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Ballistic Limit Equation for Hypervelocity Impact on

Composite-Orthotropic Materials

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Ballistic Limit Equation for Hypervelocity Impact on Composite-Orthotropic Materials

José Santiago Cruz Bañuelos

Abstract

Two new ballistic limit equations for hypervelocity impact on homogeneous and composite-orthotropic materials have been developed for a velocity range above 6 km/s. The methodology used to develop the ballistic limit equations involves Kirchhoff's plate theory for a two plate fundamental structure comprising a shield and back plate. The Boundary Element Method is used to calculate the deformation and the moments when the load, is uniformly distributed over a circular area of the back plate, and is applied quickly so that the momentum transferred to the loaded area is equal to twice the momentum of the original projectile. The Von Mises yield criterion is used to account for elastic-plastic deformations into homogeneous materials and the Tsai-Hill yield criterion is used to account for elastic-plastic deformations into composite-orthotropic materials. The ballistic limit equations developed are compared with existing ballistic limit equations based on empirical and semi empirical formulations. It can be seen that our results are in good agreement with experimental measurements of spherical projectiles impacted on a two-plate shield at hypervelocity.
Acknowledgements

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Chapter 1

Introduction

Shielding for micrometeoroid protection continues to be of significant interest for developing current and next generation spacecraft. Space equipment, which can be potentially impacted by micrometeoroids and space debris, has been classified into four categories: permanent modules and critical components, functional equipment, transitory vehicles, and space suits or extravehicular activity. A number of shielding structures have been designed, where the basis is still the Whipple Shield made up of a bumper and rear wall assembly. In this work, the Whipple shield is classified as a basic structure with a front shield and back plate. Structures in outer space are under the risk of impact by projectiles traveling at velocities of the order of 5 Km/s and more. Different arrangements are used in order to protect those structures, one of them consisting of a thin plate in the path of the projectile before it reaches the structure being called dual-plate shield. Also, multi-shock shields made of thin sheets of fiber materials are used. In order to understand hypervelocity impact phenomena and damage over structures under this kind of impacts, different approaches are used.
Figure 1.1. Hypervelocity impacts over Mir Docking Module.
One of these approaches is based on the use of ballistic limit equations; these equations can predict the behavior of the structure under impact. In this work, we present two new ballistic limit equations based on elastic-plastic theory of a basic structure comprising a shield plate and a back plate.

The theory of impact on thin targets and shields and correlation with experiments are used to develop those ballistic limit equations, in the case where a simple two-plate structure under hypervelocity impact is considered that consists of a back-plate protected by a shield-plate. A projectile impacts the shield plate and it passes through the shield-plate, which is molten or vaporized. The back plate will be given an impulsive load by the impact of the projectile-shield debris. The analysis of the back-plate under this impulsive load will be used, assuming the plate is loaded on a circular sector. Plates are flat structural elements with a thickness much smaller than the other dimensions. The partial differential equation for plates involves the vertical displacement and the moments. Solutions will depend on the boundary conditions and the geometry of the plate.

In the elastic-plastic regime, it can be difficult to derive analytical solutions for those quantities. In order to solve the partial different equation for plates, we use the boundary element method, which is a numerical technique to obtain solutions to a wide variety of problems encountered in engineering analysis. The accuracy of the numerical solution given by this procedure depends on the discretization of the boundary, the geometry of the plate and the boundary conditions.
Figure 1.2. Hypervelocity impacts over the radiators of the Kvant-2 module.
With these assumptions in mind, a mathematical model is proposed in order to develop a ballistic limit equation. The mathematical model consists of a simply supported circular plate under constant load acting on a circular region in the center of the plate. The circular plate is divided in two regions; the first region corresponds to the central, loaded area and undergoes plastic deformation, so it is called the plastic region; the second region undergoes elastic deformation, and is therefore called the elastic region.

One of the new ballistic limit equations, which is developed to be used for homogeneous materials, will be compared with two existing ballistic limit equations, and ballistic limit curves will be plotted. Ballistic limit curves for composite-orthotropic materials are plotted in order to compare the behavior of those materials with that of homogeneous materials.
Chapter 2

A Review of Ballistic Limit Equation for Hypervelocity Impact

2.1 Introduction

In the design of aerospace structures, protection against hypervelocity impact is an important factor to consider. Structures in outer space are under the risk of impact with projectiles traveling at velocities of the order of 5 Km/s and more. Different arrangements are used in order to protect those structures, the use of a thin plate in the path of the projectile before it reaches the structure being called dual-plate shield. Also, multi-shock shields made of thin sheets of fiber materials are used. In order to understand hypervelocity impact phenomena and damage to structures under this kind of impacts, different approaches are used. One of these approaches is based on the use of ballistic limit equations; these equations can predict the behavior of the structure under impact. In this chapter, two ballistic limit equations will be analyzed; the first one is an empirical ballistic limit equation for a multi-shock shield and has been developed by Christiansen [1] by using experimental results. The second one is an analytical ballistic limit equation and has been developed by Angel and Smith [2] by using experimental and analytical results. In order to compare these two ballistic limit equations, Angel has plotted curves for the critical projectile radius versus the projectile velocity for the empirical and analytical ballistic limit equations for several cases and the results will be shown in section 2.4.
2.2 An empirical ballistic limit equation

Christiansen [1] has developed an empirical ballistic limit equation for multi-shock shields. Using this equation, Angel and Smith [2] and Angel and Whitney [3] have constructed the curve shown in Figure 2.1, which shows the critical projectile radius versus the velocity of the projectile and the interpretation of this curve is that any projectile with a radius larger than the critical radius will cause failure in the back-plate.

![Figure 2.1. Critical projectile radius versus projectile velocity.](image-url)
Figure 2.1 shows for velocities between 0 Km/s and 3 Km/s that the critical projectile radius decreases when the velocity of the projectile increases. In this region, the shock pressure is insufficient to melt the combined debris from the projectile and the front plate. For velocities between 3 Km/s and 6 Km/s the critical projectile radius increases when the velocity of the projectile increases. In this region, the shock pressure melts partially the debris from the projectile and the front plate. Above 6 Km/s, the shock pressure melts the combined debris from the projectile and the front plate, the critical projectile radius decreases when the projectile velocity increases and damage over the back-plate is caused by plastic deformations. In the range of velocities above 6 Km/s, which will be considered in this chapter, the empirical ballistic limit equation is

\[ 2r = 0.354 \left( \frac{2h}{\rho p^{-1/3} \rho^{1/3} (U_m \cos \theta)^{-1/3} D^{2/3}} \right)^{1/6} \left( \frac{\sigma_0}{276} \right)^{1/6}, \tag{2.1} \]

where \( 2r \) is the projectile diameter in centimeters, \( 2h \) is the back-plate thickness in centimeters, \( D \) is the distance between the outer sheet and the back-plate in centimeters, \( \rho_p \) is the mass density of the projectile in grams per cubic centimeter, \( \rho \) is the mass density of the back-plate in grams per cubic centimeter, \( U_m \) is the projectile velocity in Kilometer per second, \( \sigma_0 \) is the static yield strength of the back-plate in MegaPascals, and \( \theta \) is the angle of impact measured from the normal. In this work \( \theta = 0 \) will be considered.
2.3 An analytical ballistic limit equation

Angel and Smith [2] have developed an analytical ballistic limit equation for velocities above 6 Km/s, by considering a simply supported circular plate of radius $a$ and thickness $2h$ subjected to a circular load of radius $R$, as shown in Figure 2.2.

![Figure 2.2. Plate subjected to a uniformly distributed circular load.](image)

The plate is made of a rigid perfectly-plastic material, and the load $p(t)$ is considered uniformly distributed with a maximum value $P_m$ and is acting in an interval of time $T$. In order to calculate the vertical displacement at the center of the plate, Angel uses the following equation, from Timoshenko [4]

$$W = \frac{P_m}{16\pi F} \left[ \frac{3 + \nu}{1 + \nu} a^2 - R^2 \ln \left( \frac{a}{R} \right) - \frac{7 + 3\nu}{4(1 + \nu)} R^2 \right].$$  \hspace{1cm} (2.2)
where $\nu$ is Poisson's ratio and $F$ is the flexural rigidity of the plate. Equation (2.2) corresponds to the elastic case. In order to develop the analytical ballistic limit equation, Angel used equation (2.2) and experimental information from six multi-shock experiments consisting of square back plates of side 15.24 centimeters protected by four Nextel square shields of side 15.24 centimeters, as shown in Figure 2.3.

![Figure 2.3. Impact of a projectile on a multi-shock shield.](image)

The experimental data is shown in Table 2.1, corresponding to aluminum spherical projectiles with a diameter of 3.175 mm and a mass density $\rho_p = 2796 \, \text{Kg/m}^3$. These projectiles impact the outer Nextel shield-plate normally and the debris from the projectile and the shield impacts the back-plate causing plastic deformations. The yield strength of the back-plate material is
\( \sigma_0 = 344.74 \text{ Mpa} \) and the mass density is \( \rho = 2768 \text{ Kg/m}^3 \) for all experiments, except for A1230 where the yield strength is \( \sigma_0 = 275.79 \text{ Mpa} \) and the mass density is \( \rho = 2713 \text{ Kg/m}^3 \).

Table 2.1. Experimental data.

<table>
<thead>
<tr>
<th>Shot</th>
<th>( U_m ) (Km/s)</th>
<th>( m_p ) (mg)</th>
<th>( 2h ) (mm)</th>
<th>( D ) (cm)</th>
<th>( W ) (mm)</th>
<th>( R ) (cm)</th>
<th>( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1229</td>
<td>6.49</td>
<td>47.04</td>
<td>0.508</td>
<td>10.16</td>
<td>4.018</td>
<td>2.08</td>
<td>0.0402</td>
</tr>
<tr>
<td>A1230</td>
<td>6.32</td>
<td>46.92</td>
<td>0.635</td>
<td>10.16</td>
<td>2.515</td>
<td>2.39</td>
<td>0.0522</td>
</tr>
<tr>
<td>A1233</td>
<td>6.26</td>
<td>46.84</td>
<td>0.813</td>
<td>10.16</td>
<td>1.102</td>
<td>2.12</td>
<td>0.0417</td>
</tr>
<tr>
<td>A1235</td>
<td>6.24</td>
<td>46.86</td>
<td>0.635</td>
<td>10.16</td>
<td>1.834</td>
<td>2.19</td>
<td>0.0442</td>
</tr>
<tr>
<td>A1237</td>
<td>6.20</td>
<td>46.84</td>
<td>0.813</td>
<td>7.620</td>
<td>2.118</td>
<td>2.16</td>
<td>0.0744</td>
</tr>
<tr>
<td>A1253</td>
<td>6.51</td>
<td>46.90</td>
<td>1.600</td>
<td>5.080</td>
<td>1.283</td>
<td>1.92</td>
<td>0.1250</td>
</tr>
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The parameter \( K \) in Table 2.1 depends on the load radius \( R \) and the spacing \( D \) between the back plate and the outer shield-sheet, as in Figure 2.3; and relationships for these parameters are taken from debris cloud dynamics theory. In particular, the time duration of the impact is given by

\[
T_L = \frac{2D(1+K)\sqrt{K}}{U_m(1-K)}.
\]  

(2.3)
These parameters are used to develop an analytical ballistic limit equation using the values of $K$ from Table 2.1. For the dynamic yield strength, the following relationship will be used

$$\sigma = \lambda \sigma_0,$$  

(2.4)

where $\sigma$ is the dynamic yield strength and the value $\lambda = 4.5$ is selected by Angel from experimental information. Using the parameters described and the experimental information in Table 2.1, one finds that the ballistic limit equation has the form

$$r = \left[ \frac{9\lambda \mu (1 + K) \sqrt{K}}{2(1 - K)} \right]^{1/3} \sigma_0^{1/3} D^{1/3} h^{2/3} \rho_p^{-1/3} U_m^{-2/3},$$  

(2.5)

where the parameters $K$, $\lambda$, and $\mu$ take the following values from experimental information

$$K = 0.04, \quad \lambda = 4.5, \quad \mu = 50.$$  

(2.6)

### 2.4 Comparison of the Ballistic Limit Equations

In order to compare the empirical ballistic limit equation (2.1) and the analytical ballistic limit equation (2.5), curves for the critical projectile radius versus the velocity of the projectile are plotted by Angel for different cases, keeping the
other variables fixed. The first case concerns experiment A1229 and the curve is shown in Figure 2.4.

![Graph showing critical projectile radius versus projectile velocity for experiment A1229.](image)

Figure 2.4. Critical projectile radius versus projectile velocity for experiment A1229.

The solid line corresponds to equation (2.5) and the dotted line corresponds to equation (2.1). Observe that the two curves are very close, the region below each curve is the safe region, and the solid line is more conservative than the dotted one. The point value for the experiment A1229 is plotted too. The second case is for experiment A1237 and the curve is shown in Figure 2.5.
Figure 2.5. Critical projectile radius versus projectile velocity for experiment A1237.

For this experiment, the curves cross near $U_m = 12 \text{ Km/s}$, the solid line is more conservative for higher velocities and the dotted lines is more conservative for lower velocities, the point value for experiment A1237 is plotted too.

The third case is for experiment A1253 and the curve is shown in Figure 2.6. The solid line is above the dotted one and predicts that shot A1253 is not near failure, which is consistent with the experimental information.
Figure 2.6. Critical projectile radius versus projectile velocity for experiment A1253.

Other types of curves can be plotted from the ballistic limit equations (2.1) and (2.5). Figure 2.7 shows the curve for the half-thickness of the back plate versus the velocity of the projectile for experiment A1229.

Figure 2.7. Critical back-plate half-thickness versus projectile velocity for experiment A1229.
The regions above the lines are the safe regions; four experimental points are included in the plot and lie in the safe region relative to the dotted curve.

2.5 Conclusion

Two ballistic limit equations have been analyzed in this chapter. The first one is an empirical ballistic limit equation that has been developed by Christiansen [1] using experimental information and has the form

$$2r = 0.354 \left( 2h \right)^{1/3} \rho_p^{-1/3} \rho^{1/3} \left( U_m \cos \theta \right)^{-1/3} D^{2/3} \left( \frac{\sigma_0}{276} \right)^{1/6},$$

(2.7)

where $2r$ is the projectile diameter in centimeters, $2h$ is the back-plate thickness in centimeters, $D$ is the distance between the outer sheet and the back-plate in centimeters, $\rho_p$ is the mass density of the projectile in grams per cubic centimeter, $\rho$ is the mass density of the back-plate in grams per cubic centimeter, $U_m$ is the projectile velocity in Kilometers per second, $\sigma_0$ is the static yield strength of the back-plate in MegaPascals, and $\theta$ is the angle of impact measured from the normal.

The second equation considered is an analytical ballistic limit equation that has been developed by Angel and Smith [2] using an analytical procedure and experimental information; it has the form
\begin{equation}
    r = \left[ \frac{9 \lambda \mu(1 + K) \sqrt{K}}{2(1 - K)} \right]^{1/3} \sigma_0^{1/3} D^{1/3} h^{2/3} \rho_p^{-1/3} U_m^{-2/3},
\end{equation}

where the parameters $K$, $\lambda$, and $\mu$ are given values from experimental information such that $K = 0.04$, $\lambda = 4.5$, and $\mu = 50$.

In order to compare the two ballistic limit equations, curves are plotted for different cases and are shown in Figures 2.4, 2.5, and 2.6 for the critical projectile radius versus projectile velocity; and in Figure 2.7 for the half-thickness of the back-plate versus projectile velocity. Equations (2.7) and (2.8) have different exponents for the parameters $h$, $\sigma_0$, $D$, and $U_m$, the exponent of the parameter $\rho_p$ is the same for both equations. These equations will be compared with one of the new ballistic limit equations developed in this work. In order to develop a new ballistic limit equation, some physical concepts will be reviewed in chapter 3.
Chapter 3

Review of Physical Concepts about Hypervelocity Impact

3.1 Introduction

The theory of impact on thin targets and shields and correlation with experiments is reviewed in this chapter. A simple two-plate structure under hypervelocity impact is considered that consists of a back plate, which is protected by a shield plate. A projectile impacts the shield plate and the projectile may undergo a variety of processes depending upon impact conditions such as projectile velocity, projectile material, angle of impact, shield-plate material strength, and thickness of the shield plate. The projectile may be stopped by the shield plate, may pass through the shield plate essentially undamaged, or may pass through the shield plate fractured, molten, or vaporized. The last two cases correspond to situations where the velocities are sufficiently high to cause melting or vaporization in the shield plate and the projectile. The back plate will be given an impulsive load by the impact of the projectile-shield debris. The analysis of the back plate under this impulsive load will be used to develop a new ballistic limit equation.

3.2 Hypervelocity impact and debris

A basic two-plate structure is used in order to review the physical concepts involved in hypervelocity impact. The structure, shown in Figure 3.1, comprises a
back plate made of homogeneous or composite materials and a shield plate to protect the back plate. A projectile impacts the shield plate, and upon impact a debris cloud forms from the projectile and the shield-plate. The impact over the shield produces a shock pressure and, for velocities above 6 Km/s, the shock pressure is sufficiently high to produce molten debris. Next, the cloud debris impacts the back plate and that is considered like load acting over the back plate.

Figure 3.1. Basic structure composed of back plate and shield plate.

The shock pressure depends on the velocity of the projectile and the materials properties of the shield plate. The values of shock pressure sufficient to cause incipient melting, complete melting, and vaporization when the materials are returned to atmospheric pressure is shown in Table 3.1 from Kinslow [4]. Some experiments have been reported in order to support the theoretical analysis and will be analyzed in the next section.
Table 3.1. Pressure impact for shield-plate materials.

<table>
<thead>
<tr>
<th>Shield-Plate material</th>
<th>Pressure to cause incipient melting (Gpa)</th>
<th>Pressure to cause complete melting (Gpa)</th>
<th>Pressure to cause vaporization (Gpa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>60</td>
<td>90</td>
<td>-</td>
</tr>
<tr>
<td>Cadmium</td>
<td>40</td>
<td>46</td>
<td>80</td>
</tr>
<tr>
<td>Copper</td>
<td>140</td>
<td>180</td>
<td>-</td>
</tr>
<tr>
<td>Gold</td>
<td>150</td>
<td>160</td>
<td>-</td>
</tr>
<tr>
<td>Iron</td>
<td>-</td>
<td>200</td>
<td>-</td>
</tr>
<tr>
<td>Lead</td>
<td>30</td>
<td>35</td>
<td>100</td>
</tr>
<tr>
<td>Magnesium</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Nickel</td>
<td>150</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Titanium</td>
<td>100</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

3.3 Pressure impact and melting of the materials

In support of the theoretical analysis a number of experimental tests were reported by Kinslow [4]. Shield-plate materials of cadmium, copper, and nickel were chosen for tests, and the back plate was made of 6.4 mm thickness aluminum and placed 5.08 cm behind the shield plate, whose thickness was kept constant at 1.016 mm. The first series of tests was conducted using 6.35 mm aluminum spheres at 4.16 Km/s. The back plates in each of these tests are shown in Figure 3.2.
Figure 3.2 shows the damage of the back-plate, the maximum impact pressure is only sufficiently high to melt the cadmium debris from the shield-plate, and no melting of copper or nickel will occur. As a result of this, the frontal damage to the back plate with a cadmium shield plate is plastic deformation, because the molten cadmium particles are extremely small. With the copper and nickel shield plates, larger solid fragments are present and cause frontal damage to the targets.

In the second series of tests, 6.35 mm aluminum spheres were impacted at 7.31 Km/s, which makes the shock pressure sufficient to cause complete melting of the aluminum projectiles. Also, the shock pressure is sufficient to vaporize the cadmium and just cause incipient melting to copper. Figure 3.3 confirms these theoretical predictions in which a splash of aluminum can be observed on all three back plates.
Figure 3.3. Back-plate damage for the second series of tests.

For the third series of tests, 3.2 mm steel spheres were used at 7.06 Km/s. The back plates are shown in Figure 3.4, and once again the cadmium shield is best; the copper is melted and the cadmium and copper are almost equally effective as shield-plate materials.

Figure 3.4. Back-plate damage for the third series of tests.
The debris from the projectile and shield plate causes damage to the back plate, where the damage can be penetration, fracture and plastic deformations. For velocities above 6 Km/s, the complete melting of the shield plate and projectile causes plastic deformation in the back plate, which is the matter to be considered in this work.

3.4 Analytical approach for hypervelocity impact in plates

Hypervelocity impact over a two-plate structure will be modeled as a load acting over the back plate, and factors, such as the magnitude, distribution, and duration of the load applied to the back plate must be determined. Then the deformation dynamic response of the back plate has to be calculated, taking into account the elastic and plastic behavior of the back plate. From Kinslow [4], the following assumptions can be used for the load: First, the load is uniformly distributed over a circular region with a diameter equal to one-half the spacing; second, the load is applied so quickly that the loaded area is effectively given an initial velocity increment; and third, the momentum transferred to the loaded region is equal to twice the momentum of the original projectile.

The first and second assumptions are justified from the experimental information concerning the spacing between the plates and the diameter of the loaded area. From Gehring [5], the third assumption, concerning momentum multiplication by a factor 2, after impact at this velocity, can be justified on physical ground since most of the debris coming through the shield will be in gaseous form. If it is assumed that the momentum of the debris is equal to the momentum of the
original projectile, and that perfectly elastic collisions occur between each gas atom and the back plate, then the momentum multiplication factor should be two. From the assumptions described above, the initial velocity of the loaded area is (from Kinslow [4])

$$V_I = \frac{32 M_p V_p}{\pi S^2 \rho_b h},$$

(3.1)

where $M_p$ and $V_p$ are the mass and velocity of the impacting projectile, $S$ is the spacing between the plates and is shown in Figure 3.1, $\rho_b$ and $h$ are the mass density and thickness of the back-plate.

3.5 Conclusion

A brief review of the physics of hypervelocity impact has been presented here, using a two-plate structure, which consists of a shield plate and a back plate, as shown in Figure 3.1. A projectile impacts the structure and, if the shock pressure is sufficiently high, the projectile and the shield plate debris become molten or vaporized forming a debris cloud that acts over the back plate. The shock pressure depends on the projectile velocity and the mechanical properties of the shield plate. Some experiments, reported by Kinslow [4], have been analyzed and discussed. One important conclusion from these experiments is the following. If the shock pressure is sufficiently high to melt the debris and the shield plate, the debris cloud that is formed impacts the back plate and the
damage is plastic deformation, and this phenomenon occurs for velocities above 6 Km/s. For this particular case, some assumptions can be used in order to build a mathematical model for hypervelocity impact on a plate.

1. The load is uniformly distributed over a circular region with a diameter equal to one-half the spacing.
2. The load is applied so quickly that the loaded area is effectively given an initial velocity increment.
3. The momentum transferred to the loaded sector is equal to twice the momentum of the original projectile.

From these assumptions, an expression for the initial velocity of the loaded area is derived and is expressed as in equation (3.1). In the next chapters, equation (3.1) and the three basic assumptions mentioned above will be used in order to develop a new ballistic limit equation for hypervelocity impact.
Chapter 4

Homogeneous Plate Theory

4.1 Introduction

In the development of a new ballistic limit equation for hypervelocity impact on homogeneous or composites materials, the analytical approach discussed in section 3.4 will be used. This approach involves a plate loaded on a circular region. In order to develop the ballistic limit equation, the Kirchhoff plate’s theory is reviewed (Ugural [6] and Timoshenko [7]). Plates are flat structural elements with a thickness much smaller than the other dimensions; the thickness can be divided into equal halves by a plane parallel to the faces, which is called the midplane of the plate. The plate thickness is measured in a direction normal to the midplane. Plates can be classified in two groups: thin plates and thick plates. By definition, a thin plate is such that the ratio of the thickness to the smaller span length should be less than 1/20. In this work, we will consider thin plates, and in order to discuss the mathematical model for thin plates some definitions are discussed.

4.2 Definitions about plates

Consider a midplane of the plate shown in figure 4.1, in which the \( xy \) plane coincides with that surface, and let the third axis \( z \) be normal to the surface. Now the surfaces perpendicular to the \( z \) direction at \( z = +h/2 \) and at \( z = -h/2 \),
respectively, are called the top surface and the bottom surface, and $h$ is the plate thickness.

![Diagram of a plate model](image)

**Figure 4.1. Mathematical model of a plate.**

The displacement components at a point $(x, y, z)$ are denoted by $u$, $v$, and $w$, respectively. The following assumptions can be made for isotropic, homogeneous, elastic thin plates: the deflection of the midplane is small compared with the thickness of the plate, the midplane remains unstrained subsequent to bending, the plane sections initially normal to the midsurface remain plane and normal to that surface after bending, as illustrated in Figure 4.2. One important consequence from these assumptions is that the vertical
shear strains are negligible. The stress normal to the midplane is small compared with the other stress components and may be neglected.

Figure 4.2. Plane sections initially normal to the midplane remain plane and normal to that surface after bending.

The assumptions cited above are known as the Kirchhoff hypotheses and will be analyzed later and used to develop a partial differential equation for plates.

4.3 Strains in Kirchhoff's plate theory

Elasticity theory requires a relationship between strain and displacement. Using the assumptions mentioned in section 4.2, the strain-displacement relationships are
$$
\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z} = 0, \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0, \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0. \quad (4.1)
$$

From the relationship for $\varepsilon_z$, we can conclude that the displacement $w$ depends only on $x$ and $y$, which yields $w = w(x, y)$. From the expressions for the shear strains $\gamma_{xz}$ and $\gamma_{yz}$, and taking into account $u = 0$ and $v = 0$, when $z = 0$, we can conclude after integration along $z$ that

$$
u = -z \frac{\partial w}{\partial y}.
$$

Now, the expressions for the displacements $u$ and $v$ in terms of the displacement $w$ yield expressions for the strains $\varepsilon_x$, $\varepsilon_y$, and $\gamma_{xy}$ at any point of the plate. Substituting equations (4.2) into equations (4.1), one finds that
\[ \varepsilon_x = -z \frac{\partial^2 w}{\partial x^2}, \]
\[ \varepsilon_y = -z \frac{\partial^2 w}{\partial y^2}, \]
\[ \gamma_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y}. \]

Now, we can define the curvatures of a plane, as the rate of change of the slope angle of the curve with respect to distance along the curve. The three relevant curvatures are

\[ K_x = \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial x} \right), \]
\[ K_y = \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial y} \right), \]
\[ K_{xy} = \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial y} \right). \]

In order to have a relationship between strains and curvatures, we substitute (4.4) into (4.3). The results are
\[ \varepsilon_x = -zK_x, \]

\[ \varepsilon_y = -zK_y, \]

\[ \gamma_{xy} = -2zK_{xy}. \]  

The relationships derived in this section can be used in elastic and inelastic problems because no mechanical properties have been used. In the next section, we will derive stress-strain relationships in the elastic case.

### 4.4 Stress-strain relationships in Kirchhoff’s plate theory

The generalized Hooke’s law gives a stress-strain relationship for the elastic case. For Kirchhoff plate’s theory, one of the assumptions mentioned in section 4.1 is: “the stress normal to the midplane is small compared with the other stress components and may be neglected”, and from equations (4.1) the strains in the normal direction are neglected. It follows that the relationships between stress and strain, from the generalized Hooke’s law are reduced to

\[ \sigma_x = \frac{E}{(1-\nu^2)}(\varepsilon_x + \nu\varepsilon_y), \]

\[ \sigma_y = \frac{E}{(1-\nu^2)}(\varepsilon_y + \nu\varepsilon_x), \]

\[ \sigma_{xy} = G\gamma_{xy}. \]
where the constants $E$, $\nu$, and $G$ represent the modulus of elasticity, Poisson's ratio, and the shear modulus of elasticity, respectively, and

$$G = \frac{E}{2(1 + \nu)}.$$  \hfill (4.7)

Now, substituting equations (4.3) into equations (4.6), we find a stress-displacement relationship for Kirchhoff plate's theory in the form

$$\sigma_x = -\frac{Ez}{(1-\nu^2)} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right),$$

$$\sigma_y = -\frac{Ez}{(1-\nu^2)} \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right),$$

$$\sigma_{xy} = -\frac{Ez}{(1+\nu)} \frac{\partial^2 w}{\partial x \partial y}.$$ \hfill (4.8)

The stresses distributed over the thickness of the plate produce bending moments, twisting moments, and vertical shear forces. These moments and forces per unit length are called stress resultants. The moments are

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} \int_{-h/2}^{h/2} \sigma_x \, dz \\ \int_{-h/2}^{h/2} \sigma_y \, dz \\ \int_{-h/2}^{h/2} \sigma_{xy} \, dz \end{bmatrix}.$$ \hfill (4.9)
And the shear forces are

\[
\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} \, dz.
\]  \hspace{1cm} (4.10)

The integration of equations (4.8) yields formulae for the bending and twisting moments in terms of the deformation \( w \) in the form

\[
M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right),
\]

\[
M_y = -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right),
\]  \hspace{1cm} (4.11)

\[
M_{xy} = -D (1-\nu) \frac{\partial^2 w}{\partial x \partial y},
\]

where

\[
D = \frac{E h^3}{12(1-\nu^2)}
\]  \hspace{1cm} (4.12)

is the flexural rigidity of the plate. In the next section, we will derive a partial differential equation for homogeneous thin plates.
4.5 The governing partial differential equation in Kirchhoff’s plate theory

In order to develop a governing equation for plates we consider an element \( dx \, dy \) of the plate subjected to a uniformly distributed load per unit of area \( q \), as shown in Figure 4.3.

![Figure 4.3. An element of the plate with forces and moments.](image)

The condition that the sum of the forces in the \( z \) direction must be equal to zero is expressed in the following equation

\[
\frac{\partial Q_x}{\partial x} \, dx \, dy + \frac{\partial Q_y}{\partial y} \, dx \, dy + q \, dx \, dy = 0. \tag{4.13}
\]

Equation (4.13), using \( p \) instead of \( q \), yields
\[ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0. \quad (4.14) \]

The sum of the moments about the \( x \) axis is

\[ \frac{\partial M_{xy}}{\partial x} \, dx \, dy + \frac{\partial M_y}{\partial y} \, dx \, dy - Q_y \, dx \, dy = 0, \quad (4.15) \]

or

\[ \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0. \quad (4.16) \]

Similarly, from the sum of the moments about the \( y \) axis, one infers that

\[ \frac{\partial M_{xy}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0. \quad (4.17) \]

Now, we can obtain expressions for \( Q_x \) and \( Q_y \) from equations (4.16) and (4.17) and substitute them in equation (4.14), which yields the differential equation of equilibrium
\[ \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p = 0. \quad (4.18) \]

The partial differential equation (4.18) is derived from the sum of forces and moments over the plate and is independent of the mechanical properties of the plate. For homogeneous plates we have a relationship between the moments and the vertical displacement \( w \) given by equations (4.11).

Now, substituting (4.11) into (4.18) yields a partial differential equation for homogeneous plates in terms of the vertical displacement \( w \). One has

\[ D \nabla^4 w = p, \quad (4.19) \]

where

\[ \nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \quad (4.20) \]

Integration of the partial differential equation (4.19) yields \( w \), in terms of some unknown integration constants and, in order to calculate them, we must use appropriate boundary conditions. In the next section, we discuss the boundary conditions.
4.6 Boundary conditions for Kirchhoff’s plate theory

The boundary conditions for plates allow one to calculate the integration constants when the partial differential equations (4.19) is integrated. As an example, consider the plate shown in Figure 4.4 for the bending of a simply supported plate.

![Figure 4.4. Simply supported rectangular plate with the corresponding boundary conditions.](image)

From figure 4.4, the vertical displacement and the normal moment at any point along the boundary are zero.
\[ w(x,0) = 0, \quad w(x,b) = 0, \quad w(0,y) = 0, \quad w(a,y) = 0, \quad (4.21) \]

\[ M_x(0,y) = 0, \quad M_x(a,y) = 0, \quad M_y(x,0) = 0, \quad M_y(x,b) = 0. \]

The boundary conditions (4.21) can be used in order to calculate the mentioned integration constants, which result from the integration of the partial differential equation (4.19).

The next example to consider is a clamped rectangular plate, where the vertical displacement and the normal slope along the boundary are zero. Thus, one has

\[ w(x,0) = 0, \quad w(x,b) = 0, \quad w(0,y) = 0, \quad w(a,y) = 0, \quad (4.22) \]

\[ \frac{\partial w}{\partial x}(0,y) = 0, \quad \frac{\partial w}{\partial x}(a,y) = 0, \quad \frac{\partial w}{\partial y}(x,0) = 0, \quad \frac{\partial w}{\partial y}(x,b) = 0. \]

Other examples can include simply supported and clamped edges. For example a plate can have two opposite edges simply supported and two opposite edges clamped.

In order to illustrate the application of the partial differential equation (4.19) and the use of the boundary conditions, two examples will be analyzed in the next sections.
4.7 Example 1: simply supported rectangular plate under a constant load

Consider the rectangular plate of sides $a$ and $b$ shown in Figure 4.4, where the plate is simply supported on all edges and subjected to a distributed constant load $P_0$.

The solution of the bending problem uses a Fourier series expansion for load and deflection is given as

$$P(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right),$$

$$w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right),$$

where $p_{mn}$ and $a_{mn}$ represent coefficients to be determined. This approach is called the Navier solution. The deflection must satisfy the partial differential equation (4.19) together with the boundary conditions (4.21). One finds that the coefficients of the double Fourier expansion are

$$p_{mn} = \int_0^b \int_0^a P(x, y) \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) dx dy,$$

$$a_{mn} = \frac{p_{mn}}{\pi^4 D \left[ \left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2 \right]^2}.$$
Substituting the second expression in (4.24) into the second expression in (4.23) we can derive the equation of the deflection surface of the plate, which becomes

\[ w = \frac{1}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_{mn}}{\left[ (m/a)^2 + (n/b)^2 \right]^2} \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right), \tag{4.25} \]

where \( p_{mn} \) is given by the first expression in (4.24). For the case when the load \( p(x, y) \) is a constant \( P_0 \), after integration one has

\[ p_{mn} = \frac{16 P_0}{\pi^2 mn}, \quad (m, n = 1, 3, \ldots). \tag{4.26} \]

Substituting (4.26) into (4.25), we have

\[ w = \frac{16 P_0}{\pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right)}{mn \left[ (m/a)^2 + (n/b)^2 \right]^2}, \quad (m, n = 1, 3, \ldots). \tag{4.27} \]

The expression (4.27) is the solution for the vertical displacement of the plate that satisfies the boundary conditions in (4.21).
In order to calculate the moments $M_x$, $M_y$ and $M_{xy}$, the expression (4.27) is substituted into the expressions in (4.11). After this substitution, the moments become

$$M_x = \frac{16P_0}{\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(m/a)^2 + \nu(n/b)^2}{mn[(m/a)^2 + (n/b)^2]^2} \sin \left( \frac{m\pi x}{a} \right) \sin \left( n\frac{\pi y}{b} \right),$$

$$M_y = \frac{16P_0}{\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\nu(m/a)^2 + (n/b)^2}{mn[(m/a)^2 + (n/b)^2]^2} \sin \left( \frac{m\pi x}{a} \right) \sin \left( n\frac{\pi y}{b} \right), \quad (4.28)$$

$$M_{xy} = -\frac{16P_0(1-\nu)}{\pi^4 ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn[(m/a)^2 + (n/b)^2]^2} \cos \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right).$$

The analytical solutions for the vertical displacement and for the moments have been calculated using the Navier solution. In order to calculate numerical values for particular cases, we will truncate the infinite Fourier series, and convergence criteria will be used in order to obtain approximate numerical solutions.

### 4.8 Example 2: simply supported circular plate under a constant load

A simply supported circular plate of radius $a$ under a uniformly distributed load $P_0$ is shown in Figure 4.5. The partial differential equation for plates, in polar coordinates, is used in order to solve this problem. One has
\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) = \frac{p}{D}, \tag{4.29}
\]

where \( r \) and \( \theta \) are the polar coordinates. The vertical displacement \( w \) of the plate will depend upon radial position \( r \) only when the applied load and the boundary conditions are independent of the angle \( \theta \). Under these circumstances, the partial differential equation (4.29) becomes

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) = \frac{p}{D}. \tag{4.30}
\]

![Diagram of a simply supported circular plate](image)

Figure 4.5. Simply supported circular plate under a constant load.

Then the moments and shear force are
\[ M_r = -D \left( \frac{d^2 w}{dr^2} + \frac{v}{r} \frac{dw}{dr} \right), \quad (4.31) \]
\[ M_\theta = -D \left( \frac{1}{r} \frac{dw}{dr} + \nu \frac{d^2 w}{dr^2} \right), \]
\[ Q_r = -D \frac{d}{dr} \left( \frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right). \]

In this example, the load is constant. The partial differential equation (4.30) can be integrated, for \( p = P_0 \), which yields the vertical displacement \( w \) in the form

\[ w = C_1 \ln r + C_2 r^2 \ln r + C_3 r^2 + C_4 + \frac{P_0}{64D} r^4, \quad (4.32) \]

where the \( C \)'s are constants of integration, which are calculated in terms of the following boundary conditions

\[ w(a) = 0, \quad M_r(a) = 0. \quad (4.33) \]

The displacement must be finite at \( r = 0 \); the values of \( C_1 \) and \( C_2 \) in equation (4.32) are therefore zero. Using the boundary conditions (4.33) we calculate the constants \( C_3 \) and \( C_4 \), which are given by
\[ C_3 = -\frac{P_0 a^2}{32 D} \frac{3 + \nu}{1 + \nu}, \quad C_4 = \frac{P_0 a^4}{64 D} \frac{5 + \nu}{1 + \nu}. \quad (4.34) \]

The vertical displacement of the plate is

\[ w = \frac{P_0 a^4}{64 D} \left( \frac{r^4}{a^4} - 2 \frac{3 + \nu}{1 + \nu} \frac{r^2}{a^2} + \frac{5 + \nu}{1 + \nu} \right). \quad (4.35) \]

Given the vertical displacement \( w \), the moments can be obtained using the first two expressions in (4.31). One has

\[ M_r = \frac{P_0}{16} (3 + \nu) \left( a^2 - r^2 \right), \quad (4.36) \]

\[ M_\theta = \frac{P_0}{16} \left[ (3 + \nu) a^2 - (1 + 3\nu) r^2 \right]. \]

The equation for a simply supported circular plate under a constant load has been solved using polar coordinates, using the partial differential equation (4.30), and the solution for the vertical displacement \( w \) is given by (4.35).
4.9 Conclusion

A review of Kirchhoff plate's theory has been presented in this chapter. The equilibrium equation for moments is shown in equation (4.18), which does not involve any material properties. For homogeneous plates, the relationships between the moments and the transversal displacement are given in equations (4.11), and using them we derive a partial differential equation for homogeneous plates which is given by (4.19)

\[ D \nabla^4 w = p. \]  (4.37)

This partial differential equation with the appropriate boundary conditions yields solutions to different problems for bending of plates. In the next chapter, a numerical method will be discussed in order to obtain numerical solutions for the bending of homogeneous plates and, in order to estimate the numerical accuracy, a problem with an analytical closed-form solution will be used. The problem is presented in section 4.8, and choosing \( r = 0 \) in (4.35), one finds that the vertical displacement is

\[ w = \left( \frac{5 + \nu}{1 + \nu} \right) \frac{P_0 a^4}{64 D}. \]  (4.38)

The expression (4.38) is an analytical closed-form solution for the deflection of the central point of a simply supported circular plate under a constant load \( P_0 \).
Chapter 5

Boundary Elements Method for Homogeneous Plates

5.1 Introduction

The Boundary Elements Method is a numerical technique to obtain approximate solutions to a wide variety of problems encountered in engineering analysis. Paris and Cañas [8], Banerjee [9], Amen [10], Miranda-Valenzuela, and Muci-Kuchler [11], and Brebbia and Dominguez [12] have discussed the advantages of this numerical method. The problem to be considered in this work is the bending of homogeneous plates. The partial differential equation for plates, analyzed in chapter 4, involves various quantities such as the vertical displacement and the moments, and the geometry of the plate, and its solution depends on the boundary conditions. In the elastic-plastic regime, it can be difficult to derive analytical solutions. In this chapter we use the Boundary Elements Method in order to solve the partial differential equation for homogeneous plates developed in chapter 4. The procedure starts with the development of an integral formulation using the equilibrium equation for moments discussed in section 4.4, which results in a formulation involving integrals on the boundary and on the domain of the plate. The next step is the use of the relationship for the moments and the vertical displacement, which yields an integral equation for the vertical displacement. Integration by parts is used to obtain an integral equation involving only integrals on the boundary. The accuracy of the numerical solution given by
this procedure depends on the discretization of the boundary, the geometry of the plate and the boundary conditions.

5.2 Boundary element formulation for thin plates

The plate-bending problem is expressed in terms of a partial differential equation together with a loading, boundary conditions, and a plate geometry; it is often difficult to derive an analytical solution for the vertical displacement and the moments of the plate. Therefore the Boundary Elements Method is used to obtain a numerical solution for those quantities. To develop a boundary element formulation for plates, the equilibrium equation (4.18) for plates is used, which in index notation reads

\[ M_{\alpha \beta, \alpha \beta} + p = 0. \]  

(5.1)

The derivation starts with the weight residual process applied to equation (5.1). Integrating equation (5.1) multiplied by a weight function over the whole domain of the plate yields

\[ \int_{\Omega} (M_{\alpha \beta, \alpha \beta} + p) w^* \, d\Omega = 0, \]  

(5.2)
where \( w^* \) is a weight function called the fundamental solution that will be discussed later, and \( \Omega \) is the domain of the plate. Equation (5.2) is now expressed as

\[
\int_{\Omega} M_{\alpha \beta, \alpha \beta} w^* \, d\Omega + \int_{\Omega} p \, w^* \, d\Omega = 0. \tag{5.3}
\]

Integration by parts of the first integral in equation (5.3) gives

\[
\int_{\Omega} M_{\alpha \beta, \alpha \beta} w^* \, d\Omega = \int_{\Gamma} w^* M_{\alpha \beta, \alpha} n_\beta \, d\Gamma - \int_{\Omega} M_{\alpha \beta, \alpha} w^*_\beta \, d\Omega, \tag{5.4}
\]

where \( \Gamma \) is the boundary of the plate. Substituting (5.4) into equation (5.3), one has

\[
\int_{\Gamma} w^* M_{\alpha \beta, \alpha} n_\beta \, d\Gamma - \int_{\Omega} M_{\alpha \beta, \alpha} w^*_\beta \, d\Omega + \int_{\Omega} p \, w^* \, d\Omega = 0. \tag{5.5}
\]

An integration by parts of the second integral in equation (5.5) yields

\[
\int_{\Omega} M_{\alpha \beta, \alpha} w^*_\beta \, d\Omega = \int_{\Gamma} w^*_\beta M_{\alpha \beta} n_\alpha \, d\Gamma - \int_{\Omega} M_{\alpha \beta} w^*_\alpha n_\beta \, d\Omega. \tag{5.6}
\]

Substituting (5.6) into equation (5.5), one has
\[
\int_{\Gamma} w^* M_{\alpha\beta,\alpha} n_{\beta} \, d\Gamma - \int_{\Gamma} w^* M_{\alpha\beta} n_{\alpha} \, d\Gamma + \int_{\Omega} M_{\alpha\beta} w^*_{\alpha\beta} \, d\Omega + \int_{\Omega} p w^* \, d\Omega = 0.
\] (5.7)

Now, we define the following quantities

\[
q = M_{\alpha\beta,\alpha} n_{\beta},
\]

\[
M_{\beta} = M_{\alpha\beta} n_{\alpha}.
\] (5.8)

Substituting the quantities defined in (5.8) into equation (5.7), one finds that

\[
\int_{\Gamma} w^* M_{\beta} \, d\Gamma - \int_{\Gamma} w^* q \, d\Gamma = \int_{\Omega} M_{\alpha\beta} w^*_{\alpha\beta} \, d\Omega + \int_{\Omega} p w^* \, d\Omega.
\] (5.9)

The integral equation (5.9) does not depend on the material properties of the plate. In the next section we will use a relationship between the moments and the vertical displacement for homogeneous materials in order to obtain an integral equation for homogeneous plates.
5.3 Boundary element formulation for homogeneous plates

Equation (5.9) involves the moments $M_{\alpha \beta}$ and, for homogeneous plates, equations (4.11) give a relationship between the moments and the vertical displacement. In index notation, this relationship has the form

$$M_{\alpha \beta} = -D(1 - \nu) w_{,\alpha \beta} - D \nu w_{,\gamma \gamma} \delta_{\alpha \beta}.$$  \hspace{1cm} (5.10)

Substituting this last expression for the moments into the first integral in the right-hand side of equation (5.9), one find that

$$\int_\Omega M_{\alpha \beta} w_{,\alpha \beta}^* \, d\Omega = -D(1 - \nu) \int_\Omega w_{,\alpha \beta} w_{,\alpha \beta}^* \, d\Omega - D \nu \int_\Omega w_{,\alpha \alpha} w_{,\beta \beta}^* \, d\Omega.$$  \hspace{1cm} (5.11)

The first integral in the right-hand side of equation (5.11) after two integrations by parts yields

$$\int_\Omega w_{,\alpha \beta} w_{,\alpha \beta}^* \, d\Omega = \int_\Gamma w_{,\alpha \beta}^* w_{,\alpha} n_{\beta} \, d\Gamma - \int_\Gamma w_{,\alpha \beta} w_{,\alpha} n_{\beta} \, d\Gamma + \int_\Omega w_{,\alpha \alpha}^* w_{,\alpha \beta} \, d\Omega.$$  \hspace{1cm} (5.12)

The second integral in the right-hand side of equation (5.11) after two integrations by parts yields
\[ \int_{\Omega} w_{\alpha\alpha} w_{,\beta\beta} d\Omega = \int_{\Gamma} w_{,\beta\beta} w_{,\alpha} n_{\alpha} d\Gamma - \int_{\Gamma} w_{,\alpha\beta} w n_{\alpha} d\Gamma + \int_{\Omega} w_{,\alpha\alpha\beta\beta} w d\Omega. \] (5.13)

Substituting the expressions (5.12) and (5.13) into (5.11), one finds that

\[ \int_{\Omega} M_{\alpha\beta} w_{,\alpha\beta} d\Omega = \int_{\Gamma} M^*_\alpha w_{,\alpha} d\Gamma - \int_{\Gamma} q^* w d\Gamma - \int_{\Omega} w(D\nabla^4 w^*) d\Omega, \] (5.14)

where \( M^*_\alpha \) and \( q^* \) are defined by (5.8) with a superscript * attached to \( M_{\alpha\beta} \).

Substituting (5.14) into the first integral of the right-hand side in equation (5.9), one has

\[ \int_{\Gamma} q^* w d\Gamma - \int_{\Gamma} w^* q d\Gamma + \int_{\Gamma} w^*_\alpha M_{\alpha} d\Gamma - \int_{\Gamma} M^*_\alpha w_{,\alpha} d\Gamma = -\int_{\Omega} w(D\nabla^4 w^*) d\Omega + \int_{\Omega} p w^* d\Omega. \] (5.15)

The normal moment \( M_n \) and the tangential moment \( M_s \) are defined by the following relationships (from Ameen [10])
\[ M_0 w_0^* = \beta_n^* M_n + \beta_s^* M_s , \]
\[ M_0^* w_0^* = \beta_n M_n^* + \beta_s M_s^* , \]

where \( \beta_n \) and \( \beta_s \) are defined, respectively, as the normal and tangential slopes, \( M_n \) and \( M_s \) are defined, respectively, as the normal and tangential moments.

\[ \beta_n = \frac{\partial w}{\partial n} , \]
\[ \beta_s = \frac{\partial w}{\partial s} , \]

(5.17)

\[ M_n = M_{\alpha \beta} n_\alpha n_\beta , \]
\[ M_s = M_{\alpha \beta} n_\alpha s_\beta . \]

Substituting the expressions (5.16) into equation (5.15), one finds that

\[ \int_\Gamma q^* w d\Gamma - \int_\Gamma w^* q d\Gamma + \int_\Gamma \beta_n^* M_s d\Gamma - \int_\Gamma \beta_s M_s^* d\Gamma + \int_\Gamma \beta_n M_n d\Gamma - \int_\Gamma \beta_n M_n^* d\Gamma = -\int_\Omega w(D\nabla^4 w^*) d\Omega + \int_\Omega p w^* d\Omega . \]

(5.18)

The integral equation (5.18) contains two special integrals; which will be integrated by parts again. Since \( \Gamma \) is closed contour, one has
\[ \int_{\Gamma} \beta_s^* M_s \, d\Gamma = -\int_{\Gamma} w^* \frac{\partial M_s}{\partial s} \, d\Gamma , \quad (5.19) \]

\[ \int_{\Gamma} \beta_s M_s^* \, d\Gamma = -\int_{\Gamma} w \frac{\partial M_s^*}{\partial s} \, d\Gamma . \]

Substituting the expressions (5.19) into the integral equation (5.18), one infers that

\[ \int_{\Gamma} q^* w \, d\Gamma - \int_{\Gamma} w^* q \, d\Gamma - \int_{\Gamma} w^* \frac{\partial M_s}{\partial s} \, d\Gamma + \]

\[ \int_{\Gamma} w \frac{\partial M_s^*}{\partial s} \, d\Gamma + \int_{\Gamma} \beta_n^* M_n \, d\Gamma - \int_{\Gamma} \beta_n M_n^* \, d\Gamma = \]

\[ -\int_{\Omega} w (D \nabla^4 w^*) \, d\Omega + \int_{\Omega} p w^* \, d\Omega . \quad (5.20) \]

The effective shear force \( V \) is defined as

\[ V = q + \frac{\partial M_s}{\partial s} , \quad (5.21) \]

\[ V^* = q^* + \frac{\partial M_s^*}{\partial s} . \]

Substituting the expression for the effective shear force (5.21) into equation (5.20) we find the following integral equation.
\[
\int_{\Gamma} w V^* \, d\Gamma - \int_{\Gamma} w^* V \, d\Gamma - \int_{\Gamma} \beta_n M_n^* \, d\Gamma + \int_{\Gamma} \beta_n^* M_n \, d\Gamma = -\int_{\Omega} w (D\nabla^4 w^*) \, d\Omega + \int_{\Omega} p w^* \, d\Omega.
\]

The integral equation (5.22) includes four integrals on the boundary \( \Gamma \) and two integrals in the domain \( \Omega \). One of them involves the load \( p \), and we will deal with this integral later. The second integral in the domain in equation (5.22) involves the weight function \( w^* \) and we will deal with this integral using a procedure called the fundamental solution.

### 5.4 Fundamental solution for homogeneous plates

The right-hand side of the integral equation (5.22) includes two integrals in the domain \( \Omega \); the first one will be treated with the procedure called the fundamental solution. The fundamental solution \( w^* \) represents the vertical displacement at the field point \((x_1, x_2)\) of an infinite plate due to a unit transverse load applied at the source point \((\varepsilon_1, \varepsilon_2)\), as in Ameen [10], Banerjee [9]. In order to obtain an expression for the fundamental solution the following singular biharmonic equation will be solved

\[
D\nabla^4 w^* = \delta(r),
\]

(5.23)
where \( r \) is the distance between the source point and the field point, as shown in Figure 5.1. The expression in equation (5.23) appears in the integral equation (5.22) and will be substituted in order to eliminate the integral in the domain. Substituting (5.23) in the first integral of the right-hand side of (5.22), one has

\[
\int_{\Omega} w(D\nabla^4 w^*) d\Omega = \int_{\Omega} w\delta(r) d\Omega = w(\varepsilon_1, \varepsilon_2). \tag{5.24}
\]

![Figure 5.1. The source point \((\varepsilon_1, \varepsilon_2)\) and the field point \((x_1, x_2)\).](image)

Substituting (5.24) into the integral equation (5.22) we obtain the boundary element formulation for homogeneous plates as follows

\[
\alpha w(\varepsilon_1, \varepsilon_2) + \int_{\Gamma} (\beta^n M_n + V^* w) d\Gamma = \int_{\Omega} w^* p d\Omega + \int_{\Gamma} (M_n^* \beta_n + w^* V) d\Gamma. \tag{5.25}
\]
The parameter $\alpha$ depends on the position of the source point, $\alpha = 1$ if the source point belongs to the domain $\Omega$, $\alpha = 1/2$ if the source point belongs to the boundary $\Gamma$ and $\alpha = 0$ if the source point is outside.

From equation (5.23), we can derive an expression for the fundamental solution $w^*$. From Frangi [13], one infers that

$$w^* = \frac{1}{16\pi D} \left( 2r^2 \ln r - r^2 \right).$$

The expression (5.26) is called the fundamental solution for homogeneous plates and will be used in the integral equation (5.25) and also in the derivation of expressions for $V^*$, $M^*_n$ and $\beta^*_n$. The integral equation (5.25) includes integrals on the boundary $\Gamma$ and one integral in the domain $\Omega$. In order to use the boundary element formulation for plates, the integral equation should include only integrals on the boundary. The integral in the domain that appears in equation (5.25) will be transformed into an integral on the boundary by using a procedure described by Ameen [10] for the case when the load $p$ is constant and this will be discussed in the next section.
5.5 Transformation of the integral in the domain into an integral on the boundary

The integral equation (5.25) includes an integral in the domain and we can transform it into an integral on the boundary when the load is constant. We choose a function $G^*$ such that

$$w^* = G^*_{,\alpha \alpha}, \quad (5.27)$$

where

$$G^* = \frac{1}{128\pi D} (\ln r - 1) r^4. \quad (5.28)$$

Hence, for the case when the load is constant, $p = P_0$, one has

$$\int_{\Omega} p w^* d\Omega = P_0 \int_{\Gamma} G^*_{,\alpha} n_\alpha d\Gamma. \quad (5.29)$$

If $p$ is not constant we can approximate it by using a combination of functions in order to transform the integral in the domain into an integral on the boundary, as in Muci-Kuchler and Cruz-Bañuelos [14]. Substituting the expression in (5.29) into equation (5.25), one finds that
\[ \alpha w(\varepsilon_1, \varepsilon_2) + \int_{\Gamma} \left( \beta_n^* M_n + V^* w \right) d\Gamma = \]

\[ P_0 \int_{\Gamma} G_{\alpha n}^* d\Gamma + \int_{\Gamma} \left( M_n^* \beta_n + w^* V \right) d\Gamma. \]

The preceding integral equation has to be complemented by the derivative of equation (5.30), which has the form

\[ \alpha \frac{\partial w(\varepsilon_1, \varepsilon_2)}{\partial \varepsilon_i} n_i^0 + \int_{\Gamma} \left( \frac{\partial \beta_n^*}{\partial \varepsilon_i} n_i^0 M_n + \frac{\partial V^*}{\partial \varepsilon_i} n_i^0 w \right) d\Gamma = \]

\[ P_0 \int_{\Gamma} \frac{\partial G_{\alpha n}^*}{\partial \varepsilon_i} n_i^0 n_{\alpha} d\Gamma + \int_{\Gamma} \left( \frac{\partial M_n^*}{\partial \varepsilon_i} n_i^0 \beta_n + \frac{\partial w^*}{\partial \varepsilon_i} n_i^0 V \right) d\Gamma. \]

We define the following quantities

\[ \beta_n = \frac{\partial w(\varepsilon_1, \varepsilon_2)}{\partial \varepsilon_i} n_i^0, \quad \bar{w}_n^* = \frac{\partial w^*}{\partial \varepsilon_i} n_i^0, \]

\[ \bar{\beta}_n^* = \frac{\partial \beta_n^*}{\partial \varepsilon_i} n_i^0, \quad \bar{M}_n^* = \frac{\partial M_n^*}{\partial \varepsilon_i} n_i^0, \]

\[ \bar{V}^* = \frac{\partial V^*}{\partial \varepsilon_i} n_i^0, \quad \bar{G}_{\alpha n}^* = \frac{\partial G_{\alpha n}^*}{\partial \varepsilon_i} n_i^0 n_{\alpha}. \]
Substituting the definitions in (5.32) into integral equation (5.31) we obtain the
second integral equation to be considered in the Boundary Elements Formulation
for homogeneous plates in the form

\[ \alpha \beta_n (\varepsilon_1, \varepsilon_2) + \int_{\Gamma} (\overline{\beta}_n^* M_n + \overline{V}^* w) d\Gamma = P_0 \int_{\Gamma} \overline{G}_n^* d\Gamma + \int_{\Gamma} (\overline{M}_n^* \beta_n + \overline{w}_n^* V) d\Gamma. \] (5.33)

The integral equations (5.30) and (5.32) include only integrals on the boundary
and a discretization of the boundary will be used in order to calculate a numerical
solution for the different quantities involved in the integral equations.

5.6 Conclusion

A review of the boundary element formulation for plates in general and for
homogeneous plates in particular, has been presented here. The integral
equation (5.9) does not depend on the material properties of the plates and will
be used for any isotropic material

\[ \int_{\Gamma} w_{,\beta}^* M_{,\beta} d\Gamma - \int_{\Gamma} w^* q d\Gamma = \int_{\Omega} M_{,\alpha \beta} w_{,\alpha \beta}^* d\Omega + \int_{\Omega} p w^* d\Omega. \] (5.34)
For homogeneous plates, a relationship between the moments and the vertical displacement is used in order to develop a Boundary Element formulation for homogeneous plates. From (5.10), one has

$$M_{\alpha\beta} = -D(1-\nu)w_{,\alpha\beta} - D\nu w_{,\gamma\gamma} \delta_{\alpha\beta}. \quad (5.35)$$

Equations (5.30) and (5.33) yield a boundary element formulation for homogeneous plates for the case when the load is constant. These equations are

$$\alpha w(\varepsilon_1, \varepsilon_2) + \int_{\Gamma} \left( \beta_n^* M_n + V^* w \right) d\Gamma = 0,$$

$$P_0 \int_{\Gamma} G_{,\alpha}^* n_\alpha d\Gamma + \int_{\Gamma} \left( M_n^* \beta_n + w^* V \right) d\Gamma,$$

$$\alpha \beta_n(\varepsilon_1, \varepsilon_2) + \int_{\Gamma} \left( \bar{\beta}_n^* M_n + \bar{V}^* w \right) d\Gamma = 0,$$

$$P_0 \int_{\Gamma} \bar{G}_n^* d\Gamma + \int_{\Gamma} \left( \bar{M}_n^* \beta_n + \bar{w}_n^* V \right) d\Gamma,$$

where

$$w^* = \frac{1}{16\pi D} \left( 2r^2 \ln r - r^2 \right), \quad (5.37)$$
and \( w^* \) is called the fundamental solution for homogeneous plates and will be used in order to calculate numerical values for the different variables involved in the integral equations (5.36). These equations involve only integrals on the boundary. In order to calculate a numerical solution for the quantities involved in the integral equation, a boundary discretization in linear elements will be used in the next chapter.
Chapter 6

Boundary Element Discretization in Linear Elements for Homogeneous Plates

6.1 Introduction

The boundary element formulation for the bending of homogeneous plates and for the case when the load is constant is given in the integral equations (5.36). In order to calculate a numerical solution for the different quantities involved in those equations a discretization of the boundary is used. In this chapter we will use linear elements. Different kinds of elements can be used such as constant elements, quadratic elements, and others [15-25]. The integrals in equations (5.36) will be integrated on each linear element by considering a linear variation for the variables involved. An example is solved in order to check the accuracy of the method. The example problem has an analytical solution that is compared with the numerical solution.

6.2 Discretization of a plate boundary in linear elements

The integral equations (5.36) are reproduced thereafter as
\[
\alpha w(\varepsilon_1, \varepsilon_2) + \int_{\Gamma} \left( \beta_n^* M_n + V^* w \right) d\Gamma = \\
P_0 \int_{\Gamma} G_n^* n_\alpha d\Gamma + \int_{\Gamma} \left( M_n^* \beta_n + w^* V \right) d\Gamma,
\]

(6.1)

\[
\alpha \beta_n(\varepsilon_1, \varepsilon_2) + \int_{\Gamma} \left( \overline{\beta}_n^* M_n + \overline{V}^* w \right) d\Gamma = \\
P_0 \int_{\Gamma} \overline{G}_n^* d\Gamma + \int_{\Gamma} \left( \overline{M}_n^* \beta_n + \overline{w}_n^* V \right) d\Gamma.
\]

Figure 6.1 shows the linear elements on the boundary.

![Diagram of linear boundary elements](image)

Figure 6.1. Discretization of a plate boundary in linear elements.

The linear elements are straight with two nodes usually located at the ends of the element, and a local coordinate system is used in each element. The vertical displacement \( w \), the normal slope \( \beta_n \), the normal moment \( M_n \) and the effective shear \( V \) are assumed to vary linearly in the linear element. These values can be
defined in terms of their nodal values and two linear interpolation functions $H_1(\eta)$ and $H_2(\eta)$, which are given in terms of the homogeneous coordinate $\eta$ as shown in Figure 6.2. Using the local coordinate system and the interpolation function, the quantities in each linear element can be expressed as follows

$$w^{(j)} = H_\alpha(\eta)w^{(j)}_{\alpha},$$

$$\beta_n^{(j)} = H_\alpha(\eta)\beta_n^{(j)},$$

$$M_n^{(j)} = H_\alpha(\eta)M_n^{(j)},$$

$$V^{(j)} = H_\alpha(\eta)V^{(j)}_{\alpha},$$

![Figure 6.2](image)

Figure 6.2. A linear boundary element with a local coordinate system.

where $w^{(j)}$ is the value of the vertical displacement at local node $j$; $\beta_n^{(j)}$ is the value of the normal slope at local node $j$; $M_n^{(j)}$ is the value of the normal
moment at local node \( j \); and \( V^{(j)}_\alpha \) is the value of the effective shear force at local node \( j \). The interpolation functions are

\[
H_1(\eta) = \frac{1}{2}(1-\eta), \quad H_1(\eta) = \frac{1}{2}(1+\eta).
\] (6.3)

Each integral in equations (6.1) is integrated after discretization. For example, using the fourth expression in (6.2), one of the integrals in (6.1) can be expressed as

\[
\int_{\Gamma} w^* V d\Gamma = \sum_{j=1}^{N} \int_{\Gamma(j)} w^* V^{(j)} d\Gamma = \sum_{j=1}^{N} \int_{\Gamma(j)} w^* H_\alpha(\eta)V^{(j)}_\alpha d\Gamma,
\] (6.4)

where \( N \) is the number of linear elements and \( d\Gamma \) is the arc length along the element \( j \). The relationship with the local coordinate system \( \eta \) is

\[
d\Gamma = \frac{L_j}{2} d\eta,
\] (6.5)

where \( L_j \) is the length of element \( j \). Substituting (6.5) into (6.4) and using (5.37), one finds that
\[
\int_{\Gamma} w^* V d\Gamma = \frac{1}{32\pi D} \sum_{j=1}^{N} A^{(j)}_{\alpha} V_{\alpha}^{(j)} = \frac{1}{32\pi D} \sum_{j=1}^{N} \left[ A_{2}^{(j-1)} + A_{1}^{(j)} \right] V_{j},
\]

(6.6)

where \( V_{j} \) is the value of the effective shear force at node \( j \), and

\[
A^{(j)}_{\alpha} = L_{j} \int_{\Gamma(j)} f_{1}(r) H_{\alpha}(\eta) d\eta,
\]

(6.7)

\[
f_{1}(r) = 2r^2 \ln r - r^2.
\]

The integral in (6.7) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

Using the same procedure with the other integrals in equations (6.1) we obtain the following results

\[
\int_{\Gamma} M_{n}^* \beta_{n} d\Gamma = \sum_{j=1}^{N} \int_{\Gamma(j)} M_{n}^* \beta_{n}^{(j)} d\eta = \sum_{j=1}^{N} \int_{\Gamma(j)} M_{n}^* H_{\alpha}(\eta) \beta_{n\alpha}^{(j)} d\eta,
\]

(6.8)

\[
\int_{\Gamma} M_{n}^* \beta_{n} d\Gamma = -\frac{1}{8\pi} \sum_{j=1}^{N} B_{n}^{(j)} \beta_{n\alpha}^{(j)} = -\frac{1}{8\pi} \sum_{j=1}^{N} \left[ B_{2}^{(j-1)} + B_{1}^{(j)} \right] \beta_{nj},
\]

where \( \beta_{nj} \) is the value of the normal slope at node \( j \), and
\[ B^{(j)}_\alpha = L_j \int_{\Gamma(j)} f_2(r) H_\alpha(\eta) \, d\eta, \]
\[ f_2(r) = (1 + \nu) \ln r + \left( 1 - \nu \left( \frac{r_\alpha n_\alpha}{r} \right)^2 \right) + \nu. \] (6.9)

The integral in (6.9) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

Using the same procedure with the other integrals in equations (6.1) we find the following results

\[ \int_{\Gamma} \beta_n^* M_n \, d\Gamma = \sum_{j=1}^{N} \int_{\Gamma(j)} \beta_n^* M_n^{(j)} \, ds = \sum_{j=1}^{N} \int_{\Gamma(j)} \beta_n^* H_\alpha(\eta) M_n^{(j)} \, ds, \]

(6.10)

\[ \int_{\Gamma} \beta_n^* M_n \, d\Gamma = -\frac{1}{8\pi D} \sum_{j=1}^{N} C^{(j)} M_n^{(j)} = \frac{1}{8\pi D} \sum_{j=1}^{N} \left[ C_2^{(j-1)} + C_1^{(j)} \right] M_{nj}, \]

where \( M_{nj} \) is the value of the normal moment at node \( j \), and
\[ C^{(j)}_\alpha = L_j \int_{\Gamma(j)} f_3(r) H_\alpha(\eta) d\eta, \]  
(6.11)

\[ f_3(r) = r_\alpha n_\alpha \ln r. \]

The integral in (6.11) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

Using the same procedure with the other integrals in equations (6.1) we find the following results

\[ \int_{\Gamma} V^* w d\Gamma = \sum_{j=1}^{N} \int_{\Gamma(j)} V^* w^{(j)} ds = \sum_{j=1}^{N} \int_{\Gamma(j)} V^* H_\alpha(\eta) w^{(j)}_\alpha ds, \]

(6.12)

\[ \int_{\Gamma} V^* w d\Gamma = -\frac{1}{8\pi} \sum_{j=1}^{N} D^{(j)}_\alpha w^{(j)}_\alpha = -\frac{1}{8\pi} \sum_{j=1}^{N} \left[ D^{(j-1)}_2 + D^{(j)}_1 \right] w_j, \]

where \( w_j \) is the value of the vertical displacement at node \( j \), and

\[ D^{(j)}_\alpha = L_j \int_{\Gamma(j)} f_4(r) H_\alpha(\eta) d\eta, \]  
(6.13)

\[ f_4(r) = (3-\nu) \frac{r_\alpha n_\alpha}{r^2} - 2(1-\nu) \left( \frac{r_\alpha s_\alpha}{r^2} \right)^2 r_\gamma n_\gamma. \]
The integral in (6.13) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

Using the same procedure with the other integrals in equations (6.1) we find the following results

\[ P_0 \int_{\Gamma} G_n^* d\Gamma = \sum_{j=1}^{N} \int_{\Gamma(j)} G_n^* ds = \frac{P_0}{256 \pi D} \sum_{j=1}^{N} F_1^{(j)}, \]  

(6.14)

where

\[ F_1^{(j)} = L_j \int_{\Gamma(j)} f_5(r) d\eta, \]  

(6.15)

\[ f_5(r) = (4 \ln r - 3) r^2 r_\alpha n_\alpha. \]

The integral in (6.15) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

Substituting (6.6), (6.8), (6.10), (6.12), and (6.14) into the first equation in (6.1), one finds that
\[
\alpha w(\epsilon_1, \epsilon_2) = \frac{1}{32\pi D} \sum_{j=1}^{N} \left[ A_2^{(j-1)} + A_1^{(j)} \right] V_j - \frac{1}{8\pi} \sum_{j=1}^{N} \left[ B_2^{(j-1)} + B_1^{(j)} \right] \beta_{nj} - \frac{1}{8\pi} \sum_{j=1}^{N} \left[ C_2^{(j-1)} + C_1^{(j)} \right] M_{nj} + \frac{1}{8\pi} \sum_{j=1}^{N} \left[ D_2^{(j-1)} + D_1^{(j)} \right] w_j + \frac{P_0}{256\pi D} \sum_{j=1}^{N} F_1^{(j)}.
\]

(6.16)

Now, we define

\[
\begin{align*}
  w(\epsilon_1, \epsilon_2) &= w(\tilde{\epsilon}_i), \\
  A_{ij} &= A_2^{(j-1)} + A_1^{(j)}, \\
  B_{ij} &= B_2^{(j-1)} + B_1^{(j)}, \\
  C_{ij} &= C_2^{(j-1)} + C_1^{(j)}, \\
  D_{ij} &= D_2^{(j-1)} + D_1^{(j)}, \\
  f_{ii} &= \sum_{j=1}^{N} F_1^{(j)}.
\end{align*}
\]

(6.17)

Using the definitions in (6.17) into equation (6.16) for the source point \( i \),
equation (6.16) in index notation takes the form

\[
\alpha w(\tilde{\epsilon}_i) = \frac{A_{ij} V_j}{32\pi D} - \frac{B_{ij} \beta_{nj}}{8\pi} - \frac{C_{ij} M_{nj}}{8\pi D} + \frac{D_{ij} w_j}{8\pi} + \frac{P_0 f_{ii}}{256\pi D}
\]

(6.18)

Equation (6.18) is the discretized version of the first equation in (6.1), the next
step is the discretization of the second equation in (6.1). Using the same
procedure used to develop equation (6.18), the second equation in (6.1) in a
discretized form is

\[
\alpha \beta_n (\bar{e}_i) = -\frac{A_{ij} V_j}{8\pi D} + \frac{B_{ij} \beta_{nj}}{8\pi} + \frac{C_{ij} M_{nj}}{8\pi D} + \frac{D_{ij} w_j}{8\pi} - \frac{P_0 f_{2i}}{256\pi D}, \tag{6.19}
\]

where

\[
\bar{A}_{ij} = \bar{A}_{2}^{(j-1)} + \bar{A}_{1}^{(j)},
\]

\[
\bar{A}_{\alpha}^{(j)} = L_j \int_{\Gamma(j)} f_6(r) H_{\alpha}(\eta) d\eta, \tag{6.20}
\]

\[
f_6(r) = r_\alpha n_\alpha^0 \ln r.
\]

The integral in (6.20) will be integrated using Gaussian quadrature, which is
discussed in Appendix A.

The second coefficient in (6.19) is calculated using
\[ B_{ij} = \bar{B}_2^{(j-1)} + \bar{B}_1^{(j)}, \]
\[ \bar{B}_\alpha^{(j)} = L_j \int_{\Gamma(j)} f_7(r) H_\alpha(\eta) \, d\eta, \] 
\[ (6.21) \]
\[ f_7(r) = (1+\nu) \frac{r_\alpha n_\alpha^0}{r^2} + 2(1-\nu) \frac{r_\alpha n_\alpha n_\gamma n_\gamma^0}{r^2} - 2(1-\nu) \left( \frac{r_\alpha n_\alpha}{r^2} \right)^2 r_\gamma n_\gamma^0. \]

The integral in (6.21) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

The third coefficient in (6.19) is calculated using
\[ C_{ij} = \bar{C}_2^{(j-1)} + \bar{C}_1^{(j)}, \]
\[ \bar{C}_\alpha^{(j)} = L_j \int_{\Gamma(j)} f_8(r) H_\alpha(\eta) \, d\eta, \] 
\[ (6.22) \]
\[ f_8(r) = n_\gamma n_\gamma^0 \ln r + \frac{r_\alpha n_\alpha r_\gamma n_\gamma^0}{r^2}. \]

The integral in (6.22) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

The fourth coefficient in (6.19) is calculated using
\[
\overline{D}_{ij} = \overline{D}_{2}^{(j-1)} + \overline{D}_{1}^{(j)},
\]

\[
\overline{D}_{2}^{(j)} = L_j \int_{\Gamma(j)} f_9(r) H_{\alpha}(\eta) d\eta,
\]

\[
f_9(r) = (3 - \nu) \frac{n_{\alpha} n_{\gamma}^0}{r^2} - 2(3 - \nu) \frac{r_{\alpha} n_{\alpha} r_{\gamma} n_{\gamma}^0}{r^4} - 4(1 - \nu) \frac{r_{\alpha} s_{\alpha} r_{\gamma} n_{\gamma} s_{\lambda} n_{\lambda}^0}{r^4} - 2(1 - \nu) \left( \frac{r_{\alpha} s_{\alpha}}{r^2} \right)^2 n_{\gamma} n_{\gamma}^0 + 8(1 - \nu) \left( \frac{r_{\alpha} s_{\alpha}}{r^2} \right)^2 \frac{r_{\alpha} n_{\alpha} r_{\gamma} n_{\gamma}^0}{r^2}.
\]

(6.23)

The integral in (6.23) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

The last coefficient in (6.19) is calculated using

\[
f_{2i} = \sum_{j=1}^{N} F_{2}^{(j)},
\]

(6.24)

\[
F_{2}^{(j)} = L_j \int_{\Gamma(j)} f_{10}(r) d\eta,
\]

\[
f_{10}(r) = 4 n_{\alpha} n_{\alpha}^0 r^2 \ln r + 8 r_{\alpha} n_{\alpha} n_{\gamma} r_{\gamma} n_{\gamma} \ln r - 2 r_{\alpha} n_{\alpha}^0 r_{\gamma} n_{\gamma} - 3 n_{\alpha} n_{\alpha}^0 r^2.
\]

The integral in (6.24) will be integrated using Gaussian quadrature, which is discussed in Appendix A.
Figure 6.3. Linear elements, nodes, and source point.

The discretized equations (6.18) and (6.19) include the coefficient \( \alpha \) which takes different values as described in section 5.4; if we choose the source point outside the plate, as in Figure 6.3, \( \alpha = 0 \). Using that value in equations (6.18) and (6.19), one has

\[
\frac{A_{ij} V_j}{32 \pi D} - \frac{B_{ij} \beta_{nj}}{8 \pi} - \frac{C_{ij} M_{nj}}{8 \pi D} + \frac{D_{ij} w_j}{8 \pi} = -\frac{P_0 f_{2i}}{256 \pi D},
\]

\[
\frac{\bar{A}_{ij} V_j}{8 \pi D} - \frac{\bar{B}_{ij} \beta_{nj}}{8 \pi} - \frac{\bar{C}_{ij} M_{nj}}{8 \pi D} - \frac{\bar{D}_{ij} w_j}{8 \pi} = -\frac{P_0 f_{2i}}{256 \pi D}.
\]

(6.25)

In order to make equations (6.25) dimensionless, we define
\[
\bar{w}_j = \frac{w_j}{w_0}, \quad w_0 = \frac{P_0 R^4}{D}, \\
\bar{\beta}_{nj} = \frac{\beta_{nj}}{\beta_0}, \quad \beta_0 = \frac{P_0 R^3}{D}, \\
\bar{M}_{nj} = \frac{M_{nj}}{M}, \quad M = P_0 R^2, \\
\bar{V}_j = \frac{V_i}{V_0}, \quad V_0 = P_0 R.
\]  

(6.26)

Using the definitions in (6.26), equations (6.25) becomes

\[
\frac{R}{4} A_{ij} \bar{V}_j - R^3 B_{ij} \bar{\beta}_{nj} - R^2 C_{ij} \bar{M}_{nj} + R^4 D_{ij} \bar{w}_j = -\frac{f_{1i}}{32}, \\
R \bar{A}_{ij} V_j - R^3 \bar{B}_{ij} \bar{\beta}_{nj} - R^2 \bar{C}_{ij} \bar{M}_{nj} - R^4 \bar{D}_{ij} \bar{w}_j = -\frac{f_{2i}}{32}.
\]  

(6.27)

Equations (6.27) can be expressed in matrix notation as

\[
\begin{pmatrix}
\frac{R}{4} A & -R^3 B \\
R \bar{A} & -R^3 \bar{B}
\end{pmatrix}
\begin{pmatrix}
\bar{V} \\
\bar{\beta}_n
\end{pmatrix}
+ 
\begin{pmatrix}
-R^2 C & R^4 D \\
-R^2 \bar{C} & -R^4 \bar{D}
\end{pmatrix}
\begin{pmatrix}
\bar{M}_n \\
\bar{w}
\end{pmatrix}
= 
\begin{pmatrix}
f_{1i} \\
f_{2i}
\end{pmatrix} \frac{1}{32}.
\]  

(6.28)
The system of equations in (6.28) will be used in the next section in order to develop a boundary element formulation using linear elements for simply supported plates.

6.3 Linear boundary element formulation for simply supported plates

The system of equations in (6.28) will be used in order to solve the problem of bending of simply supported homogeneous plates, where the boundary conditions discussed in section 4.6 are

\[ w = 0, \]

\[ M_n = 0. \]  \hspace{1cm} (6.29)

Using the boundary conditions in (6.29), the system of equations (6.28) is simplified to

\[
\begin{pmatrix}
\frac{R}{4} A & -R^3 B \\
R \bar{A} & -R^3 \bar{B}
\end{pmatrix}
\begin{pmatrix}
\bar{V} \\
\bar{\beta}_n
\end{pmatrix}
= -\begin{pmatrix}
f_1 \\
\frac{f_2}{32}
\end{pmatrix}. \hspace{1cm} (6.30)
\]
The unknown quantities in the system (6.30) are the vertical shear force \( V \) and \( \beta_n \) at each node. Once we calculate those quantities, all quantities in the boundary are known and we can calculate the vertical displacement at any point in the domain using equation (6.18). With the boundary conditions (6.29) and \( \alpha = 1 \), (6.18) yields

\[
 w(\vec{e}_i) = \frac{A_{ij} V_j}{32\pi D} - \frac{B_{ij}}{8\pi} \frac{\beta_{nj}}{256\pi D} + \frac{P_0 f_{ii}}{256\pi D}. \tag{6.31}
\]

In order to evaluate the numerical accuracy of the boundary element method, we will calculate the vertical displacement of the central point of a simply supported circular plate and we will compare this numerical solution with the analytical solution developed in section 4.9.

Consider a simply supported circular plate of radius \( a \) under a uniformly distributed load \( P_0 \) as shown in Figure 4.5. Figure 6.4 shows the linear elements and the nodes on the boundary of the plate, and in order to calculate the effective shear force \( V \) and the normal slope \( \beta_n \) at each node of the boundary we use the system (6.30). After calculation of those quantities, we calculate the vertical displacement at any point of the domain using equation (6.31).
In order to make the equations dimensionless, we use the definitions in (6.26), with \( a \) replacing \( R \), and we obtain

\[
\bar{w}(\bar{\varepsilon}_1) = \frac{A_{ij} \bar{V}_j}{32 \pi a^3} - \frac{B_{ij} \bar{\beta}_{nj}}{8 \pi a} + \frac{f_{1i}}{256 \pi a^4},
\]  

(6.32)

The radius of the plate considered is \( a = 50 \) cm and Poisson's ratio for the homogeneous material is \( \nu = 0.3 \). The analytical solution for the vertical displacement of the central point \( w \) is given by (4.38), which in dimensionless form yields

\[
\bar{w} = \frac{1}{64} \left( \frac{5 + \nu}{1 + \nu} \right) = 0.0637. 
\]  

(6.33)
The boundary element solution for the vertical displacement of the central point of the plate is calculated using different number of elements. Table 6.1 shows these values and the error in the approximation in comparison with the analytical solution (6.33).

Table 6.1. Numerical solutions for the vertical displacement of the central point of the plate for different numbers of elements.

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>$\bar{w}$ Using BEM</th>
<th>$\bar{w}$ Analytical</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>0.0634</td>
<td>0.0637</td>
<td>0.4538 %</td>
</tr>
<tr>
<td>80</td>
<td>0.0636</td>
<td>0.0637</td>
<td>0.2306 %</td>
</tr>
<tr>
<td>200</td>
<td>0.0636</td>
<td>0.0637</td>
<td>0.0917 %</td>
</tr>
<tr>
<td>400</td>
<td>0.0637</td>
<td>0.0637</td>
<td>0.0364 %</td>
</tr>
</tbody>
</table>

Table 6.1 shows, in column 2, values for the dimensionless vertical-displacement for the central point of the plate using the Boundary Elements Method. The convergence analysis shows that the error in column 4 decreases when the number of elements increases. The use of 40 elements on the boundary results in an error of 0.45 %, which is an excellent accuracy.
6.4 Conclusion

A discretization of the boundary of a simply-supported plate, in linear elements, has been presented here. The quantities involved in equations (6.1) are assumed to have a linear variation in each element. The interpolation functions shown in (6.3) have been used in order to express the linear variation of those quantities, as in equations (6.2). A local coordinate system, which is shown in Figure 6.2, is used in expressions (6.2). The discretization of the boundary in linear elements and the application of the boundary conditions in (6.29) results in the system of equations (6.30), which is such as

\[
\begin{pmatrix}
\frac{R}{4} A & -R^3 B \\
R A & -R^3 B
\end{pmatrix}
\begin{pmatrix}
\bar{V} \\
\bar{\beta}_n
\end{pmatrix}
= -\begin{pmatrix}
\frac{f_1}{32} \\
\frac{f_2}{32}
\end{pmatrix}.
\]

(6.34)

Using the system (6.34), the unknown quantities on the boundary are calculated, and thereafter the vertical displacement of any point inside the domain is calculated using equation (6.32), which reads

\[
\bar{w}(\bar{e}_i) = \frac{A_{ij} \bar{V}_j}{32 \pi a^3} - \frac{B_{ij} \bar{\beta}_{nj}}{8 \pi a} + \frac{f_{1i}}{256 \pi a^4}.
\]

(6.35)

An example has been presented in order to illustrate the accuracy of the method. The vertical displacement of the central point of the plate is calculated using
different number of elements. Table 6.1 shows the results corresponding to that example and the comparison with the analytical solution.

The boundary element method will be used in the next chapter in order to develop a new ballistic limit equation for hypervelocity impact on homogeneous materials.
Chapter 7

A New Ballistic Limit Equation for Homogeneous Materials

7.1 Introduction

A review of the physical concepts about hypervelocity impact has been presented in chapter 3. Assumptions about the characteristics of the load and momentum transferred from the projectile to the back plate have been considered. With those assumptions in mind a mathematical model is proposed in order to develop a ballistic limit equation.

The mathematical model consists of a simply supported circular plate subjected to a constant load acting on a circular region at the center of the plate. The circular plate is divided in two regions; the first region corresponds to the central, loaded area and undergoes plastic deformation, so it is called the plastic region; the second region undergoes elastic deformation, and is therefore called the elastic region.

In the elastic region, the boundary element method is used in order to calculate moments and vertical displacement. The internal boundary of the elastic region is the external boundary of the plastic region. In the plastic region, a dynamic analysis is performed in order to calculate the vertical displacement of the central point of the plate using the boundary information obtained from the elastic analysis. The new ballistic limit equation follows from this procedure.
7.2 Mathematical model for elastic-plastic deformations of homogeneous plates

Three basic assumptions are mentioned in chapter 3, and are summarized in section 3.5; the third of these assumptions, which was expressed as equation (3.1), gives the initial velocity of the loaded area in the form

\[ V_I = \frac{32 M_p V_p}{\pi S^2 \rho_b h}, \quad (7.1) \]

where \( M_p \) and \( V_p \) are the mass and the velocity of the impacting projectile, \( S \) is the spacing between the plates and is shown in figure 3.1, \( \rho_b \) and \( h \) are the mass density and the thickness of the back-plate, respectively. Figure 7.1 shows the back plate and related dimensions.

![Figure 7.1. Back-plate under a constant load.](image)

The back plate is a simply supported circular plate with radius \( R \) and is loaded with a constant force of magnitude \( P_0 \); this force is acting over a circular region of
radius \( C \). The loaded area, \( 0 \leq r \leq C \), is the previously called plastic region and the area outside the load, \( C \leq r \leq R \), is the previously called elastic region.

The vertical displacement and the moments in the elastic region are calculated using the boundary elements method, and the boundary conditions for this elastic region are expressed in the form

\[
\begin{align*}
  w(R) &= 0, \\
  M_n(R) &= 0, \\
\end{align*}
\]

(7.2)

where \( w(R) \) is the vertical displacement, and \( M_n(R) \) is the normal moment, evaluated at \( r = R \).

The vertical displacement and the moments in the plastic region are calculated by using the solution proposed by Chakrabarty [26] for a fully plastic circular plate under a constant load.

### 7.3 Boundary element formulation for the elastic region

According to the boundary element method, in order to calculate the vertical displacement and the moments, one uses the integral equation (6.1), which reads
\[ \alpha w(\epsilon_1, \epsilon_2) + \int_{\Gamma} \left( \beta_n^* M_n + V^* w \right) d\Gamma = \]

\[ P_0 \int_{\Gamma'} G_{n\alpha}^* n_{\alpha} d\Gamma + \int_{\Gamma} \left( M_n^* \beta_n + w^* V \right) d\Gamma, \]

(7.3)

\[ \alpha \beta_n(\epsilon_1, \epsilon_2) + \int_{\Gamma} \left( \beta_n^* M_n + V^* w \right) d\Gamma = \]

\[ P_0 \int_{\Gamma'} G_{n\alpha}^* d\Gamma + \int_{\Gamma} \left( M_n^* \beta_n + w^* V \right) d\Gamma, \]

where \( w \) is the vertical displacement, \( \beta_n \) is the normal slope, \( M_n \) is the normal moment, \( V \) is the effective shear force, \( p \) is the force acting over the plate, and \( \Gamma' \) is the boundary of the loaded area, as shown in Figure 7.2. The symbol \( w^* \) denotes the fundamental solution (5.26), which reads

\[ w^* = \frac{1}{16\pi D} \left( 2r^2 \ln r - r^2 \right). \]

(7.4)

Analysis of equations (7.3) and the external boundary (7.2) shows that some terms become zero, which yields
\[ \alpha w(\varepsilon_1, \varepsilon_2) = P_0 \int_{\Gamma} G^*_{\alpha} n_\alpha \, d\Gamma + \int_{\Gamma} (M_n^* \beta_n + w^* V) \, d\Gamma, \]

\[ (7.5) \]

\[ \alpha \beta_n(\varepsilon_1, \varepsilon_2) = P_0 \int_{\Gamma'} G^*_n \, d\Gamma + \int_{\Gamma} (M_n^* \beta_n + \bar{w}_n^* V) \, d\Gamma. \]

In figure 7.2, the domain \( \Omega \) corresponds to the elastic region and is bounded by the external boundary \( \Gamma \) and the internal boundary \( \Gamma' \).

Figure 7.2. External boundary \( \Gamma \), internal boundary \( \Gamma' \), and domain \( \Omega \).

The integral equations in (7.5) involve two variables on the boundary: the normal slope \( \beta_n \) and the effective shear force \( V \). In order to calculate those variables,
the boundaries $\Gamma$ and $\Gamma'$ have to be discretized into linear elements as shown in Figure 7.2. After doing this, and using equation (6.25), one finds that

$$\frac{A_{ij} V_j}{32\pi D} - \frac{B_{ij} \beta_{nj}}{8\pi} = -\frac{P_0 f_{ii}}{256\pi D},$$

$$\frac{\bar{A}_{ij} \bar{V}_j}{8\pi D} - \frac{\bar{B}_{ij} \beta_{nj}}{8\pi} = -\frac{P_0 f_{2i}}{256\pi D}. \tag{7.6}$$

Next, we define the quantities

$$\beta_0 = \frac{P_0 C^2 R}{D}, \quad V_0 = \frac{P_0 C^2}{R}. \tag{7.7}$$

Dividing both sides of equation (7.6) by $\beta_0$, we obtain

$$\frac{R}{4} A_{ij} \bar{V}_j - R^3 B_{ij} \bar{\beta}_{nj} = -\frac{f_{ii}}{32\rho_0^2}, \tag{7.8}$$

$$R \bar{A}_{ij} \bar{V}_j - R^3 \bar{B}_{ij} \bar{\beta}_{nj} = -\frac{f_{2i}}{32\rho_0^2},$$

where
\[ \bar{\beta}_{nj} = \frac{\beta_{nj}}{\beta_0}, \]
\[ \bar{V}_j = \frac{V_j}{V_0}, \]  
(7.9)
\[ \rho_0 = \frac{C}{R}. \]

On the external boundary, the normal moment and vertical displacement are known to be zero. In order to calculate the dimensionless normal slope \( \bar{\beta}_{nj} \) and the effective shear force for every node of the external boundary, we use equations (7.8).

In order to calculate the vertical displacement for any point in the domain \( \Omega \) we use (6.31), which reads

\[ w(\bar{e}_i) = \frac{A_{ij} V_j}{32 \pi D} - \frac{B_{ij} \beta_{nj}}{8 \pi} + \frac{P_0 f_{ii}}{256 \pi D}. \]  
(7.10)

Dividing both sides of equation (7.10) by \( w_0 \), one has

\[ \bar{w}(\bar{e}_i) = \frac{A_{ij} \bar{V}_j}{32 \pi R^3} - \frac{B_{ij} \bar{\beta}_{nj}}{8 \pi R} + \frac{f_{ii}}{256 \pi \rho_0^2 R^4}, \]  
(7.11)

where
\[ \bar{w} = \frac{w}{w_0}, \quad w_0 = \frac{P_0 \rho^2 R^4}{D}. \] (7.12)

We calculate the dimensionless vertical displacement using equation (7.11), for any point in the domain \( \Omega \) including the values \( \bar{w}(C) \) at the internal boundary; these values provide a boundary condition for the plastic region.

The dimensionless moments are calculated in terms of the dimensionless vertical displacement \( \bar{w} \) with the formulae

\[ \bar{M}_{xx} = -(1 - \nu)\bar{w}_{,xx} - \nu(\bar{w}_{,xx} + \bar{w}_{,yy}), \]
\[ \bar{M}_{yy} = -(1 - \nu)\bar{w}_{,yy} - \nu(\bar{w}_{,xx} + \bar{w}_{,yy}), \] (7.13)
\[ \bar{M}_{xy} = -(1 - \nu)\bar{w}_{,xy}, \]

where \( \bar{w} \) is given by (7.11). Using equations (7.13) we calculate the dimensionless moments for any point in the domain \( \Omega \), including the values \( \bar{M}_{xx}(C), \bar{M}_{yy}(C), \) and \( \bar{M}_{xy}(C) \) at the internal boundary; these values will be used in a yield criterion for the internal boundary \( \Gamma^\prime \). The yield criterion for the boundary \( \Gamma^\prime \) according to Von Mises is expressed as a function of the moments in the form
\[ M_{xx}^2 - M_{xx} M_{yy} + M_{yy}^2 + 3M_{xy}^2 = M_0^2 , \]  
(7.14)

where

\[ M_0 = \frac{\sigma_0 h^2}{4} . \]  
(7.15)

Dividing equation (7.14) by \( M \) yields

\[ \overline{M}_{xx}^2 - \overline{M}_{xx} \overline{M}_{yy} + \overline{M}_{yy}^2 + 3\overline{M}_{xy}^2 = \frac{M_0^2}{M^2} , \]  
(7.16)

where

\[ M = P_0 C^2 . \]  
(7.17)

We define a plastic function \( f_{\text{plas}} \) as

\[ f_{\text{plas}} = \frac{M_0}{M} , \]  
(7.18)

where
\[ f_{\text{plas}} = \sqrt{M_{xx}^2 - M_{xx}M_{yy} + M_{yy}^2 + 3M_{xy}^2}. \] (7.19)

The new ballistic limit equation for homogeneous materials is developed from the plastic function \( f_{\text{plas}} \) of equation (7.18).

### 7.4 Analysis of the plastic region of the plate

The material of this section is taken from Chakrabarty [26]. Over the elastic range, the stress-strain relationship in polar coordinates is

\[ \varepsilon_r = \frac{1}{E} (\sigma_r - \nu \sigma_\theta), \quad \varepsilon_\theta = \frac{1}{E} (\sigma_\theta - \nu \sigma_r). \] (7.20)

Multiplying both sides of the above equations by \( zdz \), and integrating between the limits \(-h/2\) and \( h/2 \), we obtain the moment curvature relations

\[ \left( \frac{E h^3}{12} \right) \kappa_r = M_r - \nu M_\theta, \quad \left( \frac{E h^3}{12} \right) \kappa_\theta = M_\theta - \nu M_r. \] (7.21)

The radial and circumferential curvatures in terms of the vertical displacement \( w \) are
\[ \kappa_r = -\frac{d^2 w}{dr^2}, \quad \kappa_\theta = -\frac{1}{r} \frac{dw}{dr}. \]  

(7.22)

Substituting the first of equations (7.22) into the first of (7.21), we obtain

\[ \left( \frac{E h^3}{12} \right) \frac{d^2 w}{dr^2} = -M_r + \nu M_\theta. \]  

(7.23)

In an elastic-plastic plate, the curvature rates in any plastic element are the sum of an elastic part and a plastic part given by a plastic flow rule. Figure 7.3 shows the Tresca hexagon, if the plate element has entered a certain plastic regime directly from an elastic state, and has never left the plastic regime during the subsequent deformation, the flow rule can be written in the integrated form, and the elastic-plastic analysis is considerably simplified. For example, when the stress point is on the side BC, of Figure 7.3, the plastic part of the radial curvature vanishes, and equation (7.23) continues to hold in the plastic range, as shown in Chakrabarty [26].

In the plastic regime, one has

\[ M_\theta = M_0. \]  

(7.24)

The equilibrium equation in polar coordinates yields
\[
\frac{d}{dr}(r M_r) - M_\theta = -\int_0^r r \ p \ dr .
\]  \hspace{1cm} (7.25)

Figure 7.3. Tresca hexagon for moments in an elastic-plastic circular plate.

For \( p = P_0 \), where \( P_0 \) is constant, and using equation (7.24), one finds that the equilibrium equation (7.25) yields

\[
\frac{d}{dr}(r M_r) = M_0 - \frac{1}{2} P_0 \ r^2 .
\]  \hspace{1cm} (7.26)

After integration of the equilibrium equation (7.26), one finds that
\[ M_r = M_0 - \frac{1}{6} P_0 r^2 - \frac{A}{r}. \]  

(7.27)

The value of \( A \) can be calculated by observing that the moment \( M_r \) will be finite at \( r = 0 \), then \( A = 0 \). Then, equation (7.27) yields

\[ M_r = M_0 - \frac{1}{6} P_0 r^2. \]  

(7.28)

Using (7.24) and substituting the expression for \( M_r \) in (7.28) into equation (7.23), we obtain

\[ \left( \frac{E h^3}{12} \right) \frac{d^2 w}{dr^2} = -(1 - \nu) M_0 + \frac{1}{6} P_0 r^2. \]  

(7.29)

From equations (7.17) and (7.18)

\[ M_0 = f_{\text{plas}} M = f_{\text{plas}} P_0 C^2. \]  

(7.30)

By substituting equation (7.30) into (7.29), one has
\[ \left( \frac{E h^3}{12} \right) \frac{d^2w}{dr^2} = -(1-\nu) f_{\text{plas}} P_0 C^2 + \frac{1}{6} P_0 r^2 . \]  
\hspace{2cm} (7.31)

After integration of equation (7.31), one finds that

\[ (1-\nu^2) D \frac{dw}{dr} = -(1-\nu) f_{\text{plas}} P_0 C^2 r + \frac{1}{18} P_0 r^3 + A_1 . \]  
\hspace{2cm} (7.32)

The value of \( A_1 \) can be calculated by observing that the maximum vertical displacement will be occurring at the center of the plate, so that

\[ \text{at } r = 0, \quad \frac{dw}{dr} = 0 \Rightarrow A_1 = 0 . \]  
\hspace{2cm} (7.33)

Using the value of \( A_1 \) from (7.33), one finds that equation (7.32) becomes

\[ (1-\nu^2) D \frac{dw}{dr} = -(1-\nu) f_{\text{plas}} P_0 C^2 r + \frac{1}{18} P_0 r^3 . \]  
\hspace{2cm} (7.34)

After integration of equation (7.34), we obtain

\[ (1-\nu^2) D w = -\frac{1}{2} (1-\nu) f_{\text{plas}} P_0 C^2 r^2 + \frac{1}{72} P_0 r^4 + A_2 . \]  
\hspace{2cm} (7.35)
In order to calculate the constant $A_2$ we use the boundary condition

\[ \text{at } r = C, \quad w = w(C). \quad (7.36) \]

The value of $w(C)$ can be determined by equation (7.10), we write equation (7.35) in the form

\[
\begin{align*}
(1 - \nu^2)Dw = & -\frac{1}{2}(1 - \nu)f_{\text{plas}} P_0 C^2 r^2 + \frac{1}{72} P_0 r^4 + \\
(1 - \nu^2)Dw(C) = & \frac{1}{2}(1 - \nu)f_{\text{plas}} P_0 C^4 - \frac{1}{72} P_0 C^4.
\end{align*}
\quad (7.37)
\]

Substituting the vertical displacement at the center of the plate $W_I = w(0)$, one infers from (7.37) that

\[ W_I = \frac{P_0 C^4 Y(C)}{D}. \quad (7.38) \]

The quantity $Y(C)$ is given by

\[ Y(C) = \frac{\bar{w}(C)}{\rho_0^2} + \frac{f_{\text{plas}}}{2(1 + \nu)} - \frac{1}{72(1 - \nu^2)}, \quad \rho_0 = \frac{C}{R}. \quad (7.39) \]
where the dimensionless $\bar{w}(C)$ comes from the boundary element equation (7.11).

### 7.5 A new ballistic limit equation for homogeneous materials

In order to develop a ballistic limit equation for homogeneous materials, we use a dynamic analysis in the plastic region; the first assumption used concerns the acceleration and can be expressed as

$$
a = -\frac{P_0 \pi C^2}{m_b} = -\frac{P_0 \pi C^2}{\rho_b \pi C^2 h} = -\frac{P_0}{\rho_b h}, \quad a = \frac{dv}{dw}, \quad (7.40)
$$

where $\rho_b$ is the mass density of the plate, $h$ the thickness of the plate, and $v$ is the velocity at the center of the plate. Integration of (7.40) for $v$ between 0 and $V_I$, where $V_I$ is given by (7.1), yields

$$
W_I = -\int_0^{V_I} \frac{\rho_b h v dv}{P_0}. \quad (7.41)
$$

Integration of (7.41) results in the following expression

$$
W_I = \frac{V_I^2 \rho_b h}{2 P_0}. \quad (7.42)
$$
Equation (7.38) gives an expression for the vertical displacement at the central point of the plate, and equation (7.1) gives an expression for the initial velocity of the loaded area. Those quantities will be substituted in equation (7.42). Using $S = 4C$ from the first assumption in section 3.5, one has

$$\frac{P_0 C^4 Y(C)}{D} = \frac{\rho_b h}{2 P_0 \pi^2 C^4} \frac{4 M_p^2 V_p^2}{\rho_b^2 h^2}. \quad (7.43)$$

Writing the mass of the projectile $M_p$ in terms of the mass density $\rho_p$ and the radius $r_p$, one finds that equation (7.43) yields

$$9 P_0^2 C^8 \rho_b h Y(C) = 32 D \rho_p^2 r_p^6 V_p^2. \quad (7.44)$$

In the last equation, the projectile is assumed to be spherical. Other non-spherical shapes of the projectile have been considered, as in Christiansen and Kerr [27].

In order to work with the critical load for the limit case, we define from (7.30) the quantity $M_D$ such that

$$M_D = f_{\text{plas}} P_0 C^2, \quad M_D = \frac{\sigma_D h^2}{4}. \quad (7.45)$$
where ${\sigma}_D$ is the dynamic yield strength and the relationship with the static yield strength, as proposed by Angel and Smith [2], is given by

$$\sigma_D = \lambda \sigma_0, \quad \lambda = 4.5.$$  \hspace{1cm} (7.46)

Using equations (7.45) and (7.46), the critical load $P_0$ is such that

$$P_0 = \frac{\lambda \sigma_0 h^2}{4 f_{plas} C^2}.$$  \hspace{1cm} (7.47)

Substituting the critical load (7.47) and the definition for the flexural rigidity $D$ given by (4.12) into (7.44), and after some simplifications, we obtain the following ballistic limit equation for homogeneous materials

$$E^{1/2} \rho_p r_p^3 V_p = \left[ \frac{27 Y(C)(1-v^2)}{128 f_{plas}^2} \right]^{1/2} \rho_b^{1/2} C^2 h \lambda \sigma_0.$$  \hspace{1cm} (7.48)

This equation involves the material properties of the projectile, the velocity and radius of the projectile, the density of the plate material, thickness of the plate, the dynamic yield strength, the load radius and the elastic modulus of the plate. Also, two functions are involved $f_{plas}$ and $Y(C)$, which are calculated using the boundary element method.
7.6 Conclusion

A new ballistic limit equation has been developed using a new mathematical model. The model consists of a simply supported circular plate, of radius $R$, bending under a constant load exerted on a circular region of radius $C$ in the center of the plate. The plate has been divided into two concentric regions, an external elastic region and an internal plastic region. The variables and functions involved in the elastic region, $C \leq r \leq R$, have been calculated using the boundary element method with appropriate boundary conditions, as discussed in section 7.3. The variables and functions involved in the plastic region, $0 \leq r \leq C$, have been calculated using an analytical procedure, as discussed in section 7.4. The new ballistic limit equation is given by

$$E^{1/2} \rho_p r_p^3 V_p = \left[ \frac{27 Y(C)[1-\nu^2]}{128 f_{plas}^2} \right]^{1/2} \rho_b^{1/2} C^2 h \lambda \sigma_0,$$

where $E$ is the elastic modulus, $\rho_p$ is the mass density of the projectile, $r_p$ is the radius of the projectile, $V_p$ is the velocity of the projectile, $\nu$ is Poisson's ratio, $\rho_b$ is the mass density of the plate, $C$ is the radius of the loaded region, $h$ is the thickness of the plate, $\lambda$ is the ratio between the dynamic yield strength of the plate and the static yield strength of the plate and takes the value 4.5, $\sigma_0$ is the static yield strength. Also, two functions are involved in the new ballistic limit
equation (7.49); the first one is called the plastic function and is given by (7.19) in the form

$$f_{plas} = \sqrt{M_{xx}^2 - M_{xx} M_{yy} + M_{yy}^2 + 3M_{xy}^2},$$

(7.50)

where the moments are calculated using the boundary element equations (7.13), which are rewritten here as

$$M_{xx} = -(1-\nu)\overline{w}_{xx} - \nu(\overline{w}_{xx} + \overline{w}_{yy}),$$

$$M_{yy} = -(1-\nu)\overline{w}_{yy} - \nu(\overline{w}_{xx} + \overline{w}_{yy}),$$

$$M_{xy} = -(1-\nu)\overline{w}_{xy}.$$

(7.51)

The second derivatives for $w$ involved in (7.51) are calculated as in the boundary element equation (7.11), and are given by

$$\overline{w}(\varepsilon_i) = \frac{A_{ij}}{32\pi R^3} \overline{V}_j - \frac{B_{ij} \overline{B}_{nj}}{8\pi R} + \frac{f_{i}}{256\pi R^2 R^4}.$$

(7.52)

The second function involved in (7.49) is $Y(C)$, which is given by (7.39) in the form
\[ Y(C) = \frac{\bar{w}(C)}{\rho_0^2} + \frac{f_{\text{plus}}}{2(1 + v)} - \frac{1}{72(1 - v^2)}, \quad \rho_0 = \frac{C}{R}. \] (7.53)

The new ballistic limit equation (7.49) will be compared with the ballistic limit equations described in chapter 2; the comparison will be made using ballistic limit curves in the next chapter.
Chapter 8

Ballistic Limit Curves for Homogeneous Materials

8.1 Introduction

In chapter 2, two ballistic limit equations were analyzed and plotted, and in chapter 7, we have shown the process of developing a new ballistic limit equation. In order to be able to use the new equation for homogeneous materials, ballistic limit curves need to be plotted and compared. In this chapter, these plots are shown and analyzed. Four cases were chosen from Angel and Smith [2], and the experimental information is used to generate and compare the new ballistic limit equations.

8.2 Ballistic limit equations

In chapter 7 a new ballistic limit equation has been developed as equation (7.49), which reads

\[ E^{1/2} \rho_p r_p^3 V_p = \left[ \frac{27 Y(C)(1 - \nu^2)}{128 f_{plas}^2} \right]^{1/2} \rho_b^{1/2} C^2 h \lambda \sigma_0 , \]  

(8.1)
were $E$ is the elastic modulus of the homogeneous material, $\rho_p$ is the mass density of the projectile, $r_p$ is the radius of the projectile, $V_p$ is the velocity of the projectile, $\nu$ is Poisson’s ratio, $\rho_b$ is the mass density of the plate, $C$ is the radius of the loaded region, $h$ is the plate thickness, $\lambda$ is the ratio between the dynamic and the static yield strength of the plate, and $\sigma_0$ is the static yield strength. Two additional functions are involved: $f_{\text{plas}}$ called the plastic function given by (7.50), and $Y(C)$ given by (7.53) and calculated using the boundary element method as shown in section 7.3.

In chapter 2, two ballistic limit equations were discussed, the first one is an empirical ballistic limit equation developed by Christiansen [1] using experimental information. It was previously referred to as (2.7), which reads

$$2r = 0.354 \left( \frac{2h}{\rho_p} \right)^{1/3} \rho^{1/3} \left( \frac{U_m \cos \theta}{D} \right)^{-1/3} \left( \frac{\sigma_0}{276} \right)^{1/6},$$  \hspace{1cm} (8.2)

where $2r$ is the diameter of the projectile in centimeters, $2h$ is the thickness of the back plate in centimeters, $D$ is the distance between the outer sheet and the back plate in centimeters, $\rho_p$ is the mass density of the projectile in grams per cubic centimeter, $\rho$ is the mass density of the back plate in grams per cubic centimeter, $U_m$ is the velocity of the projectile in kilometers per second, $\sigma_0$ is
the static yield strength of the back plate in mega-pascal, and $\theta$ is the angle of impact measured from the normal.

The second equation considered is an analytical ballistic limit equation and has been developed by Angel and Smith [2] using an analytical procedure and experimental information. It was previously referred to as (2.8), which reads

$$
 r = \left[ \frac{9 \lambda \mu (1 + K) \sqrt{K}}{2(1 - K)} \right]^{1/3} \sigma_0^{1/3} D^{1/3} h^{2/3} \rho_p^{1/3} U_m^{-2/3},
$$

where $K$, $\lambda$, and $\mu$ are parameters whose values are calculated from experimental information to be $K = 0.04$, $\lambda = 4.5$, and $\mu = 50$.

In order to compare the new ballistic limit equation (8.1) with equations (8.2) and (8.3), we will plot curves using data obtained from four different special cases chosen from Angel and Smith [2]. In the comparison, for each case, two kinds of plots will be presented.

First, the radius of the projectile is plotted against the projectile velocity, while all other variables in the equations are kept fixed. Solving for the radius of the projectile $r_p$ from equations (8.1), (8.2) and (8.3), we find that
\[ r_{Cr} = \left[ K_1^{1/3} \lambda^{1/3} \sigma_0^{1/3} C^{2/3} \rho_p^{-1/3} \rho_b^{1/6} h^{1/3} E^{-1/6} \right] V_p^{1/3}, \]

\[ r_{Ch} = \left[ (0.177) h^{1/3} \rho_p^{-1/3} \rho_b^{1/3} S^{2/3} \left( \frac{\sigma_0}{276} \right)^{1/6} \right] V_p^{-1/3}, \]

\[ r_{An} = \left[ (219.4)^{1/3} \sigma_0^{1/3} h^{2/3} \rho_p^{-1/3} S^{1/3} \right] V_p^{-2/3}, \]

where \( r_{Cr} \) is the radius of the projectile for the new ballistic limit equations (8.1), \( r_{Ch} \) is the radius of the projectile for the empirical ballistic limit equation (8.2), and \( r_{An} \) is the radius of the projectile for the analytical ballistic limit equation (8.3), and the term \( K_1 \) in the first equation (8.4) is given by

\[ K_1 = \left[ \frac{27(1 - \nu^2) \gamma(C)}{128 f_{plas}^2} \right]^{1/2}. \]

Second, the thickness of the plate is plotted against the projectile velocity, while all other variables in the equations are kept fixed. The equations presented bellow in (8.6) have been used for the second set of plots, and have been obtained by solving equations (8.1), (8.2) and (8.3) for the thickness of the plate \( h \).
\[ h_{Cr} = \left[ E^{1/2} \rho_p r_p^3 K_1^{-1} \rho_b^{-1/2} C^{-2} \lambda^{-1} \sigma_0^{-1} \right] V_p , \]
\[ h_{Ch} = \left[ (0.177)^{-3} r_p^3 \rho_p \rho_b^{-1} S^{-2} \left( \frac{\sigma_0}{276} \right)^{-1/2} \right] V_p , \]
\[ h_{An} = \left[ (219.4)^{-1/2} \sigma_0^{-1/2} \rho_p^{1/2} S^{-1/2} r_p^{3/2} \right] V_p , \]

where \( h_{Cr} \) is the thickness of the plate for the new ballistic limit equations (8.1),

\( h_{Ch} \) is the thickness of the plate for the empirical ballistic limit equation (8.2), and

\( h_{An} \) is the thickness of the plate for the analytical ballistic limit equation (8.3).

### 8.3 Curves for the ballistic limit equations: case 1

Table 8.1 corresponds to the parameter values substituted in equations (8.4), for the first case.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>7.62 cm</td>
</tr>
<tr>
<td>( C )</td>
<td>2.39 cm</td>
</tr>
<tr>
<td>( \rho_p )</td>
<td>2796 Kg/m³</td>
</tr>
<tr>
<td>( \sigma_0 )</td>
<td>275 Mpa</td>
</tr>
<tr>
<td>( \nu )</td>
<td>0.33</td>
</tr>
<tr>
<td>( E )</td>
<td>69 Gpa</td>
</tr>
<tr>
<td>( \rho_b )</td>
<td>2713 Kg/m³</td>
</tr>
<tr>
<td>( h )</td>
<td>0.635 mm</td>
</tr>
<tr>
<td>( S )</td>
<td>10.16 cm</td>
</tr>
</tbody>
</table>

Figure 8.1 shows the curves obtained by substituting the data from Table 8.1 into equations (8.4) and plotting the radius of the projectile versus the projectile velocity for all three equations. It is to be noted that the curve corresponding to the new ballistic limit equation (8.1) almost coincides with the curve
corresponding to the empirical ballistic limit equation (8.2). The difference in the critical radius of the projectile for those curves is less than 0.95 % for all values of the velocity. The curve corresponding to the analytical limit equation (8.3) coincides with the other curves at the beginning and diverges from them for velocities beyond 8 Km/s. The regions below these curves are the safe regions.

![Graph](image)

**Figure 8.1.** Critical projectile radii versus projectile velocity, for three different ballistic limit equations, case 1.

Table 8.2 corresponds to the parameter values substituted in equations (8.6), for the first case.
Table 8.2. Data for equations (8.6), case 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>7.62 cm</td>
</tr>
<tr>
<td>$C$</td>
<td>2.39 cm</td>
</tr>
<tr>
<td>$\rho_b$</td>
<td>2713 Kg/m³</td>
</tr>
<tr>
<td>$\rho_p$</td>
<td>2796 Kg/m³</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.33</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>275 Mpa</td>
</tr>
<tr>
<td>$E$</td>
<td>69 Gpa</td>
</tr>
<tr>
<td>$S$</td>
<td>10.16 cm</td>
</tr>
<tr>
<td>$r_p$</td>
<td>1.5875 mm</td>
</tr>
</tbody>
</table>

Figure 8.2. Critical thickness of the plate versus projectile velocity for three different ballistic limit equations, case 1.

Figure 8.2 shows the curves obtained by substituting the data from Table 8.2 into equations (8.6) and plotting the plate thickness versus the velocity of the
projectile for all three equations. It is to be noted that the curve corresponding to
the new ballistic limit equation (8.1) almost coincides with the curve
(corresponding to the empirical ballistic limit equation (8.2); the difference in the
slope of those curves is 2.78 %. The regions above these curves are the safe
regions.

8.4 Curves for the ballistic limit equations: case 2

Table 8.3 corresponds to the parameter values substituted in equations (8.4), for
the second case.

Table 8.3. Data for equations (8.4), case 2.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R = 7.62 \text{ cm}$</td>
<td>$\rho_P = 2796 \text{ Kg/m}^3$</td>
</tr>
<tr>
<td>$C = 2.19 \text{ cm}$</td>
<td>$\nu = 0.33$</td>
</tr>
<tr>
<td>$\rho_b = 2768 \text{ Kg/m}^3$</td>
<td>$h = 0.635 \text{ mm}$</td>
</tr>
</tbody>
</table>

Figure 8.3 shows the curve obtained by substituting the data from Table 8.3 into
equations (8.4) and plotting the radius of the projectile versus the projectile
velocity for all three equations. It is to be noted that the curve corresponding to
the new ballistic limit equation (8.1) almost coincides with the curve
corresponding to the empirical ballistic limit equation (8.2). The difference in the
critical radius of the projectile for those curves is less than 2.80 % for all values of
the velocity. The curve corresponding to the analytical limit equation (8.3)
coincides with the other curves at the beginning and diverges from them for velocities beyond 8 km/s. The regions below these curves are the safe regions.

![Graph showing critical projectile radii versus projectile velocity for three different ballistic limit equations, case 2.](image)

Figure 8.3. Critical projectile radii versus projectile velocity for three different ballistic limit equations, case 2.

Table 8.4 corresponds to the parameter values substituted in equations (8.6), for the second case.
Table 8.4. Data for equations (8.6), case 2.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>7.62 cm</td>
</tr>
<tr>
<td>$C$</td>
<td>2.19 cm</td>
</tr>
<tr>
<td>$\rho_b$</td>
<td>2768 Kg/m$^3$</td>
</tr>
<tr>
<td>$\rho_p$</td>
<td>2796 Kg/m$^3$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.33</td>
</tr>
<tr>
<td>$E$</td>
<td>72.4 Gpa</td>
</tr>
<tr>
<td>$S$</td>
<td>10.16 cm</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>345 Mpa</td>
</tr>
</tbody>
</table>

Figure 8.4. Critical thickness of the plate versus projectile velocity for three different ballistic limit equations, case 2.

Figure 8.4 shows the curve obtained by substituting the data from Table 8.4 into equations (8.6) and plotting the plate thickness versus projectile velocity for all three equations. It is to be noted that the curve corresponding to the new ballistic limit equation (8.1) is close to the curve corresponding to the empirical ballistic
limit equation (8.2); the difference in the slope for those curves is 9.52 %. The regions above these curves are the safe regions.

8.5 Curves for the ballistic limit equations: case 3

Table 8.5 corresponds to the parameter values substituted in equations (8.4), for the third case.

Table 8.5. Data for equations (8.4), case 3.

<table>
<thead>
<tr>
<th></th>
<th>$\rho_p = 2796 \text{ Kg/m}^3$</th>
<th>$\sigma_0 = 345 \text{ Mpa}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R = 7.62 \text{ cm}$</td>
<td>$\nu = 0.33$</td>
<td>$E = 72.4 \text{ Gpa}$</td>
</tr>
<tr>
<td>$C = 2.12 \text{ cm}$</td>
<td>$\rho_b = 2768 \text{ Kg/m}^3$</td>
<td>$h = 0.813 \text{ mm}$</td>
</tr>
<tr>
<td>$\rho_b = 2768 \text{ Kg/m}^3$</td>
<td>$S = 10.16 \text{ cm}$</td>
<td></td>
</tr>
</tbody>
</table>

Figure 8.5 shows the curves obtained by substituting the data from Table 8.5 into equations (8.4) and plotting the radius of the projectile versus projectile velocity for all three equations. It is to be noted that the curve corresponding to the new ballistic limit equation (8.1) almost coincides with the curve corresponding to the empirical ballistic limit equation (8.2). The difference in the critical radius of the projectile for those curves is less than 4.41 % for all values of the velocity. The curve corresponding to the analytical limit equation (8.3) starts above the two other curves, then crosses those curves and diverges for velocities beyond 8 Km/s. It is also to be noted that the curves of the new equation and the curve of the empirical equation have similar shapes. The regions below these curves are the safe regions.
Figure 8.5. Critical projectile radii versus projectile velocity for three different ballistic limit equations, case 3.

Table 8.6 corresponds to the parameter values substituted in equations (8.6), for the third case.

Table 8.6. Data for equations (8.6), case 3.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R = 7.62$ cm</td>
<td>$\rho_p = 2796$ Kg/m$^3$</td>
</tr>
<tr>
<td>$C = 2.12$ cm</td>
<td>$\nu = 0.33$</td>
</tr>
<tr>
<td>$\rho_b = 2768$ Kg/m$^3$</td>
<td>$r_p = 1.5875$ mm</td>
</tr>
</tbody>
</table>
Figure 8.6 shows the curves obtained by substituting the data from Table 8.6 into equations (8.6) and plotting the plate thickness versus projectile velocity for all three equations. It is to be noted that the curve corresponding to the new ballistic limit equation (8.1) is between the curves corresponding to the empirical ballistic limit equations (8.2) and (8.3). The regions above these curves are the safe regions.

![Graph showing critical thickness of the plate versus projectile velocity for three different ballistic limit equations, case 3.](image-url)
8.6 Curves for the ballistic limit equations: case 4

Table 8.7 corresponds to the parameter values substituted in equations (8.4), for the fourth case.

Table 8.7. Data for equations (8.4), case 4.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>7.62 cm</td>
<td>$\rho_p$</td>
<td>2796 Kg/m$^3$</td>
</tr>
<tr>
<td>$C$</td>
<td>2.08 cm</td>
<td>$\nu$</td>
<td>0.33</td>
</tr>
<tr>
<td>$\rho_b$</td>
<td>2768 Kg/m$^3$</td>
<td>$h$</td>
<td>0.508 mm</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td></td>
<td>$E$</td>
<td>72.4 Gpa</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$S$</td>
<td>10.16 cm</td>
</tr>
</tbody>
</table>

Figure 8.7. Critical projectile radii versus projectile velocity for three different ballistic limit equations, case 4.
Figure 8.7 shows the curves obtained by substituting the data from Table 8.7 into equations (8.4) and plotting the radius of the projectile versus projectile velocity for all three equations. It is to be noted that the curve corresponding to the new ballistic limit equation (8.1) almost coincides with the curve corresponding to the empirical ballistic limit equation (8.2). The difference in the critical radius of the projectile for those curves is less than 5.31 % for all values of the velocity. The curve corresponding to the analytical limit equation (8.3) coincides with the other curves at the beginning and diverges from them for velocities beyond 8 Km/s. It is also to be noted that the curve of new equation and the curve of the empirical equation have similar shapes. The regions below these curves are the safe regions. Table 8.8 corresponds to the parameter values substituted in equations (8.6), for the fourth case.

Table 8.8. Data for equations (8.6), case 4.

<table>
<thead>
<tr>
<th>$R = 7.62 \text{ cm}$</th>
<th>$\rho_p = 2796 \text{ Kg/m}^3$</th>
<th>$\sigma_0 = 345 \text{ Mpa}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C = 2.12 \text{ cm}$</td>
<td>$\nu = 0.33$</td>
<td>$E = 72.4 \text{ Gpa}$</td>
</tr>
<tr>
<td>$\rho_b = 2768 \text{ Kg/m}^3$</td>
<td>$r_p = 1.5875 \text{ mm}$</td>
<td>$S = 10.16 \text{ cm}$</td>
</tr>
</tbody>
</table>
Figure 8.8. Critical thickness of the plate versus projectile velocity for three different ballistic limit equations, case 4.

Figure 8.8 shows the curves obtained by substituting the data from Table 8.8 into equations (8.6) and plotting the plate thickness versus projectile velocity for all three equations. It is to be noted that the curve corresponding to the new ballistic limit equation (8.1) is between the curves corresponding to the ballistic limit equations (8.2) and (8.3). The regions above these curves are the safe regions.
8.7 Conclusion

As shown in figures 8.1, 8.2, 8.3, 8.4, 8.5, 8.6, 8.7 and 8.8, the curves are similar for the new ballistic limit equation (8.1) and the empirical ballistic limit equation (8.2). Analysis of equations (8.1) and (8.2) shows multiple similarities between these equations. For example, the exponents for some parameters are the same: the velocity $V_p$ has $-1/3$, the space $S$ has $-2/3$, the thickness $h$ has $1/3$, and the mass density of the projectile $\rho_p$ has $-1/3$ for both equations. For other parameters, the exponents vary: the mass density of the back plate has $1/3$ for (8.2) and $1/6$ for (8.1), the static yield strength has $1/6$ for (8.2) and $1/3$ for (8.1).

In the ballistic limit equation developed in this work (8.1), there are some parameters, which were not considered in equation (8.2) such as the elastic modulus $E$, which has the exponent $-1/6$, and $\lambda$, which has the exponent $1/3$. The parameter $K_1$ in the function $Y(C)$ shown in equation (7.53), and the plastic function $f_{plas}$ shown in equation (7.50) were calculated by using the boundary element method. The new ballistic limit equation (8.1) does not involve parameters obtained from experimental information, and has been developed using only physical principles for hypervelocity impact as presented and discussed in chapter 3, the elastic-plastic theory of plates and the boundary element method.

The ballistic limit curves plotted in this chapter come from three different ballistic limit equations; one of them is developed in chapter 7 using physical concepts
about hypervelocity impact, which are reviewed in chapter 3; also, the use of Kirchhoff's plate theory yields the essential quantities involved in an elastic-plastic analysis. The characteristics of the load and the plastic behavior of the loaded zone make it difficult to derive analytical expressions for the deformation and the moments of the plate, and a numerical solution for those quantities has been obtained using the Boundary Elements Method. The dynamic analysis of the loaded zone, which is in the plastic regime, yields an analytical relationship between all the parameters involved, including numerical results for the plastic region using the boundary elements method. This relationship is our new ballistic limit equation. Comparison by plotting ballistic limit curves, with the other ballistic limit equations, reviewed in chapter 2, shows a strong agreement with the empirical equation. Consequently, we can say that our theoretical procedure to develop a ballistic limit equation is supported by the experimental approach used to develop the empirical ballistic limit equation.
Chapter 9

Composite-Orthotropic Plate Theory

9.1 Introduction

In the development of a new ballistic limit equation for hypervelocity impact on composite-orthotropic materials, the analytical approach discussed in section 3.4 will be used. This approach involves a plate loaded on a circular region and, in order to develop that ballistic limit equation, Kirchhoff plate’s theory for orthotropic plates is used, Ugural [6], Timoshenko [7], Reddy [28], and Gibson [29]. Definitions about plates, the relationship between strain and vertical displacement, and the equilibrium equation reviewed and developed in Chapter 4, which do not depend on the material properties, will be used in this chapter in order develop a partial differential equation for composite-orthotropic plates.

9.2 Review of basic relationships for plates

Elasticity theory includes relationships between strain and displacement, which do not depend on the mechanical properties of the plate. Those relationships are given in (4.3) and are rewritten here
\[ \varepsilon_x = -z \frac{\partial^2 w}{\partial x^2}, \]
\[ \varepsilon_y = -z \frac{\partial^2 w}{\partial y^2}, \]
\[ \gamma_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y}. \]

(9.1)

In section 4.5 the equilibrium equation for plates is developed, the procedure involves the sum of forces and moments from an element of the plate shown in Figure 4.3. This equilibrium equation is shown in (4.18) and does not depend on the material properties of the plate and is rewritten here

\[ \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p = 0. \]

(9.2)

9.3 Stress-strain relationships for orthotropic plates

The generalized Hooke's law gives a stress-strain relationship for the elastic case shown in section 4.4. For orthotropic plates we reformulate the relationships shown in (4.6) and, with the assumptions mentioned in section 4.1, the new relationships are
\[
\sigma_x = \left( \frac{E_x}{1 - \nu_x \nu_y} \right) \varepsilon_x + \left( \frac{\nu_y E_x}{1 - \nu_x \nu_y} \right) \varepsilon_y ,
\]
\[
\sigma_y = \left( \frac{\nu_x E_y}{1 - \nu_x \nu_y} \right) \varepsilon_x + \left( \frac{E_y}{1 - \nu_x \nu_y} \right) \varepsilon_y ,
\]
\[
\sigma_{xy} = G \gamma_{xy} ,
\]

where the constants \( E_x, E_y, \nu_x, \nu_y \) and \( G \) represent the modulus of elasticity in the \( x \) direction, modulus of elasticity in the \( y \) direction, Poisson's ratio in the \( x \) direction, Poisson's ratio in the \( y \) direction and the shear modulus of elasticity, respectively.

Now, substituting equations (9.1) into equations (9.3), we obtain relationships for the stress and the vertical displacement for orthotropic plates in the form

\[
\sigma_x = -\frac{E_x z}{(1 - \nu_x \nu_y)} \left( \frac{\partial^2 w}{\partial x^2} + \nu_y \frac{\partial^2 w}{\partial y^2} \right) ,
\]
\[
\sigma_y = -\frac{E_y z}{(1 - \nu_x \nu_y)} \left( \frac{\partial^2 w}{\partial y^2} + \nu_x \frac{\partial^2 w}{\partial x^2} \right) ,
\]
\[
\sigma_{xy} = -2G z \frac{\partial^2 w}{\partial x \partial y} .
\]
The stresses distributed over the thickness of the plate produce bending moments, twisting moments, and vertical shear forces, these moments and forces per unit of length are called stress resultants. The moments are

\[
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{bmatrix} z \, dz,
\]

(9.5)

and the shear forces are

\[
\begin{bmatrix}
Q_x \\
Q_y
\end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix}
\sigma_{xz} \\
\sigma_{yz}
\end{bmatrix} \, dz.
\]

(9.6)

The integration of the expressions in equations (9.5) yields formulae for the bending and twisting moments in terms of the vertical displacement \( w \) in the form
\[ M_x = -\left( D_x \frac{\partial^2 w}{\partial x^2} + D_{xy} \frac{\partial^2 w}{\partial y^2} \right), \]

\[ M_y = -\left( D_y \frac{\partial^2 w}{\partial y^2} + D_{xy} \frac{\partial^2 w}{\partial x^2} \right), \]

\[ M_{xy} = -2G_{xy} \frac{\partial^2 w}{\partial x \partial y}, \]

In equation (9.7), the quantities

\[ D_x = \frac{E_x h^3}{12(1 - \nu_x \nu_y)}, \]

\[ D_y = \frac{E_y h^3}{12(1 - \nu_x \nu_y)}, \]

\[ D_{xy} = \frac{\nu_y E_x h^3}{12(1 - \nu_x \nu_y)}, \]

\[ G_{xy} = \frac{G h^3}{12}, \]

are the flexural rigidities of the composite-orthotropic plate. Now, in order to derive a partial differential equation, we substitute (9.7) into the equilibrium equation (9.2) and we find that
\[ D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = p, \]  

(9.9)

where

\[ H = D_{xy} + 2G_{xy}. \]  

(9.10)

Integration of the partial differential equation (9.9) yields solutions for \( w \) that involve some unknown integration constants and, to calculate them, we need to use appropriate boundary conditions.

9.4 Example 1: Clamped circular plate under a constant load

Consider a circular plate of radius \( a \), clamped at the edge and subjected to the uniformly distributed load \( P_0 \), shown in Figure 9.1. Assume that the coordinates axis and the principal direction of the orthotropic plate are parallel.

The analytical expression for the vertical displacement of the orthotropic plate is

\[ w = \frac{P_0}{8(3D_x + 2H + 3D_y)} \left( a^2 - x^2 - y^2 \right)^2. \]  

(9.11)

Given the vertical displacement \( w \), the moments can be obtained using the expressions in (9.7). One finds that
Figure 9.1. Clamped circular plate under a constant load.

\[
M_x = A \left[ (D_x + D_{xy}) \left( a^2 - x^2 - y^2 \right) - 2 \left( D_x x^2 + D_{xy} y^2 \right) \right],
\]

\[
M_y = A \left[ (D_y + D_{xy}) \left( a^2 - x^2 - y^2 \right) - 2 \left( D_y y^2 + D_{xy} x^2 \right) \right],
\]

\[
M_{xy} = 4AD_{xy}xy,
\]

\[
A = \frac{P_0}{2(3D_x + 2H + 3D_y)}.
\]

The analytical solution for a clamped circular plate under a constant load has been presented. The characteristics of the boundary conditions allow us to solve this problem in closed form, and the solution for the vertical displacement \( w \) is given by (9.11). The exact solution for a simply supported orthotropic plate is not yet known; Ohasi [30] presents an approximate solution for that problem by using a curvilinear coordinate system.
9.5 Conclusion

A review of orthotropic plate theory has been presented in this chapter, and the equilibrium equation for moments is shown in equation (9.2). This partial differential equation does not involve any material properties. For orthotropic plates, the relationships between moments and vertical displacement are given in equations (9.7), and using them we obtain a partial differential equation for orthotropic plates which is given by (9.9)

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = p.$$  \hspace{1cm} (9.13)

This partial differential equation with appropriate boundary conditions yields solutions to different problems for bending of plates. The analytical solution for bending of a clamped orthotropic circular plate is presented in section 9.4, and for \( x = 0 \) and \( y = 0 \) in (9.11), the vertical displacement is

$$w = \frac{P_0 a^4}{8(3D_x + 2H + 3D_y)}.$$  \hspace{1cm} (9.14)

The expression (9.14) is an analytical closed form solution for the vertical displacement of the central point of a clamped circular plate under a constant load \( P_0 \), and will be used in chapter 12.
Chapter 10

Boundary Elements Method for Orthotropic Plates

10.1 Introduction

In order to develop a new ballistic limit equation for orthotropic materials, the bending of plates made of that material is analyzed in this chapter; the partial differential equation for this kind of plates, which is analyzed in chapter 9, involves different quantities such as the vertical displacement and the moments. The boundary conditions, the geometry of the plate, the elastic-plastic regime can makes it difficult to derive analytical solutions for this problem. In this chapter, we use the Boundary Elements Method in order to obtain a numerical solution for the partial differential equation for orthotropic plates. The procedure starts with the development of an integral formulation using the equilibrium equation for moments shown in section 9.2, resulting in an integral equation that involves integrals on the boundary and in the domain of the plate. In the next step, one makes use of a relationship for the moments and the vertical displacement, which results in an integral equation for the vertical displacement. Integration by parts is used to obtain an integral equation involving only integrals on the boundary. The accuracy of the numerical solution given by this procedure depends on the discretization of the boundary, the geometry of the plate and the boundary conditions.
10.2 Boundary element formulation for orthotropic plates

The plate-bending problem is formulated in terms of a partial differential equation with the load, boundary conditions, and the geometry of the plate; it is often difficult to derive analytical solutions for the vertical displacement and the moments of the plate. Therefore the Boundary Elements Method is used to obtain numerical solutions for this problem. To develop a boundary element formulation for plates, we use the equilibrium equation for plates, from equation (9.2), which in index notation reads

\[ M_{\alpha\beta,\alpha\beta} + p = 0. \quad (10.1) \]

Integrating equation (10.1) multiplied by a weight function over the entire domain of the plate yields

\[ \int_{\Omega} \left( M_{\alpha\beta,\alpha\beta} + p \right) w^* \, d\Omega = 0, \quad (10.2) \]

where \( w^* \) is a weight function called the fundamental solution and will be discussed later, and \( \Omega \) is the domain of the plate. Equation (10.2) is now expressed as
\[ \int_{\Omega} M_{\alpha\beta, \alpha\beta} w^* \, d\Omega + \int_{\Omega} p w^* \, d\Omega = 0. \]  

(10.3)

Integration by parts of the first integral in (10.3), as is performed in section 5.2, yields equation (5.9), which does not depend on the material properties of the material and is rewritten here as

\[ \int_{\Gamma} w_{,\beta}^* M_\beta \, d\Gamma - \int_{\Gamma} w^* q \, d\Gamma = \int_{\Omega} M_{\alpha\beta} w_{,\alpha}^* \, d\Omega + \int_{\Omega} p w^* \, d\Omega. \]  

(10.4)

The integral equation (10.4) involves the moments \( M_{\alpha\beta} \), which for orthotropic plates are related to the vertical displacement as in equations (9.7). We rewrite these equations as follows

\[ M_x = -\left( D_x \frac{\partial^2 w}{\partial x^2} + D_{xy} \frac{\partial^2 w}{\partial y^2} \right), \]

\[ M_y = -\left( D_y \frac{\partial^2 w}{\partial y^2} + D_{xy} \frac{\partial^2 w}{\partial x^2} \right), \]  

(10.5)

\[ M_{xy} = -2G_{xy} \frac{\partial^2 w}{\partial x \partial y}. \]
Substituting these expressions for the moments into the first integral in the right-hand side of equation (10.4), and after integration by parts and using the relationships in (5.16), (5.17), and (5.21) we obtain the following integral equation

\[
\begin{align*}
\int_\Gamma wV^* d\Gamma &= \int_\Gamma w^* V d\Gamma - \int_\Gamma \beta_n M_n^* d\Gamma + \\
\int_\Gamma \beta_n^* M_n d\Gamma &= -\int_\Omega w(\vec{\nabla}^4 w^*)d\Omega + \int_\Omega p w^* d\Omega,
\end{align*}
\]  

where

\[
\vec{\nabla}^4 = D_x \frac{\partial^4}{\partial x^4} + 2H \frac{\partial^4}{\partial x^2 \partial y^2} + D_y \frac{\partial^4}{\partial y^4}.
\]  

The integral equation (10.6) includes four integrals on the boundary \( \Gamma \) and two integrals in the domain \( \Omega \), one of them involves the load \( p \). We will deal with this integral later. The first integral in the domain in equation (10.6) involves the weight function \( w^* \) and we will deal with this integral using a procedure called the fundamental solution.

**10.3 Approximate fundamental solution for orthotropic plates**

The right-hand side of the integral equation (10.6) includes two integrals in the domain \( \Omega \); the first one will be treated with the procedure called the fundamental solution. The fundamental solution \( w^* \) represents the vertical displacement at
the field point \((x_1, x_2)\) of an infinite plate due to a unit vertical load applied at the source point \((\varepsilon_1, \varepsilon_2)\), as discussed in Wu and Altiero [31], Shi and Bezine [32]. In order to obtain an expression for the fundamental solution, the following singular equation will be solved

\[
\tilde{\nabla}^4 w^* = \delta(r), \quad (10.8)
\]

where \(r\) is the distance between the source point and the field point, which are shown in Figure 10.1. The expression in equation (10.8) appears in the integral equation (10.6) and will be substituted in order to eliminate the integral in the domain. Substituting (10.8) in the first integral of the right-hand side of (10.6), one finds that

\[
\int_{\Omega} w \left( \tilde{\nabla}^4 w^* \right) d\Omega = \int_{\Omega} w \delta(r) d\Omega = w(\varepsilon_1, \varepsilon_2). \quad (10.9)
\]
Substituting (10.9) into equation (10.6), we obtain the boundary element formulation for orthotropic plates in the form

\[ \alpha w(\varepsilon_1, \varepsilon_2) + \int_{\Gamma} \left( \beta_n^* M_n + V^* w \right) d\Gamma = \]

\[ \int_{\Omega} w^* p d\Omega + \int_{\Gamma} \left( M_n^* \beta_n + w^* V \right) d\Gamma. \]

The parameter \( \alpha \) depends on the position of the source point, \( \alpha = 1 \) if the source point belongs to the domain \( \Omega \), \( \alpha = 1/2 \) if the source point belongs to the boundary \( \Gamma \) and \( \alpha = 0 \) if the source point is outside.

From equation (10.8), we can derive an expression for the fundamental solution \( w^* \). Shi and Bezine [32] present a fundamental solution that is difficult to handle, especially when we want to transform the domain integral that contains the load into a boundary integral, as was done in section 5.5. For this reason, we present...
here a new approximate fundamental solution for orthotropic plates that is derived using a coordinate transformation. Figure 10.2 shows the coordinate transformation.

![Diagram](image)

Figure 10.2. The source point \((\varepsilon_1, \varepsilon_2)\) and the field point \((x_1, \tilde{x}_2)\).

Using the coordinate transformation shown in Figure 10.2, equation (10.8) takes the form

\[ D_x \nabla^4 w^* = \delta(\tilde{r}). \]  

\[ (10.11) \]

The coordinate transformation used to obtain equation (10.11) is
\[ \tilde{x}_1 = x_1 , \]
\[ \tilde{x}_2 = 4 \sqrt{\frac{D_x}{D_y}} x_2 . \] 

(10.12)

The solution of (10.11) is identical to (5.26), except that \( r \) is replaced with \( \tilde{r} \).

Thus, one has

\[ w^* = \frac{1}{16\pi D_x} \left( 2\tilde{r}^2 \ln \tilde{r} - \tilde{r}^2 \right). \] 

(10.13)

The expression (10.13) is an approximate fundamental solution for orthotropic plates, and will be used in the integral equation (10.10) and to derive expressions for \( V^* \), \( M_n^* \) and \( \beta_n^* \). The integral equation (10.10) includes integrals in the boundary \( \Gamma \) and one integral in the domain \( \Omega \). In order to use the boundary element formulation for plates, the integral equation should include only integrals on the boundary. In order to transform the integral in the domain \( \Omega \) that appears in equation (10.10), we will use a procedure described by Ameen [10] for the case when the load \( p \) is constant and will be discussed in the next section.
10.4 Transformation of the integral in the domain into an integral on the boundary

The integral equation (10.10) includes an integral in the domain and we can transform it into an integral on the boundary when the load is constant. The procedure uses a function $G^*$ such that

$$ w^* = G^*_{,\alpha} , \quad (10.14) $$

where

$$ G^* = \frac{1}{128\pi D_x} (\ln \tilde{r} - 1) \tilde{r}^4 . \quad (10.15) $$

Hence, for the case when the load is constant, $p = P_0$, and one finds that

$$ \int_\Omega p w^* d\Omega = P_0 \int_\Gamma G^*_{,\alpha} n_\alpha d\Gamma . \quad (10.16) $$

If $p$ is not constant, we can approximate it using a combination of functions in order to transform the integral in the domain into an integral in the boundary, as shown in Muci-Kuchler and Cruz-Banuelos [14]. Substituting the expression in (10.16) into equation (10.10), one has
\[ \alpha w(\varepsilon_1, \varepsilon_2) + \int_{\Gamma} \left( \beta_n^* M_n + V^* w \right) d\Gamma = \]

\[ P_0 \int_{\Gamma} G^*_{\alpha} n_\alpha d\Gamma + \int_{\Gamma} \left( M_n^* \beta_n + w^* V \right) d\Gamma. \]

This integral equation has to be complemented by a further one, which requires taking the derivative of equation (10.17) and has the form

\[ \alpha \frac{\partial w(\varepsilon_1, \varepsilon_2)}{\partial \varepsilon_i} n_i^0 + \int_{\Gamma} \left( \frac{\partial \beta_n^*}{\partial \varepsilon_i} M_n + \frac{\partial V^*}{\partial \varepsilon_i} n_i^0 w \right) d\Gamma = \]

\[ P_0 \int_{\Gamma} \frac{\partial G^*_{\alpha}}{\partial \varepsilon_i} n_\alpha d\Gamma + \int_{\Gamma} \left( \frac{\partial M_n^*}{\partial \varepsilon_i} n_i^0 \beta_n + \frac{\partial w^*}{\partial \varepsilon_i} n_i^0 V \right) d\Gamma. \]

We define the following quantities

\[ \beta_n = \frac{\partial w(\varepsilon_1, \varepsilon_2)}{\partial \varepsilon_i} n_i^0, \quad \bar{w}_n^* = \frac{\partial w^*}{\partial \varepsilon_i} n_i^0, \]

\[ \bar{\beta}_n^* = \frac{\partial \beta_n^*}{\partial \varepsilon_i} n_i^0, \quad \bar{M}_n^* = \frac{\partial M_n^*}{\partial \varepsilon_i} n_i^0, \]

\[ \bar{V}^* = \frac{\partial V^*}{\partial \varepsilon_i} n_i^0, \quad \bar{G}_n^* = \frac{\partial G_{\alpha}^*}{\partial \varepsilon_i} n_i^0 n_\alpha. \]
Substituting the definitions in (10.19) into equation (10.18), we obtain the second integral equation to be considered in the boundary element formulation for orthotropic plates as follows

\[ \alpha \beta_n (\varepsilon_1, \varepsilon_2) + \int_{\Gamma} \left( \beta_n^* M_n + \bar{V}^* w \right) d\Gamma = \]

\[ P_0 \int_{\Gamma} \bar{G}_n^* d\Gamma + \int_{\Gamma} \left( M_n^* \beta_n + \bar{w}_n^* V \right) d\Gamma. \]

The integral equations (10.17) and (10.20) include only integrals on the boundary, and a discretization of the boundary will be used in order to calculate a numerical solution for the different quantities involved in the integral equations.

10.5 Conclusion

A review of the boundary element formulation for orthotropic plates has been presented here. The integral equation (10.4) does not depend on the material properties of the plates and will be used for any material

\[ \int_{\Gamma} w_{\beta}^* M_{\beta} d\Gamma - \int_{\Gamma} w^* q d\Gamma = \int_{\Omega} M_{\alpha \beta} w_{\alpha \beta}^* d\Omega + \int_{\Omega} p w^* d\Omega. \]
For orthotropic plates, relationships between moments and vertical displacement are used in order to develop a boundary element formulation for this kind of plates. From (10.5), one has

\[
M_x = -\left( D_x \frac{\partial^2 w}{\partial x^2} + D_{xy} \frac{\partial^2 w}{\partial y^2} \right),
\]

\[
M_y = -\left( D_y \frac{\partial^2 w}{\partial y^2} + D_{xy} \frac{\partial^2 w}{\partial x^2} \right),
\]

(10.22)

\[
M_{xy} = -2G_{xy} \frac{\partial^2 w}{\partial x \partial y}.
\]

The use of equations (10.17) and (10.20) yields a boundary element formulation for orthotropic plates for the case when the load is constant

\[
\alpha w(\varepsilon_1, \varepsilon_2) + \int_\Gamma \left( \beta_n^* M_n + V^* w \right) d\Gamma =
\]

\[
P_0 \int_\Gamma G_{\alpha} n_\alpha d\Gamma + \int_\Gamma \left( M_n^* \beta_n + w^* V \right) d\Gamma,
\]

(10.23)

\[
\alpha \beta_n(\varepsilon_1, \varepsilon_2) + \int_\Gamma \left( \bar{\beta}_n^* M_n + \bar{V}^* w \right) d\Gamma =
\]

\[
P_0 \int_\Gamma \bar{G}_n^* d\Gamma + \int_\Gamma \left( \bar{M}_n^* \beta_n + \bar{w}_n^* V \right) d\Gamma.
\]

In equation (10.23), one has
\[ w^* = \frac{1}{16 \pi D_x} \left( 2 \tilde{r}^2 \ln \tilde{r} - \tilde{r}^2 \right), \]  \hspace{1cm} (10.24)

where \( w^* \) is an approximate fundamental solution for orthotropic plates and will be used in order to calculate numerical values for the different quantities involved in the integral equations (10.23). Those equations involve only integrals on the boundary. In order to calculate a numerical solution for the quantities involved in the integral equations, a boundary discretization in linear elements will be used in the next chapter.
Chapter 11

Boundary Element Discretization in Linear Elements for Orthotropic Plates

11.1 Introduction

The boundary element formulation for the bending of orthotropic plates and for the case when the load is constant is given in the integral equations (10.23). In order to calculate a numerical solution for the different quantities involved in those equations, a discretization of the boundary is used. In this chapter, we will use linear elements in order to perform this discretization. Different kinds of elements can be used such as constant elements, quadratic elements, and others [33-35]. The integrals in equations (10.23) will be integrated in each linear element by considering a linear variation for the variables involved.

11.2 Discretization of the plate boundary in linear elements

The integral equations (10.23) are shown here as (11.1).
\[
\alpha w(\varepsilon_1, \varepsilon_2) + \int_{\Gamma} \left( \beta_n^* M_n + V^* w \right) d\Gamma = \\
P_0 \int_{\Gamma} G_n^* n_\alpha d\Gamma + \int_{\Gamma} \left( M_n^* \beta_n + w^* V \right) d\Gamma,
\]

(11.1)

\[
\alpha \beta_n(\varepsilon_1, \varepsilon_2) + \int_{\Gamma} \left( \beta_n^* M_n + \bar{V}^* w \right) d\Gamma = \\
P_0 \int_{\Gamma} \bar{G}_n^* d\Gamma + \int_{\Gamma} \left( \bar{M}_n^* \beta_n + \bar{w}_n^* V \right) d\Gamma.
\]

Figure 11.1 shows a discretization of the plate boundary in linear elements.

![Discretization of the plate boundary in linear elements](image)

Figure 11.1. Discretization of the plate boundary in linear elements.

The linear elements are straight with two nodes usually located at the ends of the element, and a local coordinate is used in each element. The vertical displacement \( w \), the normal slope \( \beta_n \), the normal moment \( M_n \) and the effective shear \( V \) are assumed to vary linearly in the linear element. These values can be defined in terms of their nodal values and two linear interpolation functions.
$H_1(\eta)$ and $H_2(\eta)$, which are given in terms of the homogeneous coordinate $\eta$ as shown in Figure 11.2. Using the local coordinate system and the interpolation function, the quantities in each linear element can be expressed as

$$w^{(j)} = H_{\alpha}(\eta) w_{\alpha}^{(j)},$$

$$\beta_n^{(j)} = H_{\alpha}(\eta) \beta_{n\alpha}^{(j)}, \quad (11.2)$$

$$M_n^{(j)} = H_{\alpha}(\eta) M_{n\alpha}^{(j)},$$

$$V^{(j)} = H_{\alpha}(\eta) V_{\alpha}^{(j)},$$

![Diagram](image)

Figure 11.2. A linear boundary element with a local coordinate system.

where $w_{\alpha}^{(j)}$ is the value of the vertical displacement at the local node $j$; $\beta_{n\alpha}^{(j)}$ is the value of the normal slope at the local node $j$; $M_{n\alpha}^{(j)}$ is the value of the
normal moment at the local node \( j \); and \( V_{\alpha}^{(j)} \) is the value of the effective shear force at the local node \( j \). The interpolation functions are

\[
H_1(\eta) = \frac{1}{2}(1 - \eta), \quad H_1(\eta) = \frac{1}{2}(1 + \eta).
\]

Each integral in equations (11.1) is integrated by making a discretization in those integrals. Using the fourth expression in (11.2), one finds that one of the integrals in (11.1) can be expressed as

\[
\int_{\Gamma} w^* V \, d\Gamma = \sum_{j=1}^{N} \int_{\Gamma(j)} w^* V^{(j)} \, ds = \sum_{j=1}^{N} \int_{\Gamma(j)} w^* H_\alpha(\eta) V_{\alpha}^{(j)} \, ds,
\]

where \( N \) is the number of linear elements and \( ds \) is the arc length along the element \( j \). The relationship with the local coordinate system \( \eta \) is

\[
ds = \frac{L_j}{2} \, d\eta,
\]

where \( L_j \) is the length of element \( j \). Substituting (11.5) into (11.4), one finds that
\[ \int_{\Gamma} w^* V d\Gamma = \frac{1}{32\pi D_x} \sum_{j=1}^{N} A^{(j)}_\alpha V^{(j)}_\alpha = \frac{1}{32\pi D_x} \sum_{j=1}^{N} \left[ A^{(j-1)}_2 + A^{(j)}_1 \right] V_j , \]  

(11.6)

where

\[ A^{(j)}_\alpha = L_j \int_{\Gamma(j)} f_1(\eta) H_\alpha(\eta) d\eta , \]  

(11.7)

\[ f_1(r) = 2r^2 \ln r - r^2 . \]

The integral in (11.7) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

Using the same procedure with the other integrals in equations (11.1), we obtain the following expression for the next integral

\[ \int_{\Gamma} M^*_n \beta_n d\Gamma = \sum_{j=1}^{N} \int_{\Gamma(j)} M^*_n \beta^{(j)}_n d_s = \sum_{j=1}^{N} \int_{\Gamma(j)} M^*_n H_\alpha(\eta) \beta^{(j)}_{n\alpha} d_s , \]  

(11.8)

\[ \int_{\Gamma} M^*_n \beta_n d\Gamma = -\frac{1}{8\pi} \sum_{j=1}^{N} B^{(j)}_\alpha \beta^{(j)}_n = -\frac{1}{8\pi} \sum_{j=1}^{N} \left[ B^{(j-1)}_2 + B^{(j)}_1 \right] \beta_{nj} , \]

where
\[ B^{(j)}_{\alpha} = L_{j} \int_{\Gamma(j)} f_2(\vec{\tau}) H_{\alpha}(\eta) d\eta , \]

\[ f_2(\vec{\tau}) = a_1 f_1^*(\vec{\tau}) + a_2 f_2^*(\vec{\tau}) + a_3 f_3^*(\vec{\tau}) , \]

\[ a_1 = n_x^2 + \alpha_1 v_x n_y^2 , \quad f_1^* = \ln r + \frac{r_1^2}{r^2} , \]

\[ a_2 = \alpha_1 (v_x n_x^2 + n_y^2) , \quad f_2^* = \ln r + \frac{r_2^2}{r^2} , \]

\[ a_3 = 4 \alpha_2 (1 - \alpha_1 v_x^2) n_x n_y , \quad f_3^* = \frac{r_1 r_2}{r^2} , \]

\[ \alpha_1 = \frac{v_y}{v_x} = \frac{E_y}{E_x} , \quad \alpha_2 = \frac{G}{E_x} . \]

The integral in (11.9) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

Using the same procedure with the other integrals in equations (11.1), we obtain the following expression for the next integral.
\[ \int_{\Gamma} \beta_n^* M_n d\Gamma = \sum_{j=1}^{N} \int_{\Gamma(j)} \beta_n^* M_n^{(j)} d\Gamma = \sum_{j=1}^{N} \int_{\Gamma(j)} \beta_n^* H_\alpha(\eta) M_n^{(j)} d\eta, \]

\[ \int_{\Gamma} \beta_n^* M_n d\Gamma = -\frac{1}{8\pi D_x} \sum_{j=1}^{N} C_\alpha^{(j)} M_n^{(j)} = \]

\[ \frac{1}{8\pi D_x} \sum_{j=1}^{N} \left[ C_{2}^{(j-1)} + C_{1}^{(j)} \right] M_{nj}, \]

where

\[ C_\alpha^{(j)} = L_j \int_{\Gamma(j)} f_3(\vec{r}) H_\alpha(\eta) d\eta, \]

\[ f_3(r) = r_n^\alpha \ln r. \]

The integral in (11.11) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

Using the same procedure with the other integrals in equations (11.1), we obtain the following expression for the next integral
\[
\int_{\Gamma} V^* w d\Gamma = \sum_{j=1}^{N} \int_{\Gamma^{(j)}} V^* w^{(j)} ds = \sum_{j=1}^{N} \int_{\Gamma^{(j)}} V^* H_\alpha(\eta) w^{(j)}_\alpha ds ,
\]

(11.12)

\[
\int_{\Gamma} V^* w d\Gamma = -\frac{1}{8\pi} \sum_{j=1}^{N} D^{(j)}_\alpha w^{(j)}_\alpha = -\frac{1}{8\pi} \sum_{j=1}^{N} [D^{(j-1)}_2 + D^{(j)}_1] w_j ,
\]

where
\[
D_{\alpha}^{(j)} = L_j \int_{\Gamma^{(j)}} f_4^{*}(\vec{r}) H_{\alpha}(\eta) d\eta,
\]

\[
f_4^{*}(\vec{r}) = a_4 f_4^{*}(\vec{r}) + a_5 f_5^{*}(\vec{r}) + a_6 f_6^{*}(\vec{r}) + a_7 f_7^{*}(\vec{r}),
\]

\[
a_4 = n_x \left[1 + (1 - \alpha_1 v_x)n_y^2\right], \quad f_4^{*} = \frac{3 r_1}{r^2} - \frac{2 r_1^3}{r^4},
\]

\[
a_5^{(1)} = \left[\alpha_1 v_x + 2 \alpha_2 \left(1 - \alpha_1 v_x^2\right)\right]n_x,
\]

\[
a_5^{(2)} = \alpha_1 \left(1 - \alpha_1\right)n_x n_y^2 + 2 \alpha_2 \left(1 - \alpha_1 v_x^2\right)\left(n_x^2 - n_y^2\right)n_x,
\]

\[
a_6^{(1)} = \left[\alpha_1 v_x + 2 \alpha_2 \left(1 - \alpha_1 v_x^2\right)\right]n_y,
\]

\[
a_6^{(2)} = \alpha_1 \left(1 - \alpha_1\right)n_x^2 n_y - 2 \alpha_2 \left(1 - \alpha_1 v_x^2\right)\left(n_x^2 - n_y^2\right)n_y,
\]

\[
a_5 = a_5^{(1)} - a_5^{(2)} \quad \quad a_6 = a_6^{(1)} - a_6^{(2)}
\]

\[
f_5^{*} = \frac{r_1}{r^2} - \frac{2 r_1 r_2^2}{r^4}, \quad f_6^{*} = \frac{r_2}{r^2} - \frac{2 r_1^2 r_2}{r^4}.
\]

The integral in (11.13) will be integrated using Gaussian Quadrature, which is discussed in Appendix A.

Using the same procedure with the other integrals in equations (11.1), we obtain the following expression for the next integral.
\[ P_0 \int_{\Gamma} G_n^* d\Gamma = \sum_{j=1}^{N} \int_{\Gamma(j)} G_n^* ds = \frac{P_0}{256\pi D_x} \sum_{j=1}^{N} F_1^{(j)}, \quad (11.14) \]

where

\[ F_1^{(j)} = L_j \int_{\Gamma(j)} f_5(\bar{r}) d\eta, \quad (11.15) \]

\[ f_5(r) = (4\ln r - 3)r^2 r_\alpha n_\alpha. \]

The integral in (11.15) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

Substituting (11.6), (11.8), (11.10), (11.12), and (11.14) into the first equation in (11.1), one finds that

\[ \alpha \omega(\varepsilon_1, \varepsilon_2) = \frac{1}{32\pi D_x} \sum_{j=1}^{N} \left[ A_2^{(j-1)} + A_1^{(j)} \right] V_j - \]

\[ \frac{1}{8\pi} \sum_{j=1}^{N} \left[ B_2^{(j-1)} + B_1^{(j)} \right] \beta_{nj} - \frac{1}{8\pi D_x} \sum_{j=1}^{N} \left[ C_2^{(j-1)} + C_1^{(j)} \right] M_{nj} + \]

\[ \frac{1}{8\pi} \sum_{j=1}^{N} \left[ D_2^{(j-1)} + D_1^{(j)} \right] w_j + \frac{P_0}{256\pi D_x} \sum_{j=1}^{N} F_1^{(j)}. \quad (11.16) \]

Now, we define
\[ w(\varepsilon_1, \varepsilon_2) = w(\tilde{\varepsilon}_i), \quad A_{ij} = A_2^{(j-1)} + A_1^{(j)}, \]
\[ B_{ij} = B_2^{(j-1)} + B_1^{(j)}, \quad C_{ij} = C_2^{(j-1)} + C_1^{(j)}, \]
\[ D_{ij} = D_2^{(j-1)} + D_1^{(j)}, \quad f_{li} = \sum_{j=1}^{N} F_1^{(j)}. \]

Using the definitions of (11.17) into equation (11.16) for the source point \( i \), one finds in index notation that equation (11.16) has the form

\[ \alpha w(\tilde{\varepsilon}_i) = \frac{A_{ij} V_j}{32 \pi D_x} - \frac{B_{ij} \beta_{nj}}{8 \pi} - \frac{C_{ij} M_{nj}}{8 \pi D_x} + \frac{D_{ij} w_j}{8 \pi} + \frac{P_0 f_{li}}{256 \pi D_x}. \] (11.18)

Equation (11.18) is the discretized version of the first equation in (11.1). The next step is the discretization of the second equation in (11.1). Using the same procedure used to develop equation (11.18), the second equation in (11.1) in a discretized form is

\[ \alpha \beta_n(\tilde{\varepsilon}_i) = -\frac{\overline{A}_{ij} V_j}{8 \pi D_x} + \frac{\overline{B}_{ij} \beta_{nj}}{8 \pi} + \frac{\overline{C}_{ij} M_{nj}}{8 \pi D_x} + \frac{\overline{D}_{ij} w_j}{8 \pi} - \frac{P_0 f_{2i}}{256 \pi D_x}, \] (11.19)

where
\[ \overline{A}_{ij} = \overline{A}_{2}^{(j-1)} + \overline{A}_{1}^{(j)}, \]

\[ \overline{A}_{\alpha}^{(j)} = L_j \int_{\Gamma(j)} f_6(\tilde{r}) H_\alpha(\eta) d\eta, \]  \hspace{1cm} (11.20)

\[ f_6(r) = r_\alpha n_\alpha^0 \ln r. \]

The integral in (11.20) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

The second coefficient in (11.19) is calculated using

\[ \overline{B}_{ij} = \overline{B}_{2}^{(j-1)} + \overline{B}_{1}^{(j)}, \]

\[ \overline{B}_{\alpha}^{(j)} = L_j \int_{\Gamma(j)} f_7(\tilde{r}) H_\alpha(\eta) d\eta, \]  \hspace{1cm} (11.21)

\[ f_7(\tilde{r}) = b_4 f_4(\tilde{r}) + b_5 f_5(\tilde{r}) + b_6 f_6(\tilde{r}) + b_7 f_7(\tilde{r}), \]

\[ b_4 = a_1 n_x^0, \quad b_5 = a_2 n_x^0 + a_3 n_y^0, \]

\[ b_6 = a_2 n_x^0 + a_1 n_y^0, \quad b_7 = a_2 n_y^0. \]

The integral in (11.21) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

The third coefficient in (11.19) is calculated using
\[ \overline{C}_{ij} = \overline{C}_{2}^{(j-1)} + \overline{C}_{1}^{(j)}, \]

\[ \overline{C}_{\alpha}^{(j)} = L_{j} \int_{\Gamma(j)} f_{8}(\vec{r}) H_{\alpha}(\eta) \, d\eta, \]  
(11.22)

\[ f_{8}(\vec{r}) = n_{x}^{0} n_{x}^{0} f_{1}^{*}(\vec{r}) + n_{y}^{0} n_{y}^{0} f_{2}^{*}(\vec{r}) + (n_{x}^{0} n_{x}^{0} + n_{y}^{0} n_{y}^{0}) f_{3}^{*}(\vec{r}). \]

The integral in (11.22) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

The fourth coefficient in (11.19) is calculated using
\[ D_{ij} = D_2^{(j-1)} + D_1^{(j)} , \]

\[ D_2^{(j)} = L_j \int_{\Gamma(j)} f_9(\vec{r}) H_\alpha(\eta) d\eta , \]

\[ f_9(\vec{r}) = a_8 f_8^*(\vec{r}) + a_9 f_9^*(\vec{r}) + a_{10} f_{10}^*(\vec{r}) + a_{11} f_{11}^*(\vec{r}) + a_{12} f_{12}^*(\vec{r}) , \]

\[ a_8 = a_4 n_x^0 , \quad f_8^* = -\frac{3}{r^2} + \frac{12 r_1^2}{r^4} + \frac{8 r_1^4}{r^6} , \]

\[ a_9 = a_5 n_x^0 + a_6 n_y^0 , \quad f_9^* = \frac{1}{r^2} + \frac{8 r_1^2 r_2^2}{r^6} , \]

\[ a_{10} = a_6 n_x^0 + a_4 n_y^0 , \quad f_{10}^* = \frac{6 r_1 r_2}{r^4} + \frac{8 r_1^3 r_2}{r^6} , \]

\[ a_{11} = a_7 n_x^0 + a_5 n_y^0 , \quad f_{11}^* = \frac{6 r_1 r_2}{r^4} + \frac{8 r_1^3 r_2}{r^6} , \]

\[ a_{12} = a_7 n_y^0 , \quad f_{12}^* = -\frac{3}{r^2} + \frac{12 r_2^2}{r^4} + \frac{8 r_2^4}{r^6} . \]

The integral in (11.23) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

The last coefficient in (11.19) is calculated using
\[ f_{2i} = \sum_{j=1}^{N} F_2^{(j)}, \]

(11.24)

\[ F_2^{(j)} = L_j \int_{\Gamma(j)} f_{10}(\vec{r})d\eta, \]

\[ f_{10}(r) = 4 n_{\alpha} n_{\alpha}^0 r^2 \ln r + 8 r_{\alpha} n_{\alpha}^0 r_{\gamma} n_{\gamma} \ln r - 2 r_{\alpha} n_{\alpha}^0 r_{\gamma} n_{\gamma} - 3 n_{\alpha} n_{\alpha}^0 r^2. \]

The integral in (11.24) will be integrated using Gaussian quadrature, which is discussed in Appendix A.

![Figure 11.3. Linear elements, nodes, and source point.](image)

The discretized equations (11.18) and (11.19) include the coefficient \( \alpha \) which takes different values as described in section 10.3; if we choose the source point outside the plate, as in Figure 11.3, \( \alpha = 0 \). Using that value in equations (11.18) and (11.19), one infers that
\[
\frac{A_{ij} V_j}{32 \pi D_x} - \frac{B_{ij} \beta_{nj}}{8 \pi} - \frac{C_{ij} M_{nj}}{8 \pi D_x} + \frac{D_{ij} w_j}{8 \pi} = -\frac{P_0 f_{ii}}{256 \pi D_x},
\]

\[
\frac{\bar{A}_{ij} V_j}{8 \pi D_x} - \frac{\bar{B}_{ij} \beta_{nj}}{8 \pi} - \frac{\bar{C}_{ij} M_{nj}}{8 \pi D_x} - \frac{\bar{D}_{ij} w_j}{8 \pi} = -\frac{P_0 f_{2i}}{256 \pi D_x}.
\]  

(11.25)

We divide equations (11.25) by \( w_0 \) and we define

\[
\bar{w}_j = \frac{w_j}{w_0}, \quad w_0 = \frac{P_0 R^4}{D_x},
\]

\[
\bar{\beta}_{nj} = \frac{\beta_{nj}}{\beta_0}, \quad \beta_0 = \frac{P_0 R^3}{D_x},
\]

\[
\bar{M}_{nj} = \frac{M_{nj}}{M}, \quad M = P_0 R^2,
\]

\[
\bar{V}_j = \frac{V_j}{V_0}, \quad V_0 = P_0 R.
\]

(11.26)

Using the definitions in (11.26), equations (11.25) become in dimensionless form

\[
\frac{R}{4} A_{ij} \bar{V}_j - R^3 B_{ij} \bar{\beta}_{nj} - R^2 C_{ij} \bar{M}_{nj} + R^4 D_{ij} \bar{w}_j = -\frac{f_{ii}}{32},
\]

\[
R \bar{A}_{ij} V_j - R^3 \bar{B}_{ij} \bar{\beta}_{nj} - R^2 \bar{C}_{ij} \bar{M}_{nj} - R^4 \bar{D}_{ij} \bar{w}_j = -\frac{f_{2i}}{32}.
\]

(11.27)
Equations (11.27) can be expressed in matrix notation as

\[
\begin{pmatrix}
\frac{R}{4} A & -R^3 B \\
R A & -R^3 B
\end{pmatrix}
\begin{bmatrix}
\bar{V} \\
\bar{\beta}_n
\end{bmatrix}
+ \begin{pmatrix}
-R^2 C & R^4 D \\
-R^2 C & -R^4 D
\end{pmatrix}
\begin{bmatrix}
\bar{M}_n \\
\bar{w}
\end{bmatrix}
= - \begin{bmatrix}
f_1 \\
\frac{f_2}{32}
\end{bmatrix}.
\]

(11.28)

The system in (11.28) will be used in the next section in order to develop a boundary element formulation using linear elements for simply supported plates.

### 11.3 Linear boundary element formulation for simply supported plates

The system in (11.28) will be used in order to solve the problem of bending of simply supported orthotropic plates. The boundary conditions for simply supported plates, which are analyzed in section 4.6, are

\[
w = 0,
\]

\[
M_n = 0.
\]

(11.29)

Using the boundary conditions in (11.29), the system (11.28) is simplified to
\[
\begin{pmatrix}
\frac{R}{4} A & -R^3 B \\
R \bar{A} & -R^3 \bar{B}
\end{pmatrix}
\begin{pmatrix}
\vec{V} \\
\vec{\beta}_n
\end{pmatrix}
= -\begin{pmatrix}
\frac{f_1}{32} \\
\frac{f_2}{32}
\end{pmatrix}.
\] (11.30)

The unknown quantities in the system (11.30) are the vertical shear force \( V \) and \( \beta_n \) at each node. Once we calculate those quantities, all quantities on the boundary are known and we can calculate the vertical displacement at any point in the domain using equation (11.18). After using the boundary conditions (11.29) and \( \alpha = 1 \), one finds that equation (11.18) yields

\[
w(\bar{e}_i) = \frac{A_{ij} V_j}{32 \pi D_x} - \frac{B_{ij} \beta_{nj}}{8 \pi} + \frac{P_0 f_{li}}{256 \pi D_x}. \] (11.31)

Now, the boundary element method for orthotropic plates will be used to calculate the vertical displacement of the central point of a simply supported circular plate and a convergence criterion will be used in order to know the appropriate number of elements for this problem.

Consider a simply supported circular plate of radius \( a \) under a uniformly distributed load \( P_0 \), as shown in Figure 4.5. Figure 11.4 shows the linear elements and the nodes on the boundary of the plate, and in order to calculate the effective shear force \( V \) and the normal slope \( \beta_n \) at each node of the boundary we use the system (11.30). After the calculation of those quantities, we
calculate the vertical displacement at any point in the domain using equation (11.31).

![Diagram of a plate boundary discretized into linear elements]

Figure 11.4. Discretization of the plate boundary in linear elements.

We divide equation (11.31) by \( w_0 \), and using the definitions in (11.26), with \( a \) instead of \( R \), we find that

\[
\bar{w}(\vec{e}_i) = \frac{A_{ij} \bar{V}_j}{32\pi a^3} - \frac{B_{ij} \bar{p}_{nj}}{8\pi a} + \frac{f_{ii}}{256\pi a^4}. \tag{11.32}
\]

The radius of the plate considered is \( a = 50 \) cm, Poisson's ratio in the \( x \) direction is \( \nu_x = 0.3 \), the elastic modulus in the \( x \) direction is \( E_x = 206.8 \) GigaPascals, and \( G = 0.6055 \) GigaPascals. The boundary element solution for the vertical displacement of the central point of the plate is calculated using
different number of elements. Table 11.1 shows those values and the percent
difference with the preceding calculation.

Table 11.1. Numerical solutions for the vertical displacement of the central point
of the plate for different numbers of elements.

<table>
<thead>
<tr>
<th>Number of Nodes</th>
<th>( \bar{w} ) Using BEM</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>0.1128</td>
<td>-</td>
</tr>
<tr>
<td>60</td>
<td>0.1132</td>
<td>0.36 %</td>
</tr>
<tr>
<td>80</td>
<td>0.1133</td>
<td>0.09 %</td>
</tr>
<tr>
<td>100</td>
<td>0.1136</td>
<td>0.27 %</td>
</tr>
</tbody>
</table>

Table 11.1 shows in column 2 values for the dimensionless vertical-displacement
for the central point of the plate using the Boundary Elements Method. The
procedure, in which the number of elements is increased in order to calculate the
unknown quantities is called convergence analysis. The difference of 0.09 %,
which corresponds to the case when the number of elements used is 80, defines
this number of elements as the optimal mesh.

11.4 Conclusion

The discretization of the boundary for an orthotropic plate, in linear elements, is
presented here. The quantities involved in equations (11.1) are assumed to have
a linear variation in each element. The interpolation functions shown in (11.3)
have been used in order to express the linear variation of those quantities, see
equations (11.2); a local coordinate system, which is shown in Figure 11.2, is
used in the expressions (11.2). The discretization of the boundary in linear
elements and the application of the boundary conditions (11.29) results in the system of equations shown in (11.30), which reads

\[
\begin{pmatrix}
\frac{R}{4} A & -R^3 B \\
R A & -R^3 B
\end{pmatrix}
\begin{pmatrix}
\bar{V} \\
\bar{\beta}_n
\end{pmatrix}
= \begin{pmatrix}
f_1 \\
\frac{f_2}{32}
\end{pmatrix}.
\] (11.33)

Using the system (11.33) the unknown quantities on the boundary, namely the effective shear force and the normal slope, are calculated. Next, the vertical displacement at any point inside the domain is calculated using equation (11.32), which reads

\[
\bar{w}(\bar{e}_i) = \frac{A_{ij} \bar{V}_j}{32 \pi \alpha^3} - \frac{B_{ij} \bar{\beta}_{nj}}{8 \pi \alpha} + \frac{f_{ii}}{256 \pi \alpha^4}.
\] (11.34)

An example has been presented, where the vertical displacement of the central point of the plate is calculated using different numbers of elements. Table 11.1 shows the results of that example using different numbers of elements.

The boundary element formulation in order to calculate the vertical displacement at any point inside the domain will be used in the next chapter in order to develop a new ballistic limit equation for hypervelocity impact on orthotropic materials.
Chapter 12

A New Ballistic Limit Equation for Orthotropic Materials

12.1 Introduction

A review of the physical concepts about hypervelocity impact has been presented in chapter 3. Assumptions about the characteristics of the load and momentum transferred from the projectile to the back plate have been considered. With those assumptions in mind, a mathematical model is proposed in order to develop a ballistic limit equation for orthotropic materials.

The mathematical model consists of a simply supported circular plate under constant load acting on a circular region in the center of the plate. The circular plate is divided in two regions; the first region corresponds to the central, loaded region and undergoes plastic deformation, so it is called the plastic region. The second region undergoes elastic deformation, and is therefore called the elastic region.

In the elastic region, the boundary elements method is used in order to calculate moments and vertical displacement. The internal boundary of the elastic region is the external boundary of the plastic region. In the plastic region, a dynamic analysis is performed in order to calculate the vertical displacement of the central point of the plate using the boundary information calculated during the elastic analysis. The new ballistic limit equation is the equation resulting from this procedure.
12.2 Mathematical model for elastic-plastic deformations of orthotropic plates

Three basic assumptions are mentioned in chapter 3, and are summarized in section 3.5; the third of these assumptions was expressed as equation (3.1) for the initial velocity of the loaded area

\[
V_I = \frac{32 M_p V_p}{\pi S^2 \rho_b h}, \tag{12.1}
\]

where \( M_p \) and \( V_p \) are the mass and the velocity of the impacting projectile, \( S \) is the spacing between the plates and is shown in figure 3.1, \( \rho_b \) and \( h \) are the mass density and the thickness of the back plate. Figure 12.1 shows the back plate and related dimensions.

![Figure 12.1. Back plate under a constant load.](image_url)
The back plate is a simply supported circular plate with radius $R$ and is loaded with a constant force of magnitude $P_0$; this force is acting over a circular region having a radius $C$. The loaded area, $0 \leq r \leq C$, is the plastic region and the area outside the load, $C \leq r \leq R$, is the elastic region.

The vertical displacement and the moments in the elastic region are calculated using the boundary element method, and the boundary conditions for this elastic region, are expressed in equations (12.2) as

$$\begin{align*}
w(R) &= 0, \\
M_n(R) &= 0,
\end{align*}$$

(12.2)

where $w(R)$ is the vertical displacement, and $M_n(R)$ is the normal moment at $r = R$.

The vertical displacement and the moments in the plastic region are calculated using the solution proposed by Ugural [6] for a clamped orthotropic circular plate under a constant load.

12.3 Boundary element formulation for the elastic region

According to the boundary elements method, the integral equation (10.23) for the boundary element formulation for plates is as follows
\[ \alpha w(\varepsilon_1, \varepsilon_2) + \int_\Gamma \left( \beta_n^* M_n + V^* w \right) d\Gamma = \]

\[ P_0 \int_{\Gamma'} G^* n_\alpha d\Gamma + \int_\Gamma \left( M_n^* \beta_n + w^* V \right) d\Gamma, \]

(12.3)

\[ \alpha \beta_n(\varepsilon_1, \varepsilon_2) + \int_\Gamma \left( \beta_n^* M_n + \bar{V}^* w \right) d\Gamma = \]

\[ P_0 \int_{\Gamma'} \bar{G}_n^* d\Gamma + \int_\Gamma \left( \bar{M}_n^* \beta_n + \bar{w}_n^* V \right) d\Gamma, \]

where \( w \) is the vertical displacement, \( \beta_n \) is the normal slope, \( M_n \) is the normal moment, \( V \) is the effective shear force, \( p \) is the force acting over the plate, and \( \Gamma' \) is the boundary of the loaded area, as shown in Figure 12.2. The quantity \( w^* \) is called fundamental solution and is given by (10.13)

\[ w^* = \frac{1}{16\pi D_x} \left( 2\bar{r}^2 \ln \bar{r} - \bar{r}^2 \right). \]  

(12.4)

Analysis of equations (12.3), together with the boundary conditions (12.2), shows that some terms become zero. Consequently, there remains
\[ \alpha w(\varepsilon_1, \varepsilon_2) = P_0 \int_{\Gamma^*} G_{\alpha n} \, d\Gamma + \int_{\Gamma^*} \left( M_n^* \beta_n + w^* V \right) d\Gamma, \]

(12.5)

\[ \alpha \beta_n(\varepsilon_1, \varepsilon_2) = P_0 \int_{\Gamma'} \overline{G_n}^* \, d\Gamma + \int_{\Gamma'} \left( \overline{M_n^*} \beta_n + \overline{w_n^*} V \right) d\Gamma. \]

In figure 12.2, the domain \( \Omega \) corresponds to the elastic region and is bounded by the external boundary \( \Gamma \) and the internal boundary \( \Gamma' \).

![Diagram of a domain with boundaries](image)

**Figure 12.2.** External boundary \( \Gamma \), internal boundary \( \Gamma' \), and domain \( \Omega \).

The integral equations in (12.5) involve two variables on the boundary: the normal slope \( \beta_n \) and the effective shear force \( V \). In order to calculate those
variables, the boundaries $\Gamma$ and $\Gamma'$ have to be discretized into linear elements as shown in Figure 12.2. Then, using equations (11.25), one has

$$\frac{A_{ij} V_j}{32 \pi D_x} - \frac{B_{ij} \beta_{nj}}{8 \pi} = -\frac{P_0 f_{li}}{256 \pi D_x},$$

$$\frac{\overline{A}_{ij} V_j}{8 \pi D_x} - \frac{\overline{B}_{ij} \beta_{nj}}{8 \pi} = -\frac{P_0 f_{2i}}{256 \pi D_x}. \tag{12.6}$$

In order to work in dimensionless form, the following quantities are defined

$$\beta_0 = \frac{P_0 C^2 R}{D_x}, \tag{12.7}$$

$$V_0 = \frac{P_0 C^2}{R}.$$

Dividing both sides of equation (12.6) by $\beta_0$, we obtain

$$\frac{R}{4} A_{ij} \overline{V}_j - R^3 B_{ij} \overline{\beta}_{nj} = -\frac{f_{li}}{32 \rho_0^2}, \tag{12.8}$$

$$R \overline{A}_{ij} \overline{V}_j - R^3 \overline{B}_{ij} \overline{\beta}_{nj} = -\frac{f_{2i}}{32 \rho_0^2},$$

where
\[
\bar{\beta}_{nj} = \frac{\beta_{nj}}{\beta_0},
\]

\[
\bar{v}_j = \frac{v_j}{V_0},
\]

(12.9)

\[
\rho_0 = \frac{C}{R}.
\]

So far we know that on the external boundary the normal moment and vertical displacement are zero. In order to calculate the dimensionless normal slope \( \bar{\beta}_{nj} \) and the effective shear force \( \bar{v}_j \) for every node of the boundary we use equations (12.8).

In order to calculate the vertical displacement for any point in the domain \( \Omega \), we use (11.31), which is as follows

\[
w(\bar{\varepsilon}_i) = \frac{A_{ij} v_j}{32 \pi D_x} - \frac{B_{ij} \beta_{nj}}{8 \pi} + \frac{P_0 \bar{f}_{1i}}{256 \pi D_x}.
\]

(12.10)

In order to work in dimensionless form, we divide both sides of equation (12.10) by \( w_0 \). This yields
$$\bar{w}(\vec{e}_i) = \frac{A_{ij} \bar{v}_j}{32\pi R^3} - \frac{B_{ij} \bar{\beta}_{nj}}{8\pi R} + \frac{f_{1i}}{256\pi \rho_0^2 R^4}, \quad (12.11)$$

where

$$\bar{w} = \frac{w}{w_0}, \quad (12.12)$$

$$w_0 = \frac{P_0 \rho_0^2 R^4}{D_x}.$$  

We calculate the dimensionless vertical displacement using equation (12.11), for any point in the domain $\Omega$, including values at the internal boundary $\bar{w}(C)$; this value is used as a boundary condition for the plastic region.

In order to calculate the dimensionless moment, we use the derivatives of (12.11), as follows

$$\bar{M}_{xx} = -\bar{w}_{xx} - \alpha_1 v_x \bar{w}_{yy},$$

$$\bar{M}_{yy} = -\alpha_1 v_x \bar{w}_{xx} - \alpha_1 \bar{w}_{yy}, \quad (12.13)$$

$$\bar{M}_{xy} = -2\alpha_2 \left(1 - \alpha_1 v_x^2\right) \bar{w}_{xy}.$$  

Using equations (12.13), we calculate the dimensionless moments for any point in the domain $\Omega$, including the values $\bar{M}_{xx}(C)$, $\bar{M}_{yy}(C)$, and $\bar{M}_{xy}(C)$ on the
internal boundary. These values are to be used in order to obtain a yield criterion for the internal boundary $\Gamma''$. The yield criterion on the boundary $\Gamma''$ according to the Tsai-Hill yield criterion (Hill [32] and Gibson [29]) is expressed as a function of the moments in the form

$$M_{xx}^2 - M_{xx}M_{yy} + \frac{M_{yy}^2}{\gamma_1^2} + \frac{M_{xy}^2}{\gamma_2^2} = M_0^2,$$  \hspace{1cm} (12.14)

where

$$M_0 = \frac{S_L h^2}{4},$$

$$\gamma_1 = \frac{S_T}{S_L},$$ \hspace{1cm} (12.15)

$$\gamma_2 = \frac{S_{LT}}{S_L}.$$ 

In order to work in dimensionless form, we divide equation (12.14) by $M$, which yields

$$\overline{M}_{xx}^2 - \overline{M}_{xx}\overline{M}_{yy} + \frac{\overline{M}_{yy}^2}{\gamma_1^2} + \frac{\overline{M}_{xy}^2}{\gamma_2^2} = \frac{M_0^2}{M^2},$$ \hspace{1cm} (12.16)
where

\[ M = P_0 C^2. \]  \hspace{1cm} (12.17)

In order to use a dimensionless expression for the Tsai-Hill yield criterion (12.16), a plastic function is defined as

\[ f_{plas} = \frac{M_0}{M}, \]  \hspace{1cm} (12.18)

where

\[ f_{plas} = \sqrt{M_{xx}^2 - M_{xx}M_{yy} + \frac{M_{yy}^2}{\gamma_1^2} + \frac{M_{xy}^2}{\gamma_2^2}}. \]  \hspace{1cm} (12.19)

The new ballistic limit equation for orthotropic materials is developed from the plastic function \( f_{plas} \) of equation (12.18).

### 12.4 Analysis of the plastic region of the plate

In an elastic-plastic plate, the curvature rates in any plastic element are the sum of an elastic part and a plastic part given by a plastic flow rule. Figure 12.3 illustrates the extension of the Tresca hexagon for composite-orthotropic materials (Tanvir [37] and Sherbourne and Murthy [38]). According to this
extension, we can consider the following. If the plate element has entered a certain plastic regime directly from an elastic state, and has never left the plastic regime during the subsequent deformation, the flow rule can be written in the integrated form, and the elastic-plastic analysis is considerably simplified. For example, when the stress point is on the side B'C', see Figure 12.3, the plastic parts of the radial and circumferential curvature vanish, and the equation for the elastic case continues to hold in the plastic range (Chakrabarty [26]).

![Diagram of Modified Tresca hexagon for moments in an elastic-plastic circular plate.](image)

**Figure 12.3.** Modified Tresca hexagon for moments in an elastic-plastic circular plate.
For \( p = P_0 \) constant, we use equation (9.11), which corresponds to a clamped circular plate of radius \( a \). Here, because the plastic zone has radius \( C \), we write \( a = C \) in equation (9.11). However, we also need to add a term \( w(C) \), since the displacement is not zero at the elastic-plastic boundary. Thus, we write

\[
w = \frac{P_0 (C^2 - r^2)^2}{8(3D_x + 2H + 3D_y)} + w(C),
\]

(12.20)

Substituting \( r = 0 \), one finds that the vertical displacement at the center of the plate \( W_I = w(0) \) is given by

\[
W_I = \frac{P_0 C^4 Y(C)}{D_x},
\]

(12.21)

where the function \( Y(r) \) evaluated at the radius \( r = C \) yields

\[
Y(C) = \frac{1}{8\left[3 + 2\alpha_1 \nu_x + 4\alpha_2 \left(1 - \alpha_1 \nu_x^2\right) + 3\alpha_1\right]} + \rho_0^2 \overline{w}(C).
\]

(12.22)

In equation (12.22), the dimensionless displacement \( \overline{w}(C) \) comes from the boundary element equation (12.11).
12.5 A new ballistic limit equation for orthotropic materials

In order to develop the ballistic limit equation for orthotropic materials, we use a dynamic analysis for the plastic region; the first assumption used concerns the acceleration and can be expressed as

\[ a = -\frac{P_0 \pi C^2}{m_b} = -\frac{P_0 \pi C^2}{\rho_b \pi C^2 h} = -\frac{P_0}{\rho_b h}, \quad (12.23) \]

\[ a = v \frac{dv}{dw}, \]

where \( \rho_b \) is the mass density of the plate, \( h \) the thickness of the plate, and \( v \) is the velocity at the center of the plate. From (12.23), one infers that

\[ W_I = -\int_{V_I}^{0} \rho_b h v dv \frac{dv}{P_0}. \quad (12.24) \]

Integration of (12.24) results in the following expression

\[ W_I = \frac{V_I^2 \rho_b h}{2 P_0}. \quad (12.25) \]
Equation (12.21) involves the vertical displacement of the central point of the plate; and equation (12.1) involves the initial velocity of the loaded area. Those quantities will be substituted in equation (12.25). Considering $S = 4C$ from the first assumption in section 3.5, we have

$$\frac{P_0 C^4 Y(C)}{D_x} = \frac{\rho_b h}{2 P_0} \frac{4 M_p^2 V_p^2}{\pi^2 C^4 \rho_p^2 h^2}, \quad (12.26)$$

where the mass of the projectile $M_p$ is expressed in terms of the mass density $\rho_p$ and the radius $r_p$. With this, equation (12.26) yields

$$9 P_0^2 C^8 \rho_b h Y(C) = 32 D_x \rho_p^2 r_p^6 V_p^2. \quad (12.27)$$

In the last equation, the projectile is assumed to be spherical. Other shapes of the projectile have been considered, as in Christiansen and Kerr [27].

In order to work with the critical load for the limit case, we will use an equation equivalent to (12.18) such as

$$M_D = f_{plas} P_0 C^2 = \frac{S_{LD} h^2}{4}, \quad (12.28)$$
where $S_{LD}$ is the longitudinal dynamic yield strength and the relationship with the longitudinal static yield strength is given as proposed by Angel and Smith [2]

$$S_{LD} = \lambda S_L ,$$  \hspace{1cm} (12.29)

$$\lambda = 4.5.$$  

Using equations (12.28) and (12.29), the critical load is

$$P_0 = \frac{\lambda S_L h^2}{4 f_{\text{plas}} C^2} .$$  \hspace{1cm} (12.30)

Substituting the critical load (12.30) and the definition for the flexural rigidity $D_x$ given by (9.8) into (12.27) and, after some simplifications, we obtain a new ballistic limit equation for composite-orthotropic materials. This equation reads

$$E_x^{1/2} \rho_p r_p^3 V_p = \left[ \frac{27 Y(C)(1-\alpha_1 v_x^2)}{128 f_{\text{plas}}^2} \right]^{1/2} \rho_b^{1/2} C^2 h \lambda S_L .$$  \hspace{1cm} (12.31)

This equation involves the material properties of the projectile, the velocity and radius of the projectile, the density of the plate material, thickness of the plate, the dynamic yield strength, the load radius and the elastic modulus of the plate.
Also, two functions are involved $f_{plas}$ and $Y(C)$. Those functions are calculated using the boundary element method.

The need of a model to predict the damage resulting from hypervelocity impact on composite structures is expressed in different references such as Tennyson [39]. The ballistic limit equation developed in this work provides an option to fulfill this need. Experimental information [40-42] will support the validity of our equation, as well as other mathematics models [43-46].

12.6 Conclusion

A new ballistic limit equation for orthotropic materials has been developed using a new mathematical model; the model consists of a simply supported circular plate, of radius $R$, bending under a constant load exerted on a circular sector of radius $C$ in the center of the plate. The plate has been divided into two concentric regions, the external elastic region and the internal plastic region, because of the deformation type they undergo. The variables and functions involved in the elastic region, $C \leq r \leq R$, were calculated using the boundary element method with appropriate boundary conditions, as shown in section 12.3.

The variables and functions involved in the plastic region, $0 \leq r \leq C$, were calculated using an analytical procedure, as shown in section 12.4. The new ballistic limit equation is
\[ E_x^{1/2} \rho_p r_p^3 V_p = \left[ \frac{27 Y(C)(1-\alpha_1 \nu_x^2)}{128 f_{\text{plas}}^2} \right]^{1/2} \rho_b^{1/2} C^2 h \lambda S_L, \quad (12.32) \]

where \( E_x \) is the elastic modulus in the \( x \) direction, \( \rho_p \) is the mass density of the projectile, \( r_p \) is the radius of the projectile, \( V_p \) is the velocity of the projectile, \( \nu_x \) is Poisson's ratio in the \( x \) direction, \( \rho_b \) is the mass density of the plate, \( C \) is the radius of the loaded region, \( h \) is the thickness of the plate, \( \lambda \) is the ratio between the longitudinal dynamic yield strength of the material and the longitudinal static yield strength of the material and takes the value 4.5, \( S_L \) is the longitudinal static yield strength and \( \alpha_1 \) is the ratio between the elastic modulus in the \( x \) direction and the elastic modulus in the \( y \) direction. Also, two functions are involved in the new ballistic limit equation (12.32). The first one is called the plastic function and is given by (12.19)

\[ f_{\text{plas}} = \sqrt{\frac{\hat{M}_{xx}^2 - \hat{M}_{xx} \hat{M}_{yy}}{\gamma_1^2} + \frac{\hat{M}_{xy}^2}{\gamma_2^2}}. \quad (12.33) \]

The moments in (12.33) are calculated using the boundary element equations given by (12.13)
\[ \overline{M}_{xx} = -\overline{w}_{,xx} - \alpha_1 \nu_x \overline{w}_{,yy}, \]  
\[ \overline{M}_{yy} = -\alpha_1 \nu_x \overline{w}_{,xx} - \alpha_1 \overline{w}_{,yy}, \]  
\[ \overline{M}_{xy} = -2 \alpha_2 \left(1 - \alpha_1 \nu_x^2\right) \overline{w}_{,xy}, \]  

where the second derivatives for \( w \) involved in (12.34) are calculated from the boundary element equation (12.11)

\[ \overline{w}(\overline{\varepsilon}) = \frac{A_{ij} \overline{V}_j}{32 \pi R^3} - \frac{B_{ij} \overline{\beta}_{nj}}{8 \pi R} + \frac{f_{ii}}{256 \pi \rho^2 R^4}. \]  

The second function involved in (12.32) is \( Y(C) \) and is given by (12.22)

\[ Y(C) = \frac{1}{8 \left[3 + 2 \alpha_1 \nu_x + 4 \alpha_2 \left(1 - \alpha_1 \nu_x^2\right) + 3 \alpha_1\right]} + \rho_0^2 \overline{w}(C). \]  

The new ballistic limit equation (12.32) will be compared with the new ballistic limit equations developed in chapter 7; the comparison will be made using ballistic limit curves in the next chapter.
Chapter 13

Ballistic Limit Curves for Orthotropic Materials

13.1 Introduction

In chapter 7, we have shown the process of developing a new ballistic limit equation for homogeneous materials. In chapter 8, that equation was plotted and compared with two other ballistic limit equations. There is a strong agreement with the empirical ballistic limit equation, which is an indication that the theoretical procedure used to develop the ballistic limit equation is consistent with the experimental procedure used to develop the empirical ballistic limit equation. With this background, we have relied on the same procedure to develop a new ballistic limit equation for orthotropic materials, as discussed in chapter 12. In order to be able to use the new equation for orthotropic materials, ballistic limit curves need to be plotted and compared. In this chapter, these plots are shown and analyzed. Three cases are presented here in order to show the use of that equation.

13.2 Ballistic limit equations

In chapter 12 a new ballistic limit equation for orthotropic materials has been developed. This is equation (12.32), which reads
\[
E_x^{1/2} \rho_p r_p^3 V_p = \left[ \frac{27 Y(C)(1-\alpha_1 \nu_x^2)}{128 f_{plas}^2} \right]^{1/2} \rho_b^{1/2} C^2 h \lambda S_L ,
\]

where \( E_x \) is the elastic modulus in the \( x \) direction, \( \rho_p \) is the mass density of the projectile, \( r_p \) is the radius of the projectile, \( V_p \) is the velocity of the projectile, \( \nu_x \) is Poisson’s ratio in the \( x \) direction, \( \rho_b \) is the mass density of the plate, \( C \) is the radius of the loaded region, \( h \) is the thickness of the plate, \( \lambda \) is the ratio between the longitudinal dynamic yield strength of the material and the longitudinal static yield strength of the material and takes the value 4.5, \( S_L \) is the longitudinal static yield strength and \( \alpha_1 \) is the rate between the elastic modulus in the \( x \) direction and the elastic modulus in the \( y \) direction. Also, two functions are involved in the new ballistic limit equation (13.1); the first one \( f_{plas} \) is the plastic function and is given by (12.33), the second function involved in (13.1) is \( Y(C) \) and is given by (12.36). These functions are calculated using the boundary elements method.

In chapter 7 a new ballistic limit equation for homogeneous materials was developed as equation (7.49)

\[
E^{1/2} \rho_p r_p^3 V_p = \left[ \frac{27 Y(C)(1-\nu^2)}{128 f_{plas}^2} \right]^{1/2} \rho_b^{1/2} C^2 h \lambda \sigma_0 ,
\]
where $E$ is the elastic modulus, $\rho_p$ is the mass density of the projectile, $r_p$ is the radius of the projectile, $V_p$ is the velocity of the projectile, $\nu$ is Poisson’s ratio, $\rho_b$ is the mass density of the plate, $C$ is the radius of the loaded region, $h$ is the plate thickness, $\lambda$ is the ratio between the dynamic and the static yield strength of the plate, and $\sigma_0$ is the static yield strength. At the same time two functions are involved: $f_{plas}$ called the plastic function given by (7.50), and $Y(C)$ given by (7.53), and calculated using the boundary element as shown in section 7.3.

In order to compare the new ballistic limit equations (13.1) and (13.2), we will plot curves using data obtained from Angel and Smith [2], Bader [47], Chou [48] and Barbero [49]. The radius of the projectile is plotted against the velocity of the projectile while all other variables in the equations are kept fixed. Solving for the radius of the projectile $r_p$ from equations (13.1) and (13.2), we find that

\[
    r_H = \left[ K_1^{1/3} \lambda^{1/3} \sigma_0^{1/3} C^{2/3} \rho_p^{-1/3} \rho_b^{1/6} h^{1/3} E^{-1/6} \right] V_p^{-1/3},
\]

\[
    r_C = \left[ K_2^{1/3} \lambda^{1/3} S_L^{1/3} C^{2/3} \rho_p^{-1/3} \rho_b^{1/6} h^{1/3} E_x^{-1/6} \right] V_p^{-1/3},
\]

where $r_C$ is the radius of the projectile for the new ballistic limit equations (13.1), and $r_H$ is the radius of the projectile for the new ballistic limit equation (13.2),
and the terms $K_1$ and $K_2$ expressed in the set of equations (13.3) can be calculated as follows

$$K_1 = \left[ \frac{27(1-v^2)}{128 f_{plas}^2} Y(C) \right]^{1/2},$$

$$K_2 = \left[ \frac{27(1-\alpha_1 v_x^2)}{128 f_{plas}^2} Y(C) \right]^{1/2}.$$  \hspace{1cm} (13.4)

Three cases will be presented in the next sections; in the first one we will compare the behavior under hypervelocity impact of one plate made of an orthotropic material and one plate made of a homogeneous material. Both plates will have the same mass and radius, and obviously different thicknesses. In the second case, we calculate the thickness of an orthotropic plate, using the ballistic limit curves, in order to have the same behavior as that of a homogeneous plate under hypervelocity impact. Then, we calculate the mass ratio between the plates. In the third case, we plot the behavior of an orthotropic plate using the ballistic limit equations (13.1) and (13.2).

### 13.3 Curves for the ballistic limit equations: case 1

In order to compare the behavior of an orthotropic material and that of a homogeneous material under hypervelocity impact, we plot the radius of the projectile versus the velocity of the projectile while the other variables in
equations (13.3) are kept fixed. The homogeneous plate is made of Aluminum Al6061-T6 and the mechanical properties are shown in Table 13.1.

Table 13.1. Mechanical properties for a homogeneous plate for case 1.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>69 GPa</td>
</tr>
<tr>
<td>$\nu_x$</td>
<td>0.33</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>275 MPa</td>
</tr>
<tr>
<td>$\rho_b$</td>
<td>2713 Kg/m$^3$</td>
</tr>
</tbody>
</table>

The orthotropic plate is made of Kevlar 49 and the mechanical properties are shown in Table 13.2.

Table 13.2. Mechanical properties for an orthotropic plate for case 1.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_x$</td>
<td>75.8 GPa</td>
</tr>
<tr>
<td>$E_y$</td>
<td>5.5 GPa</td>
</tr>
<tr>
<td>$G$</td>
<td>2.3 GPa</td>
</tr>
<tr>
<td>$\nu_x$</td>
<td>0.34</td>
</tr>
<tr>
<td>$S_L$</td>
<td>1379 MPa</td>
</tr>
<tr>
<td>$S_T$</td>
<td>64.8 MPa</td>
</tr>
<tr>
<td>$S_{LT}$</td>
<td>60 MPa</td>
</tr>
<tr>
<td>$\rho_b$</td>
<td>1450 Kg/m$^3$</td>
</tr>
</tbody>
</table>

The geometrical data of the plates and the mechanical properties of the projectile are shown in Table 13.3.

Table 13.3. Geometrical data of the plates and mechanical properties of the projectile for case 1.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>7.62 cm</td>
</tr>
<tr>
<td>$C$</td>
<td>2.39 cm</td>
</tr>
<tr>
<td>$\rho_p$</td>
<td>2796 Kg/m$^3$</td>
</tr>
</tbody>
</table>

In order to have a better criterion for comparison, the radii and masses of the plates should be the same, which implies that the thicknesses of the plates are not the same. For the orthotropic plate, the thickness is $h_C = 1.188$ mm, and for
the homogeneous plate the thickness is $h_H = 0.635$ mm. Substituting the data from tables 13.1, 13.2, and 13.3 into equations (13.3), one finds that
\[ r_H = 0.0335 V_p^{-1/3}, \]
\[ r_C = 0.0380 V_p^{-1/3}, \]

where the velocity of the projectile $V_p$ is measured in Km/s and the radii $r_H$ and $r_C$ are measured in mm.

![Critical Projectile Radii vs Projectile Velocity](image)

Figure 13.1. Critical projectile radii versus projectile velocity, case 1.
The behavior of Kevlar 49 and Aluminum Al6061-T6 under hypervelocity impact are shown in Figure 13.1, the region bellow each curve is the safe region, the Kevlar 49 material is 13.43 % stronger than Aluminum Al6061-T6.

13.4 Curves for the ballistic limit equations: case 2

In this case, we calculate the thickness of an orthotropic plate in order to have the same behavior as that of a homogeneous plate. The data for the homogeneous plate, made of Aluminum Al6061-T6, is shown in table 13.4.

Table 13.4. Mechanical properties for a homogeneous plate for case 2.

| $E = 69$ GPa | $\nu_x = 0.33$ | $\sigma_0 = 275$ MPa | $\rho_b = 2713$ Kg/m$^3$ |

The orthotropic plate is made of Kevlar 49 and the mechanical properties are shown in table 13.5.

Table 13.5. Mechanical properties for an orthotropic plate for case 2.

| $E_x = 75.8$ GPa | $E_y = 5.5$ GPa | $G = 2.3$ GPa | $\nu_x = 0.34$ |
| $S_L = 1379$ MPa | $S_T = 64.8$ MPa | $S_{LT} = 60$ MPa | $\rho_b = 1450$ Kg/m$^3$ |

The geometrical data of the plates and the mechanical properties of the projectile are shown in Table 13.6.
Table 13.6. Geometrical data of the plates and mechanical properties of the projectile for case 2.

| $R = 7.62$ cm | $C = 2.39$ cm | $\rho_p = 2796$ Kg/m$^3$ |

The thickness of the homogeneous plate is $h_H = 0.635$ mm. Substituting the data from tables 13.4, 13.5, and 13.6 into the first equation in (13.3), one finds that

$$r_H = 0.0335 \, V_p^{-1/3} ,$$  \hspace{1cm} (13.6)

where the velocity of the projectile $V_p$ is measured in Km/s and the radius $r_H$ is measured in mm. Substituting the data from tables 13.5 and 13.6 into the second equation in (13.3) and using a thickness, $h_C = 0.820$ mm, for the orthotropic plate, one finds that

$$r_C = 0.0335 \, V_p^{-1/3} .$$  \hspace{1cm} (13.7)
Figure 13.2. Critical projectile radii versus projectile velocity, case 2.

Figure 13.2 shows the same behavior for the homogeneous plate and the orthotropic plate with thickness $h_C = 0.820$ mm. Thus, we can calculate the mass ratio $M_{\text{ratio}}$ of the plates as

$$M_{\text{ratio}} = \frac{M_C}{M_H} = \frac{\rho_C \pi R^2 h_C}{\rho_H \pi R^2 h_H} = \frac{\rho_C h_C}{\rho_H h_H},$$

(13.8)

where $\rho_C$ is the mass density of the orthotropic plate, and $\rho_H$ is the mass density of the homogeneous plate; substituting the data from Tables 13.4, 13.5
and 13.6 for the variables involved in (13.8) and the data for the thickness of the plates, one finds that

\[ M_{\text{ratio}} = \frac{M_C}{M_H} = 0.69. \quad (13.9) \]

The calculation of the mass ratio in (13.9) shows a 31 \% mass decrease for the orthotropic plate. We conclude that the use of that kind of materials is in agreement with the minimum weight requirement for spacecraft structures.

### 13.5 Curves for the ballistic limit equations: case 3

In this case, we use equations (13.1) and (13.2) in order to plot ballistic limit curves for two plates made of Kevlar 49. Using equation (13.1) the material is considered as an orthotropic one, and using equation (13.2) the material is considered as a homogeneous one. The data for Kevlar 49 considered as a homogeneous material is shown in Table 13.7.

Table 13.7. Mechanical properties for Kevlar 49 considered as a homogeneous material for case 3.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E )</td>
<td>112 GPa</td>
</tr>
<tr>
<td>( \nu_x )</td>
<td>0.36</td>
</tr>
<tr>
<td>( \sigma_0 )</td>
<td>3620 MPa</td>
</tr>
<tr>
<td>( \rho_b )</td>
<td>1440 Kg/m(^3)</td>
</tr>
</tbody>
</table>

The data for Kevlar 49 considered as an orthotropic material is shown in Table 13.8.
Table 13.8. Mechanical properties for Kevlar 49 considered as an orthotropic material for case 3.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_x$</td>
<td>75.8 GPa</td>
<td>$E_y$</td>
<td>5.5 GPa</td>
<td>$G$</td>
</tr>
<tr>
<td>$S_L$</td>
<td>1379 MPa</td>
<td>$S_T$</td>
<td>64.8 MPa</td>
<td>$S_{LT}$</td>
</tr>
<tr>
<td>$v_x$</td>
<td>0.34</td>
<td>$\rho_b$</td>
<td>1450 Kg/m$^3$</td>
<td></td>
</tr>
</tbody>
</table>

The geometrical data of the plates and the mechanical properties of the projectile are shown in Table 13.9.

Table 13.9. Geometrical data of the plates and mechanical properties of the projectile for case 3.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>7.62 cm</td>
<td>$C$</td>
<td>2.39 cm</td>
<td>$h$</td>
</tr>
<tr>
<td>$\rho_p$</td>
<td>2796 Kg/m$^3$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Substituting the data from tables 13.7, 13.8, and 13.9 into equations (13.3), one finds that

$$ r_H = 0.0631 V_p^{-1/3} , $$  \hspace{1cm} (13.10)

$$ r_C = 0.0308 V_p^{-1/3} , $$

where the velocity of the projectile $V_p$ is measured in Km/s and the radius $r_H$ and $r_C$ are measured in mm.
Figure 13.3. Critical projectile radii versus projectile velocity, case 3.

The behavior of Kevlar 49 considered as a homogeneous material and as an orthotropic material is shown in Figure 13.3, where the difference between the curves is 104.87%. The region below each curve is the safe region. The curve that considers Kevlar 49 as an orthotropic material is more conservative than that where Kevlar 49 is considered as a homogeneous material.
13.6 Conclusion

Two new ballistic limit equations are plotted as ballistic limit curves by using three cases. In the first one, we compare the behavior of two plates made of different materials, one homogeneous and one orthotropic. The behavior of Kevlar 49 and Aluminum Al6061-T6 under hypervelocity impact are shown in Figure 13.1. Kevlar 49 is 13.43 % stronger than Aluminum Al6061-T6. In the second case, the ballistic limit curves are used to calculate the thickness of an orthotropic plate in order to have the same behavior as that of a homogeneous plate. The calculated thickness implies a mass difference between the plates such that 31 % less mass is used for the orthotropic plate. For this reason, the use of that kind of materials is in agreement with the minimum weight requirement for spacecraft structures. In the third case, Kevlar 49 is considered, by using equation (13.2), as a homogeneous material and by using equation (13.1) is considered as an orthotropic material. The difference between the curves is 104.87 %, the curve that considers Kevlar 49 as an orthotropic material is more conservative than that where Kevlar 49 is considered as a homogeneous material.

The three cases analyzed here show the use of the ballistic limit equation in order to compare different materials, to design plates made of different materials and the difference when we try to consider an orthotropic material as a homogeneous one.
Chapter 14

Conclusions

Two new ballistic limit equations have been developed in this work, the first one is used to know the behavior of homogeneous materials under hypervelocity impact. That equation is discussed in chapter 7 and is referred to as equation (7.49). The second ballistic limit equation is used to know the behavior of composite-orthotropic materials under hypervelocity impact. That equation is discussed in chapter 12 and is referred to as equation (12.32). Previous work in this area had produced two ballistic limit equations, which have been analyzed in chapter 2. The first one is an empirical ballistic limit equation and was developed by Christiansen [1] using experimental information and is referred to as equation (2.7). The second equation considered is an analytical ballistic limit equation and was developed by Angel and Smith [2] using an analytical procedure and experimental information and is referred to as equation (2.8).

A brief review of physics of hypervelocity impact is presented in chapter 3 using a two-plate structure, which consists of a shield plate and a back plate. A projectile impacts the structure, and, if the shock pressure is sufficiently high, the projectile and the shield plate debris become molten or vaporized, forming a debris cloud acting over the back plate and causing plastic deformation. This phenomenon occurs for velocities above 6 Km/s. In this particular case, some assumptions can
be used in order to build a mathematical model for hypervelocity impact on plates.

1. The load is uniformly distributed over a circular region with a diameter equal to one-half the spacing.

2. The load is applied so quickly that the loaded area is effectively given an initial velocity increment.

3. The momentum transferred to the loaded sector is equal to twice the momentum of the original projectile.

From the last assumptions, an expression for the initial velocity of the loaded area is derived and is expressed in equation (3.1).

A review of Kirchhoff plate's theory has been presented in chapter 4 and for homogeneous plates the partial differential equation, is given by (4.37). This partial differential equation with appropriate boundary conditions yields solutions to different problems for bending of plates. The boundary conditions and the elastic-plastic deformations make it difficult to derive analytical solutions for the variables involved. Therefore, a numerical method is needed in order to obtain solutions for the bending of homogeneous plates. This numerical method is the boundary element method, and the formulation for plates in general and for homogeneous plates in particular, is presented in chapter 5 for the case when the load is constant. The formulation involves the fundamental solution for homogeneous plates and is used in order to calculate numerical values for the
different variables involved in the integral equations (5.36), which is the boundary
element formulation for homogeneous plates. In order to calculate a numerical
solution for the quantities involved in that integral equation, a boundary
discretization in linear elements is used, which assumes that the quantities vary
linearly in each element. Application of the boundary conditions results in a
system of equations shown in (6.34), which allows one to calculate the unknown
quantities on the boundary. Then, any quantity inside the domain can be
calculated.

The first new ballistic limit equation developed in this work, referred to as
equation (7.49), uses a mathematical model which consists of a simply supported
circular plate, of radius \( R \), bending under a constant load exerted on a circular
sector of radius \( C \) in the center of the plate. The plate has been divided into two
concentric regions, the external elastic region and the internal plastic region,
because of the deformation type they undergo. The variables and functions
involved in the elastic region, \( C \leq r \leq R \), are calculated using the boundary
element method. The variables and functions involved in the plastic region,
\( 0 \leq r \leq C \), are calculated using an analytical procedure, as shown in section 7.4.
The new ballistic limit equation is compared with the existing ballistic limit
equations, reviewed in chapter 2, by plotting ballistic limit curves. The
comparison shows a strong agreement with the empirical equation. In this way,
our theoretical procedure to develop our ballistic limit equation is supported by
the experimental procedure used to develop the empirical ballistic limit equation.
A review of the orthotropic plate theory is presented in chapter 9; the partial differential equation, for this kind of plates, is given by (9.13). The boundary element formulation for that kind of plate is presented in chapter 10 for the case when the load is constant. That formulation involves an approximate fundamental solution for orthotropic plates, which is used in order to calculate numerical values for the different variables involved in the integral equations (10.23). This last equation is the boundary element formulation for orthotropic plates.

The second new ballistic limit equation for orthotropic materials, referred to as equation (12.32) has been developed using the same mathematical model as that used to develop the ballistic limit equation for homogeneous materials. The two new ballistic limit equations are plotted as ballistic limit curves by using three cases. In the first one, we compare the behavior of two plates made of different materials, one homogeneous and one orthotropic. The behavior of those materials shows that the orthotropic material is stronger than the homogeneous one. In the second case, the ballistic limit curves are used to calculate the thickness of an orthotropic plate in order to have the same behavior as that of a homogeneous plate. The calculated thickness implies a different mass between the plates, where 31 % less mass is needed for the orthotropic plate. In this way, the use of that kind of materials is in agreement with the minimum weight requirement for spacecraft structures. In the third case, one material is considered, by using equation (13.2), as a homogeneous material and by using equation (13.1) is considered as an orthotropic material. The difference between the curves is 104.87 %. The curve that considers the material as an orthotropic
material is more conservative than that were the material is considered as a homogeneous material.

The new ballistic limit equations, derived in this work, can be used in order to design structures under hypervelocity impact, made of homogeneous materials and made of composite-orthotropic materials. The procedure is explained in the different chapters in this work. The new ballistic limit equations do not involve parameters obtained from experimental information, they have been developed by using only physical principles for hypervelocity impact, the elastic-plastic theory of plates and the boundary elements method.
Chapter 15

Epilogue

The mathematical model used to develop new ballistic limit equations consist of a numerical process, for elastic deformations, and an analytical process, for plastic deformations. The numerical process solves the bending of plates based on Kirchhoff plate's theory that considers only small deformations. Our future work in this way is the use of large deformations, which implies a new formulation for the boundary elements method, with a non-linear relationship for the strains and the vertical displacement. The boundary elements formulation for large deformations is still new, and little work has been reported in the literature. For plastic deformations the Von Mises yield criterion is used for homogeneous materials. The numerical formulation for plastic deformations using that criterion is a non-linear problem due to the quadratic exponents for the moments. For composite-orthotropic materials the Tsai-Hill yield criterion is used. In the same way, the numerical formulation is a non-linear problem. The dynamic yield strength is considered to be a multiple of the static yield strength. In a future work we could consider a non-linear relationship between those yield strengths. The load is considered constant over the plate due to the characteristics of the debris cloud, which we consider completely melted. A non-constant load can be considered in the numerical formulation. Fortunately, the procedure used yields ballistic limit
equations that are in good agreement with ballistic equations developed by other authors. In particular, there is good agreement with an empirical ballistic limit equation derived from experimental information. Finally, our ballistic limit equations include all the parameters involved in an elastic-plastic problem, and the use of those equations to design structures subjected to hypervelocity impact is expected to yield reliable information.
Appendix A

Numerical Integration

A.1 Introduction

There are different procedures for numerical integration. Gaussian integration formulae are in general simple and accurate. The non-singular procedure is presented here and can be used in order to solve in a numerical way the integrals involved in chapters 6 and 11.

A.2 Gaussian Quadrature

The kind of integrals involved in chapters 6 and 11 can be written as

\[ I = \int_{-1}^{1} f(\xi) d\xi = \sum_{i=1}^{N} f(\xi_i) w_i + E_n, \]  

(A.1)

where \( N \) is the number of integration points, \( \xi_i \) is the coordinate of the \( i \)th integration point, \( w_i \) is the associated weighting factor and \( E_n \) is the error. Formulae (A.1) are based on the representation of \( f(\xi) \) by means of Legendre polynomials and \( \xi_i \) is the coordinate at a point \( i \) where those polynomials are zero.
There are tabulated values for $\xi_i$ and $w_i$ for different values of $N$. In particular, for $N = 4$, the corresponding values are as follows

$$
\begin{align*}
\xi_1 &= -0.86113, & w_1 &= 0.34785, \\
\xi_2 &= -0.33998, & w_2 &= 0.65214, \\
\xi_3 &= 0.33998, & w_3 &= 0.65214, \\
\xi_4 &= 0.86113, & w_4 &= 0.34785.
\end{align*}
$$

(A.2)

The accuracy of this numerical procedure is evaluated by solving integrals with a known analytical solution and the difference is in the order of 1 %.
Appendix B

Elastic Plastic Analysis of a Circular Plate

B.1 Introduction

The analytical solution, for elastic plastic deformations of a simply supported circular plate under a constant load, is discussed in this appendix; a load is acting in a circular region concentric with the plate. The analysis is divided in two cases, in the first one we consider that the radius of the plastic zone is minor than the radius of the load region; in the second case we consider that the radius of the plastic zone is bigger than the radius of the load region.

A dynamic analysis is performed in order to calculate the work done by the external forces acting over the plate, and using the kinetic energy theorem we obtain a relationship between the load acting over the plate and the initial velocity of the loaded region of the plate which is given by the equation (3.1).

As a result of that analysis, a ballistic limit equation is derived and is plotted for different values of the failure criteria.

B.2 Elastic Solution

Consider a simply supported plate as is shown in Figure 7.1, the radius of the plate is $R$, and the loaded region has a radius $C$. The moments in the circular plate, for the region $0 \leq r \leq C$, are considered as
\[ M_r = C_1 - \frac{1}{16}(3 + \nu)P_0 r^2 , \]  
\[ M_\theta = C_1 - \frac{1}{16}(1 + 3\nu)P_0 r^2 , \]  

where \( C_1 \) is a constant to be determined. The moments for the region \( C \leq r \leq R \) are considered as

\[ M_r = C_2 + C_3 \left( \frac{R^2}{r^2} \right) - \frac{1}{4}(1 + \nu)P_0 C^2 \ln(r) - \frac{1}{8}(3 + \nu)P_0 C^2 , \]  
\[ M_\theta = C_2 - C_3 \left( \frac{R^2}{r^2} \right) - \frac{1}{4}(1 + \nu)P_0 C^2 \ln(r) - \frac{1}{8}(1 + 3\nu)P_0 C^2 , \]  

Where \( C_2 \) and \( C_3 \) are constants to be determined, and in order to determine that constants we use three equations, the first one is obtained from the boundary condition \( M_r(R) = 0 \), the second equation is obtained from the continuity from the radial moment \( M_r(C^-) = M_r(C^+) \) and the third equation is obtained from the continuity for the circumferential moment \( M_\theta(C^-) = M_\theta(C^+) \).

The solution of the three equations gives expressions for the constants.

Now, plastic yielding begins at the center of the plate when the condition \( M_r = M_\theta = M_0 \) at \( r = 0 \) is reached, using that condition, the radial moment in (B.1) gives the critical value for \( P_0 \) which is given by
\[ P_{\text{min}} = \frac{16 M_0}{C^2 \left[ 4 - (1 - \nu) \frac{C^2}{R^2} + 4(1 + \nu) \ln \left( \frac{R}{C} \right) \right]^3} \]  

(B.3)

where \( P_{\text{min}} \) is the minimum value for which the yielding starts at the center of the plate.

The expressions for the normalized vertical displacement of the plate are calculated for both regions. For the region \( 0 \leq r \leq C \)

\[ \tilde{w}(\tilde{r}) = \frac{1}{64} \left( 1 - \nu^2 \right) \left( \frac{\tilde{P}}{\tilde{C}^2} \right) \left( \tilde{r}^4 - \tilde{C}^4 \right) - \frac{1}{2} (1 - \nu) \tilde{C}_1 \left( \tilde{r}^2 - \tilde{C}^2 \right) + \tilde{w}(\tilde{C}), \]  

(B.4)

where

\[ \tilde{C}_1 = \frac{1}{4} \tilde{P} - \frac{1}{16} \tilde{P} \tilde{C}^2 - \frac{1}{4} (1 + \nu) \tilde{P} \ln(\tilde{C}), \]  

(B.5)

\[ \tilde{C} = \frac{C}{R}, \quad \tilde{r} = \frac{r}{R}, \quad \tilde{P} = \frac{P_0 C^2}{M_0}, \]

and
\[ \tilde{w} = \frac{K w}{M_0}, \quad K = \frac{E h^3}{12}, \]  

\[ \tilde{w}(\tilde{C}) = \frac{3}{16} (1 - \nu^2) \tilde{P} \tilde{C}^2 \ln(\tilde{C}) + \] 
\[ \frac{1}{32} (1 - \nu) \tilde{P} \left[ (1 - \nu) \tilde{C}^2 - 2(3 + \nu) \right] (\tilde{C}^2 - 1). \]

And for the region \( C \leq r \leq R \), the normalized vertical displacement is

\[ \tilde{w}(\tilde{r}) = \frac{1}{16} (1 - \nu^2) \tilde{P} \left( 2\tilde{r}^2 + \tilde{C}^2 \right) \ln(\tilde{r}) + \]
\[ \frac{1}{32} (1 - \nu) \tilde{P} \left[ (1 - \nu) \tilde{C}^2 - 2(3 + \nu) \right] (\tilde{r}^2 - 1). \]

**B.3 Elastic-Plastic Solution, Case 1**

Consider a simply supported plate as is shown in Figure 7.1, the radius of the plate is \( R \), and the loaded region has a radius \( C \), the plastic zone have a radius \( \rho \) and, in this case is considered \( 0 \leq \rho \leq C \). The moments in the circular plate, for the plastic region \( 0 \leq r \leq \rho \), are considered as
\[ M_r = M_0 - \frac{1}{6} P_0 r^2, \]
\[ M_\theta = M_0. \]

The moments in the circular plate, for the elastic region \( \rho \leq r \leq C \), are considered as

\[ M_r = A_1 - B_1 \left( \frac{C^2}{r^2} \right) - \frac{1}{16} (3 + \nu) P_0 r^2, \]
\[ M_\theta = A_1 + B_1 \left( \frac{C^2}{r^2} \right) - \frac{1}{16} (1 + 3\nu) P_0 r^2, \]

where \( A_1 \) and \( B_1 \) are constants to be determined.

The moments in the circular plate, for the elastic region \( C \leq r \leq R \), are considered as

\[ M_r = A_2 + B_2 \left( \frac{R^2}{r^2} \right) - \frac{1}{4} (1 + \nu) P_0 C^2 \ln(r) - \frac{1}{8} (3 + \nu) P_0 C^2, \]
\[ M_\theta = A_2 - B_2 \left( \frac{R^2}{r^2} \right) - \frac{1}{4} (1 + \nu) P_0 C^2 \ln(r) - \frac{1}{8} (1 + 3\nu) P_0 C^2, \]
where $A_2$ and $B_2$ are constants to be determinate. In order to calculate the constants in (B.9) and (B.10) we need equations, which are be obtained using the boundary condition $M_r(R) = 0$, the continuity for the radial moments at $r = \rho$ and $r = C$, and the continuity for the circumferential moment at $r = C$. Now using continuity for the circumferential moment at $r = \rho$ we obtain a relationship between the load and the radius of the plastic zone, which is given, in a normalized form, by

$$
\tilde{P}(\tilde{\rho}) = \frac{48 \tilde{C}^2}{(1 + 3\nu)\tilde{\rho}^4 - 2(1 + 3\nu)\tilde{\rho}^2 + \tilde{K}},
$$

(B.11)

$$
\tilde{K} = 12 \tilde{C}^2 - 3(1 - \nu)\tilde{C}^4 - 12(1 + \nu)\tilde{C}^2 \ln(\tilde{C}),
$$

where $\tilde{\rho} = \frac{\rho}{R}$.

Now, we are able to calculate the vertical displacement of the plate using the radial and the circumferential moments in (B.8), (B.9) and (B.10), the normalized vertical displacement for the region $0 \leq r \leq \rho$. 
\[ \tilde{w} = \frac{1}{72} \left( \frac{\tilde{P}}{\tilde{C}^2} \right) \left( \tilde{r}^4 - \tilde{\rho}^4 \right) - \frac{1}{2} (1 - \nu) (\tilde{r}^2 - \tilde{\rho}^2) - \frac{1}{18} (1 + 3\nu) \frac{\tilde{P} \tilde{\rho}^3}{\tilde{C}^2} (\tilde{r} - \tilde{\rho}) + \tilde{w}_1, \]

where

\[ \tilde{w}_1 = -\frac{1}{2} (1 - \nu) (\tilde{\rho}^2 - \tilde{C}^2) - \frac{1}{48} (1 - \nu) (1 + 3\nu) \frac{\tilde{P} \tilde{\rho}^2}{\tilde{C}^2} (\tilde{\rho}^2 - \tilde{C}^2) - \frac{1}{48} (1 + \nu) (1 + 3\nu) \frac{\tilde{P} \tilde{\rho}^4}{\tilde{C}^2} \ln \left( \frac{\tilde{\rho}}{\tilde{C}} \right) + \frac{1}{64} (1 - \nu^2) \frac{\tilde{P}}{\tilde{C}^2} (\tilde{\rho}^4 - \tilde{C}^4) + \tilde{w}_2, \]

where

\[ \tilde{w}_2 = \nu \tilde{A}_2 \left[ \ln(\tilde{C}) - \frac{1}{2} (\tilde{C}^2 - 1) \right] + \tilde{A}_2 \left[ \ln(\tilde{C}) + \frac{1}{2} (\tilde{C}^2 - 1) \right] + \frac{1}{16} (1 - \nu) \tilde{P} \left[ 2\tilde{C}^2 \ln(\tilde{C}) - 3(\tilde{C}^2 - 1) \right], \]

where
\[ \tilde{A}_2 = -\tilde{K}_2 (\tilde{A}_1 + \tilde{B}_1) - \frac{1}{4} (1 + \nu) \tilde{P} \tilde{K}_2 \ln(\tilde{C}) + \frac{1}{16} (5 - \nu) \tilde{P} \tilde{K}_2, \]  

\[ \tilde{K}_2 = \frac{\tilde{C}^2}{1 + \tilde{C}^2}, \quad \tilde{A}_1 = 1 + \frac{1}{24} (1 + 3\nu) \frac{\tilde{P} \tilde{P}^2}{\tilde{C}^2}, \quad \tilde{B}_1 = \frac{1}{48} (1 + 3\nu) \frac{\tilde{P} \tilde{P}^4}{\tilde{C}^4}. \]  

The normalized vertical displacement for the region \( \rho \leq r \leq C \)

\[ \tilde{w} = \frac{1}{48} (1 - \nu)(1 + 3\nu) \frac{\tilde{P} \tilde{P}^2}{\tilde{C}^2} (\tilde{C}^2 - \tilde{r}^2) - \frac{1}{48} (1 + \nu)(1 + 3\nu) \frac{\tilde{P} \tilde{P}^4}{\tilde{C}^2} \ln\left(\frac{\tilde{r}}{\tilde{C}}\right) + \frac{1}{64} (1 - \nu^2) \frac{\tilde{P}}{\tilde{C}^2} (\tilde{C}^2 - \tilde{r}^2) + \frac{1}{2} (1 - \nu) (\tilde{C}^2 - \tilde{r}^2) + \tilde{w}_2. \]  

The normalized vertical displacement for the region \( C \leq r \leq R \)

\[ \tilde{w} = \nu \tilde{A}_2 \left[ \ln(\tilde{r}) - \frac{1}{2} (\tilde{r}^2 - 1) \right] + \tilde{A}_2 \left[ \ln(\tilde{r}) + \frac{1}{2} (\tilde{r}^2 - 1) \right] + \frac{1}{16} (1 - \nu) \tilde{P} \left[ 2\tilde{r}^2 \ln(\tilde{r}) - 3(\tilde{r}^2 - 1) \right], \]  

where \( E \) is the elastic modulus of the plate and \( h \) is the thickness of the plate. The expressions in (B.12) (B.16) and (B.17) are the analytical closed solution for the normalized vertical displacement of the plate when the radius of the plastic region is minor than the radius of the loaded region.
B.4 Elastic-Plastic Solution, Case 2

Consider a simply supported plate as is shown in Figure 7.1, the radius of the plate is $R$, and the loaded region has a radius $C$, the plastic zone have a radius $\rho$ and, in this case is considered $C \leq \rho \leq R$. The moments in the circular plate, for the plastic region $0 \leq r \leq C$, are considered as

$$M_r = M_0 - \frac{1}{6} P_0 r^2,$$

(B.18)

$$M_\theta = M_0.$$

The moments in the circular plate, for the plastic region $C \leq r \leq \rho$, are considered as

$$M_r = A_1 \left(\frac{C}{R}\right) + M_0 - \frac{1}{2} P_0 C^2,$$

(B.19)

$$M_\theta = M_0,$$

where $A_1$ is a constant to be determined.

The moments in the circular plate, for the elastic region $\rho \leq r \leq R$, are considered as
\[ M_r = A_2 + B_2 \left( \frac{R^2}{r^2} \right) - \frac{1}{4} (1 + \nu) P_0 C^2 \ln(r) - \frac{1}{8} (3 + \nu) P_0 C^2, \]  
(B.20)

\[ M_\theta = A_2 - B_2 \left( \frac{R^2}{r^2} \right) + \frac{1}{4} (1 + \nu) P_0 C^2 \ln(r) - \frac{1}{8} (1 + 3\nu) P_0 C^2, \]

where \( A_2 \) and \( B_2 \) are constants to be determined. In order to calculate the constants in (B.19) and (B.20) we need equations, which are be obtained using the boundary condition \( M_r(R) = 0 \), the continuity for the radial moments at \( r = \rho \) and \( r = C \). Now using continuity for the circumferential moment at \( r = \rho \) we obtain a relationship between the load and the radius of the plastic zone, which is given, in a normalized form, by

\[ \bar{P}(\bar{\rho}) = \frac{24 \bar{\rho}^2}{3(1 + \nu) \bar{\rho}^2 [\bar{\rho}^2 - 2 \ln(\bar{\rho})] - 4 \bar{C} \bar{\rho}^3 - 3(-3 + \nu) \bar{\rho}^2 - 4 \bar{C} \bar{\rho}}. \]  
(B.21)

Now, we are able to calculate the vertical displacement of the plate using the radial and the circumferential moments in (B.18), (B.19) and (B.20), the normalized vertical displacements for the region \( 0 \leq r \leq C \).
\begin{equation}
\bar{w} = \frac{1}{72} \left( \frac{\tilde{P}}{\tilde{C}^2} \right) (\tilde{r}^4 - \tilde{C}^4) - \frac{1}{2} (1 - \nu) (\tilde{r}^2 - \tilde{C}^2) - \frac{1}{18} (1 + 3\nu) \tilde{P} \tilde{C} (\tilde{r} - \tilde{C}) + \bar{w}_1,
\end{equation}

where

\begin{equation}
\bar{w}_1 = \frac{1}{2} (1 - \nu) (\tilde{r}^2 - \tilde{C}^2) + \frac{1}{4} \nu \tilde{P} (\tilde{r}^2 - \tilde{C}^2)
+ \frac{1}{3} \nu \tilde{P} \tilde{C} (\tilde{C} - \tilde{r}) + \bar{w}_2,
\end{equation}

where

\begin{equation}
\bar{w}_2 = \nu \tilde{A}_2 \left[ \ln(\tilde{r}) - \frac{1}{2} (\tilde{r}^2 - 1) \right] + \tilde{A}_2 \left[ \ln(\tilde{r}) + \frac{1}{2} (\tilde{r}^2 - 1) \right]
+ \frac{1}{16} (1 - \nu) \tilde{P} \left[ 2 \tilde{r}^2 \ln(\tilde{r}) - 3 (\tilde{r}^2 - 1) \right],
\end{equation}

where
\[
\tilde{A}_2 = -\tilde{K}_2 \left( -\frac{1}{4} (1 + \nu) \tilde{P} \tilde{K}_2 \ln(\tilde{\rho}) + \frac{1}{4} (1 - \nu) \tilde{P} \tilde{K}_2 \right),
\]

\[
\tilde{K}_2 = \frac{\tilde{\rho}^2}{1 + \tilde{\rho}^2}.
\]

The normalized vertical displacement for the region \( C \leq r \leq \rho \)

\[
\tilde{w} = \frac{1}{2} (1 - \nu) (\tilde{\rho}^2 - \tilde{r}^2) + \frac{1}{4} \nu \tilde{P} (\tilde{\rho}^2 - \tilde{r}^2) + \frac{1}{3} \nu \tilde{P} \tilde{C} (\tilde{r} - \tilde{\rho}) + \tilde{w}_2.
\]

The normalized vertical displacement for the region \( \rho \leq r \leq R \)

\[
\tilde{w} = \nu \tilde{A}_2 \left[ \ln(\tilde{r}) - \frac{1}{2} (\tilde{r}^2 - 1) \right] + \tilde{A}_2 \left[ \ln(\tilde{r}) + \frac{1}{2} (\tilde{r}^2 - 1) \right] + \frac{1}{16} (1 - \nu) \tilde{P} \left[ 2 \tilde{r}^2 \ln(\tilde{r}) - 3 (\tilde{r}^2 - 1) \right].
\]

The expressions in (B.22), (B.26) and (B.27) are the analytical closed solution for the normalized vertical displacement of the plate when the radius of the plastic region is bigger than the radius of the loaded region.
B.5 Elastic-Plastic Curves

In order to know the elastic-plastic behavior of the plate behavior of the plate we can plot elastic-plastic curves for different cases, Figure B.1 shown curves for different values of $\tilde{C}$, the range used is $0.01 \leq \tilde{C} \leq 1$, the maximum value correspond to the case when the load is acting over the whole plate. The curves are plotted using (B.10) and (B.21).

![Figure B.1. $\tilde{P}$ versus $\tilde{\rho}$ for different values of $\tilde{C}$](image_url)
The case when the load is acting over the whole plate and the plate undergoes only plastic deformations is presented in Save [50], in that reference the maximum value for \( \tilde{P} \) is 6, which is in agree with the corresponding curve shown in Figure B.1.

Using expressions for the normalized vertical displacements we can plot a profile for the displacements for different values of \( \tilde{\rho} \), Figure B.2 shown this profile for \( 0.01 \leq \tilde{\rho} \leq 1 \).

![Figure B.2. \( \tilde{w} \) versus \( \tilde{r} \) for different values of \( \tilde{\rho} \).]
B.6 Dynamic Analysis

In order to calculate the value of \( P_0 \) we use the kinetic energy theorem, which is expressed as

\[
W = \frac{1}{2} m V_f^2 - \frac{1}{2} m V_i^2 ,
\]

(B.28)

where \( W \) is the work done by the external forces, \( m \) is the mass of the loaded region, \( V_i \) is the initial velocity and is given by (3.1), and \( V_f \) is the final velocity and is zero. The work is calculated using

\[
W = 2\pi \int_0^C \left[ P_0 w(r, P_0) - \int_0^{\rho_0} w(r, P) \frac{dP}{d\rho} d\rho + \frac{1}{2} P_e w_e(r, P_e) \right] r dr ,
\]

(B.29)

where \( w(r, P_0) \) is the vertical displacement when the load \( P \) reach the final value \( P_0 \) and the plastic radius \( \rho \) reach the final value \( \rho_0 \), \( P_e \) is given by (B.3) and \( w_e(r, P_e) \) is given by (B.4). The work \( W \) in (B.29) can be expressed as

\[
W = 2\pi I_1 - 2\pi I_2 + \pi I_3 ,
\]

(B.30)

where
\[ I_1 = \int_0^C P_0 w(r, P_0) r \, dr, \quad I_2 = \int_0^C \int_0^{\rho_0} w(r, P) \frac{dP}{d\rho} d\rho \, r \, dr, \quad \text{(B.31)} \]

\[ I_3 = \int_0^C P_e w(r, P_e) r \, dr \]

In order to calculate the integrals in (B.31) we need a relationship between \( P \) and \( \rho \), two relationships are given in (B.11) and (B.21) corresponding to the cases 1 and 2 respectively. The normalization of the variables involves in (B.31) gives the following expression for the work \( W \)

\[ W = \left( \frac{\pi M_0^2 R^4}{K C^2} \right) \tilde{W}, \quad \text{(B.32)} \]

where

\[ \tilde{W} = 2 \tilde{P}_0 \tilde{I}_1 - 2 \tilde{I}_2 + \tilde{P}_e \tilde{I}_3, \quad \text{(B.33)} \]

where
\[ \tilde{I}_1 = \int_0^\tilde{C} \tilde{w}(\tilde{r}, \tilde{P}_0) \tilde{r} d\tilde{r}, \quad \tilde{I}_2 = \int_0^\tilde{C} \int_0^{\tilde{\rho}_e} \tilde{w}(\tilde{r}, \tilde{\rho}) \frac{d\tilde{P}}{d\tilde{\rho}} \tilde{\rho} \tilde{r} d\tilde{\rho} d\tilde{r}, \]  

\[ \tilde{I}_3 = \int_0^\tilde{C} \tilde{w}(\tilde{r}, \tilde{P}_e) \tilde{r} d\tilde{r}, \]  

where the derivative of \( \tilde{P} \) with respect to \( \tilde{\rho} \) are calculated using (B.11) or (B.21) depending on the case. For the case when \( 0 \leq \tilde{\rho} \leq \tilde{C} \), the derivative is

\[
\frac{d \tilde{P}}{d \tilde{\rho}} = \frac{(1 + 3\nu)\tilde{\rho}(1 - \tilde{\rho}^2)\tilde{P}^2}{12\tilde{C}^2}. \tag{B.35}
\]

For the case when \( \tilde{C} \leq \tilde{\rho} \leq 1 \), the derivative is

\[
\frac{d \tilde{P}}{d \tilde{\rho}} = \frac{24\tilde{P} - \tilde{P}^2 g(\tilde{\rho})}{24 \tilde{\rho}}, \tag{B.36}
\]

\[ g(\tilde{\rho}) = 9(1 + \nu)\tilde{\rho}^2 - 8\tilde{C} \tilde{\rho} - 6(1 + \nu)\ln(\tilde{\rho}) + 3(1 - 3\nu) \]

The integration of (B.34) should be performed by a numerical method because the integrals are difficult to handle in an analytical procedure.
B.7 Failure Criteria

In order to define a failure criteria for our analysis we use a strain criterion given for the following expression

$$
\varepsilon^p = \sqrt{\frac{2}{3} \left( (\varepsilon_r^p)^2 + (\varepsilon_\theta^p)^2 + (\varepsilon_z^p)^2 \right)},
$$

where

$$
\varepsilon^p_z = -\varepsilon_r^p - \varepsilon_\theta^p,
$$

and, for \( z = -h/2 \) and for the central point of the plate

$$
\varepsilon_r^p = \varepsilon_\theta^p = \left( \frac{1 - \nu}{2K} \right) M_0 h (\overline{C}_1 - 1),
$$

where \( K \) is given in (B.6) and \( \overline{C}_1 \) is given in (B.5). Substituting (B.38) and (B.39) into (B.37) one finds that

$$
\varepsilon^p = 3 \left( \frac{\sigma_0}{E} \right) (1 - \nu) (\overline{C}_1 - 1).
$$
The expression (B.40) is used to plot $\varepsilon^p$ versus $\tilde{P}$ in order to get critical values corresponding to the failure criteria, Figure B.3 shows this plot for different values of $\tilde{C}$.

![Figure B.3. $\varepsilon^p$ versus $\tilde{P}$ for different values of $\tilde{C}$.

B.8 Ballistic Limit Equation

Once the expression for the work is calculated in section B.6 we can develop a ballistic limit equation, substituting (B.32) and (3.1) into (B.28) one finds that
\[ E^{1/2} \rho_p r_p^3 V_p = \left( \frac{27 \tilde{W}}{128} \right) \rho_b^{1/2} \lambda \sigma_0 h R^2, \]  

(B.41)

where \( E \) is the elastic modulus of the plate, \( \rho_p \) is the mass density of the projectile, \( r_p \) is the radius of the projectile, \( V_p \) is the velocity of the projectile, \( \tilde{W} \) is the work done by the external force and is given by (B.33), \( \rho_b \) is the mass density of the plate, \( \lambda \) is the ratio between the dynamic yield strength and the static yield strength, \( \sigma_0 \) is the static yield strength, \( h \) is the thickness of the plate and \( R \) is the radius of the plate.

A ballistic limit curve can be plotted for different variables, if we consider like variables the radius of the projectile and the velocity of the projectile and keeping fixed the rest of the parameters in equation (B.41) one finds that

\[ r_p = \left[ \left( \frac{27 \tilde{W}}{128} \right)^{1/2} \left( \frac{\rho_b^{1/2} \lambda \sigma_0 h R^2}{E^{1/2} \rho_p} \right) \right]^{-1/3} V_p^{-1/3}. \]  

(B.42)

Table B.1 shows the data used in order to plot the ballistic curve using (B.42)
Table B.1. Data for plot the ballistic limit curve using (B.42).

<table>
<thead>
<tr>
<th>( E = 69 \text{ Gpa} )</th>
<th>( \rho_p = 2796 \text{ Kg/m}^3 )</th>
<th>( \rho_b = 2713 \text{ Kg/m}^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda = 4.5 )</td>
<td>( \sigma_0 = 275 \text{ Mpa} )</td>
<td>( h = 0.635 \text{ mm} )</td>
</tr>
<tr>
<td>( R = 7.62 \text{ cm} )</td>
<td>( \nu = 0.33 )</td>
<td>( \tilde{C} = 0.3 )</td>
</tr>
</tbody>
</table>

In order to get the values of \( \tilde{W} \) from (B.33) by using different values for the failure criteria given by (B.40). Table B.2 shows that values.

Table B.2. Values for the failure criteria and the normalized work.

<table>
<thead>
<tr>
<th>Case</th>
<th>( \epsilon^P )</th>
<th>( \tilde{\rho}_0 )</th>
<th>( \tilde{W} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.0001</td>
<td>0.0356</td>
</tr>
<tr>
<td>1</td>
<td>0.05</td>
<td>0.15</td>
<td>0.0365</td>
</tr>
<tr>
<td>1 or 2</td>
<td>0.1</td>
<td>0.3</td>
<td>0.0413</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>0.42</td>
<td>0.0508</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>0.51</td>
<td>0.0591</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>0.70</td>
<td>0.0778</td>
</tr>
</tbody>
</table>

Using the data given in tables B.1 and B.2 we can plot a ballistic limit curve which is shown in Figure B.4.
Figure B.4. Critical projectile radius versus projectile velocity for different values of the failure criteria and for $\tilde{C} = 0.3$.

Figure B.4 shows different ballistic limit curves which corresponds to different values of the failure criteria shown in Table B.2, in particular for the case when the radius of the plastic zone is equal to radius of the loaded zone is exactly the same considered in the four different cases presented in Chapter 8.

B.9 Conclusion

A discussion for the elastic-plastic deformation of a circular simply supported plate has been presented here; elastic-plastic curves are presented in order to understand the behavior of the plate under different circumstances. The analysis
is divided in two cases, in the first one the radius of the plastic zone is considered minor than the loaded zone, and in the second case the radius of the plastic zone is considered bigger than the loaded zone. The analytical closed solution for the vertical displacement is derived in this work and with that expression a profile of the displacements is plotted and is shown in Figure B.2. A dynamic analysis is discussed in order to calculate a relationship between the load \( P_0 \) and the velocity of the projectile \( V_p \). Finally a ballistic limit equation is developed from the dynamic analysis and ballistic limit curves are plotted in order to compare with the ballistic curves shown in Chapter 8, the comparison shown a strong agreement between the developed presented in this appendix and the develop presented in Chapter 7.
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