RICE UNIVERSITY

Harmonic Maps and the Geometry of Teichmüller Space

by

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A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

Doctor of Philosophy

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July, 2003
Abstract

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In this thesis work, we investigate the asymptotic behavior of the sectional curvatures of the Weil-Petersson metric on Teichmüller space. It is known that the sectional curvatures are negative. Our method is to investigate harmonic maps from a nearly noded surface to nearby hyperbolic structures, hence to study the Hopf differentials associated to harmonic maps and the analytic formulas resulting from the harmonicity of the maps.

Besides providing a quantitative result, our estimates imply that even though the sectional curvatures are negative, they are not staying away from zero. In other words, we show that when the complex dimension of Teichmüller space $\mathcal{T}$ is greater than one, then there is no negative upper bound for the sectional curvature of the Weil-Petersson metric. During the proof, we also give the explicit description of a family of tangent planes which are asymptotically flat.
Acknowledgments

I would like to express my deepest gratitude to my thesis advisor, Michael Wolf, for suggesting this problem, many stimulating conversations and for his incredible patience. Mike, you are the greatest, as an advisor and as a dear friend.

I am grateful for enormous help and encouragement from many professors here at Rice, especially Dr. Tim Cochran, Dr. Robin Forman, Dr. Bob Hardt, Dr. John Hemple, Dr. Frank Jones, and Dr. Bill Veech. Thanks also go to Dr. Bob Hardt and Dr. Steve Cox who kindly served in my thesis committee.

I would like to thank my fellow graduate students, especially Aaron Heap, Robert Huff, Connie Leidy, Tantiana Marinenko, Matt McLelland and Jimmy Peterson. You guys are great, my life won’t be the same without everyone of you, I love you all!

Special thanks to Dr. Adria Baker, Ms. Somarine Diep and Ms. Lily Lam in OISS, to Math Moms Marie Magee and Maxine Turner, for their heartfelt kindness; to Niki Serakiotou, who kindly fixed every problem after I abused my computer.

My work was kindly supported by a Rice Graduate Fellowship (1998-2002) and a Nettie S. Autrey fellowship (2002-2003). It was also partially supported by NSF grants 9971563 and 0139887.

Lastly, I would like to acknowledge everyone in my family. I cannot thank my parents enough for their support throughout my entire life. Also I thank my dear brother who takes care of my parents on the other side of the Earth.
I dedicate my thesis to Zhen.

Without her, I might be able to graduate sooner,

but definitely much less happier.
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Chapter 1

Introduction

1.1 Statement of the Problem and Main Results

Let $\Sigma$ be a smooth, closed Riemann surface of genus $g$, with $n$ punctures and $3g - 3 + n > 1$. The Riemann Moduli problem is to describe the isomorphism classes of all Riemann surfaces. Let $M_g$ be Riemann moduli space of genus $g$, that is, the set of all conformal equivalence classes of closed Riemann surfaces of genus $g$. Since every closed Riemann surface of genus zero is conformal to the Riemann sphere, hence $M_0$ consists of one single point. It is not hard to show that $M_1$ is identified with the complex plane.

This moduli problem becomes more interesting for the more complicated Riemann surfaces. By the Uniformization Theorem of Klein, Poincare and Koebe, these Riemann surfaces are covered by the upper half-plane, hence are hyperbolic. While studying this moduli problem, Teichmüller proposed a modification of the moduli problem that gives rise to now so-called Teichmüller space.
Let $\Pi_{-1}$ be the space of metrics of constant curvature $-1$ on the Riemann surface $\Sigma$. The group, $Diff_0(\Sigma)$, of diffeomorphisms homotopic to the identity acts by pull back on $\Pi_{-1}$.

**Definition 1.1.** Teichmüller space $\mathcal{T}_{g,n}$ is the quotient space $\Pi_{-1}/Diff_0(\Sigma)$.

Note that the constant $3g - 3 + n$ is fundamental in Teichmüller theory: it is the complex dimension of Teichmüller space $\mathcal{T}_{g,n}$, or more topologically, the number of curves in any pair-of-pants decomposition of $\Sigma$.

The Weil-Petersson metric on Teichmüller space has been heavily studied. It is defined as a cocometric by the $L^2$ inner product on the space of holomorphic quadratic differentials, identified as the cotangent space of Teichmüller space. Equipped with this metric, Teichmüller space has many curious properties: for example, in 1960s, Ahlfors ([2]) showed that Teichmüller space is a Kählerian manifold; Teichmüller space is geodesically incomplete, showed by Chu ([7]) and Wolpert ([29]) in 1970s; Tromba ([24]) and Wolpert ([30]) showed in 1980s that Teichmüller space has negative sectional curvature; in the same paper, Wolpert ([30]) proved that there is an negative upper bound, $-\frac{1}{2\pi(g-1)}$, for the holomorphic sectional curvature and Ricci curvature, a result conjectured by Royden ([20]). However, there is no known negative upper bound for the sectional curvatures. So a fundamental question is:

*Is there a negative upper bound for the sectional curvature of the Weil-Petersson metric on Teichmüller space?*

Whether such a negative bound exists makes a huge difference on studying the Weil-Petersson geometry of Teichmüller space. Recently Brock and Farb ([5]) showed
that the Weil-Petersson metric on $\mathcal{T}$ is Gromov hyperbolic if and only if $\text{dim}_G(\mathcal{T}) \leq 2$. They pointed out that if the Weil-Petersson metric on $\mathcal{T}$ had curvature pinched from above by a negative constant, its Gromov hyperbolicity in the case of the surface being doubly-punctured torus or 5-punctured sphere would be an immediate consequence of the comparison theorems. In the end of their paper, they asked the following question: if $\text{int}(\Sigma)$ is homeomorphic to a doubly-punctured torus or 5-punctured sphere, are the sectional curvatures of the Weil-Petersson metric bounded away from zero? In this work, we prove that even though the sectional curvatures are negative, they are not bounded away from zero, hence giving a negative answer to above question. More specifically, we prove the following:

**Main Theorem.** If the complex dimension of Teichmüller space $\mathcal{T}$ is greater than 1, then there is no negative upper bound for the sectional curvature of the Weil-Petersson metric.

The proof of the main theorem also provides a quantitative estimate for the sectional curvature, namely,

**Theorem 1.2.** Let the complex dimension of Teichmüller space $\mathcal{T}$ is greater than 1, and let $l$ be the length of shortest geodesic along a path to a boundary point in Teichmüller space, then there exists a family of tangent planes with Weil-Petersson sectional curvature of the order $O(l)$. 
1.2 Ideas to Prove Main Results

To estimate the curvature, we study harmonic maps between Riemann surfaces, especially when the target surface is developing nodes.

The moduli space of Riemann surfaces admits a compactification, known as the Deligne-Mumford compactification ([17]), and any element of the compactification divisor can be thought of as a Riemann surface with nodes, a connected complex space where points have neighborhoods complex isomorphic to either \( \{ |z| < \varepsilon \} \) (regular points) or \( \{ zw = 0; |z|, |w| < \varepsilon \} \) (nodes). We can think of noded surfaces arising as elements of the compactification divisor through a pinching process: fix a family of simple closed curves on \( \Sigma \) such that each component of the complement of the curves has negative Euler characteristic. The noded surface is topologically the result of identifying each curve to a point, the node ([4]).

Naturally associated to a harmonic map \( w : (\Sigma, \sigma|dz|^2) \to (\Sigma, \rho|dw|^2) \) is a quadratic differential \( \Phi(\sigma, \rho)dz^2 \), which is holomorphic with respect to the conformal structure of \( \sigma \). This association of a quadratic differential to a conformal structure then defines a map \( \Phi : \mathcal{T} \to QD(\Sigma) \) from Teichmüller space \( \mathcal{T} \) to the space of holomorphic quadratic differentials \( QD(\Sigma) \). Work of Wolf ([26]) showed that this map is in fact a homeomorphism.

As an important computational tool in geometry of Teichmüller theory, the method of harmonic maps has been studied by many people. In particular, Tromba ([24]) showed that the second variation of the energy of the harmonic map \( w = w(\sigma, \rho) \) with respect to the domain structure \( \sigma \) at \( \sigma = \rho \) yields the Weil-Petersson metric
on $\mathcal{T}$, and Wolf ([26]) proved that the second variation of the energy of the harmonic map $w = w(\sigma, \rho)$ with respect to the image structure $\rho$ at $\sigma = \rho$, again, yields the Weil-Petersson metric on $\mathcal{T}$. By this method, Jost ([14]), and Wolf ([26]) also re-established Tromba-Wolpert’s curvature formula of the Weil-Petersson metric.

In this work, we investigate the asymptotic behavior of the sectional curvatures of the Weil-Petersson metric on Teichmüller space. Our method is to investigate harmonic maps from a nearly noded surface to nearby hyperbolic structures. One of the difficulties in estimating Weil-Petersson curvatures is working with the operator $D = -2(\Delta - 2)^{-1}$, which appears in Tromba-Wolpert’s curvature formula. Our approach is to study the Hopf differentials associated to harmonic maps and the analytic formulas resulting from the harmonicity of the maps. There is a natural connection between the operator $D$ and the local variations of the energy of a harmonic map between surfaces (see Lemma 2.5). From this connection, we estimate the solutions to some ordinary differential equations to derive our curvature estimates when the surface is a pair of cylinders. Then we will construct families of maps which are close to the harmonic maps between surfaces and hence we can adapt previous estimates and prove the main theorem.

1.3 Organization of This Paper

The organization of this paper is as follows.

In chapter 2, we will study the theory of harmonic maps between Riemann surfaces: In section 1, we give the necessary background, define our terms and introduce
the notations. In section 2, we will study explicitly harmonic maps between a pair of hyperbolic cylinders, which will play an important role in our further arguments. We call those maps "cylinder maps" when we refer to them in later chapters. In section 3, we will discuss the local variation of the energy functional of the harmonic map, and present the variational formulas which will initiate the connection to the Weil-Petersson geometry of Teichmüller space.

In chapter 3, we will turn our attention to Weil-Petersson geometry of Teichmüller space: In section 1, again, we give the necessary background and collect some known results on this subject. In section 2, we will study the Tromba-Wolpert curvature formula for the Weil-Petersson metric on Teichmüller space, and work out the estimates of some terms in the formula. In section 3, we study the connection between the operator $D = -2(\Delta - 2)^{-1}$, and local variations of the energy of the harmonic maps between Riemann surfaces.

Chapter 4 is devoted to proving our main theorem. In this chapter, we will extensively focus on the case where our surface is a pair of hyperbolic cylinders. In section 1, we describe a family of 2-dimensional subspaces in the tangent space of Teichmüller space. We will have to show this family is asymptotically flat, a statement sufficient to imply our main theorem. In section 2, we start our calculations by estimating the term $\Gamma$ in the curvature formula. In section 3, we mainly work in the cylindrical region $M_0$, where one geodesic is pinched, and we will use the connection revealed in last chapter to establish an ordinary differential equation and estimate the solutions. In section 4, we work on the other cylindrical region $M_1$, where the second geodesic is pinched, to estimate the rest term in the curvature formula.
We continue the proof of the main theorem in chapter 5 to treat the general case. In section 1, we construct families of maps which have small tension. In section 2, we show that the constructed family is reasonably close to the family of harmonic maps resulting from the pinching process. In section 3, we work in the general situation and adapt the estimates in last chapter to this case. In section 4, we remark on the case when the surfaces are punctured hence complete the proof of the main theorem.
Chapter 2

Harmonic Maps Between Riemann Surfaces

2.1 Energy Functional and Harmonic Maps

Recall that $\Sigma$ is a fixed, oriented, $C^\infty$ surface of genus $g \geq 1$, and $n \geq 0$ punctures where $3g - 3 + n > 1$. We denote two hyperbolic metrics on $\Sigma$ by $\sigma|dz|^2$ and $\rho|dw|^2$, where $z$ and $w$ are conformal coordinates on $\Sigma$.

**Definition 2.1.** For a Lipschitz map $w : (\Sigma, \sigma|dz|^2) \rightarrow (\Sigma, \rho|dw|^2)$, we define the energy density of $w$ at a point to be

$$e(w; \sigma, \rho) = \frac{e(w(z))}{\sigma(z)} |w_z|^2 + \frac{e(w(z))}{\sigma(z)} |w_{\bar{z}}|^2$$

and the total energy $E(w; \sigma, \rho) = \int_{\Sigma} e(w; \sigma, \rho) \sigma dz d\bar{z}$.

**Definition 2.2.** A critical point of $E(w; \sigma, \rho)$ is called a harmonic map; it satisfies the Euler-Lagrange equation, namely,
\[ w_{zz} + \frac{2\pi}{\rho} w_z w_{\bar{z}} = 0. \]

The Euler-Lagrange equation for the energy is the condition for the vanishing of the tension, which is, in local coordinates,

\[ \tau(w) = \Delta w^\gamma + \frac{N}{4} \Gamma_{\alpha\beta}^\gamma w_\alpha w_\beta w^\gamma = 0 \]

The work of Al'ber ([3]) and Eells-Sampson ([9]) showed that given two conformal structures \( \sigma, \rho \) on the surface \( \Sigma \), there exists a harmonic map \( w : (\Sigma, \sigma) \to (\Sigma, \rho) \) homotopic to the identity map of \( \Sigma \). The uniqueness was showed by Hartman ([11]), and, independently, Schoen-Yau ([23]) and Sampson ([21]) showed that this map is a diffeomorphism.

Thus we have a quadratic differential \( \Phi dz^2 = \rho w_z \bar{w}_z dz^2 \), which is holomorphic when \( w \) is harmonic. Evidently, we also easily see that

\[ \Phi = 0 \iff w \text{ is conformal} \iff \sigma = \rho. \]

where \( \sigma = \rho \) means that \( (\Sigma, \sigma) \) and \( (\Sigma, \rho) \) represent the same point in Teichmüller space \( \mathcal{T} \) (see definition 1.1). This describes \( \Phi \) as a well-defined map \( \Phi : \mathcal{T} \to QD(\Sigma) \) from Teichmüller space \( \mathcal{T} \) to the space of holomorphic quadratic differentials \( QD(\Sigma) \). In fact, this map is a homeomorphism ([26]). Also note that the map \( w \) can be extended to surfaces with finitely many punctures.

We now define two auxiliary functions as following:

\[ \mathcal{H} = \mathcal{H}(\sigma, \rho) = \frac{\rho(w(z))}{\sigma(z)} |w_z|^2 \]

\[ \mathcal{L} = \mathcal{L}(\sigma, \rho) = \frac{\rho(w(z))}{\sigma(z)} |w_{\bar{z}}|^2 \]
Hence the energy density can be written as } e = \mathcal{H} + \mathcal{L}. \text{ Many things connected with a harmonic map between surfaces can be written in terms of } \mathcal{H} \text{ and } \mathcal{L} \text{ (and } \Phi). \text{ We denote on } (\Sigma, \sigma | dz |^2 )

\[ \Delta = \frac{4}{\sigma} \frac{\partial^2}{\partial z \partial \bar{z}}, \quad K(\rho) = -\frac{2}{\rho} \frac{\partial^2}{\partial w \partial \bar{w}} \log \rho, \quad K(\sigma) = -\frac{2}{\sigma} \frac{\partial^2}{\partial z \partial \bar{z}} \log \sigma, \]

where } \Delta \text{ is the Laplacian, and } K(\rho), K(\sigma) \text{ are curvatures of the hyperbolic metrics } \rho \text{ and } \sigma, \text{ respectively.}

The Euler-Lagrange equation gives

\[ \Delta \log \mathcal{H} = -2K(\rho)\mathcal{H} + 2K(\rho)\mathcal{L} + 2K(\sigma). \]

**Remark 2.3.** We observe that since the map } w \text{ is a diffeomorphism, the expression } \mathcal{H} - \mathcal{L} \text{ is the Jacobian and that } H(w) > 0, \text{ so that } \log \mathcal{H} \text{ is well defined.}

When we restrict ourselves to the situation } K(\rho) = K(\sigma) = -1, \text{ we will have the following facts:

**Lemma 2.4.** (Wolf [26]): \text{ With above notations, we have}

1. } \mathcal{H} > 0;

2. \text{ The Beltrami differential } \mu = \frac{w_\bar{z}}{w_z} = \frac{\bar{\Phi}}{\sigma \mathcal{H}},

3. \Delta \log \mathcal{H} = 2\mathcal{H} - 2\mathcal{L} - 2.

### 2.2 Cylinder Maps

In this section, we consider the boundary value problem of harmonically mapping the cylinder.
\[
M = [l^{-1} \sin^{-1}(l), \pi l^{-1} - l^{-1} \sin^{-1}(l)] \times [0, 1]
\]

with the following boundary identification

\[
\left[\frac{\sin^{-1}(l)}{l}, \frac{\pi}{l} - \frac{\sin^{-1}(l)}{l}\right] \times \{0\} = \left[\frac{\sin^{-1}(l)}{l}, \frac{\pi}{l} - \frac{\sin^{-1}(l)}{l}\right] \times \{1\}
\]

where the hyperbolic length element on \(M\) is \(l \csc(lx) dz\), to the cylinder

\[
N = [L^{-1} \sin^{-1}(L), \pi L^{-1} - L^{-1} \sin^{-1}(L)] \times [0, 1]
\]

with boundary identification

\[
\left[\frac{\sin^{-1}(L)}{L}, \frac{\pi}{L} - \frac{\sin^{-1}(L)}{L}\right] \times \{0\} = \left[\frac{\sin^{-1}(L)}{L}, \frac{\pi}{L} - \frac{\sin^{-1}(L)}{L}\right] \times \{1\}
\]

where the hyperbolic length element on \(N\) is \(L \csc(Lu) dw\). Here \(l\) and \(L\) are the lengths of the simple closed core geodesics in the corresponding cylinders.

Let \(w = u + iv\) be a harmonic map between cylinders \(M\) and \(N\), where

\[
u(l, L; x, y) = u(l, L; x), \, v(x, y) = y.
\]

Then the Euler-Lagrange equation becomes

\[
u'' = L \cot(Lu)(u'^2 - 1).
\]

with the following boundary conditions

\[
u\left(\frac{\sin^{-1}(l)}{l}\right) = \frac{\sin^{-1}(L)}{L}, \, u\left(\frac{\pi}{2L}\right) = \frac{\pi}{2L}.
\]

Note that both cylinders \(M\) and \(N\) admit an anti-isometric reflection about the curves \(\{\frac{\pi}{2L}\} \times [0, 1]\) and \(\{\frac{\pi}{2L}\} \times [0, 1]\).

Since the quadratic differential \(\Phi = \rho w_{\bar{z}} \bar{w}_z = \frac{1}{4} L^2 \csc^2(Lu)(u'^2 - 1)\) is holomorphic in \(M\), we have
\[ 0 = \frac{\partial}{\partial x} \left( \rho^2 w_x \tilde{w}_x \right) = \frac{\partial}{\partial x} \left( \frac{1}{5} L^2 c_s^2 (Lu)(u'' - 1) \right) \]

Therefore \( L^2 c_s^2 (Lu)(u'' - 1) = c_0(l, L) \), where \( c_0(l, L) \) is independent of \( x \), and \( c_0(l, l) = 0 \) since \( u(x) \) is the identity map when \( L = l \). When fixing \( l \), let \( L = L(t) \) vary so that \( L(0) = l \), then \( c_0(t) = c_0(l, L(t)) \) is independent of \( x \) and \( c_0(0) = 0 \).

Now we have

\[ u' = \sqrt{1 + c_0(l, L)L^{-2} \sin^2(Lu)} \]

with boundary conditions

\[ u\left( \frac{\sin^{-1}(l)}{l} \right) = \frac{\sin^{-1}(L)}{L}, \quad u\left( \frac{\pi}{2l} \right) = \frac{\pi}{2L}. \]

Thus the solution to the Euler-Lagrange equation can be derived from the equation

\[ \int_{L^{-1} \sin^{-1} L}^{u} \frac{du}{\sqrt{1 + c_0(l, L)L^2 \sin^2(Lu)}} = x - l^{-1} \sin^{-1}(l) \]

with \( c_0(l, L) \) chosen such that \( \int_{\frac{\pi}{2L}}^{L^{-1} \sin^{-1} L} \frac{du}{\sqrt{1 + c_0(l, L)L^2 \sin^2(Lu)}} = \frac{\pi}{2l} - l^{-1} \sin^{-1} l. \)

It is not hard to show that when \( l \to 0 \), the solution \( u(l, L; x) \) converges to a solution \( u(L; x) \) to the “noded” problem, i.e., \( M = [1, +\infty) \), where we require \( u(L; 1) = L^{-1} \sin^{-1}(L) \) and \( \lim_{x \to +\infty} u(L; x) = \frac{\pi}{2L}. \)

This “noded” problem has the explicit solution as following ([27])

\[ u(L; x) = L^{-1} \sin^{-1} \left\{ \frac{\sqrt{1 - \frac{(L - 1)^2}{4L(1-L)}}}{1 + \frac{(L - 1)^2}{4L(1-L)}} e^{L(1-x)} \right\} \]

with the holomorphic energy

\[ H_0(L; x) = \frac{L^2 \pi^2}{4} \left| \frac{1 + \sqrt{1 - \frac{(L - 1)^2}{4L(1-L)}} e^{L(1-x)}}{1 - \sqrt{1 - \frac{(L - 1)^2}{4L(1-L)}} e^{L(1-x)}} \right|^2 \]
2.3 Local Variation of the Energy

We consider a family of harmonic maps \( w(t) \) for \( t \) small, where \( w(0) = \text{id} \), the identity map. Denote by \( \Phi(t) \) the family of holomorphic quadratic differentials (also called Hopf differentials) determined by \( w(t) \). We can rewrite (3) of Lemma 2.4 as

\[
\Delta \log \mathcal{H}(t) = 2\mathcal{H}(t) - \frac{2|\Phi(t)|^2}{\sigma^2 \mathcal{H}(t)} - 2
\]

For this equation, we see that the maximum principle will force all the odd order \( t \)-derivatives of \( \mathcal{H}(t) \) to vanish, since the above equation only depends on the modulus of \( \Phi(t) \) and not on its argument ([26]), and \( \mathcal{H}(t) \) is real-analytic in \( t \) ([27]).

Wolf computed the \( t \)-derivative of various geometric quantities associated with the harmonic maps, and we collect these local variational formulas into our next lemma:

**Lemma 2.5.** (Wolf [26]) For the above notations, we have

1. \( \mathcal{H}(t) \geq 1 \), and \( \mathcal{H}(t) \equiv 1 \iff t = 0 \);
2. \( \dot{\mathcal{H}}(t) = \partial/\partial t^a |_{0} \mathcal{H}(t) = 0 \);
3. \( \dot{\mu} = \partial/\partial t^a |_{0} \mu(t) = \Phi_\alpha/\sigma \);
4. \( \dddot{\mathcal{H}}(t) = \frac{\sigma^2}{\partial t^a \partial t^b} |_{0} \mathcal{H}(t) = D \frac{\Phi_\alpha \Phi_\beta}{\sigma^2} \), where \( D = -2(\Delta - 2)^{-1} \).

**Remark 2.6.** Evidently \( D \) is a self-adjoint compact integral operator with a positive kernel, and it is the identity on constant functions. Hence with (4) in Lemma 2.5, we obtain a partial differential equation about \( \dot{\mathcal{H}}(t) \):

\[
(\Delta - 2)(\dddot{\mathcal{H}}(t)) = -2 \frac{\Phi_\alpha \Phi_\beta}{\sigma^2}
\]

Hence we can see the local variations of the energy of harmonic maps are related to holomorphic quadratic differentials.
Chapter 3

The Weil-Petersson Geometry of

Teichmüller Space

3.1 Teichmüller Space of Riemann Surfaces

A hyperbolic Riemann surface is both a space of constant curvature, and a one-dimensional complex manifold. These two natures are reflected in Teichmüller theory.

In studying Riemann surfaces, one is naturally led to a study of various space of moduli of Riemann surfaces. Teichmüller space is the space of hyperbolic structures on the closed surface $\Sigma$, where two structures are equivalent if there is a conformal map in the homotopy class of the identity map between these two structures.

By the uniformization theorem, the set of all similarly oriented hyperbolic structures $\Pi_{-1}$ on $\Sigma$ can be identified with the set of all conformal structures on $\Sigma$ with the given orientation. Equivalently, this is the same as the set of all complex structures on $\Sigma$ with the given orientation. As in definition 1.1, the Teichmüller space $\mathcal{T}$ is
defined to be

\[ \mathcal{T} = \Pi_{-1}/Diff_0(\Sigma). \]

Modern Teichmüller theory began in the 1950s and 1960s with the work of Ahlfors and Bers, who founded a theory of Teichmüller spaces, Riemann surfaces, and Fuchsian groups upon the elliptic partial differential equation \( w_z = \mu w_z \) associated to a quasiconformal map \( w \). Ahlfors ([2]) showed that Teichmüller space \( \mathcal{T} \) has a complex structure, hence it is a complex manifold and the cotangent space at a point \( \Sigma \in \mathcal{T} \) is identified as the space of holomorphic quadratic differentials \( \Phi dz^2 \) on \( \Sigma \).

On \( \Sigma \), there is a natural pairing of quadratic differentials and Beltrami differentials \( \mu(z) \frac{dz}{dz} \) given by

\[ < \mu, \Phi > = \text{Re} \int_\Sigma \mu(z)\Phi(z)dzd\bar{z} \]

The tangent space at \( \Sigma \) is naturally represented by a space of Beltrami differentials \( \{ \mu = \frac{w_z}{w} \} \) that are harmonic with respect to the non-Euclidean metric on \( \Sigma \). In other words, it is the space of Beltrami differentials modulo the ones such that \( < \mu, \Phi > = 0 \).

**Definition 3.1.** The Weil-Petersson metric on \( \mathcal{T} \) is obtained by duality from the \( L^2 \)-inner product on \( QD(\Sigma) \)

\[ < \phi, \psi > = \int_\Sigma \frac{\phi\bar{\psi}}{\sigma} dzd\bar{z} \]

where \( \sigma dzd\bar{z} \) is the hyperbolic metric on \( \Sigma \).

The Weil-Petersson cometric on Teichmüller space is one of the most important and interesting metrics one can put on Teichmüller space. With this metric, Teichmüller space is a Kählerian manifold ([2]) with negative sectional curvature ([24],
[30]), and it is well known ([7], [29]) that it is incomplete since not every Weil-
Petersson geodesic can be extended infinitely. When a Weil-Petersson geodesic can-
not be further extended, a nontrivial closed geodesic shrinks in length (with respect
to the hyperbolic metric) to zero, thus developing a node.

3.2 Weil-Petersson Curvatures

The curvature tensor of the Weil-Petersson metric is given by ([30])

\[ R_{\alpha\beta\gamma\delta} = (\int_D D(\mu_\alpha \dot{\mu}_\beta) \dot{\mu}_\gamma \dot{\mu}_\delta dA) + (\int_D D(\mu_\alpha \mu_\delta) \mu_\gamma \mu_\delta dA) \]

where \(dA\) is the area element, and \(\mu_\alpha, \mu_\beta, \mu_\gamma, \mu_\delta\) are Beltrami differentials. This
formula is called the Tromba-Wolpert curvature formula.

Let \(\Omega\) be a 2-dimensional subspace of tangent space of the Teichmüller space,
spanned by \(\dot{\mu}_0\) and \(\dot{\mu}_1\), then the curvature of \(\Omega\) is \(R/\Gamma\) ([30]), where

\[ R = R_{0101} - R_{0110} - R_{1001} + R_{1010} \]

and

\[ \Gamma = 4 \langle \dot{\mu}_0, \dot{\mu}_0 \rangle \langle \dot{\mu}_1, \dot{\mu}_1 \rangle - 2 | \langle \dot{\mu}_0, \dot{\mu}_1 \rangle |^2 - 2 \text{Re}(\langle \dot{\mu}_0, \dot{\mu}_1 \rangle)^2 \]

\[ = 4 \langle \dot{\mu}_0, \dot{\mu}_0 \rangle \langle \dot{\mu}_1, \dot{\mu}_1 \rangle - 4 | \langle \dot{\mu}_0, \dot{\mu}_1 \rangle |^2 \]

Now we deliver an estimate on \(R\).

**Lemma 3.2.** When we have above notation, and both \(\dot{\mu}_0\) and \(\dot{\mu}_1\) are real functions,
then \(|R| \leq 4 \int_D D(\mu_0^2) |\dot{\mu}_1|^2 \sigma dxdy\)

**Proof.** Note that \(D = -2(\Delta - 2)^{-1}\) is self-adjoint, hence we have
\[ \int_{\Sigma} D(|\mu_0|^2)|\mu_1|^2 \sigma dxdy = \int_{\Sigma} D(|\mu_1|^2)|\mu_0|^2 \sigma dxdy \]

Therefore,

\[ R = R_{01\bar{0}1} - R_{0\bar{1}0\bar{1}} - R_{1\bar{0}0\bar{1}} + R_{1\bar{0}1\bar{0}} \]

\[ = 2 \int_{\Sigma} D(\hat{\mu}_0 \hat{\mu}_1) \mu_0 \mu_0 \sigma dxdy + 2 \int_{\Sigma} D(\mu_1 \hat{\mu}_0) \mu_1 \mu_0 \sigma dxdy \]

\[ - \int_{\Sigma} D(|\mu_0|^2)|\mu_1|^2 \sigma dxdy - \int_{\Sigma} D(|\mu_1|^2)|\mu_0|^2 \sigma dxdy \]

\[ - \int_{\Sigma} D(\mu_0 \hat{\mu}_1) \mu_1 \mu_0 \sigma dxdy - \int_{\Sigma} D(\hat{\mu}_0 \mu_1) \mu_0 \mu_1 \sigma dxdy \]

\[ = 2 \int_{\Sigma} D(\hat{\mu}_0 \hat{\mu}_1) \mu_0 \mu_0 \sigma dxdy - 2 \int_{\Sigma} D(|\mu_0|^2)|\mu_1|^2 \sigma dxdy \]

The last equality follows from that here \( \hat{\mu}_0 \) and \( \hat{\mu}_1 \) are real functions. Now from lemma 4.3 of [30], we have \( |D(\hat{\mu}_0 \hat{\mu}_1)| \leq |D(|\mu_0|^2)|^{1/2}|D(|\mu_1|^2)|^{1/2} \). Then an application of Hölder inequality shows that

\[ |\int_{\Sigma} D(\hat{\mu}_0 \hat{\mu}_1) \mu_0 \mu_0 dA| \leq \int_{\Sigma} |D(|\mu_0|^2)|^{1/2}|D(|\mu_1|^2)|^{1/2} \hat{\mu}_1 \mu_0 dA \]

\[ \leq \left( \int_{\Sigma} D(|\mu_0|^2)|\mu_1|^2 dA \right)^{\frac{1}{2}} \left( \int_{\Sigma} D(|\mu_1|^2)|\mu_0|^2 dA \right)^{\frac{1}{2}} \]

\[ = \int_{\Sigma} D(|\mu_0|^2)|\mu_1|^2 \sigma dxdy \]

So we will have

\[ |R| \leq 4 \int_{\Sigma} D(|\mu_0|^2)|\mu_1|^2 \sigma dxdy \] (3.1)

which completes the proof of this lemma. \( \Box \)

**Remark 3.3.** From this lemma and the curvature formula, we can see that the key computation is to estimate the integral \( \int_{\Sigma} D(|\mu_0|^2)|\mu_1|^2 \sigma dxdy \).
3.3 Connection with Harmonic Maps

From section 2.1, for a harmonic map \( w(\sigma, \rho) \) between Riemann surfaces, we have a holomorphic quadratic differential \( \Phi dz^2 = \rho w_2 \bar{w}_2 dz^2 \). This describes \( \Phi \) as a homeomorphism \( \Phi : \mathcal{T} \to QD(\Sigma) \), from Teichmüller space \( \mathcal{T} \) to the space of Hopf differentials \( QD(\Sigma) \). This observation furnishes the link between harmonic maps and Teichmüller theory.

To explore this relation, one investigates the variations of the hyperbolic structure \( \sigma \) on the domain surface and the hyperbolic structure \( \rho \) on the image surface. The work of Tromba ([24]) and Wolf ([26]) showed that the second variation of the energy of the harmonic map \( w = w(\sigma, \rho) \) at \( \sigma = \rho \) yields the Weil-Petersson metric on \( \mathcal{T} \), with respect to either \( \sigma \) or \( \rho \). Hence it is possible to develop the geometric theory of Teichmüller space systematically in terms of harmonic maps.

Furthermore, we see that the operator \( D = -2(\Delta - 2)^{-1} \) appears in the Tromba-Wolpert curvature formula, it also appears in (4) of Lemma 2.5, where we consider local variation of the energy of the family of harmonic maps \( w(t) = w(\sigma, \rho(t)) \). We indicated in remark 3.3, that the key computation to estimate the curvature is to estimate the integral \( \int_{\Sigma} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 \sigma dx dy \), in other words, to estimate \( D(|\dot{\mu}_0|^2) \).

One realizes, from (4) of the Lemma 2.5 and remark 2.6, that one needs to estimate the solution of the differential equation in remark 2.5 for the choice of family of harmonic maps. In particular, for a family of cylinder maps that we studied in the section 2.2, we will have more explicit forms of the differential equations about the local variation of the energy.
Chapter 4

Proof of the Main Results, Model Case

In this chapter, we will start to prove the main theorem of this work, namely,

Main Theorem. If the complex dimension of Teichmüller space $\mathcal{T}$ is greater than 1, then there is no negative upper bound for the sectional curvature of the Weil-Petersson metric.

Schumacher showed that there is no lower bound for the sectional curvature ([22]).

Combined with his result, We have

Corollary 4.1. There are no negative bounds for the sectional curvature of the Weil-Petersson metric when $\dim_C(\mathcal{T}) > 1$.

In this chapter, we will estimate all the terms in the curvature formula in the model case, i.e., the surface is a pair of cylinders. For the sake of simplicity of exposition,
we assume that our surface has no punctures. We will comment on punctured case in section 5.3.

4.1 Asymptotically Flat Planes

Consider the surface $\Sigma$ which is developing two nodes, i.e., we are pinching two nonhomotopic closed geodesics $\gamma_0$ and $\gamma_1$ on $\Sigma$ to two points, say $p_0$ and $p_1$. We denote $M_0$ and $M_1$ their pinching neighborhoods, i.e., two cylinders described as $M$ in section 2.2 centered at $\gamma_0$ and $\gamma_1$, respectively.

We define $\Sigma(l_0, l_1)$ be the surface with two of the Fenchel-Nielsen coordinates, namely, the hyperbolic lengths of $\gamma_0$ and $\gamma_1$, being $l_0$ and $l_1$, respectively. When we set the length of these two geodesics equal to $l$ simultaneously, we will have a point $\Sigma(l) = \Sigma(l, l)$ in Teichmüller space $T_g$. Note that as $l$ tends to zero, the surface is developing two nodes.

At the point $\Sigma(l)$, we look at two directions. First, we fix $\gamma_1$ in $M_1$ having length $l$, and pinch $\gamma_0$ in $M_0$ into length $L = L(t)$, where $L(0) = l$. So the $t$-derivative of $\mu_0(t)$, the Beltrami differential of the resulting harmonic map, at $t = 0$ represents a tangent vector, $\dot{\mu}_0$, of the Teichmüller space $T_g$ at $\Sigma(l)$; we denote the resulting family of harmonic maps by $W_0(t) : \Sigma(l, l) \to \Sigma(L(t), l)$. Then we fix $\gamma_0$ in $M_0$ having length $l$, and pinch $\gamma_1$ in $M_1$ into length $L = L(t)$, so the $t$-derivative of $\mu_1(t)$ at $t = 0$ represents another tangent vector, $\dot{\mu}_1$, at $\Sigma(l)$; we denote the resulting family of harmonic maps by $W_1(t) : \Sigma(l, l) \to \Sigma(l, L(t))$. These two tangent vectors $\dot{\mu}_0$ and $\dot{\mu}_1$, at $\Sigma(l)$, will span a two dimensional subspace of the tangent space $T_{\Sigma(l)}T_g$ to $T_g$. 
hence we obtain a family, denoted by $\Omega_t$, of two dimensional subspaces of the tangent space of $T_g$.

**Theorem 4.2.** The Weil-Petersson sectional curvatures of $\Omega_t$ tend to 0 as $l \to 0$.

**Remark 4.3.** It is immediate that this theorem implies our main theorem. We will postpone the proof of theorem 4.2 until next chapter but to concentrate on estimating the terms in the curvature formula when the surface is a pair of cylinders. In this chapter, we ignore the compact part in between $M_0$ and $M_1$ for now. Throughout this chapter, our surface is a pair of cylinders $M_0$ and $M_1$.

We denote $\phi_0(t)$ as the family of Hopf differentials corresponding to the family of cylinder maps $w_0(t)$ in $M_0$, and $\phi_1(t)$ as the family of Hopf differentials corresponding to the family of cylinder maps $w_1(t)$ in $M_1$. Here $w_0(t) : \Sigma(l) \to \Sigma(L(t))$ and $w_1(t) : \Sigma(l) \to \Sigma(L(t))$ are families of harmonic maps described as $w = u(x) + iy$ in section 2.2. In $M_1$, we abuse our notation slightly, still using $\phi_0$ to denote the family of Hopf differentials corresponding to the family of harmonic maps $W_0(t) : \Sigma(l, l) \to \Sigma(L(t), l)$; while in $M_0$, we still use $\phi_1$ as the family of Hopf differentials corresponding to the family of harmonic maps $W_1(t) : \Sigma(l, l) \to \Sigma(l, L(t))$. We also denote $\mu_0$ and $\mu_1$ as the corresponding Beltrami differentials to $\phi_0$ and $\phi_1$.

## 4.2 Estimate of the Term $\Gamma$

We denote $a = a(l) = l^{-1} \sin^{-1}(l)$, and $b = b(l) = \pi l^{-1} - l^{-1} \sin^{-1}(l)$. And in this paper, $A \sim B$ means $A/C < B < CA$ for some constant $C > 0$, while $A = O(B)$ and $A = o(B)$ are in the usual sense.
In the hyperbolic cylinder $M_0$, as in section 2.2, we have a family of cylinder maps $w_0(t) = u(x, t) + iy$, then we choose $L(t)$ so that $\dot{c}_0 = \frac{d}{dt}|_{t=0}c_0(t) = \frac{d}{dt}|_{L=0}c_0(l, L) = 4$. Here we recall that $c_0(t) = c_0(l, L(t)) = L^2\csc^2(Lu)(u^2 - 1)$ is independent of $x$. We notice that $\dot{c}_0$ is never zero for all $l > 0$, otherwise we would have $\dot{w}_0 = 0$ as $\dot{c}_0 = \sigma \dot{w}_0$. Hence $w_0$ is a constant map by rotational invariance of the map, and then we have $\dot{c}_0 = \frac{d}{dt}|_{t=0}\left(\frac{1}{2}c_0(t)\right) = 1$ in $M_0$.

Also, we see that $|\dot{\phi}_1|_{M_0} = \zeta(x, l)$ for $x \in [a, b]$, where $\zeta(x, l)$ satisfies that

1. $\zeta(x, l) \leq C_1 x^{-4}$ for $x \in [a, \pi/2l]$;
2. $\zeta(x, l) \leq C_1 (\pi/l - x)^{-4}$ for $x \in [\pi/2l, b]$;
3. $\zeta(x, 0)$ decays exponentially in $[1, +\infty]$.

Here $C_1 = C_1(l)$ is positive and bounded as $l \to 0$. To see this, notice that $\dot{\phi}_1$ is holomorphic and $|\dot{\phi}_1|$ is positive, so $\log|\dot{\phi}_1|$ is harmonic in the cylinder $M_0$. Hence we can express $\log|\dot{\phi}_1|$ in a Fourier series $\Sigma a_n(x) \exp(-iny)$, and we compute $0 = \Delta \log|\dot{\phi}_1| = \Sigma (a''_n - n^2 a_n) \exp(-iny)$. We will see, in section 5.1, that $\dot{\phi}_1$ is close to 0 in $M_0$. Hence we conclude the properties $\zeta$ has. Similarly, we have $\dot{\phi}_1|_{M_1} = 1$ and $|\dot{\phi}_0|_{M_1} = \zeta(x, l)$ for $x \in [a, b]$.

**Remark 4.4.** Note that these estimates imply, informally, that most of the mass of $|\phi_0|$ resides in the thin part associated to $\gamma_0$, and most of the mass of $|\phi_1|$ resides in the thin part associated to $\gamma_1$.

Now in $M_0$, the corresponding Beltrami differential is

$$\dot{\mu}_0 = \frac{d}{dt}|_{t=0}\left(\frac{\dot{w}_0}{w_0}\right) = \dot{\phi}_0/\sigma$$

and $|\dot{\mu}_0|^2_{M_0} = |\dot{\phi}_0/\sigma|^2_{M_0} = l^{-4} \sin^4(lx)$. 

Also, in that same neighborhood $M_0$,

$$\hat{\mu}_1 = \frac{\hat{\phi}_1}{\sigma}$$

Hence

$$|\hat{\mu}_1|^2|_{M_0} = |\frac{\hat{\phi}_1}{\sigma}|^2|_{M_0} = l^{-4} \sin^4(lx) \zeta^2(x, l).$$

Similarly, in $M_1$, we have

$$|\hat{\mu}_0|^2|_{M_1} = l^{-4} \sin^4(lx) \zeta^2(x, l)$$

$$|\hat{\mu}_1|^2|_{M_1} = l^{-4} \sin^4(lx)$$

We recall that $\Gamma = 4 < \hat{\mu}_0, \hat{\mu}_0 > < \hat{\mu}_1, \hat{\mu}_1 > -4| < \hat{\mu}_0, \hat{\mu}_1 > |^2$. We have the following estimate of this $\Gamma$ in terms of $l$.

**Lemma 4.5.** $1/\Gamma = O(l^3)$.

**Proof.** Now we compute the asymptotics in $l$ of each term in $\Gamma$. Using $|\hat{\mu}_0|^2|_{M_0} = l^{-4} \sin^4(lx)$, and noting that $a = a(l) = l^{-1} \sin^{-1}(l)$, and $b = b(l) = \pi l^{-1} - l^{-1} \sin^{-1}(l)$,

we have that

$$< \hat{\mu}_0, \hat{\mu}_0 >|_{M_0} = \int_{M_0} |\hat{\mu}_0|^2\sigma dxdy$$

$$= \int_0^1 \int_a^b |\hat{\mu}_0|^2\sigma dxdy$$

$$= \int_0^1 \int_a^b l^{-2} \sin^2(lx) dxdy$$

$$= \frac{\pi}{2} l^{-3} - l^{-3} \sin^{-1}(l)$$

$$\sim l^{-3}$$
And using $|\hat{\mu}_1|^2|_{M_0} = l^{-4} \sin^4(lx)\zeta^2(x,l)$, and $\zeta(x,l) \leq C_1 x^{-4}$ for $x \in [a, \pi/2l]$, we have

$$<\hat{\mu}_1, \hat{\mu}_1> |_{M_0} = \int_{M_0} |\hat{\mu}_1|^2 \sigma dxdy$$
$$= 2 \int_0^{\pi/2l} \int_a^1 l^{-4} \sin^4(lx)\zeta^2(x,l) \sigma dxdy$$
$$\leq 2C_1^2 \int_0^{\pi/2l} \int_a^1 l^{-2} \sin^2(lx)x^{-8} dxdy$$
$$= O(1)$$

Also $<\hat{\mu}_0, \hat{\mu}_1> = \int_{\Sigma} \hat{\mu}_0 \hat{\mu}_1 dA$, hence,

$$<\hat{\mu}_0, \hat{\mu}_1> |_{M_0} = \int_{M_0} \hat{\mu}_0 \hat{\mu}_1 \sigma dxdy$$
$$\leq C_1 \int_{M_0} l^{-2} \sin^2(lx)x^{-4} dxdy$$
$$= O(1)$$

Similarly,

$$<\hat{\mu}_0, \hat{\mu}_0> |_{M_1} = O(1)$$
$$<\hat{\mu}_1, \hat{\mu}_1> |_{M_1} \sim l^{-3}$$
$$<\hat{\mu}_0, \hat{\mu}_1> |_{M_1} = O(1)$$

Note that $\Gamma \geq (4 <\hat{\mu}_0, \hat{\mu}_0> <\hat{\mu}_1, \hat{\mu}_1> - 4(<\hat{\mu}_0, \hat{\mu}_1>)^2)|_{M_0} \sim l^{-3}$, which completes the proof of Lemma 4.5.

\[ \Box \]

**Remark 4.6.** From Lemma 4.5, we have

$$|R|/\Gamma = O(|R|/(l^{-3})) = O(|R|l^3) \quad (4.1)$$

Therefore together with Lemma 3.2, to show theorem 4.2, it is enough to show that

$$\int_{\Sigma} D(|\hat{\mu}_0|^2)|\hat{\mu}_1|^2 \sigma dxdy = o(l^{-3}).$$
In fact, we will show that

**Lemma 4.7.** \( \int_S D(|\hat{\mu}_0|^2)|\hat{\mu}_1|^2 \sigma dxdy = O(l^{-2}) \)

Since the surface is a pair of cylinders \( M_0 \) and \( M_1 \), we will split the integral in Lemma 4.7 into two integrals \( \int_{M_0} D(|\hat{\mu}_0|^2)|\hat{\mu}_1|^2 \sigma dxdy \) and \( \int_{M_1} D(|\hat{\mu}_0|^2)|\hat{\mu}_1|^2 \sigma dxdy \). We will estimate them in the next two sections, respectively.

### 4.3 Calculations in \( M_0 \)

In this section, we will give an estimate on \( \int_{M_0} D(|\hat{\mu}_0|^2)|\hat{\mu}_1|^2 \sigma dxdy \), in other words, we will have the following lemma:

**Lemma 4.8.** \( \int_{M_0} D(|\hat{\mu}_0|^2)|\hat{\mu}_1|^2 \sigma dxdy = O(l^{-2}) \)

**Proof.** Firstly, from Lemma 2.5, we have

\[
D(|\hat{\mu}_0|^2) = -2(\Delta - 2)^{-1} \frac{|\dot{\phi}_0|^2}{\sigma^2}
\]

We recall in the hyperbolic cylinder \( M_0 \), the Hopf differential \( \phi_0 \) is corresponding to the cylinder map \( w_0(t) : (M_0, \sigma) \to (M_0, \rho(t)) \), and \( |\dot{\phi}_0| = 1 \). As in section 2.1, we write \( \mathcal{H} = \frac{\theta(w(z))}{\sigma(z)} |w_z|^2 \), therefore we can write \( D(|\hat{\mu}_0|^2) = \tilde{\mathcal{H}} \). Then in \( M_0 \), we have

\[
(\Delta - 2)\tilde{\mathcal{H}} = -2 \frac{\phi_0^2}{\sigma^2} = -2l^{-4} \sin^4(lx)
\]  

(4.2)

A maximum principle argument implies that \( \tilde{\mathcal{H}} \) is positive, since at the minimum, \( \Delta \tilde{\mathcal{H}} > 0 \), then \( \tilde{\mathcal{H}} > l^{-4} \sin^4(lx) > 0 \).

And we can see that this \( \tilde{\mathcal{H}} \) converges to the holomorphic energy \( \tilde{\mathcal{H}}_0 \) in the "noded" problem (of section 2.2) when \( x \) is fixed but sufficiently large in \([a, \pi/2l]\). The convergence suggests that \( \tilde{\mathcal{H}} \) is bounded on the compacta in \([a, b]\). Thus we can assume that
\( \tilde{H} \) is bounded on the boundary of \([a, b]\), i.e., \( A_1(l) = \tilde{H}(a) = \tilde{H}(l^{-1}\sin^{-1}l) = O(1) > 0 \).

Then \( \tilde{H}(x) \) solves the following differential equation

\[
(l^{-2}\sin^2(lx))\tilde{H}'' - 2\tilde{H} = -2l^{-4}\sin^4(lx) \tag{4.3}
\]

with the conditions

\[
\tilde{H}(l^{-1}\sin^{-1}l) = A_1(l), \tilde{H}'(\pi/2l) = 0
\]

Recall from section 2.3 that all the odd order \( t \)-derivatives of \( \mathcal{H}(t) \) vanish. We also observe that

\[
J(x) = \frac{\sin^2(lx)}{2l^4}
\]

is a particular solution to equation (4.3). And Noticing that \( \zeta(x, l) \leq C_1x^{-4} \) in \([a, \pi/2l]\), we have

\[
\int_a^b \int_0^1 J(x)(l^{-2}\sin^2(lx))\zeta^2(x, l)dxdy \leq 2C_1^2 \int_a^b \int_0^{\pi/2l} J(x)l^{-2}\sin^2(lx)x^{-8}dxdy
\]

\[
\leq 2C_1^2 \int_a^b \int_0^{\pi/2l} \frac{\sin^2(lx)}{2l^4} l^{-2}\sin^2(lx)x^{-8}dxdy
\]

\[
= O(l^{-2})
\]

Hence we can check, by the method of reduction of the solutions, the general solution to equation (4.3) with the assigned conditions has the form

\[
\tilde{H}(l; x) = J(x) + A_2\cot(lx) + A_3(1 - lx\cot(lx))
\]

where coefficients \( A_2 = A_2(l) \) and \( A_3 = A_3(l) \) are constants independent of \( x \), and we can check, by substituting the solution into the assigned conditions, that they satisfy that

\[
A_2 = \frac{\pi}{2} A_3 = O(l^{-1})
\]
We also have that
\[
\int_0^1 \int_0^b A_2 \cot(lx)(l^{-2} \sin^2(lx)) \zeta^2(x, l) \, dx \, dy \leq 2C_1^2 A_2 l^{-2} \int_0^{\pi/2l} \cot(lx) \sin^2(lx) x^{-8} \, dx = O(l^{-2})
\]
and
\[
\int_0^b A_3 (1 - lxcot(lx))(l^{-2} \sin^2(lx)) \zeta^2(x, l) \, dx \leq 2C_1^2 A_3 \int_0^{\pi/2l} (1 - lxcot(lx)) x^{-6} \, dx = O(l^{-2})
\]
Now we can compute the following
\[
\int_{M_0} \tilde{\mathcal{H}}(x) |\tilde{\mu}_1|^2 \sigma \, dx \, dy = \int_0^1 \int_0^b \tilde{\mathcal{H}}(x)(l^{-2} \sin^2(lx)) \zeta^2(x, l) \, dx \, dy \\
\leq 2C_1^2 \left( \int_0^1 \int_0^{\pi/2l} J(x) l^{-2} \sin^2(lx) x^{-8} \, dx \, dy \right) \\
+ \int_0^1 \int_a^{\pi/2l} A_2 \cot(lx) l^{-2} \sin^2(lx) x^{-8} \, dx \, dy \\
+ \int_0^1 \int_a^{\pi/2l} A_3 (1 - lxcot(lx)) l^{-2} \sin^2(lx) x^{-8} \, dx \, dy \\
= O(l^{-2}) + O(l^{-2}) + O(l^{-2}) \\
= O(l^{-2})
\]
Therefore
\[
\int_{M_0} D(|\tilde{\mu}_0|^2) |\tilde{\mu}_1|^2 \sigma \, dx \, dy = \int_a^b \tilde{\mathcal{H}} |\tilde{\mu}_1|^2 \sigma \, dx \\
= O(l^{-2}) \quad (4.4)
\]
This completes the proof of Lemma 4.8. \(\square\)

### 4.4 Calculations in \(M_1\)

Now let us look at the term \(\int_{M_1} D(|\tilde{\mu}_0|^2) |\tilde{\mu}_1|^2 \sigma \, dx \, dy\). We show that:
Lemma 4.9. \( \int_{M_1} D(|\mu_0|^2)|\hat{\mu}|^2 \sigma dx dy = O(l^{-2}) \)

Proof. Note that in \( M_1 \), which we identify with \([a, b] \times [0, 1] \), we have \( D(|\mu_0|^2) = \tilde{\mathcal{H}} \), here \( \mu_0(t) \) and \( \mathcal{H} \) come from the harmonic map \( W_0(t) : \Sigma(l, l) \to \Sigma(L(t), l) \), where \( |\hat{\mu}_0|_{M_1} = l^{-2} \sin^2(lx) \zeta(x, l) \), and \( \mathcal{H}(t) \) solves

\[
(l^{-2} \sin^2(lx)) \tilde{\mathcal{H}}'' - 2 \tilde{\mathcal{H}} = -2l^{-4} \sin^4(lx) \zeta^2(x, l)
\]  

(4.5)

with the conditions

\[
\tilde{\mathcal{H}}(l^{-1} \sin^{-1} l) = B_1(l), \quad \tilde{\mathcal{H}}(\pi/l - l^{-1} \sin^{-1} l) = B_2(l).
\]

Here \( B_1(l) \) and \( B_2(l) \) are positive and bounded as \( l \to 0 \), since \( \tilde{\mathcal{H}} \) converges to the holomorphic energy in the “noded” problem (of section 2.2, when \( M_1 = [1, \infty) \)). We recall that \( |\tilde{\phi}|_{M_0} = \zeta(x, l) \), where \( \zeta(x, l) \leq C_1 x^{-4} \) for \( x \in [a, \pi/2l] \) and \( \zeta(x, l) \leq C_1 (\pi/l - x)^{-4} \) for \( x \in [\pi/2l, b] \).

Consider the equation

\[
(l^{-2} \sin^2(lx)) Y'' - 2 Y = 0
\]  

(4.6)

with the boundary conditions that satisfy

\[
Y(l^{-1} \sin^{-1} l) = O(1) \text{ and } Y(\pi/l - l^{-1} \sin^{-1} l) = O(1), \text{ as } l \to 0.
\]

We claim that there exists some \( h = O(l) \) such that \( \tilde{\mathcal{H}} - h \) is a supersolution to (4.6) for \( x \in [l^{-1/4}, b - l^{-1/4}] \). To see this, we notice that \( 2|\tilde{\phi}_0|^2_{M_1} l^{-4} \sin^4(lx) \) decays rapidly as \( x \to \pi/2l \) for small \( l \). So there is a positive number \( h = O(l) \) such that \( 2|\tilde{\phi}_0|^2_{M_1} l^{-4} \sin^4(lx) < 2h \) for \( x \in [l^{-1/4}, b - l^{-1/4}] \).
Now for \( x \in [l^{-1/4}, b] \) we have

\[
(l^{-2}\sin^2(lx))(\ddot{\mathcal{H}} - h)'' - 2(\ddot{\mathcal{H}} - h) = 2h - 2|\phi_0|^2_{M_1} l^{-4}\sin^4(lx) > 0
\] (4.7)

Notice that for any constant \( \lambda \), we have that if \( Y(x) \) solves the equation (4.6), then so does \( \lambda Y(x) \). So up to multiplying by a bounded constant, we have \( Y|_{\partial M_1} > (\ddot{\mathcal{H}} - h)|_{\partial M_1} \). Hence \( \ddot{\mathcal{H}} - h \) is a subsolution to (4.6) for \( x \in [l^{-1/4}, b - l^{-1/4}] \). We can check the solutions to (4.6) have the form of

\[
Y(l; x) = B_3\text{cot}(lx) + B_4(1 - lx\text{cot}(lx))
\]

where constants \( B_3 = B_3(l) \) and \( B_4 = B_4(l) \) satisfy, from the boundary conditions for the equation (4.6), that

\[
B_3 = O(l) \text{ and } B_4 = O(l)
\]

and we also have

\[
\int_a^b B_3\text{cot}(lx)(l^{-2}\sin^2(lx))dx = O(l^{-2})
\]

\[
\int_a^b B_4(1 - lx\text{cot}(lx))(l^{-2}\sin^2(lx))dx = O(l^{-2})
\]

Therefore in \([l^{-1/4}, b - l^{-1/4}]\), we have \( \ddot{\mathcal{H}} \leq h + Y(x) \). Now,

\[
\int_{M_1} Y(x)|\dot{\mu}_1|^2 \sigma dxdy = \int_0^1 \int_a^b Y(x)(l^{-2}\sin^2(lx))dxdy
\]

\[
= \int_0^1 \int_a^b B_3\text{cot}(lx)(l^{-2}\sin^2(lx))dxdy
\]

\[
+ \int_0^1 \int_a^b B_4(1 - lx\text{cot}(lx))(l^{-2}\sin^2(lx))dxdy
\]

\[
= O(l^{-2}) + O(l^{-2})
\]

\[
= O(l^{-2})
\]
Also for \( x \in [a, l^{-1/4}] \cup [b - l^{-1/4}, b] \), we apply the maximum principle to equation (4.5) and find that

\[
\hat{H} \leq \max(l^{-4} \sin^4(lx_0) \zeta^2(x_0, l), \sup(\hat{H}|_{\partial([a, l^{-1/4}] \cup [b - l^{-1/4}, b] \times [0, 1]))
\]

for some \( x_0 \in [a, l^{-1/4}] \). Hence using the properties of \( \zeta(x, l) \) in \([a, b]\), a direct computation shows that

\[
\int_0^1 \int_a^{l^{-1/4}} \hat{H}(x)|\hat{\mu}_1|^2dA = O(l^{-3/4}) = o(l^{-1})
\]
\[
\int_0^1 \int_{b - l^{-1/4}}^{b} \hat{H}(x)|\hat{\mu}_1|^2dA = O(l^{-3/4}) = o(l^{-1})
\]

With this and using \( \hat{H} \leq h + Y(x) \) for \( x \in [l^{-1/4}, b - l^{-1/4}] \), we have

\[
\int_{M_0} D(|\hat{\mu}_0|^2)|\hat{\mu}_1|^2dA = \int_0^1 \int_a^b \hat{H}(x)|\hat{\mu}_1|^2dA = \int_0^1 \int_{l^{-1/4}} b \hat{H}(x)|\hat{\mu}_1|^2dA + \int_0^1 \int_a^{l^{-1/4}} \hat{H}(x)|\hat{\mu}_1|^2dA + \int_0^1 \int_{b - l^{-1/4}} b \hat{H}(x)|\hat{\mu}_1|^2dA \\
\leq \int_0^1 \int_a^b (Y(x) + h)|\hat{\mu}_1|^2\sigma dxdy + o(l^{-1}) = O(l^{-2}) + O(l^{-2}) + o(l^{-1}) = O(l^{-2}).
\]

Now combine the estimates of Lemma 3.2, (4.4), (4.8), we will have

\[
|R| \leq 4 \int \Sigma D(|\hat{\mu}_0|^2)|\hat{\mu}_1|^2\sigma dxdy = 4\left(\int_{M_0} D(|\hat{\mu}_0|^2)|\hat{\mu}_1|^2dA + \int_{M_1} D(|\hat{\mu}_0|^2)|\hat{\mu}_1|^2dA\right) = O(l^{-2})
\]
\[ |R|/\Gamma = O(t^{-2})l^3 = O(l) \]

namely, \( |R|/\Gamma \) tends to zero as we pinch the geodesics.

**Remark 4.10.** Essentially, in each pinching neighborhood \( M_i \), we used the cylinder map \( w_i \) instead of the harmonic map \( W_i \) during the computation, for \( i = 0, 1 \). In the next chapter, we will continue the proof of theorem 4.2 by showing that cylinder maps are actually not too far away from the restriction of the actual harmonic maps in the cylinders.
Chapter 5

Proof of the Main Results, General Case

5.1 Construction

Now we are in the general setting, i.e., the surface $\Sigma$ develops two nodes. In this section, we will construct a family of maps $G_i$ to approximate the harmonic map $W_i$, for $i = 0, 1$, and the essential parts of this family are the identity map of the surface restricted in the non-cylindrical part and the cylinder map in each pinching neighborhood $M_i$, for $i = 0, 1$. We will also show that this constructed family $G_i$ is reasonably close to the harmonic maps $W_i$, for $i = 0, 1$; hence we can use the estimates we obtained in the previous chapter to the general situation.

We recall some of the notations from previous chapter. We still set $M_0$ and $M_1$ to be the pinching neighborhoods of the nodes $p_0$ and $p_1$, respectively. Also $W_0(t)$ is the harmonic map corresponding to fixing $\gamma_1$ in $M_1$ having length $l$, and pinching $\gamma_0$
in $M_0$ into length $L = L(t)$, where $L(0) = l$. Similarly $W_1(t)$ is the harmonic map corresponding to fixing $\gamma_0$ in $M_0$ into length $l$, and pinching $\gamma_1$ in $M_1$ having length $L = L(t)$. Let $w_0(t)$ and $w_1(t)$ be cylinder maps in the model case, we want to show that $W_0(t)$ is close to $w_0(t)$ in $M_0$ and is close to identity map outside of $M_0$.

We denote subsets $\Sigma_0 = \{ p \in \Sigma : \text{dist}(p, \partial M_0) > 1 \}$, and $\Sigma_1 = \{ p \in \Sigma : \text{dist}(p, \partial M_1) > 1 \}$. Define the 1-tube of $\partial M_0$ as $B(\partial M_0, 1) = \{ p \in \Sigma : \text{dist}(p, \partial M_0) \leq 1 \}$, and the 1-tube of $\partial M_1$ as $B(\partial M_1, 1) = \{ p \in \Sigma : \text{dist}(p, \partial M_1) \leq 1 \}$. We can construct a $C^{2,\alpha}$ map $G_0 : \Sigma \to \Sigma$ such that

$$G_0(p) = \begin{cases} w_0(t)(p), & p \in M_0 \cap \Sigma_0 \\ p, & p \in (\Sigma_0 \setminus M_0) \\ g_t(p), & p \in B(\partial M_0, 1) \end{cases}$$

where $g_t(p)$ in $B(\partial M_0, 1)$ is constructed so that it satisfies:

1. $g_t(p) = p$ for $p \in \partial (\Sigma_0 \setminus (M_0 \cup B(\partial M_0, 1)))$, and $g_t(p) = w_0(t)(p)$ for $p \in \partial (M_0 \cap \Sigma_0);
2. g_t$ is the identity map when $t = 0$;
3. $g_t$ is smooth and the tension of $g_t$ is of the order $O(t)$.

We note that $G_0$ consists of 3 parts. It is the cylinder map of $M_0$ in $M_0 \cap \Sigma_0$, the identity map in $\Sigma_0 \setminus M_0$, and a smooth map in $B(\partial M_0, 1)$. Among 3 parts of the constructed map $G_0(t)$, two of them, the identity map and the cylinder map, are harmonic hence have zero tension; thus the tension of $G_0(t)$ is concentrated in $B(\partial M_0, 1)$. From section 2.2, for the cylinder map $w_0 = u(x) + iy$, we have $u' = \sqrt{1 + c_0(t)L^{-2} \sin^2 Lu}$, where $c_0(0) = 0, \dot{c}_0(0) = 4$. Hence for $x \in [l^{-1} \sin^{-1}(l), l^{-1} \sin^{-1}(l) + 1]$, 

$$w_{0,z}(x, y) = \frac{1}{2}(u'(x) + 1) = \frac{1}{2}((1 + O(1)c_0(t))^{\frac{1}{2}} + 1) = 1 + O(1)t + O(t^2)$$

$$|w_{0,z}(x, y) - 1| = O(t) \to 0, (t \to 0)$$
\[ |w_{0,z}(x,y)| = |\frac{1}{2} u''(x)| = O(|L \cot(Lu)(u'^2 - 1)|) = O(t) \]

Thus we can require that \(|g_{t,z}| \leq C_2 t\) and \(|g_{t,z,l}| \leq C_2 t\), and the constant \(C_2 = C_2(t, l)\) is bounded in both \(t\) and \(l\) since the coefficient of \(t\) for \(c_0(t)\) is bounded for small \(t\) and small \(l\). With the local formula of the tension in section 2.1, we have \(\tau(G_0(t))\), the tension of \(G_0(t)\), is of the order \(O(t)\).

Similarly, we have a \(C^{2,\alpha}\) map \(G_1 : \Sigma \to \Sigma\). Note that these constructed maps \(G_0\) and \(G_1\) are not necessarily harmonic.

### 5.2 Comparison of the Maps

In this section, we are about to compare the constructed family \(G_0(t)\) and the family of harmonic maps \(W_0(t)\). In order to do this, we will consider the following function \(Q_0 = \cosh(\text{dist}(W_0, G_0)) - 1\).

**Lemma 5.1.** \(\text{dist}(W_0, G_0) \leq C_3 t\) in \(B(\partial M_0, 1)\), where the constant \(C_3 = C_3(t, l)\) is bounded for small \(t\) and \(l\).

**Proof.** First, we want to show that \(Q_0\) is a \(C^2\) function. Notice that both the harmonic map \(W_0(t)\) and the constructed map \(G_0(t)\) are the identity map when \(t = 0\), and both families vary smoothly without changing homotopy type in \(t\) for sufficiently small \(|t|\) ([8]). For all \(l > 0\), and for any \(\varepsilon > 0\), there exists a \(\delta\) such that for \(|t| < \delta\), we have \(|W_0(t) - W_0(0)| < \frac{\varepsilon}{2}\) and \(|G_0(t) - G_0(0)| < \frac{\varepsilon}{2}\). Therefore the triangular inequality implies that \(|W_0(t) - G_0(t)| < \varepsilon\). Since \(l\) is positive, the Collar Theorem ([6]) implies that the surface has positive injectivity radius \(r\) bounded below, and we choose our \(\varepsilon << r\), so \(Q_0\) is well defined and smooth.
We follow an argument in [12]. For any unit \( v \in T^1(B(\partial M_0, 1)) \), the map \( G_0 \) satisfies the inequality \( ||dG_0(v)||\gamma_0 = O(t) \), hence \( |dG_0(v)|^2 > 1 - \varepsilon_0 \) where \( \varepsilon_0 = O(t) \), then we find that for any \( x \in \Sigma \),

\[
\Delta Q_0 \geq \min \{|dG_0(v)|^2 : dG_0(v) \perp \gamma_x\} Q \\
- \langle \tau(G_0), \nabla d(\bullet, W_0)|_{G_0(x)} \rangle > \sinh(d(W_0, G_0))
\]  

(5.1)

where \( \gamma_x \) is the geodesic joining \( G_0(x) \) to \( W_0(x) \) with initial tangent vector \(- \nabla d(\bullet, W_0)|_{G_0(x)} \) and terminal tangent vector \( \nabla d(G_0(x), \bullet)|_{W_0(x)} \).

If \( G_0(t) \) does not coincide with \( W_0(t) \) on \( B(\partial M_0, 1) \), we must have all maxima of \( Q_0(t) \) on the interior of \( B(\partial M_0, 1) \), at any such maximum, we apply the inequality \( |dG_0(v)|^2 > 1 - \varepsilon_0 \) to (5.1) to find

\[
0 \geq \Delta Q_0 \geq (1 - \varepsilon_0)Q_0 - \tau(G_0)(\sinh(d(W_0, G_0))
\]

so that at a maximum of \( Q_0 \), we have

\[
Q_0 \leq \frac{\tau(G_0)\sinh d(W_0, G_0)}{(1 - \varepsilon_0)}
\]

We notice that \( Q_0 \) is of the order \( \text{dist}^2(W_0, G_0) \) and \( \sinh \text{dist}(W_0, G_0) \) is of the order \( \text{dist}(W_0, G_0) \), this implies that \( \text{dist}(W_0, G_0) \) is of the order \( O(t) \) in \( B(\partial M_0, 1) \), which completes the proof of this lemma.

\[ \square \]

**Remark 5.2.** Lemma 5.1 implies that \( Q_0(t) \) is of the order \( O(t^2) \) in \( B(\partial M_0, 1) \). Similarly, we can consider the function \( Q_1 = \cosh(\text{dist}(W_1, G_1)) - 1 \). A similar argument gives us \( Q_1(t) \) is of the order \( O(t^2) \) in \( B(\partial M_1, 1) \), and \( \text{dist}(W_1, G_1) \leq C_3't \) in \( B(\partial M_1, 1) \), where the constant \( C_3' = C_3'(t, l) \) is bounded for small \( t \) and \( l \).
Note that $B(\partial M_0, 1)$ contains the boundary of the cylinder $M_0 \cap \Sigma_0$, which we identify with $[a+1, b-1] \times [0, 1]$, where, again, $a = a(l) = l^{-1} \sin^{-1}(l)$, and $b = b(l) = \pi l^{-1} - l^{-1} \sin^{-1}(l)$. While in the cylinder $M_0 \cap \Sigma_0$, we have the inequality

$$\Delta Q_0 \geq (1 - \varepsilon_0)Q_0 - \tau(G_0)(\sinh(dist(W_0, G_0)))$$

$$= (1 - \varepsilon_0)Q_0 - \tau(G_0)(\tanh(dist(W_0, G_0)))(1 + Q_0)$$

$$= (1 - \varepsilon_0 - \tau(G_0))Q_0 - \tau(G_0)(\tanh(dist(W_0, G_0)))$$

$$\geq 1/2Q_0 - C_4t^2$$

where the constant $C_4$ is bounded for small $t$ and $l$. Therefore we find that $Q_0(z, t)$ decays rapidly in $z = (x, y)$ for $x$ close enough to $\pi/2l$. Hence we can assume that $dist(W_0, G_0)$ is at most of order $C't$ in $[a + 1, b - 1]$, here $C' = C'(x, l)$ is no greater that $C_5x^{-2}$ for $x \in [a + 1, \pi/2l]$, and no greater than $C_5(\pi/l - x)^{-2}$ for $x \in [\pi/2l, b - 1]$, where $C_5$ is bounded for small $t$ and $l$. Both maps $W_0$ and $G_0$ are harmonic in $M_0 \cap \Sigma_0$, so they are also $C^1$ close ([8]), i.e. we have $|W_{0, z} - G_{0, z}| \leq C_5x^{-2}t$ for small $t$ and $l$, when $x \in [a + 1, \pi/2l]$. Thus we see that $|\dot{W}_{0, z} - \dot{G}_{0, z}| = C_5x^{-2}$, for $x \in [a + 1, \pi/2l]$. Also, $|\ddot{W}_{0, z} - \ddot{G}_{0, z}| = C_5(\pi/l - x)^{-2}$, for $x \in [\pi/2l, b - 1]$.

As before we denote $\phi_0$ and $\phi_1$ as families of Hopf differentials corresponding to families of harmonic maps $W_0(t)$ and $W_1(t)$, respectively. We also denote $\mu_0$ and $\mu_1$ as the corresponding families of Beltrami differentials in $M_0 \cap \Sigma_0$ and $M_1 \cap \Sigma_1$, respectively. Write $\phi_{G_0} = \rho G_{0, z}(t)\dot{G}_{0, z}(t)$ and $\phi_{G_1} = \rho G_{1, z}(t)\dot{G}_{1, z}(t)$.

**Lemma 5.3.** $|\dot{\phi}_{G_0} - \dot{\phi}_{W_0}| = O(1)$ for $x \in [a + 1, b - 1]$.

**Proof.** Notice that in $M_0 \cap \Sigma_0$, map $G_0$ is the cylinder map hence harmonic, so $\phi_{G_0}$ is the Hopf differential corresponding to $G_0$. When $t = 0$ we have $W_0 = G_0 = \text{identity}$
and $\rho = \sigma$, hence we can differentiate $\phi_G = \rho G_{0,t}(t)\bar{G}_{0,t}(t)$ in $t$ at $t = 0$, and find that
\[
|\dot{\phi}_G - \dot{\phi}_W| = \sigma |\dot{W}_0 - \dot{G}_0| \leq C_5 t^2 x^{-2} \csc^2(lx) = O(1) \text{ for } x \in [a + 1, \pi/2l],
\]
\[
|\dot{\phi}_G - \dot{\phi}_W| \leq C_5 t^2 (\pi/l - x)^{-2} \csc^2(lx) = O(1) \text{ for } x \in [\pi/2l, b - 1].
\]

\[\square\]

**Remark 5.4.** Similarly we have $|\dot{\phi}_0 - \dot{\phi}_G|_{M_t \cap \Sigma_1} = O(1)$, and $|\dot{\phi}_1 - \dot{\phi}_G|_{M_t \cap \Sigma_0} = O(1)$, also $|\dot{\phi}_1 - \dot{\phi}_G|_{M_t \cap \Sigma_1} = O(1)$.

### 5.3 General Situation

In last section, we constructed families of maps and found that they are reasonably close to the corresponding families of harmonic maps between Riemann surfaces. Now we are ready to adapt the estimates in the model case to the general setting, and prove theorem 4.2 in this situation, which will imply our Main theorem.

Recall from section 3.2 that the sectional curvature is $R/\Gamma$. It is easy to see that Lemma 3.2 still holds. In other words, we still have that $|R| \leq 4 \int_{\Sigma} D(\mu_0^2) |\mu_1|^2 \sigma dx dy$.

We proved Lemma 4.5 in the model case, now we need a similar estimate:

**Lemma 5.5.** $1/\Gamma = O(1^3)$, in other words, Lemma 4.5 still holds.

**Proof.** We recall, from section 3.2, that
\[
\Gamma = 4 < \hat{\mu}_0, \hat{\mu}_0 > < \hat{\mu}_1, \hat{\mu}_1 > - 4| < \hat{\mu}_0, \hat{\mu}_1 >|^2
\]

Recall from the proof of Lemma 4.5, we have
\[ < \hat{\mu}_0^G, \hat{\mu}_0^G > |_{M_0} = \int_{M_0} |\hat{\mu}_0^G|^2 \sigma dx dy = (\pi/2 - \sin^{-1}l)l^{-3} \sim l^{-3} \]
\[ < \hat{\mu}_1^G, \hat{\mu}_1^G > |_{M_0} = \int_{M_0} |\hat{\mu}_1^G|^2 \sigma dx dy = O(1) \]
\[ < \hat{\mu}_0^G, \hat{\mu}_1^G > |_{M_0} = O(1) \]

Also from the proof of Lemma 5.3, we have, in \( M_0 \),
\[ |\dot{\phi}_0 - \dot{\phi}_0^G| \leq C_5 l^2 x^{-2} \csc^2 (lx) \text{ for } x \in [a + 1, \pi/2l], \]
\[ |\dot{\phi}_0 - \dot{\phi}_0^G| \leq C_5 l^2 (\pi/l - x)^{-2} \csc^2 (lx) \text{ for } x \in [\pi/2l, b - 1]. \]

Therefore we find that
\[
\int_{M_0} |\hat{\mu}_0 - \hat{\mu}_0^G|^2 \sigma dx dy = \int_{M_0} \frac{|\dot{\phi}_0 - \dot{\phi}_0^G|^2}{\sigma} dx dy
\]
\[
= \int_0^1 \int_{\pi/2l}^{a+1} \frac{|\dot{\phi}_0 - \dot{\phi}_0^G|^2}{\sigma} dx dy + \int_0^1 \int_{\pi/2l}^{\pi/2l} \frac{|\dot{\phi}_0 - \dot{\phi}_0^G|^2}{\sigma} dx dy
\]
\[
+ \int_0^1 \int_{\pi/2l}^{b-1} \frac{|\dot{\phi}_0 - \dot{\phi}_0^G|^2}{\sigma} dx dy + \int_0^1 \int_{\pi/2l}^{b-1} \frac{|\dot{\phi}_0 - \dot{\phi}_0^G|^2}{\sigma} dx dy
\]
\[
= O(1) + O(1) + O(1) + O(1)
\]
\[
= O(1)
\]

Now from the triangle inequality, we have
\[ (|\hat{\mu}_0^G| - |\hat{\mu}_0 - \hat{\mu}_0^G|)^2 \leq |\hat{\mu}_0|^2 \leq 2|\hat{\mu}_0^G|^2 + 2|\hat{\mu}_0 - \hat{\mu}_0^G|^2 \]

Thus,
\[
< |\hat{\mu}_0|, |\hat{\mu}_0| > |_{M_0} = \int_{M_0} |\hat{\mu}_0|^2 \sigma dx dy
\]
\[
\leq 2 \int_{M_0} |\hat{\mu}_0^G|^2 \sigma dx dy + 2 \int_{M_0} |\hat{\mu}_0 - \hat{\mu}_0^G|^2 \sigma dx dy
\]
\[
\leq (\pi - 2\sin^{-1}l)l^{-3} + O(1)
\]
\[
= O(l^{-3})
\]
\[
<|\hat{\mu}_0|, |\hat{\mu}_0|> |_{M_0} = \int_{M_0} |\hat{\mu}_0|^2 \sigma dx dy \\
\geq \int_{M_0} (|\hat{\mu}_0^G| - |\hat{\mu}_0 - \hat{\mu}_0^G|)^2 \sigma dx dy \\
\geq (\pi/2 - sin^{-1} l) l^{-3} + O(1)
\]

Hence \(<|\hat{\mu}_0|, |\hat{\mu}_0|> |_{M_0} \sim l^{-3}\), and \(1/\Gamma = O(l^{-3})\).

\[\square\]

**Remark 5.6.** With this lemma, we have, again, as in last chapter, \(|R|/\Gamma = O(l^3)|R|\), so it will be sufficient to show that \(|R| = o(l^{-3})\). so we will show that Lemma 4.7 still holds, which implies desired curvature estimate immediately.

Now we are about to estimate \(\int_{\Sigma} D(|\hat{\mu}_0|^2)|\hat{\mu}_1|^2 \sigma dx dy\), which breaks into 3 integrals as follows:

\[
\int_{\Sigma} D(|\hat{\mu}_0|^2)|\hat{\mu}_1|^2 \sigma dx dy = \int_{M_0 \cap \Sigma_0} D(|\hat{\mu}_0|^2)|\hat{\mu}_1|^2 \sigma dx dy \\
+ \int_{M_1 \cap \Sigma_1} D(|\hat{\mu}_0|^2)|\hat{\mu}_1|^2 \sigma dx dy \\
+ \int_{K} D(|\hat{\mu}_0|^2)|\hat{\mu}_1|^2 \sigma dx dy \tag{5.2}
\]

where \(K\) is the compact set disjoint from \((M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1)\).

For the third integral, from the previous discussion, because of the convergence of the harmonic maps to the harmonic maps of “noded” problem, we have both \(|\hat{\mu}_0|\) and \(|\hat{\mu}_1|\) are bounded. The maximum principle implies that \(D(|\hat{\mu}_0|^2) = \tilde{H} \leq sup\{|\hat{\mu}_0|^2\}\). Note that \(K\) is compact, hence we have the third integral is the order of \(O(1)\).
From Lemma 2.5, we have \((\Delta - 2)\tilde{H} = -2\frac{\dot{\phi}_0^2}{\sigma^2}\), so we can rewrite (5.2) as

\[
\int_{\Sigma} D(|\mu_0|^2)|\dot{\mu}_1|^2\sigma dxdy = \int_{M_0 \cap \Sigma_0} \tilde{H}|\dot{\mu}_1|^2\sigma dxdy + \int_{M_1 \cap \Sigma_1} \tilde{H}|\dot{\mu}_1|^2\sigma dxdy + O(1) \tag{5.3}
\]

Now we will look at the first and the second integrals in (5.3). Recall that \((\Delta - 2)\tilde{H}^G = -2\frac{\dot{\phi}_0^{G^2}}{\sigma^2}\), where \(\tilde{H}^G\) is the holomorphic energy corresponding to the model case when the harmonic map is the cylinder map, and \(\phi_0^G\) is the quadratic differential corresponding to the constructed map \(G_0\). We also denote \(\mu_0^G\) to be the Beltrami differential corresponding to \(\phi_0^G\). We assign similar meanings for \(\phi_1^G\) and \(\mu_1^G\).

Also recall that \(|\dot{\phi}_0 - \dot{\phi}_0^G| = O(1)\) and \(|\dot{\phi}_1 - \dot{\phi}_1^G| = O(1)\) in \((M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1)\), here \(O(1)\) is bounded in \(l\) for small \(l\). So we can set some \(\lambda = O(1)\) (bounded in \(l\) for small \(l\)) such that \(|\dot{\phi}_0|^2 < \lambda^2|\dot{\phi}_0^G|^2\) and at the boundary of \((M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1)\) satisfies \(\tilde{H} < \lambda \tilde{H}^G\). For example, we can take \(\lambda = 1 + \max_{\partial K} (\frac{\tilde{H}}{\tilde{H}^G}, \frac{|\dot{\phi}_0|}{|\dot{\phi}_0^G|})\), this \(\lambda = O(1)\) because at \(\partial K = \partial((M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1))\), both \(\tilde{H}, \tilde{H}^G\), and \(\frac{|\dot{\phi}_0|}{|\dot{\phi}_0^G|}\) are bounded. Therefore,

\[
(\Delta - 2)(\tilde{H} - \lambda \tilde{H}^G) = 2\frac{\lambda^2|\dot{\phi}_0^G|^2 - |\dot{\phi}_0|^2}{\sigma^2} > 0
\]

So \((\tilde{H} - \lambda \tilde{H}^G)\) is a subsolution to the equation \((\Delta - 2)Y = 0\), as in section 4.4, whose solutions have the form of \(Y(l; x) = B_3\cot(lx) + B_4(1 - lx\cot(lx))\), where constants \(B_3\) and \(B_4\) satisfy that \(B_3 = O(l)\) and \(B_4 = O(l)\).

Hence in \((M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1)\), we have \(\tilde{H} \leq \lambda \tilde{H}^G + B_3\cot(lx) + B_4(1 - lx\cot(lx))\).
We apply this into (5.3) and find that,

\[
\int_{\Sigma} D(|\mu_0|^2)|\mu_1|^2 \sigma dxdy = \int_{(M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1)} \mathcal{H}_l|\mu_1|^2 \sigma dxdy + O(1)
\]

\[
\leq \int_{M_0 \cap \Sigma_0} (\lambda \mathcal{H}_l^G + Y(l; x))(|\mu_1|^2) \sigma dxdy
\]

\[
+ \int_{(M_1 \cap \Sigma_1)} (\lambda \mathcal{H}_l^G + Y(l; x))(|\mu_1|^2) \sigma dxdy + O(1)
\]

\[
\leq 2 \int_{M_0 \cap \Sigma_0} (\lambda \mathcal{H}_l^G + Y(l; x))(|\mu_1|^2) \sigma dxdy + |\mu_1 - \mu_1^G|^2 dA
\]

\[
+ 2 \int_{M_1 \cap \Sigma_1} (\lambda \mathcal{H}_l^G + Y(l; x))(|\mu_1|^2) \sigma dxdy + |\mu_1 - \mu_1^G|^2 dA + O(1)
\]

\[
\leq 2\lambda \int_{\Sigma} \mathcal{H}_l^G |\mu_1|^2 dA + 2 \int_{M_0} Y(l, x)|\mu_1|^2 dA
\]

\[
+ 2 \int_{M_1} Y(l, x)|\mu_1|^2 dA + 2\lambda \int_{(M_0 \cap \Sigma_0)} \mathcal{H}_l^G |\mu_1 - \mu_1^G|^2 dA
\]

\[
+ 2\lambda \int_{(M_1 \cap \Sigma_1)} \mathcal{H}_l^G |\mu_1 - \mu_1^G|^2 dA + O(1)
\]

Lemma 4.7 indicates that

\[
\int_{\Sigma} \mathcal{H}_l^G |\mu_1|^2 dA = O(l^{-2}).
\]

Recalling the computation in section 4.3 and section 4.4, we have also,

\[
\int_{M_0} Y(l, x)|\mu_1|^2 dA = O(1),
\]

\[
\int_{M_1} Y(l, x)|\mu_1|^2 dA = O(l^{-2}).
\]

From the proof of Lemma 5.3, we have \(|\mu_1 - \mu_1^G| = |\phi_1 - \phi_l^G|/\sigma\), where \(|\phi_1 - \phi_l^G| \leq C_5 l^2 x^{-2} csc^2(lx)\) for \(x \in [a + 1, \pi/2l]\), and \(|\phi_0 - \phi_l^G| \leq C_5 l^2 (\pi/2l - x) csc^2(lx)\) for \(x \in [\pi/2l, b - 1]\). Then a straight computation gives the following:

\[
\int_{M_0 \cap \Sigma_0} (\mathcal{H}_l^G + Y(l; x))(|\mu_1 - \mu_1^G|^2) dA = o(l^{-2})
\]

\[
\int_{M_1 \cap \Sigma_1} (\mathcal{H}_l^G + Y(l; x))(|\mu_1 - \mu_1^G|^2) dA = o(l^{-2})
\]
Therefore $\int_{\Sigma} D(\mu_0, \mu_1)^2 \sigma dxdy = O(l^{-2})$, in other words, Lemma 4.7 still holds. With Lemma 3.2, we have $|R| = O(l^{-2})$. Apply this into the curvature formula, we have $|R|/\Gamma = O(l) \to 0$ as $l \to 0$, which completes the proof of theorem 4.3, hence completes the proof of our main theorem.

5.4 Remark on Punctured Surfaces

For the case of punctures surfaces, the existence of a harmonic diffeomorphism between punctured surfaces has been investigated by Wolf ([27]) and Lohkamp ([15]). In particular, Lohkamp ([15]) showed that a homeomorphism between punctured surfaces is homotopic to a unique harmonic diffeomorphism with finite energy, and the holomorphic quadratic differential corresponding to the harmonic map in the homotopy class of the identity is a bijection between Teichmüller space of punctured surfaces and the space of holomorphic quadratic differentials.

In this case, the set $\Sigma \setminus ((M_0 \cap \Sigma_0) \cup (M_1 \cap \Sigma_1))$ is no longer compact. Let $K_0$ be a compact surface with finitely many punctures, and $\{K_m\}$ be a compact exhaustion of $K_0$. We now estimate $\int_{K_0} D(\mu_0, \mu_1)^2 dA$. Let $H(t)$ be the holomorphic energy corresponding to the harmonic map $\omega(t) : (K_0, \sigma) \to (K_0, \rho(t))$, then $H(t)$ is bounded above and below, and has nodal limit 1 near the punctures ([27]), hence both $|\mu_0|^2$ and $|\mu_1|^2$ have the order $o(1)$ near the punctures. To see this, we consider $K_0$ as the union of $K_m$ and disjoint union of finitely many punctured disks, each equipped hyperbolic metric $d\bar{z}^2 / z^2 \log^2 z$. Then $|\mu_0| = O(|z| \log^2 z) \to 0$ as $z$ tends to the puncture, since the quadratic differential has a pole of at most the first order. A similar result
holds for $|\dot{\mu}_1|$. We notice that $\partial K_0$ is the boundary of the cylinders, where the harmonic maps converge to a solution to the "noded" problem as $\ell \to 0$, hence $D(|\dot{\mu}_0|^2) = \ddot{H}(t)$ is bounded on $\partial K_0$. Therefore we apply Omori-Yau maximum principle ([19] [32]) to $(\Delta - 2)\ddot{H} = -2|\dot{\mu}_0|^2$ on $K_0$ and obtain that $\sup(D(|\dot{\mu}_0|^2)) \leq \max(\sup(|\dot{\mu}_0|^2), \max(D(|\dot{\mu}_0|^2))|_{\partial K_0}) = O(1)$. Hence we have

$$\int_{K_0} D(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA \leq \int_{K_0} \sup(|\dot{\mu}_0|^2)|\dot{\mu}_1|^2 dA$$

$$\leq O(1)O(1)Vol(K_0)$$

$$= O(1)$$

In other words, our proof carries over to the punctured case, which completes the proof of our Main theorem.
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