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Strong S-equivalence of Ordered Links

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Abstract

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Recently Swatee Naik and Theodore Stanford proved that two S-equivalent knots are related by a finite sequence of doubled-delta moves on their knot diagrams. We show that classical S-equivalence is not sufficient to extend their result to ordered links. We define a new algebraic relation on Seifert matrices, called Strong S-equivalence, that better respects the boundary components of a link’s Seifert surface, and we prove that two oriented, ordered links $L$ and $L'$ are related by a sequence of doubled-delta moves if and only if they are Strongly S-equivalent. We also show that this is equivalent to the fact that $L'$ can be obtained from $L$ through a sequence of Y-clasper surgeries, where each clasper leaf has total linking number zero with $L$. 
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Chapter 1

Introduction

1.1 Background

A fundamental goal of knot theory is the classification of knots and links by type: When are two knots or links the same and when are they different? In the absence of a rigorous method for a complete classification, many attempts have been made to sort knots and links into families, or equivalence classes, using both geometric and algebraic equivalence relations. Often the two approaches collide. The goal of this paper is to examine the similarities and reconcile some differences between one such geometric relation, the doubled-delta move, and an algebraic relation, S-equivalence.

The doubled-delta move is a move on knot or link diagrams, depicted in figure 2.3. S-equivalence is an equivalence relation on matrices, and we say that two knots or links are S-equivalent if they have S-equivalent Seifert matrices. In 1999, S. Naik and T. Stanford proved a correspondence between the doubled-delta move and S-equivalence for knots:
Theorem 1.1 (Naik-Stanford) Two knots are $S$-equivalent if and only if they are equivalent by a sequence of doubled-delta moves.

In this thesis, we demonstrate that Theorem 2.2 is false for links, and we introduce definitions to state the hypotheses necessary to prove an analogous statement for links.

1.2 Results

In order to answer the question, "When are two links related by a sequence of doubled-delta moves?" we will define the notion of an ordered Seifert matrix and a condition called Strong $S$-equivalence that will allow us to prove for ordered links an analogous theorem to that of S. Naik and T. Stanford, as well as a correspondence to $Y$-clasper surgery:

Theorem 7.1 Consider two oriented, ordered $m$-component links $L_0$ and $L_1$. The following four statements are equivalent:

i. $L_1$ can be obtained from $L_0$ through a sequence of doubled-delta moves.

ii. $L_0$ and $L_1$ are related by a sequence of $Y$-clasper surgeries, where each leaf of each clasper has total linking number zero with the link.

iii. $L_0$ and $L_1$ are Strongly $S$-equivalent.

iv. For some choice of Seifert Surfaces $\Sigma_0$ and $\Sigma_1$ and bases of $H_1(\Sigma_i)$, $L_0$ and $L_1$ have identical ordered Seifert Matrices.
1.3 Layout of Thesis

Two geometric moves on knot and link diagrams are introduced in chapter 2, the delta move and the doubled-delta move. Also discussed is the notion of Borromean surgery on a knot or link. Borromean surgery is a special case of clasper surgery, which is defined and described in detail in chapter 3. The algebraic relation of S-equivalence is presented in chapter 4, following a discussion of Seifert surfaces and the Seifert matrix. Connections between the geometric moves and the algebraic relations are made in chapter 5, where we prove that Naik-Stanford's Theorem 1.1 is false for links. Stronger hypotheses are needed for an analogous theorem for links, and this stronger version of S-equivalence is defined and discussed in chapter 6. Finally, the main theorem is stated and proved in chapter 7.
Chapter 2

Moves on Link Diagrams

A link $L$ with $m$-components is a subset of $S^3$ (or of $\mathbb{R}^3$), that consists of $m$ disjoint, piecewise linear, simple closed curves. A link with one component is a knot. The components of a link may be intertwined in a non-trivial way, and it is often helpful to quantify the amount of linking between components.

**Definition 2.1** The linking number $lk(J,K)$ between oriented components $J$ and $K$ is the total number of times $J$ crosses over $K$, counted with sign as below:

![Crossing Diagram](image)

Figure 2.1: Positive and Negative Crossings

To compute $lk(J,K)$, we first need to choose an orientation for each component of the link. Then we look at only the crossings where $J$ passes over $K$, adding one
for each positive crossing and subtracting one for each negative crossing. It is easy to verify that \( \text{lk}(J, K) = \text{lk}(K, J) \) and that \( \text{lk}(J, K) = -\text{lk}(J, rK) \), where \( rK \) is \( K \) taken with the opposite orientation.

### 2.1 The Delta Move

Before defining the doubled-delta move, we look first at the simpler (single) delta move. The *delta move* shown in Figure 2.2 is a particular move on knot or link diagrams. Given a link containing the tangle in Figure 2.2a, replace the tangle with

![Figure 2.2: The Delta Move](image)

that of Figure 2.2b in such a way that respects the free ends. In the case of a link, the strands depicted do not directly correspond to particular link components. This move is not a planar isotopy; even though it looks similar to the third Reidemeister move, it *does* change the knot or link type.

It is easy to see that the delta move preserves the pairwise linking numbers of any components it affects, since for each pair of strands, the move changes neither the crossing nor the orientation. Though somewhat less obvious, the converse is also true:
Theorem 2.2 (Murakami-Nakanishi) Two links have the same sets of pairwise linking numbers if and only if they are equivalent under delta moves.

Since a knot is a one-component link, it follows that any two knots are equivalent under delta moves, since their sets of pairwise linking numbers are empty.

2.2 The Doubled-Delta Move

The doubled-delta move is similar to the single delta move, with each of the three strands being replaced by a pair of oppositely oriented strands. Again, the link

![Diagram](image)

Figure 2.3: The Doubled-Delta Move

components that comprise the six affected strands are irrelevant. The doubled-delta move is more restrictive than the (single) delta move, but it still preserves the pairwise linking numbers between any affected link components.

2.3 Borromean Surgery

The delta move and doubled-delta move categorize knots and links into equivalence classes. Two links are said to be in the same class if they are related by a sequence of such moves. However, unlike many of the basic moves that change a knot or link
diagram, the delta and doubled-delta move are closely related to other operations used by topologists in a variety of contexts. In Figure 2.4, we see that the effect of the delta move is the same as "adding in" Borromean rings. Another name for

\[ \begin{array}{c}
\begin{array}{c}
\text{Figure 2.4: The Delta-Move is Borromean Surgery}
\end{array}
\end{array} \]

the operation depicted in the figure is Borromean surgery. We will see in Section 3.2 how Borromean surgery relates to claspers, and thus to grope cobordisms, finite-type invariants, and other ideas of low-dimensional topology.
Chapter 3

Clasper Surgery

Under what conditions are two links related by a sequence of doubled-delta moves? This question is interesting to many topologists in the context of claspers, a special case of which is the Borromean surgery described in Section 2.3. Claspers were first defined by K. Habiro [H], where they arose in the context of finite-type invariants. Habiro demonstrated how the theory of claspers can provide an alternative calculus under which one can study finite-type invariants of knots and 3-manifolds. They also were implicit in the work of Goussarov [Gu1][Gu2]. Today, claspers are studied across various fields of low-dimensional topology. In addition to applications of finite-type invariants, P. Teichner and J. Conant examine their relationships to groppe cobordisms [CT], and S. Garoufalidis uses clasper surgery [GL] to better understand the Kontsevich integral and concordance classes of knots.

3.1 Claspers

A clasper is a compact surface constructed from the following three types of pieces:
• *edges*, or bands that connect the other two types of pieces

• *nodes*, or disks with three incident edges

• *leaves*, or annuli with one incident edge.

The annuli that comprise the leaves may be twisted with any number of full twists. We call this number the *framing* of the leaf. Figure 3.1 shows a clasper with zero-framed leaves.

![Diagram showing a Y-clasper with associated graph(s) and link.]

Figure 3.1: A Y-clasper, with its associated graph(s) and link.

A clasper $C$ collapses to a graph $\Gamma_C$ with two types of trivalent vertices, those that come from nodes and those that come from leaves. We commonly ignore the leaves and only consider the node vertices to be trivalent. The leaf vertices are then univalent, and the graph is called unitrivalent. The only clasper information lost in the graph $\Gamma_C$ is the twisting of the leaves; to remedy this we assign an integer framing to each leaf of the graph.
3.2 Clasper Surgery

Assume that a clasper $C$ is embedded in a 3-manifold $M$. To do surgery on the clasper means to remove a handlebody neighborhood of the trivalent graph $\Gamma_C$ from a 3-manifold $M^3$, and glue it back in a prescribed way according to the clasper and its framing. In particular, we associate a link $L_C$ to the clasper $C$ as in Figure 3.1 using the following substitutions: each node of the clasper is replaced by a copy of the positive, zero-framed Borromean rings, and each edge is replaced by a positive Hopf link. Leaves of the clasper do not contribute additional link components, but the framing of a leaf does determine the framing of the Hopf link component corresponding to that leaf's adjacent edge. Clasper surgery, then, is really integer surgery on the associated framed link.

An important property of clasper surgery is that it preserves the homology of the affected manifold. When each leaf of the clasper "clasps" a knot or link and at least one leaf bounds a disk in $M^3 - \text{clasper}$, the result of the surgery is a new knot or link in the same 3-manifold. The handle slide move from Kirby calculus is useful in understanding the effect of clasper surgery on a knot.

Surgery on the simplest clasper—the strut, with a single edge and two leaves—results in a single crossing change in the original knot. Since the basic crossing change is an unknotted relation, all knots are related by strut-clasper surgery. The simplest interesting clasper, then, is the $Y$-clasper. One may even argue that the $Y$-clasper is the most interesting clasper, since any larger clasper may be realized as several $Y$-claspers by expanding edges of the larger clasper into Hopf-linked pairs of leaves [H].
3.3 Null Claspers

In [GR], S. Garoufalidis and L. Rozansky discuss null claspers, those whose leaves are null-homologous links in $M - K$ for a 3-manifold $M$ and knot $K$. Such leaves are sent to zero under the map $\pi_1(M - K) \to H_1(M - K) \cong \mathbb{Z}$; in other words, each leaf has algebraic linking number zero with the knot. There they explain that null claspers have been used to explain a rational version of the Kontsevich integral, and can be used to define a notion of finite-type invariants. In Lemma 1.3 of [GR], they show that surgery on null claspers preserves not only the homology of a knot complement, but also the Alexander module and Blanchfield linking form. Furthermore, null clasper surgery describes a move on the set of knots in integral homology spheres that directly corresponds to the doubled-delta move.

For the pair $(M, L)$, where $M$ is a 3-manifold and $L$ is a link, a null clasper would be one whose leaves each have linking number zero with each component. Unfortunately, the doubled-delta move for links acts on strands chosen independently of link components. Therefore, in considering the relationship between claspers and the doubled-delta move, our interest is with a slightly larger class of claspers: those in which each leaf clasps several strands of the link in such a way that the total linking number with all link components is zero.

We claim that when the leaves of a zero-framed $Y$-clasper each have total linking number zero with the link, the $Y$-clasper surgery has the same effect on the link as a finite sequence of doubled-delta moves, as well as that of Borromean surgery. This is one of the implications of our main theorem, and is illustrated in Figure 7.1.
Chapter 4

The Seifert Matrix

Many of the basic algebraic invariants of knot theory, including the Alexander module, the Conway polynomial, and the signature, depend on the Seifert matrix of a knot or link, which is readily computable but not uniquely defined.

For every $m$-component link $L$ in $S^3$, there is a Seifert surface $\Sigma$ associated to the link, where $\Sigma$ is a connected, oriented, embedded surface with the components of $L$ as its boundary. Given a basis $\{b_i\}$ of $H_1(\Sigma)$, we can associate a Seifert matrix $M$ to the link $L$, where the entries of $M$ are defined from the linking number of two basis elements. In particular, $M_{i,j} = lk(b_i, b_j^+)$, where $b_j^+$ is a pushoff of $b_j$ in the positive normal direction.

4.1 Disk-Band Form for a Seifert Surface

Diverging momentarily, it will often be useful to look at the Seifert surface in disk-band form. Any Seifert surface for a knot is homeomorphic to a disk with $2g$ bands attached; more importantly, we can always isotope our surface to one in disk-band
form as in Figure 4.1, where the $2g$ bands are interlaced in pairs. Within the dotted

![String link on bands](image)

Figure 4.1: Disk-Band Form for a Knot’s Seifert Surface

box, the bands of our surface may be knotted, twisted, or intertwined as long as the bands entering the top of the box match up with those that leave the bottom of the box. We call this a string link on the bands. A string link is similar to a pure braid, except that unlike a braid, the strands do not have to be steadily descending. Note that, because of the interlacing of the bands, the surface in Figure 4.1 has exactly one boundary component. An advantage of viewing the Seifert surface $\Sigma$ in this way is that there is an obvious choice for a symplectic basis $\{b_i\}$ of $H_1(\Sigma)$, i.e. one whose intersection form $V_{i,j} = \langle b_i, b_j \rangle$ is the block sum $\bigoplus_{j=1}^{2g} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Like knots, links also have Seifert surfaces that can be isotoped into disk-band form, with $2g$ interlaced bands and $m - 1$ non-interlaced bands as in Figure 4.2. There is still a natural choice of basis corresponding to the bands, but, unlike the Seifert surface for a knot, the basis for this Seifert surface is not symplectic. We coin the term semi-symplectic to refer to this natural choice of basis. The $2g$ interlaced bands yield basis elements with the properties of a symplectic basis, and the $m - 1$
non-interlacing bands yield basis elements that are pushoffs of the respective boundary components and have no intersection with any other basis elements.

4.2 Classical S-equivalence

The Seifert matrix of a knot or link may vary for two primary reasons. First, the Seifert surface $\Sigma$ of $L$ is not uniquely defined, since for example, we can always add genus to the surface by attaching 1-handles, thereby increasing the dimension of the basis of $H_1(\Sigma)$ by two and hence increasing the size of the Seifert matrix. Second, a different choice of basis $\{b_i\}$ of $H_1(\Sigma)$ would lead to a completely different matrix. To resolve these deficiencies, we often consider Seifert matrices up to S-equivalence.

S-equivalence is a notion that has been widely considered for both knots and links [Go] [K] [Li]. Two square integral matrices $M$ and $N$ are said to be $S$-equivalent if $M$ can be transformed into $N$ by a finite sequence of integral congruences (that is, $M = A^tNA$ for some integral matrix $A$ with $\det(A) = \pm 1$) and row or column enlargements of the form.
\[ N = \begin{pmatrix} M & \overrightarrow{y^t} & 0 \\ \overrightarrow{x} & z & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad N = \begin{pmatrix} M & \overrightarrow{y^t} & 0 \\ \overrightarrow{x} & z & \overrightarrow{y} \\ 0 & 0 & 0 \end{pmatrix}. \]

The congruences resolve the differences in Seifert matrices that arise from ambient isotopies of the Seifert surface \( \Sigma \) and the choice of basis for \( H_1(\Sigma) \). The row and column enlargements compensate for any 1-handle additions on the Seifert surface by enlarging the Seifert matrix by 2 rows and columns at a time. This yields a new square matrix that corresponds to a genus \( g + 1 \) surface, where \( g \) is the genus of the original surface. Nonzero entries in the vectors \( \overrightarrow{x}, \overrightarrow{y}, \) and \( z \) account for the possibility of twisting, knotting, and linking between the new handle and the existing basis elements. Up to S-equivalence, Seifert matrices are well-defined for knots and links.
Chapter 5

Geometry meets Algebra

Despite being a purely algebraic relation, S-equivalence has geometric implications for knots. As stated in Theorem 1.1, Swatee Naik and Theodore Stanford showed that S-equivalence of classical knots is the same as the equivalence relation of the doubled-delta move on knot diagrams [NS].

One might assume that a similar statement should be true of links. Is the S-equivalence of two links enough to guarantee that they are equivalent under a sequence of doubled-delta moves? This question was posed by Stavros Garoufalidis in the context of clasper surgery, and given that all the definitions leading up to S-equivalence are the same for links as they are for knots, it seems that the analogous proof for links should follow in a straightforward fashion from the proof of Naik-Stanford. The question seems plausible, but in fact it is false. We prove by counterexample in Proposition 5.1 that two S-equivalent links are not necessarily related by a sequence of doubled-delta moves.

**Proposition 5.1** S-equivalence is not a sufficient condition for two links to be related by a sequence of doubled-delta moves.
Proof. The two links $L_0$ and $L_1$ depicted in figure 5.1 are S-equivalent but not related by a sequence of doubled delta moves.

Figure 5.1: Two 3-component Links

Note that the pairwise linking numbers of the first link are \{-1, 2, 2\} while for the second link they are \{1, 0, 0\}. Since the doubled-delta move preserves pairwise linking numbers, these two links cannot be related by doubled-delta moves.

The fact that the two links are S-equivalent can be seen by choosing Seifert Surfaces as in figure 5.2, with bases \{a, b\} and \{c, d\}. Each of the surfaces can be viewed as a punctured 2-sphere with a Y-shaped band glued along the boundary, where each of the three strips of the Y is twisted according to the desired linking numbers. The
Seifert matrices are then $M_1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, respectively. The matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ satisfies the condition $A^t M_2 A = M_1$, demonstrating that the two links are in fact S-equivalent. □

Finding the sufficient conditions to extend the Naik-Stanford theorem will require some new definitions, treated in Chapter 6.
Chapter 6

Strong S-equivalence

As shown in Proposition 5.1, the classical definition of S-equivalence is inadequate for the proof of our main theorem, and actually it seems inappropriate for links in many contexts. Treating a link as a disjoint knot, S-equivalence regards the entire link as a whole, without reference to the individual components. Strong S-equivalence, defined in this chapter, better respects the boundary components of a Seifert surface, and hence the components of the link.

6.1 New Definitions

Let \( L = \{L_1, L_2, ..., L_m\} \) be an oriented, ordered link in \( S^3 \). Let \( \Sigma \) be a Seifert surface for \( L \) with \( m \) boundary components, and let \( g \) denote the genus of \( \Sigma \). That is, \( \Sigma \subseteq S^3 \) is an oriented surface with \( \partial \Sigma = L \). Construct an ordered basis \( \beta = \{\ell_1, \ell_2, ..., \ell_{m-1}, \beta_1, \beta_2, ..., \beta_{2g}\} \) for \( H_1(\Sigma) \), where \( \ell_i \) is represented by the \( i \)th component \( L_i \) of \( L \). Define the Seifert pairing \( \sigma(a, b) = lk(a, b^+) \), where \( b^+ \) is a pushoff of \( b \) in the positive normal direction. We introduce the definition:
Definition 6.1 A matrix $M$ representing $\sigma$ with respect to an ordered basis $\beta$ of the form above is called an Ordered Seifert Matrix for the oriented ordered link $L$.

Remark 6.2 $M$ has the form of a block matrix $\begin{pmatrix} \lambda & A \\ B & C \end{pmatrix}$, where $\lambda$ is an $(m-1) \times (m-1)$ block and $C$ is a $2g \times 2g$ block. The block $\lambda$ is completely determined by the pairwise linking numbers of $L$, and has the following properties:

i. $\lambda_{i,i} = -\sum_{i \neq j} \lambda_{i,j} - lk(L_i, L_m)$ for $1 \leq i, j \leq m-1$, and

ii. $\lambda_{i,j} = \lambda_{j,i} = lk(L_i, L_j)$ if $i \neq j$ and $1 \leq i, j \leq m-1$.

The first of these properties is somewhat less obvious. If we let $\ell_m$ be the homology class represented by $L_m$, then the class $\sum_{j=1}^{m} \ell_j = 0 \in H_1(\Sigma)$, since it is represented by the boundary of $\Sigma$. Then

$$0 = \sigma\left(\ell_i, \sum_{j=1}^{m} \ell_j\right)$$

$$= \sum_{j=1}^{m} \sigma(\ell_i, \ell_j)$$

$$= \sum_{i \neq j} \sigma(\ell_i, \ell_j) + \sigma(\ell_i, \ell_m) + \sigma(\ell_i, \ell_i) \quad \text{for } 1 \leq i, j \leq m-1$$

$$= \sum_{i \neq j} \lambda_{i,j} + lk(L_i, L_m) + \lambda_{i,i} \quad \text{for } 1 \leq i, j \leq m-1.$$

The following two definitions can be found in [K], as they are essential to the classical definition of S-equivalence.

Definition 6.3 We say two integral square matrices $V$ and $W$ are congruent if $V = P^tWP$ for some integral matrix $P$ with $\det(P) = \pm 1$.

Definition 6.4 For integral square matrices $V$ and $W$, we say that $W$ is an enlargement of $V$, or $V$ is a reduction of $W$ if
\[
W = \begin{pmatrix}
V & y^t & 0 \\
\bar{x} & z & 1 \\
0 & 0 & 0
\end{pmatrix}
\quad \text{or} \quad
W = \begin{pmatrix}
V & y^t & 0 \\
\bar{x} & z & 0 \\
0 & 1 & 0
\end{pmatrix}.
\]

With a slight modification of these two definitions, we define:

**Definition 6.5** Two matrices \( V \) and \( W \) are Strongly S-equivalent if \( V \) is equivalent to \( W \) under a finite sequence of

- Congruences \( (\cong) \) that fix the upper-left \((m - 1) \times (m - 1)\) block of the matrix. That is, there exists an integral matrix \( A = \begin{pmatrix} I & z \\ 0 & * \end{pmatrix} \), so that \( A^tVA = W \) and \( \det(A) = \pm 1 \), where \( I \) is the \((m - 1) \times (m - 1)\) identity matrix.

- Enlargements \( (\Join) \) and reductions \( (\backslash) \) where the first \((m - 1)\) elements of the vectors \( \bar{x} \) and \( \bar{y} \) are equal, and where the reduced matrix \( V \) is \( n \times n \) for \( n \geq m - 1 \).

Note that reductions should not be allowed to reduce the size of the matrix smaller than \((m - 1) \times (m - 1)\), for in the case of Seifert matrices, this would effectively eliminate link components.

Also note that the first \((m - 1)\) elements of the vectors \( \bar{x} \) and \( \bar{y} \) are equal because these entries should be zero in the intersection form \( W - W^t \) of the enlarged matrix \( W \), since boundary components will not intersect any other basis elements.

**Definition 6.6** We say that two links \( L \) and \( L' \) are Strongly S-equivalent if, for some choice of Seifert surfaces and ordered bases, \( L \) and \( L' \) have ordered Seifert matrices \( M \) and \( M' \) that are Strongly S-equivalent.
One might object that Strong S-equivalence imposes the “restriction” that $V$ and $W$ agree on their upper-left $(m-1) \times (m-1)$ blocks. However, since homeomorphisms of $\Sigma$ from the pure mapping class group preserve the boundary components pointwise, any change of basis of $M$ must fix the $\ell_1, \ell_2, \ldots, \ell_{m-1}$ basis elements. With Strong S-equivalence, we’re not “restricting” our definition so much as respecting the boundary components of a link.

6.2 New Results

In order to distinguish when two links are not Strongly S-equivalent, we need an invariant.

Proposition 6.7 Pairwise linking number is an invariant of Strong S-equivalence.

Proof. If two links $L$ and $L'$ are Strongly S-equivalent then, regardless of the choice of ordered Seifert matrices $M$ and $M'$, the upper left $(m-1) \times (m-1)$ blocks of $M$ and $M'$ will necessarily agree. That is, for $i \neq j$, $1 \leq i, j < m$,

$$lk(L_i, L_j) = M_{i,j} = M'_{i,j} = lk(L'_i, L'_j).$$

Furthermore, by property $i.$ of Remark 6.2,

$$lk(L_i, L_m) = - \sum_{j=1}^{m-1} M_{i,j} = - \sum_{j=1}^{m-1} M'_{i,j} = lk(L'_i, L'_m).$$

Thus all the pairwise linking numbers for $L$ agree with those for $L'$.
**Proposition 6.8** Any two ordered Seifert matrices for an oriented ordered link $L$ are Strongly $S$-equivalent.

**Proof.** Let $M_1$ and $M_2$ be two ordered Seifert matrices for $L$ with respect to Seifert surfaces $\Sigma_1$ and $\Sigma_2$ and bases $\beta_1$ and $\beta_2$.

By Lemma 5.2.4 of [K], any two connected Seifert surfaces $\Sigma_1$ and $\Sigma_2$ of a link $L$ are ambient isotopic after modifying them by a finite sequence of 1-handle enlargements. In other words, there are two ambient isotopic surfaces $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ such that $\widetilde{\Sigma}_i$ is the result of several 1-handle enlargements of $\Sigma_i$. Without loss of generality, we can consider just the following case: suppose some surface $\widetilde{\Sigma}_2$ is ambient isotopic to $\widetilde{\Sigma}_1$, a single 1-handle enlargement of $\Sigma_1$. Let $a_2$ be a meridian (technically, the belt sphere) of the 1-handle and choose a closed curve $a_1$ that intersects the meridinal disk (technically, the co-core) exactly once. Then one of

$$
\widehat{M}_1 = \begin{pmatrix}
M_1 & \frac{1}{y^t} & 0 \\
- x & - z & 1 \\
0 & 0 & 0
\end{pmatrix}
$$
or

$$
\widetilde{M}_1 = \begin{pmatrix}
M_1 & \frac{1}{y^t} & 0 \\
- x & - z & 0 \\
0 & 1 & 0
\end{pmatrix}
$$

is a Seifert matrix for $\widetilde{\Sigma}_1$ with respect to the basis $\beta_1 \cup \{a_1, a_2\}$ for $H_1(\widetilde{\Sigma}_1)$. The matrix on the left, $\widehat{M}_1$, corresponds to a 1-handle attached so that its core lies on the negative side of the surface, while $\widetilde{M}_1$ corresponds to a 1-handle with its core on the positive side of the surface. We will treat only the $\widehat{M}_1$ case and note that the $\widetilde{M}_1$ case follows similarly. The zeros in the last column of $\widehat{M}_1$ come from the fact that a positive pushoff of the meridian, $a_2$, will not link any of the basis elements except $a_1$. The last row of $\widehat{M}_1$ is all zero because a negative pushoff of the meridian
a_2 will not link any of the basis elements. Since for any Seifert matrix \( N \), \( N - N^t \) is an intersection form, the first \( m - 1 \) entries of the vectors \( \vec{x} \) and \( \vec{y} \) must be equal. Otherwise \( a_1 \) would be intersecting a boundary component, which is impossible. The rest of the entries of \( \vec{x} \) and \( \vec{y} \) are freely determined by the particular embedding of the new 1-handle and the choice of curve \( a_1 \), as always with \( M_{i,j} = \text{lk}(\beta_i, \beta_j^+) \). The entry \( z \) is, of course, \( \text{lk}(a_1, a_i^+) \). Note that with more restrictions on the choice of \( a_1 \), the allowable entries of \( \vec{x} \), \( \vec{y} \), and \( z \) could be defined to have more structure.

By definition of enlargement and reduction, \( \widetilde{M}_1 \) and \( M_2 \) are Strongly S-equivalent. This process can be iterated for any finite number of 1-handle enlargements so that \( \widetilde{M}_1 \) is Strongly S-equivalent to \( M_1 \), and similarly, \( M_2 \) can be shown to be Strongly S-equivalent to \( \widetilde{M}_2 \), the similarly constructed Seifert matrix for \( \Sigma_2 \).

Since \( \widetilde{\Sigma}_1 \) is ambient isotopic to \( \Sigma_2 \), \( \widetilde{M}_1 \) and \( \widetilde{M}_2 \) differ only by a choice of bases. Furthermore, since the upper left blocks of both \( \widetilde{M}_1 \) and \( \widetilde{M}_2 \) are determined by the link \( L \) and are equal (note that \( \widetilde{M}_i \) has the same upper-left block as \( M_i \)), \( \widetilde{M}_1 \) and \( \widetilde{M}_2 \) are related by a change of basis that preserves the first \( m - 1 \) basis elements, and they are thus Strongly S-equivalent. Therefore, by transitivity \( M_1 \) is Strongly S-equivalent to \( M_2 \).

Like its classical S-equivalence analogue, the converse of Proposition 6.8 is not true. If \( N \) is a Seifert matrix for the link \( L \) with respect to Seifert surface \( \Sigma \), and \( M \) is Strongly S-equivalent to \( N \), \( M \) is not necessarily a Seifert matrix for \( L \). The difficulty arises when \( M \) is a reduction of \( N \), since it may not be possible to find a corresponding 1-handle reduction of the Seifert surface \( \Sigma \). The converse is true, however, in the special cases of congruences and enlargements. As mentioned previously, matrix congruence corresponds to change of basis for \( H_1(\Sigma) \). In order to
explicitly understand how any given matrix enlargement corresponds to a 1-handle enlargement of a Seifert surface, we prove the following proposition:

**Proposition 6.9** If $N$ is an ordered Seifert matrix for the link $L$ with respect to Seifert surface $\Sigma$, and $M$ is an enlargement of $N$, then $M$ is an ordered Seifert matrix for $L$ with respect to a 1-handle enlargement $\widehat{\Sigma}$ of $\Sigma$.

**Proof.** Let $\Sigma$ be a Seifert surface for $L$ with basis $\beta$ of $H_1(\Sigma)$ that induces the Seifert matrix $N$. Since $M$ is an enlargement of $N$, either

$$M = \begin{pmatrix} N & \frac{1}{y_t} & 0 \\ - & x & z \\ - & 0 & 0 \end{pmatrix} \text{ or } M = \begin{pmatrix} N & \frac{1}{y_t} & 0 \\ - & x & z \\ - & 0 & 1 \end{pmatrix}.$$

We will construct the 1-handle enlargement $\widehat{\Sigma}$ of $\Sigma$ that has $M$ as its Seifert matrix. As in Proposition 6.8, the matrix on the left corresponds to attaching the 1-handle so that its core lies on the negative side of the surface, while the matrix on the right corresponds to a 1-handle with its core on the positive side of the surface.

Let $\{b_1, \ldots, b_{2g+m-1}\} \subset \Sigma$ be a set of representative curves for the basis $\beta$. Find two small disks in $\Sigma$ that are disjoint from the curves $\{b_1, \ldots, b_{2g+m-1}\}$. These disks will be the attaching region for the 1-handle. Designate two points, $p$ and $q$, one point on the boundary of each disk. For the two new basis elements $a_1$ and $a_2$, the enlarged matrix $M$ determines the Seifert form—that is, a combination of their linking and intersection numbers—with curves $\{b_1, \ldots, b_{2g+m-1}\}$. The 1-handle we construct will have $a_2$ as a meridian and $a_1$ running parallel to the core. First let
us look at $a_1$, breaking it into two parts, $\gamma$ and $\delta$, as in figure 6.1. In order to be consistent with the Seifert matrix $M$, we need $a_1$ to intersect and link the curves $\{b_1, \ldots, b_{2g+m-1}\}$ according to the entries of the vectors $\overline{x}$ and $\overline{y}$. We can extract the intersection information from the intersection form $M - M'$, or in particular, from the vector entries $\overline{x_i} - \overline{y_i}$ [R]. Running through the surface $\Sigma$, $\gamma$ will first take care of any intersections, while the handle itself, and hence $\delta$, will be free to link the basis elements. Together, $\gamma \cup \delta = a_1$ will then satisfy each of the entries $lk(a_1, b_i^+)$. 

![1-handle](image)

Figure 6.1: 1-handle

To find $\gamma \subset \Sigma$, the half of $a_1$ that will travel from $p$ to $q$ along the surface $\Sigma$, let us look at the intersection form $M - M'$. The individual entries of $\overline{x} - \overline{y}$ determine how $\gamma$ should intersect the curves $\{b_1, \ldots, b_{2g+m-1}\}$; since $\delta \subset a_1$ does not intersect $\beta$ at all, $\langle \gamma, b_i \rangle = \langle a_1, b_i \rangle = \overline{x_i} - \overline{y_i}$. The first $m - 1$ entries of $\overline{x} - \overline{y}$ are zero, which is consistent with the fact that $\gamma$ cannot cross a boundary component. For the last $2g$ curves $b_i$, choose $\gamma$ so that $\langle \gamma, b_i \rangle = \overline{x_i} - \overline{y_i}$. This is possible. As described in section 4.1, the basis $\beta$ is in correspondence with a semi-symplectic basis. For ease of construction, choose a set of curves $\{c_i\}$ in $\Sigma$ to be representatives of this semi-symplectic basis with the same algebraic intersection properties. If, for any set of integers $\{k_i\}$, we can construct a curve $\gamma$ that has algebraic intersection
number \( k_i \) with each \( c_i \), then the same can be done for the set of curves \( \{b_i\} \) via this correspondence. Start with a curve \( \gamma_1 \) from \( p \) to \( q \) that does not intersect any of the curves \( c_i \). For each desired \( (\pm) \) intersection with a particular curve \( c_{2k} \), take \( \gamma_2 = \gamma_1 \pm c_{2k-1} \). (For each \( (\pm) \) intersection with \( c_{2k-1} \), take \( \gamma_2 = \gamma_1 \mp c_{2k} \).) Continue this until all intersections are achieved, and call the new curve \( \gamma \). Figure 6.2 demonstrates how a path \( \gamma \) from \( p \) to \( q \) can be chosen to intersect \( c_1 \) exactly one time and intersect \( c_4 \) exactly \(-2\) times.

![Diagram of construction of 1-handle](image)

**Figure 6.2: Construction of 1-handle**

After the intersection properties are established between \( \gamma \) and the \( b_i \), we shift our focus to the linking properties. As with the intersections, the linking numbers of \( \gamma \) with the \( b_i \) are in correspondence with those of \( \gamma \) with the \( c_i \), so we will focus our construction on the semi-symplectic basis \( \{c_i\} \). The core of the 1-handle can be chosen in the complement of \( \Sigma \) so that it links each of the curves \( c_i \) the desired
number of times. This is easy to see in the disk-band representation of $\Sigma$. The core can then be fattened up to a solid handle, the surface of which is the 1-handle enlargement of $\Sigma$. We choose a curve $\delta$ on the surface of the handle parallel to the core, and let $a_1 = \gamma + \delta$ be one of our new basis elements. Figure 6.2 demonstrates how the core can be chosen to link $c_2$ one time and link $c_3$ negative one time.

Lastly, we must adjust the path $\delta$ along the 1-handle so that $\text{lk}(a_1, a_1^+) = z$. The remedy is simple: each repeated positive or negative full twist to the 1-handle will increase or decrease $\text{lk}(a_1, a_1^+)$ by one without affecting any of the other basis elements. Figure 6.2 illustrates $\delta$ chosen so that $z = -2$.

We have explicitly constructed a new Seifert surface $\hat{\Sigma}$ for $L$ and a new basis for $H_1(\hat{\Sigma})$ such that $M$ is an ordered Seifert matrix for $L$ with respect to them. $\Box$
Chapter 7

Proof of the Main Theorem

**Theorem 7.1** Consider two oriented, ordered m-component links $L_0$ and $L_1$. The following four statements are equivalent:

i. $L_1$ can be obtained from $L_0$ through a sequence of doubled-delta moves.

ii. $L_0$ and $L_1$ are related by a sequence of $Y$-clasper surgeries, where each clasper has total linking number zero with the link.

iii. $L_0$ and $L_1$ are Strongly $S$-equivalent.

iv. For some choice of Seifert Surfaces $\Sigma_0$ and $\Sigma_1$ and bases of $H_1(\Sigma_i)$, $L_0$ and $L_1$ have the same ordered Seifert Matrix.

The proofs of each implication $i. \implies ii. \implies iii. \implies iv. \implies i.$ are treated individually below:

7.1 **Proof of $i. \implies ii.$**

*Proof.* Doubled-delta moves correspond to “Borromean surgery,” which is exactly the effect of $Y$-clasper surgery, where each leaf clasps pairs of oppositely oriented
strands. This is depicted in figure 7.1. In this figure, the relationship $a$ introduces

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure71.png}
\caption{Y-Clasper surgery is Doubled-Delta Move}
\end{figure}

the desired $Y$-clasper, $b$ transforms the clasper into its associated link, $c$ depicts the effect of surgery via handle slide moves, $d$ is merely the second Reidemeister move, and $e$ is the doubled-delta move.

7.2 Proof of $ii. \implies iii.$

Next, we actually prove a stronger implication than $ii. \implies iii.$, namely that the clasper surgery described in $ii.$ does not alter the ordered Seifert matrix.
Proof. Consider a neighborhood of the clasp, as in Figure 7.2a. There are $2k$ strands passing through the leaf, with $k$ in each direction. We can assume the strands’ directions alternate within this neighborhood. If they do not alternate, we can permute the strands by introducing inverse braids just above and below the $2k$ braid in question.

![Figure 7.2: Enlarged View of Clasper Leaf](image)

Temporarily cut the strands at the neighborhood’s boundary and fill in the alternating arcs to get bands as in Figure 7.2b.

Outside the neighborhood, apply Seifert’s algorithm to the resulting link, then reattach the new bands at the neighborhood’s boundary to obtain a Seifert surface for the original link where the leaf of the clasper actually grabs $k$ bands of the Seifert surface. Now, clasper surgery is equivalent to tying these bands into Borromean rings, which doesn’t affect the linking of the strands. If a basis element of $H_1(\Sigma)$ runs through a band, its pairwise linking with the other basis elements is unchanged; if not, it is completely unaffected by the surgery. Thus the clasper surgery leaves the Seifert matrix unchanged. $\square$
7.3 Proof of iii. $\implies$ iv.

With almost no alteration, the following proof of iii. $\implies$ iv. can also be used to show that two classically S-equivalent knots or links have a Seifert matrix in common.

Proof. If $L$ and $L'$ are Strongly S-equivalent then, by definition, there are Seifert matrices $M$ and $M'$ respectively such that $M$ is equivalent to $M'$ under a finite sequence of enlargements ($\nearrow$), reductions ($\searrow$), and congruences ($\cong$). That is, we can write a sequence of the form

$$M \searrow M_1 \nearrow M_2 \cong M_3 \searrow M_4 \cong M_5 \nearrow M_6 \searrow M'. \quad (*)$$

In order to prove that $L$ and $L'$ have a Seifert matrix in common, it will be helpful to rewrite the sequence (*) so that all the enlargements precede all the reductions, as in

$$M \nearrow \widehat{M_1} \cong \widehat{M_2} \nearrow \widehat{M_3} \cong \widehat{M_4} \nearrow \widehat{M_5} \cong \widehat{M_6} \searrow \widehat{M_7} \searrow \widehat{M_8} \cong M'. \quad (**)$$

The following lemmas establish that such an ordered sequence exists for every pair of strongly S-equivalent matrices and show how the ordered sequence completes the proof of iii. $\implies$ iv..

**Lemma 7.2** $M_1 \searrow M_2 \nearrow M_3 \implies M_1 \nearrow M_4 \cong M_5 \searrow M_3$.

Before proving this lemma, we first note that the simpler implication $M_1 \searrow M_2 \nearrow M_3 \implies M_1 \nearrow M_4 \searrow M_3$ is not true. In the right-hand side, a reduction immediately follows an enlargement, so the two actions cancel each other and $M_1 = M_3$, which is not necessarily true in the left-hand side.

Proof. Assume that
\[
M_1 = \begin{pmatrix}
V & \frac{1}{y_1^t} & 0 \\
\rightarrow x_1 & \frac{1}{y_1^t} & 1 \\
0 & 0 & 1 \\
\end{pmatrix}
,\quad M_2 = V, \quad \text{and} \quad M_3 = \begin{pmatrix}
V & \frac{1}{y_3^t} & 0 \\
\rightarrow x_3 & \frac{1}{y_3^t} & 1 \\
0 & 0 & 1 \\
\end{pmatrix}.
\]

Then let
\[
M_4 = \begin{pmatrix}
V & \frac{1}{y_1^t} & \frac{1}{y_3^t} & 0 \\
\rightarrow x_1 & \frac{1}{y_1^t} & \frac{1}{y_3^t} & 1 \\
\rightarrow x_3 & \frac{1}{y_1^t} & \frac{1}{y_3^t} & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

If \( M_4 \) is a \( k \times k \) matrix, then let \( M_5 \) be the result of a change of basis that permutes the \( k \)th basis element with the \( k - 2 \)nd, and the \( k - 1 \)st basis element with the \( k - 3 \)rd. Then
\[
M_5 = \begin{pmatrix}
V & \frac{1}{y_3^t} & \frac{1}{y_1^t} & 0 \\
\rightarrow x_3 & \frac{1}{y_3^t} & \frac{1}{y_1^t} & 0 \\
\rightarrow x_1 & \frac{1}{y_3^t} & \frac{1}{y_1^t} & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
\]

which reduces to \( M_3 \).

\[\square\]

**Lemma 7.3** \( M_1 \cong M_2 \not\cong M_3 \implies M_1 \not\cong M_4 \cong M_3 \).
Proof. If $M_2 = P^t M_1 P$, and

$$M_3 = \begin{pmatrix} P^t M_1 P & \frac{1}{y_3 t} & 0 \\ \overrightarrow{x_3} & \overrightarrow{z_3} & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

let $M_4 = \begin{pmatrix} M_1 & \frac{1}{(P^t)^{-1} \cdot \overrightarrow{y_3^t}} & 0 \\ \overrightarrow{x_3} \cdot P^{-1} & \overrightarrow{z_3} & 0 \\ 0 & 1 & 0 \end{pmatrix}$

be an enlargement of $M_1$. Now let $Q = \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}$. Then

$$Q^t M_4 Q = \begin{pmatrix} P^t & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} M_1 & \frac{1}{(P^t)^{-1} \cdot \overrightarrow{y_3^t}} & 0 \\ \overrightarrow{x_3} \cdot P^{-1} & \overrightarrow{z_3} & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} P^t M_1 & P^t \cdot \frac{1}{[(P^t)^{-1} \cdot \overrightarrow{y_3^t}]} & 0 \\ \overrightarrow{x_3} \cdot P^{-1} & \overrightarrow{z_3} & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} P^t M_1 P & P^t \cdot \frac{1}{[(P^t)^{-1} \cdot \overrightarrow{y_3^t}]} & 0 \\ \overrightarrow{x_3} \cdot P^{-1} \cdot P & \overrightarrow{z_3} & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} P^t M_1 P & \frac{1}{\overrightarrow{y_3^t}} & 0 \\ \overrightarrow{x_3} & \overrightarrow{z_3} & 0 \\ 0 & 1 & 0 \end{pmatrix} = M_3$$

$\square$
Lemma 7.4 Any sequence of relations between Strongly S-equivalent matrices can be rewritten so that all enlargements come before all reductions, as in the sequence (**).

Proof. The proof of Lemma 7.4 follows from an induction argument using Lemmas 7.2 and 7.3. For simplicity, we will use strings of the symbols \( \langle, \rangle, \equiv \) to denote enlargements, reductions, and congruences, respectively, while omitting explicit reference to the matrices.

- Base Case: \( \& \langle \rangle \implies \langle \rangle \equiv \& \) (Lemma 7.2).

- Inductive Step: Find the first enlargement with a reduction preceding it. Note that the sequence preceding this enlargement is arranged as desired. There are two cases:

1. If this enlargement is immediately preceded by a congruence, apply Lemma 7.3 to replace \( \equiv \langle \rangle \) with \( \langle \rangle \equiv \). Repeat Lemma 7.3 until the enlargement is immediately preceded by a reduction.

2. If this enlargement is immediately preceded by a reduction, then by Lemma 7.2, \( \& \langle \rangle \) can be replaced with \( \langle \rangle \equiv \& \).

Continue the two steps above until the enlargement is immediately preceded by another enlargement.

We have reduced the number of out-of-order \( \langle, \&, \equiv \rangle \) by one without increasing the number of enlargements or reductions. Continue the inductive step until there is no enlargement preceded by a reduction.

We can now finish the proof of iii. \( \implies \) iv. If \( L \) and \( L' \) are Strongly S-equivalent, then by Lemma 7.4 we may assume there exists a sequence of relations \( (\langle, \&, \equiv) \)
between $M$ and $M'$ where all enlargements precede all reductions. Using the following two facts:

- If $M$ is an ordered Seifert matrix for the link $L$ with respect to Seifert surface $\Sigma$ and basis $\beta$, and if $M \not\rightarrow M'$, then $M'$ is also an ordered Seifert matrix for $L$ with respect to a 1-handle enlargement $\hat{\Sigma}$ of $\Sigma$ and the corresponding new basis $\beta \cup \{a_1, a_2\}$. This follows from Proposition 6.9.

- If $M$ is an ordered Seifert matrix for the link $L$, and $M \cong M'$, then $M'$ is also an ordered Seifert matrix for $L$, with respect to the same Seifert surface and a new basis as prescribed by the congruence.

we can work inwards from both ends of the ordered sequence to show that $L$ and $L'$ have a common Seifert matrix (though not necessarily $M$ or $M'$) for some Seifert surfaces and bases. Starting with $M$ and working from left to right, each enlargement or congruence yields a new Seifert matrix for $L$. This terminates when we reach the first reduction. Similarly, since $M_i \searrow M' \iff M' \nearrow M_i$, starting with $M'$ and working from right to left, each reduction or congruence yields a new Seifert matrix for $L'$. In the ordered sequence labelled (**) above, $\hat{M}_6$ is a common Seifert matrix for $L_1$ and $L_2$, as is $\hat{M}_6$.

\[\square\]

7.4 Proof of $i\forall \implies i$.

The following proposition contains the bulk of content of the Main Theorem. This is the primary step that distinguishes the proof of $i\forall \implies i$. from Naik-Stanford's proof of Theorem 1.1 and uses the extra hypotheses of Strong S-equivalence:
Proposition 7.5 If two m-component links $L_0$ and $L_1$ have the same ordered Seifert matrix $M$ with respect to Seifert surfaces $\Sigma_0$ and $\Sigma_1$ and ordered bases $\beta_0$ and $\beta_1$ of $H_1(\Sigma_0)$ and $H_1(\Sigma_1)$, respectively, then it is possible to arrange $\Sigma_0$ and $\Sigma_1$ into disk-band form and to find new semi-symplectic bases $\gamma_0$ and $\gamma_1$ for $\Sigma_0$ and $\Sigma_1$ that give rise to a new shared ordered Seifert matrix $N$ for both $L_0$ and $L_1$.

Proof of Proposition. We start with surfaces $\Sigma_0$ and $\Sigma_1$ and bases $\beta_0$ and $\beta_1$ of $H_1(\Sigma_0)$ and $H_1(\Sigma_1)$, respectively. In transforming $\Sigma_0$ and $\Sigma_1$ into disk-band form, it is important that we keep track of the new bases in terms $\beta_0$ and $\beta_1$. This means understanding the homeomorphisms involved in the transformation.

Both $\Sigma_0$ and $\Sigma_1$ have $m$ boundary components, and since they share a Seifert matrix, both have the same genus, say $g$. Let $F_g$ be an abstract surface of genus $g$ with $m$ boundary components, specifically realized as a disk with bands as in Figure 7.3 (i.e. the string link from Figure 4.2 is trivial), where the $\{a_i\}$ form an ordered basis for $H_1(F_g)$. The intersection form for $\{a_i\}$ is represented by the block matrix $X = \begin{pmatrix} 0 & 0 \\ 0 & Sym \end{pmatrix}$, where $Sym = \bigoplus_{j=1}^{g} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Choose orientation-preserving homeomorphisms $\Phi_i : F_g \to \Sigma_i$ from the pure mapping class group, i.e. the boundary components are fixed, pointwise. Assume that
the boundary component of $F_g$ that is parallel to $a_j$ is sent by $\Phi_i$ to the $j$th component of $L_i$. Then

$$\Phi_i(\{a_i\}) = \{\Phi_i(a_1), \ldots, \Phi_i(a_{m-1+2g})\}$$

$$= \{\beta_{i,1}, \beta_{i,2}, \ldots, \beta_{i,m-1}, \Phi_i(a_m), \ldots, \Phi_i(a_{m-1+2g})\}$$

is also a basis for $H_1(\Sigma_i)$. So there exist invertible matrices $A_i = (\begin{smallmatrix} I & \ast \\ 0 & \ast \end{smallmatrix})$ such that $N_i = A_i^t M A_i$ denotes the Seifert matrix of $\Sigma_i$ with respect to the new basis $\Phi_i(\{a_i\})$. These homeomorphisms describe an explicit way to put $\Sigma_0$ and $\Sigma_1$ into disk band form as in Figure 4.2, where the two surfaces differ only by a string links $\Lambda_0$, $\Lambda_1$ of bands of the Seifert surfaces. The new Seifert matrices $N_0$ and $N_1$ correspond to these "new" surfaces, but in their construction process, we lost the critical hypothesis that the ordered Seifert matrices for $L_0$ and $L_1$ were equal. We need to show that $N_0$ is also a Seifert matrix for $\Sigma_1$, with respect to a different basis.

By definition, if we have a Seifert matrix $N$ determined by a given basis, then $N - N^t$ is the intersection form for that same basis. Since homeomorphisms of surfaces preserve the intersection properties of their basis elements, $\Phi_i(\{a_i\})$ will have the same intersection properties as $\{a_i\}$, and so we have that $N_i - N_i^t = X = (\begin{smallmatrix} 0 & 0 \\ 0 & \text{sym} \end{smallmatrix})$. We use this fact to construct a matrix $C = A_1^{-1} A_0$ that we show stabilizes $X$, in that $X = C^t X C$. First we show that $N_0 = C^t N_1 C$:

$$N_0 = A_0^t M A_0$$

$$= A_0^t ((A_1^t)^{-1} M A_1^{-1}) A_0$$

$$= (A_1^{-1} A_0)^t N_1 (A_1^{-1} A_0)$$

$$= C^t N_1 C$$
Then,

\[ X = N_0 - N_0^t \]
\[ = C^t N_1 C - C^t N_1^t C \]
\[ = C^t (N_1 - N_1^t) C \]
\[ = C^t X C. \]

Now the key step is that we'd like to find a homeomorphism of the pure mapping class group that induces \( C \) with respect to the ordered basis \( \Phi_1(\{a_i\}) \). Actually the pure mapping class group, which fixes the boundary components pointwise, is stronger than we need. Fixing the boundary component-wise would be sufficient; however, the ordinary mapping class group only fixes the boundary set-wise. For the case of knots, this key step is easy. \( C \) is symplectic, and thus is well-known to be induced by such a homeomorphism, since the map from the pure mapping class group to the symplectic group is surjective [MKS, pp. 178, 355-6]. For links though, it takes several steps to find a homeomorphism inducing \( C \).

**Lemma 7.6** The matrix \( C \) is of the form \( \begin{pmatrix} I & B \\ 0 & S \end{pmatrix} \) where \( S \) is a symplectic matrix.

**Proof.** By construction, \( C = A_1^{-1} A_0 \). We know by definition that \( A_0 \) and \( A_1 \) are of the form \( A_i = \begin{pmatrix} I & * \\ 0 & * \end{pmatrix} \).

First we want to show that \( A_1^{-1} \) is also of the form \( \begin{pmatrix} I & * \\ 0 & * \end{pmatrix} \). Let \( A_1 = \begin{pmatrix} I & Y \\ 0 & Z \end{pmatrix} \). Suppose \( \begin{pmatrix} P & Q \\ R & T \end{pmatrix} \) is an inverse for \( A_1 \). Then

\[
\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} P & Q \\ R & T \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & Z \end{pmatrix} = \begin{pmatrix} P & P Y + Q Z \\ R & R Y + T Z \end{pmatrix},
\]
so \( R = 0 \) and \( P = I \). Thus \( A_1^{-1} \) must be of the form \( \begin{pmatrix} I & * \\ 0 & \end{pmatrix} \).

From this it is easy to see that

\[
C = A_1^{-1}A_0 = \begin{pmatrix} I & Q \\ 0 & T \end{pmatrix} \begin{pmatrix} I & V \\ 0 & W \end{pmatrix} = \begin{pmatrix} I & V + QW \\ 0 & TW \end{pmatrix}
\]

is also of the form \( \begin{pmatrix} I & * \\ 0 & \end{pmatrix} \).

Now we need to demonstrate that the lower right block of \( C \) is symplectic. For this, we let \( C = \begin{pmatrix} I & B \\ 0 & S \end{pmatrix} \). Then

\[
C^t X C = \begin{pmatrix} I & 0 \\ B^t & S^t \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & S_{ym} \end{pmatrix} \begin{pmatrix} I & B \\ 0 & S \end{pmatrix} \\
= \begin{pmatrix} 0 & 0 \\ 0 & S_{ym}^t \end{pmatrix} \begin{pmatrix} I & B \\ 0 & S \end{pmatrix} \\\n= \begin{pmatrix} 0 & 0 \\ 0 & S_{ym}^t S_{ym} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & S_{ym} \end{pmatrix} = X.
\]

Since \( S_{ym}^t S_{ym} = S_{ym} \), \( S \) is a symplectic matrix. \( \square \)

**Lemma 7.7** The matrix \( D = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix} \), relative to any fixed semi-symplectic basis, is induced by a homeomorphism of the pure mapping class group.

*Proof.* We want to consider \( D \) to represent an action on \( H_1(\Sigma_1) \). If \( \Sigma \) were a once-punctured surface (in other words, if our link were a knot), then \( D \) would be symplectic and this action would then be induced by a homeomorphism from the pure mapping class group [MKS].
Our Seifert surface $\Sigma_1$ is an $m$-punctured surface. Choose a representative curve $\bar{\beta}_i \subset \Sigma_1$ for each of the last $2g$ basis elements of $\beta_1$, where $m \leq i \leq 2g + m - 1$, and such that the geometric intersections among representatives respect the algebraic intersections determined by the ordered Seifert matrix.

Temporarily cap off the $m$ boundary components with disks $D_1, \ldots, D_m$, forming a new closed surface $\Sigma$. Consider a larger disk $U \subset \Sigma$ that encompasses all of the smaller disks $D_1, \ldots, D_m$, but does not intersect any of the $2g$ curves $\bar{\beta}_m, \ldots, \bar{\beta}_{2g+m-1}$. This is possible. Since the $2g$ basis elements are semi-symplectic, we can cut along these curves to obtain an $m$-punctured $2g$-gon. The disk $U$ can be chosen in the interior of this $2g$-gon such that it encompasses all of the disks $D_i$ (a pushoff of the $2g$-gon’s boundary will suffice).

Now $\Sigma - U \subset \Sigma_1$ is a once-punctured surface with symplectic basis $\bar{\beta}_m, \ldots, \bar{\beta}_{2g+m-1}$. Where $D = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$ represents an action on $H_1(\Sigma_1)$, $S$ represents an action on $H_1(\Sigma - U)$ and corresponds to a homeomorphism $\bar{g}$ of the pure mapping class group by the surjective map mentioned above.

We can extend $\bar{g}$ to $g$, which agrees with $\bar{g}$ on $\Sigma - U \subset \Sigma_1$ and fixes the remaining $U - \bigcup_{j=1}^{m} D_j \subset \Sigma_1$. We now have that the homeomorphism $g$ induces the action $D = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$ on $H_1(\Sigma_1)$. □

**Lemma 7.8** There is a matrix $E$ such that $C = DE$, and $E$ can be taken to be a product of elementary matrices $E(i,j)$ where each $E(i,j)$ is induced by a homeomorphism of the pure mapping class group.

**Proof.** The matrix $C = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$ can be taken to the matrix $D = \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$ via right-multiplication by $\begin{pmatrix} I & -B \\ 0 & I \end{pmatrix}$. That is, $D = C \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix}$, or $C = D \left( \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix} \right)^{-1} = D \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$. 

Let $E = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$. Right-multiplication by this matrix $E$ can be achieved by a product of several elementary matrices $E_{i,j}$, which we will proceed to define.

To better illustrate the construction, we first let $B_{i,j}$ be the $(m-1) \times 2g$ matrix that has the $b_{i,j}$ entry of $B$ in the $i,j$th spot and zeros elsewhere. Then we can verify that $E = \prod_{i,j=1}^{m-1} \begin{pmatrix} I & B_{i,j} \\ 0 & I \end{pmatrix}$. Define an elementary $(2g + m - 1) \times (2g + m - 1)$ matrix $E_{i,j}$ to be the identity matrix with $+1$ in the $i,j + m - 1$ spot, where $1 \leq i \leq m - 1$, and $1 \leq j \leq 2g$. For example, if $m = 4$ and $g = 2$ then

$$E_{1,2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It should also be noted that for any integer $k$,

$$(E_{1,2}) = \begin{pmatrix} 0 & k & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

Then $C = DE$, where $E = \prod_{i=1}^{m-1} \prod_{j=1}^{2g} (E_{i,j})^{b_{i,j}}$.

To show that these elementary matrices $E_{i,j}$ are induced by homeomorphisms of the pure mapping class group, note that each $E_{i,j}$ fixes the first $m - 1$ basis elements and sends $b_{m-1+j}$ to $b_{m-1+j} + b_i$. An example of this is shown below for $E_{1,3}$ where $m = 4$ and $g = 2$. 


We need to find a homeomorphism that induces $E_{i,j}$, that is, one that exactly takes $b_k$ to $b_k + b_i$ for $k = j + m - 1$ (and $1 \leq i \leq m - 1, 1 \leq j \leq 2g$), while fixing all the other basis elements. These homeomorphisms can be realized as simple Dehn twists, and are illustrated in figure 7.4. In each of these figures, the surface $\Sigma_1$ is represented schematically as a genus $2g$ surface with $m$ boundary components. Pairs of the standard symplectic basis elements are labelled $\{a_i, b_i\}$, while the boundary basis elements are labelled $c_i$. The notation $D_b(a)$ is used to represent the effect on the curve $a$ of performing a Dehn twist about the curve $b$. Using this notation, we show that $a_i$ can be taken to $a_i + c_j$ by the composition of Dehn twists $D_{-b_i}(D_{b_i+c_j}(a_i)) = a_i + c_j$. Similarly, $D_{-a_i}(D_{a_i+c_j}(b_i)) = b_i + c_j$.

Since each $E_{i,j}$ fixes the first $m - 1$ basis elements, $E_{i,j}$ is induced by a homeomorphism $f_{i,j}$ that fixes the first $m - 1$ boundary components of $\Sigma_i$ and hence fixes all $m$ boundary components. All that is required for the sake of $C$ is that the boundary be fixed, component-wise. However, as the $f_{i,j}$ are constructed from these simple Dehn
Figure 7.4: Dehn twists \( D_{-b_i}(D_{b_i+c_j}(a_i)) = a_i + c_j \).

twists, they can easily be taken to fix the boundary components point-wise. Now, 
\( E \), as a product of elementary matrices \( E_{i,j} \), is induced by the corresponding composition of the \( f_{i,j} \). That is, the action of \( E \) on \( H_1(\Sigma_1) \) is induced by the composition 

\[
f = \circ_{i=1}^{m-1} \circ_{j=1}^{2g} \circ_{k=1}^{b_{i,j}} f_{i,j}
\]

of homeomorphisms \( f_{i,j} \).

Together, Lemmas 7.7 and 7.8 imply that \( C \) is induced by a homeomorphism \( h = f \circ g \) of the pure mapping class group.
Now we can use the homeomorphisms \( \Phi_0 : F_g \to \Sigma_0 \) and \( h \circ \Phi_1 : F_g \to \Sigma_1 \) to put \( \Sigma_0 \) and \( \Sigma_1 \) into disk and band form as in Figure 7.5. The basis elements \( h \circ \Phi_1(\{a_i\}) \) are the columns of the matrix \( C \). Where \( N_1 \) was the Seifert matrix for \( \Sigma_1 \) with respect to the basis \( \Phi_1(\{a_i\}) \), now \( N_0 = h^* N_1 h_* = C^t N_1 C \) is the Seifert matrix for \( \Sigma_1 \) with respect to the basis \( h \circ \Phi_1(\{a_i\}) \). Thus the Seifert surfaces \( \Sigma_0 \) and \( \Sigma_1 \), together with the semi-symplectic bases \( \Phi_0(\{a_i\}) \) and \( h \circ \Phi_1(\{a_i\}) \), respectively, give rise to a shared ordered Seifert matrix \( N_0 \) for \( L_0 \) and \( L_1 \). 

Finally, the tools are in place to prove the final implication of the Main Theorem.

Proof of iv. \( \Rightarrow \) i.. Suppose \( L_0 \) and \( L_1 \) have the same ordered Seifert matrix \( M \) with respect to Seifert surfaces \( \Sigma_0 \) and \( \Sigma_1 \) and ordered bases \( \beta_0 \) and \( \beta_1 \) of \( H_1(\Sigma_0) \) and \( H_1(\Sigma_1) \), respectively. Proposition 7.5 states that the Seifert surfaces \( \Sigma_0 \) and \( \Sigma_1 \) can be arranged in disk-band form as in Figure 7.5, with new semi-symplectic bases \( \gamma_0 \) and \( \gamma_1 \) that each give rise to a shared ordered Seifert matrix \( N \) for \( L_0 \) and \( L_1 \).

![String link diagram](image_url)

Figure 7.5: \( \Sigma_0 \) and \( \Sigma_1 \) differ by string links \( \Lambda_0 \) and \( \Lambda_1 \) on their bands.

In their new disk-band form, the Seifert surfaces \( \Sigma_0 \) and \( \Sigma_1 \) differ only by the string links \( \Lambda_0 \) and \( \Lambda_1 \) of bands. Because \( \gamma_0 \) and \( \gamma_1 \) have been chosen to be semi-symplectic, the new basis elements run straight through each band. The sets of pairwise linking numbers for \( \Lambda_0 \) and \( \Lambda_1 \) are equal, both being determined by the matrix \( N \). By
Murakami-Nakanishi’s Theorem 2.2, \( \Lambda_0 \) and \( \Lambda_1 \) are related by a sequence of (single) delta moves. Therefore our original links \( L_0 \) and \( L_1 \) are related by a sequence of doubled-delta moves, having two oppositely oriented strands for each band of \( \Lambda_0 \) or \( \Lambda_1 \).

Although the issue of framing on the bands of \( \Sigma_0 \) and \( \Sigma_1 \) has not been addressed, it is clear that the delta move doesn’t change the framing on any strand of a string link, and the doubled-delta move doesn’t alter the self-linking of any of the bands. Moreover, the framing of each band corresponds to the self-linking of the basis element running through that band, and is thus the same for \( \Sigma_0 \) as for \( \Sigma_1 \), both being determined by the diagonal entries of \( N \). □
Bibliography


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