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Flat Structures, Soap Films, and Capillary Surfaces

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Abstract

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A technique is presented by way of example for proving the existence of minimal surfaces bounded by straight line segments and planar curves along which the surface meets the plane of the curve at a constant angle. Set in complex analysis, this technique provides a way to construct new examples of soap films and capillary surfaces. The soap films established are a soap film spanning five edges of a regular tetrahedron and a soap film spanning a rectangular prism. The examples of capillary graphs over a square presented here were previously shown to exist by Concus, Finn, and McCuan. However, with this new approach, we are able to examine the behavior of the graphs at the corners of the square. More precisely, we construct two one-parameter families of capillary graphs. The first family provides examples of capillary graphs that have continuous unit normal up to the corner, but the graphing function is not $C^2$ at the corner. The second family consists of capillary graphs with contact angle data in $D_2^+ \cup D_2^-$ such that the graphing function has a finite jump discontinuity at each corner.
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Chapter 1

Introduction

In the mid-1800’s, the physicist Plateau conducted his famous soap film experiments, where a wire frame is dipped into a soap solution and removed to produce a film clinging to the wire. One thing that makes soap films interesting is that they locally minimize area. That is, if you perturb the soap film inside a small 3-dimensional ball disjoint from the wire, the area of the film will increase. Now, given a wire frame, it is in some cases (a rectangular prism, for example) possible to observe more than one spanning soap film. Thus, there is no guarantee that a Plateau experiment will produce a least-area soap film. We will call such a least-area film a solution to Plateau’s problem.

Plateau observed two types of singularities in soap films:

(1) Y-singular curves: Curves along which three surfaces intersect at 120°

and

(2) T-singularities: points at which four Y-singular curves meet at 109°.

In 1976, a very good mathematical model of soap films came out of geometric measure theory in the form of Almgren’s [Alm76] \((M, 0, \delta)\) minimal sets. These sets are good models because Taylor [Tay76] proved that 2-dimensional \((M, 0, \delta)\) min-
imal sets are minimal surfaces except for the two types of singularities observed by Plateau (here, the approximate angle $109^\circ$ is replaced with the exact value $\cos^{-1}(-1/3)$). In chapters 3 and 4, we will construct sets fitting this description, so we will be interested in the converse to Taylor's theorem. Progress in this area has been made by Lawlor and Morgan [LM96], who have shown that Y-singular curves locally minimize area. Fortunately, the sets we construct are free of T-singularities, so this theorem is sufficient. It should also be noted that, as an example of theory leading to observation, the soap film spanning a rectangular prism in chapter 4 has been produced experimentally by John McCuan.

Chapter 5 concerns capillary surfaces, which are produced by placing a cylindrical tube in a fluid, causing the level of the fluid inside the tube to rise or fall. The interface between the air and the fluid inside the tube is a surface that is a graph over the cross section of the tube, and this graph meets the tube at a constant angle, called the contact angle. The contact angle depends only on the material of the tube and the type of fluid. In particular, it does not depend on the shape of the cross section.

Mathematically, capillary surfaces are modeled as graphs of constant mean curvature $H$ over a domain $\Omega \subset \mathbb{R}^2$ such that the graph exhibits contact angle behavior
on $\partial \Omega \times \mathbb{R}$. We will be concerned with the case where the cross section $\Omega$ contains a wedge of angle $\alpha$ formed by two curves $\Sigma_1$ and $\Sigma_2$ and the capillary surface has contact angles $\gamma_m$ along $\Sigma_m$ (the physical interpretation for the subcase $\gamma_1 \neq \gamma_2$ is a tube made of two different materials). In this case, a solution to the capillary problem will be a surface that can be written as a graph of a function $u$ such that

$$\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 2H \text{ on } \Omega,$$

$$\left\langle \nu, \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right\rangle = \cos \gamma_1 \text{ on } \Sigma_1,$$

and

$$\left\langle \nu, \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right\rangle = \cos \gamma_2 \text{ on } \Sigma_2,$$

where $\nu$ is the exterior unit normal to $\Omega$ and $H$ is a constant. To discuss some of the results in this case, we refer to Figure 1.3. If $\Sigma_1$, $\Sigma_2$ are straight line segments and the ordered pair of angles $(\gamma_1, \gamma_2)$ lies in the interior of the shaded rectangle $R$, then Concus and Finn [CF96] have shown that for any $H$, there exists a solution to the capillary problem. Moreover, such a solution is continuous and admits a continuous unit normal at the corner $P = \Sigma_1 \cap \Sigma_2$. In the first part of chapter 5, the existence of a one-parameter family of capillary graphs $S_\gamma$, $\pi/4 < \gamma < \pi/2$, over a square ($\alpha = \pi/2$) with contact angle data $(\pi - \gamma, \gamma) \in \text{interior}(R)$ is established. The
graphing function $u_\gamma$ in this case is not $C^2$ at the corner $P$, providing an upper bound for the regularity theory. A second result of Concus and Finn [CF96] states that if $(\gamma_1, \gamma_2)$ lies in the interior of $D_2^+ \cup D_2^-$, then solutions to the capillary problem may exist, but they cannot have continuous normal vector up to $P$. It is conjectured by Chen, Finn, and Miersemann [CFM98] that all such solutions are discontinuous at $P$. Concus, Finn, and McCuan [CFM99] examined the conjecture in the case of a minimal graph over a square. In particular, they prove the existence of a one-parameter family of such graphs with contact angles alternating from $\gamma$ and $\pi - \gamma$ on adjacent sides. Their calculations indicate a finite jump discontinuity at the corner.

In the second part of chapter 5, the existence of this one-parameter family is proves from a different point of view, which allows us to prove the existence of the expected finite jump discontinuity at the corner.
Chapter 2

Background

2.1 Minimal Surfaces and Soap Films

A two dimensional manifold whose transition maps are biholomorphisms is called a Riemann surface. Given a Riemann surface $\mathcal{R}$, where $z = x + iy \in \mathcal{R}$, a minimal immersion $X : \mathcal{R} \to \mathbb{E}^3$ is an immersion whose mean curvature is identically zero. Furthermore, $X$ is said to be conformal if the Euclidean metric in $\mathbb{E}^3$ is pulled back by $X$ to a conformal metric on $\mathcal{R}$. That is, if

$$\left| \frac{\partial X}{\partial x} \right|_{\mathbb{E}^3} = \left| \frac{\partial X}{\partial y} \right|_{\mathbb{E}^3} \quad \text{and} \quad \left\langle \frac{\partial X}{\partial x}, \frac{\partial X}{\partial y} \right\rangle_{\mathbb{E}^3} = 0.$$

In the case where $X$ is injective on $\mathcal{R}$, we replace “immersion” by “embedding”.

A minimal surface is a set $M \subset \mathbb{E}^3$ such that, for each $p \in M$, there is an open ball $B(p, r)$ such that $M \cap B(p, r)$ is the image of a minimal embedding from a disk $D \subset \mathbb{R}^2$. An important property of minimal surfaces is that they are sets that locally minimize area.

A more general class of area-minimizing sets is the class of soap films. First studied experimentally by Plateau, soap films can be modeled mathematically by
the \((M, 0, \delta)\)-minimal sets of Almgren [Alm76]. A set \(S \subset \mathbb{E}^n\) spanning a closed set \(B \subset \mathbb{E}^3\) is called \((M, 0, \delta)\) minimal with respect to \(B\) if \(S\) is nonempty and bounded, \(\mathcal{H}^m(S) < \infty\), and for every Lipschitz transformation \(\eta\) of \(\mathbb{E}^n\) which differs from the identity map only in a \(\delta\)-ball disjoint from \(B\), it follows that

\[
\mathcal{H}^m(S) < \mathcal{H}^m(\eta(S)),
\]

where \(\mathcal{H}^m\) is \(m\)-dimensional Hausdorff measure. A 2-dimensional \((M, 0, \delta)\) minimal set is called a soap film. Jean Taylor in 1976 [Tay76] proved that soap films are minimal surfaces except for two types of singularities:

(1) curves along which three minimal surfaces meet at an angle of 120°

and

(2) points where four curves of type (1) meet at the angle \(\cos^{-1}(-1/3)\).

Curves of type (1), which were later shown by [KNS78] to be real analytic, will be called Y-singular, while points of type (2) will be referred to as T-singularities. These singularities are exactly the ones observed in Plateau’s soap film experiments. Thus, Taylor’s work established Almgren’s \((M, 0, \delta)\) minimal sets as very good models of soap films.

As a converse to Taylor’s theorem, Lawlor and Morgan [LM96] proved that Y-singular curves are locally area minimizing. If, instead of a minimal surface, the set \(S - \{\text{singularities}\}\) is a countable collection of images of minimal immersions, we will say \(S\) is an immersed soap film.

### 2.2 Weierstrass data

Let

\[
X = (X_1, X_2, X_3) : \mathcal{R} \to \mathbb{E}^3
\]
be a non-planar conformal minimal immersion. The coordinate functions \( X_m \) of such a parametrization are harmonic, which implies that the one-forms

\[
\Phi_m := dX_m + i \star dX_m
\]

are holomorphic (Note that \( \Phi_m = 2 \frac{\partial X_m}{\partial z} dz \) in local coordinates). Thus, for any fixed \( p_0 \in \mathcal{R} \), the immersion \( X \) can be expressed by the formula

\[
X(p) = \int_{p_0}^{p} \left( \text{Re}(\Phi_1, \Phi_2, \Phi_3) \right) + X(p_0).
\] (2.1)

Since the one-forms \( \Phi_m \) are holomorphic, it follows that the function \( g : \mathcal{R} \rightarrow \hat{\mathbb{C}} \) given by \( g := -\frac{(\Phi_1 + i\Phi_2)}{\Phi_3} \) is a meromorphic function on \( \mathcal{R} \). Also, the conformality of \( X \) implies that

\[
\Phi_1^2 + \Phi_2^2 + \Phi_3^2 \equiv 0,
\] (2.2)

which allows us to express the one-forms \( \Phi_1, \Phi_2 \) in terms of \( g \) and \( \Phi_3 \) as follows:

\[
\Phi_1 = \frac{1}{2}(g - g^{-1})\Phi_3 \quad \text{and} \quad \Phi_2 = \frac{i}{2}(g + g^{-1})\Phi_3.
\] (2.3)

Thus, (2.1) can be written as

\[
X(p) = \int_{p_0}^{p} \left( \text{Re} \left( \frac{1}{2}(g^{-1} - g)\Phi_3, \frac{i}{2}(g^{-1} + g)\Phi_3, \Phi_3 \right) \right) + X(p_0).
\] (2.4)

To explain the significance of the function \( g \), let \( \sigma : S^2 \rightarrow \hat{\mathbb{C}} \) denote stereographic projection from the north pole of \( S^2 \) onto the (extended) equatorial plane \( \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0\} \). Then \( \sigma^{-1}(g) = \frac{1}{|g|^2 + 1}(2\text{Re}(g), 2\text{Im}(g), |g|^2 - 1) \). From (2.4), we see that

\[
\frac{\partial X}{\partial x} = \text{Re} \left( \frac{1}{2}(g^{-1} - g) \frac{\partial X_3}{\partial z}, \frac{i}{2}(g^{-1} + g) \frac{\partial X_3}{\partial z}, \frac{\partial X_3}{\partial z} \right)
\]

and

\[
\frac{\partial X}{\partial y} = -\text{Im} \left( \frac{1}{2}(g^{-1} - g) \frac{\partial X_3}{\partial z}, \frac{i}{2}(g^{-1} + g) \frac{\partial X_3}{\partial z}, \frac{\partial X_3}{\partial z} \right).
\]
Thus, it follows that \( \left< \frac{\partial X}{\partial x} - i \frac{\partial X}{\partial y}, \sigma^{-1}(g) \right>_C = \left< \left( \frac{1}{2}(g^{-1} - g), \frac{i}{2}(g^{-1} + g), 1 \right) \frac{\partial X_3}{\partial z}, \left( 2 \text{Re}(g), 2 \text{Im}(g), |g|^2 - 1 \right) \frac{1}{|g|^2 + 1} \right>_C = 0 \in \mathbb{C}, \)
and thus \( \sigma^{-1}(g) \) is a normal map on \( \mathcal{R} \).

So far, we have shown that a non-planar conformal minimal immersion can be expressed in terms of the meromorphic function \( g \), which is a stereographic projection of a normal map on the underlying Riemann surface, and the holomorphic one-form \( \Phi_3 \), which is commonly replaced by the notation \( dh \) (Note: the notation \( dh \) is misleading, as this one-form is not necessarily exact). This expression, given by (2.4), is known as the \textit{Weierstrass representation} in terms of the \textit{Weierstrass data} \( g \) and \( dh \). For the purposes of this paper, we will also be interested in the converse. That is, given a Riemann surface \( \mathcal{R} \), a non-constant meromorphic function \( g \) on \( \mathcal{R} \), and a holomorphic one-form \( dh \neq 0 \) on \( \mathcal{R} \), does (2.4) define a conformal minimal immersion?

To answer this question, we first define the holomorphic one-forms \( \Phi_1, \Phi_2, \) and \( \Phi_3 \) by (2.3), where \( \Phi_3 := dh \). If \( X \) is to be well-defined on all of \( \mathcal{R} \), it must be true that, for \( m = 1, 2, 3 \),
\[
\int_{\gamma} \text{Re}(\Phi_m) = 0 \text{ for every closed curve } \gamma \subset \mathcal{R}. 
\] (2.5)
Now, \( X \) is an immersion if and only if
\[
0 < |\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 = \frac{1}{2} \left( \frac{1}{|g| + |g|} \right)^2 |dh|^2 \] (2.6)
on \( \mathcal{R} \) (Note: one-half of the right hand side of (2.6) is the conformal factor of the metric on \( \mathcal{R} \) obtained by pulling back the Euclidean metric on \( \mathbb{E}^3 \) via \( X \)). Thus, from (2.6) we see that (2.4) defines an immersion if and only if
\[
g \text{ has a zero or pole of order } n \text{ wherever } dh \text{ has a zero of order } n. \] (2.7)
Once well-definedness and regularity have been established, conformality and minimality follow immediately. The immersion is conformal because, by construction, the one-forms $\Phi_m$ satisfy (2.2). Therefore, the map $X$ is minimal because each coordinate function is harmonic since it is the real part of a holomorphic function. To summarize, we state the Weierstrass representation theorem, with emphasis on its global formulation due to Osserman [Oss86].

**Theorem 2.1 Weierstrass representation theorem.** Let $\mathcal{R}$ be a Riemann surface, and let $X = (X_1, X_2, X_3) : \mathcal{R} \to \mathbb{E}^3$ be a non-planar conformal minimal immersion. Then, if we set $\Phi_m := dX_m + i \ast dX_m$, $m = 1, 2, 3$, and $g := -\frac{(\Phi_1 + i\Phi_2)}{\Phi_3}$, it follows that, for any fixed base point $p_0 \in \mathcal{R}$, the map $X$ can be expressed by the formula

$$X(p) = \int_{p_0}^{p} \left( \text{Re} \left( \frac{1}{2}(g^{-1} - g)dh, \frac{i}{2}(g^{-1} + g)dh, dh \right) \right) + X(p_0),$$

where $dh := \Phi_3$ and $g$ is a stereographic projection of a normal map on $\mathcal{R}$.

Conversely, let $g : \mathcal{R} \to \hat{\mathbb{C}}$ be a non-constant meromorphic function on $\mathcal{R}$ and $dh \neq 0$ be a holomorphic one-form on $\mathcal{R}$ such that $g$ has a pole or zero of order $n$ wherever $dh$ has a zero of order $n$. If $dh$ is denoted by $\Phi_3$ and $\Phi_1$ and $\Phi_2$ are defined by (2.3), then (2.4) defines a (non-planar) conformal minimal immersion if

$$\int_{\gamma} \text{Re}(\Phi_i) = 0 \text{ for every closed curve } \gamma \subset \mathcal{R}.$$

### 2.3 The second fundamental form and extremal length

Take $\mathcal{R}$ to be a domain in $\mathbb{C}$, and let $X : \mathcal{R} \to \mathbb{E}^3$ be an immersion of class $C^3(\mathcal{R}, \mathbb{E}^3)$. Then, we have the following well known proposition (e.g., see [DHKW92]).
Proposition 2.1 Suppose \( c \subset X(\mathcal{R}) \) is contained in a plane \( E \). Then \( c \) is a line of curvature on \( X(\mathcal{R}) \) if and only if \( X(\mathcal{R}) \) intersects \( E \) along \( c \) at a constant angle \( \gamma \).

Suppose further that \( X \) is a nonplanar conformal minimal immersion, and let \( g, dh \) be the Weierstrass data for \( X \). Denoting the second fundamental form on \( \mathcal{R} \) by \( II \), it can be shown (for example, see [HK97]) that

\[
\frac{dg(v)dh(v)}{g} = II(v,v) - iII(v,iv). \tag{2.8}
\]

As an immediate consequence of (2.8), we have the following proposition.

Proposition 2.2 Given a curve \( c \) on \( X(\mathcal{R}) \),

(i) \( c \) is a line of curvature \( \iff \frac{dg(c)dh(c)}{g} \in \mathbb{R} \)

and

(ii) \( c \) is asymptotic \( \iff \frac{dg(c)dh(c)}{g} \in i\mathbb{R} \).

Consider now the one-form \( \sqrt{dg \frac{dh}{g}} \) and the developing map

\[
\zeta(z) = \int_z^x \sqrt{\frac{dg \frac{dh}{g}}{g}}, \tag{2.9}
\]

where we take \( \sqrt{1} = 1 \). Note that \( \sqrt{\frac{dg \frac{dh}{g}}{g}} \) is well-defined on \( \mathcal{R} \) so long as the zeros and poles of \( g \) are simple. The term “developing map” comes from the fact that \( \zeta \) is local isometry between \( \mathcal{R} \) equipped with the conformal metric \( \left| \frac{dg \frac{dh}{g}}{g} \right| \) and \( \mathbb{C} \) equipped with the Euclidean metric. Thus, the map \( \zeta \) “develops” the surface \( X(\mathcal{R}) \) into the Euclidean plane \( \mathbb{E}^2 \). From Proposition 2.2 we have

\[
\zeta \text{ maps lines of curvature into horizontal or vertical lines} \tag{2.10}
\]

and

\[
\zeta \text{ maps asymptotic lines into lines in the direction } e^{i\pi/4} \text{ or } e^{3\pi/4}. \tag{2.11}
\]
This map $\zeta$ has a key role in the method outlined in this paper. Each application will involve proving the existence of a minimal surface bounded by a simple closed curve consisting of straight line segments (asymptotic) and planar curves along which the surface meets the plane at a constant angle (lines of curvature, by Proposition 2.1). Hence, (2.10) and (2.11) will indicate $\zeta$ maps the surface onto a Euclidean polygon. Further properties of the surface will allow us to determine the angles of this polygon as well as the orientation of each edge (e.g., whether or not the image of a line of curvature is horizontal or vertical). To find a parametrization for this surface, we need a parameter domain $\mathcal{R}$ and Weierstrass data $g, dh$ on $\mathcal{R}$. By construction, the domain $\mathcal{R}$ and the function $g$ will be known. For, $dh$, we use (2.9) to conclude
\[
dh = \frac{g(d\zeta)^2}{dg}.
\]
Thus, we can write down a parametrization for our surface in terms of known quantities as follows:
\[
X(z) = \int^z \text{Re} \left( (1 - g^2, i(1 + g^2), 2g \frac{(d\zeta)^2}{2dg}) \right).
\]

In chapters 4 and 5, we will encounter situations where $\mathcal{R}$ is a curvilinear quadrilateral and $\zeta(\mathcal{R})$ is a Euclidean quadrilateral. The conformal map $\zeta$ will then be a vertex preserving map between these two quadrilaterals. To prove such a map exists, we will need to use some facts about the conformal invariant extremal length.

Given a domain $\Omega \subset \mathbb{C}$ and sets $A, B \subset \overline{\Omega}$, let $\rho$ be a nonnegative Borel measurable function on $\Omega$ such that
\[
\int_{\Omega} \rho^2 \leq 1.
\]
Such a $\rho$ is called admissible. For admissible $\rho$, we consider the quantity
\[
L(\rho, \Omega) = \inf_{c} \int_{c} \rho,
\]
where the infimum is taken over all curves $c : [a, b] \to \Omega$ such that $c(a) \in A$, $c(b) \in B$, and $c((a, b)) \subset \Omega - A \cup B$. The extremal length of $\Omega$ with respect to $A$ and $B$ is then defined to be

$$Ext_{\Omega}(A, B) = \sup L(\rho, \Omega),$$

where the supremum is taken over all admissible $\rho$. By definition, extremal length is invariant under conformal bijections on $\Omega$.

A crucial step in proving the existence of some of the soap films and capillary surfaces in this paper is to establish the existence of a conformal map between two curvilinear quadrilaterals that takes edges to edges. To do this, we will make use of the following fact about extremal length (e.g., see [Ahl73]).

**Proposition 2.3** Let $Q, \tilde{Q}$ be curvilinear quadrilaterals with sides $A_i, \tilde{A}_m$, respectively. Choose a pair of opposite sides $(A_v, A_v'), (\tilde{A}_m', \tilde{A}_m'')$ from $Q$ and $\tilde{Q}$, respectively, and suppose

$$Ext_Q(A_v, A_v') = Ext_{\tilde{Q}}(\tilde{A}_m', \tilde{A}_m'').$$  \hspace{1cm} (2.12)

Then, there is an edge-preserving conformal map $\zeta$ between $Q$ and $\tilde{Q}$ taking $A_v$ to $\tilde{A}_m'$.

In order to verify that hypothesis (2.12) is satisfied, we will need the following three propositions (again, see [Ahl73]).

**Proposition 2.4** Let $A, B$ be sides of the curvilinear polygon $P$, and let $\tilde{A}, \tilde{B}$ be sides of the curvilinear polygon $\tilde{P}$. If $A, B, P$ are strictly contained in $\tilde{A}, \tilde{B}, \tilde{P}$, respectively, then $Ext_P(\tilde{A}, \tilde{B}) < Ext_P(A, B)$.

**Proposition 2.5** If $A$ is a vertex and $B$ is a non-adjacent side to $A$ on a curvilinear polygon $P$, then

$$Ext_P(A, B) = \infty.$$
Proposition 2.6 If $A$ and $B$ are adjacent sides on a curvilinear polygon $P$, then

$$\text{Ext}_P(A, B) = 0.$$

2.4 The Schwarz reflection principle, the Maximum principle, and Radó’s theorem

Once the existence of the minimal surface is established, we will want to extend it by rigid motions of $\mathbb{E}^3$ to a capillary surface or soap film. In order to do this, we will make use of the following two symmetry properties of minimal surfaces, discovered by H.A. Schwarz:

Theorem 2.2 (The Schwarz Reflection Principle for minimal surfaces)

(i) Every straight line contained in a minimal surface is an axis of symmetry of the surface.

(ii) If a minimal surface intersects some plane $E$ perpendicularly, then $E$ is a plane of symmetry of the surface.

This theorem is an immediate consequence of the following lemma, found in for example [DHKW92].

Lemma 2.1 Let $X : \mathcal{R} \to \mathbb{E}^3$, $z = x + iy \in \mathcal{R} \subset \mathbb{C}$, be a minimal immersion, where the domain $\mathcal{R}$ contains some interval $I$ that lies on the real axis.

(i) If $X(I) \subset x_1 - \text{axis}$, then we have

$$X_1(\bar{z}) = X_1(z), \quad X_2(\bar{z}) = -X_2(z), \quad \text{and} \quad X_3(\bar{z}) = -X_3(z).$$

(ii) If $X(I) \subset x_1x_2$ plane, and if $X(\mathcal{R})$ intersects $E$ orthogonally at $X(I)$, then it follows that

$$X_1(\bar{z}) = X_1(z), \quad X_2(\bar{z}) = X_2(z), \quad \text{and} \quad X_3(\bar{z}) = -X_3(z).$$
In addition to Theorem 2.2, Lemma 2.1, which relates the extension of the surface in $\mathbf{E}^3$ to the parametrization, will be needed in our existence proofs.

Now, in our applications, we will initially only prove the existence of a conformal minimal immersion. In order to show that the immersion is injective, we will need the following generalization of Radó's theorem referred to in [DHKW92].

**Theorem 2.3** If a curve $\Gamma \subset \mathbf{E}^3$ is mapped bijectively by parallel projection onto a planar convex curve $c$, except for segments perpendicular to the plane of projection, then there is a unique minimal surface whose boundary is $\Gamma$, and this is a graph over the planar domain determined by $c$.

Finally, to show that extending our minimal surface by Schwarz reflection to a soap film or capillary surface doesn't introduce intersections, we will use the following fact about harmonic functions, which will be applied to the components $X_m, \ m = 1, 2, 3$, of our conformal minimal immersion.

**Theorem 2.4** (The Maximum Principle) Let $\Omega \subset \mathbf{C}$ be a bounded domain, and let $h : \Omega \to \mathbf{R}$ be a harmonic function such that $h$ is continuous on $\overline{\Omega}$. If $h$ achieves its maximum at an interior point of $\Omega$, then $h$ is constant on $\Omega$.

Applying Theorem 2.4 to $-h$, we see also that a nonconstant harmonic function can not achieve its minimum at an interior point.
Chapter 3

A soap film spanning five edges of a regular tetrahedron

The cone over a regular tetrahedron is a soap film. Consisting of six planar sheets, four Y-singular curves, and one T-singularity, it is the area-minimizer among 2-dimensional sets dividing the tetrahedron into four regions [LM94]. However, the question of whether or not there is a smaller soap film dividing the tetrahedron into $n$ regions, $n \neq 4$, is still open.

Figure 3.1: The cone over a regular tetrahedron
In this chapter, we will prove the existence of a soap film $S^{5}_L$ spanning $T^{5}_L$, where $T^{5}_L$ consists of five edges of a regular tetrahedron of side length $L$. Consisting of two nonplanar sheets, one planar sheet, one Y-singular curve, and no T-singularities, the film $S^{5}_L$ divides the solid tetrahedron into three regions. Of course, this soap film cannot be considered as a candidate for the above open question, as it does not span the tetrahedron. A sketch of what we think $S^{5}_L$ should look like is shown in Figure 3.2.

![Figure 3.2: A sketch of $S^{5}_L$ and Schwarz's surface](image)

This sketch is motivated by Schwarz’s surface, a minimal surface spanning the quadrilateral $T^{4}_L$ formed by removing a pair of dual edges from a regular tetrahedron (see Figure 3.2). Note that $T^{5}_L$ is obtained by reinstating one of the two missing edges of $T^{4}_L$, and thus, our soap film can be thought of as Schwarz’s surface with a fin. Here, the fin, denoted by $F$ in Figure 3.2, is a planar region bounded by the reinstated edge and the Y-singular curve $a_2$.

For computational purposes, orient $T^{5}_L$ so that

\[ F \text{ is contained in the } x_2x_3 \text{ plane} \]

and

\[ \text{the plane perpendicular to } F \text{ containing the missing edge of } T^{5}_L \text{ is the } x_1x_3 \text{ plane.} \]
From the above sketch it appears our soap film has two symmetries: reflection across the $x_2x_3$ plane and reflection across the $x_1x_3$ plane. Ignoring the fin and taking the quotient by these symmetries, we obtain a minimal surface $M_L^5$ bounded by a simple closed curve $\Gamma = a_1 \cup a_2 \cup a_3$ (see Figure 3.3). If $N$ is the outward pointing normal on $\overline{M_L^5}$, then the fact that $a_2$ is a Y-singular curve implies

(i) along $a_2 \subset x_2x_3$ plane, $N$ makes a constant angle of $\pi/3$ with $(1,1,0)$.

For $a_1$ and $a_3$, we have

(ii) $a_1 \subset x_1x_3$ plane, and $M_L^5$ intersects the $x_1x_3$ plane orthogonally at $a_1$, and

(iii) $a_3$ is a line segment in the direction $(-1, -1, \sqrt{2})$.

![Figure 3.3: A sketch of $M_L^5$](image)

We assume $N$ is 1-1 on $\overline{M_L^5}$, so that (i) - (iii) imply $\sigma \circ N$ maps $M_L^5$ conformally onto a curvilinear triangle with sides $\hat{a}_m = \sigma \circ N(a_m)$ such that

\[ \hat{a}_1 \subset \mathbb{R} \times \{0\}, \]  
(3.1)

\[ \hat{a}_2 \subset \partial B(2, \sqrt{3}), \]  
(3.2)

and

\[ \hat{a}_3 \subset \partial B(e^{i\pi/4}, \sqrt{2}) \]  
(3.3)
(We say that a domain is mapped \textit{conformally onto} another domain if the map in question is a conformal bijection). Furthermore, upon closer examination of the sketch, we can determine enough about the behavior of $\sigma \circ N$ on $\overline{M_L^5}$ to conclude $\sigma \circ N(M_L^5)$ is the bounded triangle $\mathcal{R}$ with vertices
\[ \hat{v}_{12} = 2 + \sqrt{3}, \quad \hat{v}_{13} = \frac{1 + \sqrt{3}}{\sqrt{2}}, \quad \text{and} \quad \hat{v}_{23} = (1 + \sqrt{2})e^{i\pi/4}, \]
where $\hat{v}_{mn} = \hat{a}_m \cap \hat{a}_n$ (see Figure 3.4). As a result of this construction, when the parametrization $(\sigma \circ N)^{-1}$ on $\mathcal{R}$ is expressed in terms of the Weierstrass data $g$ and $dh$, we have
\[ g(z) = z. \]

![Figure 3.4: The domain $\mathcal{R}$](image)

Combined with Proposition 2.1, statements (i) and (ii) imply
\[ a_1 \text{ and } a_2 \text{ are lines of curvature}, \]
while from statement (iii) we have
\[ a_3 \text{ is asymptotic}. \]
So, from Proposition 2.2 we expect \( \zeta = \int \sqrt{dgdh/g} \) to map \( M_1^e \) conformally onto the interior of a Euclidean triangle with sides \( \tilde{a}_m = \zeta(a_m) \) such that

\[
\text{each of } \tilde{a}_1, \tilde{a}_2 \text{ is horizontal or vertical} \tag{3.5}
\]

and

\[
\tilde{a}_3 \text{ is in the direction } e^{i\pi/4} \text{ or } e^{i3\pi/4}. \tag{3.6}
\]

In particular, the triangle \( \zeta(M_1^e) \) has angles

\[
\tilde{\phi}_{12} = \frac{\pi}{2} \text{ and } \tilde{\phi}_{13} = \tilde{\phi}_{23} = \frac{\pi}{4}, \tag{3.7}
\]

where \( \tilde{\phi}_{mn} \) is the angle on \( \zeta(M_1^e) \) between \( \tilde{a}_m \) and \( \tilde{a}_n \).

To determine the orientation of \( \zeta(M_1^e) \), note from the sketch that as we move along \( a_1 \) from \( v_{12} \) towards \( v_{13} \), where \( v_{mn} = a_m \cap a_n \), the \( x_1 \) coordinate appears to increase. Parametrizing \( \tilde{a}_1 \) from \( \tilde{v}_{12} \) to \( \tilde{v}_{13} \) by

\[
z_1(t) = -t, \quad -(2 + \sqrt{3}) < t < -\left(1 + \frac{\sqrt{3}}{\sqrt{2}}\right) \tag{3.8}
\]

we assume

\[\tilde{a}_1 \text{ is horizontal.}\]

Then

\[dg(\tilde{z}_1) \equiv -1 \text{ and } d\zeta(\tilde{z}_1)^2 > 0,\]

so that, since \( t^2 > 1 \), we have that

\[
d(\sigma \circ N)^{-1}_1(\tilde{z}_1) = \text{Re} \left(1 - z_1^2\right) \frac{d\zeta(\tilde{z}_1)^2}{2dg(\tilde{z}_1)} = -\frac{1}{2} d\zeta(\tilde{z}_1)^2(1 - t^2) > 0. \tag{3.9}
\]

Hence, the function \( (\sigma \circ N)^{-1}_1(z_1) \) is increasing as desired, supporting the assumption that \( \tilde{a}_1 \) is horizontal. As

\[
\tilde{\phi}_{12} = \frac{\pi}{2},
\]
it follows immediately that
\[ \tilde{a}_2 \text{ is vertical.} \]

To determine the direction of \( \tilde{a}_3 \), observe that as we move along \( a_3 \) from \( v_{13} \) towards \( v_{23} \), the \( x_3 \) coordinate is increasing. Thus, parametrizing \( \hat{a}_3 \) from \( \hat{v}_{13} \) to \( \hat{v}_{23} \) by
\[ z_3(t) = e^{i\pi/4} + \sqrt{2}e^{it}, \tag{3.10} \]
suppose
\[ \tilde{a}_3 \text{ is in the direction } e^{i\pi/4}. \]

Then
\[ dg(\hat{z}_3) = i\sqrt{2}e^{it} \text{ and } \frac{1}{i}d\zeta(\hat{z}_3)^2 > 0, \]
so that, since \( t \in (-\pi/6, \pi/4) \) and \( \cos(t) + \sin(t) \) is positive on \( (-\pi/4, 3\pi/4) \), we have that
\[ d(\sigma \circ N)^{-1}_3(\hat{z}_3) = Re \left( z_3 \frac{d\zeta(\hat{z}_3)^2}{dg(\hat{z}_3)} \right) = \frac{1}{i4}d\zeta(\hat{z}_3)^2(\cos(t) + \sin(t)) > 0. \tag{3.11} \]

Hence, the function \((\sigma \circ N)^{-1}_3(z_3)\) is increasing, justifying the assumption that \( \tilde{a}_3 \) is in the direction \( e^{i\pi/4} \). So, choosing \( \hat{v}_{12} \) to be the base point of integration, it follows that, for some \( \lambda > 0 \), the function \( \zeta \) maps \( M^5_L \) conformally onto one of the Euclidean triangles \( \pm \Delta_\lambda \), where \( \Delta_\lambda \) has vertices \( \hat{v}_{mn} = \zeta(v_{mn}) \) given by
\[ \hat{v}_{12} = 0, \quad \hat{v}_{13} = -\lambda, \quad \text{and} \quad \hat{v}_{23} = i\lambda. \tag{3.12} \]

Translating so that \( v_{12} = 0 \), we conclude that for some \( \lambda > 0 \), the minimal surface \( M^5_L \) can be parametrized on \( \mathcal{R} \) by
\[ X^\lambda(z) = \int_{\hat{v}_{12}}^z Re \left( (1 - z^2, i(1 + z^2), 2z) \frac{(d\xi_\lambda)^2}{2dz} \right), \tag{3.13} \]
where \( \xi_\lambda \) maps \( \mathcal{R} \) conformally onto \( \Delta_\lambda \) in such a way that \( \hat{v}_{mn} = \xi_\lambda(\hat{v}_{mn}) \) (see Figure 3.5).
Figure 3.5: The map \( \xi_{\lambda} \) on \( \mathcal{R} \)

Now, the equation (3.13) gives the parametrization we expect for \( M^2_\mathcal{L} \) should it exist. To show that this intuition is correct, we have the following theorem.

**Theorem 3.1** The surface \( X^\lambda(\mathcal{R}) \) extends to a soap film \( S^5_L \) spanning some \( T^5_L \), where \( L = 2X^\lambda(\hat{v}_{13}) \). In addition, \( S^5_L \) is free of T-singularities; it consists of two nonplanar sheets, one planar sheet, and one Y-singular curve.

**proof:** It follows immediately that \( X^\lambda \) is a conformal minimal immersion. By examining \( X^\lambda \) on \( \partial \mathcal{R} \), we show that \( X^\lambda(\mathcal{R}) \) extends to a soap film spanning \( T^5_L \). To begin with, the Schwarz reflection principle for holomorphic functions can be used to extend \( \xi_{\lambda} \), and hence, the immersion \( X^\lambda \), across each edge \( \hat{a}_m \). For the vertices, first note that

\[
\hat{\phi}_{12} = \frac{\pi}{2}, \quad \hat{\phi}_{13} = \frac{\pi}{3}, \quad \text{and} \quad \hat{\phi}_{23} = \cos^{-1}(1/\sqrt{3}),
\]

where \( \hat{\phi}_{mn} \) is the angle on \( \mathcal{R} \) between \( \hat{a}_m \) and \( \hat{a}_n \). Thus, additional applications of the Schwarz reflection principle extend \( X^\lambda \) to a neighborhood of \( \hat{v}_{12} \). It follows from [Car54] that

\[
\frac{(d\xi_{\lambda})^2}{dz^2} \sim (z - \hat{v}_{13})^{-1/2},
\]
at \( \hat{v}_{13} \) and
\[
\frac{(d\xi_\lambda)^2}{dz^2} \sim (z - \hat{v}_{23})^\alpha,
\]
at \( \hat{v}_{23} \), where
\[
\alpha = \frac{\cos^{-1}(\sqrt{2}/\sqrt{3}) - \cos^{-1}(1/\sqrt{3})}{\cos^{-1}(1/\sqrt{3})} > -1
\]
(By the notation \((d\xi_\lambda)^2/dz^2 \sim (z - \hat{v}_{13})^{-1/2}\), for example, we mean that
\[
\lim_{z \to \hat{v}_{13}} \frac{(d\xi_\lambda)^2/dz^2}{(z - \hat{v}_{13})^{-1/2}}
\]
events and is nonzero). Therefore, the boundary behavior of \( \xi_\lambda \) implies that the one-form \((d\xi_\lambda)^2/dz\) is integrable on \( \overline{R} \). Hence, it follows that
\[
X^\lambda \text{ is continuous on } \overline{R}
\] (3.14)
and, as expected, we have that
\[
\phi_{12} = \frac{\pi}{2} \text{ and } \phi_{13} = \frac{\pi}{6},
\] (3.15)
where \( \phi_{mn} \) is the angle on \( X^\lambda(\overline{R}) \) between \( a_m \) and \( a_n \) (\( a_m = X^\lambda(\hat{a}_m) \)). In addition, we also have
\[
\phi_{23} = \cos^{-1}(\sqrt{2}/\sqrt{3}),
\] (3.16)
which could not be discerned from our sketch.

Proceeding with our analysis of \( X^\lambda \) on \( \partial\overline{R} \), parametrize interior(\( \hat{a}_1 \)) from \( \hat{v}_{12} \) to \( \hat{v}_{13} \) by \( z_1(t) = -t \). Then \( dz(z_1) \equiv -1 \) and \( d\xi_\lambda(z_1)^2 > 0 \), so that a calculation as in (3.9) shows
\[
dX_1^\lambda(z_1) > 0.
\] (3.17)
Furthermore, we have
\[
dX_2^\lambda(z_1) = Re \left(-i(1 + t^2)\frac{d\xi_\lambda(z_1)^2}{2} \right) = 0
\] (3.18)
and
\[ dX_3^\lambda(\hat{z}_1) = Re \left( td\xi_\lambda(\hat{z}_1)^2 \right) > 0. \]  \hspace{1cm} (3.19)

Next, let \( z_2(t) = 2 + \sqrt{3}e^{it} \) parametrize \textit{interior}(\hat{a}_2) from \( \hat{v}_{12} \) to \( \hat{v}_{23} \). Then \( dz(\hat{z}_2) = i\sqrt{3}e^{it} \) and \( d\xi_\lambda(\hat{z}_2)^2 < 0 \), so that
\[ dX_1^\lambda(\hat{z}_2) = -\frac{1}{2\sqrt{3}} d\xi_\lambda(\hat{z}_2)^2 Re \left( ie^{-it}(3 + 4\sqrt{3}e^{it} + 3e^{2it}) \right) = 0, \]  \hspace{1cm} (3.20)
\[ dX_2^\lambda(\hat{z}_2) = \frac{1}{2\sqrt{3}} d\xi_\lambda(\hat{z}_2)^2(8\cos(t) + 4\sqrt{3}) < 0, \]  \hspace{1cm} (3.21)

and
\[ dX_3^\lambda(\hat{z}_2) = -\frac{1}{\sqrt{3}} d\xi_\lambda(\hat{z}_2)^2 Re \left( ie^{-it}(2 + \sqrt{3}e^{it}) \right) = -\frac{2}{\sqrt{3}} d\xi_\lambda(\hat{z}_2)^2 \sin(t) > 0. \]  \hspace{1cm} (3.22)

Inequalities (3.21) and (3.22) hold since \( t \in (0, 5\pi/6) \), on which the functions \( 8\cos(t) + 4\sqrt{3} \) and \( \sin(t) \) are positive. Finally, parametrize \textit{interior}(\hat{a}_3) from \( \hat{v}_{13} \) to \( \hat{v}_{23} \) by \( z_3(t) = e^{i\pi/4} + \sqrt{2}e^{it} \). Then \( dz(\hat{z}_3) = i\sqrt{2}e^{it} \) and \( \frac{1}{i} d\xi_\lambda(\hat{z}_3)^2 > 0 \), so that
\[ dX_1^\lambda(\hat{z}_3) = dX_2^\lambda(\hat{z}_3) = -\frac{1}{\sqrt{2}} dX_3^\lambda(\hat{z}_3) = -\frac{1}{2} d\xi_\lambda(\hat{z}_3)^2(\cos(t) + \sin(t) + 2) < 0. \]  \hspace{1cm} (3.23)

Using the fact that \( X^\lambda(\hat{v}_{12}) = 0 \), from (3.17) - (3.19) it follows
\[ X^\lambda(\hat{a}_1) \subset x_1x_3 \text{ plane, } X_1^\lambda(z_1) \text{ is increasing, and } X_3^\lambda(z_1) \text{ is decreasing}, \]  \hspace{1cm} (3.24)
while (3.20) - (3.22) imply
\[ X^\lambda(\hat{a}_3) \subset x_2x_3 \text{ plane, } X_2(z_2) \text{ is decreasing, and } X_3(z_2) \text{ is increasing}. \]  \hspace{1cm} (3.25)

Lastly, (3.23) implies
\[ X^\lambda \text{ maps } \hat{a}_3 \text{ bijectively onto a segment in the direction } (-1, -1, \sqrt{2}). \]  \hspace{1cm} (3.26)

The length \( L \) of this segment is given by
\[ L^2 = |X^\lambda(\hat{v}_{23}) - X^\lambda(\hat{v}_{23})|^2 = \sum_{m=1}^{3} (X_1^\lambda(\hat{v}_{23}) - X_1^\lambda(\hat{v}_{23}))^2. \]
From (3.26) we have

\[(X_1^\lambda(\hat{v}_{23}) - X_1^\lambda(\hat{v}_{13}))^2 = (X_2^\lambda(\hat{v}_{23}) - X_2^\lambda(\hat{v}_{13}))^2 = \frac{1}{2} (X_3^\lambda(\hat{v}_{23}) - X_3^\lambda(\hat{v}_{13}))^2,\]

so that

\[L^2 = 4(X_1^\lambda(\hat{v}_{23}) - X_1^\lambda(\hat{v}_{13}))^2.\]

By (3.25), we know that \(X_1^\lambda(\hat{v}_{23}) = 0\), and (3.24) implies \(X_1^\lambda(\hat{v}_{13}) > 0\). Hence, we have that

\[L = 2X_1^\lambda(\hat{v}_{13}). \quad (3.27)\]

From (3.24) - (3.26) it follows that under projection onto the \(x_1x_2\) plane, the boundary curve \(X^\lambda(\partial \mathcal{R})\) maps bijectively onto a Euclidean triangle. Thus, Theorem 2.3 implies \(X^\lambda(\mathcal{R})\) is a graph over this triangle.

Combined with the facts \(g(z) = z\) and \(\hat{a}_1 \subset \mathbb{R} \times \{0\}\), statement (3.24) implies \(X^\lambda(\mathcal{R})\) meets the \(x_1x_2\) plane along \(a_1\) at a constant angle of \(\pi/2\).

Thus, using the Schwarz reflection principle for minimal surfaces, the minimal surface \(X^\lambda(\mathcal{R})\) can be extended across \(a_1\) by reflection through the \(x_1x_3\) plane to a minimal surface \(2X^\lambda(\mathcal{R})\). As Lemma 2.1 implies this reflection corresponds to extending \(X^\lambda\) across \(\hat{a}_1\), we have that \(a_2\) extends smoothly across \(v_{12}\) to a curve \(2a_2\).

Next, reflect \(2X^\lambda(\mathcal{R})\) across the \(x_2x_3\) plane, and denote the union of the two resulting nonplanar sheets by \(4X^\lambda(\mathcal{R})\). Then \(S_L^5 = 4X^\lambda(\mathcal{R}) \cup F\) is a set spanning some \(T_L^5\), where \(F\) is the planar disk bounded by \(2a_2\) and the line segment connecting its two endpoints.

From (3.24) - (3.26), we can use the maximum principle to prove \(S_L^5\) has no self intersections. To show \(S_L^5\) is a soap film, it remains to prove \(a_2\) is a Y-singular curve. For this, choose \(p \in \hat{a}_2\) and let \(C_p\) be the plane passing through \(X^\lambda(p)\) that is
perpendicular to the tangent line to \(a_2\) at \(X^\lambda(p)\). Denote by \(c_1, c_2, c_3 : [a, b] \to \mathbb{E}^3\), \(c_m(a) = X^\lambda(p)\), the three curves in \(C_p \cap S^5_L\) meeting at \(X^\lambda(p)\), where

\[
c_1((a, b)) \subset F,
\]

\[
c_2((a, b)) \subset X^\lambda(\mathcal{R}),
\]

and

\[
c_3((a, b)) \text{ is the image of } c_2((a, b)) \text{ under reflection across the } x_2x_3 \text{ plane.}
\]

Since \(c_1((a, b)) \subset F\), it follows that the angle between \(\dot{c}_1(a)\) and \((1, 0, 0)\) is \(\pi/2\). From the facts \(g(z) = z\) and \(\dot{a}_2 \subset \partial B(2, \sqrt{3})\), we have that the surface normal \(\sigma^{-1}(p)\) makes an angle of \(\pi/3\) with \((1, 0, 0)\). Now, from (3.25) we have that \(\dot{c}_1(a)\) has positive \(x_3\) coordinate. As \(|p| > 1\), it follows that \(\sigma^{-1}(p)\) has positive \(x_3\) coordinate, and thus, since \(\dot{c}_1(a), \sigma^{-1}(p),\) and \((1, 0, 0)\) all lie in the plane passing through the origin and perpendicular to the tangent line to \(a_2\) at \(X^\lambda(p)\), we can use angle addition to conclude that the angle between \(\dot{c}_1(a)\) and \(\sigma^{-1}(p)\) is \(\pi/6\). Therefore, as the angle between \(\sigma^{-1}(p)\) and \(\dot{c}_2(a)\) is \(\pi/2\), it follows that the angle \(\alpha\) between \(\dot{c}_1(a)\) and \(\dot{c}_2(a)\) is either \(2\pi/3\) or \(\pi/3\). If \(\alpha = \pi/3\), then there is a point \(z \in \mathcal{R}\) such that \(X^\lambda_1(z) < 0\), which violates the maximum principle since (3.24) - (3.26) imply \(X^\lambda_1 \geq 0\) on \(\partial \mathcal{R}\). It follows that

\[
\alpha = \frac{2\pi}{3},
\]

and the theorem is proved. Q.E.D.
Chapter 4

Soap films spanning a rectangular prism

By conducting a Plateau experiment with a cube, one can observe the soap film of Figure 4.1. This film contains four planar sheets and eight nonplanar sheets.

Figure 4.1: The suspended “square” soap film
emanating from the edges, eight Y-singular curves, and four T-singularities. These twelve sheets support a planar “square” in the center of the cube (it’s not actually a square, as the angles between any pair of edges is $\cos^{-1}(-1/3)$). Together, the minimal sheets and the planar “square” divide the solid cube into six regions.

In this chapter, we give a parametrization for a different soap film spanning a rectangular prism with a square base. This film, which can be produced experimentally, consists of four planar sheets, four nonplanar sheets, four Y-singular curves, and no T-singularities. Furthermore, as shown in Figure 4.2, this soap film divides the solid prism into five regions.

Figure 4.2: A sketch of the soap film $S_{Lh}$

As with $S_5$ in the last chapter, our sketch is obtained by adding planar fins to a minimal surface bounded by polygonal components (in this case, the two square bases). Then, as before, we can use the sketch as a guide to derive the properties the soap film should possess, and use these properties to arrive at a parametrization. To begin with, let $P_{Lh}$ be a rectangular prism with height $h$ and square base of side
length $L$, and let $S_{Lh}$ denote both the sketch and the soap film it is assumed to represent.

The first thing to note is that $S_{Lh}$ has symmetries. For computational purposes, orient $P_{Lh}$ so that the planes of reflectional symmetry parallel to the faces of the prism are the coordinate planes, and the diagonal planes of reflectional symmetry are

$$E_1 = \{(x_1, x_2, x_3) \mid x_2 = x_1\} \text{ and } E_2 = \{(x_1, x_2, x_3) \mid x_2 = -x_1\}.$$ 

Thus, ignoring the fins and taking the quotient of $S_{Lh}$ by its symmetry group, we obtain a minimal surface $M_{Lh}$ (indicated in Figure 4.3) bounded by a simple closed curve $\Gamma = a_1 \cup a_2 \cup a_3 \cup a_4$, where

(i) $a_1 \subset x_1x_2$ plane, and $M_{Lh}$ intersects the $x_1x_2$ plane orthogonally at $a_1$,

(ii) along the Y-singular curve $a_2 \subset E_2$, the map $N$ makes a constant angle of $\pi/3$ with $(1, 1, 0)$ (the map $N$ is the outward pointing normal on $M_{Lh}$),

(iii) $a_3$ is a line segment in the direction $(0, 1, 0)$,

and

(iv) $a_4 \subset x_1x_3$ plane, and $M_{Lh}$ intersects the $x_1x_3$ plane orthogonally at $a_4$.

Figure 4.3: The minimal surface $M_{Lh}$
Assuming as in chapter 3 that $N$ is 1-1 on $\overline{M_{Lh}}$, statements (i) - (iv) indicate $\sigma \circ N$ maps $M_{Lh}$ conformally onto a curvilinear quadrilateral with sides $\hat{a}_m = \sigma \circ N(a_m)$ such that

\begin{align*}
\hat{a}_1 & \subset \partial B(0,1), \\
\hat{a}_2 & \subset \partial B(2e^{i\pi/4},\sqrt{3}),
\end{align*}

and

\begin{align*}
\hat{a}_3, \hat{a}_4 & \subset \mathbb{R} \times \{0\}.
\end{align*}

Upon further inspection of the behavior of $N$ on $\overline{M_{Lh}}$, we conclude $\sigma \circ N(M_{Lh})$ has vertices

\begin{align*}
\hat{v}_{12} = \frac{1}{2\sqrt{2}}(\sqrt{3} + 1 - i(\sqrt{3} - 1)), \quad \hat{v}_{14} = 1, \quad \hat{v}_{23} = 1 + \sqrt{2},
\end{align*}

and

\begin{align*}
1 < \hat{v}_{34} < 1 + \sqrt{2}
\end{align*}

(see Figure 4.4). Notice here that $\hat{v}_{34}$ cannot be determined completely, as it does not seem possible by visual inspection to ascertain how much $N$ rotates along $a_3$. Thus, we have a one parameter family

\begin{align*}
\{\mathcal{R}_{\delta}\}, 1 < \delta < 1 + \sqrt{2}
\end{align*}

of possibilities for $\sigma \circ N(M_{Lh})$, where $\mathcal{R}_{\delta}$ is a curvilinear quadrilateral whose sides are given by (4.1) – (4.3) and whose vertices are given by (4.4), with $\hat{v}_{34} = \delta$. Additionally, this inability to locate $\hat{v}_{34}$ lends support to the conjecture that the amount of rotation depends on the height of the prism in relation to the side length of the base, as this dependence relation is certainly not something we could expect to derive from our rough sketch!

Together with Proposition 2.1, statements (i), (ii) and (iv) imply

\begin{align*}
a_1, a_2, \text{ and } a_4 \text{ are lines of curvature,}
\end{align*}
while (iii) implies

\[ a_3 \text{ is asymptotic.} \]

Therefore, by Proposition 2.2 we expect \( \zeta = f \sqrt{dgdh/g} \) to map \( M_{Lh} \) conformally onto the interior of a Euclidean quadrilateral with sides \( \tilde{a}_m = \zeta(a_m) \) such that

\[ \text{each of } \tilde{a}_1, \tilde{a}_2, \text{ and } \tilde{a}_4 \text{ is horizontal or vertical} \tag{4.5} \]

and

\[ \tilde{a}_3 \text{ is in the direction } e^{i\pi/4} \text{ or } e^{i3\pi/4}. \tag{4.6} \]

Unlike the situation in chapter 3, statements (4.5) and (4.6) are not enough to determine the angles \( \tilde{\phi}_{mn} \) completely. However, we do know that

\[ \hat{\phi}_{12} = \hat{\phi}_{14} = \frac{\pi}{2}, \quad \hat{\phi}_{34} = \pi, \quad \text{and} \quad \hat{\phi}_{23} = \cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right), \]

and from our sketch we see that

\[ \phi_{12} = \phi_{14} = \phi_{34} = \frac{\pi}{2}, \]
so that we can use the equations

\[ 2 \left( \frac{\tilde{\phi}_{mn}}{\tilde{\phi}_{mn}} - 1 \right) + 1 = \frac{\phi_{mn}}{\phi_{mn}} \]

to conclude

\[ \tilde{\phi}_{12} = \tilde{\phi}_{14} = \frac{\pi}{2} \text{ and } \tilde{\phi}_{34} = \frac{3\pi}{4}. \]

Thus, it follows that

\[ \tilde{\phi}_{23} = \frac{\pi}{4}, \]

and hence, we have

\[ \phi_{23} = \cos^{-1} \left( \frac{1}{\sqrt{3}} \right). \tag{4.7} \]

Now that the angles are known, it remains to determine the orientation of \( \zeta(M_{Lh}) \).

Beginning with \( \tilde{a}_1 \), note from the sketch that as we move along \( a_1 \) from \( v_{12} \) towards \( v_{14} \), the \( x_2 \) coordinate appears to increase. So, parametrizing \( \tilde{a}_1 \) from \( \tilde{v}_{12} \) to \( \tilde{v}_{14} \) by

\[ z_1(t) = e^{it}, \tag{4.8} \]

let’s suppose

\( \tilde{a}_1 \) is horizontal.

Then

\[ dg(\dot{z}_1) = ie^{it} \text{ and } d\zeta(\dot{z}_1)^2 > 0, \]

so that

\[ d(\sigma \circ N)^{-1}_2(z_1) = \text{Re} \left( i(1 + e^{2it}) \frac{d\zeta(\dot{z}_1)^2}{2ie^{it}} \right) = d\zeta(\dot{z}_1)^2 \cos(t) > 0. \tag{4.9} \]

Hence, the function \( (\sigma \circ N)^{-1}_2(z_1) \) is increasing as desired, thereby supporting the assumption that \( \tilde{a}_1 \) is horizontal. Since

\[ \tilde{\phi}_{12} = \tilde{\phi}_{14} = \frac{\pi}{2}, \]
it then follows immediately that

\[ \tilde{a}_2 \text{ and } \tilde{a}_4 \text{ are vertical}. \]

Finally, to determine the direction of \( \tilde{a}_3 \), note that as we move along \( a_3 \) from \( v_{23} \) towards \( v_{34} \), the \( x_2 \) coordinate is again increasing. Parametrizing \( \tilde{a}_3 \) from \( \hat{v}_{23} \) to \( \hat{v}_{34} \) by

\[ z_3(t) = -t, \quad (4.10) \]

let’s assume

\[ \tilde{a}_3 \text{ is in the direction } e^{i \pi/4}. \]

Then

\[ dg(\hat{z}_3) \equiv -1 \text{ and } \frac{1}{i} d\zeta(\hat{z}_3)^2 > 0, \]

so that

\[ d(\sigma \circ N)^{-1}_2(\hat{z}_3) = Re \left( -\frac{i}{2} (1 + t^2) d\zeta(\hat{z}_3)^2 \right) = -\frac{i}{2} d\zeta(\hat{z}_3)^2 (1 + t^2) > 0. \quad (4.11) \]

Thus, the function \((\sigma \circ N)^{-1}_2(z_3)\) is increasing, supporting the claim that \( \tilde{a}_3 \) is in the direction \( e^{i \pi/4} \). Therefore, choosing \( \hat{v}_{14} \) to be the base point of integration, we conclude that, for some \( \lambda, s > 0 \), the function \( \zeta \) maps \( M_{Lh} \) conformally onto one of the Euclidean quadrilaterals \( \pm Q_{\lambda s} \), where \( Q_{\lambda s} \) has vertices

\[ \hat{v}_{14} = 0, \quad \hat{v}_{12} = \lambda, \quad \hat{v}_{34} = i s, \quad \text{and} \quad \hat{v}_{23} = \lambda + i(\lambda + s). \quad (4.12) \]

Putting together the information obtained up to this point, we have that for some \( 1 < \delta < 1 + \sqrt{2} \) and some \( \lambda, s > 0 \), the minimal surface \( M_{Lh} \) can be parametrized on \( \mathcal{R}_\delta \) by

\[ X^{\lambda s}(z) = \int_{\hat{v}_{14}}^z Re \left( (1 - z^2, i(1 + z^2), 2z) \frac{(d\xi_{\lambda s})^2}{2dz} \right) + \mathcal{K}^{\lambda s}, \quad (4.13) \]
where $\mathcal{K}^{\lambda s} = v_{14}$ and $\xi_{\lambda s}$ maps $\mathcal{R}_\delta$ conformally onto $Q_{\lambda s}$ in such a way that

$$\tilde{v}_{mn} = \xi_{\lambda s}(\hat{v}_{mn}) \quad \text{(see Figure 4.5)}.$$ 

To express the constant $\mathcal{K}^{\lambda s}$ in terms of the known quantities $g(z) = z$ and $\zeta = \pm \xi_{\lambda s}$, recall that by applying the symmetries of $P_{Lh}$ and adding the fins, the minimal surface $M_{Lh}$ should extend to a soap film spanning $P_{Lh}$. In particular, we should have

$$\mathcal{K}_2^{\lambda s} = \mathcal{K}_3^{\lambda s} = 0 \quad \text{(4.14)}$$

and

$$\text{the length of } a_3 \text{ is equal to the distance of } v_{34} \text{ from the } z \text{ axis.} \quad \text{(4.15)}$$

Stated in terms of the parametrization, the statement (4.15) means

$$-X_2^{\lambda s}(\hat{v}_{23}) = X_1^{\lambda s}(\hat{v}_{34}).$$

Thus, we have

$$\mathcal{K}_1^{\lambda s} = X_1^{\lambda s}(\hat{v}_{34}) + \left(X_1^{\lambda s}(\hat{v}_{14}) - X_1^{\lambda s}(\hat{v}_{34})\right) = -X_2^{\lambda s}(\hat{v}_{23}) + \left(X_1^{\lambda s}(\hat{v}_{14}) - X_1^{\lambda s}(\hat{v}_{34})\right)$$

$$= \int_{\hat{v}_{23}}^{\hat{v}_{34}} \text{Re} \left( i(1 + z^2) \frac{(d\xi_{\lambda s})^2}{2dz} \right) - \int_{\hat{v}_{14}}^{\hat{v}_{34}} \text{Re} \left( (1 - z^2) \frac{(d\xi_{\lambda s})^2}{2dz} \right). \quad \text{(4.16)}$$

We are now ready to state the existence theorem for $S_{Lh}$. Note that here, unlike in

![Figure 4.5: The map $\xi_{\lambda s}$ on $\mathcal{R}_\delta$](image)

Theorem 3.1, some work must be done to prove the existence of the vertex-preserving conformal map $\xi_{\lambda s}$. 
Theorem 4.1 Given $1 < \delta < 1 + \sqrt{2}$, there exists a map $\xi_{\lambda s}$ as in (4.13), and $X^{\lambda s}(R_\delta)$ extends to an immersed soap film $S_{Lh}$ spanning $P_{Lh}$, where $L = -2X^{\lambda s}_2(\hat{v}_{23})$ and $h = -2X^{\lambda s}_3(\hat{v}_{34})$. Furthermore, $S_{Lh}$ consists of four planar sheets, four nonplanar sheets, four Y-singular curves, and has no T-singularities.

proof: Let $B_{\lambda s}$ be the edge of $Q_{\lambda s}$ connecting the vertices $\lambda$ and $\lambda + i(\lambda + s)$, and let $D_{\lambda s}$ be the edge connecting 0 and $i s$. For $1 < \delta < 1 + \sqrt{2}$, choose $\lambda, s$ so that

$$\text{Ext}_{R_\delta}(\hat{a}_2, \hat{a}_4) = \text{Ext}_{Q_{\lambda s}}(B_{\lambda s}, D_{\lambda s})$$

(This can be done since, for fixed $\lambda + s$, $\text{Ext}_{Q_{\lambda s}}(B_{\lambda s}, D_{\lambda s})$ increases from 0 to $\infty$ as $\lambda$ increases from 0 to $\lambda + s$). Thus, by Proposition 2.3, there is a map $\xi_{\lambda s}$ from $R_\delta$ onto $Q_{\lambda s}$ as in (4.13).

It is clear that $X^{\lambda s}$ is a conformal minimal immersion. To show $X^{\lambda s}(R_\delta)$ extends to an immersed soap film spanning $P_{Lh}$, we analyze $X^{\lambda s}$ on $\partial R_\delta$. First of all, notice that by the Schwarz reflection principle for holomorphic functions, the map $\xi_{\lambda s}$, and hence, the immersion $X^{\lambda s}$, can be extended across each edge $\hat{a}_m$. For the vertices, we again use the Schwarz reflection principle to extend $X^{\lambda s}$ to an entire neighborhood of $\hat{v}_{12}$ and $\hat{v}_{14}$. Using [Car54], we have

$$\frac{d\xi_{\lambda s}(z)^2}{dz^2} \sim (z - \hat{v}_{34})^{-1/2},$$

at $\hat{v}_{34}$, while

$$\frac{d\xi_{\lambda s}(z)^2}{dz^2} \sim (z - \hat{v}_{23})^{\alpha},$$

at $\hat{v}_{23}$, where

$$\alpha = \frac{\cos^{-1}(1/\sqrt{3})}{\cos^{-1}(\sqrt{2}/\sqrt{3})} - 1 > 0.$$ 

Therefore, the boundary behavior of $\xi_{\lambda s}$ implies that the one-form $(d\xi_{\lambda s})^2/dz$ is integrable on $\overline{R_\delta}$. Hence, it follows that

$$X^{\lambda s} \text{ is continuous on } \overline{R_\delta}$$

(4.17)
and, as desired, the angles on the surface are given by

\[ \phi_{12} = \phi_{14} = \phi_{34} = \pi/2 \text{ and } \phi_{23} = \cos^{-1}(1/\sqrt{3}). \]  

(4.18)

Continuing our analysis of the boundary, if we parametrize \(\text{interior}(\hat{a}_1)\) by \(z_1(t) = e^{it}\), then a calculation as in (4.9) shows

\[ dX_2^{\lambda s}(\hat{z}_1) > 0. \]  

(4.19)

Additionally, we compute

\[ dX_1^{\lambda s}(\hat{z}_1) = \text{Re} \left( \left(1 - e^{2it}\frac{d\xi_{\lambda s}(\hat{z}_1)^2}{2ie^{it}} \right) \right) = -d\xi_{\lambda s}(\hat{z}_1)^2 \sin(t) > 0 \]  

(4.20)

and

\[ dX_3^{\lambda s}(\hat{z}_1) = \text{Re} \left( e^{-it}\frac{d\xi_{\lambda s}(\hat{z}_1)^2}{ie^{it}} \right) = 0. \]  

(4.21)

For \(a_2\), let \(z_2(t) = 2e^{it/4} + \sqrt{3}e^{it}\) be a parametrization of \(\text{interior}(\hat{a}_2)\) from \(\hat{v}_{12}\) to \(\hat{v}_{23}\). Here, \(t_1 < t < t_2\), where \(t_1, t_2\) are such that

\[ \cos(t_1) = \frac{1 - \sqrt{3}}{2\sqrt{2}}, \text{ sin}(t_1) = -\frac{1 + \sqrt{3}}{2\sqrt{2}}, \cos(t_2) = \frac{1}{\sqrt{3}}, \text{ sin}(t_2) = -\frac{\sqrt{2}}{\sqrt{3}}, \]

so that the functions

\[ b_1(t) = \sqrt{6} + 2(\text{sin}(t) + \text{cos}(t)), \quad b_2(t) = \text{cos}(t) - \text{sin}(t) \]

(encountered below) are positive on the interval \((t_1, t_2)\). So, using

\[ dz(\hat{z}_2) = i\sqrt{3}e^{it} \quad \text{and} \quad d\xi_{\lambda s}(\hat{z}_2)^2 < 0, \]

we calculate

\[ dX_1^{\lambda s}(\hat{z}_2) = -dX_2^{\lambda s}(\hat{z}_2) = -\frac{1}{\sqrt{3}}d\xi_{\lambda s}(\hat{z}_2)^2(\sqrt{6} + 2(\text{sin}(t) + \text{cos}(t))) > 0 \]  

(4.22)

and

\[ dX_3^{\lambda s}(\hat{z}_2) = \frac{\sqrt{2}}{\sqrt{3}}d\xi_{\lambda s}(\hat{z}_2)^2(\text{cos}(t) - \text{sin}(t)) < 0. \]  

(4.23)
Next, parametrize $\text{interior}(\hat{a}_3)$ by $z_3(t) = -t$ and compute as in (4.11) to find

$$dX_2^{\lambda s}(\hat{z}_3) > 0.$$  \hfill (4.24)

Continuing, we also have

$$dX_1^{\lambda s}(\hat{z}_3) = \text{Re} \left( -(1 - t^2) \frac{d\xi_{\lambda s}(\hat{z}_3)^2}{2} \right) = -\frac{1}{2t} d\xi_{\lambda s}(\hat{z}_3)^2 \text{Re}(i(1 - t^2)) = 0,$$  \hfill (4.25)

and

$$dX_3^{\lambda s}(\hat{z}_3) = \text{Re} \left( t \frac{d\xi_{\lambda s}(\hat{z}_3)^2}{2} \right) = \frac{1}{2t} d\xi_{\lambda s}(\hat{z}_3)^2 \text{Re}(it) = 0.$$  \hfill (4.26)

Finally, parametrize $\text{interior}(\hat{a}_4)$ from $\hat{v}_{14}$ to $\hat{v}_{34}$ by

$$z_4(t) = t,$$

so that

$$dz(\hat{z}_4) \equiv 1$$

and

$$d\xi_{\lambda s}(\hat{z}_4)^2 < 0.$$  

Calculating, we find

$$dX_1^{\lambda s}(\hat{z}_4) = \text{Re} \left( (1 - t^2) \frac{d\xi_{\lambda s}(\hat{z}_4)^2}{2} \right) = \frac{1}{2} d\xi_{\lambda s}(\hat{z}_4)^2(1 - t^2) > 0$$  \hfill (4.27)

(since $t^2 > 1$),

$$dX_2^{\lambda s}(\hat{z}_4) = \text{Re} \left( i(1 + t^2) \frac{d\xi_{\lambda s}(\hat{z}_4)^2}{2} \right) = 0,$$  \hfill (4.28)

and

$$dX_3^{\lambda s}(\hat{z}_4) = \frac{t}{2} d\xi_{\lambda s}(\hat{z}_4)^2 < 0.$$  \hfill (4.29)

As $K_3^{\lambda s} = 0$, statements (4.19) – (4.21) imply

$$a_1 \subset x_1 x_2 \text{ plane, and } X_1^{\lambda s}(z_1), X_2^{\lambda s}(z_1) \text{ are increasing.}$$  \hfill (4.30)

Furthermore, since $g(z) = z$ and $\hat{a}_1 \subset \partial B(0, 1)$, it follows that

$$X^{\lambda s}(R_5) \text{ meets the } x_1 x_2 \text{ plane along } a_1 \text{ at a constant angle of } \pi/2.$$  \hfill (4.31)
Similarly, statements (4.27) – (4.29) combined with the fact that $\mathcal{K}_2^{\lambda s} = 0$ imply
\[ a_4 \subset x_1x_3 \text{ plane, } X_1^{\lambda s}(z_4) \text{ is increasing, and } X_3^{\lambda s}(z_4) \text{ is decreasing.} \quad (4.32) \]

Adding that $g(z) = z$ and $\hat{a}_4 \subset \mathbb{R} \times \{0\}$, we have
\[ X^{\lambda s}(\mathcal{R}_4) \text{ meets the } x_1x_3 \text{ plane along } a_4 \text{ at a constant angle of } \pi/2. \quad (4.33) \]

To describe the edge $a_2$, statements (4.22) and (4.23) imply
\[ X_1^{\lambda s}(z_2) = -X_2^{\lambda s}(z_2) \text{ is increasing, and } X_3^{\lambda s}(z_2) \text{ is decreasing.} \quad (4.34) \]

To show $a_2 \subset E_2$, we use (4.16) to compute
\[
X_1^{\lambda s}(\hat{v}_{23}) = \int_{\hat{v}_{14}}^{\hat{v}_{34}} \text{Re} \left( (1 - z^2) \frac{(d\xi_{\lambda s})^2}{2dz} \right) + \int_{\hat{v}_{34}}^{\hat{v}_{23}} \text{Re} \left( (1 - z^2) \frac{(d\xi_{\lambda s})^2}{2dz} \right) \\
+ \int_{\hat{v}_{23}}^{\hat{v}_{34}} \text{Re} \left( i(1 + z^2) \frac{(d\xi_{\lambda s})^2}{2dz} \right) - \int_{\hat{v}_{14}}^{\hat{v}_{34}} \text{Re} \left( (1 - z^2) \frac{(d\xi_{\lambda s})^2}{2dz} \right).
\]

From (4.25), the second integral is zero, so that
\[ X_1^{\lambda s}(\hat{v}_{23}) = \int_{\hat{v}_{23}}^{\hat{v}_{34}} \text{Re} \left( i(1 + z^2) \frac{(d\xi_{\lambda s})^2}{2dz} \right). \]

Then, by (4.28) and the fact that $\mathcal{K}_2^{\lambda s} = 0$, it follows
\[ X_2^{\lambda s}(\hat{v}_{23}) = \int_{\hat{v}_{23}}^{\hat{v}_{34}} \text{Re} \left( i(1 + z^2) \frac{(d\xi_{\lambda s})^2}{2dz} \right) = -X_1^{\lambda s}(\hat{v}_{23}) \]
so that
\[ a_2 \subset E_2. \quad (4.35) \]

Lastly, for $a_3$, statements (4.24) – (4.26) imply
\[ X^{\lambda s} \text{ maps } \hat{a}_3 \text{ bijectively onto a line segment in the direction } (0, 1, 0). \quad (4.36) \]

We now extend $X^{\lambda s}(\mathcal{R}_3)$ to a set $S_{\lambda h}$ spanning $P_{\lambda h}$. First, using (4.31) and the Schwarz reflection principle for minimal surfaces, we extend $X^{\lambda s}(\mathcal{R}_3)$ by reflection
through the $x_1x_2$ plane (call this extended surface $2X^{ls}(\mathcal{R}_\delta)$). As Lemma 2.1 implies this reflection corresponds to extending $X^{ls}$ across $\hat{a}_1$, it follows that $a_2$ and $a_4$ extend smoothly across $v_{12}$ and $v_{14}$ to curves $2a_2$ and $2a_4$, respectively. Similarly, we can extend $2X^{ls}(\mathcal{R}_\delta)$ by reflection through the $x_1x_3$ plane to a minimal surface $4X^{ls}(\mathcal{R}_\delta)$. To obtain the other three nonplanar sheets, reflect $4X^{ls}(\mathcal{R}_\delta)$ through the $x_2x_3$ plane and each of the planes $E_1$ and $E_2$ to obtain a set $16X^{ls}(\mathcal{R}_\delta)$. Finally, note that $2a_2$ together with the vertical edge of length $h$ connecting the two endpoints of $2a_2$ bound a disk $F$ in $E_2$. Thus, by adding to $16X^{ls}(\mathcal{R}_\delta)$ the disk $F$ and its images under the above rigid motions, we obtain a set $S_{Lh}$ spanning $P_{Lh}$ consisting of four planar sheets, four nonplanar sheets, and no T-singularities.

To finish the proof, we need to show $a_2$ is a Y-singular curve. So, given $p \in \hat{a}_2$, let $c_1, c_2, c_3 : [a, b] \to \mathbf{B}^3, c_m(a) = X^{ls}(p)$, be the three curves meeting at $X^{ls}(p)$ obtained by intersecting $S_{Lh}$ with the plane $C_p$ passing through $X^{ls}(p)$ and perpendicular to the tangent line to $a_2$ at $X^{ls}(p)$, where

$$c_1((a, b)) \subset F, \quad c_2((a, b)) \subset X^{ls}(\mathcal{R}_\delta),$$

and

$$c_3((a, b)) \text{ is the image of } c_2((a, b)) \text{ under reflection through } E_2.$$
However, from (4.30), (4.32), (4.34), and (4.36) we have that

\[ X_1^{\lambda s} + X_2^{\lambda s} \geq 0 \text{ on } \partial R_\delta. \]

Hence, by the maximum principle,

\[ X_1^{\lambda s}(z) + X_2^{\lambda s}(z) \geq 0, \]

a contradiction. Therefore, it follows

\[ \alpha = \frac{2\pi}{3}, \]

and thus, \( \alpha_2 \) is a \( Y \)-singular curve. Q.E.D.
Chapter 5

Capillary Surfaces

Using the methods of chapters 3 and 4, we construct a one parameter family \( \{ S_{\gamma_1} \} \), \( \pi/4 < \gamma_1 < \pi/2 \), of capillary graphs over a square such that \( S_{\gamma_1} \) has alternating contact angles \( \gamma_1 \) and \( \pi - \gamma_1 \). The graph \( S_{\gamma_1} \) will be an example of a capillary surface over a wedge domain (the wedge is formed by two adjacent sides of the square) with wedge angle \( \alpha = \pi/2 \) and contact angle data \( (\pi - \gamma_1, \gamma_1) \in \text{interior}(R) \) such that the graphing function \( u_{\gamma_1} \) is not \( C^2 \) at the corners.

![Figure 5.1: A sketch of \( S_{\gamma_1} \)](attachment:figure51.png)

Using Figure 5.1, we assume \( S_{\gamma_1} \) is a graph over the square \([-L/2, L/2]^2\) for some \( L > 0 \). Also from the sketch, we see that \( S_{\gamma_1} \) is symmetric with respect to reflections...
through the $x_1 x_3$ and $x_2 x_3$ planes and with respect to rotations of 180° about the lines $x_2 = \pm x_1$ in the $x_1 x_2$ plane. Taking the quotient of $S_{\gamma_1}$ by these symmetries, we obtain a minimal surface $M_{\gamma_1}$ bounded by a simple closed curve $\Gamma = a_1 \cup a_2 \cup a_3$ (see Figure 5.2), where

(i) $a_1 \subset x_1 x_3$ plane, and $M_{\gamma_1}$ meets the $x_1 x_3$ plane orthogonally at $a_1$,

(ii) along $a_2 \subset \{(x_1, x_2, x_3) \mid x_1 = L/2\}$, the downward pointing normal $N$ makes a constant angle of $\gamma_1$ with $(1, 0, 0)$,

and

(iii) $a_3$ is a line segment in the direction $(1, -1, 0)$.

Assuming $N$ is 1-1 on $M_{\gamma_1}$, statements (i) - (iii) imply $\sigma \circ N$ maps $M_{\gamma_1}$ conformally onto a curvilinear triangle with sides

\[ \hat{a}_1 \subset \mathbb{R}, \quad (5.1) \]

\[ \hat{a}_2 \subset \partial B(\sec \gamma_1, \tan \gamma_1), \quad (5.2) \]

and

\[ \hat{a}_3 \subset \{z \mid Re(z) = Im(z)\}. \quad (5.3) \]

Figure 5.2: A sketch of $M_{\gamma_1}$

Upon closer inspection of the sketch of $M_{\gamma_1}$, we conclude further that $\sigma \circ N(M_{\gamma_1})$ is
the bounded triangle $\mathcal{R}_{\gamma_1}$ with vertices

\[ \hat{v}_{12} = \sec \gamma_1 - \tan \gamma_1, \quad \hat{v}_{13} = 0, \quad \hat{v}_{23} = \left( \frac{\sec \gamma_1 - \sqrt{\sec^2 \gamma_1 - 2}}{\sqrt{2}} \right) e^{i\pi/4} \]  \hspace{1cm} (5.4)

(see Figure 5.3).

![Diagram](image)

Figure 5.3: The domain $\mathcal{R}_{\gamma_1}$

Along with Proposition 2.1, statements (i) and (ii) imply

\[ a_1 \text{ and } a_2 \text{ are lines of curvature}, \]

while from statement (iii) we have

\[ a_3 \text{ is asymptotic}, \]

so that from Proposition 2.2 we expect $\zeta = f \sqrt{dgdh/g}$ to map $M_{\gamma_1}$ conformally onto the interior of a Euclidean triangle with sides $\tilde{a}_m$ such that

\[ \text{each of } \tilde{a}_1, \tilde{a}_2 \text{ is horizontal or vertical} \]  \hspace{1cm} (5.5)

and

\[ \tilde{a}_3 \text{ is in the direction } e^{i\pi/4} \text{ or } e^{i3\pi/4}. \]  \hspace{1cm} (5.6)
In particular, the triangle $\zeta(M_{11})$ has angles

$$\tilde{\phi}_{12} = \frac{\pi}{2} \quad \text{and} \quad \tilde{\phi}_{23} = \tilde{\phi}_{13} = \frac{\pi}{4}. \quad (5.7)$$

To determine the orientation of $\zeta(M_{11})$, note from the sketch that as we move along $a_1$ from $v_{13}$ towards $v_{12}$, the $x_1$ coordinate appears to increase. Parametrizing $\tilde{a}_1$ from $\tilde{v}_{13}$ to $\tilde{v}_{12}$ by

$$z_1(t) = t, \quad 0 < t < \sec \gamma_1 - \tan \gamma_1 \quad (5.8)$$

we assume

$$\tilde{a}_1 \text{ is horizontal.}$$

Then

$$dg(\tilde{z}_1) \equiv 1 \quad \text{and} \quad d\zeta(\tilde{z}_1)^2 > 0,$$

so that, since $t < 1$, it follows that

$$d(\sigma \circ N)^{-1}_1(\tilde{z}_1) = Re \left( (1 - z_1^2) \frac{d\zeta(\tilde{z}_1)^2}{dg(\tilde{z}_1)} \right) = \frac{1}{2} d\zeta(\tilde{z}_1)^2 (1 - t^2) > 0. \quad (5.9)$$

Hence, the function $(\sigma \circ N)^{-1}_1(z_1)$ is increasing as desired, supporting the assumption that $\tilde{a}_1$ is horizontal. As

$$\tilde{\phi}_{12} = \frac{\pi}{2},$$

it follows immediately that

$$\tilde{a}_2 \text{ is vertical.}$$

To determine the direction of $\tilde{a}_3$, observe that as we move along $a_3$ from $v_{13}$ towards $v_{23}$, the $x_1$ coordinate is increasing. Thus, parametrizing $\tilde{a}_3$ from $\tilde{v}_{13}$ to $\tilde{v}_{23}$ by

$$z_3(t) = t + it \quad (5.10)$$

suppose

$$\tilde{a}_3 \text{ is in the direction } e^{i\pi/4}. $$
Then
\[ dg(\tilde{z}_3) \equiv 1 + i \text{ and } \frac{1}{i} d\zeta(\tilde{z}_3)^2 > 0, \]
so that, since \( t > 0 \), we compute that
\[
d(\sigma \circ N)^{-1}_1(\tilde{z}_3) = \text{Re} \left( (1 - z_3^2) \frac{d\zeta(\tilde{z}_3)^2}{2dg(\tilde{z}_3)} \right) = \frac{1}{i4} d\zeta(\tilde{z}_3)^2(1 + 2t) > 0. \tag{5.11}\]

Hence, the function \((\sigma \circ N)^{-1}_1(\tilde{z}_3)\) is increasing, justifying the assumption that \( \tilde{a}_3 \) is in the direction \( e^{i\pi/4} \). So, choosing \( \tilde{v}_{13} \) to be the base point of integration, it follows that, for some \( \lambda > 0 \), the function \( \zeta \) maps \( M_{\gamma_1} \) conformally onto one of the Euclidean triangles \( \pm \Delta_\lambda \), where \( \Delta_\lambda \) has vertices \( \tilde{v}_{mn} \) given by
\[
\tilde{v}_{13} = 0, \quad \tilde{v}_{12} = \lambda, \quad \text{and} \quad \tilde{v}_{23} = \lambda + i\lambda. \tag{5.12}\]

Thus, we conclude that for some \( \lambda > 0 \), the minimal surface \( M_{\gamma_1} \) can be parametrized on \( R_{\gamma_1} \) by
\[
X^\lambda(z) = \int_{\tilde{v}_{13}}^z \text{Re} \left( (1 - z^2, i(1 + z^2), 2z) \frac{(d\xi_\lambda)^2}{2dz} \right), \tag{5.13}\]
where \( \xi_\lambda \) maps \( R_{\gamma_1} \) conformally onto \( \Delta_\lambda \) in such a way that \( \tilde{v}_{mn} = \xi_\lambda(\tilde{v}_{mn}) \) (see Figure 5.4).

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\hat{a}_2
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\tilde{a}_1
\
\tilde{a}_2
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\lambda + i\lambda
\
\lambda
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\tilde{a}_1
\
\tilde{a}_2
\end{array}
\end{array}\]

Figure 5.4: The map \( \xi_\lambda \) on \( R_{\gamma_1} \).

We are now ready to state the following existence theorem.
Theorem 5.1 The surface $X^\lambda(\mathcal{R}_{\gamma_1})$ extends to a minimal graph $S_{\gamma_1}$ over the square $[-L/2, L/2]^2$, where $L = 2X^\lambda_1(\hat{v}_{12})$. The graph $S_{\gamma_1}$ has alternating contact angles $\pi - \gamma_1$ and $\gamma_1$, and the graphing function $u_{\gamma_1}$ is not $C^2$ at the corners of the square.

proof: It follows immediately that $X^\lambda$ is a conformal minimal immersion. By examining $X^\lambda$ on $\partial\mathcal{R}_{\gamma_1}$, we show that $X^\lambda(\mathcal{R}_{\gamma_1})$ extends to a minimal graph over $[-L/2, L/2]^2$. To begin with, the Schwarz reflection principle for holomorphic functions can be used to extend $\xi_{\lambda}$, and hence, the immersion $X^\lambda$, across each edge $\hat{a}_m$. For the vertices, first note that

$$\hat{\phi}_{12} = \frac{\pi}{2}, \quad \hat{\phi}_{13} = \frac{\pi}{4}, \quad \text{and} \quad \hat{\phi}_{23} = \cos^{-1}(\frac{1}{\sqrt{2}} \csc \gamma_1). \quad (5.14)$$

Thus, from (5.7) and (5.14), we see that additional applications of the Schwarz reflection principle extend $X^\lambda$ to a neighborhood of $\hat{v}_{12}$ and a neighborhood of $\hat{v}_{14}$. In a neighborhood $\mathcal{U} \subset \mathcal{R}_{\gamma_1}$ of $\hat{v}_{23}$, it follows from [Car54] that

$$\frac{(d\xi_{\lambda})^2}{dz^2} \sim (z - \hat{v}_{23})^\alpha,$$

where

$$\alpha = \frac{\pi/2 - 2 \cos^{-1}(\frac{1}{\sqrt{2}} \csc \gamma_1)}{\cos^{-1}(\frac{1}{\sqrt{2}} \csc \gamma_1)} > 0.$$ 

Therefore, the boundary behavior of $\xi_{\lambda}$ implies that the one-form $(d\xi_{\lambda})^2/dz$ is integrable on $\mathcal{R}_{\gamma_1}$ (In addition, we have that $d\xi_{\lambda}$ has a zero at $\hat{v}_{23}$, a fact that will be exploited later in the proof). Hence, it follows that

$$X^\lambda \text{ is continuous on } \overline{\mathcal{R}_{\gamma_1}}. \quad (5.15)$$

Proceeding with our analysis of $X^\lambda$ on $\partial\mathcal{R}_{\gamma_1}$, parametrize $\text{interior}(\hat{a}_1)$ from $\hat{v}_{13}$ to $\hat{v}_{12}$ by $z_1(t) = t$. Then $dz(z_1) \equiv 1$ and $d\xi_{\lambda}(z_1)^2 > 0$, so that a calculation as in (5.9) shows

$$dX^\lambda_1(z_1) > 0. \quad (5.16)$$
Furthermore, we have

$$dX_2^\lambda(\hat{z}_1) = \Re \left( i(1 + t^2) \frac{d\xi_d(\hat{z}_1)^2}{2} \right) = 0. \quad (5.17)$$

Next, let $z_2(t) = \sec \gamma_1 + (\tan \gamma_1)e^{it}$, $t_0 < t < \pi$ parametrize interior($\hat{a}_2$) from $\hat{v}_{23}$ to $\hat{v}_{12}$, where

$$\cos(t_0) = -\frac{1}{2} \csc \gamma_1 \left( 1 + \sqrt{-\cos(2\gamma_1)} \right) \quad \text{and} \quad \sin(t_0) = \frac{1}{2} \csc \gamma_1 \left( 1 - \sqrt{-\cos(2\gamma_1)} \right).$$

Then $dz(\hat{z}_2) = i(\tan \gamma_1)e^{it}$ and $d\xi_d(\hat{z}_2)^2 < 0$, so that

$$dX_1^\lambda(\hat{z}_2) = \frac{1}{2} d\xi_d(\hat{z}_2)^2 \Re \left( ie^{-it}(\tan \gamma_1 + 2 \sec \gamma_1 e^{it} + (\tan \gamma_1)e^{2it}) \right) = 0. \quad (5.18)$$

Continuing, we have

$$dX_2^\lambda(\hat{z}_2) = \csc \gamma_1 d\xi_d(\hat{z}_2)^2(\tan \gamma_1 + \sec \gamma_1 \cos(t)).$$

Note here that $f(t) = \tan \gamma_1 + \sec \gamma_1 \cos(t)$ is decreasing on $(t_0, \pi)$ and

$$f(t_0) = -\frac{\cos(2\gamma_1) + \sqrt{-\cos(2\gamma_1)}}{2 \sin \gamma_1 \cos \gamma_1} < 0,$$

so that

$$f(t) < 0 \text{ on } (t_0, \pi).$$

Thus, it follows that

$$dX_2^\lambda(\hat{z}_2) > 0. \quad (5.19)$$

Finally, parametrize interior($\hat{a}_3$) from $\hat{v}_{13}$ to $\hat{v}_{23}$ by $z_3(t) = t + it$. Then $dz(\hat{z}_3) = 1 + i$ and $\frac{1}{t}d\xi_d(\hat{z}_3)^2 > 0$, so that

$$dX_1^\lambda(\hat{z}_3) = -dX_2^\lambda(\hat{z}_3) = \frac{1}{4i} d\xi_d(\hat{z}_3)^2(1 + 2t^2) > 0 \quad (5.20)$$

and

$$dX_3^\lambda(\hat{z}_3) = \Re \left( (t + it) \frac{d\xi_d(\hat{z}_3)^2}{1 + i} \right) = \Re(t d\xi_d(\hat{z}_3)^2) = 0. \quad (5.21)$$
Putting it all together, we have from equations (5.15) - (5.21) that \( X^\lambda(\partial R_{\gamma_1}) \) is a simple closed curve \( \Gamma = a_1 \cup a_2 \cup a_3 \), where \( a_m = X^\lambda(\hat{a}_m) \), such that, under projection onto the \( x_1x_2 \) plane, the curve \( \Gamma \) is mapped bijectively onto the boundary of a Euclidean triangle with vertices

\[
0, \ (L/2, 0, 0), \ \text{and} \ (L/2, -L/2, 0).
\]

So, from Theorem 2.3, it follows that \( X(R_{\gamma_1}) \) is a graph over this triangle. Now, \( a_1 \) is a planar curve of symmetry and \( a_3 \) is a straight line segment. Therefore, we can use the Schwarz reflection principle for minimal surfaces to extend \( X^\lambda(R_{\gamma_1}) \) to a minimal graph over the square \([-L/2, L/2]^2\]. By construction, this graph has contact angle data \((\pi - \gamma_1, \gamma_1)\).

Denote the graphing function by \( u_{\gamma_1} \). To show that \( u_{\gamma_1} \) is not \( C^2 \) in the corners, note that

\[
u_{\gamma_1} \left( X_1^\lambda, X_2^\lambda \right) = X_3^\lambda.
\] (5.22)

Calculating as in (5.18) and (5.19), we have

\[
d X_3^\lambda(\hat{z}_2) = -\csc \gamma_1 d \xi_\lambda(\hat{z}_2)^2 \sin(t).
\] (5.23)

Differentiating (5.22), it follows that

\[
\frac{d}{dy} u_{\gamma_1} \left( X_1^\lambda(z_2(t)), X_2^\lambda(z_2(t)) \right) = \frac{d X_3^\lambda(\hat{z}_2)}{d X_2^\lambda(\hat{z}_2)} = -\frac{\sin(t)}{\tan \gamma_1 + \sec \gamma_1 \cos(t)}.
\] (5.24)

Hence, we differentiate (5.24) to obtain

\[
\frac{d^2}{dy^2} u_{\gamma_1} \left( X_1^\lambda(z_2(t)), X_2^\lambda(z_2(t)) \right) = -\frac{\sin \gamma_1}{d \xi_\lambda(\hat{z}_2)^2} \frac{\tan \gamma_1 \cos(t) + \sec \gamma_1}{(\tan \gamma_1 + \sec \gamma_1 \cos(t))^3}.
\] (5.25)

Now, as \( t \to t_0 \), the pair of functions \( \left( X_1^\lambda(z_2(t)), X_2^\lambda(z_2(t)) \right) \) approaches the corner \((L/2, -L/2)\) and the function \( z_2(t) \) approaches \( \hat{v}_{23} \). Hence, since \( d \xi_\lambda \) has a zero at \( \hat{v}_{23} \), equation (5.25) shows that

\[
\left| \frac{d^2}{dy^2} u_{\gamma_1} \left( X_1^\lambda(z_2(t)), X_2^\lambda(z_2(t)) \right) \right| \to \infty \text{ as } t \to t_0.
\]
Therefore, we conclude that $u_{\gamma_1}$ is not $C^2$ at the corner, finishing the proof of the theorem. Q.E.D.

For $0 < \gamma_2 < \pi/4$, the graph $S_{\gamma_2}$ has contact angle data $(\pi - \gamma_2, \gamma_2) \in D^2_+ \cup D^2_2$. Here, we would like $S_{\gamma_2}$ to have a jump discontinuity at the corners. A sketch with this desired discontinuity is produced from $S_{\gamma_1}$ by replacing $v_{23}$ in $M_{\gamma_1}$ with a vertical line segment (see Figure 5.5). Now, we know that

$$\hat{a}_3 \subset \partial B(0,1),$$

(5.26)

so that we expect the image of $M_{\gamma_2}$ under $\sigma \circ N$ to be the curvilinear quadrilateral formed by replacing $\hat{v}_{23}$ in $R_{\gamma_1}$ with an arc of $\partial B(0,1)$ (see Figure 5.6). The two endpoints of this arc are

$$\hat{v}_{23} = e^{i\gamma_2} \text{ and } \hat{v}_{34} = e^{i\pi/4}. \quad (5.27)$$

As $a_3$ is an asymptotic curve, we expect $\zeta(M_{\gamma_2})$ to be formed by replacing $\hat{v}_{23}$ in $\zeta(M_{\gamma_1})$ with a line segment in the direction $e^{i3\pi/4}$. So, we conclude that for some $\lambda, s$ such that $0 < s < \lambda$, we have

$$\zeta = \pm \xi_{\lambda s},$$

where $\xi_{\lambda s}$ maps $R_{\gamma_2}$ conformally onto the Euclidean quadrilateral $Q_{\lambda s}$ with vertices

$$\bar{v}_{12} = \lambda, \; \bar{v}_{14} = 0, \; \bar{v}_{23} = \lambda + is, \text{ and } \bar{v}_{34} = \frac{\lambda + s}{\sqrt{2}} e^{i\pi/4} \quad (5.28)$$
Figure 5.6: The domain $\mathcal{R}_{\gamma_2}$

(see Figure 5.7). Hence, our expectation is that $M_{\gamma_2}$ can be parametrized on $\mathcal{R}_{\gamma_2}$ by

Figure 5.7: The map $\xi_{\lambda s}$ on $\mathcal{R}_{\gamma_2}$

$$X^{\lambda s}(z) = \int_{\mathcal{R}_{\gamma_2}} \text{Re} \left( (1 - z^2, i(1 + z^2), 2z) \frac{(d\xi_{\lambda s})^2}{2dz} \right), \quad (5.29)$$

and so we state the following theorem. Note that in this case some work must be done to establish the existence of the vertex-preserving conformal map $\xi_{\lambda s}$.

**Theorem 5.2** Given $0 < \gamma_2 < \pi/4$, a map $\xi_{\lambda s}$ as in (5.29) exists, and the surface $X^{\lambda s}(\mathcal{R}_{\gamma_2})$ extends to a minimal graph $S_{\gamma_2}$ over the square $[-L/2, L/2]^2$, where $L =$
2X^{\lambda s}(\hat{v}_{12}). The graph \( S_{\gamma_2} \) has alternating contact angles \( \pi - \gamma_2 \) and \( \gamma_2 \), and the graphing function \( u_{\gamma_2} \) has a finite jump discontinuity at each corner.

**proof:** Let \( B_{\lambda s} \) be the side of \( Q_{\lambda s} \) joining the vertices \( \lambda \) and \( \lambda + is \), and let \( D_{\lambda s} \) be the side joining 0 and \( \frac{\lambda + s}{\sqrt{2}} e^{i\pi/4} \). Given \( 0 < \gamma_2 < \pi/4 \), choose \( \lambda, s \) so that

\[
\text{Ext}_{R_{\gamma_2}}(\hat{a}_2, \hat{a}_4) = \text{Ext}_{Q_{\lambda s}}(B_{\lambda s}, D_{\lambda s}).
\]

(This can be done since, for fixed \( \lambda \), the function \( \text{Ext}_{Q_{\lambda s}}(B_{\lambda s}, D_{\lambda s}) \) increases from 0 to \( \infty \) as \( s \) decreases from \( \lambda \) to 0.) Thus, by Proposition 2.3, there exists a map \( \xi_{\lambda s} \) from \( R_{\gamma_2} \) onto \( Q_{\lambda s} \) as in (5.29).

Arguing as in the proof of Theorem 5.1, we know that \( X^{\lambda s} \) is a conformal minimal immersion on \( \overline{R}_{\gamma_2} - \{ \hat{v}_{23}, \hat{v}_{34} \} \). The fact that

\[
\hat{\phi}_{34} = \bar{\phi}_{34} = \frac{\pi}{2}
\]

implies \( X^{\lambda s} \) can be extended to a neighborhood of \( \hat{v}_{34} \). At \( \hat{v}_{23} \), the fact that

\[
\hat{\phi}_{23} = \frac{\pi}{2} \quad \text{and} \quad \bar{\phi}_{23} = \frac{3\pi}{4}
\]

implies

\[
\frac{(d\xi_{\lambda s})^2}{dz^2} \sim (z - \hat{v}_{23}).
\]

Therefore, the one-form \( (d\xi_{\lambda s})^2/dz \) is integrable on \( \overline{R}_{\gamma_2} \), and thus, we have that

\[
X^{\lambda s} \text{ is continuous on } \overline{R}_{\gamma_2}.
\]

(5.30)

The behavior of \( X^{\lambda s} \) on \( \hat{a}_1, \hat{a}_2, \) and \( \hat{a}_4 \) is the same as the behavior of \( X^\lambda \) (from Theorem 5.1) on \( \hat{a}_1, \hat{a}_2, \) and \( \hat{a}_3 \), respectively. For \( a_3 \), parametrize \( \text{interior}(\hat{a}_3) \) from \( \hat{v}_{23} \) to \( \hat{v}_{34} \) by

\[
z_3(t) = e^{it}.
\]
Then \(dz(\hat{z}_3) = ie^{it}\) and \(\frac{1}{i}d\xi_{\lambda^s}(\hat{z}_3)^2 < 0\), so that

\[
dX_{1\lambda^s}(\hat{z}_3) = \text{Re} \left( 1 - e^{i2t} \frac{d\xi_{\lambda^s}(\hat{z}_3)^2}{2ie^{it}} \right) = \frac{1}{2t}d\xi_{\lambda^s}(\hat{z}_3)^2 \text{Re}(e^{-it} - e^{it}) = 0, \tag{5.31}
\]

\[
dX_{2\lambda^s}(\hat{z}_3) = \text{Re} \left( i(1 + e^{i2t}) \frac{d\xi_{\lambda^s}(\hat{z}_3)^2}{2ie^{it}} \right) = \frac{1}{2t}d\xi_{\lambda^s}(\hat{z}_3)^2 \text{Re}(i(e^{-it} + e^{it})) = 0, \tag{5.32}
\]

and

\[
dX_{3\lambda^s}(\hat{z}_3) = \text{Re} \left( e^{it} \frac{d\xi_{\lambda^s}(\hat{z}_3)^2}{i e^{it}} \right) = \frac{1}{i}d\xi_{\lambda^s}(\hat{z}_3)^2 < 0. \tag{5.33}
\]

Thus, it follows that \(X^{\lambda^s}\) maps \(\hat{a}_3\) bijectively onto a vertical line segment. It follows that, except for this vertical segment, the boundary curve \(X^{\lambda^s}(\partial R_{\gamma_2})\) projects bijectively onto a Euclidean triangle \(\Delta_L\) with vertices \(0, (L/2, 0, 0)\), and \((L/2, -L/2, 0)\). By Theorem 2.3, the surface \(X^{\lambda^s}(R_{\gamma_2})\) is a graph over \(\Delta_L\), and we use the Schwarz reflection principle for minimal surfaces to extend \(X^{\lambda^s}(R_{\gamma_2})\) to a minimal graph \(S_{\gamma_2}\) over the square \([-L/2, L/2]^2\). As in the proof of Theorem 5.1, the graph \(S_{\gamma_2}\) has alternating contact angles \(\pi - \gamma_2\) and \(\gamma_2\), and as a result of the vertical line segment \(a_3\), it follows that the graphing function \(u_{\gamma_2}\) has a finite jump discontinuity at the corners. Q.E.D.
Bibliography


