fact, we will show that these singularities cancel out with each other and leave a finite conductivity. Second, once the local moments have formed, the exchange coupling among them is expected to lead to a magnetic ordering. Such an ordering will have a finite rigidity against a small Zeeman coupling. The system will then remain metallic for a finite range of Zeeman coupling.

When the density of states vanishes at the chemical potential, the electron self-energy \( \Sigma(k, \omega) \) is singular. Due to the diffusive origin, we expect that the most singular term of \( \Sigma(k, \omega) \) takes the following form,

\[
\Sigma(k, \omega) = f(k)(\omega - \mu)^{-\alpha}
\]

Here \( \mu \) is the chemical potential and \( f(k) \) varies smoothly with \( k \). The exponent \( \alpha \) characterizes the way the density of states goes to zero. The one-loop analysis[32]
gives $\alpha = 1$, but our result will not be sensitive to the specific value of $\alpha$ (so long as it is positive).

With such a singular self-energy and the expression for conductivity as shown in Eq. (2.37,2.38), we can analyze whether it gives a metallic phase ($\sigma(T = 0) \neq 0$) or an insulating phase ($\sigma(T = 0) = 0$).

We will first focus on $\sigma_2$. With a singular self-energy, Eq. (4.1), the expression for $Q$ simplifies to

$$Q(k; q = 0 \epsilon \rightarrow 0) \approx 2i \frac{1}{\Sigma(k, \omega)} \text{Im} \frac{1}{\Sigma(k, \omega)}$$  \hspace{1cm} (4.2)

The factor $A = 1 - Q \tilde{\gamma}_{21}$ in Eq. (2.38) is of order unity and nonzero provided that $\phi_{\text{inc}} I \sim -1$. In the strong coupling regime, $\phi_{\text{inc}} \sim -G_+ G_-$. Moreover $G_- G_+ I \sim 1$ is valid in the perturbation regime and is expected to hold in the strong coupling regime as well. Using the Ward identity, Eq.(2.23), we see that $\Lambda^2$ is of the order of $\Sigma^2$, which compensates the $\frac{1}{\Sigma^2}$ form of $Q$. The result is a finite contribution to the conductivity.

We now turn to $\sigma_1$, which contains two terms. The second term is confined to $\omega = 0$. The first term involves an integration over $\omega$. However, non-analyticity only arises at the infrared, where $\Phi_{\text{inc}} \sim \frac{1}{\Sigma^2}$ and $\tilde{\Lambda}_1 \sim \tilde{\Lambda}_2 \sim \Sigma$. So, in the end, $\sigma_1$ vanishes.

The total conductivity is therefore finite at zero temperature. In other words, the ground state of disordered interacting electrons in 2D at the strong coupling fixed point is a metal.
In addition to providing an understanding of the origin of the metallic phase at zero temperature, our results also allow us to address the transport properties in a parallel magnetic field\cite{35}. A small Zeeman coupling, meaning that the Zeeman coupling is much less than certain critical value, will leave the ordering of the local moments intact. The system, then, remains in the strong coupling regime and the conductivity remains finite.

A large Zeeman coupling, on the other hand, starts to gap out two of the three triplet channels: Only the longitudinal ($S_z = 0$) part of the spin diffusion remains important, together with the charge diffusion. When the screening length is shorter than the mean free path, this problem was already considered in detail in Refs. \cite{4, 29} (see also Ref. \cite{31}). The triplet interaction amplitude no longer grows to infinity (and neither does the energy renormalization factor). The system becomes an insulator.

When the Zeeman coupling is even larger than the energy scale corresponding to the scattering rate due to impurity, the two triplet channels with $|S_z| = 1$, are completely gapped out, and the magnetoresistivity saturates.

Combining the results in the two limits gives rise to the schematic picture shown in Fig.B.10. The system turns from being metallic at small Zeeman couplings to being insulating at large Zeeman couplings. The details of how the system evolves between the two regimes go beyond the scope of this thesis.
Chapter 5
Summary

We have constructed a generalized transport equation for disordered interacting electrons in 2D. The singularities due to interaction and weak-localization are consistently incorporated into the irreducible interaction and into the self-energy.

We have first applied the transport equation perturbatively to the leading order in interaction and disorder. As a check of formalism, the known perturbative result is recovered.

We have then applied the transport equation to the critical regime, where a pseudo-gap in density of states exists at the Fermi energy. The singular self-energy implied by the pseudo-gap leads to an infinitesimal Green function at low energy; the propagating probability for a single particle at low energy is infinitesimal. Nevertheless, the same singularity comes into vector vertex as well, leading to a singular vertex; the current carried by each particle is singular. The compensation of the two singularities gives rise to a finite conductivity.

We have also considered the effect of a parallel magnetic field. When the magnetic field is small, the system remains metallic. For magnetic fields above certain threshold value, it becomes insulating.
Part II
Orthogonality Catastrophe in Bose-Einstein Condensates
Chapter 6
Introduction and Motivation

In the area of correlated electrons, there is a long history of studying quantum impurity problems such as the Kondo effect [40]. A broad range of phenomena arise depending on the type of impurity, the nature of the bulk electron system, and the way they are coupled. The conceptual basis for this richness is provided by the orthogonality catastrophe [41]. It deals with the effect of a single scattering potential on the many-body states of an ideal $N$-fermion system. The many-body ground state wavefunction is a Slater determinant of the $N$ lowest single particle wavefunctions. Since the impurity potential affects each and every one of the single-particle states, its effect on the many-body state is significantly amplified: in the thermodynamic limit the ground state wavefunction in the presence of the impurity potential ($|gs'\rangle$) is orthogonal to its counterpart in the absence of the impurity potential ($|gs\rangle$). For large but finite $N$, the wavefunction overlap, $S \equiv | < gs | gs' \rangle |$, depends on $N$ in an algebraic form:

\[ S_{\text{fermion}} \sim N^{-\alpha}. \]  

(6.1)

The exponent $\alpha = \frac{\delta^2}{d\pi^2}$ when the scattering contains only an s-wave component; here, $\delta$ is the s-wave scattering phase shift of the single-particle state at the Fermi energy, and $d$ the dimensionality. The initial work of Anderson provided an upper
bound. An exact solution was later achieved in the work of Nozières and collaborators [42]. For an illuminating discussion, including the connection with the Friedel sum rule, see Ref. [43].

Cold atoms provide a setting to engineer many-boson systems with a variety of quantum phases [44]. It is natural to ask what happens when local defects are introduced into these systems. Here we consider the orthogonality effect in a Bose-Einstein condensate (BEC). For ideal homogeneous bosons, the problem will be solved exactly in a very simple way. For the cases with interactions and/or confining background potentials, we will carry out analysis perturbatively in the impurity potential.

This part of the thesis is organized as the following. In Chapter 7, we will consider bosons in a homogeneous background with and without interactions. In Chapter 8, the effect of a harmonic oscillator potential is studied, for both the ideal and interacting bosons. We elucidate how the orthogonality catastrophe in bosons affects the absorption spectrum in Chapter 9. In the last Chapter of this part (Chapter 10), we discuss the experimental implications of our results.
Chapter 7
Bosons in a Uniform Background

7.1 Ideal bosons in a uniform background

In this case, the effect of an impurity can be determined exactly. The ground state condensate wavefunction is

$$|gs >= \varphi_0(x_1)\varphi_0(x_2)...\varphi_0(x_N), \quad (7.1)$$

where $\varphi_0$ is the single-particle ground state wavefunction. $|gs'>$ has the same form, with $\varphi_0$ being replaced by $\varphi_0'$, the single-particle ground state wavefunction in the presence of the impurity. The overlap, $S$, is then simply

$$S = s^N \quad (7.2)$$

$$s \equiv <\varphi_0 |\varphi_0'> \quad (7.3)$$

Consider first the case of three dimensions. We choose a spherical box of radius $R$, and a fixed boundary condition $\varphi(r = R) = 0$. The impurity is placed at the center:

$$\Delta H = V\Theta(a - r) \quad (7.4)$$

where $a$ is its size. Define $\psi$ as the radial part of the single-particle wavefunction multiplied by $r$. In the region $0 < r < a$, $\psi_I = A\sin(k_I r)$, with $k_I^2 = \frac{2m}{\hbar^2}(\epsilon - V)$. In the region $a < r < R$, $\psi_{II} = B\sin(k_{II} r + \delta)$, with $k_{II}^2 = \frac{2m}{\hbar^2}\epsilon$, where $\delta$ is the phase
shift. The fixed boundary condition at \( r = R \) yields \( k_{II} = \frac{\pi - \delta}{R} \). The continuity of \( \psi \) and its derivative at \( r = a \) gives rise to

\[
\frac{\tan(k_I a)}{k_I} = \frac{\tan(k_{II} a + \delta)}{k_{II}} \tag{7.5}
\]

Without a loss of generality, we choose the potential to be repulsive.

The phase shift for the single-particle ground state in the thermodynamic limit (\( R/a \to \infty \)) is found to be

\[
\delta_{3d} = -\frac{a}{R}f(V) \tag{7.6}
\]

where \( f(V) = \pi(1 - \frac{\tan(V)}{V}) \), with \( \tilde{V} \equiv (\frac{2mVa^2}{\hbar^2})^{1/2} \). The \( a/R \) dependence reflects the fact that the single-particle state sits at the edge of the band, where the density of states is proportional to \( \sqrt{\epsilon} \propto a/R \). The numerical result [Fig. B.11] shows the validity of this form already for \( R/a \) of the order \( 10^3 \).

From the forms of the wavefunctions in the presence and in the absence of the impurity, we determine the overlap of the single-particle wavefunctions

\[
s = 1 - \frac{1}{6}(1 + \frac{3}{4\pi^2})\delta_{3d}^2 \tag{7.7}
\]

where we have used the fact that \( \delta_{3d} \) is always small in the thermodynamic limit. Combining Eqs. (7.2, 7.7, 7.6), we find a stretched-exponential form for the overlap of the many-body ground states:

\[
S_{3d} = \exp[-\lambda(V)N^{1/3}] \tag{7.8}
\]
where $\lambda(V) = \frac{1}{6}(1 + \frac{3}{4\pi^2})a^2 f^2(V) \left(\frac{4\pi n_0}{3}\right)^{\frac{3}{2}}$, with $n_0$ being the particle density.

Consider next the case of two dimensions – a disk of radius $R$. The ground state wavefunction $\psi$ is now a linear combination of the zeroth order Bessel and Neumann functions: $\psi_I = A J_0(k_I r)$ and $\psi_{II} = B[j_0(k_{II} r) + \tan(\delta)N_0(k_{II} r)]$. The fixed boundary condition at $r = R$ now gives $k_{II} = \frac{3\pi + \delta}{R}$. The continuity at $r = a$ of $\psi$ and its derivatives leads to

$$\frac{k_I J_1(k_I a)}{J_0(k_I a)} = \frac{k_{II}[J_1(k_{II} a) + \tan(\delta)N_1(k_{II} a)]}{J_0(k_{II} a) + \tan(\delta)N_0(k_{II} a)}$$  \hspace{1cm} (7.9)

Using the limiting forms of the Bessel and Neumann functions appropriate for $a/R \to 0$, we find the following phase shift of the single-particle ground state

$$\delta_{2d} = \frac{\pi}{2} \frac{1}{\ln(R/a)}. \hspace{1cm} (7.10)$$

Note that it is independent of the potential strength. The finite value of the density of states at the band edge would have implied a phase shift that is independent of $R/a$. The exact result, on the other hand, contains a logarithmic correction factor.

In Fig. 2, we plot the numerical solution which confirms the logarithmic factor.

Taking these results together we find the overlap of the many-body ground states in two dimensions:

$$S_{2d} = \exp[-\alpha N/\ln^2 N], \hspace{1cm} (7.11)$$

where $\alpha = \frac{\pi}{2} \left[ \frac{\int B^2(x)}{\int J_0^2(k_{II} x)} - 5 \left( \frac{\int J_0(3\pi x)B(x)}{\int J_0(3\pi x)} \right)^2 \right]$, with $B(x) = N_0(3\pi x) - J_1(3\pi x)x$ and $\int f(x) \equiv \int_0^1 f(x)x dx$. 


So far, the results are exact. In order to generalize to situations in which exact solutions are not readily available (see below), we now turn to a perturbative treatment of the impurity potential. To the second order in a linked cluster expansion, the overlap between the many-body ground state wavefunctions is [45]:

\[
S = \exp[-C]
\]

\[
C = \frac{1}{2} \sum_{n \neq \mathbf{p}} \frac{|\langle n | \Delta H | \mathbf{g} \mathbf{s} \rangle|^2}{(E_n - E_\mathbf{g})^2}.
\]  

(7.12)

The impurity scattering can be rewritten as \( \Delta H = V a^\dagger(\mathbf{x} = 0)a(\mathbf{x} = 0) \), where \( a^\dagger(\mathbf{x}) \) and \( a(\mathbf{x}) \) are the boson field operators. The off-diagonal long-range order of a BEC implies that we can set \( a_0 \) and \( a_0^\dagger \) to be \( \sqrt{N} \). The impurity Hamiltonian becomes

\[
\Delta H = \Delta H_1 + V/L^d(N + \sum_{\mathbf{p} \neq 0 \neq \mathbf{p}'} a_{\mathbf{p}}^\dagger a_{\mathbf{p}'}),
\]

where the part of concern to us is

\[
\Delta H_1 = V\frac{\sqrt{N}}{L^d} \sum_{\mathbf{p} \neq 0} (a_{\mathbf{p}}^\dagger + a_{\mathbf{p}})
\]  

(7.13)

Here, \( L \) is the linear dimension of the system. Combining Eqs. (7.12,7.13), and using the dispersion relation \( \epsilon(\mathbf{p}) = p^2/2m \), we have

\[
C = \frac{V^2}{2L^d (2\pi \hbar)^d} \int_{h\pi/L}^\infty dp \ p^{-4}.
\]  

(7.14)

In three dimensions, \( C \) has a divergence of the form \( L \propto N^{1/3} \), which is the same as the exact result [Eq. (7.8)]. (For the spherical box geometry considered earlier, the perturbative result is found to fully agree with what comes out of Eq. (7.8) when \( \lambda(V) \) is expanded to order \( V^2 \).) In two dimensions, the divergence becomes \( L^2 \propto N \).
Compared with the exact result [Eq. 7.11], the perturbative result recovers the $N$ factor in the exponential but misses the multiplicative logarithmic correction.

The perturbative treatment also provides the physical picture for our results. To see this, we rewrite the expression for $C$ in terms of the dissipation spectral function, $\text{Im}\chi_0^{-1}(\omega)$, as follows,

\begin{equation}
C = \frac{1}{2} \int_{(E_1-E_{gs})}^{\infty} d\omega \frac{\text{Im}\chi_0^{-1}(\omega)}{\omega^2} \quad (7.15)
\end{equation}

\begin{equation}
\text{Im}\chi_0^{-1}(\omega) = \sum_{n\neq gs} |<n|\Delta H|gs>|^2 \times
\end{equation}

\begin{equation}
\times \delta(\omega - E_n + E_{gs}) \quad (7.16)
\end{equation}

where $E_1$ is the energy of the first excited state with a non-zero $<n|\Delta H|gs>$. The form of the impurity potential, Eq. (7.13) implies that the dissipation spectral function is simply proportional to the single-particle density of states:

\begin{equation}
\text{Im}\chi_0^{-1}(\omega > 0) \propto \omega^{\frac{d}{2}-1}. \quad (7.17)
\end{equation}

The exponent $(\frac{d}{2} - 1)$ reflects the quadratic nature of the dispersion of the low-lying excitations [47]. It is less than 1 for both three and two dimensions; in the terminology adopted in the dissipated two-level system literature [46], both are sub-ohmic. This abundance of low-lying excitations is responsible for the strong orthogonality in a BEC.
7.2 Weakly interacting bosons in a uniform background

Unlike for fermions, even weak interaction is a relevant perturbation for bosons. We use the standard Bogoliubov transformation [48],

$$a_p = \frac{1}{\sqrt{1 - L_p^2}}[b_p + L_p b_{-p}^\dagger].$$  \hspace{1cm} (7.18)

Under this transformation, the quadratic part of the Hamiltonian becomes

$$H_0 = \sum_{p \neq 0} \xi(p) b_p^\dagger b_p.$$  \hspace{1cm} (7.19)

For $p \gg 2mu$ (where $u = \sqrt{U n_0/m}$ is the sound velocity, with $U$ being the effective contact interaction amplitude), $L_p$ nearly vanishes and we recover the non-interacting limit, including Eq. (7.13) and $\xi(p) \approx \epsilon(p)$. For $p \ll 2mu$, on the other hand, $L_p \approx -1 + p/mu$, and

$$\xi(p) = up$$

$$\Delta H_1 = V \frac{\sqrt{N}}{L^d} \sum_{p \neq 0} \sqrt{\frac{p}{2mu}} (b_p^\dagger + b_p),$$  \hspace{1cm} (7.20)

where the prime denotes that the summation is up to about $mu$. Eq. (7.14) is then replaced by

$$C \approx \frac{V^2 N}{2} \frac{1}{L^d (2\pi \hbar)^d} \frac{(2m)^2 \times}{(2mu)^3 \int_{\hbar p/L}^{2mu} d^d p\ p^{-1} + \int_{2mu}^{\infty} d^d p\ p^{-d}}.$$  \hspace{1cm} (7.21)
It is convergent in both three and two dimensions, implying a finite wavefunction overlap. The resulting overlap in the thermodynamic limit depends on the interaction $U$ in the following (stretched-)exponential forms:

$$S_{U,3d} = \exp[-\lambda'/\sqrt{U}]$$
$$S_{U,2d} = \exp[-\alpha'/U]$$  \hspace{1cm} (7.22)

where $\lambda' \approx \frac{V^2 \sqrt{m_0 m^{3/2}}}{4\pi^2 \hbar^3}$ and $\alpha' \approx \frac{3V^2 m}{8\pi \hbar^2}$.

This conclusion can also be seen through the form of the dissipative-bath spectral function, which, at low-energies, now takes the form [cf. Eqs. (7.16,7.20)]

$$\text{Im} \chi_0^{-1}(\omega > 0) \propto \omega^d.$$  \hspace{1cm} (7.23)

Its super-ohmic nature in two and three dimensions implies that $C$ [cf. Eq. (7.15)] is non-singular, as extensively discussed in the context of the spin-boson problem [46].
Chapter 8
Bosons in a Harmonic Confining Potential

8.1 Ideal bosons in a harmonic confining potential

Consider an isotropic harmonic potential with frequency $\Omega_0$. The thermodynamic limit is defined by keeping

$$N\Omega_0^d = \text{const.} \quad (8.1)$$

as both $N$ and $1/\Omega_0$ go to infinity [49]. (For instance, it ensures a finite energy per particle for ideal fermions.)

We consider an impurity located at the center of the confining potential: $\Delta H = V a_\mathbf{x}^\dagger(x = 0)a(x = 0)$. Using $a(x) = \sum_n \varphi_n(x)a_n$, where $\mathbf{n} \equiv (n_1, \cdots, n_d)$, and the single-particle eigenfunctions are

$$\varphi_n(x) = \left(\frac{m\Omega_0}{2\hbar}\right)^{d/4} \Pi_{i=1}^d (n_i!)^{-\frac{1}{2}} \left(\frac{m\Omega_0}{2\hbar}\right)^{\frac{n_i}{2}} \times$$

$$\times \left(x_i - \frac{\hbar}{m\Omega_0} \frac{d}{dx_i}\right)^{n_i} e^{-m\Omega_0 x^2/2\hbar}, \quad (8.2)$$

we can write the linear part of the impurity Hamiltonian as

$$\Delta H_1 = V \sqrt{N\Omega_0^d} \sum_{n \neq 0} A_{2n} (a_{2n}^\dagger + a_{2n}). \quad (8.3)$$

Here, we have defined

$$A_{2n} \equiv \frac{\varphi_{2n}^2(0)\varphi_0(0)}{\Omega_0^{d/2}} = \frac{(-1)^{\sum_{i=1}^d n_i}}{(\pi\hbar/m)^{d/2}} \Pi_{i=1}^d \left[\frac{(2n_i - 1)!!}{(2n_i)!!}\right]^{1/2}. \quad (8.4)$$
From Eq. (7.12), we have

\[ C = \frac{V^2}{2} \left( \frac{N \Omega_0^d}{\hbar \Omega_0} \right)^2 \frac{1}{\Omega_0^2} \sum_{n \neq 0} \frac{A_{2n}^2}{(2n)^2} \]  \hspace{1cm} (8.5)

where \( n = \sum_{i=1}^{d} n_i \). At large \( n \), we find that \( \sum_{n_2, \ldots, n_d} A_{2n}^2 \) is proportional to \( \sqrt{n} \) in three dimensions and approaches a constant in two dimensions [50], so the summation over \( n \) in Eq. (8.5) is convergent at ultraviolet. \( C \) is then proportional to \( \Omega_0^{-2} \), which, using the thermodynamic limit, is equivalent to \( N^{2/d} \). The overlap between the ground state wavefunctions is then

\[ S_{ho,3d} \propto \exp[-\lambda_{ho} N^{2/3}] \]  \hspace{1cm} (8.6)

\[ S_{ho,2d} \propto \exp[-\alpha_{ho} N] \]  \hspace{1cm} (8.7)

where \( \lambda_{ho} = \frac{V^2}{4\pi} \zeta(\frac{3}{2})(m/\pi \hbar^2)^3(N \hbar^3 \Omega_0)^{\frac{1}{2}} \) and \( \alpha_{ho} = (m/\hbar^2)^2 / 48 \).

### 8.2 Weakly-interacting bosons in a harmonic confining potential

For a system confined in a harmonic oscillator potential, in the framework of Bogoliubov theory, the exact field operator can be separated into two terms, \( a(x) = \Psi(x) + \hat{\phi}(x) \), where \( \Psi(x) \) represents the condensate wave function and \( \hat{\phi}(x) \) characterizes the remaining non-condensate fluctuation [59]. Following this, the corresponding impurity perturbation is \( \delta H = V[n_0(0) + \sqrt{n_0(0)}(\hat{\phi}(0) + \hat{\phi}_I(0))] \), where \( n_0(0) = |\Psi(0)|^2 \) and assuming \( \Psi(0) \) real. In analogy with Bogoliubov’s theory on Bose gas in a uniform background, it is shown in A. Fetter’s work on confined Bose
gas that the fluctuation field can be expanded in a set of normal modes \([60, 61]\), i.e.
\[
\hat{\phi}(r) = \sum' \left[ u_j(r) \alpha_j - v_j^*(r) \alpha_j^\dagger \right],
\]
where the prime means that the sum runs over all excited states. It is worth to point out that in Fetter’ work, a time-dependent field operator is considered and we, on the other hand, only consider, the special case, its static form. in terms of the quasi-particle operator, the perturbation of our interest is as

\[
\Delta H = V \sqrt{n_0} \sum'_j \left[ (u_j(0) - v_j(0)) \alpha_j + (u_j(0) - v_j(0))^\dagger \alpha_j^\dagger \right]. \quad (8.8)
\]

In the limit of \(Na/a_{ho} \gg 1\), where \(a\) is the scattering length and \(a_{ho} = (\hbar/m\omega)^{1/2}\) is the harmonic oscillator length, the quasi-particle can be described by the collisionless hydrodynamics, provided that the interaction is strong enough so that the density profile is smooth and one can neglect the quantum pressure in the Gross-Pitaevskii Equation [59]. In this regime, we know that, at the origin,

\[
n'_j = \sqrt{n_0}(u_j - v_j) \quad (8.9)
\]
\[
\Phi'_j = \frac{i}{2M\sqrt{n_0}}(u_j + v_j), \quad (8.10)
\]

where \(n'_j\) and \(\Phi'_j\) are the eigenvalues of density fluctuation operator and perturbation in phase operator, and \(\Psi(0)\) is assumed real as \(\sqrt{n_0}\) at the origin. Therefore, in the framework of hydrodynamic theory, with Eqs. (8.9,8.8), one can write down,
\[ C' = \frac{V^2}{2} \sum_j \frac{n_j^2(0)}{\hbar^2 \omega_j} \quad (8.11) \]

where \( \omega_j^2 = \Omega_0^2 |j(2j + 3)| \) is the energy spectrum of the normal modes and \( n_j' \) is the corresponding eigenfunctions. In fact, \( n_j' \) is defined in the region \( 0 < r < R \) and has the form of Jacobi Polynomial. \( R \), the size of the condensate, is defined as \( \mu = \frac{1}{2} m \Omega_0^2 R^2 \) according to Thomas-Fermi approximation.

Combining the normalization \( \int d^3r (|u_j|^2 - |v_j|^2) = 1 \) and Eqs. (8.10, 8.9), the value of \( n_j'(0) \) can be found as

\[ n_j'(0) = \left( \frac{\hbar \omega_j}{8 \pi g R^2 J_j} \right)^{1/2} \quad (8.12) \]

where \( J_j = \int_0^1 P_j(x^2) x^2 dx \) and \( P_j(x) \) is the \( j \)th order of Jacobi polynomial [51]. \( J_j \) decreases as \( j \) becomes larger. Therefore, the contribution to the exponent factor \( C' \) due to hydrodynamic normal modes is

\[ C' = \lambda_{hy} \sum_j \frac{1}{J_j[j(2j + 3)]^{1/2}} \quad (8.13) \]

where \( \lambda_{hy} = \frac{V^2}{16 \pi g} \left( \frac{m}{2\mu R^3} \right)^{3/2} (N \hbar^3 \Omega_0^3)^{2/3} N^{-2/3} \). This is the result for 3D. So, \( C' \), the contribution of hydrodynamic normal modes, clearly depends on the ultraviolet cutoff \( j_c \) in the summation. Below which the hydrodynamic theory is valid. In fact we know that the hydrodynamic description only works for low-lying excitations.

Naturally, the question, now, is what are the excitations going beyond hydrodynamic theory and how these excitations contribute to the exponent factor \( C \). Let’s
first recall the criteria for which the hydrodynamic description is valid. The condition
is that the spatial variance of the collective modes’ density is larger than the cohe-
rence length or healing length [62]. In other words, \( R \gg \frac{\hbar}{(2\pi m n_0 U)^{1/2}} \) or \( n_0 U \gg \frac{\hbar^2}{2m} \left( \frac{1}{R} \right)^2 \),
where \( R \) is size of the condensate and \( j \) is the number of radial nodes. This gives
\( j_{c1} = \frac{2\mu}{\hbar \Omega_0} \) at the origin, which is similar to the cutoff \( p \gg 2\mu \) or \( \epsilon(p_\text{c}) = \mu = n_0 U \) in
the homogeneous case. In fact, there exists a more intuitively direct way to deduce
this cutoff number or the threshold energy. The hydrodynamic theory only works
when the interaction energy is much larger than the kinetic energy. And interaction
energy per particle in this case can be characterized by \( U n_0 \) and kinetic energy per
particle is proportional to \( j \hbar \Omega_0 \) for high energy excitations. So, from these, we simply
get \( j_{c2} = \frac{\mu}{\hbar \Omega_2} \). We see that the critical number we obtained in two ways are of the
same order of magnitude, and we use \( j_{c1} \) in the following calculations as the cutoff for
the hydrodynamic regime. It is reasonable to believe that for high-energy excitations
beyond the cutoff, the interaction part can be neglected and the particles behave in
a trapped-free-particle manner and the corresponding dispersion relation and eigen-
functions recover the harmonic oscillator results. And we have the overlap due to
high-energy free-particle-like excitations as

\[
S_{ho,3d} = \exp \left( -\lambda_{ho}^3 N^{1/2} \right) \tag{8.14}
\]

\[
S_{ho,2d} = \exp \left( -\alpha_{ho}^2 N^{1/2} \right) \tag{8.15}
\]
where \( \chi'_\text{ho} = \frac{V^2}{2\sqrt{4\pi\mu}}(N\hbar^2\Omega_0^3)^{1/2}(\frac{m}{\hbar^2})^3 \) and \( \alpha'_\text{ho} = \frac{V^2}{ln\mu}(N\hbar^2\Omega_0^2)^{1/2}(\frac{m}{\hbar^2})^2 \). It is seen that the orthogonality, though reduced by interparticle interactions, remains. They behave in the same asymptotic way in both 2D and 3D in the thermodynamic limit. The dependence on the strength of the interaction \( U \), interestingly, in 2D and 3D, are the same as the homogeneous case.

The factor, \( C' \), due to the contribution of excitations described by hydrodynamic theory can be calculated in 3D as well. Using the ultraviolet cutoff \( j_c = \frac{2\mu}{\hbar m_0} \) and asymptotic form of \( J_j \propto j^{-2} \), we find

\[
C' \propto \frac{V^2(m/\hbar^2)^{3/2}}{U^{1/2}n_0^{-1/2}}.
\] (8.16)

This means that the overlap integral due to hydrodynamic excitations is a constant in the thermodynamic limit. The dependence of \( C' \) on the boson-boson interaction \( U \) shares the same form as in the homogeneous case and as in the high energy free-particle-like case. This provides a consistency check of the theory.

Similar results apply to two dimensions. The contribution of hydrodynamic excitations to \( C'_{2d} \) in the thermodynamic limit is a constant as well. The dependence of \( C'_{2d} \) on boson-boson interaction is also \( C'_{2d} \propto \frac{V^2}{U} \) as in the homogeneous case. Likewise the high energy contribution in the free-particle-like case.
Chapter 9
Change of Absorption Spectrum due to Orthogonality

In analogy with the infrared divergence problems in metals due to orthogonality catastrophe in fermions [43], we expect different photo-absorption spectra between two inner levels in an atom with or without a surrounding Bose-Einstein Condensate.

We consider the most elementary kind of transition between two bound levels on an atom, normally which would just be a sharp delta function in the absence of BEC environment. Now, consider a condensate co-existing with a localized different species atom at the center. The two atomic levels of the local atoms will provide different scattering potentials. For simplicity, it is assumed that the localized atoms have no interaction with the BEC atoms, when they are in the ground state; and the scattering length between the localized atoms and the BEC atoms is $a_{12}$, when the former are in the excited states. In the more realistic case, it is the difference between the scattering potential that matters.

It is easy to see according to Fermi’s Golden Rule that, for the aforementioned setup, the corresponding intensity of the photo-absorption spectrum is

$$P(E) = |W|^2 \sum_{n=0}^{\infty} b_n^2 \cdot \delta(E - (E_n - E_0^0) - \Delta).$$

(9.1)

where $W$ is the transition matrix element between the two inner-levels of the lo-
calized atoms and \( b_n = | \langle \Psi_n | \Phi_0 \rangle | \) is the overlap integral between the unperturbed ground state wavefunction and the wavefunction of the perturbed states. By perturbation, we simply mean that the local atoms are pumped to the excited state, and therefore can be seen by the BEC as a local impurity and cause the orthogonality catastrophe. \( E_n \) is the eigenvalue of the perturbed Hamiltonian and \( E_0^0 \) is the ground state energy of the unperturbed Hamiltonian, and \( \Delta \) is the energy gap between the two inner-levels of the localized atoms.

If we measure the frequency relative to the threshold \( E_{th} = (E_0 - E_0^0) + \Delta \), and define \( \omega = E - E_{th} \), then we have

\[
P(\omega) = |W|^2 \sum_{n=0}^{\infty} b_n^2 \cdot \delta(\omega - (E_n - E_0)).
\]

(9.2)

Perturbation theory to second order of local impurity potential \( V = \frac{4\pi\hbar^2\alpha_{12}}{m} \) yields

\[
b_0^2 = 1 - \sum_{n=0}^{\infty} \frac{| \langle \Phi_0 | \Delta H_1 | \Phi_n \rangle |^2}{(E_n^0 - E_0^0)^2}
\]

(9.3)

\[
b_n^2 = \sum_{n=0}^{\infty} \frac{| \langle \Phi_0 | \Delta H_1 | \Phi_n \rangle |^2}{(E_n^0 - E_0^0)^2}.
\]

(9.4)

If Eq. (9.3) for \( b_0 \) is written in a exponential form for impurity potential is small, then the orthogonality catastrophe is recovered. Now plug Eqs. (9.3,9.4) into Eq. (9.2) and perform Fourier transform on it, we get

\[
p(t) \propto \exp \left[ -\sum_{n=0}^{\infty} \frac{| \langle \Phi_0 | \Delta H_1 | \Phi_n \rangle |^2}{(E_n^0 - E_0^0)^2} (1 - e^{-i(E_n-E_0)t}) \right].
\]

(9.5)
First, we will consider the absorption spectrum in the homogenous case. The evaluation of the summation is quite similar to what we did in the previous orthogonality case. The difference is that the factor $e^{-i(E_n - E_0)t} \simeq e^{-ic(p_n)}$ in the summation makes the cutoff in the infrared limit $p_n \simeq \sqrt{\frac{2m}{|t|}}$ as opposed to in the orthogonality case, where the infrared cutoff is given by the quantum level spacing $p \simeq \frac{\hbar}{L}$. We get, in 3D and 2D,

\begin{align}
 p_{3d}(t) &\propto \exp \left[ -\lambda_{sp} \sqrt{|t/\hbar|} \right] \\
p_{2d}(t) &\propto \exp \left[ -\alpha_{sp} |t/\hbar| \right]
\end{align}

where $\lambda_{sp} = 4\pi V^2 n_0 \left( \frac{2m}{\hbar^2} \right)^{3/2}$ and $\alpha_{sp} = \pi V^2 n_0 \frac{2m}{\hbar^2}$. Now, we Fourier transform the spectrum back into the frequency space again and end up with,

\begin{align}
P_{3d}(\omega) &\propto \frac{2}{\pi \lambda_{sp}^2} e^{-\lambda_{sp} \sqrt{1/\omega}} (e^{\lambda_{sp} \sqrt{1/\omega}} - 1 - \lambda_{sp} \sqrt{1/\omega}) \\
P_{2d}(\omega) &\propto \frac{1}{\alpha_{sp} \left( \frac{1 + (\omega/\alpha_{sp})^2}{\alpha_{sp}} \right)}
\end{align}

In the evaluation for $P_{3d}$, we have taken the Fourier factor $e^{i\omega t/\hbar}$ into account by introducing a cutoff at $t/\hbar \sim 1/\omega$. The function for 3D looks complicated, but in fact its plot as shown in Fig. (B.13) is quite simple. The local perturbation determines the height and width of the function, and the smaller the perturbation is, the more it looks like a \textit{delta} function. The spectrum for 2D is simply a Lorentzian function. As shown in Fig. (B.14 $\alpha_{sp}$ decides its width and its inverse gives the height. So,
in 2D, similar to the case in 3D, the smaller the localized potential is, the more the spectrum looks like a δ function.

Secondly, we will consider the case in which the BEC is confined in a Harmonic Oscillator trap. Similar to what we did in the orthogonality case, we can evaluate Eq. (9.5) by plugging in the dispersion relation and wave-functions for Harmonic Oscillator and we get,

$$p_{ho}(t) \propto \exp \left[ -\frac{V^2}{h} \sum_{n \neq 0} \frac{A^2_{2n}}{(2n)^2} \left( 1 - e^{-i(\Omega_0 \frac{1}{2} + n) t} \right) \right]$$  \hspace{1cm} (9.10)

Recall that $A_{2n,3d} = \left( \frac{m}{\hbar} \right)^3 \frac{2\sqrt{n}}{\sqrt{\pi}}$ and $A_{2n,2d} = \left( \frac{m}{\hbar} \right)^2$, and that the factor $e^{-i(\Omega_0 \frac{1}{2} + n) t}$ provides a cutoff on the infrared limit $n_c \simeq \frac{1}{\Omega_0 t}$ for the summation, it is straightforward that

$$p_{ho,3d}(t) \propto \exp \left[ -\lambda_{sp,ho} |t|^{1/2} \right]$$ \hspace{2cm} (9.11)

$$p_{ho,2d}(t) \propto \exp \left[ -\alpha_{sp,ho} |t| \right],$$ \hspace{2cm} (9.12)

where $\lambda_{sp,ho} = \frac{V^2 \hbar^3 \Omega_0^3}{\pi^2 \hbar^2 \sqrt{\Delta_0}} \left( \frac{m}{\hbar} \right)^3 \sqrt{\Omega_0}$ and $\alpha_{sp,ho} = \frac{V^2 \hbar^3 \Omega_0^2}{4 (\hbar \Omega_0)^2} \left( \frac{m}{\hbar} \right)^2 \Omega_0$. Perform Fourier Transform on it, and we get in frequency space that

$$P_{ho,3d}(\omega) \propto \frac{2}{\pi \lambda_{sp,ho}^2} e^{-\lambda_{sp,ho} \sqrt{\frac{T}{\omega}}} \left( e^{\lambda_{sp,ho} \sqrt{\frac{T}{\omega}}} - 1 - \lambda_{sp,ho} \sqrt{\frac{T}{\omega}} \right)$$ \hspace{2cm} (9.13)

$$P_{ho,2d}(\omega) \propto \frac{1}{\pi \alpha_{sp,ho} \left( \frac{\omega}{\alpha_{sp,ho}} \right)^2} \frac{1}{1 + \left( \frac{\omega}{\alpha_{sp,ho}} \right)^2}$$ \hspace{2cm} (9.14)
So, apparently, the absorption spectrum line-shapes in the homogeneous case and harmonic oscillator trap case have exactly the same form both in 3D and 2D. The difference lies in that the factors $\lambda$ and $\alpha$, in the thermodynamic limit, approach different limits. While in the homogeneous case, they are constants, in the harmonic oscillator case, they go to infinity in the form of $\lambda_{sp,ho} \propto \Omega_0^{-3/2}$ and $\alpha_{sp,ho} \propto \Omega_0^{-1}$ as $\Omega_0$ goes to zero. Therefore, the absorption spectrum line-shapes are far from their counterpart in the homogeneous case as shown in Fig. (B.13,B.14) or $\delta$ function in the thermodynamic limit. As a matter of fact, they should become more and more flat as $\Omega_0$ goes to zero. We expect experiments performed on such cases and show the distinctiveness of line-shapes in the two cases in the thermodynamic limit. So far, it is the kind of BEC made of ideal bosons that we are concerning. the interacting bosons in homogeneous case will not affect the spectrum due to a spoiled orthogonality effect and the spectrum of confined interacting bosons is a more subtle issue which deserve further more research.
Chapter 10
Experimental Implications

In addition to the theoretical significance, the orthogonality effect may also be directly probed in experiments. One way is to perform the analog of the x-ray edge problem in metals [43, 45]. Consider a condensate co-existing with a separate species of atoms that are considerably more dilute and localized (by a deep optical potential well that only these atoms see). The photo-absorption or luminescence corresponding to a transition between two levels of this second species of atoms would ordinarily be a sharp delta function. (In practice, the spectral width of a hyperfine transition for atoms in a BEC can be as narrow as \( \sim 100 \) Hz or even \( \sim 10 \) Hz [54].) However, the two atomic levels will provide different scattering potentials for the atoms of the condensate. The weight of the delta function – which is precisely the overlap of the condensate wavefunctions corresponding to these two different potentials – would then have to vanish in the thermodynamic limit due to the orthogonality catastrophe. [For an order of magnitude estimate, we may use the perturbative expression, Eq. (8.6).]

Taking a scattering length \( a_{12} \) associated with the impurity potential \( V \) to be such that \( \hbar/ma_{12}^2 \approx 2\pi \times 1 \) MHz, and a harmonic frequency \( \Omega_0 \approx 2\pi \times 100 \) Hz, Eq. (8.6) yields [55] an overlap \( S_{ho,3d} \approx \exp(-2 \times 10^4 N) \), which is already \( e^{-200} \) for a condensate particle number of \( 10^6 \).] The strong orthogonality effect will be manifested in the
absorption spectrum as we proposed in the previous chapter. With current typical BEC experiments conditions, however, the coefficients $\lambda_{sp}^2$ and $\alpha_{sp}$ in the homogeneous case are as tiny as $10^{-36}$Hz. Therefore, it is impossible to distinguish the spectra from a $\delta$-function. So is it in the case of HO confinement in 2D. In which, the coefficient $\alpha_{sp,ho} \sim 10^{-8}$Hz. But, in the case of HO confining potential in 3D, with current experimental reality, we see that the coefficient $\lambda_{sp,ho}^2 \sim 10^4$Hz. This is sufficiently big to see the corresponding spectrum is dramatically different from $\delta$-function due to the large width caused by the $\lambda_{sp,ho}^2$. We cordially hope that experiments are performed in this case to check the theory.

The orthogonality may also be manifested in the time evolution of a condensate after a sudden introduction of a local potential. The orthogonality makes it rather hard for the system to evolve into the new ground state. In other words, the density distribution will tend to keep its initial profile; the impurity is “hardly visible” to the condensate.

Yet another implication may be seen through a Kondo effect. Localized atoms can be introduced by deep optical potentials. An effective Kondo problem arises when both the localized atoms and the itinerant ones forming a condensate contain real or pseudo- spin degrees of freedom [57, 58]. Strong orthogonality makes the spin flip processes entirely incoherent.

To summarize, we have studied the orthogonality effect in Bose-Einstein conden-
sates. For ideal bosons, the overlap of the ground state wavefunctions when a single local scattering potential changes from one value to another vanishes in a stretched-exponential form in the thermodynamic limit. With interactions, the overlap becomes finite but is small for weak interactions; its dependence on the interaction strength is typically stretched-exponential as well. These effects can be probed using spectroscopical experiments in cold atoms, which can be tuned from being essentially ideal to strongly interacting. These effects also have significant implications for the coherence and decoherence phenomenon in bosonic systems.
Part III

Effects of a Collective Spin Resonance Mode on the STM Spectra of D-Wave Superconductors
Chapter 11
Introduction and Motivation

A prominent feature in the excitation spectrum of the high-$T_c$ superconductors is the “41 meV” collective spin resonance mode, seen by inelastic neutron scattering experiments in most of the cuprate families [63, 64, 65, 66]. The physics of this resonance mode – including its microscopic origin, its connections with other physical properties, as well as its role on superconductivity itself – has been the subject of considerable debate. Given the recent developments of the atomic resolution scanning tunneling microscopy (STM) [67, 68, 69, 70, 71], it is timely to address the possible manifestation of this resonance mode in the local density of states (LDOS). A number of theoretical works [72, 73, 74, 75] have addressed the effect of related spin fluctuations on the LDOS. These works focused on the pinning of the spin fluctuations by impurity: a dynamical spin mode centered around the wavevector $Q$ leads to a $2Q$ spatial modulation in the low-energy LDOS. This result is smoothly connected to what happens in the case of a static spin-density-wave ordering [76, 77]. However, for the resonance mode – which is sharply peaked at $Q = (\pi, \pi)$ – such effects would not be manifested [since $2Q = (2\pi, 2\pi)$ is equivalent to $(0, 0)$]. There are also works about quasiparticle scattering from nonmagnetic impurities [78, 79].

In this paper, we show that the coupling of $d$-wave quasiparticles to the resonance
mode does produce spatial modulations in the LDOS around an impurity. The feature is located at relatively high energies, $\approx \pm (\Delta_0 + \Omega_0)$, where $\Delta_0$ is the maximum superconducting energy gap and $\Omega_0$ the resonance energy. In addition, the wavevector of the LDOS modulation is close to $Q = (\pi, \pi)$. Our predictions could be observable by the STM experiments. Such STM studies represent a new means to characterize the coupling between the electronic excitations and the resonance mode. The STM feature we discuss relates to the “peak-dip-hump” structure of the angle resolved photoemission spectroscopy (ARPES) [80, 81, 82, 83, 84, 85]; the inference about the electron-spin coupling from the ARPES and related spectroscopies is a topic of recent controversy [83, 82] and we hope that the STM studies we propose will shed new light on this important issue.

This part of the thesis has three chapters. In Chapter 12, we formulate the problem and in Chapter 13, we show our numerical treatment and results. In Chapter 14, we summarize this part of the work.
Chapter 12
General Formulation

We start with a model Hamiltonian describing two-dimensional electrons coupled to a collective spin mode and in the presence of a single-site impurity: \( \mathcal{H} = \mathcal{H}_{BCS} + \mathcal{H}_{sp} + \mathcal{H}_{imp} \). Here the BCS-type Hamiltonian for a uniform \( d \)-wave superconductor is given by \( \mathcal{H}_{BCS} = \sum_{k,\sigma} (\varepsilon_k - \mu) c_{k\sigma}^\dagger c_{k\sigma} + \sum_k (\Delta_k c_{k1}^\dagger c_{-k1} + \Delta_k^* c_{-k1} c_{k1}) \), where \( c_{k\sigma}^\dagger \) (\( c_{k\sigma} \)) creates (annihilates) a conduction electron of spin \( \sigma \) and wavevector \( k \), \( \varepsilon_k \) is the normal state energy dispersion for the conduction electrons, \( \mu \) the chemical potential, and \( \Delta_k = \frac{\Delta_0}{2}(\cos k_x - \cos k_y) \) the \( d \)-wave superconducting energy gap. The coupling between the electrons and the resonance mode is modeled by an interaction term \( \mathcal{H}_{sp} = g \sum_i \mathbf{S}_i \cdot \mathbf{s}_i \), where the quantities \( g, \mathbf{s}_i, \) and \( \mathbf{S}_i \) are the coupling strength, the electron spin operator at site \( i \), and the operator for the collective spin degrees of freedom, respectively. The dynamics of the collective mode will be specified below. The impurity scattering is given by \( \mathcal{H}_{imp} = U_0 \sum_\sigma c_{i\sigma}^\dagger c_{i\sigma} \), where without loss of generality we have taken a single-site impurity of strength \( U_0 \) located at the origin, \( \mathbf{r}_i = 0 \). By introducing a two-component Nambu spinor operator, \( \Psi_k = (c_{k1}, c_{-k1}^\dagger)^T \), the matrix Green's function for the \( d \)-wave BCS Hamiltonian \( \mathcal{H}_{BCS} \) is determined by

\[
G_0^{-1}(k; i\omega_n) = \begin{pmatrix}
  i\omega_n - \xi_k & -\Delta_k \\
  -\Delta_k & i\omega_n + \xi_k
\end{pmatrix},
\]

where \( \xi_k = \varepsilon_k - \mu \) and \( \omega_n = (2n + 1)\pi T \) is the fermionic Matsubara frequency. We have also assumed that the \( d \)-wave pair
potential is real.

For a homogeneous system, where only the inelastic scattering of quasiparticles from the collective mode occurs, we calculate the self-energy to the second order in the coupling constant (see Fig. B.15b):

\[
\tilde{\Sigma}(k; i\omega_n) = \frac{3g^2T}{4} \sum_q \sum_{i\Omega_l} \chi(q; i\Omega_l) G_0(k - q; i\omega_n - i\Omega_l),
\]

where \(\chi(q; i\Omega_l)\) is the dynamical spin susceptibility \(\chi_{ij}(\tau) = \langle T_r(S_r^x(\tau)S^y_0(0))\rangle\) and \(\Omega_l = 2l\pi T\) the bosonic Matsubara frequency. The dressed Green’s function is:

\[
G^{-1}_0(k; i\omega_n) = \begin{pmatrix}
  i\omega_n - \xi_k - \Sigma_{11} & -\Delta_k - \Sigma_{12} \\
  -\Delta_k - \Sigma_{21} & i\omega_n + \xi_k - \Sigma_{22}
\end{pmatrix}.
\]

The corresponding real-space dressed Green’s function \(G_0(i, j; i\omega_n)\) is obtained through a Fourier transform with respect to \(r_i - r_j\). For the \(d\)-wave pairing symmetry, one can show that the local Green’s function, \(\hat{g}_0(i\omega_n) = G_0(i, i; i\omega_n)\) is diagonal. In the presence of a single-site impurity at \(r_1 = 0\) with potential strength \(U_0\), the site-dependent Green’s function can be written in terms of the \(T\)-matrix:

\[
G(i, j; E) = G_0(i, j; E)
+ \sum_{lm} G_0(i, l; E) \hat{T}_{lm}(E) G_0(m, j; E).
\]

Due to the vertex corrections induced by the coupling to the collective modes (Fig. B.15a), the \(T\) matrix in general contains site-off-diagonal terms. We will first carry out the calculation without the vertex corrections, in which case \(\hat{T}_{lm} = \hat{T}\delta_{l,0}\delta_{m,0}\), with
\hat{T}^{-1} = U_0^{-1} \sigma_3 - \hat{g}_0, \text{ where } \sigma_3 \text{ is the } z\text{-component of the Pauli matrix. The LDOS at}
the \(i\)-th site, summed over two spin components, is

\[ \rho(\mathbf{r}_i, E) = -\frac{2}{\pi} \text{Im} G_{11}(i, i; E + i\gamma), \]  \hspace{1cm} (12.3) 

where \(\gamma = 0^+\).
Chapter 13
Numerical Treatment

Up to now, our discussion and formulation are quite general and can be used to study the effects of any dynamic mode once the susceptibility $\chi$ is known. We treat the susceptibility in a phenomenological form (based on the inelastic neutron scattering observations), see also [84]:

$$\chi(\mathbf{q}; i\Omega_l) = -\frac{\delta_{0,Q}}{2} \left[ \frac{1}{i\Omega_l - \Omega_0} - \frac{1}{i\Omega_l + \Omega_0} \right], \quad (13.1)$$

where we denote the wavevector $\mathbf{Q} = (\pi, \pi)$ and the mode energy by $\Omega_0$. This form is especially suitable for the optimally doped YBa$_2$Cu$_3$O$_{6+y}$ (YBCO) compounds in the superconducting phase, where the observed neutron resonance peak is almost resolution-limited in energy and fairly sharp in wavevector. The resonance peak in BSCCO is broadened in both energy and wavevector. The finite width in the wavevector space might be important for the ARPES lineshape in general and in particular the understanding of the ARPES spectra away from the $M$ points [$\mathbf{k} = (\pi, 0)$ and symmetry-related points] of the Brillouin zone [84], but should not change the qualitative conclusion of our work: the LDOS effects we will discuss arise from the fact that the dominant effects of the resonance mode on the single-electron spectral functions occur near the $M$ points which is expected to remain to be the case beyond our simplified form for the susceptibility. In addition, given that the peak in BSCCO
is still quite sharp in energy, we expect that the main effect of the broadening in energy of the resonance mode is to extend the bias window for the LDOS feature we will discuss. We have also neglected the incommensurate peaks seen in the inelastic neutron scattering experiments in YBCO (the part that disperses “downward” away from the resonance peak) [86, 87, 88, 89], since their spectral weight is significantly smaller than that of the resonance mode. For the normal-state energy dispersion, we use \( \varepsilon_k = -2t(\cos k_x + \cos k_y) - 4t' \cos k_x \cos k_y \), where \( t \) and \( t' \) are the nearest and next-nearest neighbor hopping integral. Unless specified explicitly, the energy is measured in units of \( t \). We choose \( t' = -0.2 \) to model the band structure of the hole-doped cuprates. Since the maximum energy gap for most of the cuprate superconductors at the optimal doping is about 30 meV while the resonance mode energy is in the range between 35 and 47 meV, we take \( \Delta_0 = 0.1 \) and \( \Omega_0 = 0.15 \) (i.e., \( 1.5\Delta_0 \)). The chemical potential is tuned to give an optimal doping value 0.16. To mimic the intrinsic life time broadening, in our numerical calculation we take \( \gamma \) of Eq. (12.3) to be \( 0.04\Delta_0 \). A system size of \( N_x \times N_y = 1000 \times 1000 \) is taken in the numerical calculation.

In the absence of impurities, the density of states is the summation of the spectral function, \( A_k(E) = -\frac{2}{\pi} \text{Im} G_{\alpha,11}(k; E + i\gamma) \), over all wavevectors \( k \). Fig. B.16(a) shows the spectral function at an \( M \) point of the Brillouin zone. Without the electron-mode coupling, as is well known, the spectral function is peaked at the maximum gap edges \( \pm \Delta_0 \). As the electron-mode coupling is switched on, new peaks emerge at the energies...
\( \pm E_1 \approx \pm (\Delta_0 + \Omega_0) \). (For simplicity, we have neglected the broad "background" part of the single-electron spectral function in our consideration.) The peaks in \( A_k \) originate from the poles of the Green's function \( G_{0,11} \). Note that the weight of the peak at \(-\Delta_0\) is larger than at \(\Delta_0\) because the van Hove singularity is below the Fermi energy. Since the spectral weight of the spin resonance mode [i.e., \( \text{Im}\chi(q;\omega) \)] is peaked at \( Q \), the feature of the quasiparticle self-energy is the strongest around the \( M \) points of the zone because they are connected by \( Q \). The singularity in the quasiparticle self-energy causes additional poles in the Green's function. As the coupling constant \( g \) increases, these peaks are shifted to higher energies and, in addition, their spectral weight is enhanced; simultaneously, the weight of the superconducting coherent peaks is reduced to obey the sum rule. The shift of states due to inelastic scattering is expected in DOS and is also expected for scattering off local mode [90]. Fig. B.16(b) plots the density of states and clearly shows that the high energy peaks still occur around \( \pm E_1 \). In other words, the contributions from near the \( M \) points dominate the wavevector summation for the density of states, reflecting the flat nature of the normal state band near this point. Furthermore, the highly asymmetrical structure in the DOS at energies \(-E_1\) and \(E_1\) comes from the singular structure in the quasiparticle self-energy. These results, for the clean case, are consistent with earlier studies of the ARPES [80, 81, 82, 83, 84] and DOS [85].

We are now in a position to address the LDOS in the presence of a single nonmag-
netic impurity. For concreteness, we take the on-site potential $U_0 = 100\Delta_0$. Fig. B.17 shows the LDOS directly at the impurity site, as well as at its nearest neighbor. The near-zero energy resonant state triggered by the quasiparticle scattering from the impurity [91] is robust against the electron-mode coupling. Our key new results are two-fold. First, the impurity modifies the shape of the spectral features at $\pm E_1$, which can now be either a dip or a peak. Second, and more importantly, these high energy features at $\pm E_1$ exhibit a spatial dependence. At the impurity site, the LDOS displays a dip at $-E_1$ but a peak at $+E_1$. The behavior is reversed at the site closest to the impurity.

To explore this spatial variation of the LDOS in more detail, we have calculated the LDOS, with the energy fixed at $-E_1$, in the vicinity of the impurity with and without the mode coupling. In the absence of mode coupling, we obtain the results (not shown) which are consistent with previous studies [91]: the LDOS exhibits a Friedel oscillation along the diagonals of CuO$_2$ plane but no other non-trivial features. When the mode coupling is turned on, in addition to the Friedel oscillation along the diagonals, the LDOS displays a new type of modulation with a period of $2a$ in the wide regions along the bond direction. This new modulation can be more easily seen Fig. B.18(a) when the pre-dominant Friedel oscillation is filtered away. It occurs in four disconnected triangles in the field of view.

Also in order to highlight the new modulation, we find it useful to perform a filtered
Fourier transform, \( \rho(q, E) = \sum' e^{iq \cdot r_i} \rho(r_i, E) \), where \( \sum' \) denotes a summation over all the sites in four triangles in Fig. B.18(a). The resulting Fourier-transformed image [92] is given in Fig. B.18(b), which unambiguously shows that the new feature induced by the coupling to the spin resonance mode has a spatial modulation wavevector \((\pi, \pi)\).

This new type of LDOS modulation with a wavevector close to \( Q = (\pi, \pi) \) reflects the dominance of the collective mode effect near the \( M \) points of the Brillouin zone. It follows from Eqs. (12.2,12.3), with the form of the \( T \)-matrix under consideration and when the chosen field of view has an inversion symmetry with respect to the impurity site, that the Fourier transformed LDOS for any finite \( q \) is,

\[
\rho(q, E) = -\frac{2}{\pi} \text{Im} \int dp \left[ G_0(p + q, E) \hat{T}(E) G_0(p, E) \right]_{11} .
\]  \hspace{1cm} (13.2)

At positive energies, the poles shown in Fig. B.16 for \( G_0(p, E) \) are located [80, 81, 82, 83, 84] at \( E_p \) and \( E_{p-Q} + \Omega_0 \), respectively. (Here, \( E_p \equiv \sqrt{\xi_p^2 + \Delta_p^2} \).) Likewise, \( G_0(p + q, E) \) is the sum of two poles, one at \( E_{p+q} \) and the other at \( E_{p+q-Q} + \Omega_0 \). At \( E = E_1 \), Eq. (13.2) is dominated by the convolution between the poles at \( E_{p-Q} + \Omega_0 \) and \( E_{p+q-Q} + \Omega_0 \). This term in turn is dominated by the contributions corresponding to when both \( p - Q \) and \( p + q - Q \) are near to the \( M \) points, leading to a \( \rho(q, -E_1) \) that is peaked near \((\pi, \pi)\).

Similar phase-phase considerations show that the vertex correction terms lead to a similar momentum dependence. In the presence of vertex corrections, the \( T \)-matrix satisfies an integral equation in the wavevector space. The vertex correction to the
$T$-matrix to the same order ($g^2$) of our calculation for the self-energy is shown in Fig. B.15(a). It involves a wavevector convolution of a form similar to that given in Eq. (13.2). We therefore expect [93] an additive contribution to $\rho(q, -E_1)$ that is also peaked near ($\pi, \pi$).
Chapter 14
Discussion and Summary

Finally, we remark on issues which go beyond the idealized model we have considered so far: (i) In the presence of a gap inhomogeneity at the nanoscale, the mode signature would appear at $\langle \Delta_0 \rangle + \omega_0$, with a slight smearing. Here, $\langle \Delta_0 \rangle$ is the spatially averaged superconducting gap. The fact that a well-defined peak-dip-hump structure appears in the break-junction tunneling spectrum [94] implies that the smearing is not too large; (ii) The mode signature depends on the detailed band structure. In the cuprates, the band is flat near the antinodal points (i.e., $(\pi, 0)$ etc.), and a mode with a momentum significantly different from $(\pi, \pi)$ will have a weaker effect compared to that of the $(\pi, \pi)$ mode we addressed; (iii) Tunneling matrix elements need to be taken into account in order to understand the detailed spatial variation of the LDOS as observed around zinc impurities in BSCCO [95, 96]. Such a filtering effect, however, will not affect our conclusion on the momentum of the LDOS modulation.

To summarize, we have studied the effects of the magnetic resonance mode on the tunneling spectrum in the presence of a nonmagnetic impurity. The LDOS around the impurity displays resonant features at relatively high energies [close to $\pm (\Omega_0 + \Delta_0)$], which modulates in space with a wavevector close to $(\pi, \pi)$. Our prediction can be tested straightforwardly by operating the existing high resolution STM at a relatively
high energy window. Such studies should shed considerable new light on the physics of the spin resonance mode and its coupling to the electronic excitations.
References

22. Steven M. Girvin. cond-mat/9907002.
34. This is supported by the one-loop RG result of Refs. [4, 29], which indeed has a singular self-energy but a finite conductivity. Note that the singlet-only problem considered in Refs. [4, 29] is very different. From the RG analysis, the pseudo-gap will not be developed until the system has already scaled to the strongly localized regime. In that case, our formulation no longer applies.
35. A parallel field also induces some orbital effect, due to the finite confinement length in the third direction (S. Das Sarma and E. H. Hwang, Phys. Rev. Lett. 84, 5596 (2000)). This effect, however, has been demonstrated to be small in Si-MOSFETs (Ref. [18]).


38. See also P. Schwab and C. Castellani, Phys. Rev. Lett. 84, 4779 (2000); Q. Si and C. M. Varma, 84, 4780 (2000).


47. The sub-ohmic form also arises from spin waves of a ferromagnet, which come into play in a different context (S. Kirchner, L. Zhu, Q. Si, and D. Natelson, in preparation).


50. It is important to use the strong form of Sterling’s formula, \( n! \approx \sqrt{2\pi n}n^n e^{-n} \), which implies \( (2n - 1)!!/(2n)!! \approx 1/\sqrt{\pi n} \).

51. \[ P_j(x) = \frac{x^j}{(-1)^j j! (j+3/2)!} P_j^0(0, 1/2)(2x^2 - 1) \] and \( P_j^0(0, 1/2)(x) \)


54. R. Hulet, private communications.

55. We take $V^2(m/\pi \hbar^2)^3 \approx (16/\pi)(\hbar^2/ma_0^2)^{-1}$.


69. C. Howald et al., cond-mat/0201546.


75. S.A. Kivelson et al., cond-mat/0210683.
92. Similar procedure for the LDOS in the absence of a mode coupling yields a Fourier-transformed LDOS that is featureless near $(\pi, \pi)$. 
93. We have numerically calculated the leading order vertex correction contribution to the LDOS, finding that it is indeed peaked near $(\pi, \pi)$.


Appendix A
Details of Calculation

A.1 Ward identities

In this appendix we show that, to the order of \( \frac{1}{g} \ln \) and \( \frac{V}{g} \ln \), a) both the Ward identity for the vector vertex, Eq. (2.18), and that for the scalar vertex, Eq. (2.17), are valid; and b) the vector vertex in the static limit (\( \epsilon \rightarrow 0 \) first and then \( q \rightarrow 0 \)) and the dynamic limit (\( q \rightarrow 0 \) first and then \( \epsilon \rightarrow 0 \)) are equal to each other [Eq. (2.20)].

We start with the Ward identity for the vector vertex, Eq. (2.18), as well as the equality of the vector vertex in the two limits, Eq. (2.20). To the order \( \frac{1}{g} \ln \) and \( \frac{V}{g} \ln \), the vector vertex can be explicitly calculated and has the following form,

\[
\Lambda_\alpha(k, \omega; q, \epsilon) = \Lambda^{(0)}_\alpha + \int dk' d\omega' \left( (I_{14} \bar{y}(\omega', \epsilon) + I_{22} \delta(\omega - \omega')) \bar{y}(\omega, \epsilon) + I_{21}^c \delta(\omega - \omega') y(\omega, \epsilon) \right) \Lambda^{(0)}_\alpha + O.T. \tag{A.1}
\]

This is just Eq.(3.7) without letting \( q \rightarrow 0 \). O.T. stands for other terms which are non-zero for generic \( q \) and \( \epsilon \) but vanish in either the static limit or the dynamic limit. The first three correction terms are illustrated in Fig.B.8. They correspond to \( \Lambda_{\alpha,14}, \Lambda_{\alpha,22}, \) and \( \Lambda^{c}_{\alpha,21} \) introduced in Eq. (3.10). Note that \( I_{21}^c \) here means only the Cooperon contribution. Written explicitly, the three correction terms are
\[ \Lambda_{a,22} = \int (d\Omega) \int (dk') \int (dQ) \frac{k'_x}{m} G_{0,-}(k',\omega) G_{0,-}(k' + q,\omega + \epsilon) \]
\[ \cdot G_{0,+}(k' + q + Q,\omega + \epsilon + \Omega) G_{0,+}(k + q + Q,\omega + \epsilon + \Omega) \]
\[ \cdot \frac{i V \theta(-\epsilon + \omega)(\omega + \epsilon + \Omega)) \theta(-\omega(\omega + \epsilon + \Omega))}{2\pi N_F \tau^4(-i\Omega + DQ^2)^2(-i(\epsilon + \Omega) + D(q + Q)^2)} \]

\[ \Lambda_{a,14} = \int (d\Omega) \int (dk') \int (dQ) \frac{k'_x}{m} G_{0,-}(k',\omega) G_{0,+}(k' + Q,\omega + \Omega) \]
\[ \cdot G_{0,+}(k' + q + Q,\omega + \epsilon + \Omega) G_{0,+}(k + q + Q,\omega + \epsilon + \Omega) \]
\[ \cdot \frac{i V \theta(-\omega(\omega + \Omega)) \theta(-\epsilon + \omega)(\omega + \epsilon + \Omega)) \theta(-\omega(\omega + \epsilon + \Omega))}{2\pi N_F \tau^4(-i\Omega + DQ^2)^2(-i(\epsilon + \Omega) + D(q + Q)^2)} \]
\[ + \int (d\Omega) \int (dQ) \frac{k_x}{m} G_{0,+}(k + q + Q,\omega + \epsilon + \Omega) \]
\[ \cdot G_{0,+}(k + Q,\omega + \Omega) \theta(-\omega(\omega + \Omega)) \theta(-\omega + \epsilon(\omega + \epsilon + \Omega)) \]

\[ \Lambda_{a,21} = \int (dQ) G_{0,+}(k + Q,\omega + \epsilon) G_{0,-}(k + q + Q,\omega) \frac{1/2\pi N_F \tau^2}{-i\epsilon + DQ^2} \]

(A.2)

\( G_0 \) is still the Green function for the noninteracting impurity system. Without losing generality, we choose \( \omega \) negative. We leave the expression as it is without making any computation, so that when the limits are taken, we can see clearly that the result are not sensitive to how the limits are approached. This point is actually easy to see, because neither the Green functions nor the corrected interaction( the fraction terms above) are sensitive to the commutation of the limits. Therefore, the results we will obtain at either limit are the same. Or, in mathematical language
\[ \Lambda_\alpha(k, \omega; q \to 0, \epsilon = 0) = \Lambda_\alpha(k, \omega; q = 0, \epsilon \to 0). \] (A.3)

In both the static and dynamic limit, the sum of \( \Lambda_{\alpha,22} \) and the first term of \( \Lambda_{\alpha,14} \) is proportional to \( \epsilon \). (More generally, this linear relationship occurs so long as \( q \) goes to zero.) Therefore, it goes to zero either way. \( \Lambda_{\alpha,21}^\infty \) is proportional to \( y(\omega, \epsilon) \), so it vanishes in the dynamic limit. In addition, the factor multiplying \( y(\omega, \epsilon) \) is only logarithmically singular in terms of \( \epsilon \). Therefore, \( \Lambda_{\alpha,21}^\infty \) vanishes even in the static limit. So, in the end, only the second term of \( \Lambda_{\alpha,14} \) gives rise to a finite contribution. In addition, the result is the same for the static and dynamic limit. Two conclusions follow. First, we have proven the equality of the vector vertex in the two limits, as specified by Eq. (2.20). Second, the perturbative expression for the vector vertex for both the static and dynamic limit is given by

\[ \Lambda_\alpha(q = 0, \epsilon = 0) = \Lambda_\alpha^0 + \int (d\Omega) \int (dQ) \frac{iV}{(-i\Omega + DQ^2)^2 \tau^2 m} k_x \cdot G_0,+(k + Q, \omega + \Omega)^2 \theta(-\omega(\omega + \Omega)) \] (A.4)

Recall the expression for self energy as in Eq.(3.1), take derivative with respect to \( k_x \) and we find exactly the same result as the correction obtained above. Therefore Eq.(2.18) is proved perturbatively.

We now work on the Ward Identity defined in Eq.(2.17), which basically connects
the derivative of self energy with respect to frequency to the scalar vertex in the
dynamic limit. We will prove it in different regimes, \( \theta(-\omega(\epsilon + \omega)) \) and \( \theta(-\omega(\epsilon + \omega)) \),
because different diagrams make the correction in the two opposite regimes. First,
in the \( \theta(-\omega(\epsilon + \omega)) \) regime, let us focus on the correction to scalar vertex to the
first order in interaction and \( 1/g \), which to the zeroth order in \( \epsilon \), are composed of
the diagrams(a-f,A-E) in Fig. B.9. One point worth to be pointed out is that both
diffuson and cooperon exists only in the vicinity of fermi surface in the frequency
space, or \( |\omega| < \frac{1}{\tau} \), and this actually gives the cutoff in infinity. In the computation,
only the leading singularity of \( \delta(\omega) \ln[\frac{-1}{\omega \tau}] \) are kept, and terms with only a \( \delta \) function
or a \( \ln[\frac{-1}{\omega \tau}] \) are dropped for not as singular as the former term, which can easily been
seen after integrating them over \( \omega \). We give the final result of the diagrams as follows,
assuming \( \omega \to 0^- \) which is actually a result of the assumption \( \epsilon > 0 \).

\[
a + b + c + d + e = \frac{-2iV_1}{g\tau} \ln\left[\frac{-1}{\omega \tau}\right] \delta(\omega)
\]
\[
c' + d' + e' = 0
\]
\[
e'' = \frac{iV_1}{2g\tau^3} G_{0,+,0,-} \ln\left[\frac{-1}{\omega \tau}\right] \delta(\omega)
\]
\[
f = \frac{iV_1}{g\tau} \ln\left[\frac{-1}{\omega \tau}\right] \delta(\omega)
\]
\[
A + B = A' + B' = A'' + B'' = 0
\]
\[
C + D = C' + D' = C'' + D'' = 0
\]
\[ E = \frac{i}{\tau} \delta(\omega) \] (A.5)

The RHS, which involves self energy derivative, can be easily obtained from taking derivative of Eq.(3.1) with respect to \( \omega \) and only keeping the most singular term, which is to the order of \( \delta(\omega) \ln[\frac{1}{\omega \tau}] \). One should be careful with the derivative and it is easier to perform by using the definition \( \frac{M(\omega + \epsilon) - M(\omega)}{\epsilon} \) at \( \theta(-\omega(\theta + \omega)) \). It is worth to point out that the only thing involves in derivative is the sign function and the logarithmic function is treated as \( \omega \) independent for as \( \omega \rightarrow 0 \) the thermal fluctuation will come in as the cutoff. We show the result as follows,

\[ \frac{\partial M}{\partial \omega} = \frac{-i}{\tau} \delta(\omega) + \frac{iV_l}{g \tau} \ln[\frac{1}{\omega \tau}] \delta(\omega) + \frac{-iV_l}{2g \tau^3} G_{0,+} G_{0,-} \ln[\frac{1}{\omega \tau}] \delta(\omega) \] (A.6)

Therefore, Eq.(2.17) is satisfied at \( \theta(-\omega(\epsilon + \omega)) \).

On the opposite regime, the correction diagrams(g-j) are as shown in Fig.B.9, where the most singular term is proportional to \( \omega^{-1} \).

\[ g + h = 0 \]

\[ i = \frac{-V_l}{2g \tau^2} G_{0,+}(k,0) \frac{1}{\omega} \]

\[ j = \frac{iV_l}{2g \tau} \text{sgn}(\omega) \frac{1}{\omega} \] (A.7)

The derivative of self energy involves the logarithmic terms only because the sign of \( \omega + \epsilon \) and \( \omega \) are the same, and therefore gives singularity of \( \omega^{-1} \). It is easy to see it
is just the opposite of the sum of the two terms above. So, Eq.(2.17) holds as well in \( \theta(\omega(\omega + \epsilon)) \).

### A.2 Ward identity due to nonanalytic interactions

In this appendix, we will prove the relationship between the dynamic component of the vector vertex and the derivative of the scalar vertex, as specified by Eq.(2.23). We first compute contributions to the LHS with leading order in \( \epsilon \), i.e., zeroth order. In order to do that, let's revisit Eq.(3.11) and write down the perturbative expression for \( \Lambda_2 \).

\[
\Lambda_2 = \Lambda_{\alpha,21}^\epsilon - \Lambda_{\alpha,14} - \Lambda_{\alpha,22}
\]

(A.8)

The corresponding correction diagrams for \( \Lambda_2 \) defined as above are shown in Fig.B.8. One should be aware that the diagrams involved \( I_{14} \) and \( I_{22} \) do NOT exist at \( y(\omega, \epsilon) \), on the contrary they exist only at \( \tilde{y} \). The reason they appear in \( \lambda_2 \) lies in that we rewrite \( \tilde{y} \) as \( 1 - y \), and the \( y(\omega, \epsilon) \) part leads to \( \Lambda_2 \). So,

\[
\Lambda_2 = \frac{kV_1}{2gmT^2} \ln \left| \frac{1}{-\omega_T} \right| G_{0,-}^2 + \frac{-ik}{2gmT^2} \ln \left| \frac{1}{\epsilon_T} \right| G_{0,+}G_{0,-} + O(\epsilon).
\]

(A.9)

where the contribution actually comes from the third and fourth diagram in Fig.B.8. The contribution from the first two diagrams is proportional to \( \epsilon \), therefore dropped.
Now that the RHS of Eq.(2.23) involves first order derivative of scalar vertex with respect to \( q \) at \( q = 0 \), we need to look for diagrams which is odd function of \( q \). Also, the scalar vertex to our interest should have a \( y(\omega, \epsilon) \) as a factor and apart form the \( y \) function be of the order of \((-1)\) in \( \epsilon \). Therefore, we end up with diagram \( e'' \), \( A, B, C \), and \( D \) as shown in Fig.B.9, which all have a Green function left unintegrated on the particle line, and it is in fact this Green function that gives the odd dependence in \( q \). In fact, diagram \( B \) with the unintegrated Green Function on the hole line also contributes due to the special structure of cooperon. Calculation shows that

\[
\frac{\partial e''}{\partial q} \bigg|_{q=0} = \frac{kV_1}{2g\mu \tau^2} \ln[\frac{1}{-\omega \tau}]G_{0,-}^2 y(\omega, \epsilon)
\]

\[
\frac{\partial (A' + B')}{\partial q} \bigg|_{q=0} = \frac{-ik}{2g\mu \tau^2} \ln[\frac{1}{\epsilon \tau}]G_{0,+}G_{0,-} y(\omega, \epsilon)
\]

\[
C'' + D' = 0 \quad \text{(A.10)}
\]

Comparing the result with \( \Lambda_2 \), we just proved Eq.(2.23).

### A.3 Correction to conductivity

In this section, the details of computing correction to conductivity will be shown. Now that we have the expression for \( \tilde{\Lambda}_\alpha \), \( \Lambda_\alpha \) and Green function in first order of interaction and \( 1/g \), the current-current correlation function can been written as

\[
S_{\alpha\alpha} = S_{1,\alpha\alpha} + S_{2,\alpha\alpha}
\]
\begin{align*}
S_{1,aa} &= \Lambda_\alpha^{(0)} \Phi_{inc} \tilde{\Lambda}_1 + y(\omega, \epsilon) \Lambda_\alpha^{(0)} \Phi_{inc} \tilde{\Lambda}_2 \\
S_{2,aa} &= [\Lambda_\alpha^{(0)} + \Lambda_\alpha^{(20)} - \int dk^' I_{20} Q^{(0)} \Lambda_\alpha^{(0)}] Q y(\omega, \epsilon) [\Lambda_\alpha^{(0)} + \Lambda_\alpha^{(20)}] \\
&= \Lambda_\alpha^{(0)} Q y(\omega, \epsilon) \Lambda_\alpha^{(0)} + \Lambda_\alpha^{(0)} G_0(\omega + \epsilon) G_0(\omega) I_{21} \Lambda_\alpha^{(0)} G_0(\omega + \epsilon) G_0(\omega) y(\omega, \epsilon) \Lambda_\alpha^{(0)}
\end{align*}

(A.11)

where \( \tilde{\Lambda}_1 \) and \( \tilde{\Lambda}_2 \) are defined in Eq.(3.11). One may find that in the calculation with \( I_{21} \) involved, the Green functions are directly reduced to zeroth order in interaction. The reason for that lies in, first, calculation dealing with cooperon and diffusion-corrected interaction present at the same time is inconsistent with the self-energy we considered, therefore violate the gauge invariance. \( S_1 \) and \( S_2 \) lead to the contribution of \( \sigma_1 \) and \( \sigma_2 \) respectively.

Recall that we mentioned previously that the correction to self-energy has an extra pole in it, therefore the Normal Fermi-liquid theory treatment doesn’t apply here. Fortunately, if we in first order of perturbation take \( G = G^0 + G^0 \delta M G^0 \), where \( \delta M \) is the sum of the last two terms in \( M \), then the extra pole will be treated correctly and give right correction to conductivity. The reason to justify such an operation lies in that \( \frac{V_1}{g} \) is viewed as small quantity. Keeping all the terms with logarithmic singularity and to first order in \( V_1 \) and \( 1/g \), we have

\[ S_{1,aa} = S_{11} + S_{12} \]
\[
S_{11} = -\frac{n_0}{m} \\
S_{12} = \frac{-iV_i}{g} \sigma_0 \epsilon \ln\left[\frac{1}{\epsilon\tau}\right] \\
S_{2,aa} = S_{21} + S_{22} \\
S_{21} = -i\epsilon\sigma_0 + \frac{2iV_i}{g} \epsilon\sigma_0 \ln\left[\frac{1}{\epsilon\tau}\right] \\
S_{22} = \frac{i}{g} \epsilon\sigma_0 \ln\left[\frac{1}{\epsilon\tau}\right] \\
\text{(A.12)}
\]

where, \(\sigma_0\) is the conductivity of a pure noninteracting electron system. \(S_{11}\) and \(S_{12}\) correspond to the first and second terms in \(S_1\), and \(S_{21}\) and \(S_{22}\) are defined similarly.

Plug these expressions back into Eq.(2.35), we obtain the conductivity in first order perturbation at zero temperature.

\[
\sigma_{aa} = \sigma_0(1 - \frac{V_i}{g} \ln\left[\frac{1}{\epsilon\tau}\right] - \frac{1}{g} \ln\left[\frac{1}{\epsilon\tau}\right]) + O\left(\frac{V}{g^2}, \frac{1}{g^2}\right) \\
\text{(A.13)}
\]

We give further detailed calculation in the following. Plugging in the expression for \(\tilde{\Lambda}_1\), we have

\[
S_{11} = \Lambda^{(0)}_{a,\phi_{inc}}\Lambda^{(0)}_{a} + \Lambda^{(0)}_{a,\phi_{inc}}\Lambda_{a,34} + \Lambda^{(0)}_{a,\phi_{inc}}\Lambda_{a,22} \\
\text{(A.14)}
\]

The first term is in fact

\[
\Lambda^{(0)}_{a}G(\omega + \epsilon)G(\omega; \text{sgn}(\omega + \epsilon))\Lambda^{(0)}_{a} = A + 2B \\
A = \Lambda^{(0)}_{a}G_0(\omega + \epsilon)G_0(\omega; \text{sgn}(\omega + \epsilon))\Lambda^{(0)}_{a} \\
B = \Lambda^{(0)}_{a}G_0^2(\omega + \epsilon)G_0(\omega; \text{sgn}(\omega + \epsilon))\delta M(\omega + \epsilon)\Lambda^{(0)}_{a} \\
\text{(A.15)}
\]
where, in first order of interaction, $G = G_0 + G_0 \delta M G_0$.

\[
A = (-i) \int_{-\epsilon}^{\infty} (d\omega) \int (dk) \left( \frac{k_x}{m} \right)^2 \frac{1}{\omega + \epsilon - \epsilon_k + \alpha + \frac{i}{2\tau}} \frac{1}{\omega - \epsilon_k + \alpha + \frac{i}{2\tau}} \\
+ (-i) \int_{-\infty}^{-\epsilon} (d\omega) \int (dk) \left( \frac{k_x}{m} \right)^2 \frac{1}{\omega + \epsilon - \epsilon_k + \alpha - \frac{i}{2\tau}} \frac{1}{\omega - \epsilon_k + \alpha - \frac{i}{2\tau}} \\
= \frac{-n_0}{m}
\]  

(A.16)

where $n_0$ is the carrier's density. In fact, an extra prefactor of $(-i)$ is due to the fact that the correlation function makes a fermion loop.

Recall the expression for $\delta M$, which is the self-energy due to diffusion-corrected interaction, we have

\[
B = (-i) \int_{-\epsilon}^{\frac{1}{\tau}} (d\omega) \int (dk) \left( \frac{k_x}{m} \right)^2 G_{0,+}^{3} G_{0,-}^{3} \left( \frac{V_1}{2g\tau^2} \right) \ln \left[ \frac{1}{(\omega + \epsilon)\tau} \right] \\
+ (-i) \int_{-\frac{1}{\tau}}^{-\epsilon} (d\omega) \int (dk) \left( \frac{k_x}{m} \right)^2 G_{0,-}^{3} G_{0,+}^{3} \left( \frac{V_1}{2g\tau^2} \right) \ln \left[ \frac{1}{(\omega + \epsilon)\tau} \right] \\
= 0
\]  

(A.17)

In the calculation above, the second term of $\delta M$ is also plugged in, yet it makes no contribution due to the fact that all the Green functions involved are on the same side of fermi surface [3]. The zero, in the end, results from that the integration ranges from the lower cutoff to upper cutoff, leaving no nonanalyticity. And for the same reason the contributions from the last two terms in $S_{11}$ are zero.
As for $S_{12}$,

$$S_{12} = -\Lambda^{(0)}_\alpha \phi_{\text{inc} y}(\omega, \epsilon) \Lambda_{\alpha, 14} - \Lambda^{(0)}_\alpha \phi_{\text{inc} y}(\omega, \epsilon) \Lambda_{\alpha, 22} + \Lambda^{(0)}_\alpha \phi_{\text{inc} y}(\omega, \epsilon) \Lambda^{c}_{\alpha, 21}(\Phi_{\text{inc}}) \tag{A.18}$$

The first term can be broken into two parts, corresponding to $\Lambda_{\alpha, 14, 1}$ and $\Lambda_{\alpha, 14, 2}$ respectively, which are second and third diagram in Fig.B.8. We find

$$\Lambda^{(0)}_\alpha \phi_{\text{inc} y}(\omega, \epsilon) \Lambda_{\alpha, 34, 1} = \frac{-2iV_1}{g} \sigma_0 \epsilon \ln\left[\frac{1}{\epsilon \tau}\right]$$

$$\Lambda^{(0)}_\alpha \phi_{\text{inc} y}(\omega, \epsilon) \Lambda_{\alpha, 34, 2} = \frac{2iV_1}{g} \sigma_0 \epsilon \ln\left[\frac{1}{\epsilon \tau}\right] \tag{A.19}$$

Therefore we reach the result of zero contribution from the first term.

$$\Lambda^{(0)}_\alpha \phi_{\text{inc} y}(\omega, \epsilon) \Lambda_{\alpha, 22} = \frac{-iV_1}{g} \sigma_0 \epsilon \ln\left[\frac{1}{\epsilon \tau}\right] \tag{A.20}$$

The second term therefore makes contribution of $-\frac{iV_1}{g} \sigma_0 \epsilon \ln\left[\frac{1}{\epsilon \tau}\right]$. The last term in $S_{12}$ is zero for that the four Green’s functions have poles on the same side of the complex plane of $\varepsilon_k$, and their integration range in frequency space is finite. At this point, we may give the analytic expression for $S_1$, the same as shown in Eq.(A.12).

As for $S_2 = S_{21} + S_{22}$, we have, first, the expression for

$$S_{21} = \Lambda^{(0)}_\alpha G_0(\omega + \epsilon) G_0(\omega) I_{21}^c G_0(\omega + \epsilon) G_0(\omega) \Lambda^{(0)}$$

$$= \frac{i}{g} \epsilon_0 \ln\left[\frac{1}{\epsilon \tau}\right] \tag{A.21}$$
then the analytic expression for

$$S_{22} = \Lambda^{(0)}_\alpha Q_y(\omega, \epsilon)\Lambda^{(0)}_\alpha$$

$$= \Lambda^{(0)}_\alpha G_0(\omega + \epsilon)G_0(\omega)y(\omega + \epsilon)\Lambda^{(0)}_\alpha$$

$$+ 2\Lambda^{(0)}_\alpha G_0^2(\omega + \epsilon)\delta M(\omega + \epsilon)G_0(\omega)y(\omega + \epsilon)\Lambda^{(0)}_\alpha$$

$$- 2\Lambda^{(0)}_\alpha G_0^2(\omega + \epsilon)\delta M(\omega + \epsilon)G_0(\omega; sgn(\omega + \epsilon))y(\omega + \epsilon)\Lambda^{(0)}_\alpha$$

$$= F + 2G - 2H \quad (A.22)$$

where, F, G and H are the three terms in second line.

$$F = (-i) \int_{-\epsilon}^{0} (d\omega) \int (dk) \left( \frac{k_z}{m} \right)^2 \frac{1}{\omega + \epsilon - \epsilon_k + \frac{i}{2\tau}} \frac{1}{\omega - \epsilon_k - \frac{i}{2\tau}} = (-i)\sigma_0\epsilon$$

$$G = (-i) \int_{-\epsilon}^{0} (d\omega) \int (dk) \left( \frac{k_z}{m} \right)^2 G_{0, +}^2 G_{0, -} \left( -\frac{V_1}{2g\tau^2} \ln\left[ -\frac{1}{\omega\tau} \right] G_{0, -} + \frac{iV_1}{2g\tau} \ln\left[ \frac{1}{\omega\tau} \right] \right)$$

$$= \frac{iV_1}{2g} \sigma_0\epsilon \ln\left[ \frac{1}{\epsilon\tau} \right]$$

$$H = (-i) \int_{-\epsilon}^{0} (d\omega) \int (dk) \left( \frac{k_z}{m} \right)^2 G_{0, +}^3 \left( -\frac{V_1}{2g\tau^2} \ln\left[ -\frac{1}{\omega\tau} \right] G_{0, -} \right)$$

$$= -\frac{iV_1}{2g} \sigma_0\epsilon \ln\left[ \frac{1}{\epsilon\tau} \right] \quad (A.23)$$

Therefore, in the end, we obtain the final expression for the full correlation function.

$$S = (-i)\sigma_0\epsilon \left( 1 - \frac{1 + \frac{V_1}{g} \ln\left[ \frac{1}{\epsilon\tau} \right]}{m} \right) + \frac{n_0}{m} \quad (A.24)$$
Appendix B
Figures
Figure B.1  Temperature dependence of resistivity for different carriers’ densities, from Kravchenko et. al., Phys. Rev. B 51, 7038(1995).
Figure B.2  Resistivity grows dramatically as the parallel magnetic field increases, from Simonian et. al., Phys.Rev. Lett. 79, 2304(1997).

\[ \text{Figure B.3} \quad \text{The product of two Green's functions. The upper and lower propagators are for one particle and one hole, respectively.} \]
Figure B.4  Self-energy of single particle to the first order in interaction and disorder. Only the contributions from the exchange channel are displayed.

Figure B.5  Transport equation in general. Here, $\gamma$ is the interaction vertex, $\Lambda$ the field vertex, and $S_{\mu\nu}$ the current-current correlation function.
Figure B.6  Isolation of the nonanalyticity in vertex and transport equation.
Figure B.7  Irreducible interaction vertices to the first order in interaction and disorder. Only those that have a logarithmic singularity are kept.
Figure B.8  Vector vertices which are nonzero at either the static or the dynamic limit. The first two diagrams are accompanied by those obtained by exchanging the particle and hole lines. Both the second and third diagrams belong to $\Lambda_{34}$. 
Figure B.9  Diagrams for scalar vertices to the first order in interaction and disorder. Diagrams linear or higher order in $\epsilon$ are neglected.
Figure B.10  Schematics of the resistivity as a function of parallel field at different temperatures. The system is metallic at small fields, but becomes insulating at high fields. The dashed lines represent interpolations between the two limits.
Figure B.11  Phase shift as a function of the system size in three dimensions, with $\bar{V} = 0.1$. Inset: the phase shift vs. $a/R$.

Figure B.12  The analogous plot for two dimensions, with $\bar{V} = 1$. Inset: phase shift vs. $1/\ln(R/a)$. 
**Figure B.13** Absorption spectrum as a function of frequency in 3 dimensions, where $\lambda_{sp} = 0.25$.

**Figure B.14** Absorption Spectrum as a function of frequency in 2 dimensions, which is a Lorentzian function. $\alpha_{sp} = 0.10$. 
Figure B.15  a) Diagrams for $G(p, p + q; \omega$). The thick solid line represents the full conduction electron Green’s function in the coupling to an impurity, specified by a cross. The wavy line denotes the propagator of the collective mode; b) Self-energy diagrams for the conduction electrons in the absence of impurity, $\Sigma$. The thin solid line is the bare (BCS) conduction electron Green’s function, $G_0$.

Figure B.16  Spectral function $A_k(E)$ at the wavevector $k = (\pi, 0)$ for $g/\Delta_0 = 3/\sqrt{2}$ (a). The density of states, $\sum_k A_k(E)$, is shown in (b), for various values of $g/\Delta_0 = 0, 1/\sqrt{2}, 2/\sqrt{2},$ and $3/\sqrt{2}$ (from lower to upper); here, for easier viewing, the consecutive curves are shifted by 0.2 along the vertical axis.
Figure B.17  Local density of states at the impurity site (a) and at its nearest neighbor (b). The coupling constant $g/\Delta_0 = 3/\sqrt{2}$.

Figure B.18  The spatial variation of the LDOS around the impurity with the mode coupling $g/\Delta_0 = 3/\sqrt{2}$ at $E = -E_1$ (a), and its corresponding Fourier spectrum $\rho(q, -E_1)$ (b). The density of states is measured with respect to its spatial average value. The image window of $25 \times 25$ plaquettes is taken from a system of size $1000 \times 1000$. Here the impurity-induced Friedel oscillation along the diagonals have been filtered away; see main text for details.