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Double-adiabatic MHD Theory of a Thin Filament in the Geotail and possible Applications to Bursty Bulk Flows and Substorms

by

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ABSTRACT

Double-adiabatic MHD Theory of a Thin Filament in the Geotail and possible Applications to Bursty Bulk Flows and Substorms

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During fast fluid flows in Earth's magnetotail, the plasma distribution function often takes the form of one beam flowing through another, which raises the question of whether Bursty Bulk Flows (BBF's) can reasonably be represented in terms of single-fluid magnetohydrodynamics (MHD), either in global MHD codes or in thin-filament theory. An exact kinetic solution is compared with exact fluid solutions for a simplified case of cold, collisionless particles in a pipe, under conditions where there are countstreaming beams similar to the ones that often occur in Earth's magnetotail. The results from kinetic theory differ from standard fluid theory but are exactly consistent with Chew-Goldberger-Low double-adiabatic fluid theory. Double-adiabatic MHD equations are derived for the motion of a thin filament through a medium. Simulation results are presented for a double-adiabatic filament that starts out with lower gas pressure than nearby flux tubes and also for plasma ejected earthward from a patch of reconnection at $X \sim -25 R_E$. As in earlier calculations for the isotropic case, in both cases the near-equatorial part of the filament moves rapidly earthward. A compressional shock wave forms in the filament near the equatorial plane and propagates earthward. The near-equatorial region of the filament exhibits characteristics similar to a flow burst, while the behavior far from the equatorial plane resembles that of earthward-streaming plasma-sheet boundary layer.
In both cases, the double-adiabatic filament becomes firehose unstable after the shock wave reflects from the earthward boundary of the simulation and propagates back into the tail. The tailward-propagating compressional wave, which brakes the earthward flow in the filament, is thus characterized by strong magnetic fluctuations. Within the context of the Near-Earth-Neutral-Line model of substorms, we suggest that firehose instability might cause the intense magnetic-field fluctuations that are observed in the inner plasma sheet at substorm onset. Additional simulations have been carried out to confirm the robustness of our principal conclusion that fast earthward flows in the Earth’s plasma sheet should lead to firehose instability.
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I. INTRODUCTION

This introductory chapter is intended to provide some background information for the non-specialist reader and to set the stage for the later chapters, which will describe the thesis research in detail. The chapter starts with a brief introduction to plasma physics, particularly the elements that are essential to this thesis. Next comes a basic description of the Earth’s magnetosphere and particularly of the plasma sheet and the phenomena of convection and substorms. The chapter concludes with a summary of the paper of Chen and Wolf [1999] (subsequently referred to as C&W99), which forms the immediate basis of the present effort.

A. Elementary plasma physics

1. What is plasma?

According to Chen [1984], “it has often been said that 99% of the matter in the universe is in the plasma state; that is, in the form of an electrified gas with the atoms dissociated into positive ions and negative electrons.” The definition of a plasma as an ionized gas is insufficiently precise, because there is always some small degree of ionization in any gas. A more precise and often more useful definition is as follows: “A plasma is a quasineutral gas of charged and neutral particles which exhibits collective behavior.”

Define the number density of ionized atoms to be \( n_i \), and the number density of electrons to be \( n_e \). “Quasineutral” means the gas is “...neutral enough that one can take \( n_i = n_e = n \), where \( n \) is a common density called the plasma density, but not so neutral that all the interesting electromagnetic forces vanish” [Chen, 1984]. By “collective behavior”
we mean motions that depend not only on local conditions (for example, the collisions between the particles in ordinary gas) but on the state of the plasma in remote regions as well through long-range Coulomb and magnetic forces [Chen, 1984].

2. Single-particle motions

Chen [1984] explains the motivation for studying single-particle motions as follows. “What makes plasmas particularly difficult to analyze is the fact that the densities fall in an intermediate range. Fluids like water are so dense that the motions of individual molecules do not have to be considered. Collisions dominate, and the equations of ordinary fluid dynamics suffice. At the other extreme, only single-particle trajectories need be considered in very low-density devices like the alternating-gradient synchrotron; collective effects are often unimportant. Plasmas behave sometimes like fluids, and sometimes like a collection of individual particles. The first step in learning how to deal with this schizophrenic personality is to understand how single particles behave in electric and magnetic fields.”

2.1 Larmor motion

First consider the motion of a charged particle in a uniform magnetic field and zero electric field. The equation of motion is

\[ \frac{\vec{v}}{\overline{m}} \frac{dv}{dt} = q \vec{v} \times \vec{B}, \]  

(I.A.1)

where \( \overline{m} \) is the mass and \( q \) is the charge of that particle.
Taking unit vector \( \hat{z} \) to be in the direction of \( \mathbf{B} (\mathbf{B}=\mathbf{B} \hat{z}) \), we have

\[
\overline{m} \ddot{v}_x = qBv_y, \quad \overline{m} \ddot{v}_y = -qBv_x, \quad \overline{m} \ddot{v}_z = 0,
\]

\[
\ddot{v}_x = \frac{qB}{m} \ddot{v}_y = \left( \frac{qB}{m} \right)^2 v_z,
\]

\[
\ddot{v}_y = -\frac{qB}{m} \ddot{v}_x = \left( \frac{qB}{m} \right)^2 v_y.
\]  \hspace{1cm} (I.A.2)

This describes a simple harmonic oscillator at the cyclotron frequency

\[
\omega_c = \frac{|q|B}{m},
\]  \hspace{1cm} (I.A.3)

and the gyration is also called Larmor motion. The particle moves in a circle, about a point called the “guiding center” (Figure 1). The radius of the gyrational motion is called the Larmor radius:

\[
r_L = \frac{v_\perp}{\omega_c} = \frac{\overline{m}v_\perp}{|q|B},
\]  \hspace{1cm} (I.A.4)

where \( v_\perp \) is particle’s velocity component perpendicular to magnetic field.
2.2 $\mathbf{E} \times \mathbf{B}$ drift

If now we allow an electric field $\mathbf{E}$ to be present, the equation of motion becomes

$$\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Let the electric field be perpendicular to $\mathbf{B}$, and choose the coordinates so that $\mathbf{E} = E \hat{x}$. Thus

$$\begin{align*}
\dot{v}_x &= \frac{q}{m} E \pm \omega_c v_y, \quad \dot{v}_y = \mp \omega_c v_x, \quad \dot{v}_z = 0, \\
\ddot{v}_x &= -\omega_c^2 v_y, \\
\ddot{v}_y &= -\omega_c^2 \left(v_y + \frac{E}{B}\right),
\end{align*}$$

where $\pm$ stands for the sign of the charge. We can rewrite the $y$ component as
\[
\frac{d^2}{dt^2} \left( v_y + \frac{E}{B} \right) = -\omega_C^2 \left( v_y + \frac{E}{B} \right).
\] (I.A.7)

The quantities \( v_x \) and \( v_y + E/B \) thus execute simple harmonic motion. The particle executes the sum of two motions: circular Larmor gyration and a drift of the guiding center perpendicular to the constant \( E \) and \( B \) fields. The kind of drift is called \( E \times B \) drift. The guiding center moves with velocity \( v_G \), such that:

\[ v_G \times B + E = 0, \]

or

\[ v_G = \frac{E \times B}{B^2}. \] (I.A.8)

One thing is important: the drift velocity describes the motion of the particle's guiding center: the particle itself is always accelerating in its gyrational motion.

In fact, if a charged particle feels a constant force \( F \) in the uniform \( B \) field, the resulting drift velocity is given by:

\[ v_d = \frac{1}{q} \frac{F \times B}{B^2}. \] (I.A.9)

Notice that if the sign of \( F \) is the same for both positive and negative charges, they will drift in opposite directions, resulting in an electrical current.
2.3 $\nabla |B| \perp \mathbf{B}$: GRAD-$B$ drift

Now consider the motion of particles in inhomogeneous field — a $\mathbf{B}$ field which varies in space.

![Figure 2: the drift of a gyrating particle in a nonuniform magnetic field. Figure is from Chen [1984].](image)

We further assume $r_L \ll L$, where $L$ is the scale length of the variation of $|\mathbf{B}|$. Under this assumption, we could take the undisturbed gyration orbit to calculate the average force that the charged particles feel in that non-uniform field. As shown in Figure 2, the $\mathbf{B}$ field is in the $z$ direction; but it is not uniform and the gradient of its magnitude is in the $y$ direction. From the symmetry it is clear that $\bar{F}_x = 0$, where the overbar indicates an average over a gyration. For motion in the $y$ direction we have (first order expansion):

$$F_y = -qv_{\perp} B_z(y) = -qv_{\perp} \cos(\omega_e t) \left[ B_0 \pm r_L \cos(\omega_e t) \frac{\partial B}{\partial y} \right]. \tag{I.A.10}$$

Averaging over a gyration gives

$$\bar{F}_y = \pm \frac{1}{2} qv_{\perp} r_L \frac{\partial B}{\partial y}. \tag{I.A.11}$$
Substituting (I.A.11) in (I.A.9) and using the fact that the choice of the y-direction for the gradient was arbitrary, we find the expression for the gradient-drift velocity:

\[ v_g = \pm \frac{1}{2} v_r r_L \frac{\mathbf{B} \times \nabla \mathbf{B}}{B^2}. \]  

(I.A.12)

As before ± stands for the sign of the charge.

2.4 Curved \( \mathbf{B} \): Curvature Drift

![Diagram of curved magnetic field]

Figure 3: a curved magnetic field. Figure is from Chen [1984].

When the lines of force have curvature, the particles moving along them will feel a centrifugal force as shown in Figure 3. The average force on the guiding center is

\[ \mathbf{F}_{cf} = \frac{m v_{gi}^2}{R_c} \mathbf{\hat{r}} = \frac{m v_{gi}^2}{R_c^2} \frac{\mathbf{R}_c}{R_c^2}, \]  

(I.A.13)
where $v_\parallel$ is particle’s velocity component parallel to magnetic field. Substituting (I.A.13) in (I.A.9), we easily obtain the curvature drift velocity

$$v_c = \frac{1}{q} \frac{F_{\parallel}}{B^2} = \frac{m v_\parallel^2}{q B^2} \frac{R_c \times B}{R^2}.$$  \hspace{1cm} (I.A.14)

Note that curvature drift, like gradient drift, is in opposite directions for particles of opposite signs.

2.5 $\nabla B \parallel B$: Magnetic Mirrors

Now consider a magnetic field which is pointed primarily in the $z$ direction and whose magnitude varies in the $z$ direction. Let the field be axisymmetric, with $B_\theta = 0$ and $\partial B / \partial \theta = 0$ as shown in Figure 4.

![Diagram](image.png)

Figure 4: Drift of a particle in a magnetic mirror field. Figure is from Chen [1984].
We get $B_r$ from the Maxwell equation $\nabla \cdot \mathbf{B} = 0$:

$$\frac{1}{r} \frac{\partial}{\partial r} (r B_r) + \frac{\partial B_z}{\partial z} = 0.$$  \hspace{1cm} (I.A.15)

If we further assume $\partial B_z / \partial z$ does not vary much with $r$, we have approximately

$$B_r = - \frac{1}{2} r \left. \frac{\partial B_z}{\partial z} \right|_{z=0}.$$  \hspace{1cm} (I.A.16)

Now the components of the force acting on the charged particle are

$$\begin{align*}
F_r &= q (v_r B_z - v_z B_r), \\
F_\theta &= q (-v_\theta B_z + v_z B_r), \\
F_z &= q (v_\theta B_\theta - v_\phi B_\phi).
\end{align*}$$  \hspace{1cm} (I.A.17)

As $B_\theta = 0$, the first two will give out the common Larmor motion. Substituting (I.A.16) into the third equation in (I.A.17) we get

$$F_z = \frac{1}{2} q v_\phi r \frac{\partial B_z}{\partial z}.$$  \hspace{1cm} (I.A.18)

Now consider a particle whose guiding center lies on the axis. It is easy to show that $|v_\theta| = v_\perp$ and $r = r_L$, thus...
\[ \vec{F}_i = \mp \frac{1}{2} q v_{\perp} r_c \frac{\partial B}{\partial z} = \mp \frac{1}{2} q \frac{v_{\perp}^2}{\omega_c} \frac{\partial B}{\partial z} = -\frac{1}{2} \frac{m v_{\perp}^2}{B} \frac{\partial B}{\partial z}. \]  

(I.A.19)

We define the magnetic moment of the gyrating particle to be

\[ \mu = \frac{1}{2} \frac{m v_{\perp}^2}{B}. \]  

(I.A.20)

Thus the equation of motion in the direction parallel to \( B \) is

\[ \frac{m}{\mu} \frac{dv_{\parallel}}{dt} = -\mu \frac{\partial B}{\partial s}. \]  

(I.A.21)

Multiplying the above equation by \( v_{\parallel} \) gives

\[ \frac{m v_{\parallel}}{\mu} \frac{dv_{\parallel}}{dt} = \frac{d}{dt} \left( \frac{1}{2} m v_{\parallel}^2 \right) = -\mu \frac{\partial B}{\partial s} \frac{ds}{dt} = -\mu \frac{dB}{dt}. \]

(I.A.22)

The conservation of particle’s kinetic energy gives

\[ \frac{d}{dt} \left( \frac{1}{2} m v_{\parallel}^2 + \frac{1}{2} m v_{\perp}^2 \right) = \frac{d}{dt} \left( \frac{1}{2} m v_{\parallel}^2 + \mu B \right) = 0. \]

(I.A.23)

Together with (I.A.22) we have

\[ -\mu \frac{dB}{dt} + \frac{d}{dt} (\mu B) = 0, \]

(I.A.24)
so that

\[
\frac{d\mu}{dt} = 0. \quad \text{(I.A.25)}
\]

The magnetic moment \( \mu \) is thus a constant of the motion. Thus if a particle with non-zero \( \mu \) sees an increasing \( B \), its \( v_\perp \) must increase in order to keep \( \mu \) constant. The magnetic mirror happens when \( B \) reaches the value \( B = \frac{E_0}{E_\perp} B_0 \), where \( E_0 \), \( E_\perp \) and \( B_0 \) are the total kinetic energy of the particle, the initial kinetic energy in the plane perpendicular to \( B \) and the initial magnitude of \( B \) field. At the mirroring point, the particle’s parallel velocity component is zero; all the kinetic energy is in perpendicular motion.

Above we have only discussed the situation where fields vary in space but not in time. The general result is that \( \mu \) is conserved if the parameters of the gyromotion (mainly \( B \)) change on a time scale that is slow compared to the gyroperiod and a length scale that is long compared to the gyroradius [Landau and Lifshitz, 1994]. \( \mu \) is also called the first adiabatic invariant of the charged particle’s motion; it describes the periodic motion in the plane perpendicular to the magnetic field \( B \). Another important adiabatic invariant related to this paper is the second invariant — the longitudinal invariant \( J \). It is defined by \( J = \oint m v_\parallel ds \), where \( ds \) is an element of path length of the guiding center along a field line. This invariant is defined for those particles trapped between two magnetic mirrors (like those in the Earth’s magnetosphere). The trapped particles bounce between the two mirrors and form a periodic motion with which \( J \) is associated.
3. Plasma as fluid — Ideal MHD and double-adiabatic MHD

In a plasma the situation is much more complicated than our discussion of single-particle motion; the $\mathbf{E}$ and $\mathbf{B}$ fields are not prescribed but are determined by the positions and motions of the charges themselves. To get an accurate result, one must solve a self-consistent problem; that is, find a set of particle trajectories and field patterns such that the particles will generate the fields as they move along their orbits and the fields will cause the particles to move in those exact orbits. The typical plasma density in Earth’s magnetosphere is $\sim 1$ ion-electron pair per cm$^3$, and the volume of the magnetosphere is $\sim 10^{30}$ cm$^3$. It is hopeless for anyone to try to follow the motion of $10^{30}$ particles. To avoid this problem, we treat plasmas as fluids, which means the identity of the individual particle is neglected and only the motion of fluid elements is taken into account. In an ordinary fluid, frequent collisions between particles keep the particles in a fluid element moving together. It is surprising that such a model works for magnetospheric plasmas, which generally have infrequent collisions. The reason is that the wave-particle interactions act like collisions to keep the particles together; the magnetic field also plays a big role in helping the plasma act like a fluid by strongly constraining the motions across field lines.

The differential equations of MHD are the combination of Maxwell equations and the hydrodynamic fluid equations.

In the vacuum the Maxwell’s equations are

\begin{align*}
\nabla \cdot \mathbf{E} &= \rho_e / \varepsilon_0, \\
\nabla \times \mathbf{E} &= -\partial \mathbf{B} / \partial t,
\end{align*}

(I.A.26a) (I.A.26b)
\[ \nabla \cdot \mathbf{B} = 0, \]  
(IA.26c)

\[ \nabla \times \mathbf{B} = \mu_0 j + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \]  
(IA.26d)

where \( \rho_c \) is the charge density, \( \mathbf{j} \) is the current density, \( \mu_0 \) is the magnetic permittivity of free space and \( \varepsilon_0 \) is the electric polarizability of free space.

The hydrodynamic fluid equations are as following

\[ \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \rho + \rho \nabla \cdot \mathbf{v} = 0, \]  
(IA.27a)

\[ \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\frac{1}{\rho} \nabla \cdot \tilde{\mathbf{P}} + \frac{\mathbf{F}}{\rho}, \]  
(IA.27b)

\[ \frac{\partial}{\partial t} \left( U + \frac{1}{2} \rho v^2 \right) + \nabla \cdot \left[ \mathbf{v} \left( U + \frac{1}{2} \rho v^2 \right) + \mathbf{v} \cdot \tilde{\mathbf{P}} + \Phi_{ch} \right] = \mathbf{H}, \]  
(IA.27c)

where \( \mathbf{F} \) is the force on unit volume, \( U \) is the density of internal energy, \( \tilde{\mathbf{P}} \) is the gas pressure tensor, \( \mathbf{H} \) is the rate at which electromagnetic forces feed energy into particles, and \( \Phi_{ch} \) is conductive heat flux. They are the mass conservation, momentum and energy equations of a fluid.

If we assume that the force on the fluid is electromagnetic, so that \( \mathbf{F} = \rho_c \mathbf{E} + \mathbf{j} \times \mathbf{B} \) in (IA.27b), use Maxwell’s equations to express \( \rho_c \) and \( \mathbf{j} \) in terms of \( \mathbf{E} \) and \( \mathbf{B} \), and perform some mathematical manipulations, we get an alternate momentum equation

\[ \rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla \cdot (\tilde{\mathbf{P}} - \mathbf{M}) - \frac{\partial}{\partial t} (\varepsilon_0 \mathbf{E} \times \mathbf{B}), \]  
(IA.28)
where \(\tilde{M}\) is the Maxwell stress tensor given by:

\[
\tilde{M} = \varepsilon_0 \text{EE} + \text{BB}/\mu_0 - \left[ \frac{1}{2} \varepsilon_0 E^2 + B^2/2\mu_0 \right] I. \tag{I.A. 29}
\]

Setting the heat-transfer rate \(H\) equal to \(j \cdot E\), using Maxwell's equations, and performing some mathematical manipulations leads to the form

\[
\frac{\partial}{\partial t} \left( U + \frac{1}{2} \rho \nu^2 + \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} B^2 \right) + \nabla \cdot \left( \Phi_e + \Phi \left( U + \frac{1}{2} \rho \nu^2 \right) + \nu \cdot \vec{P} + \frac{\text{E} \times \text{B}}{\mu_0} \right) = 0, \tag{I.A.30}
\]

for the energy equation. In ideal MHD, we further assume that the plasma is a perfect conductor, which means

\[
\nu \times \text{B} + \text{E} = 0, \tag{I.A.31}
\]

where \(\nu\) is the velocity of fluid. Equation (I.A.31) is equivalent to the statement that the electric field is zero in the rest frame of the fluid. Note that the assumption of perfect conductivity (I.A.31) is equivalent to assuming that the fluid \(\text{E} \times \text{B}\) drifts (cf. Equation I.A.8) and that \(\text{E} \cdot \text{B} = 0\). The physical reason for assuming that there is no electric field parallel to \(\text{B}\) is that charged particles can move easily along magnetic field lines. If there is an electric field in the direction of \(\text{B}\), positive charges will accelerate in the direction of \(\text{E}\) and negative charges will accelerate in the opposite direction, both acting to cancel the applied electric field.
As long as (I.A.31) is true and \(|v|<<c\), we can ignore all the terms that contain \(E\) (or \(E\)) in the Maxwell stress Tensor, as those terms are small relativistic corrections. If we further simplify the problem by treating the plasma as an isotropic gas that reduces the pressure tensor to a scalar pressure, then we get the ideal MHD equations. The conservation of mass is the same as before, and the ideal MHD equations are:

\[
\frac{d\rho}{dt} + \rho \nabla \cdot v = 0, \tag{I.A.32a}
\]

\[
\rho \frac{dv}{dt} = -\nabla \left( P \hat{1} - \bar{M} \right) = -\nabla \left( P + \frac{B^2}{2\mu_0} \right) + \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B}, \tag{I.A.32b}
\]

\[
\frac{d}{dt} \left( \frac{P}{(\gamma - 1)\rho} \right) + \frac{1}{\rho} \rho \nabla \cdot v = \frac{D}{\rho} \nabla \cdot v^2 T, \tag{I.A.32c}
\]

\[
\frac{dB}{dt} - \mathbf{B} \cdot \nabla v + \mathbf{B} \nabla \cdot v = 0, \tag{I.A.32d}
\]

\[
P = \rho RT, \tag{I.A.32e}
\]

where \(\frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla\) is the convective derivative, and \(D\) is the thermal diffusivity; in the simulations we will be discussing, we assume \(D\) is a constant. The Maxwell stress tensor without electric field terms is

\[
\bar{M} = \mathbf{BB} / \mu_0 - B^2 / 2\mu_0 \mathbf{1}. \tag{I.A.33}
\]

In the MHD momentum equation (I.A.32b), the term \(B^2 / 2\mu_0\) acts the same way as the isotropic gas pressure \(P\), and we call \(B^2 / 2\mu_0\) magnetic pressure; the term \(\mathbf{BB} / \mu_0\) on the
other hand, representing a force along the field line that is transmitted along the magnetic line, is called magnetic tension.

As the left-hand side of equation (I.A.31) is just the total $\mathbf{E}$ field in the fluid frame, equation (I.A.31) corresponds to assuming zero resistivity. This relation together with Maxwell equations (I.A.26) leads to following key results [Wolf, 1993]:

1. “Suppose that you paint a closed ring of particles at time $t$, and then let the fluid move in time, the painted ring always moving with the fluid. One can show that the amount of magnetic flux threading through the closed ring does not change in time. This is often called the FROZEN-IN FLUX THEOREM.”

2. “If two fluid elements lie on the same magnetic field line at some time $t$, then they will forever lie on the same field line. That is, the two fluid elements will move and change shape, and the magnetic lines that thread through them will also change their shape. But the two elements will always share the same field line. Thus the fluid elements are said to be FROZEN to the magnetic field line (and vice versa).”

The assumption that the pressure is isotropic, i.e., $\tilde{P} = P \tilde{I}$, eliminates viscous effects. It also means that the mean thermal motion (temperature) parallel to the field lines is the same as the temperature perpendicular to $\mathbf{B}$. This assumption is generally valid for a dense plasma with frequent collisions, which randomize the particle velocities. However, it can be very unrealistic for a collisionless plasma. Sometimes in a collisionless plasma, it is better to assume that motions parallel and perpendicular to $\mathbf{B}$ are almost decoupled.

The classic way to express the idea that motions parallel and perpendicular to $\mathbf{B}$ are decoupled is to use the Chew-Goldberger-Low relations (CGL relations) [Chew et al., 1965]. To get the energy equation for perpendicular motion, consider the special case of a two-dimensional configuration in which nothing varies with distance along the field line; and there are no collisions of any kind to transfer internal energy between parallel and perpendicular motion. Consider first the perpendicular pressure. Expansions of the system parallel to $\mathbf{B}$ decrease the density, but not the thermal energy per particle in
perpendicular motion; thus $P_\perp / \rho$ stays constant in such expansions, as does $B$. In expansions perpendicular to $B$, from section 2.4 above we know that the magnetic moment of a charged particle is the first adiabatic invariant, and for the case where $B$ changes slowly relative to the gyroperiod, the magnetic moment is conserved, so that

$$\mu = \frac{\langle m v_\perp^2 \rangle}{2B} \approx \frac{P_\perp}{\rho B} = \text{constant},$$  \hspace{1cm} \text{(I.A.34)}$$

so for either type of expansion we have the relation

$$\frac{d}{dt} \left( \frac{P_\perp}{\rho B} \right) = 0.$$  \hspace{1cm} \text{(I.A.35)}$$

For the motion parallel to the magnetic field, the parallel thermal energy per particle does not change under perpendicular expansions, so that $P_\parallel / \rho = \text{constant}$; the ratio $B / \rho$ also remains constant as the result of the frozen-in-flux condition. Under parallel expansions, $P_\parallel / \rho^3 = \text{constant}$, because a gas with one degree of freedom has $\gamma=3$, also $B$ doesn’t change in such an experiment. Thus an expression that holds for both perpendicular and parallel expansions is:

$$\frac{d}{dt} \left( \frac{P_B B^2}{\rho^3} \right) = 0.$$  \hspace{1cm} \text{(I.A.36)}$$
(I.A.35) and (I.A.36) together are called the CGL relations. Double-adiabatic MHD is obtained by replacing the energy equation by these two relations and keeping the tensor form of gas pressure in momentum equation. The relation \( P = \rho \mathcal{R} T \) is still valid, but we need apply this relation separately to the parallel and perpendicular directions, and the pressure and temperature are no longer scalars: the pressure tensor now has a gyrotropic form \( \bar{P} = P_\perp \mathbb{I} + (P_\parallel - P_\perp) \mathbb{B} \mathbb{B} \). And the momentum equation is:

\[
\rho \frac{d\mathbf{v}}{dt} = -\nabla \left( P_\perp + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left[ (P_\perp - P_\parallel) \mathbb{B} \mathbb{B} \right] + \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B}
\]  

(I.A.37)

Equation (I.A.37) together with (I.A.32a,d), (I.A.35) and (I.A.36) are the equations of double-adiabatic MHD.

4. Grad-Shafranov Equation

Consider an isotropic plasma that is in force equilibrium, with the pressure gradient just balancing magnetic forces. If we further assume the plasma flow is zero or very slow, we can neglect the inertial term in momentum equation (I.A.32b), that is to say to assume the left side hand of (I.A.32b) equals zero. So we get

\[
0 = -\nabla \left( P + \frac{B^2}{2\mu_0} \right) + \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B},
\]  

(I.A.38a)

which becomes, after some mathematical manipulation
\[ \nabla P = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}. \] (I.A.38b)

From equation (I.A.38b) we notice that in equilibrium the gradient of isotropic gas pressure is perpendicular to the magnetic field; in other words, \( P \) should be constant along any magnetic field line. In the simulations we will be discussing, the background is just a 2D equilibrium medium, so we consider specifically the case of 2D configurations in \( xz \) plane. Since \( \nabla \cdot \mathbf{B} = 0 \) (equation (I.A.26c)), we can express the \( \mathbf{B} \) field as the curl of some vector field, i.e.

\[ \mathbf{B} = \nabla \times \mathbf{A}, \] (I.A.39)

and this vector field is called the vector potential. In our 2D configuration, if we assume the vector potential has the form

\[ \mathbf{A} = A(x,z) \hat{y}, \] (I.A.40)

then (I.A.39) implies \( B_x = -\partial A / \partial z \) and \( B_z = \partial A / \partial x \) and we have

\[ \mathbf{B} = \nabla A \times \hat{y}. \] (I.A.41)

This equation tells us that \( A \) is constant along each magnetic field line. Substituting (I.A.39) into (I.A.38b) gives
\[ \nabla P = -\frac{1}{\mu_0} (\nabla^2 A) \nabla A, \quad (I.A.42) \]

As equation (I.A.38b) implies that \( P \) is constant along each magnetic field line, we can write \( P \) as a function just of \( A \), i.e.,

\[ P = P(A). \quad (I.A.43) \]

Substituting this into (I.A.42) we have

\[ \nabla P = \frac{dP}{dA} \nabla A = -\frac{1}{\mu_0} (\nabla^2 A) \nabla A, \quad (I.A.44) \]

and we have the Grad-Shafranov equation

\[ \nabla^2 A = -\mu_0 \frac{dP(A)}{dA}. \quad (I.A.45) \]

It is a very neat result — just a one-component equation represents the essence of equations (I.A.38b) and (I.A.26c). The background in our simulations satisfies the Grad-Shafranov equation. The vector potential \( A \) is chosen to have a certain form so that the field lines are stretched out to resemble those in geotail. The background isotropic pressure is determined by \( A \). The detailed expression of \( A \) is given in chapter III.
B. Description of the magnetosphere, convection, plasma sheet, and substorms

Figure 5: Plasma regions viewed in a noon-midnight meridian plane. Figure is from Wolf [1993].

Figure 6: Plasma regions viewed in the Earth’s magnetic equatorial plane. The hollow arrows indicate the flow velocity. Figure is from Wolf [1993].
The plasma regions around the Earth are shown in Figures 5 and 6. Figure 5 is a view in the "noon-midnight meridian plane", which contains the Earth's dipole axis and the center of the Sun. For simplicity, the Figure assumes the dipole axis is perpendicular to the Earth-Sun line. Figure 6 shows the same system in the magnetic equatorial plane which is perpendicular to the Earth's magnetic dipole axis.

1. Solar Wind, Bow Shock and Magnetosheath

The "solar wind" or "interplanetary medium" is a magnetized plasma originating from the outer atmosphere of the Sun. It flows approximately radially outward from the Sun, and it is supersonic.

The "Interplanetary Magnetic Field" (IMF) is the magnetic field frozen in the solar wind. As one end of the IMF line is frozen to the surface of the Sun, the rotation of the Sun and the radially outward motion of the plasma make the IMF form a spiral pattern in space. The IMF makes an angle of about 45 degrees with the Earth-Sun line near Earth.

Figure 7: Idealized picture of the heliospheric current sheet. Arrows represent magnetic field directions. The sheet separates magnetic field lines originating in opposite polar coronal holes. Figure is from Crooker and Siscoe [1986].
Another important feature of the IMF is the presence of a "neutral sheet" region, where the magnetic field is much weaker than in neighboring regions. A cartoon is shown in Figure 7. Note that the magnetic fields are in opposite directions on the two sides of the current sheet, on one side pointing out of one pole of the Sun and on the other side pointing into the other pole. Since the neutral sheet is not exactly planar, the Earth is sometimes in the "toward sector" and sometimes in the "away sector" as the Sun rotates.

Solar wind plasma emanating from the Sun is far from uniform; its speed, temperatures and densities fluctuate both temporally and spatially. Some average solar-wind parameters near the Earth are the following: the average northward component of the IMF at the Earth’s orbit is near zero; the average number density is about 6/cm³; the average speed is about 400 km/s; and the average intensity of $B$ is about 10 nT (nano-Tesla). Most of the energy of the solar wind is in the bulk flow energy $\rho v^2/2$, which also is called solar wind ram pressure; the average value of that is about 2 nP (nano-Pascals).

The solar wind transports solar energy to the Earth’s magnetosphere and ionosphere, and, in the process, it transmits information about the dynamic activity occurring at the solar surface. The solar wind drives magnetospheric dynamics.

When the solar wind encounters the magnetosphere of the Earth, the region where the Earth’s magnetic field is dominant, a shock wave is formed, called the bow shock. It curves around the magnetosphere as shown in Figures 5 and 6.

The internal structure of the bow shock is very complicated partly because the gas (plasma) is collisionless. As the bow shock can not dissipate energy through collisions, it does so by wave-particle interactions.
After passage through the bow shock, the solar wind enters a region between the bow shock and magnetosheath, which is called the magnetosheath. This region is hotter and denser than the solar wind. Various kinds of waves exist in this region, and the shocked IMF exhibits a high level of fluctuations [Tascione, 1998]. Some subtle mechanisms that violate the ideal MHD approximation transport particles, momentum and energy to the magnetosheath. Though these processes are minor for the magnetosheath, they are crucial for the magnetosphere.

2. Magnetopause and Magnetosphere

The intrinsic magnetic field of the Earth is predominantly a dipole, although higher order magnetic moments do exist. The dipole is roughly perpendicular to the solar wind flow direction, but the angle between the dipole and the Sun-Earth line changes with time. Pressed by the solar wind, the dayside magnetosphere is limited to a dimension of about 10 Earth radius ($R_E$), on average, while the nightside magnetosphere extends hundreds of $R_E$ [Tascione, 1988].

The boundary of the magnetosphere is called the magnetopause. It separates the magnetic field of the Earth from the IMF in the magnetosheath. In the ideal MHD approximation, it is impossible to mix fluid between IMF field lines and magnetospheric field lines, i.e. ideal MHD implies that no magnetic field can across the magnetopause. Thus the plasmas on the two sides of this boundary have different origins and different properties. Real life is not so simple though. Mother Nature finds ways to transport mass, tangential momentum and energy across the magnetopause, to make the magnetopause dynamically active. Solar-wind and magnetospheric field lines undergo reconnection
(explained in section 4 later this chapter) at the magnetopause, and open field lines thread through that boundary, allowing plasma from the magnetosheath (shocked solar wind) to flow into the polar cusp region of the magnetosphere (Figure 5). As the magnetic field is different inside and outside the magnetosphere, currents flow on the magnetopause to maintain the difference. These currents, called Chapman-Ferraro currents, are shown in Figure 8, together with other major currents in the magnetosphere.

![Diagram of magnetosphere currents](image)

Figure 8: Major types of current in the Earth’s magnetosphere. The view is from slightly above the magnetic equatorial plane. Figure is from Wolf [1993].

Inside the magnetopause there are two boundary layers located at different latitudes. One is the low latitude boundary layer, which forms at lower latitudes as shown in Figures 5 and 6. The motion in this layer is anti-sunward. The other boundary layer, called the plasma mantle, is located at higher latitudes on open lines. The solar wind
drags the Earth’s open field lines through this region in the anti-sunward direction, and the plasma flow in the mantle region is approximately anti-sunward.

The tail lobe is the region inside the plasma mantle. It contains a highly ordered magnetic field and very low plasma pressure.

The plasma sheet is the region located between the north magnetotail lobe and the south magnetotail lobe as shown in Figure 5, with an increasing thickness from midnight to each flank. It is a remarkably interesting and dynamic plasma regime. It contains a large fraction of the total particle energy in the magnetosphere. It is also the primary population involved in the magnetospheric substorm, which is generally regarded as the fundamental dynamic process of the magnetosphere [Wolf, 1993]. Also most of the aurora occur on field lines that are connected to the plasma sheet. In the plasma sheet, $T_i \gg T_e$, where subscript “i” denotes ion and subscript “e” denotes electron, so ions dominate the pressure and the dynamics there.

The tail current flows westward through the plasma sheet and changes the direction of the magnetic field from sunward in the north tail lobe to anti-sunward in the south tail lobe. The tail current flows across the tail from dawn to dusk and connects to the magnetopause current at the flanks of the magnetosphere. The plasma sheet is the most active region of the magnetosphere. At the center of plasma sheet is the neutral sheet, where the magnetic field is quite weak, however, the magnetic field there generally has a weak northward component. The beta value (ratio of thermal energy density to magnetic energy density) is very high in the neutral sheet. Between the plasma sheet and the lobe is the plasma sheet boundary layer, distinguished by its multi-peak distribution function, representing interpenetrating beams.
The inner magnetosphere consists of Van Allen Belts (radiation belts), ring current, and plasmasphere as shown in Figure 6. The plasmasphere contains cold and dense plasma that comes from the ionosphere, while the Van Allen Belts consist of much less dense energetic particles (typically Mev). The radiation belts, ring current, and plasmasphere occupy the same region of space.

3. Ionosphere

The ionosphere is the ionized component of the atmosphere that coexists with the thermospheric region of the upper neutral atmosphere. The magnetic field lines in the magnetosphere nearly all pass through the ionosphere to the solid Earth. The ionosphere acts as both a source and a sink of magnetospheric plasma.

The main difference between magnetosphere and ionosphere is that, while the magnetosphere could be treated as a fully ionized collisionless plasma, or in other words, neutrals have negligible direct effect on the electrodynamics on the magnetospheric charged particles, we must include the effects of the neutrals in the ionosphere, i.e., collisions with neutrals have a big effect on the motions of the charged particles in the ionosphere. The collisions produce finite conductivities in the ionosphere in directions perpendicular to magnetic field. These conductivities are known as the Pedersen conductivity, which carries current in the direction of \( E_x \), and the Hall conductivity, which carries current in the direction of \(-E \times B\). This conducting layer carries current horizontally, connecting field lines that carry current to and from the magnetosphere (horizontal current arrows in Figure 8). Since charged particles move rather easily along
magnetic field lines, so that $|E_\parallel| < |E_\perp|$ most (but not all) places in the ionosphere and magnetosphere.

4. Magnetospheric Convection

The magnetosphere has a basic flow pattern that is called magnetospheric convection and is driven by solar wind. The outer part of the magnetosphere moves in an anti-sunward direction while the interior part moves in a sunward direction. The open model suggested by Dungey [1961] is widely accepted today as describing the primary driver of magnetic convection.

In the open model, some open field lines are generated by a process called reconnection, in which interplanetary field lines become connected to the Earth. Reconnection is most likely to happen when the magnetic fields are anti-parallel across the current sheet. During this process the topology and connectivity of field lines change, which means this process violates ideal MHD, as the frozen-in-flux condition mentioned above pre-excludes this kind of change in topology and connectivity. During the reconnection process, magnetic energy is converted to heat, bulk-flow energy, and the acceleration of energetic particles. Also large electric currents and electric fields are created, as well as shock waves and filamentation, all of which may help to accelerate the fast particles [Priest and Forbes, 2000]. A simple cartoon of reconnection is shown in Figure 9. From the geometry shown in the Figure 9, we notice the just reconnected field line is highly stretched. And from the discussion following (I.A.36) we know there will be a strong magnetic tension to accelerate the plasma left and right, away from the reconnection region. As the reconnection works best when magnetic fields on opposite
sides of the current sheet are antiparallel, reconnection proceeds rapidly at the magnetopause when the IMF is southward, the case is shown in Figure 10a. The dayside reconnection generates open field lines, which are dragged tailward by the solar wind and become part of the anti-sunward part motion just inside the magnetopause. Reconnection also occurs in the tail at the far-tail X-line (Figure 10a), resulting in sunward flow in the plasma sheet.

![Diagram of reconnection sequence]

Figure 9: Reconnection sequence, starting from the top panel. Figure is from Chen [1994].

5. Magnetospheric Substorms

A magnetospheric substorm is a sudden and explosive release of energy in the plasma sheet [Kan et al., 1991]. Most substorms consist of three phases: the growth
phase, the expansion phase and the recovery phase. When the IMF turns southward to signal the beginning of the growth phase, reconnection takes place at the dayside magnetopause as shown in Figure 10a. The newly reconnected flux tubes are pulled to the night side and piled up in the tail lobe. As this process proceeds, the inner plasma sheet thins and the magnetotail field lines become stretched as they store energy taken from the solar wind. At the same time, in the ionosphere the equatorward edge of the diffuse aurora starts to move equatorward, and a stable auroral arc, called the equatorward most arc becomes evident in the lower latitude part of the auroral zone. This phase is called the growth phase.

The growth phase usually ends when some solar-wind disturbance (like a northward turning of IMF or increase of ram pressure $\rho v^2$) triggers the quick and explosive release of the magnetic energy that had been stored in the lobe field. This is called the onset of the expansion phase. Much of the energy is transferred into kinetic energy in the form of plasma bulk motion, acceleration of energetic particles and heat. At the same time the equatorward-most arc brightens at onset, and that auroral form becomes turbulent and moves poleward. The region of bright aurora expands along with accompanying strong ionospheric currents in the auroral zone. At the same time in the plasma sheet a substorm current wedge (Figure 10b) forms and expands in local time, the magnetic field lines within the current wedge become more dipolar. This period is called the expansion phase.

During the recovery phase, everything gradually recovers to approximately the pre-substorm condition.
Figure 10: (a) Cartoon showing the open model of the magnetosphere. Figure is from Wolf [1993]. (b) Illustration of substorm current wedge. Figure is from McPherron et al. [1973].
Identification of the physical mechanism responsible for substorm onset represents one of the major remaining unsolved problems of magnetospheric physics. A variety of conceptual models have been proposed. One class of models places the essential onset mechanism about 8 $R_E$ behind the Earth. This class includes the current disruption (CD) model [Lui et al., 1988, 1990, 1992; Lui, 1996] as well as ballooning and configurational instability models [e.g., Roux et al., 1991; Erickson et al., 2000; Crabtree et al., 2003; Cheng and Lui, 1998]. The basic idea of the current disruption model is that some instability associated with the very high cross-field current density in the inner plasma sheet (about 8 $R_E$ behind the Earth) during the growth phase disrupts the current there, resulting in the creation of a substorm current wedge and the accompanying dipolarization of the field lines there. Reconnection plays only a secondary role in the models that place onset near 8 $R_E$. Figure 11 shows a typical Current Disruption (CD) event at the expansion onset of a substorm that occurred on August 28, 1986, presented in the local dipole VDH coordinate system in which $\hat{e}_H$ (north) is anti-parallel to the geomagnetic dipole, $\hat{e}_D$ (east) is perpendicular to the dipole meridian of the spacecraft, and $\hat{e}_V = \hat{e}_D \times \hat{e}_H$. The polar angles are defined as $\theta \equiv \sin^{-1}(B_H/B_T)$, $\phi \equiv \tan^{-1}(B_D/B_V)$ where $B_T$ is the magnitude of the field [Takahashi et al., 1987]. Notice the time before 1152 UT is the growth phase characterized by stretched magnetic field: $\theta$ is far from 90 degrees and the magnetic field is weak. The field is dipolarized by 1157 UT, with $\theta$ near 90 degrees and a stronger magnetic field.

The Near-Earth-Neutral-Line (NENL) model, on the other hand, identifies substorm onset with the formation of an X-line about 20-30 $R_E$ behind the Earth [Nagai et al., 1998; Fairfield et al., 1998]. Reconnection causes mass, magnetic flux and energy to be
transported away from the X-line at high speed, both earthward and tailward. The reconnection at about 25 $R_E$ behind the Earth in the NENL model offers a natural explanation of the dipolarization of inner plasma sheet field lines, which have been stretched and become tail-like in the growth phase. Reconnection also offers a natural interpretation of the substorm current wedge, as it creates flux tubes with smaller $pV^Y$ than their neighbors (The definition and discussion of $pV^Y$ are in the next subsection). In fact, the NENL model predicted the existence of tailward moving plasmoids in the distant magnetotail, years before they were apparently observed by ISEE-3; at the same time Near-Earth neutral lines nearly always form in the 3D global MHD simulations, for conditions of southward IMF [Wolf, 1993]. The NENL picture is summarized in Figure 12.
Figure 11: Observation of highly variable magnetic field on August 28, 1986, by the AMPTE-CCE satellite. The unit of magnetic field is nanoTesla, and a dipole $VDH$ coordinate system is used to display the magnetic field measurements. The fluctuation amplitude is comparable to the ambient field strength and the timescale is comparable to the proton gyroperiod. The spacecraft is located about 8.1 $R_E$ behind the Earth and slightly before the magnetic midnight. Figure is from Takahashi et al. [1987].
Figure 12.a: Beginning of cartoon series representing the Near-Earth-Neutral-Line model of substorms. "N" represents the pre-substorm far-tail neutral line, while "N'" represents the new substorm associated neutral line in the inner plasma sheet. Five specific flux tubes are followed in time. Configuration 1 occurs in the growth phase of the substorm, when the inner plasma sheet has begun to stretch. After stretching continues, a neutral line forms in the inner plasma sheet at ~ 15 R_E, as shown in Configuration 2; this occurs at or near the end of the growth phase. A plasmoid is formed, consisting of field lines that are not connected to either the Earth or the interplanetary medium. The plasmoid gradually grows in time (Configurations 3 and 4). Note that reconnection has also created additional closed quasi-dipolar flux tubes in the inner plasma sheet.
Figure 12.b: End of cartoon series representing the Near-Earth-Neutral-Line model of substorms. When the system reaches Configuration 5, reconnection has “eaten up” all of the closed field lines that had passed \( N' \) and the plasmoid is no longer bound to the Earth by the magnetic tension of closed field lines that surround it. In Configuration 6, reconnection continues at \( N' \), now converting previously open tail-lobe field lines into field lines that are connected with the interplanetary medium at both ends, and there is now a magnetic-tension force pulling the plasmoid tailward. In Configuration 7 and 8, the plasmoid moves off down the tail. In Configuration 9, the plasmoid has escaped from the tail, and reconnection has somehow created a new plasma sheet. Both Figures are from Hones [1977].
In the NENL model, the violent disruptions observed at 6-10 \( R_E \) are attributed to the effects of plasma that was ejected earthward from the X-line crashing into the strong-field region near the Earth. This explanation is called the "pileup" or "braking" model [Hesse and Birn, 1991; Shiokawa et al., 1997]. However, it is not clear how the pileup model can explain the observed very large magnetic fluctuations, and recent studies suggest that the mechanism is more complicated than simple pileup [Ohtani et al., 2002].

**Convection in the Plasma Sheet**

In the traditional point of view, the plasma flows systematically earthward through the Earth's plasma sheet in the region within 40 \( R_E \) of the Earth, with an average speed of about 15 km/s, except in substorms, when the steady configuration may be disrupted. If we further assume that some process (most likely chaotic ion motion in the central current sheet where the ion gyroradius is comparable to the minimum radius of field-line curvature) keeps the pitch-angle distribution isotropic, then \( pV^\gamma \) is approximately constant along a flux tube's drift path [Wolf, 1983], where \( p \) is the thermal pressure along the field line, \( V = \int ds/B \) is the volume of a unit magnetic flux tube, and \( \gamma \) is the ratio of specific heats. If \( pV^\gamma \) is conserved, one would expect the particle pressure \( p \) to vary with position in the plasma sheet in approximate proportion to \( V^{-\gamma} \), with \( \gamma = 5/3 \) for isotropic monatomic gas. Suppose one calculates \( pV^\gamma \) in the middle plasma sheet, \( \sim 20 R_E \) from Earth, using a statistical, observation-based magnetic field model, and suppose one estimates \( p \) at \( \sim 10 R_E \) by assuming that \( pV^\gamma \) is the same at 10 as at 20 \( R_E \), and using the same field model to compute \( V \) at 10 \( R_E \). An extremely large particle pressure results, too much to be balanced by the magnetic pressure in the tail lobe as estimated from the same
magnetic field model. This theoretical problem [Erickson and Wolf, 1980] is usually called the “pressure-balance inconsistency” or “pressure crisis”. Experts have tried to resolve this problem by considering the particle loss due to gradient/curvature drift; Kivelson and Spence [1988] and Spence and Kivelson [1993] showed that that is possible for weak convection. For average conditions or strong convection, gradient/curvature drift can’t cause the majority of the plasma sheet to drift out the sides of the tail [Erickson, 1992], assuming that the convection is uniform across the tail.

The pressure-balance inconsistency suggests that slow, steady, uniform sunward convection in the Earth’s plasma sheet probably occurs only rarely, if ever. At the same time, observation shows the background plasma sheet has $pV'\gamma$ decreasing earthward [Spence et al., 1989]. Pontius and Wolf [1990] suggested underpopulated earthward-moving flux tubes as a mechanism for resolving the pressure balance inconsistency. They suggested that in the far tail there might be some small isolated local density depletion (bubbles) as the result of a non-uniform loading process in the far tail. These bubbles would be displaced earthward by an interchange process (next section). Low-$pV'$ bubbles spend little time speeding through the outer plasma sheet before coming nearly to rest in the inner plasma sheet. This would cause the inner plasma sheet to have smaller average $pV'\gamma$ than the outer plasma sheet.

At about the same time, Baumjohann et al., [1989, 1990] identified short time period (10 minutes) of high speed flow in the plasma sheet. These periods, called “bursty bulk flows” (BBF’s) contain series of flow bursts with flow velocity $>400\text{km/s}$. Further studies showed that BBFs are responsible for most of the particle, energy and magnetic
flux transport in the plasma sheet [Angelopoulos et al., 1992, 1994], even during time intervals of Steady Magnetic Convection events [Sergeev et al., 1996].

**Interchange Instability**

The energy principle of MHD [Schmidt, 1979] requires that in a stable configuration, the unit magnetic flux tube with the largest volume \( V \) should also have the largest \( PV^r \) value. The magnetosphere is basically stable against this interchange instability, because the value of \( PV^r \) tends to increase with increasing geocentric distance.

Now consider the interchange instability based on charged particle motions in non-uniform magnetic field. Assume we have a flux tube with smaller \( PV^r \) but the same shape as its neighbors in the Earth’s magnetotail. How will this flux tube behave?

From the discussion above about the curvature and gradient drift, we know the charged particles in the plasma sheet will drift in dawn-dusk direction. As the drift velocities depend on the sign of particles’ charge, ions and electrons will drift in opposite directions. It is easy to imagine that there will be a current flowing dawn to dusk as the result of the drifts. Though we don’t know the ratio of those two drifts (the curvature drift and the gradient drift), in most cases they are of the same order of magnitude. From equations (I.A.4), (I.A.12), and (I.A.14), we can estimate the gradient-drift current

\[
|\mathbf{J}_s| \sim \frac{1}{2} n_e \langle v_{\perp} \rangle \frac{\overline{m} \langle v_{\perp} \rangle}{B} \frac{\mathbf{B} \times \nabla \mathbf{B}}{B^2} \approx \frac{P}{BR_c},
\]

and the curvature-drift current
\begin{align*}
|J_c| &\sim n_i \frac{\bar{m} (v_i)^2}{B^2} \frac{R_c \times B}{R_c^2} \approx \frac{P}{BR_c}. \tag{I.B.2}
\end{align*}

So the total drift current density is

\begin{align*}
|J| &\approx \frac{P}{BR_c}. \tag{I.B.3}
\end{align*}

The bubble flux tube under consideration has the same $R_c$ as its neighbors as they have the same shape; at the same time, its density is lower and $B$ is higher than background. From equation (I.B.3), we know that this means that a filament with smaller $pV^7$ value has a weaker current density, the sides of filament will be charged, and a new $E$ field will build up as the result (See Figure 13).

Figure 13: a bubble in plasma sheet, viewed in the equatorial plane. The thick arrows denote the currents in the system. Because the westward cross-tail current is weak inside the bubble, charges build up on the east and west sides of the bubble, driving currents along the field lines, to and from the ionosphere. Figure is from Wolf [1993].
Notice that this polarization $\mathbf{E}$ field in the bubble adds to the convection $\mathbf{E}$ field, and the bubble undergoes a stronger earthward $\mathbf{E} \times \mathbf{B}$ drift than its neighbors. Thus bubbles (filaments with reduced $pV'$) should rush earthward.

C. Bursty bulk flows, filaments and the work of C&W99

![Diagram](image)

Figure 14: Cartoon of a plasma-sheet bubble. Its shape, as viewed in the xz-plane (top diagram), becomes different from its neighbors. Figure is from C&W99.

Following this idea, C&W99 developed an isotropic MHD model for a “bubble”, expressing the idea of Pontius and Wolf [1990] in quantitative form. They set up an idealized filament with smaller $pV'$ value by depleting a background flux tube (the ordinary MHD thin-filament calculations have been used in solar physics and astrophysics by Parker [1981] and Semenov [2000]). They represented the depleted flux tube as a thin filament and used their filament theory as a quantitative framework for a
discussion of BBFs in the magnetotail. As discussed earlier in the paper, the background plasma sheet has $pV^r$ decreasing earthward. A bubble rushes earthward to find its new equilibrium position near Earth, where its $pV^r$ matches the background. Figure 15 shows the time evolution of the filament’s position. Figure 16 shows the component of filament velocity perpendicular to $B$ at different times. Figure 17 shows the parallel velocity component and Figure 18 shows the evolution of plasma pressure at different times.

From these results we can see that the “bubble” is consistent with many aspects of BBFs, like the fast earthward flow, dipolarization of field line, and equatorward motion of the ionospheric end of the magnetic field line.

Figure 15: The top panel shows the shapes of the filament at different times. The filament moves earthward from the initial equatorial position at 40 $R_E$. The positions are shown in the top panel for times 0, 1, 2, 3, 4, 5, 6, 7, 8, 13, 18, 23, 28, 33, 38, 43 minutes. The middle and bottom panels are enlarged views of the tailward and earthward portions of the top panels. The final position is shown in the third panel (the darker line). For all the panels unit is $R_E$. Figure is from C&W99.
Figure 16: The perpendicular velocity component of the filament (km/s). The abscissa is the x-coordinate in $R_E$. Figure is from C&I99.
Figure 17: The parallel velocity component. Figure is from C&W99.
Figure 18: Plasma pressure along the filament, for times 0, 1, 2, 3, 4 and 5 minutes. The solid line represents $P$ inside the filament while the dashed line represents pressure $P_{\text{med}}$ in the surrounding medium, just outside the filament. Figure is from C&W99.
D. Motivation for the research and the organization of this thesis

Parks et al. [2001] pointed out that the ion distribution functions observed in fast flows often exhibit two or more interpenetrating beams moving at different velocities and don’t look anything like a Maxwellian. They questioned the applicability of (single fluid) ideal MHD to such a situation. This thesis was originally motivated by the need to modify the C&W99 theory in response to the objections of Parks et al. [2001].

The reason for interpenetrating beams in fast flows is shown in Figure 19a. The particles that have been accelerated in the current sheet and been shot out from the reconnection site. Together with the background particles that haven’t been accelerated they naturally form the counter-streaming situation as that observed by Parks et al. [2001]. Figure 19b shows the fluid viewpoint of the same situation where the effects of the fast beam shot out from the current sheet are represented by a slow shock.

![Diagram](image)

(a) Particle picture
(b) Fluid picture

Figure 19. Two ways of looking at a patch of reconnection. In both diagrams, the solid lines are magnetic field lines. The Earth and Sun are to the left. The left diagram illustrates a patch of reconnection from a particle viewpoint, showing drift trajectories of two ions that arrive simultaneously at point P. Ion A has been accelerated in the current sheet, while ion B has not. The right diagram shows a fluid picture of the same situation, with slow shocks represented by dashed lines.
We have determined that, though the isotropic assumption may be inaccurate, it is still possible to treat the filament as a single fluid. In section II, by considering a simple Gedanken experiment, we first demonstrate that double-adiabatic MHD, which is a single fluid MHD, can represent interpenetrating-beam distributions much better than isotropic MHD. Then the double-adiabatic MHD theory for thin filaments is developed, followed by the simulated evolution of two different plasma-sheet bubbles generated by different mechanisms (section III). Our simulation results show many of the same features as C&W99, but they also exhibit a dramatically new feature: firehose instability. In section IV.A, we first demonstrate the agreement between the double-adiabatic MHD calculation and the kinetic calculation for the situations similar to those presented in section III. Section IV.B presents a number of runs aimed at determining the sensitivity of the results to the initial conditions and various other input parameters together with the kinetic calculation for some crucial cases. Section V summarizes the research and discusses possible areas for future work.
II. GENERAL THEORY

As we pointed out in Chapter I, observations indicate that ideal MHD may not be a proper way to treat the bursty bulk flows, as the ion distribution functions observed in fast flows often exhibit two or more interpenetrating beams and don’t look anything like a Maxwellian. Rigorous treatment of these situations would require use of either kinetic theory or particle simulations, but applying either of those approaches to bursty bulk flows would represent a very ambitious computational project. A fluid formalism would be much more desirable, if it can provide a reasonably accurate representation of the situation. In this chapter, we will first use a Gedanken experiment to show that double-adiabatic MHD is better than isotropic MHD in representing situations with multi-peak distribution functions; then we will derive the equations of motion for a thin filament in double-adiabatic MHD and discuss linear-wave solutions.

A. The first Gedanken experiment

In this section we analyze a simple case to gain insight into the question of whether a single-fluid method can provide a reasonable representation of interpenetrating beams. The case is simple enough to yield exact analytic solutions in both the kinetic and the fluid formations, which can easily be compared. Yet it contains the essential element of the fast-tail-flow situation shown in Figure 19: a single-fluid theory is being used to represent a situation with two or more interpenetrating particle beams.
Figure 20: Sketch of particle trajectories near an almost-neutral sheet, which is the yz plane. The yz plane is not quite magnetically neutral but has a small positive $B_z$. The sketch is illustrative and not to scale. Figure is from Speiser [1965].

Figure 20 shows the typical trajectories of charged particles near the neutral sheet. After entering the neutral sheet, both ions and electrons oscillate about it, accelerating in opposite directions and are turned toward the same direction in the plane of the neutral sheet by the small magnetic field component perpendicular to the neutral sheet. When the particles are turned 90 degrees, they are ejected from the neutral sheet. While in the sheet, both electrons and ions move in the direction of $qE$, where $q$ is the charge, and therefore
gain kinetic energy. However, the low-mass electrons turn much sharper in the $yz$ plane, move a smaller distance in the direction of $q\mathbf{E}$, and therefore gain less energy. The ejected particles are the type A particles in Figure 19a.

1. Set up of gedanken experiment

![Diagram of shock front and moving piston](image)

Figure 21. An idealization of magnetic-field-aligned plasma flow in a magnetic filament. The moving piston generates interpenetrating particle beams in the pipe.

Consider the simple situation where a piston moves with constant speed $V_0$ along a straight pipe, toward a cap at the left end (Figure 21). The pipe is at rest in the lab frame and has constant cross section. It is filled with collisionless gas, and there is no coupling between motions parallel to the pipe ($x$-direction) and perpendicular to it. At $t=0$ the parallel temperature $T_\parallel$ of the gas is zero and the particles are at rest in the lab frame. Here $T_\parallel$ is defined as $\frac{m}{2}\left(\langle v_x^2 \rangle - \langle v_x \rangle^2\right)$. At $t=0$ the piston is at the position $x=L_0$. The gas is constrained between the piston and the cap and there is no energy transfer to the walls of the pipe. The particles bounce elastically off the piston and the end cap. This pipe represents a simplification of the BBF problem. The cap represents the strong-field region near the Earth where a magnetic flux tube narrows down by a factor $\sim 10^4$. The piston represents the neutral-sheet region, which generates a beam of left-streaming particles. From the kinetic point of view, it is clear that an observer in the pipe will see a
double-peaked distribution, once the particles that have hit the piston are observed. The number of beams will grow with time as particles hit the cap and piston multiple times.

2. Kinetic calculation

Since the particles bounce elastically off the piston, the left-moving beam has the same density as the background gas. As there are assumed to be no particle-particle collisions, the individual particles will move at constant velocity except when they bounce off the piston and cap. In the lab frame the leading edge of the beam will gain $2V_0$ of speed every time it hits the piston. The direction of the motion of the leading edge will reverse every time it hits either end (piston or cap). If the last end the leading edge hit was the piston (cap), the average velocity of the particles between the piston (cap) and the leading edge is the same as the velocity of the piston (cap).

Consider the situation after the $n$th contact of the leading edge with the piston but before its $n$th reflection from the end cap. Each reflection of the original beam creates a new beam of density $\rho_0$, where the $\rho_0$ is the total density at $t=0$. In the region between the piston and the leading edge, there are $n$ beams heading right at speeds $0, 2V_0, \ldots, 2(n-1)V_0$, and $n$ beams heading left at $2V_0, 4V_0, \ldots, 2nV_0$, all in the lab frame. The average velocity in the lab frame is $V_0$ left. Just after the leading edge reflects from the piston for the $n$th time, the density between the piston and the leading front is $2n\rho_0$, and the speed of the leading edge in the lab frame is given by $V_{\text{leading-edge}}^{\text{pe}} = 2nV_0$.

View the situation in the rest frame of the leading edge, and label the gas ahead of the leading edge "upstream", the gas behind the leading edge "downstream". Since the average velocity of the downstream gas is the same as that of the piston, the average
downstream speed in the leading edge rest frame is \((2n-1)V_o\). Define a parallel pressure
\[ P_{\parallel} = \int f \cdot \bar{m} (v_x - \langle v_x \rangle)^2 d\vec{v}, \]
where \(\bar{m}\) is the mass of the particles, and \(f\) is the distribution function.

Use tilde ("\~") to denote the leading-edge rest frame. Two beams are present next to the piston after the first reflection, so the density is \(2\rho_0\); after the second reflection from the piston, four beams are present and the density is \(4\rho_0\); after the \(n\)th reflection from the piston, the density is \(2n\rho_0\). After the \(n\)th reflection of the leading edge from the piston but before \(n\)th reflection from the cap, conditions on the downstream side are described by

\[ \rho_{\text{down}}^{pn} = 2n\rho_0 \quad \text{(II.A.1a)} \]

and

\[ \bar{V}_{\text{down}}^{pn} = (2n-1)V_o, \quad \text{(II.A.1b)} \]

where the superscript "\(pn\)" means that the leading edge has just been reflected from the piston for the \(n\)th time, and the subscript "\(\text{down}\)" means downstream. From

\[ P_{\text{down}}^{pn} = 2\rho_0 \sum_{i=1}^{n} (2i-1)^2 V_o^2, \]

we get

\[ P_{\text{down}}^{pn} = \frac{2}{3} n(4n^2 - 1)\rho_0 V_o^2. \quad \text{(II.A.1c)} \]
In equation (II.A.1c) and in the rest of Chapter II, the symbol \( P \) refers to parallel pressure, unless otherwise indicated.

On the upstream side the corresponding results are the following:

\[
\rho_{up}^{\text{pn}} = (2n-1)\rho_0, \quad (\text{II.A.1d})
\]

\[
\tilde{V}_{up}^{\text{pn}} = 2nV_0. \quad (\text{II.A.1e})
\]

From

\[
P_{up}^{\text{pn}} = 2\rho_0 \sum_{i=1}^{n-1} (2i)^2 V_0^2,
\]

we get

\[
P_{up}^{\text{pn}} = \frac{4}{3} n(n-1)(2n-1)\rho_0 V_0^2. \quad (\text{II.A.1f})
\]

The speed of the leading edge in the lab frame is given by

\[
V_{\text{leading-edge}}^{\text{pn}} = 2nV_0. \quad (\text{II.A.1g})
\]

After the leading edge is reflected from the cap for the \( n \)th time, but before it hits the piston for the \( n+1 \) time, the corresponding results are, for the downstream side,

\[
\rho_{down}^{\text{cn}} = (2n+1)\rho_0, \quad (\text{II.A.2a})
\]

\[
\tilde{V}_{down}^{\text{cn}} = 2nV_0. \quad (\text{II.A.2b})
\]
Now the pressure is given by

\[ P_{down}^{cn} = 2\rho_0 \sum_{i=1}^{n} (2i)^2 V_0^2, \]

\[ P_{down}^{cn} = \frac{4}{3} n(n+1)(2n+1)\rho_0 V_0^2. \]  

(II.A.2c)

For the upstream side, the corresponding results are

\[ \rho_{up}^{cn} = 2n\rho_0. \]  

(II.A.2d)

and

\[ \tilde{V}_{up}^{cn} = (2n+1)V_0. \]  

(II.A.2e)

From

\[ P_{up}^{cn} = 2\rho_0 \sum_{i=1}^{\tilde{x}} (2i-1)^2 V_0^2, \]

we obtain

\[ P_{up}^{cn} = \frac{2}{3} n(4n^2 - 1)\rho_0 V_0^2. \]  

(II.A.2f)

Now the speed of the leading edge in the lab frame is given by

\[ V_{leading-edge}^{cn} = 2nV_0. \]  

(II.A.2g)
3. Fluid calculation

**Condition before shock hits end cap**

At time $t=0$ the piston begins to move at constant speed $V_0$. Since the initial temperature is zero, a shock forms at the piston and moves toward the cap, ahead of the piston. From fluid dynamics (e.g., Wu, 1983) we know that, in the rest frame where the shock is stationary and flows are perpendicular to the shock, the jump conditions for velocity, density, and pressure are

\[
\begin{align*}
\left( \frac{\tilde{V}_{\text{down}}}{\tilde{V}_{\text{up}}} \right)_{\text{fluid}} &= \frac{\gamma - 1}{\gamma + 1} + \frac{2}{\tilde{M}_{\text{up}}^2 (\gamma + 1)} \\
\left( \frac{\rho_{\text{down}}}{\rho_{\text{up}}} \right)_{\text{fluid}} &= \frac{\tilde{M}_{\text{up}}^2 \frac{\gamma + 1}{2}}{1 + \tilde{M}_{\text{up}}^2 \frac{\gamma - 1}{2}} \\
\left( \frac{P_{\text{down}}}{P_{\text{up}}} \right)_{\text{fluid}} &= \tilde{M}_{\text{up}}^2 \frac{2\gamma - \gamma - 1}{\gamma + 1}
\end{align*}
\]

where $\tilde{M}_{\text{up}}^2 = \rho_{\text{up}} \tilde{V}_{\text{up}}^2 l (\gamma P_{\text{up}})$, and $\gamma$ is the ratio of specific heats for the parallel motion. As the gas inside the pipe is initially cold, we have $P_{\text{up}} = 0$, $\tilde{M}_{\text{up}}^2 = +\infty$, so for the first shock propagating down the tube:

\[
\left( \frac{\tilde{V}_{\text{down}}}{\tilde{V}_{\text{up}}} \right)_{\text{fluid}} = \frac{\gamma - 1}{\gamma + 1},
\]

\[\text{(II.A.4a)}\]

and
\[
\left( \frac{\rho_{\text{down}}^{p1}}{\rho_{\text{up}}^{p1}} \right)_{\text{fluid}} = \frac{\gamma + 1}{\gamma - 1}; \quad (\text{II.A.4b})
\]

while the downstream pressure is

\[
P_{\text{down}}^{p1} = \rho_0 \left( \frac{V_{\text{up}}^{p1}}{V_{\text{shock}}^{p1}} \right)^2 \frac{2}{\gamma + 1}, \quad (\text{II.A.5})
\]

where the superscript \(p1\) indicates conditions after the shock left the piston for the first time.

To find the speed of the shock front, transform from the shock-rest frame into the lab frame, where the shock front has speed \(V_{\text{shock}}^{p1}\). The upstream flow velocity is zero and the downstream flow velocity is \(V_0\) toward the cap. Equation (II.A.4a) becomes

\[
\left( \frac{V_{\text{down}}^{p1}}{V_{\text{up}}^{p1}} \right)_{\text{fluid}} = \frac{V_{\text{shock}}^{p1} - V_0}{V_{\text{shock}}^{p1}} = \frac{\gamma - 1}{\gamma + 1}, \quad (\text{II.A.6})
\]

or

\[
V_{\text{shock}}^{p1} = \frac{(\gamma + 1)V_0}{2}. \quad (\text{II.A.7})
\]

**Condition after shock first hits endcap**

The shock reverses direction after hitting the endcap the first time, and the fluid behind it is at rest in the lab frame. In the shock rest frame, (II.A.3) is still valid, but now
\[
\left( \tilde{M}_{up}^c \right)^2 = \frac{\left( \tilde{V}_{up}^c \right)^2 \rho_{up}^c}{\gamma P_{up}^c} = \frac{\left( \tilde{V}_{up}^c \right)^2 \rho_{down}^p}{\gamma P_{down}^p}. \tag{II.A.8a}
\]

Substituting (II.A.4) and (II.A.5) in (II.A.8a) gives

\[
\left( \tilde{M}_{up}^c \right)^2 = \frac{\left( \tilde{V}_{up}^c \right)^2 \rho_0 \gamma + 1}{\gamma - 1} \cdot \frac{2 \rho_0 \left( \tilde{V}_{p1}^c \right)^2}{\gamma + 1}. \tag{II.A.8b}
\]

Because \( V_{up}^p = 0 \), \( \tilde{V}_{up}^p \) should equal \( V_{shock}^p \), and equation (II.A.7) implies that

\[
\left( \tilde{M}_{up}^c \right)^2 = \frac{2 \left( \tilde{V}_{up}^c \right)^2}{\gamma (\gamma - 1) V_0^2} = \frac{2 \left( \tilde{V}_{shock}^c + V_0 \right)^2}{\gamma (\gamma - 1) V_0^2}. \tag{II.A.8c}
\]

The velocity jump condition (II.A.3a) can be expressed in terms of lab frame velocities as

\[
\left( \frac{\tilde{V}_{down}^c}{\tilde{V}_{up}^c} \right)_{\text{fluid}} = \frac{V_{shock}^c}{V_{shock}^c + V_0} = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{\left( \tilde{M}_{up}^c \right)^2 (\gamma + 1)}. \tag{II.A.9a}
\]

Substituting (II.A.8c) in (II.A.9a) gives

\[
\frac{V_{shock}^c}{V_{shock}^c + V_0} = \frac{\gamma - 1}{\gamma + 1} \left[ 1 + \frac{\gamma V_0^2}{\left( V_{shock}^c + V_0 \right)^2} \right], \tag{II.A.9b}
\]

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or

\[ V_{\text{shock}}^{cl} = (\gamma - 1)V_0. \]  \hspace{2cm} (II.A.10a)

Thus we get 
\[ \tilde{V}_{up}^{cl} = V_{\text{shock}}^{cl} + V_0 = \gamma V_0 \] and, using (II.A.8c),

\[ (\tilde{M}_{up}^{cl})^2 = \frac{2\gamma}{(\gamma - 1)}. \]  \hspace{2cm} (II.A.10b)

4. Comparison

Table 1 compares the shock velocities, parallel pressures and densities calculated from the kinetic method to those calculated from fluid theory. For simplicity we only compare these values for the first two "shocks". We consider two cases: \( \gamma = 5/3 \), which corresponds to a 3-D isotropic gas; and \( \gamma = 3 \), which corresponds to a 1-D gas.

In table 1, the fluid results for \( \gamma = 3 \) all agree exactly with the kinetic theory results, while for \( \gamma = 5/3 \), the fluid results disagree with kinetic theory by significant factors. Further study shows that the fluid (\( \gamma = 3 \)) results agree exactly with kinetic theory even if the leading-edge/shock bounces back and forth many times (see appendix). Note that no thermalization is needed for the "kinetic shock"; counter-streaming does it all.

Figure 22 compares the evolution of the simple fluid system for the \( \gamma = 3 \) and \( \gamma = 5/3 \) cases; where \( L_0 \) (the initial position of the piston) is \( 10^4 \) units, and \( V_0 \) (the velocity of the piston) is 1 unit. The curves show parameters measured next to the piston, and the jumps show the effects of the shock waves. Note that the difference between the densities as functions of time is modest and mostly temporary: the pipe-average densities are the same in both cases, since the volume of the cylinder is the same in both cases, as a
function of time. However, shock waves propagate faster in the $\gamma=3$ case, causing the two calculations to show substantial differences when the densities are plotted against the number of shock-wave reflections. The parallel pressure builds up faster with time in the $\gamma=3$ case, because all of the flow energy is fed into parallel thermal motions. As the shock waves get weaker after many reflections, the system becomes more adiabatic, as illustrated in Figure 22c. Of course, the adiabatic laws are different ($P_\| \rho^{-3} = \text{constant for } \gamma=3$, $P_\| \rho^{-5/3} = \text{constant for } \gamma=5/3$).

Figure 22. (a) $\log(P_\|/\rho_0 V_0^2)$ vs. time $t$; (b) $\log(\rho/\rho_0)$ vs. time $t$; (c) $\log \left( \left( P_\|/\rho_0 V_0^2 \right)(\rho/\rho_0)^{-\gamma} \right)$ vs. $n$, where $n$ is the number of times the shock has been reflected from the piston. All logarithms are base $e$. 
Table 1. Comparison of shock velocities, parallel pressures, and densities.

<table>
<thead>
<tr>
<th>Physical Parameters</th>
<th>Kinetic</th>
<th>Fluid ($\gamma=5/3$)</th>
<th>Fluid ($\gamma=3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shock/leading-edge</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>velocity in lab frame</td>
<td>Leaving piston 1(^{st}) time</td>
<td>$2V_0$ (II.A.1g)</td>
<td>$4/3V_0$ (II.A.7)</td>
</tr>
<tr>
<td></td>
<td>Leaving cap 1(^{st}) time</td>
<td>$2V_0$ (II.A.2g)</td>
<td>$2/3V_0$ (II.A.9a)</td>
</tr>
<tr>
<td>Density</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(downstream)</td>
<td>Leaving Piston 1(^{st}) time</td>
<td>$2\rho_0$ (II.A.1a)</td>
<td>$4\rho_0$ (II.A.4b)</td>
</tr>
<tr>
<td></td>
<td>Leaving cap 1(^{st}) time</td>
<td>$3\rho_0$ (II.A.2a)</td>
<td>$10\rho_0$ (II.A.3)(II.A.9b)</td>
</tr>
<tr>
<td>Pressure($P$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(downstream)</td>
<td>Leaving Piston 1(^{st}) time</td>
<td>$2\rho_0V_0^2$ (II.A.1c)</td>
<td>$4/3\rho_0V_0^2$ (II.A.5)</td>
</tr>
<tr>
<td></td>
<td>Leaving cap 1(^{st}) time</td>
<td>$8\rho_0V_0^2$ (II.A.2c)</td>
<td>$8\rho_0V_0^2$ (II.A.3)(II.A.9b)</td>
</tr>
</tbody>
</table>

B. Basic Equations For a Thin Filament in Double-Adiabatic MHD

1. Momentum equation

Following $C\&W$99, a filament is divided into a number of mass elements, and the vector momentum equation is applied to each element. The motion of a mass element is determined by the pressure gradient, Lorentz force and drag force as shown below:

$$ \rho \frac{du}{dt} = -\nabla \cdot \bar{P} + \mathbf{J} \times \mathbf{B} + F_d. $$  \hspace{1cm} (II.B.1)

where $F_d$ is an abstraction of the interaction between the filament and the background medium. In our simulation shown in following chapters, we have included the friction force and also the viscous force associated with motion parallel to the magnetic field and gradients parallel to that field.
\[ \mathbf{F}_q = \frac{-2F_c \eta_r \mathbf{u}}{R^2} + f_h \eta_v \frac{\partial^2}{\partial S^2} u_B \mathbf{b}, \]  \hspace{1cm} (II.B.2)

where \( F_c \) is defined in equation (14.a) of C&W99, \( \eta_f \) and \( \eta_v \) are viscosities, \( f_h \) is the fraction of viscosity that would be considered in the simulation \( (0 < f_h < 1) \). The reason we use two different viscosities is because \( \eta_f \) is mostly based on the mean free path in the perpendicular direction while \( \eta_v \) is for parallel motion.

The \( \eta_f \) has the same definition as \( \eta \) in C&W99 except that we have already incorporated the constant \( f_e \) into \( F_c \).

\[ \eta_f = \frac{1}{6} \left[ \left( \rho u_\perp \bar{\lambda} \right) + \left( \rho u \bar{\lambda} \right) \right], \]  \hspace{1cm} (II.B.3)

where \( u_\perp \) is the average thermal speed in the plane perpendicular to \( \mathbf{B} \); we take the mean free path \( \bar{\lambda} = 2r_L \). After substituting the expression for \( \bar{u} \) and \( r_L \), we get:

\[ \eta_f = \frac{\pi}{6} \left[ \left( \frac{P \bar{m}}{eB} \right) + \left( \frac{T \rho}{eB} \right) \right]. \]  \hspace{1cm} (II.B.4)

For the value of \( \eta_v \), we add in the parallel thermal speed as follows:

\[ \eta_v = \frac{1}{6} \left[ \left( \rho (u_\perp \langle \Delta u \rangle)^{1/2} \bar{\lambda} \right) + \left( \rho (u_\perp \langle \Delta u \rangle)^{1/2} \bar{\lambda} \right) \right], \]  \hspace{1cm} (II.B.5)
where \( \langle \Delta \tilde{u}_\| \rangle \) is the average thermal speed in direction parallel to the magnetic field.

Substituting the expressions for those average speeds we get

\[
\eta_v = \frac{1}{6} \left[ \frac{(2P_{\parallel}P_{\perp})^{1/2}}{eB} \bar{m} + \sqrt{2} \left( \frac{T_P}{eB} \right)_{\text{med}} \right].
\]  

(II.B.6)

The double-adiabatic momentum equation differs from C&W99 by replacing the isotropic pressure by the pressure tensor. Assuming the motion is nonrelativistic and using the Maxwell equations, we can rewrite (II.B.1) as

\[
\rho \frac{du}{dt} = -\nabla \cdot \left( \frac{2\bar{P} + B^2}{2\mu_0} \bar{1} \right) + \left( \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{\mu_0} \right)_\mathbf{F}_d,
\]  

(II.B.7)

where \( \rho, \mathbf{u}, \mathbf{B} \) and \( \bar{P} \) are the density, velocity, magnetic field of the mass element and gas pressure tensor inside the filament. We further assume that the pressure tensor takes the gyrotrropic form, that is \( \bar{P} = P_{\parallel} + (P_{\perp} - P_{\parallel}) \mathbf{b} \mathbf{b} \), where \( \mathbf{b} \) is unit vector along \( \mathbf{B} \). In the \( y \)-direction, we assume the total pressure balances between the filament and the background medium. The mathematical expression is

\[
P_{\perp} + \frac{B^2}{2\mu_0} = P_{\text{med}} + \frac{B_{\text{med}}^2}{2\mu_0},
\]  

(II.B.8)

where subscript "med" means the undisturbed background medium. Following C&W99, we define \( \Delta \mathbf{B} = \mathbf{B} - \mathbf{B}_{\text{med}} \), and using the assumption of background force balance

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\[-\nabla \left( P_{\perp,\text{med}} + \frac{B_{\text{med}}^2}{2\mu_0} \right) + \frac{(\mathbf{B}_{\text{med}} \cdot \nabla)\mathbf{B}_{\text{med}}}{\mu_0} = 0; \tag{II.B.9}\]

we can rewrite the momentum equation as

\[
\rho \frac{du(m,t)}{dt} = -\nabla \cdot [(P_{\parallel} - P_{\perp})\mathbf{b}\mathbf{b}] + \frac{(\mathbf{B} \cdot \nabla)\Delta\mathbf{B}}{\mu_0} + \frac{(\Delta\mathbf{B} \cdot \nabla)\mathbf{B}_{\text{med}}}{\mu_0} + F_d \tag{II.B.10}
\]

We further define the third term on the right side of (II.B.10) as \textbf{term3}:

\[
\text{term3} = \frac{(\Delta\mathbf{B} \cdot \nabla)\mathbf{B}_{\text{med}}}{\mu_0}, \tag{II.B.11}
\]

Because the derivatives in term3 are taken analytically, they introduce little numerical error, which is very important as the code is always running at the edge of numerical stability.

2. Energy equation

Unlike the isotropic case, now we have two energy equations for the motion perpendicular and parallel to the \(\mathbf{B}\) field. For the perpendicular part, we assume the particle magnetic moments are conserved. Thus we get

\[
\frac{P_{\perp}(m,t)}{\rho(m,t)B(m,t)} = \frac{P_{\perp}(m,t = 0)}{\rho(m,t = 0)B(m,t = 0)}. \tag{II.B.12}
\]
This is also the first Chew-Goldberger-Low relation for the perpendicular part [Chew et al., 1956]. For the parallel part, we assume there is a finite heat conductance in the parallel direction

\[
\frac{\partial (T_\parallel B^2 / \rho^2)}{\partial t} = D \frac{B^2}{\rho^2} \frac{\partial^2 T_\parallel}{\partial s^2},
\]  

(II.B.13)

where \( T_\parallel = P_\parallel \overline{m} / \rho \), \( \overline{m} \) is the average ion mass, \( D \) is the thermal diffusivity (\( D \) is constant in the simulations in following chapters) and \( s \) is the length along the field line. If we put \( D \) equal to zero, the parallel energy equation is then equivalent to the second Chew-Goldberger-Low relation.

3. Conservation of mass and Magnetic field

Since the filament is defined to be along the magnetic field, we have the relation:

\[
\mathbf{B} = B \frac{\partial \chi(m,t)}{\partial s}.
\]  

(II.B.14)

Conservation of mass is built into the notation. The relation between the volume along the filament and the length along the filament is given by:

\[
dV = \Phi \frac{ds}{B},
\]  

(II.B.15)
where $\Phi$ is the magnetic flux (e.g., C&W99). From (II.B.15) we further get the relation between mass and the length along the filament:

$$dm = \Phi \rho \frac{ds}{B}.$$  \hfill (II.B.16)

Substituting (II.B.16) back into (II.B.14), we have

$$B = \Phi \rho \frac{\partial x(m,t)}{\partial m}.$$  \hfill (II.B.17)

Thus we get a closed set of equations to describe the motion of a thin filament. The equations are summarized in table 2. Together with the relationships $v = dx/dt$, $T_\parallel = P_\parallel m / \rho$, this equation set determines the evolution of parameters $x$, $v$, $B$, $P_\parallel$, $P_\perp$, $\rho$ and $T_\parallel$.

Table 2: Equations for the motion of a thin filament in double-adiabatic MHD

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
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<tr>
<td>Momentum equation in $xz$ plane</td>
<td>(II.B.10), two components.</td>
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C. Linear Waves

To get a better understanding of the filament, we need to investigate the linear wave modes existing in it. Following the common linearization method, we study the filament with plasma parameter values given in the form $B + \delta B$, $P_\parallel + \delta P_\parallel$, $P_\perp + \delta P_\perp$, $\rho + \delta \rho$, $T_\parallel + \delta T_\parallel$, where $B$, $P_\parallel$, $P_\perp$, $\rho$, and $T_\parallel$ are the values in the surrounding homogenous medium. For simplicity we will not consider the heat-conductance here. Assume that the wave perturbations $\delta X$ have the time and space dependence $\delta X = \delta X_0 \exp(i \omega t - i \mathbf{k} \cdot \mathbf{x})$.

The linearization of equation (II.B.12) gives

$$\frac{\delta P_\perp}{P_\perp} - \frac{\delta \rho}{\rho} - \frac{\delta B}{B} = 0. \quad (II.C.1)$$

(II.B.8) gives

$$P_\perp \frac{\delta P_\perp}{P_\perp} + \frac{B^2}{\mu_0} \frac{\delta B}{B} = 0. \quad (II.C.2)$$

Linearization of (II.B.17) gives

$$\delta B = \delta \rho \Phi \frac{\partial x}{\partial m} + \rho \Phi \frac{\partial \delta x}{\partial m}, \quad (II.C.3a)$$

or

$$\delta B = \delta \Phi \mathbf{b} + B \delta \mathbf{b}, \quad (II.C.3b)$$
which also implies

$$\delta B_\perp = \rho \Phi \frac{\delta x_\perp}{\partial m}. \quad (\text{II.C.4})$$

From the relation (II.B.16), we can get the relation

$$\frac{ds}{dm} = \frac{B}{\Phi \rho}, \quad (\text{II.C.5})$$

Combining with equation (II.C.4), we get

$$\delta b = \frac{\delta B_\perp}{B} = \frac{\partial \delta x_\perp}{\partial s}. \quad (\text{II.C.6})$$

As $\delta B = b \cdot \delta B$, dotting both sides of (II.C.3a) with $b$ and substituting (II.B.17) and (II.C.5) gives

$$\delta B = \delta \rho b \cdot B + Bb \cdot \frac{\partial \delta x_\perp}{\partial s}.$$ 

which simplifies to

$$\frac{\delta B}{B} = \frac{\delta \rho}{\rho} + \frac{\partial \delta x_\perp}{\partial s}. \quad (\text{II.C.7})$$
Substituting equation (II.C.7) into (II.C.1), we have

\[
\frac{\delta P_\perp}{P_\perp} - 2 \frac{\delta B}{B} = -\frac{\partial \delta x_\parallel}{\partial s}.
\] (II.C.8)

Substituting (II.C.8) into (II.C.2) gives

\[
\frac{\delta B}{B} = \frac{\beta_\perp}{2(1 + \beta_\perp)} \frac{\partial \delta x_\parallel}{\partial s},
\] (II.C.9)

where \( \beta_\perp = 2\mu_0 P_\perp / B^2 \). Substituting (II.C.9) into (II.C.8) gives

\[
\frac{\delta P_\perp}{P_\perp} = -\frac{1}{1 + \beta_\perp} \frac{\partial \delta x_\parallel}{\partial s}.
\] (II.C.10)

Substituting (II.C.9) and (II.C.10) in (II.C.1) gives:

\[
\frac{\delta p}{\rho} = \frac{-2 - \beta_\perp}{2(1 + \beta_\perp)} \frac{\partial \delta x_\parallel}{\partial s}.
\] (II.C.11)

Now let’s consider the parallel energy equation. With \( D=0 \), linearizing equation (II.B.13) gives

\[
\frac{\delta P_\parallel}{P_\parallel} - 3 \frac{\delta p}{\rho} + 2 \frac{\delta B}{B} = 0..
\] (II.C.12)
Substituting (II.C.9), (II.C.10) and (II.C.11) into (II.C.12) we have

\[
\frac{\delta P_{\perp}}{P_{\parallel}} = \frac{6 + 5 \beta_{\perp}}{2(1 + \beta_{\perp})} \frac{\partial \delta x_{\perp}}{\partial s}. \tag{II.C.13}
\]

The linearization of (II.B.5) with uniform background \( \mathbf{B} \) (no drag force) gives

\[
\rho \frac{\partial^2 \delta x}{\partial t^2} = -\nabla \cdot [(P_{\parallel} - P_{\perp}) \mathbf{b} \mathbf{b}] + \frac{(\mathbf{B} \cdot \nabla) \delta \mathbf{B}}{\mu_0},
\]

\[
= \mathcal{B} \left( \frac{1}{\mu_0} + \frac{P_{\perp} - P_{\parallel}}{B^2} \right) \frac{\partial \delta \mathbf{B}}{\partial s} + \mathbf{b} \frac{\partial (\delta P_{\perp} - \delta P_{\parallel})}{\partial s} - 2 \frac{B}{B^2} (P_{\perp} - P_{\parallel}) \frac{\partial \delta \mathbf{B}}{\partial s}. \tag{II.C.14}
\]

Using equation (II.C.3) and (II.C.4), the component of the momentum equation perpendicular to \( \mathbf{B} \) is

\[
\rho \frac{\partial^2 \delta x_{\perp}}{\partial t^2} = \mathcal{B} \left( \frac{1}{\mu_0} + \frac{P_{\perp} - P_{\parallel}}{B^2} \right) \frac{\partial \delta \mathbf{b}_{\perp}}{\partial s},
\]

\[
= \left( \frac{B^2}{\mu_0} + P_{\perp} - P \right) \frac{\partial \delta \mathbf{b}_{\perp}}{\partial s}, \tag{II.C.15}
\]

\[
= \left( \frac{B^2}{\mu_0} + P_{\perp} - P \right) \frac{\partial^2 \delta x_{\perp}}{\partial s^2}.
\]

If we further define
\[
\beta_\parallel = \frac{P_\parallel \cdot 2\mu_0}{B^2}, \\
C_\parallel^2 = \frac{B^2}{\mu_0 \rho}, \\
C_\parallel^2 = \frac{\gamma \rho P_\parallel}{\rho}.
\] (II.C.16)

Equation (II.C.15) describes a transverse wave with speed given by

\[
V_{\text{trans}} = C_A \left[1 - \frac{1}{2} (\beta_\parallel - \beta_\perp) \right]^3,
\] (II.C.17)

The parallel part of (II.C.14) is

\[
\rho \frac{\partial^2 \delta x_\parallel}{\partial t^2} = B^2 \left( \frac{1}{\mu_0} - \frac{P_\perp - P_\parallel}{B^2} \right) \frac{\partial (\delta B/B)}{\partial s} + P_\perp \frac{\partial (\delta P_\perp/P_\parallel)}{\partial s} - P_\parallel \frac{\partial (\delta P_\parallel/P_\parallel)}{\partial s}.
\] (II.C.18)

Substituting (II.C.9), (II.C.10) and (II.C.13) into (II.C.18) gives

\[
\rho \frac{\partial^2 \delta x_\parallel}{\partial t^2} = \left( 3P_\parallel + \frac{P_\perp}{2 \left( 1 + \beta_\perp \right)} \right) \frac{\partial^2 \delta x_\perp}{\partial s^2}.
\] (II.C.19)

Equation (II.C.19) describes a longitudinal wave with speed given by

\[
V_{\text{long}} = C_\parallel \left[1 - \frac{\beta_\perp^2}{6\beta_\parallel(1 + \beta_\perp)} \right]^\frac{1}{3}.
\] (II.C.20)
Comments:

1. Equation (II.C.17) is the same as the expression for the velocity of an Alfvén wave in double-adiabatic MHD, for a wave with \( \mathbf{k} \parallel \mathbf{B} \). (See, e.g., Siscoe 1983.) Though the condition of constant total pressure (equation (II.B.8)) is not an assumption of the usual treatment of MHD plane waves, the intermediate (Alfvén) wave turns out to have that property. Since there are no \( z \)-gradients involved in a plane wave propagating in the \( x \)-direction, all of the accelerating force in such a plane wave comes from magnetic tension; a filament feels only the same tension force, because there is no gradient in the total external pressure, and the filament is assumed to slip between background field lines.

2. Equation (II.C.20) is the same as \( \omega / k_z \) for a slow-mode plane wave that is propagating almost perpendicular to the magnetic field (e.g., Siscoe, 1983). For such a wave, the acceleration force is in the \( x \) direction, and the magnetic tension force is negligible. Consequently, the total pressure is constant in such a plane wave, in agreement with the assumption (II.B.8). (Of course, total pressure is not constant in a slow plane wave propagating in other directions.)

3. The longitudinal wave speed is never more than \( C_{s_{\parallel}} \), which is the sound speed of a one-dimensional ideal gas. If the magnetic field is quite strong, which corresponds to small \( \beta_{\perp} \), the longitudinal wave speed approaches its upper limit \( C_{s_{\parallel}} \). Of course that wave speed (\( \sqrt{3P_{\parallel}/\rho} \)) differs from the slow-mode speed in a low-\( \beta \) plasma with isotropic pressure (\( \sqrt{5P_{\parallel}/3\rho} \)) even if \( P_{\parallel} = P_{\perp} = P \), because now the perpendicular and parallel parts are treated separately, so that the wave energy is no longer distributed evenly between the different degrees of freedom.
4. Although in ideal MHD the transverse wave always propagates faster than the longitudinal (slow) wave, the same is not true for a double-adiabatic filament. From (II.C.17) and (II.C.20) using (II.C.16)

$$\frac{V_{\text{long}}}{V_{\text{trans}}} = \left\{ \frac{\beta_{\perp} [6 \beta_{\parallel} + \beta_{\perp} (6 \beta_{\parallel} - \beta_{\perp})]}{2(1 + \beta_{\perp}) [\beta_{\perp} (\beta_{\perp} + 2) - \beta_{\parallel} \beta_{\perp}]} \right\}^{\frac{1}{2}}. \quad \text{(II.C.21)}$$

Figure 23 shows a contour plot of equation (II.C.21).

![Contour plot](image-url)

Figure 23. Contour plot of $V_{\text{long}}/V_{\text{trans}}$ on the $\beta_{\parallel} - \beta_{\perp}$ plane.
D. Unstable modes

From the dispersion relation (II.C.17), the transverse mode is unstable if

\[ 1 - \frac{1}{2} (\beta_\parallel - \beta_\perp) < 0. \]  (II.D.1)

This is the standard double-adiabatic MHD firehose criterion. When (II.D.1) is satisfied, the transverse wave velocity \( V_{\text{trans}} = \omega/k \) is imaginary, according to (II.C.17), and independent of \( k \). The growth rate of the instability is thus proportional to \( k \).

For the longitudinal wave mode there exists a corresponding unstable mode. Equation (II.D.21) indicates that the filament slow mode exhibits the usual mirror instability in double-adiabatic MHD if

\[ 6\beta_\parallel (1 + \beta_\perp) < \beta_\perp^2. \]  (II.D.2)

Note that this MHD criterion differs from the kinetic criterion derived from kinetic theory [Siscoe, 1983; Kulsrud, 1983]:

\[ \beta_\parallel (1 + \beta_\perp) < \beta_\perp^2. \]  (II.D.3)

However, in our filament simulations the mirror instability is unlikely to occur; the firehose instability turns out to be more important for filaments in the tail and will affect the evolution of those filaments.
III. SIMULATION RESULTS AND COMPARISONS

In this chapter, we will be simulating two cases that result in a filament moving rapidly earthward using double-adiabatic MHD. One simulation is set up, as much as possible, to be the same as the MHD simulation of C&W 99, so that a comparison of the new and old results will clearly show the effects of the use of anisotropic MHD; that simulation was designed to represent a bubble, i.e., a filament that is, for some reason, substantially under-populated relative to its neighbors. The other simulation represents a closed filament created by a patch of reconnection in the middle plasma sheet near $x=25 R_E$. These cases attempt to represent two different ways of creating an interchange-unstable situation in the plasma sheet, resulting in an earthward-rushing filament. After presenting the numerical results from the first case, we compare with C&W 99 and then give a short theoretical argument to show that firehose instability seems inevitable in situations where a beam of particles is shot out from current sheet and streams through the background medium. The presentation of numerical results from the second case is followed by a discussion of implications for substorms.

A. Boundary conditions and initial conditions

In our simulation we still adopt the same 2D background configuration as in C&W99 (thin curves in Figure 24), which satisfies the Grad-Shafranov condition (I.A.45)

$$\nabla^2 A_{\text{med}} = -\mu_0 \frac{dP_{\text{med}}}{dA_{\text{med}}}$$  \hspace{1cm} (III.A.1)
[Voigt, 1986]. Here $A_{\text{med}}$ is the vector potential in the background medium, and $P_{\text{med}}$ is the corresponding pressure, which has been assumed to be isotropic. We give these parameters the same analytic forms as in equation (36) of C&W99, specifically

$$A_{\text{med}}(x,z) = \frac{\pi A_0}{2\alpha \Delta} \sin \left[ \alpha |z|-\Delta \right] \exp(-\alpha x) \quad |z|>\Delta \quad \text{(III.A.2a)}$$

$$A_{\text{med}}(x,z) = -A_0 \cos \left( \frac{\pi z}{2\Delta} \right) \exp(-\alpha x) \quad |z|\leq\Delta \quad \text{(III.A.2b)}$$

$$P(A_{\text{med}}) = 0 \quad |z|>\Delta \quad \text{(III.A.2c)}$$

$$P(A_{\text{med}}) = \left( \frac{A_{\text{med}}}{2\mu_0} \right)^2 \left( \frac{\pi}{2\Delta} \right)^2 - \alpha^2 \quad |z|\leq\Delta \quad \text{(III.A.2.d)}$$

Given the expression for the vector potential $A_{\text{med}}$ (III.A.2), the expression for $B_{\text{med}}$ for $|z|<\Delta$ is:

$$B_{\text{med},x} = -\frac{\pi A_0}{2\Delta} \sin \left( \frac{\pi z}{2\Delta} \right) \exp(-\alpha x),$$

$$B_{\text{med},z} = A_0 \alpha \cos \left( \frac{\pi z}{2\Delta} \right) \exp(-\alpha x).$$

\text{(III.A.3a)}$$

If $|z|>\Delta$, the expression for $B_{\text{med}}$ is:

$$B_{\text{med},x} = \text{sign}(z) \frac{\pi A_0}{2\Delta} \cos \left[ \alpha (|z| - \Delta) \right] \exp(-\alpha x),$$

$$B_{\text{med},z} = -\frac{\pi A_0}{2\Delta} \sin \left[ \alpha (|z| - \Delta) \right] \exp(-\alpha x).$$

\text{(III.A.3b)}$$
where \( \text{sign}(x) \) is a step function; \( \text{sign}(x) = -1 \) if \( x < 0 \), \( \text{sign}(0) = 0 \) and \( \text{sign}(x) = 1 \) if \( x > 0 \).

![Graph](image)

**Figure 24:** The shapes of the background magnetic field lines. The line spacing does not indicate field strength.

For the runs described in this chapter, all of the adjustable parameters in (III.A.2) are given the same values as in C&W99: specifically, \( A_0 = 150 \text{ nT}/R_E \); \( \alpha = 0.0208 \; R_E^{-1} \), \( \Delta \), which is the half thickness of the plasma sheet is set to \( 4 \; R_E \). The shapes of the background magnetic field lines with these values are shown in the Figure 24. For the runs discussed in this chapter, the background temperature is assumed to be 5 keV and uniform. Some runs for other conditions are included in the sensitivity studies of Chapter 4.

**Initial condition for depletion case**

We have set the initial conditions to be the same as C&W99, so that comparison of the new results with the earlier paper can clearly show the differences between ordinary MHD and double-adiabatic MHD. We select a background flux tube with equatorial crossing distance \( 40 \; R_E \) (the outermost thin curve in Figure 24), and define that as the filament. Then we decrease the pressure inside that filament by 30% (in chapter IV, we will vary this value and denote this parameter by \textit{percent}), keeping the shape the same, i.e., the initial pressure components and density inside the filament are set to
\((1 - \text{percent}) \cdot X_{\text{med}}\), where \(X_{\text{med}}\) is the parameter value of background medium; the y-force balance is obtained by increasing the filament's magnetic field. The initial temperature is taken to be 5 keV everywhere in the filament. The heat-diffusion coefficient \(D\) was set equal to \(V_{\text{thermal}} \cdot \bar{\lambda}_\parallel\), where \(V_{\text{thermal}}\) is the ion thermal velocity at 5 keV, and \(\bar{\lambda}_\parallel\) is set at 0.1 \(R_E\). We use 500 equal mass points to represent the filament. Each mass point has mass \(\Delta m\).

Since the gas pressure \(p\) is lower in the filament than in the surroundings and \(|\mathbf{B}|\) is higher, \(pV^{5/3}\) is clearly smaller. (Specifically, \(pV^{5/3}\) in the filament is 0.34 of the value in the neighboring flux tubes.)

**Initial condition for reconnection case**

![Initial condition for reconnection case](image)

Figure 25: the initial shape of the filament (thick line), in the reconnection case. The thin curves represent the background magnetic field, for both the depletion and the reconnection cases. The line spacing does not indicate field strength.

We wish to represent a situation in which a patch of reconnection occurs at \(X \sim 25 R_E\) (from now on, we use capital \(X\) to indicate the position is measured in the SM coordinate except \(X_0, X_1,\) and \(X_R\) in this chapter and \(X_{\text{min}}, X_{\text{max}}\) in Chapter IV) and to follow the subsequent evolution of a magnetic filament that undergoes reconnection and is ejected earthward. Therefore, the initial filament configuration is chosen so that the filament
crosses the equatorial plane at \( X_R = 25 \, R_E \). Far from the reconnection site, it resembles a background field line that extends to \( X_0 (= 35 \, R_E) \), further out in the tail. (See Figure 25.)

For the sake of simplicity, the filament pressure is taken to be uniform, isotropic, and equal to value in the background filament that extends to \( X_0 \). The initial filament temperature is 5 keV, and the initial velocity is zero. The heat-diffusion coefficient \( D \), as in the depletion case, was set equal to \( V_{\text{thermal}} \cdot \bar{A}_\|= \), with \( \bar{A}_\| = 0.1 \, R_E \). We use 250 equal mass points to represent the filament in the reconnection case.

Mathematically, we describe the initial filament shape \( z_i(x) \) as a linear combination of \( z_0(x) \), which represents a background field line that crosses the equatorial plane at \( X_0 \), and \( z_R(x) \), which describes a background field line that extends only to \( X_R \):

\[
z_i(x) = a(x)z_0(x) + \frac{z_0(X_1)}{z_R(X_1)}[1 - a(x)]z_R(x)
\]  

(III.A.4)

with

\[
a(x) = \begin{cases} 
1 & \text{for } x < X_1 \\
\frac{1}{e^{k_\| (x - X_1)^2}} & \text{for } x \geq X_1 
\end{cases}
\]

(III.A.5)

where \( k_\|= 0.2 \, R_E^{-2} \) and \( X_1 = 19 \, R_E \). Equations (III.A.4) and (III.A.5), while admittedly containing a high degree of arbitrariness, ensure that \( z_i(x) = z_0(x) \) for \( x < X_1 \), that \( z_i(X_R) = 0 \), and that \( z_i \) and \( dz_i/dx \) are continuous at \( x = X_1 \). Figure 25 shows the initial filament shape with background field lines.
Boundary conditions

The filament intersects the model boundaries at the equatorial plane \((z=0)\) and at the left side \((x=5\, R_E)\). We assume the symmetry conditions \(V_z=0, B_z=0\) at the equatorial plane.

The left boundary condition is a bit more complex. As the direction of the \(B\) field inside the filament is different from the \(B\) field of adjacent background, there should be field-aligned current on both sides of the filament as shown in Figure 13. The earthward boundary condition is designed to represent the effect of the conducting ionosphere. We assume that the current that flows from the modeling region into the boundary is completed by Pedersen current, with a conductance of 4 mhos, the same as in \(C\&W'99\). This conducting-boundary condition relates the current into the boundary to the \(z\) component of the filament velocity and the \(E\) field \([C\&W'99, \text{equation (34)}]\), specifically

\[
E_y = \frac{\delta B_z}{\mu_0 \Sigma_p} = -\dot{z} B_x \tag{III.A.6}
\]

The \(x\) component of the filament velocity is set to zero at the ionosphere boundary.

B. Numerical Method

The numerical scheme is Trapezoidal-Leapfrog/Central-Difference, with automatic time step adjustment. We have put a lower limit of \(10^{-6}\) s for the time step; when the time step drops below that value we terminate the simulation. To get the first two time-steps for our three-level schemes, either an Euler scheme is used, or a combination of Euler and second-order Adams-Bashforth or Leapfrog. The same procedures are used when the time-step is adjusted.
(i) momentum equation

We rewrite the momentum equation (II.B.7) for convenience:

\[
\rho \frac{du(m,t)}{dt} = - \nabla \cdot \left[ \frac{(P_\parallel - P_\perp)}{B^2} \mathbf{BB} \right] + \frac{(\mathbf{B} \cdot \nabla)\Delta\mathbf{B}}{\mu_0} + \frac{(\Delta\mathbf{B} \cdot \nabla)\mathbf{B}_{\text{med}}}{\mu_0} + \mathbf{F}_d.
\]

As \( \mathbf{B} \cdot \nabla = B \partial / \partial s \) and \( \nabla \cdot \mathbf{B} = 0 \), we can rewrite the above equation in the form

\[
\frac{du}{dt} = \frac{B}{\rho \partial s} \left[ \frac{(P_\parallel - P_\perp)}{B^2} \right] \mathbf{B} + \frac{B}{\rho \mu_0} \frac{\partial \Delta\mathbf{B}}{\partial s} + \frac{\rho \mu_0}{\rho} \frac{(\Delta\mathbf{B} \cdot \nabla)\mathbf{B}_{\text{med}}}{\mu_0} + \frac{\mathbf{F}_d}{\rho}.
\]  

(III.B.1a)

If we further transform \( \partial / \partial s \) into \( \partial / \partial m \) using (II.C.5), we have:

\[
\frac{du}{dt} = \Phi \frac{\partial}{\partial m} \left[ \frac{(P_\parallel - P_\perp)}{B^2} \right] \mathbf{B} + \frac{\Phi}{\mu_0} \frac{\partial \Delta\mathbf{B}}{\partial m} + \frac{(\Delta\mathbf{B} \cdot \nabla)\mathbf{B}_{\text{med}}}{\rho \mu_0} + \frac{\mathbf{F}_d}{\rho}.
\]  

(III.B.1b)

Notice that the first term on the RHS of (III.B.1b) comes from the anisotropy of the gas, and is zero in the isotropic case.

The Trapezoidal-Leapfrog scheme can be separated into two steps: the common Leapfrog step or prediction step, and the correction step. The general form of this scheme is [Durran, 1999]:

\[
U^* = U^{n-1} + 2\Delta t F(U^n),
\]  

(III.B.2a)
\[ U^{n+1} = U^n + \frac{\Delta t}{2} \left[ F(U^*) + F(U^n) \right]. \]  

(III.B.2b)

As usual, the superscript denotes time, the symbol "*" denotes the prediction step; the subscript denotes the mass element number along the filament. We also use second order Adam-Bashforth when the time-step is adjusted. The general formula for this scheme is [Durran, 1999]:

\[ U^{n+1} = U^n + \frac{\Delta t}{2} \left[ 3F(U^n) - F(U^{n-1}) \right]. \]  

(III.B.3)

The prediction step of the momentum equation is:

\[ \frac{u^*_{i} - u^{n-1}_i}{2\Delta t} = \Phi \frac{1}{2\Delta m} \left[ \left( \frac{P_i - P_j}{B^2} \right) B_{i+1}^n - \left( \frac{P_i - P_j}{B^2} \right) B_{i-1}^n \right] 
+ \Phi \frac{1}{\mu_0 2\Delta m} [\Delta B_{i+1}^n - \Delta B_{i-1}^n] + \frac{\text{term3}_i^n}{\rho_i^n} + \frac{(F_d)_i^n}{\rho_i^n} \]

or

\[ u^*_{i} = u^{n-1}_i + 2\Delta t\Phi \frac{1}{2\Delta m} \left[ \left( \frac{P_i - P_j}{B^2} \right) B_{i+1}^n - \left( \frac{P_i - P_j}{B^2} \right) B_{i-1}^n \right] 
+ 2\Delta t \frac{1}{\mu_0 2\Delta m} [\Delta B_{i+1}^n - \Delta B_{i-1}^n] + 2\Delta t \frac{\text{term3}_i^n}{\rho_i^n} + 2\Delta t \frac{(F_d)_i^n}{\rho_i^n} \]  

(III.B.4a)

\[ = u^{n-1}_i + 2\Delta t \cdot a^*_{i}. \]
As Matlab (the software we adopt to do the simulation) can do array operations, we keep
the equations in vector form. The correction step is:

\[
\mathbf{u}_{i}^{n+1} = \mathbf{u}_{i}^{n} + \frac{\Delta t}{2} \Phi \frac{1}{2\Delta m} \left[ \left( \frac{P_{i}-P_{i}}{B^{2}} \right)_{t+1}^{*} \mathbf{B}_{t+1}^{*} - \left( \frac{P_{i}-P_{i}}{B^{2}} \right)_{1-t} \right] + \frac{\Delta t}{2} \frac{\Phi}{\mu_{0} 2 \Delta m} [\Delta \mathbf{B}_{i+1}^{\ast} - \Delta \mathbf{B}_{i-1}^{\ast}] + \frac{\Delta t}{2} \frac{\text{term}3^{*}}{\rho_{i}^{n}} + \frac{\Delta t}{2} \frac{(F_{d})^{*}}{\rho_{i}^{n}} + \frac{\Delta t}{2} \mathbf{a}_{i}^{n}.
\]  

(III.B.4b)

\[= \mathbf{u}_{i}^{n} + \frac{\Delta t}{2} (\mathbf{a}_{i}^{n} + \mathbf{a}_{i}^{*}).\]

(ii) energy equations

The prediction step for the perpendicular energy equation (II.B.12) is

\[
\left( \frac{P_{\perp}}{\rho B} \right)_{i}^{*} = \left( \frac{P_{\perp}}{\rho B} \right)_{i}^{0},
\]  

(III.B.5a)

and the correction step is

\[
\left( \frac{P_{\perp}}{\rho B} \right)_{i}^{n+1} = \left( \frac{P_{\perp}}{\rho B} \right)_{i}^{0}.
\]  

(III.B.5b)

For the parallel energy equation (II.B.13), the prediction step is
\[
\left( \frac{T_i B^2}{\rho^2} \right)^* \left( \frac{T_i B^2}{\rho^2} \right)^{n-1} + 2\Delta t \left( \frac{B^2}{\rho^2} \right)^n \left( \frac{\partial^2 T_i}{\partial s^2} \right)^n
\]

\[
= \left( \frac{T_i B^2}{\rho^2} \right)^{n-1} + 2\Delta t \Phi^2 \left( \frac{B}{\rho} \right)^n \left[ \left( \frac{\rho}{B} \right)_{i+1}^n - \left( \frac{\rho}{B} \right)_{i-1}^n \right] \left( T_i \right)_{i+1}^n - \left( T_i \right)_{i-1}^n \left( 2\Delta m \right)^2
\]

\[
+ 2\Delta t \Phi^2 \left[ \left( T_i \right)_{i+1}^n - 2\left( T_i \right)_i^n + \left( T_i \right)_{i-1}^n \right] \left( \Delta m \right)^2
\]

\[
= \left( \frac{T_i B^2}{\rho^2} \right)^{n-1} + 2\Delta t \cdot \text{HeatConduct}^n_i.
\]

(III.B.6a)

The correction step is

\[
\left( \frac{T_i B^2}{\rho^2} \right)^{n+1}_i = \left( \frac{T_i B^2}{\rho^2} \right)^n_i + \frac{\Delta t}{2} \cdot \left( \text{HeatConduct}^n_i + \text{HeatConduct}^*_i \right).
\]

(III.B.6b)

(iii) force balance in perpendicular direction

The force balance equation (II.B.8) can be expressed in prediction-correction form as:

\[
\left( P_i + \frac{B^2}{2\mu_0} \right)^* = P_{\text{med}} \left( x_i^* \right) + \frac{B_{\text{med}} \left( x_i^* \right)^2}{2\mu_0},
\]

(III.B.7a)

and

\[
\left( P_i + \frac{B^2}{2\mu_0} \right)^{n+1}_i = P_{\text{med}} \left( x_i^{n+1} \right) + \frac{B_{\text{med}} \left( x_i^{n+1} \right)^2}{2\mu_0}.
\]

(III.B.7b)
(iv) magnetic field components:

The equation for the magnetic field inside the filament (II.B.17) is expressed in finite difference form as

$$\left( \frac{\mathbf{B}}{\rho} \right)_{i}^{n+1} = \Phi \left( \frac{\mathbf{x}_{i+1}^{n} - \mathbf{x}_{i-1}^{n}}{2\Delta m} \right).$$

(III.B.8a)

and

$$\left( \frac{\mathbf{B}}{\rho} \right)_{i}^{n+1} = \Phi \left( \frac{\mathbf{x}_{i+1}^{n+1} - \mathbf{x}_{i-1}^{n+1}}{2\Delta m} \right).$$

(III.B.8b)

(v) position of filament:

The prediction step is:

$$\mathbf{x}_{i}^{\ast} = \mathbf{x}_{i}^{n-1} + 2\Delta t \mathbf{u}_{i}^{n},$$

(III.B.9a)

and the correction step is:

$$\mathbf{x}_{i}^{n+1} = \mathbf{x}_{i}^{\ast} + \frac{\Delta t}{2} (\mathbf{u}_{i}^{n} + \mathbf{u}_{i}^{\ast}).$$

(III.B.9b)

(vi) perfect gas law for parallel direction:

For the relation between $P_{\parallel}$, $T_{\parallel}$ and density, the prediction step is:
\[(P_i^*)_i = \frac{(T_i \rho_i)_i}{\bar{m}}, \quad \text{(III.B.10a)}\]

and the correction step is:

\[ (P_i^{n+1})_i = \frac{(T_i \rho_i)^{n+1}_i}{\bar{m}}. \quad \text{(III.B.10b)} \]

We have tested the accuracy of the numerical method by comparing the simulation results with known analytic results for linear waves. For the transverse wave mode and for long wavelengths (250 grid points per wavelength), second order Adams-Bashforth gives 0.072\% error in period and the Trapezoidal-Leapfrog scheme produces 0.028\% error. For short wavelengths (10 grid points per wavelength), the second order Adams-Bashforth gives a 5.95\% error, while Trapezoidal Leapfrog gives 6.56\%. For the longitudinal wave mode, the corresponding errors are: 0.046\%, 0.041\%, 5.75\% and 5.75\%. Tests were also run for firehose unstable situations, and the error in linear growth rate was about 1\% for intermediate wavelength waves (about 40 ~ 70 grid points per wavelength). So the code is very accurate for the long wavelength waves, but the numerical errors rise as the wavelength becomes shorter and shorter.
C. Simulation results

1. Depletion case

Figure 26 shows how the filament field-line changes shape during the first 200 seconds of the simulation, and Figures 27–30 show the evolution of various other parameters at five different times. All show general features that were also present in the isotropic case considered by C&W99. The equatorial part of the field line moves rapidly earthward. A transverse wave and a compressional wave both propagate earthward. In the present case, the compressional wave reflects from the earthward boundary between $t = 120$ and $t = 160$ seconds, and begins traveling tailward again.

Figure 26. Shapes and positions of the filament at $t = 0, 40, 80, 120, 160, 200$ seconds. Different scales are used for the $x$ and $z$ axes. The length unit is $R_E$. The dashed curve represents the $t=180$ sec field line from the isotropic-MHD calculation (Fig. 6 of C&W99).
The results shown in Figures 26 to 30 show some obvious differences from the isotropic simulations of C&W99, from which the key figures were reproduced in Chapter I:

1. The equatorial part of the filament exhibits slightly slower earthward motion than in the isotropic case (Figure 26).

2. The perpendicular pressure is smaller than the isotropic case for approximately the same shock location, as most of the released magnetic energy now goes into parallel motion. (Compare Figure 29 with Figure 18.)

3. The parallel pressure is higher than in the isotropic case. The effect is most dramatic after the compressional shock has reflected from the earthward boundary. (Compare Figure 30 with Figure 18.)

4. Though the compressional shock propagates earthward faster than in the isotropic case, the transverse wave propagates much slower, and field line motion is largely confined to large distances from Earth, within the period of the present simulations. The line shape for isotropic MHD is considerably different (Figure 26).

5. Firehose instability, which cannot occur in isotropic MHD, begins at $x=15 R_G$, at around $t=200$ s. The wave phase velocity $[\left( P_p + B^2/2m_e \right) / \rho]^{1/2}$ decreases to zero as the firehose instability approaches. The instability occurs in this run before the Alfven wave hits the ionosphere boundary. Consequently, the Alfven wave never reached the ionosphere boundary in this simulation, which means that the ionospheric foot point of the filament hardly moves during this simulation.
Figure 27. The perpendicular component of velocity, $V_\perp$ (unit 100km/s) vs. $x$ (unit $R_E$). Time $t$ is shown in each panel.

Figure 28. The parallel component of velocity, $V_\parallel$ (unit 100km/s) vs. $x$. 
Figure 29. The perpendicular component of pressure tensor, $P_\perp$ (unit $10^{-10}$ Pa) vs. $x$.

Figure 30. The parallel component of pressure tensor, $P_\parallel$ (unit $10^{-10}$ Pa) vs. $x$. 
Figure 31. Detailed evolution of $V_\perp$ around $t=200$ second. Format is the same as Figure 27.

Figure 31 shows the nonlinear development of $V_\perp$ near $t = 200$ second with more time detail. After the compressional shock reflects from the near-Earth region and starts tailward again, it moves the region downstream of it toward firehose instability. Figure 32 shows the firehose parameter
\[ C_f = 1 - \frac{1}{2} (\beta_\parallel - \beta_\perp) \]  

where \( C_f < 0 \) implies firehose instability. Note that the filament is firehose stable until the compressional wave reflects from the earthward boundary and propagates tailward beyond \( x = 10R_e \).

About the time the firehose instability criterion is first satisfied in this simulation, the shock encounters the Alfven wave that has been propagating earthward (slowly, due to large \( P_\parallel \)). That Alfven wave provides a large seed amplitude for the firehose instability, which develops rapidly. The linear growth rate of the firehose instability is proportional to the wave number, so short-wavelength waves dominate. Large wave amplitudes develop on the scale of the distance between mass points, and we stopped the simulation because it had lost accuracy.

We cannot follow the nonlinear development of the firehose instability reliably within the present formalism but would like to suggest that observers watch for signs of firehose instability on field lines that exhibit counter-streaming flows in a region where the magnetic field is not large.
Figure 32. Evolution of firehose parameter $C_f$ vs. $x$. Values of $C_f<0$ indicate linear firehose instability.
2. Theoretical Consideration of Firehose Instability

Since the accuracy of single-fluid, double-adiabatic MHD in treating these field lines has not been rigorously demonstrated, it seems prudent to consider the question of firehose instability from another theoretical viewpoint, specifically viewing the plasma in terms of particles in beams. Assume that an earthward-moving beam is generated near the equatorial plane with velocity given by

$$V_b^2 = \frac{B_E^2}{\mu_0 \rho_E},$$  \hspace{1cm} (III.C.2)

where \(B_E\) and \(\rho_E\) are typical 'external' values. (Reconnection generates such a beam, as does the creation of a substantially underpopulated (bubble) flux tube like the one simulated above.) For simplicity, assume that the background gas and the beam have the same velocity distribution function (relative to different mean velocities). However, we still treat the parallel and perpendicular motions separately; we assume that the total pressure is balanced in the \(y\)-direction, as represented by equation (II.B.3). Let \(f_0(v_{||})\) be the distribution function for the parallel velocity component.

Consider first the case where the plasma consists of one beam (designated by the subscript \(b\)) and a background plasma (designated by subscript \(0\)). Assume the beam has density \(\rho_b\) and that the background gas has density \(\rho_0\). The temperatures of the beam and background are assumed to be isotropic and equal. We write
\[ P_1 = \int m f(v) \left( v - \overline{v}_1 \right)^2 dv_1 d\tilde{x} \]
\[ = \rho_0 \int f_0(v) \left( v^2 - 2v \overline{v}_1 + \overline{v}_1^2 \right) dv_1 + \rho_b \int f_0(\tilde{v}) \left( \tilde{v}^2 - 2\tilde{v} \overline{v}_1 + \overline{v}_1^2 \right) d\tilde{v} \]
\[ = \left( 1 + \frac{\rho_b}{\rho_0} \right) P_{\parallel 0} + \rho_0 \overline{v}_1^2 + \rho_b \overline{v}_1^2, \]  
(III.C.3)

where the mean velocity of the whole distribution is given by

\[ \overline{v}_1 = \frac{\rho_b}{\rho_0 + \rho_b} V_b, \]  
(III.C.4)

We have also used the definitions

\[ \hat{v}_1 = \frac{\rho_0}{\rho_0 + \rho_b} V_b, \]  
(III.C.5)

and

\[ \tilde{v}_1 = v_1 - V_b. \]  
(III.C.6)

With (III.C.5) and (III.C.3), the tension along the field line can be written

\[ \frac{B^2}{\mu_0} + P_\perp - P_\parallel = \frac{B^2}{2\mu_0} + P_{\text{med}} \left( 1 + \frac{\rho_b}{\rho_0} \right) P_{\parallel 0} - \left( \rho_0 \overline{v}_1^2 + \rho_b \overline{v}_1^2 \right), \]  
(III.C.7)

which becomes, with (III.C.2), (III.C.5) and (III.C.6)
\[ \frac{B^2}{\mu_0} + P_\perp - P_\parallel = \frac{B^2}{2\mu_0} + P_\text{med}^{\text{total}} - \left(1 + \frac{\rho_b}{\rho_0}\right) P_\parallel - \frac{\rho_0 P_0}{\rho_b + \rho_0} \cdot \frac{B_E^2}{\mu_0 \rho_E}. \quad (\text{III.C.8}) \]

Under the same assumption (III.C.2), but for the case where there is a background plus two identical beams, with the same speed but opposite directions, the tension is given by

\[ \frac{B^2}{\mu_0} + P_\perp - P_\parallel = \frac{B^2}{2\mu_0} + P_\text{med}^{\text{total}} - \left(1 + \frac{2\rho_b}{\rho_0}\right) P_\parallel - 2\rho_b \cdot \frac{B_E^2}{\mu_0 \rho_E}. \quad (\text{III.C.9}) \]

Of course, firehose instability corresponds to negative tension, or negative values on the right side of (III.C.8) or (III.C.9).

Comments:

1. The region near the equatorial plane, where all of the terms on the right side of (III.C.8) and (III.C.9) are comparable, tends to be close to firehose unstable.
2. In the low-\(\beta\) region near the Earth, the positive terms dominate, and the filament tends to be firehose stable.
3. The system has the strongest tendency to be unstable for the situation where the second beam has reached the weak-field region near the equatorial plane.

3. Reconnection case

In the NENL model of substorms, an X-line at \(X\sim 25 \, R_E\) creates a jet of sunward-streaming plasma, which crashes into the strong-field region at \(X\sim 8 \, R_E\). Various onset phenomena that seem centered at \(X\sim 8 \, R_E\) are, within the context of the NENL model,
attributed to this “magnetic pileup” process [Hesse and Birn, 1991; Shiokawa et al., 1997]. However, it has never been clear how this magnetic pileup process could cause the very strong magnetic fluctuations that are observed at X≈ 8 R_E (e.g., Figure 11) [Ohtani et al., 2002].

In the simulation described above, a bubble-type filament that initially stretched to 40 R_E but was substantially depleted relative to its neighbors resulted in the filament rushing earthward and the creation of firehose instability and large-amplitude magnetic fluctuations at X≈ 15 R_E. We therefore set up a simulation to test whether a filament that results from a patch of reconnection at X= -25 R_E could result in firehose instability at X≈ 8 R_E.

Simulation results with X_0=35 R_E, X_R =25 R_E, X_1 =19 R_E, and F_c=0.06 are shown in Figures 33~40. The heat-diffusion coefficient D has the same value as in above depletion case. And we use 250 mass points to represent the filament in this simulation.

Figure 33: the shapes of the filament at different times, in seconds.
Figure 34: $V_\perp$ (in units of 100 km/s) vs. $x$ (in units of $R_E$). The times, from the top, are 0, 32, 64, 96, and 128 s.

Figure 35: $V_\parallel$ (in units of 100 km/s) vs. $x$ (in units of $R_E$). The times, from the top, are 0, 32, 64, 96, and 128 s.
Figure 36: $P_\perp$ (in units of $10^{10}$ Pa) vs. $x$. The times, from the top, are 0, 32, 64, 96, and 128 s.

Figure 37: $P_\parallel$ (in units of $10^{10}$ Pa) vs. $x$. The times, from the top, are 0, 32, 64, 96, and 128 s.
Figure 38: $B$ (in nT) vs. $x$. The times, from the top, are 0, 32, 64, 96, and 128 s.

Figure 39: $B_z$ (in nT) vs. $x$. The times, from the top, are 0, 32, 64, 96, and 128 s.
Figure 40: Firehose factor $C_f$ vs. $x$. The times, from the top, are 0, 32, 64, 96, and 128 s.

Several features of Figures 33–39 are similar to those in the bubble simulations of C&W99:

1. Fast earthward flow, which shows up in the plots as large $V_\perp$ near the equatorial plane and large $V_\parallel$ along most of the field-line length. Of course, it could be viewed as an interchange effect as the reconnection flux tube has a smaller $pV_\parallel$ than its undisturbed neighbors.

2. The $V_\perp$ plot shows a transverse wave propagating from the equatorial region to the earthward conducting boundary. After the wave reaches that boundary, the filament’s left end (which represents the ionospheric footprint) starts to move equatorward. This is different from the depletion case discussed earlier, in which the firehose instability occurred before the transverse wave reached the earthward boundary.
3. Figures 35-37 show a compressional shock that propagates to the left boundary. The reflected, tailward-propagating shock brings the flow almost to a halt. That is how “flow braking” shows up in these filament simulations. The small “overshoot” that appears just downstream of the compressional shock is a numerical artifact.

4. The amplitude of disturbance in $B_z$ quickly becomes very large, but for $|B|$, the disturbance is very small. Also we should notice that $|B|$ does not vary as dramatically in the unstable region as other parameters do.

As in the depletion case above, we get firehose instability, which begins after the compressional wave reflects from the near-Earth boundary. This simulation was terminated $\sim 30$ s after onset of the instability, because the automatic time-step dropped below a pre-set minimum of $10^6$ s. The simulation had anyway lost its fidelity to the original differential equations, because the wavelength of the dominant simulated firehose ripples is $\sim 4$ mass points and the dominant mode is clearly affected by the numerics. This is inevitable with the double-adiabatic-MHD-filament equations, because the linear firehose growth rate is proportional to the wave number [Ji and Wolf, 2003]. More detailed theoretical work by Horton et al. [2003] suggests that the dominant firehose ripples should have $k^1$--ion gyroradius or inertial length, which are comparable for $\beta \sim 1$ and $\sim 0.05 \, R_E$ for our case, $\sim 1/3$ of the distance between our adjacent mass points.

Our simulation results also suggest that flow bursts are likely to cause firehose instability. The two main mechanisms that have been suggested for generating bursty bulk flows are (1) flux tubes created with smaller $pV^T$ than others [Pontius and Wolf, 1990]; (2) patches of reconnection in the middle plasma sheet [Baumjohann et al., 1990].
The simulations presented in this chapter suggest that both of these circumstances lead to firehose instability. Of course, the simulations do not suggest that firehose occurs where the bulk flow is fast enough for an in situ spacecraft to call it a flow burst, but rather in a later stage, when the flow is braking back down to low speed.

In the simulations, the firehose instability saturates in a simple way: crinkling of the filament increases its length, which, through the relevant Chew-Goldberger-Low equation, reduces $P_\parallel$ and increases $C_f$ back near zero, the marginal-stability level.

We should also remark that, aside from the short-wavelength ripples, the simulated filament has taken a contorted shape by $t = 128s$ in the region where $C_f \approx 0$ — a shape that looks unusual for a $\beta$-1 situation. It can occur because, at the firehose limit, there is no tension to straighten the field lines.

Our simulation suggests that reconnection occurring about $X = -25 R_E$ may generate large amplitude magnetic fluctuations at about $X = -10 R_E$. This is physically different from mechanisms that have been suggested previously for explaining those fluctuations, like the ballooning instability [Roux et al., 1991; Cheng and Lui, 1998] or the cross-field current instability [Lui et al., 1992]. Our fluctuations at $X = -10 R_E$ were caused by reconnection at $X = -25 R_E$.

Our wrinkled field lines represent a "friendly amendment" to the pile-up model [Hesse and Birn, 1991; Shiokawa et al., 1997]. If the disturbance in the near Earth region basically results from the fast ejection from the reconnection site, the accompanying firehose instability generates large-amplitude fluctuations. In the simulation, the firehose instability occurs just behind the tailward-propagating compressional shock that slows the flow. Though our simulation was terminated before the shock reached the equatorial
plane, the shock and its accompanying magnetic fluctuations must inevitably reach that region.

The creation of situations with $P_{\parallel}$ substantially greater than $P_{\perp}$ appears inevitable for the reconnection geometry considered here, as long as the gas is collisionless and pitch-angle scattering is not strong enough to effectively isotropize the distribution. The reason is essentially the one discussed quantitatively in Section III.C.2. The reconnection process creates a beam traveling earthward along the field line, and most of that beam must reflect from the strong-field region near the Earth, creating a distribution with two counter-streaming beams and an ambient population. By the time the tailward-directed beam reaches the weak-field region near the equatorial plane, firehose instability seems easy to achieve.

Our conclusions are subject to several theoretical uncertainties:

1. Although the physical reason for development of the firehose instability is clear, the detailed calculations are subject to the inaccuracies inherent in using double-adiabatic MHD in a collisionless plasma [e.g., Ferriere and Andre, 2001]. Additional calculations described in Chapter IV indicate that this uncertainty is insufficient to change the conclusion, although it may change significant details.

2. The location of firehose-instability onset is sensitive to the choice of the initial condition. Some initial conditions lead to weak firehose instability near the beginning of the simulation as the compression wave leaves the equatorial region. In other cases, the instabilities do not develop until the reflected wave gets back to the near-equatorial region. Sensitivity studies are described in Chapter IV.
3. We have assumed that the particle moves in the $xz$ plane and have not considered $y$-structure, e.g., twisting.

4. The chaotic motion of ions in the plasma sheet also helps to isotropize the distribution function. Detailed calculation shows that for the configuration we considered in this chapter, most ions have chaotic motion in the current sheet.

5. Because our simulations do not contain the physics that determines the dominant wavelength of the firehose, we cannot reliably follow the nonlinear development to the point where the instability reaches the equatorial plane.

6. Wave-particle interactions may isotropize the distribution function and prevent it from becoming firehose unstable.

Only the last uncertainty on the list holds clear potential for reversing our basic conclusion.

Although there are many observations of beams in the plasma sheet and plasma-sheet boundary layer [e.g., Parks et al., 2001], firehose-unstable particle distributions are more difficult to find in the data and do not show up in statistical averages [e.g., Kaufmann et al., 2000]. On the other hand, the simulation suggests that firehose-unstable pitch-angle distributions might be difficult to observe. In the simulation, the compressional shock creates strong firehose instability on a time scale of a few seconds. Once the instability develops, the large, rapid magnetic fluctuations make it very difficult to measure pitch-angle distributions. Furthermore, the instability in any region typically saturates in less than 20 seconds. Data from the Geotail measurements seem to indicate that there are many instances of firehose unstable pitch-angle distributions at $X \sim -10$ $R_E$ [R. L.
Kaufmann, private communication]. The results are shown in Figure 41, where the parameter $F$ is defined as

$$F = \frac{1}{2} (\beta_{\parallel} - \beta_{\perp}) \quad \text{(III.C.10)}$$

where $\beta_{\parallel}$ and $\beta_{\perp}$ are defined in (II.C.9) and (II.C.16). So the plasma is firehose unstable when $F>1$, and $F=0$ means isotropy.

![Figure 41: The histogram of firehose occurrences. The data shown in the Figure 41 are taken from region $-28 \, R_E < X < -10 \, R_E$ and $-15 < Y < 15 \, R_E$; and there are total about 14 thousand cases, the data are 65-second averaged. Data is from R. L. Kaufmann.](image)

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IV. SUPPORTING SIMULATIONS

This chapter describes two sets of computer experiments that were undertaken to evaluate the validity and robustness of double-adiabatic-MHD simulations described in Chapter 3. The first part describes a Gedanken experiment designed to test how well double-adiabatic MHD can approximate the more rigorous kinetic-theory approach, for a situation like the filaments discussed in Chapter 3, and also suggests how the predictions of a full kinetic-theory treatment might differ from MHD.

The second part of the chapter is devoted to sensitivity studies. We run the fluid code for different sets of parameter values, focusing on the relations between those parameters and the occurrence of the firehose instability.

A. Another Gedanken experiment

The experiment in chapter II showed that double-adiabatic MHD is much better than isotropic MHD in dealing with the multi-peak distribution function. Though the double-adiabatic calculation agrees exactly with the kinetic calculation in that experiment, there are two major differences between the conditions of the experiment and the filaments we are trying to represent in chapter III:

1. Although in Section II.A, we considered a pipe of constant cross section, the particles shot out from the current sheet in the magnetic tail by the Speiser mechanism will see an increasing magnetic field (corresponding to a gradually decreasing pipe cross-section).

2. The particles on the flux tube are not necessarily cold. (In the simulations described in Chapter III, the initial temperature is 5 keV.) So the magnetic moments are not
zero. From the discussion about magnetic mirroring in Chapter I, we know these particles will feel a mirroring force, which implies energy transfer between parallel and perpendicular motions. For the magnetospheric cases discussed in Chapter III, the parallel motion and the piston velocity are of the same order of magnitude as the average initial thermal speed in the simulation; thus the effects of finite initial temperature are not negligible. With non-zero initial temperature, we would not expect a kinetic calculation to show an Earthward-moving beam with a sharp leading edge. However a slow shock always forms in our fluid calculations, and its propagation and reflection from the Earthward boundary affect the firehose instability greatly.

To get a better feel for how much error is caused by using the double-adiabatic-MHD approximation, especially in the low $\beta$ region near the earthward boundary, where we found firehose instability in the reconnection case, we consider a second Gedanken experiment. We have found a case that is simple enough that the kinetic-theory solution can be worked out analytically but includes finite temperatures and a pipe of varying cross section.

1. Setup of the second experiment

We study the dynamics of gas inside a pipe with a piston at one end and a solid wall at the other, as shown in Figure 42.
Figure 42. An idealization of magnetic-field-aligned plasma flow in a magnetic filament. The moving piston generates interpenetrating particle beams in the pipe. The main difference between this filament and the one in Chapter II is that this pipe has varying cross section. The magnetic flux in the pipe is conserved, so the magnitude of magnetic field is proportional to the reciprocal of the cross section.

We assume that the gas initially has zero flow velocity, isotropic pressure, and temperature 5 keV; the initial distribution function is taken to be Maxwellian:

\[ f_0(v|, v_\perp) = M_i \exp \left( \frac{-m(v|^2 + v_\perp^2)}{2T} \right) \]

\[ = M_i \exp \left( \frac{-mv|^2}{2T} \right) \exp \left( \frac{-\mu B}{T} \right) \]

where \( \mu \) is the magnetic moment of the particle, which is a constant during the motion, and \( m \) is the mass of the particle. The magnetic field inside the pipe is not constant but rather a linear function of position \( x \)

\[ B(x) = B_{\text{min}} + kx, \quad x > 0 \]  \hspace{1cm} (IV.A.1)

Here the numerical values of the constants \( B_{\text{min}} \) and \( k \) are taken from simulations in Chapter III: \( B_{\text{min}} \) is the initial magnetic field strength at the equatorial crossing point of
the filament, and \( k \) is obtained by a linear regression of the initial magnetic field and position data from the simulations of Chapter III. We also want to point out that the coordinate system in this section is reversed from what was used in Chapter II, III, and IV.B, i.e., the bigger the \( x \) is, the closer the position to the ionospheric boundary. Assuming a time-independent magnetic field greatly simplifies the kinetic-theory calculation, and, in the simulations shown in Chapter III, magnetic field strength does not show dramatic changes with time; the time changes in \( B^2 \) play a less important role in determining the evolution of the filament than the time changes in \( P_\parallel \) and \( P_\perp \). We thus hope that by fixing \( B \) we will get a relatively easy case to study that is still close enough to the magnetospheric-filament situation that we can judge the validity of the double-adiabatic MHD for such situations.

We have set up two configurations to represent both the reconnection and depletion cases in Chapter III. In the first configuration, which represents the reconnection-case filament simulation, the piston speed \( V_0 \) is set to 500km/s, a typical Earthward speed of the filament's equatorial part in the reconnection simulation; the length of pipe is 20 \( R_E \), with piston initially at \( X_{\text{min}}=5 \ R_E \) and the cap at \( X_{\text{max}}=25 \ R_E \) at time \( t=0 \). In the second configuration, \( V_0 \) is 600km/s, the length of pipe is 35 \( R_E \), with piston initially at \( x=5 \ R_E \) and the cap at \( x=40 \ R_E \). For both cases, the initial density is taken from the simulation results in Chapter III.

2. Kinetic theory

We assume that the particle's total kinetic energy is conserved during its motion, except when it hits the piston. We also assume that the first invariant is conserved during
the motion, and the gas inside the pipe is collisionless. From kinetic theory we know that, for a collisionless gas, the distribution function should be a constant along any particle’s trajectory [Nicholson, 1992]:

$$\frac{df(v_{\parallel},v_{\perp})}{dt} = 0,$$  \hspace{1cm} (IV.A.2a)

$$f(v_{\parallel},v_{\perp})_{t=t_i} = f(v_{\parallel},v_{\perp})_{t=0}.$$ \hspace{1cm} (IV.A.2b)

where $v_{\parallel}$ and $v_{\perp}$ in (IV.A.2b) are velocity components for the same particle at different times. If we can find the mapping from the velocity space at $t = t_i$ to the velocity space at $t = 0$, we can easily calculate the distribution function at $t = t_i$, and from it the density, average velocity and pressure tensor at $t = t_i$ using relation (IV.A.2b).

From conservation of the particle’s kinetic energy we have

$$v_{\parallel} = \frac{ds}{dt} = \pm \left(2 \frac{E_k - \mu B}{m}\right)^{1/2}$$ \hspace{1cm} (IV.A.3)

where $E_k$ is the total kinetic energy of the particle. We should notice that $E_k$ is conserved between the particle’s two consecutive collisions with the piston. The ‘-‘ sign stands for the motion toward the piston and the ‘+’ sign stands for the motion toward the cap.

If we further define
\[ y = 2 \left( \frac{E_k - \mu B}{m} \right) = 2 \left( \frac{E_k - \mu B_{\text{min}} - \mu k x}{m} \right), \]  

we have

\[ dy = 2d \left( \frac{E_k - \mu B_{\text{min}} - \mu k x}{m} \right) = \frac{-2\mu k}{m} dx. \]  

Substituting (IV.A.4b) into (IV.A.3) (note \( x = s \)), gives

\[ \frac{m}{2\mu k} \frac{dy}{dt} = \mp y^{1/2}, \]  

or

\[ \frac{m}{\mu k} d\left(y^{1/2}\right) = \mp dt. \]  

And the integral of (IV.A.5b) gives

\[ \left(y_{1/2} - y_{2/2}\right) = \mp C_1 (t_1 - t_2), \]  

where \( C_1 \) is defined as

\[ C_1 = \frac{\mu k}{m}. \]  

It is very obvious that if \( y^{1/2} \) is real, then \( y^{1/2} = |V_\parallel|. \)

We can get the mapping between the velocity spaces by tracing the particle’s motion backward in time. In the following we call the trace-back of a particle with negative \( V_\parallel \)
the "negative tracing branch" and the trace-back of one with positive $V_{\parallel}$ as a "positive tracing branch". We use subscript 1 to denote the values at time $t_1$ (the current time) and subscript 0 to denote the values at time $t_0 = 0$, the subscript 2 denotes the time when the sign of the $V_{\parallel}$ changes, i.e., when the particle mirrors or collides with the ends.

First we consider motion toward the piston, in other words the motion with negative $V_{\parallel}$. The information we have at $t_1$ consists of the particle's parallel velocity component $V_{\parallel}$, the current position $x_i$ and of course the particle's magnetic moment $\mu$. For a particle that has negative $V_{\parallel}$ at time $t_1$, there are three possible pasts. The following list gives details on how we trace particles of each historical type backwards in time for the reconnection configuration:

1. The particle hit the cap at some time $0 < t_2 < t_1$ and was reflected from it. In this case the particle gained no energy from the collision, but its $V_{\parallel}$ changed sign: from '+' to '-'. We can calculate the $V_{\parallel}$ at the cap, flip the sign and then keep tracing using the positive tracing branch. We further denote the $B$ value at the right boundary by $B_{\text{max}}$, i.e., $B_{\text{max}} = B(x_{\text{max}})$. Using equation (IV.A.6), and taking the branch corresponding to negative $V_{\parallel}$ (that is the '+' sign), the parallel velocity just before the collision is given by

$$\langle V_{\parallel} \rangle = \left(\frac{2E_k - 2\mu B_{\text{max}}}{m}\right)^{1/2},$$  \hspace{1cm} (IV.A.8)

and the time of collision is
\[ t_2 = t_1 + \left( \frac{2E_k - 2\mu B_{\text{max}}}{m} \right)^{1/2} - \left| V_{\|} \right| \bigg| C_1^{-1} \bigg|. \] (IV.A.9)

The position of the collision is \( X_{\text{max}} \). The parameters \( (V_{\|})_2 \), and \( t_2 \) and the collision position \( x_2 = X_{\text{max}} \) constitute the input for the positive tracing branch. Note that a solution \( t_2 > 0 \) does not exist either if the right side of (IV.A.9) is complex (meaning that the particle mirrored), or if the right side is negative (meaning that the particle trace would have encountered the cap if the trace had been followed further back in time). We now discuss those two cases.

2. The particle mirrored at some time \( 0 < t_2 < t_1 \). We need to find the time and position of mirroring. Notice that \( V_{\|} = 0 \) at the mirror point, so that the time from equation (IV.A.6) is

\[ t_2 = t_1 - \left| V_{\|} \right| C_1^{-1}. \] (IV.A.10)

And the position is determined by equation

\[ E_k = \mu B_2 = \mu (B_{\text{min}} + kx_2), \] (IV.A.11)

from which we have

\[ x_2 = \frac{E_k}{k\mu} - \frac{B_{\text{min}}}{k}. \] (IV.A.12)
The particle gains no energy. The parameters \( (V_0) = 0, t_2 \) and \( x_2 \) are the inputs for the positive tracing branch.

3. If neither (IV.A.9) nor (IV.A.10) has a solution for \( t_z \) inside \( [0, t_1] \), then we can map the values at \( t_1 \) directly to \( t_0 = 0 \). The velocity is given by

\[
|V_0| = |V_1| - \frac{\mu k t_1}{m}.
\]  

(IV.A.13)

The position is given by

\[
|V_0|^2 = \frac{2E_k - 2\mu B_0}{m} = \frac{2E_k - 2\mu B_{\text{min}} - 2\mu k x_0}{m}
\]

\[
x_0 = \frac{2E_k - 2\mu B_{\text{min}} - m(V_0)^2}{2\mu k}
\]  

(IV.A.14)

Comments:

- The first case, which calculates the reflection time, should be calculated first, before the mirror case, because otherwise the calculation of the mirror case will include some situations where the particle mirrors beyond \( X_{\text{max}} \) at some time \( 0 < t_z < t_1 \).

- We need the position, because it provides mapping information that is required for calculation of the perpendicular velocity. We know that the total kinetic energy may not be constant for a particle, if that particle gains energy from the piston. So we always track both the parallel velocity and the position of the particle.
• For the first two cases, the tracing is not completed, and we still need to keep tracing
  the particle's motion back in time; however, the results obtained in the first two
  cases are the inputs for the additional tracing required to get back to the initial
  condition.

• Also for the first two cases, the traced-back parallel velocity has a smaller absolute
  value than had at \( t_i \), because \( B \) is weaker at \( t_i \).

Now consider the case where the parallel velocity is positive, i.e., the particle is
moving toward the cap. There are only two possible pasts for this kind of particle, with or
without a collision with the piston.

1. First we calculate the collision time for particles with positive \( V_\parallel \). Taking the
   positive \( V_\parallel \) branch in equation (IV.A.6) (that is, the '−' sign), assume the collision
   happens at some time \( 0 < t_2 < t_1 \); we have

   \[
   y_2^{1/2} - y_1^{1/2} = |V_2| - |V_1| = C_1(t_1 - t_2); \tag{IV.A.15}
   \]

   notice that we have omitted the '∥' sign in the velocity. Substituting (IV.A.4) into
   (IV.A.15) we have

   \[
   \left( \frac{2E_k - 2\mu R_2}{m} \right)^{1/2} = |V_1| + C_1(t_1 - t_2). \tag{IV.A.16}
   \]

   If we further define
\[ Z_1 = |V_1| + C_1 t_1. \]  \hspace{0.5cm} \text{(IV.A.17)}

substitute the expression for the magnetic field (IV.A.1) into (IV.A.16), and set the piston position \( x_2 = V_0 t_2 \), we get a quadratic equation for \( t_2 \):

\[ (C_1 t_2)^2 + 2C_1 (V_0 - Z_1) t_2 + Z_1^2 - \frac{2E_k - 2\mu B_{\text{min}}}{m} = 0. \] \hspace{0.5cm} \text{(IV.A.18)}

The roots are

\[ t_2 = \frac{-(V_0 - Z_1) \pm \sqrt{V_0^2 - 2Z_1 V_0 + \frac{2E_k - 2\mu B_{\text{min}}}{m}}}{C_1}. \] \hspace{0.5cm} \text{(IV.A.19a)}

The root with '+' sign corresponds to the collision in the future when piston catches up with the particle and the root with '-' sign corresponds to the collision in the past. Only past collisions are relevant here, so the collision time is given by

\[ t_2 = \frac{-(V_0 - Z_1) - \sqrt{V_0^2 - 2Z_1 V_0 + \frac{2E_k - 2\mu B_{\text{min}}}{m}}}{C_1}. \] \hspace{0.5cm} \text{(IV.A.19b)}

The velocity just after the collision is given by

\[ |V_2| = C_1 (t_1 - t_2) + |V_1|. \] \hspace{0.5cm} \text{(IV.A.20)}
and the velocity just before collision is given by

$$(V_2)_{\text{before}} = 2V_0 - |V_2|.$$  \hspace{1cm} (IV.A.21)

The position is given by

$$x_2 = \frac{2E_k - 2\mu B_{\text{pin}} - mV_2^2}{2\mu k}.$$  \hspace{1cm} (IV.A.22)

What to do next depends on whether $(V_2)_{\text{before}}$ is positive or negative. If it is positive then we need to use the positive tracing branch again, but if it is negative, we need to switch to the negative tracing branch to continue the trace.

2. If we could not find a positive $t_2$, which means there has been no collision between the particle and the piston in the past, we can directly map the parallel velocity back to time $t=0$. The mapping is just equation (IV.A.17), and $Z_t$ is the velocity at time $t=0$.

3. Fluid theory

As before, we represent the gas inside the pipe by lots of mass points (100 for the reconnection case and 200 for the depletion case). All the mass points have the same mass. Since the magnetic field is now time-independent, or, equivalently, the pipe is rigid, equation (II.B.3), which is the momentum equation in the perpendicular direction,
and (II.B.4), which is the force balance with the background medium, are no longer applicable. For the test, we further ignore the friction force in the parallel momentum equation, obtaining

$$
\mathbf{b} \cdot \left( \rho \frac{d\mathbf{u}}{dt} \right) = \mathbf{b} \cdot \left\{ -\nabla \cdot \left[ P_\perp \mathbf{1} + (P_\parallel - P_\perp) \mathbf{b}\mathbf{b} \right] - \nabla \left( \frac{B^2}{2\mu_0} \right) + \frac{(\mathbf{B} \cdot \nabla)\mathbf{B}}{\mu_0} \right\}
$$

(IV.A.23)

Notice that the last two terms on the right side cancel each other (because $\mathbf{b} \cdot \mathbf{J} \times \mathbf{B} = 0$).

Equation (IV.A.23) simplifies to

$$
\rho \frac{du_\parallel}{dt} = -\frac{\partial P_\parallel}{\partial x} + \frac{(P_\parallel - P_\perp)}{B} \frac{\partial B}{\partial x} = -\frac{\partial P_\parallel}{\partial x} + \frac{(P_\parallel - P_\perp)}{B} \frac{\partial B}{\partial x}
$$

(IV.A.24)

This equation together with (II.B.6), (II.B.17), and the original second CGL equation

$$
\left( \frac{P_\parallel B^2}{\rho^3} \right)_{j=1_{1}} = \left( \frac{P_\parallel B^2}{\rho^3} \right)_{j=0}
$$

(IV.A.25)

determines the parallel motion. The speed of the piston is constant during the simulation.

The rightmost mass point is fixed.

4. Simulation results
Figures 43-46 show the results for the first configuration designed to represent the reconnection case from the kinetic and fluid calculations. Figures 47-50 show the results for the configuration representing the depletion case. For both cases, we have shown the evolution of the following parameters: $V_\parallel$, the average velocity in parallel direction; $P_\perp$, the pressure component in the plane perpendicular to the magnetic field; $P_\parallel$, the pressure component parallel to magnetic field; and $C_f$, the firehose factor.

The speed of the piston is set to 500km/s for the reconnection configuration and 600km/s for the depletion configuration, which are the typical velocities at the time the filaments became firehose unstable (Chapter III). For the reconnection-like configuration $B_{\text{max}}$ is 20.5 nT and the slope $k$, which is defined in equation (IV.A.1), is 1.2nT/$R_E$. For the depletion-like configuration $B_{\text{min}}$ is 25.2 nT $k$ is 0.96nT/$R_E$. The initial pressures and densities are taken from Chapter III.
Results for the configuration representing the reconnection case

Figure 43: $V_\parallel$ vs. $x$, for the fluid-vs.-kinetic-theory test, with reconnection parameters. The time interval between each panel is 20 s and the times are, from the top, 20, 40, 60, 80, 100, and 120 s. The dots represent the results from the kinetic calculation; the solid lines represent the results from the fluid calculation.
Figure 44: $P_{\perp}$ vs. $x$, for the fluid-vs.-kinetic-theory test, with reconnection parameters. The times and formats of the panels are the same as in Figure 43.

Figure 45: $P_{\parallel}$ vs. $x$, for the fluid-vs.-kinetic-theory test, with reconnection parameters. The times and formats of the panels are the same as in Figure 43.
Figure 46: Firehose factor $C_f$ vs. $x$, for the fluid-vs.-kinetic-theory test, with reconnection parameters. Negative $C_f$ means that the firehose criterion is satisfied. The time interval between panels is 10 s; the top panel in part (a) shows the results at $t=10$ s, and the top panel in part (b) shows the results at $t=70$ s.
Results for the configuration representing the depletion case

Figure 47: $V_\parallel$ vs. $x$, for the fluid-vs.-kinetic-theory test, with depletion parameters. The times, from the top, are 40, 80, 120, 160, and 200 s. As before the dots represent the results from the kinetic calculations and the solid line represent the results of fluid calculations.
Figure 48: $P_\perp$ vs. $x$, for the fluid-vs.-kinetic-theory test, with depletion parameters. The times and formats of the panels are defined in Figure 47.

Figure 49: $P_\parallel$ vs. $x$, for the fluid-vs.-kinetic-theory test, with depletion parameters. The times and formats of the panels are defined in Figure 47.
Figure 50: Firehose factor $C_f$ vs. $x$, for the fluid-vs.-kinetic-theory test, with depletion parameters. Negative $C_f$ means that the firehose criterion is satisfied. The time interval between those panels is 20 s for both part (a) and (b). The top panel in (a) shows the results at $t=20$ s and the top panel in (b) shows the results at $t=100$ s.
5. Discussion

For those four plasma parameters shown in Figures 43–50, the fluid results presented above are generally similar to the filament reconnection and depletion simulation results from Chapter III, which lends confidence that the results of the second gedanken experiment provide a useful estimate of the accuracy of the fluid-model filament simulations. As we haven’t allowed the pipe (magnetic field lines) to bend in the gedanken experiment, firehose instability does not develop even though the firehose instability criterion is satisfied. Because of that, we see that the slow shock reflected from the cap is able to make its way back to the piston and get reflected again, generating an even higher $P_\parallel$ downstream from the shock, a region with $C_\parallel$ far below zero in both test cases.

The kinetic results, as expected, do not exhibit a sharp jump in those plasma parameters like ones that occur in the fluid simulations. In another words, we don’t get shock waves. Instead, the parameters change gradually on a much longer length scale. The thickness of the structures in the kinetic simulations increases with time, eventually becoming as large as the simulation region. Nevertheless, for the time period we studied, the kinetic calculation agrees with fluid calculation in the following respects:

1. For both reconnection and depletion cases, the characteristic flow velocity is almost the same in the kinetic and fluid calculations. For example, in the reconnection configuration, from the evolution of $P_\parallel$, we see that the region with higher $P_\parallel$ is expanding at a speed about $15 R_\parallel$ per minute in both kinetic and fluid calculations. But we should also notice that in the kinetic case, particles with high $v_\parallel$ travel faster
than the majority of particles accelerated by the piston and smooth out the parameter profile.

2. \( P_\parallel \) exhibits approximately the same behavior in the kinetic and fluid calculations except there is no shock wave in the kinetic calculation. The magnitudes of \( P_\parallel \) are in rough agreement, though the fluid calculation predicts a slightly larger \( P_\parallel \) downstream of the slow shock after it has reflected from the cap, the \( P_\parallel \) computed from kinetic theory eventually catches up.

3. For \( P_\perp \), things are more complicated. The fluid and kinetic roughly agree before the fluid-version slow shock reaches the cap, but after that the shapes of the curves differ considerably. But we should also notice that high accuracy in \( P_\perp \) is not crucial to the firehose calculation, since the value of \( P_\perp \) is much smaller than the value of \( P_\parallel \) in both calculations.

4. For the firehose factor \( C_f \), we notice that before the slow shock reflects from the cap and reaches the piston again, the results of kinetic and fluid calculations agree very well except that the region near the piston becomes unstable first in the kinetic calculation; this is clearer in the reconnection case.

As the firehose instability is the main concern of the paper, we want to make sure that double-adiabatic MHD gives roughly the right answer about when and where the firehose instability will occur. A detailed comparison between the fluid and kinetic calculation of the firehose factor \( C_f \) is given in Figure 51 for the configuration representing the reconnection case, near the time when the filament first became firehose unstable in the kinetic calculation.
Firehose instability occurs earlier in the fluid simulation, beginning fairly near the end cap (earthward boundary). The instability starts ~ 15 s later in the kinetic calculation but then spreads in a few seconds to engulf nearly the entire filament.

Now consider the question of where firehose instability occurs. In the reconnection configuration we see that after the firehose instability is well developed, the unstable region given by the kinetic calculation covers the unstable region given by the fluid calculation. In the depletion case, the agreement is not as good as in the reconnection case, but still the unstable region given by the kinetic calculation roughly covers the unstable region in fluid calculation (the last panel in Figure 50b). We should also notice that there is a major difference between fluid and kinetic calculations: in the kinetic calculation, the first and most unstable region is right next to the piston (equatorial region of plasma sheet); in the fluid calculation, it is usually just downstream of the slow shock. In both cases, the instability eventually engulfs nearly the entire filament.

Concerning the magnitude of $C_f$, which indicates the growth rate of the firehose instability, the fluid method agrees with the kinetic method in the downstream regions. In the region upstream of the shock, there is a significant difference, regardless of whether the shock has been reflected from the cap. In the upstream region, the $C_f$ given by the fluid method tends to be larger (more stable) than its counterpart in the kinetic calculation. That is because, in the kinetic calculation, particles with high $v_\parallel$ could reach the upstream region and build up $P_\parallel$ there, while in the fluid calculation the information travels at the speed of shock, and $P_\parallel$ is built up only after the shock wave passes. Downstream of the reflected shock, after the overshoot has passed, the curve of $C_f$ given
by the fluid method tends to lie slightly below the kinetic one at beginning; then after
some time, it gradually rises above the kinetic curve.

So the overall agreement between the kinetic and fluid calculations is reasonably
good when we focus on the firehose factor $C_f$. The basic conclusion about the tendency
toward firehose instability remains valid in the kinetic treatment, which predicts firehose-
instability criterion about as strongly as fluid treatment. The most serious problem with
the fluid calculation, according to this test, is that it artificially delays the occurrence of
firehose near the equatorial plane.

To get a better understanding of the difference between the fluid and the kinetic
calculation, we show the normalized distribution function given by the kinetic
calculation. ("Normalized" means that the velocity-space integral of the distribution
function is unity.) Figure 52 shows the distribution at three different positions at time
t=40 s, when the piston is at $x = 8.14 \, R_E$. Figure 52a shows the distribution function very
close to the piston at $x = 8.2 \, R_E$; and we see a clear double-peaked distribution with the
second peak at $v_\parallel = 1000 \text{km/s}$. This corresponds to the fact that $P_{||}$ has increased near the
piston in the top panel of Figure 45. When we look at (b), which is further out at $x = 14
R_E$, we see a very thin second peak, consisting of only a few particles that have high
enough $v_\parallel$ to reach $14 \, R_E$ that have been accelerated by the piston in the last 40 s.
Figure 51: the firehose factor $C_f$ vs. position for the reconnection case. The firehose criterion is satisfied when $C_f < 0$. This Figure shows the detailed evolution of firehose parameter $C_f$ around the time the filament first became firehose unstable. The dotted lines represent the kinetic results and the solid lines represent the fluid results. The time period is from 94 to 108 seconds, the time interval between two successive panels is 2 s.
Figure 52: Normalized distribution functions in $v_l$-$v_{\perp}$ plane at time $t = 40$ s, for the reconnection case. Part (a) shows the distribution just ahead the piston at position $x = 8 \ R_\odot$; (b) shows the distribution at position $x = 14 \ R_\odot$; (c) shows the distribution at position $x = 24.5 \ R_\odot$ near the cap.
It is not surprising that the near-cap distribution function shown in Figure 52c is just an isotropic one except for a very weak peak at 3000 km/s, consisting of a few particles with large initial $v_\parallel$ that have hit the piston and reached the cap. At time $t = 40$ s, there is no place that is firehose unstable.

Figure 53 shows the distribution function at time $t = 80$ s for three different positions. The distribution shown in Figure 53a, like the one shown in Figure 52a, is very close to the piston, which is at $x = 11.27 \, R_e$. It looks almost the same as Figure 52a, except for the feather-like feature between those two peaks, which represents the particles that are in resonance with the piston (discussed below); and two very thin peaks at $v_\parallel = -2700$ km/s and $v_\parallel = 3700$ km/s. The peak at $v_\parallel = -2700$ km/s corresponds to the particles which have traveled about $33.5 \, R_e$ in the past 80 seconds; they were accelerated by the piston near time $t = 0$, traveled $20 \, R_e$ and were reflected from the cap, then traveled $13.5 \, R_e$ to reach the current position. The peak at $v_\parallel = 3700$ km/s corresponds to the same kind of particles, except that these have hit the piston again and gained another 1000 km/s from the collision. Figure 53b just shows three peaks: one is the isotropic background distribution; one represents the particles accelerated by the piston, and the third one represents those particles with high initial $v_\parallel$ that were reflected from the cap. Part (c) shows many fast (i.e., large initial $v_\parallel$) particles have reached the cap, but not enough to make the plasma there firehose unstable (Figure 45).
Figure 53: Normalized distribution functions in $v_y$-$v_\perp$ plane at time $t = 80$ s, for the reconnection case. Part (a) shows the distribution just ahead the piston at position $x = 11.5 \, R_E$; (b) shows the distribution at position $x = 19 \, R_E$; (c) shows the distribution at position $x = 24.5 \, R_E$ near the cap.
From Figure 46, we know there is no firehose instability in the kinetic calculation at time $t = 80$ s. From Figure 53, we see that the particles accelerated by the piston at some earlier time are observed all along the pipe, but their densities are not big enough to make the plasma firehose unstable.

Figure 53a shows an interesting small population of high-$V_\perp$ particles; in $V_\parallel$, it is centered at $V_0$ (=500 km/s), the speed of the piston, between the two major peaks. This population consists of particles in resonance with the piston, as we mentioned before. Consider a particle near the piston that has $v_\parallel$ just a little bit smaller than $V_0$, then the particle will gain a speed of $2(V_0 - v_\parallel)$ colliding with the piston. As $v_\parallel$ is very close to $V_0$, the increase in speed $2(V_0 - v_\parallel)$ is also very small. As the particle is going toward a region with stronger magnetic field, it will feel a mirror force, which decelerates it. If the magnetic moment of the particle is big enough, soon the particle’s velocity will drop below $V_0$, and then another collision will occur. During this process, which may be repeated many times, the piston keeps adding energy to the perpendicular motion of the particle. This motion pattern will repeat again and again. And the results of the process are shown in the Figure 53a: a symmetric population centered at $V_0$ with high average thermal energy in the plane perpendicular to the magnetic field.
Figure 54: Normalized distribution functions in $v_{||}-v_{\perp}$ plane at time $t = 110$ s, for the reconnection case. Part (a) shows the distribution just ahead the piston at position $x = 14 \, R_\odot$; (b) shows the distribution at position $x = 20 \, R_\odot$; (c) shows the distribution at position $x = 24.5 \, R_\odot$ near the cap.
From Figure 46 we know at time $t = 110$ s, most of the filament satisfies the firehose instability criterion in both the kinetic and fluid calculations. Figure 54 shows the distribution function at this time. Part (a) shows the distribution at $x = 14 \, R_E$, just ahead the piston, which is at $13.62 \, R_E$; the distribution has four strong and two very weak peaks. Part (b) shows the distribution at $x = 20 \, R_E$, with three strong and two weak peaks. Part (c) shows the distribution at $x = 24.5 \, R_E$, which also consists of three strong and two weak peaks. Clearly at time $t = 110$ s, the beams are strong enough to produce a $P_\perp$ much bigger than the $P_\parallel$. Clearly for Figures 54a and 54b, the two weak peaks contribute little to $P_\parallel$; $P_\parallel$ is dominated by the beam that has just been accelerated by the piston and the reflection of that beam from the cap. In Figure 54a, the beams that have hit the piston twice are not negligible as they were in 54b and 54c.

To understand better which beam is the "the last straw" that leads to instability, we further study the evolution of distributions just ahead of the piston and also at a fixed position far away from the piston. Figure 55 shows the results for the time period during which the filament first became firehose unstable near the piston in the kinetic calculation (Figure 46). Clearly there is little change for the two peaks centered at $v_{\parallel} = 0$ and $v_{\parallel} = 1000$ km/s. But we know that the plasma becomes firehose unstable from Figure 55a to 55c. As the magnitude of the magnetic field is increasing Figure 55a to 55c, the only reason that plasma there becomes firehose unstable is the strengthening of the two peaks centered at $v_{\parallel} = -2000$ km/s and $v_{\parallel} = 3000$ km/s. The peak at $v_{\parallel} = -2000$ km/s corresponds to the particles with large initial $v_{\parallel}$ that have been accelerated by the piston once and also reflected from the cap. The peak at 3000 km/s consists of particles that have hit the piston a second time.
Figure 55: Normalized distribution functions in $v_\parallel$-$v_\perp$ plane at a position just ahead the piston ($0.2 \, R_E$). The positions are, from the top, 12.25, 12.88 and 13.5 $R_E$. The times are, from the top, 90, 98, and 106 seconds. And the corresponding firehose factors $C_f$ are: 0.1143, −0.1945, and −0.5539.
Figure 56: Normalized distribution functions at $x = 20 R_E$. The times, from the top, are 90, 98, and 106 seconds. The corresponding firehose factors $C_f$ are: 0.1346, $-0.0279$, and $-0.2433$. 

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Figure 56 shows the distribution functions at fixed position $x = 20 \, R_g \ (5 \, R_g$ from the end cap) for the time when the filament first became firehose unstable in the kinetic calculation. From Figure 56, it is very clear that particles with $|v_\parallel| > 3000 \, \text{km/s}$ contribute only a small fraction to $P_\parallel$. Clearly for larger $x$, those peaks centered at high $v_\parallel$ will become even weaker: thus we know that, except for the region near the piston, for most regions the beams that consist of particles that have hit the piston more than once (corresponding to particles that have experienced the equatorial plane more than once) do not play an essential role in the firehose instability. This is very important for physical consistency with the fluid calculation, because in that calculation the firehose instability happens before the shock could reach the piston and get reflected again.
B. Sensitivity Studies

As we pointed out earlier, the main difference between our double-adiabatic simulation and the isotropic simulation in C&W99 is the firehose instability. The discussion in the above subsection confirms the ability of our double-adiabatic-MHD model to calculate pressure anisotropy and thus firehose instability. We pointed out in Chapter III that the firehose instability could help us understand some phenomena that are observed near $x = 10 R_E$ during a substorm expansion phase. It is important for us to know the effects of different input parameters on our model results, especially with regard to when and where the firehose instability will occur. In this subsection we focus on the sensitivity of the firehose instability to the values of free parameters in the simulation. Through this search, we hope to determine which input parameters are most important in governing firehose instability.

For the reconnection and depletion cases, we have the following free parameters:

1. $F_o$, which determines the strength of friction force; see equation (14.a) of C&W99.
2. $D$, which determines the strength of heat conduction in parallel direction (equation II.B.13).
3. $f_h$, which determines the strength of viscosity due to gradients along the magnetic field (equation II.B.2).
4. $T$, the background temperature.
5. $\alpha$, which determines the shape of background flux tubes (equation III.A.2).
6. $X_0$, the equatorial crossing distance; in the reconnection case, it is the crossing distance of the field line before reconnection occurs.
7. *percent*, which for depletion case, determines the degree of depletion of the initial flux tube.

We will base our sensitivity study on the nominal reconnection and depletion cases discussed in Chapter III, varying one parameter at a time. We will first present a very brief discussion of the effects of the transport coefficients (parameters 1-3); the results are, on the whole, less sensitive to those parameters than to the MHD parameters 4-7, which will be discussed in more detail.

For simplicity and to save time, we have reduced the spatial resolution for these sensitivity studies, as compared to the nominal cases discussed in Chapter III; we use 300 mass points in the depletion test case (instead of 500). For the reconnection case we still use 250 mass points, the same as the simulation in the Chapter III. The time resolution of stored data, unless otherwise indicated, is 2 seconds though the actual time step may be much smaller. The following discussion, otherwise specifically indicated, is for both the reconnection and the depletion configurations.

Before we go to the detailed discussion of those parameters, we should point out that there actually is another important parameter in the numerical simulation: the number of mass points $n$. As we mentioned in Chapter II, the linear firehose growth rate is proportional to the wave number, and the number of mass points places an upper limit on the wave number. The general trend for different values of $n$ is the following: the bigger the $n$ is, the more sensitive the filament to those negative firehose factor $C_f$, and more quickly the simulation reaches the lower limit of time step. Despite all that, the position and time of firehose instability's occurrence with more mass points are almost identical.
to the simulation results carried out with fewer mass points. The results of runs with different number of mass points are shown in Figure 57.

![Graphs showing time and filament properties vs. number of mass points](image)

Figure 57: Dependence of firehose-instability onset on the number of mass points. Part (a) shows the results of the reconnection case. The first panel shows the times when the filaments first become firehose unstable; the second shows the position of the crossing point of the filament on the equatorial plane (circles) and the position of the firehose unstable region (triangles) at the times shown in the first panel. Part (b) shows the results of depletion case.

We have introduced three transport parameters to describe the non-MHD processes during the evolution of the thin filament. The friction coefficient \( F_c \), which is defined in
equation (14.a) of C&W99, is a simple idealization to represent the interaction between the filament and the background medium. The heat conduction coefficient $D$ is to represent the heat conduction in the collisionless plasma. The coefficient of the viscosity in the parallel direction $f_p$, defined in (II.B.2), is to represent the strength of viscosity due to gradients along the magnetic field.

We add transport coefficients to our MHD calculations for two reasons: they represent crude approximations to non-ideal-MHD processes that occur in the nearly collisionless plasma sheet, and they help suppress numerical instabilities. Because the equation set we used to do the simulation is highly non-linear, and because of numerical errors introduced by the Trapezoidal Leapfrog and Central Difference schemes, the code tends to be very unstable especially in the region downstream of the slow shock. In runs without transport terms, results typically show large overshoots just downstream of the shock. Fortunately the transport terms described above tend to stabilize the code and greatly reduce those overshoots.

As before, though the friction term (defined in equation (II.B.2)) helps to dissipate the filament’s kinetic energy into the background medium, its effects on stabilization of the code are limited; it helps to damp the overshoot a little bit, but not much. Recent 3D MHD simulation [Birn et al., 2003] shows that in the 3D environment, the bubble-like structure has to push background flux tubes out of the way. Comparison of the calculations of Birn et al. [2003] to filament calculations [C. X. Chen, private communication, 2003] shows that the force exerted on the narrow bubble by the background flux tubes is about the same as the friction force in the C&W99 simulation. So in our parameter searching, we just test several values near the default value of $F_c$.
(0.06) used in C&W99. For those different values, with all other parameters fixed, the simulation results look almost identical, with a small difference in the evolution speeds: the larger the $F_c$ is, the slower the evolution is. The value of $F_c$ only affects the occurrence of firehose instability slightly. The results are shown in Figure 58.

![Figure 58](image)

Figure 58: Dependence of firehose instability onset on the friction coefficient $F_c$, defined in equation (14.a) of C&W99. For all the panels the $x$-axis is the value of $F_c$. Part (a) shows the results for the reconnection case. The first panel shows the times when the filaments first become firehose unstable; the second shows the position of the crossing point of the filament on the equatorial plane (circles) and the position of the firehose unstable region (triangles) at the times shown in the first panel. Part (b) shows the results for the depletion case.
The heat conduction term plays a crucial role in shock waves, which typically form in our filament simulations. It is also more important for stabilizing the code than the friction term; it is the main factor that stabilizes the downstream region of the slow shock. The real mean free path along the magnetic field line is very large in the magnetotail, and our parameter search covered mean free paths \( \vec{\lambda}_q \) between 0.1 \( R_E \) and 2 \( R_E \).

Figure 59: Dependence of firehose onset on the mean free path \( \vec{\lambda}_q \) assumed in the calculation of heat conductance. The x-axis is \( \vec{\lambda}_q \) in unit \( R_E \). Part (a) shows the results of the reconnection case. The first panel shows the times when the filaments first become firehose unstable; the second shows the position of the crossing point of the filament on the equatorial plane (circles) and the position of the firehose unstable region (triangles) at the times shown in the first panel. Part (b) shows the results of depletion case.
The results are insensitive to the value of the heat diffusion coefficient, except for the details like the overshoots and the thickness of the slow shock. The results are shown in Figure 59. Larger $\lambda_{\parallel}$ (and thus larger $D = V_{\text{thermal}} \cdot \lambda_{\parallel}$) lead to smoother profiles of various parameters and thicker shocks; the occurrence of firehose does not seem to be affected by the assumed collisional mean free path $\lambda_{\parallel}$.

It is easy to understand why firehose instability is relatively insensitive to the heat conductivity. Firehose instability occurs when (II.D.1) holds, and that condition is determined by the thermal energy densities in the parallel and perpendicular directions and by the magnetic energy density. A larger heat-conduction coefficient for parallel motion can only smooth the thermal energy density profile in parallel motion, and it has no direct effect on either the thermal energy density in perpendicular motion or the magnetic energy density. Consequently, its effects on firehose instability are modest.

Now consider the effect of the viscosity on the parallel motion. Our numerical experiments indicate that if $f_h$ is bigger than some crucial value (which depends on the values of the other parameters, but in most cases is about 0.1), the simulation becomes unstable even when the filament is firehose stable. For values smaller than the threshold, the simulation results are insensitive to the value of $f_h$. As adding viscosity introduces numerical errors, and firehose instability is insensitive to this parameter, in most cases we just set the viscosity to zero.

Now consider the initial filament temperature $T$. We have carried out a series of filament simulations, all with the same initial pressure and magnetic field but different temperatures (and thus different densities). We have done tests for initial temperatures between 2 and 8 keV, covering the average temperatures that correspond to the range of
AE indices considered by Huang and Frank [1998]. The tests have been carried out for both the depletion and reconnection cases discussed in Chapter III.

Some results are shown in Figure 60. Though the time scale depends on initial temperature, other characteristics do not depend significantly on initial temperature. The reason can easily be seen. From equations (II.C.16), (II.C.17) and (II.C.20) we know that increasing (decreasing) the value of \( T \) will increase (decrease) both transverse and longitudinal wave speeds, which determine the evolution speed of the filament; however, it will not change the ratio of those two wave speeds, which determines the position of firehose instability’s occurrence. The times shown in the first panels of parts (a) and (b) of Figure 60 are approximately proportional to \( T^{-1/2} \), which supports this statement; also the third panel shows that the parallel and perpendicular pressures depend very little on \( T \), which suggests that the pressure increase in the compressional shock is approximately independent of \( T \). In fact, if we think of the process in term of energy densities, the effect becomes much clearer: the magnetic energy density and the thermal energy densities, which are proportional to \( P_\parallel \) and \( P_\perp \), are all independent of temperature \( T \). So the initial temperature \( T \) just scales the evolution time.

Notice that the positions shown in the Figure 60 are not exactly independent of \( T \). One reason is that we recorded results with a time resolution of 2 seconds, i.e., there is at least 2 seconds uncertainty in the time; the use of discrete mass points also contributes to the uncertainty, but has much smaller effects, as the distance between adjacent mass points is much smaller than the distance they travel in 2 s. This also an important factor in understanding why the positions shown in Figures 57–59 are not exactly independent of \( x \).
Figure 60: Dependence of firehose onset on initial temperature. Part (a) shows the results for the reconnection case. The first panel shows the times when the filaments first become firehose unstable; the second shows the position of the crossing point of the filament on the equatorial plane (circles) and the position of the firehose unstable region (triangles) at the times shown in the first panel; the third panel shows the pressure components at those times at the ionospheric boundary (circles for $P_{||}$ and the triangles for $P_{\perp}$). Part (b) shows the results for the depletion case.
Now consider the sensitivity of the firehose instability to the parameter $\alpha$, which determines the shape of the background flux tubes. Figure 61 shows the background magnetic field configurations for three different values of $\alpha$. From the figure we see that the smaller $\alpha$ is, the more stretched the magnetic field lines become. As in the reconnection case we assume the filament in consideration is generated by reconnection at $x=25\, R_E$, we want to keep the field line sufficiently stretched near $x=25\, R_E$ so that reconnection could take place there. In our study, the range of $\alpha$ is from $0.0104/R_E$ to $0.0624/R_E$, i.e., 0.5 to 3 times the $\alpha$ value in Chapter III.

![Graphs showing background magnetic field configurations](image)

Figure 61: Background magnetic field configurations for three different $\alpha$ values. The line spacing does not indicate field strength.
Figure 62: Dependence of various model results on the tail-field-stretch parameter $\alpha$. Part (a) shows the results of reconnection runs with different values of $\alpha$, and part (b) shows the results of depletion case. The first panel shows the times when the filament first became firehose unstable; the second panel shows the positions of the crossing point (circles) and the place that is unstable (triangles); the third panel shows the average speed of the crossing point; the fourth panel shows the ratio of the filament’s $pV^r$ after the reconnection/depletion to the value of undisturbed background flux tube; the fifth panel shows the initial intensities of magnetic field at the left boundary (circles) and the right boundary (triangles) in units of nT; the sixth panel shows the initial pressure inside the filament in direction parallel to the magnetic field.
Figure 62(a) displays the sensitivity of the firehose instability to the $\alpha$-parameter, for the reconnection case. The main conclusion is that the instability occurs in every case, and in the same general region (between 6 and $9 \, R_E$). However, the earthward motion of the filament is more violent if $\alpha$ is large, because the initial filament pressure is lower, it is initially further from equilibrium and interchange stability, and flow velocities are therefore higher.

The fourth panel of Figure 62a gives a quantitative indication of how far the initial filament is from interchange stability. It shows the initial ratio of $pV^{5/3}$ in the filament to the $pV^{5/3}$ of the background flux tube with the same equatorial crossing point. The lower that ratio, the further the flux tube has to move earthward to achieve interchange stability, and the faster it moves.

For the depletion case, we first notice that the firehose instability only occurs very near the equatorial plane for large $\alpha$. From the fourth panel we know that may come from the fact that for the large value of $\alpha$, the initial ratio of $pV^{5/3}$ in the filament to the $pV^{5/3}$ of the background flux tube with the same equatorial crossing point is getting bigger. The initial filaments with large value of $\alpha$ are less interchange unstable than those with smaller $\alpha$; thus they have a weaker slow shock.

Another major difference between the reconnection and the depletion cases is the initial intensities of magnetic field where the filament crosses the equatorial plane. In the depletion case, the magnetic field is very weak compared to its counterpart in the reconnection case. There is too much mass in the equatorial part of filament, the acceleration is small and the slow shock is weak early in the simulation.
Now consider the dependence of firehose instability on the parameter $X_0$, which is the initial distance of the equatorial crossing point for the filament. (For the reconnection case, it is the equatorial crossing distance before reconnection occurs.) Figure 63 shows the results for the reconnection and depletion cases with different value of $X_0$; all other parameters have the same value as in the simulations shown in Chapter III.
Figure 63: Simulation results for different values of $X_0$, the equatorial crossing point of the filament before depletion or reconnection. Part (a) shows the results for the reconnection configuration and part (b) shows the results for depletion. For both parts, the first panel shows the times when the filaments first become firehose unstable; the second panel shows the positions of crossing point on the equatorial plane (circles) and the positions of those firehose unstable places (triangles) at times given in the first panel; the third panel shows the values of $\beta_\parallel$ vs. the value of $X_0$ at those unstable positions; and the forth panel shows the average speed of the crossing point; the fifth panel shows the ratio of the value of $pV'$ of the filament after the reconnection/depletion to the value of undisturbed background flux tube.
For the reconnection case, we vary \( X_0 \) from 28 to 40 \( R_E \). We limit \( X_0 \) to a maximum of 40 \( R_E \) to make sure the undisturbed flux tubes (i.e. the flux tubes before reconnection) are not too far away from the neutral sheet at \( x=25 R_E \), where the reconnection takes place. For the depletion case, we vary the value of \( X_0 \) from 30 to 80 \( R_E \).

From Figure 63a we notice that for the reconnection case, the further \( X_0 \) is from 25 \( R_E \), the further the initial filament is from equilibrium, and the more violent the earthward motion is. More violent earthward motion tends to increase \( P_\parallel \) and lead to firehose instability closer to the left boundary. From the second and third panels in part (a) we see that when \( X_0 \) is very close to the reconnection place, the slow shock is so weak that the firehose hasn’t happened till the reflected slow shock reached a place where is the magnetic field is very weak; but for larger \( X_0 \), firehose instability occurred right after the slow shock reflected from the ionospheric boundary.

Finally let’s consider the effects of parameter \textit{percent}, which determines the degree of depletion to generate our filament out of a background flux tube. Figure 64 shows the results of depletion case with different degree of depletion. All the parameters have the same values as in the depletion case shown in Chapter III except the value of \textit{percent}. Figure 64 shows the results with \textit{percent} changes from 0.1 to 0.8.

As the initial pressure components and density inside the filament are set to \((1 - \textit{percent}) \cdot X_{med}\), where \( X_{med} \) is the parameter value of background medium, and the y-force balance is obtained by increasing the filament’s magnetic field. The larger the value \textit{percent} is, the smaller the initial filament density is, the stronger the initial magnetic field has to be, and thus the stronger magnetic tension force is. This explains why the evolution becomes faster when \textit{percent} becomes larger. If we take the interchange
instability point of view, then the bigger the \textit{percent} is, the smaller the $pV'$ is, compared to its neighbor (fifth panel), and the stronger the interchange effect is.

We notice that the larger the \textit{percent} value is (from 0.1 to 0.5) the further from the current sheet firehose first occurs. For simulations with a \textit{percent} greater than 0.6, the firehose occurs first at the left boundary. We can also see that the slow shock is stronger when \textit{percent} becomes larger.
Figure 64: Results for different degrees of depletion. (percent = 0 and 1 mean no depletion and full depletion, respectively.) The first panel shows the time when the filament first becomes firehose unstable vs. the degree of depletion. The second panel shows positions of the initial instability (triangles) and of the equatorial crossing point at that time (circles). The third panel shows the values of $P_{||}$ (circles) and $P_{\perp}$ (triangles) at those initial-instability positions; the fourth panel shows the average speed of the equatorial crossing point until the times shown in the first panel; the fifth panel shows the ratio of the filament’s initial depleted $pV^\gamma$ to the value for a neighboring background flux tube.
V. SUMMARY AND SUGGESTIONS FOR FUTURE WORK

A. Summary

Both bursty bulk flows and substorms are magnetotail phenomena that involve high flow speeds, and such flows are generally treated theoretically using ordinary MHD. But the validity of using ordinary MHD has been questioned, because the measured distribution functions in fast magnetotail flows often exhibit two or more interpenetrating beams moving at different velocities and don’t look anything like a Maxwellian. With our the first gedanken experiment (Chapter II), we showed that double-adiabatic MHD, which is a single-fluid MHD, can sometimes provide a remarkably accurate representation of a physical situation, even when the plasma distribution function is far from a classic single-fluid distribution function: a quasi-Maxwellian distribution centered about the mean velocity. Then we derived the equations of motion of a double-adiabatic MHD filament.

In Chapter III, we applied double-adiabatic MHD to two cases where the filament moves rapidly earthward. In one case, the filament was set up initially to represent a depleted bubble that extended to \( X = -40 R_E \) and in the other case to represent reconnection about 25 \( R_E \) behind the Earth, which is observed to occur in substorms. Both of these double-adiabatic simulations exhibited several of the same characteristics that were noted by C&W99 for their isotropic-MHD simulations of a plasma-sheet bubble, particularly the high earthward speed and the dipolarization of the filament; these are both characteristics of bursty bulk flows. Both isotropic- and double-adiabatic MHD simulations exhibit a compressional wave that propagates from tail to earthward.
boundary, reflects, and then propagates tailward. The tailward-propagating wave brakes the earthward motion of the filament.

But our double-adiabatic-MHD simulations also exhibited a dramatic feature that is not present in isotropic MHD: firehose instability. In our simulations firehose instability occurs after the compressional shock wave reflects from the earthward boundary. In Chapter III, we also offered a short theoretical discussion to show that the firehose instability seems inevitable for these fast flows.

The braking of the fast flows in the plasma sheet is thus characterized by strong magnetic fluctuations caused by the firehose instability. Within the context of the Near-Earth-Neutral-Line model of substorms, we thus suggest that firehose instability might cause the intense magnetic field fluctuations that are observed in the inner plasma sheet at substorm onset.

Chapter IV described additional simulations that were carried out to confirm the accuracy and robustness of our principal conclusion based on double-adiabatic simulations. A second gedanken experiment was carried out to further explore the accuracy of double-adiabatic MHD by comparing its predictions to the kinetic-theory solutions for idealized situations that are similar to those presented in Chapter III. These computer experiments confirmed the usefulness of double-adiabatic MHD: kinetic theory and double-adiabatic MHD agree well on the crucial prediction of the occurrence of firehose instability, though the results indicate that the fluid calculation may be incorrect on some details, such as the location where the instability occurs first. However, both kinetic and fluid calculations predict that the firehose instability quickly expands to cover a large fraction of the filament.
A sensitivity study described in Chapter IV shows that some details of the firehose instability are sensitive to the assumed initial condition and the background field configuration. However, the firehose instability occurred in all of the test cases, which covered a wide range of different parameter values. So our conclusion about the occurrence of firehose instability seems quite robust. Since firehose instability always occurred after the reflection of the compressional shock from the near Earth boundary, and as the reflected compressional shock brakes the earthward motion of the filament, we thus suggest that the death of a bursty bulk flow tends to produce firehose instability.

One obvious way to test our theoretical prediction that fast earthward flows generate firehose instability would be to search plasma-sheet observations for firehose-unstable pitch-angle distributions. The observational results obtained from Geotail by Richard Kaufmann do show some firehose unstable cases, but the unstable cases count amount to less than 5%, and their statistical significance has not been demonstrated. Also, we want to point out that for most cases, the firehose instability saturates in less than 20 s, which is less than the time it takes present instruments to complete one scan of the pitch-angle distribution. Furthermore, the small spatial scale of the ripples in magnetic field means the spacecraft is likely to move through several wavelengths in one scan, which makes the observational situation even more difficult. It will take more sophisticated measurements to test our theory.
B. Suggestions for Future Work

1. A full-particle or hybrid simulation could provide a way to test our theory in a formulation that entails fewer simplifying assumptions and approximations than our double-adiabatic MHD approach.

2. It would be useful to find a way to include, in the fluid simulations, an approximation to the effects of chaotic ion motion in the current sheet.

3. A better understanding of kinetic instabilities associated with the counter-streaming beams would be very helpful, because waves generated by those instabilities, may act to isotropize the plasma. Understanding these unstable waves and their effects on isotropy is very important.
APPENDIX

More general proof of equivalence

We now show that the kinetic solution satisfies the fluid-shock jump condition for \( \gamma = 3 \) for the \( n \)th bounce of shock off the piston. For that case, kinetic theory (equations (II.A.1) and (II.A.2)) gives

\[
\begin{pmatrix}
\rho_{\text{down}}^{pn} \\
\rho_{\text{up}}^{pn}
\end{pmatrix}_{\text{kinetic}} = \frac{2n}{2n-1} \\
\begin{pmatrix}
\tilde{V}_{\text{down}}^{pn} \\
\tilde{V}_{\text{up}}^{pn}
\end{pmatrix}_{\text{kinetic}} = \frac{2n-1}{2n} \\
\begin{pmatrix}
P_{\text{down}}^{pn} \\
P_{\text{up}}^{pn}
\end{pmatrix}_{\text{kinetic}} = \frac{2n+1}{2(n-1)}
\]

As in earlier sections, the symbol \( P \) represents the parallel component of the pressure tensor, unless otherwise indicated. To check against the fluid jump conditions, we need to derive an upstream Mach number from kinetic theory. After the leading edge has been reflected from the piston, the upstream fluid velocity in the lab frame is zero, and the leading edge propagates to the left. According to (II.A.1g) the lab frame velocity of the leading edge, after it bounced off the piston for the \( n \)th time, is \( 2nV_0 \). Using (II.A.2) and (II.A.1g), the Mach number is given by

\[
\left( \tilde{M}_{\text{up}}^{pn} \right)^2 = \frac{(2nV_0)^2 \cdot \rho_{\text{down}}^{c(n-1)}}{\gamma \rho_{\text{down}}^{c(n-1)}}
\]
using (II.A.2a) and (II.A.2c)

\[
\left( \tilde{M}_{\text{up}}^{\text{pn}} \right)^2 = \frac{(2nV_0)^2 (2n-1)\rho_0}{\frac{4}{3} n(n-1)(2n-1)\rho_0 V_0^2} = \frac{3n}{\gamma(n-1)},
\]

(A.2a)

or

\[
\left( \tilde{M}_{\text{up}}^{\text{pn}} \right)^2 = \begin{cases} 
\frac{n}{(n-1)} & \gamma = 3, \\
\frac{9n}{5(n-1)} & \gamma = 5/3.
\end{cases}
\]

(A.2b)

In fluid theory, the shock jump conditions can be derived from (II.A.3) and (A.2b),

\[
\begin{align*}
\left( \frac{\rho_{\text{down}}^{\text{pn}}}{\rho_{\text{up}}^{\text{pn}}} \right)_{\text{fluid}} &= \frac{\left( \tilde{M}_{\text{up}}^{\text{pn}} \right)^2 \gamma + 1}{2} = \begin{cases} 
\frac{2n}{2n-1} & \gamma = 3 \\
\frac{12n}{8n-5} & \gamma = 5/3
\end{cases} \\
\left( \frac{\tilde{v}_{\text{down}}^{\text{pn}}}{\tilde{v}_{\text{up}}^{\text{pn}}} \right)_{\text{fluid}} &= \frac{\gamma - 1}{\gamma + 1} + \frac{2}{\left( \tilde{M}_{\text{up}}^{\text{pn}} \right)^2 (\gamma + 1)} = \begin{cases} 
\frac{2n-1}{2n} & \gamma = 3 \\
\frac{8n-5}{12n} & \gamma = 5/3
\end{cases} \\
\left( \frac{p_{\text{down}}^{\text{pn}}}{p_{\text{up}}^{\text{pn}}} \right)_{\text{fluid}} &= \left( \tilde{M}_{\text{up}}^{\text{pn}} \right)^2 \frac{2\gamma}{\gamma + 1} \frac{\gamma - 1}{\gamma + 1} = \begin{cases} 
\frac{2n+1}{2(n-1)} & \gamma = 3 \\
\frac{8n+1}{4n-4} & \gamma = 5/3
\end{cases}
\end{align*}
\]

(A.3)

Comparing (A.1) and (A.3), it is clear that the kinetic theory solutions satisfy the shock jump conditions exactly for \( \gamma = 3 \).
Now we need to show that the lab frame velocities of the leading edge in kinetic theory agree with the fluid shock velocities for $\gamma=3$. As we assume that the gas ahead of the shock front is at rest in the lab frame for this case, we have in the lab frame, for the fluid equation (II.A.3),

$$
\left( \frac{\tilde{V}_{dn}}{\tilde{V}_{up}} \right)_{\text{fluid}} = \frac{V_{shock}^{pn}}{V_{shock}^{up}} = \frac{\gamma-1}{\gamma+1} + \frac{2}{\left( M_{up}^{pn} \right)^2 (\gamma+1)}.
$$

(A.4a)

Substituting (A.2b) in (A.4a) gives

$$
\left( \frac{\tilde{V}_{dn}}{\tilde{V}_{up}} \right)_{\text{fluid}} = \begin{cases} 
\left( \frac{2n-1}{2n} \right)_{\gamma=3} \\
\left( \frac{8n-5}{12n} \right)_{\gamma=5/3}
\end{cases}.
$$

(A.4b)

Thus we get

$$
V_{shock}^{pn} = 2nV_0 \quad \text{for} \quad \gamma = 3
$$

(A.5a)

$$
V_{shock}^{pn} = \frac{12nV_0}{4n+5} \quad \text{for} \quad \gamma = 5/3.
$$

(A.5b)

Equation (A.5a) agrees with equation (II.A.1g), which implies that, for the simple pipe problem, the velocity of the shock wave calculated from fluid theory for $\gamma=3$ agrees exactly with the velocity of the leading edge of particles computed from kinetic theory. We have demonstrated the result explicitly only for the times when the shock wave and
leading edge are traveling to the left, from the piston toward the cap. A similar proof can easily be constructed for times when the leading edge and shock are traveling to the right, but in those cases \( (\tilde{M}_{up}^{pn})^2 = (2n+1)/(2n-1) \) if \( \gamma = 3 \) and \( (\tilde{M}_{up}^{pn})^2 = 9(2n+1)/[5(2n-1)] \) if \( \gamma = 5/3 \). Note that all of the Mach numbers exceed one.

This completes the proof that, for our idealized pipe problem, the fluid theory for \( \gamma = 3 \) gives densities, parallel pressures and velocities that are in exact agreement with the values computed from kinetic theory. Of course, since the perpendicular motions are decoupled, the perpendicular pressures are simply proportional to density in both cases, so they also agree. If one defines effective temperatures as ratio of pressure to density, the parallel and perpendicular temperatures also agree exactly.
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