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Radiative Transfer Solution with Discrete Wavelets in the Angular Domain

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ABSTRACT

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Radiative heat transfer analysis requires the consideration of seven independent variables, three spatial directions, two angular directions, frequency and time. Consequently, the solution of radiative transfer problems demands specialized numerical methods in order to deal with each of these independent variables. In this study, a new numerical scheme is developed, which employs wavelet analysis in the evaluation of radiative intensity in the angular domain. The formulation of the method for one- and two-dimensional problems is presented, and the accuracy and effectiveness of the method are tested by case studies. The wavelet analysis is implemented in the treatment of transient radiative transfer problems and as a test case, imaging of inhomogeneties within an absorbing, scattering medium exposed to short-pulse laser irradiation is studied.
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CHAPTER 1

INTRODUCTION

Thermal energy transport by radiation is the source of life on earth. Without solar energy, no life, including humankind, would have flourished. Radiative heat transfer has been utilized in the daily life in variety of ways since the ancient times. However, only by the beginning of the twentieth century, have significant steps been taken in understanding the mechanism of radiative transfer. Lord Rayleigh and Sir James Jean attempted to predict the spectrum of the Sun. Later, Wilhelm Wien reported an accurate spectral distribution of blackbody emissive power over the large part of the spectrum. Finally, Max Planck hypothesized the existence of quantized energy states and found the spectral blackbody emissive power distribution, now commonly known as Planck’s law. Thanks to the pioneers of radiation research, the theoretical basis of the radiative heat transfer is now well established [1-4].

In the modern era, radiation has found many engineering applications such as boilers of power generating equipment, fossil fuel-fired industrial furnaces for material processing, high-temperature heat recovery equipment, combustors and rocket engines, hyperbolic propulsion, entry and re-entry vehicle protection. With the advent of short pulse lasers, radiation applications have extended to ocean and atmosphere remote sensing, optical tomography, laser surgery and combustion product characterization [1-5].
1.1 Basic Concepts of Radiation Phenomena

All materials continuously emit and absorb electromagnetic waves, or photons, by virtue of changes in their molecular energy levels. When a photon is emitted from a particle, it travels in a certain direction with a certain frequency at the speed of light.

Before embarking on the analysis of radiative heat transfer, the differences of radiation and the other possible modes of heat transfer should be addressed. In both conduction and convection, a physical medium must be present to carry energy: conduction is achieved by free electrons carrying energy through an atomic lattice, by phonon-phonon interactions in solids or by molecule to molecule collisions in liquids and gases. Convection is the consequence of flow of the molecules of different kinetic energies. On the other hand, a medium does not have to be present between two locations for radiative exchange to occur. This fact makes radiation the only means of thermal energy transfer in vacuum and space applications. The other distinguishing feature of radiation from the other modes of energy transfer is the difference in its temperature dependency. This can be illustrated by heat flux correlations for each of the three the heat transfer modes. For the majority of conduction applications, conductive heat flux is calculated by Fourier's law as

\[ q_x = -k \frac{dT}{dx}, \]  

where \( q_x \) is the heat flux in the \( x \)-direction, \( T \) is the temperature and \( k \) is the thermal conductivity. Similarly, convective heat transfer is usually evaluated from a correlation such as
\[ q = h(T - T_\infty), \quad (2) \]

where \( h \) is the convective heat transfer coefficient and \( T_\infty \) is the reference temperature. As seen from Eq.'s (1) and (2), conduction and convection are linearly proportional to the temperature. However, radiative heat transfer is generally proportional to fourth power of the temperature, i.e.

\[ q \propto T^4 - T^4_\infty. \quad (3) \]

Therefore, the importance of the radiative heat transfer intensifies with rising temperatures and it may completely dominate conduction and convection in high temperature applications.

Another difference between radiation and the other two means of energy transfer emerges when calculating the energy profiles for a given system. Radiation analysis requires the consideration of seven independent variables, three spatial directions, two angular directions, frequency and time, while conduction and convection only include four independent variables, three spatial directions and time.

1.2 Radiative Transfer Equation

An energy balance for the radiative energy traveling in the direction of \( \hat{s} \) at location \( s \) (Fig. 1) can be written as [1]

\[
\frac{1}{c} \frac{\partial I_\eta}{\partial t} + \frac{\partial I_\eta}{\partial s} = \kappa_\eta I_{bn} - \kappa_\eta I_\eta - \sigma_{sn} I_\eta + \frac{\sigma_{sn}}{4\pi} \int I_\eta(\hat{s}_i) \Phi_\eta(\hat{s}_i, \hat{s}) d\Omega_i \quad (4)
\]

In Eq. (4), \( I_\eta \) is the spectral intensity, defined as

\( I_\eta \equiv \text{radiative energy flow / time / area normal to rays / solid angle / wavelength.} \)
In Eq. (4), \( I_{bn} \) is the black body intensity; \( c \) is the speed of light; \( \kappa_\eta \) and \( \sigma_\eta \) are absorption and scattering coefficients of the medium. \( \Omega \) represents the total solid angle. \( \Phi \) is called the scattering phase function and describes the probability of a ray from one direction, \( \hat{s}_1 \), will be scattered into a certain other direction, \( \hat{s} \). In this equation all quantities may vary by location in space, time and frequency while the intensity and phase function also depend on direction \( \hat{s} \) (and \( \hat{s}_1 \)). The subscript \( \eta \) is the wavenumber and stands for frequency dependency. And in the case of the gray medium assumption the intensity is independent of the frequency and therefore the subscript \( \eta \) is dropped. The gray medium assumption will be retained throughout the remainder of this thesis.

The first term on the left side of Eq. (4) is the transient term and might be neglected for most engineering applications since it is inversely proportional to the speed of light (1/c is a very small number). However, as will be seen in Chapter 4, for some of the recent
radiation applications such as short pulse lasers, this term is significant and cannot be ignored. The second term on the left hand side represents change in the intensity in a certain direction. The first and last expressions on the right hand side of Eq. (4) correspond to augmentation of intensity by local emission and in-scattering respectively. The second and third terms on the right hand side correspond to the extinction of the intensity due to local absorption and out-scattering, respectively. Equation 4 is the general mathematical model for the radiative heat transfer phenomenon and is called the Radiative Transfer Equation (RTE), which is often rewritten in terms of nondimensional optical thickness

$$\tau = \int_{0}^{s} (\kappa + \sigma_s) \, ds = \int_{0}^{s} \beta \, ds$$  \hspace{1cm} (5)$$

where $\beta = \kappa + \sigma_s$ is the extinction coefficient, leading to

$$\frac{1}{\beta c} \frac{\partial I}{\partial t} + \frac{\partial I}{\partial \tau} + I = S(\tau, \hat{s})$$ \hspace{1cm} (6)$$

where $S(\tau, \hat{s})$ is the source function and defined as

$$S(\tau, \hat{s}) = (1 - \omega)I_b + \frac{\omega}{4\pi} \int \Phi(\hat{s}_i, \hat{s}) d\Omega_i.$$ \hspace{1cm} (7)$$

In Eq. (7), $\omega$ is the single scattering albedo defined as

$$\omega = \frac{\sigma_s}{\kappa + \sigma_s} = \frac{\sigma_s}{\beta}.$$ \hspace{1cm} (8)$$

The boundary conditions for RTE can be expressed in its most general form as

$$I(t, s_B, \Omega) = \varepsilon(s_B)I_b(s_B) + \frac{\rho(s_B)}{\pi} \int_{\hat{n} \cdot \hat{s}_i < 0} I(s_B, \hat{s}_i) |\hat{n} \cdot \hat{s}_i| d\Omega_i$$ \hspace{1cm} (9)$$
where $\varepsilon$ and $\rho$ are emissivity and reflectivity of surfaces, respectively.

The problem is not completely specified with RTE because blackbody intensity and all the other radiative properties are temperature dependent. In order to complete the definition of the problem, the temperature field of the system either needs to be specified or calculated through the energy equation. If the other modes of heat transfer are present, an iterative procedure is utilized by solving RTE after assuming a temperature profile and updating the heat source terms of nonzero radiation in the full energy equation. If radiation is the dominant mode and radiative equilibrium holds, the energy equation takes the form of

$$\nabla \cdot q = 0$$

(10)

where $q$ is the radiative heat flux. Then the RTE and energy equation can be solved simultaneously.

The analytical solution to the RTE is extremely difficult except for all but very simplified situations. The numerical solution of the RTE is also problematic owing to dependency of radiative intensity to seven parameters (time, 3 space, 2 directional and frequency). Handling the directional dependency especially has been the biggest challenge for numerical solution methods.

1.3 Literature Survey

Various solution techniques have been proposed for radiative transfer problems. These include Monte Carlo [1,2], zonal [1,2], spherical harmonics ($P_N$ approximation) [1,2],
discrete ordinates ($S_N$) [1,2,6-12] and finite-volume [13,14] methods. The Monte Carlo and zonal methods are very accurate for the calculation of radiative heat transfer. However, it is now well established that both methods are computationally intensive and they are difficult to incorporate into other numerical methods for conduction and convection heat transfer [1,2]. $P_N$ methods have received much attention, yet the $P_1$ approximation is inaccurate and the formulation of higher order approximations is complicated. Their implementation leads to important computational times without substantial gain in accuracy [1,2]. Among others, the discrete ordinates method has had the most attention owing to its simple formulation, relatively good accuracy and compatibility with existing computer codes used in the transport processes involved in many convective transport problems. However, discrete ordinates predictions suffer from some shortcomings such as “ray effects”, occurrence of negative intensities during the solution process, and “false scattering” (a numerical phenomenon arising from the chosen spatial discretization scheme) [9]. A basic formulation of the discrete ordinates method and concepts of ray effect and false scattering is given in Appendix A.

In one of the early studies of two-dimensional radiative heat transfer, Fiveland [6] presented some preliminary studies involving the discrete ordinates method with $S_2$, $S_4$ and $S_6$ approximations. Results displaying the presence of ray effects were presented, and the use of a fine spatial mesh in problem areas was suggested as a remedy. In another early attempt to alleviate ray effects, Truelove [8] pointed out the importance of quadrature selection based on half-range moments to improve accuracy of low order discrete ordinate approximations.
Cheong and Song [10] incorporated cubic interpolation into the standard discrete ordinates method (SDO), considering numerical accuracy and grid dependence. They showed that the discrete ordinates interpolation method (DOIM) mitigates the errors introduced by false scattering. Nonetheless, the DOIM does not improve the results in terms of ray effects.

Recently, in order to deal with ray effects, Ramankutty and Crosbie [11] introduced modified discrete ordinates method (MDO), a semi-analytical method. They split the intensity into direct and diffuse components where the direct component is determined analytically, and the diffuse transport equation is solved numerically by conventional discrete ordinates procedure. MDO decreased the anomalies caused by ray effects, yet there can be observed some anomalies for small aspect ratios.

The finite-volume method (FVM) is a discrete ordinates type of method. The FVM can be regarded as the most sophisticated scheme among the currently available schemes. The main advantage of the FVM procedure is that the user has the complete flexibility in laying out the spatial and angular grids that best capture the physics of a given problem. However, there is a great complexity in extending its application to three dimensional enclosures. Ray effects and false scattering encountered in the discrete ordinates method are also encountered with the FVM [13,14].
1.4 Basic Concepts of Wavelet Analysis

In the past decade, the theory of wavelet analysis has been developed and applied to various fields such as signal processing, the solution of partial differential equations [15] and integro-differential equations [16]. Wavelet analysis can be viewed as a multi-resolution analysis which consists of a sequence of successive approximation spaces. Donoho [17] showed that wavelets are unconditional bases for a very wide set of function classes. Particularly, when the functions exhibit localized variations, wavelets provide very good approximations.

In order to construct a wavelet function $\psi$, Daubechies [18] started from the dilation equation for the scaling function $\varphi$,

$$
\varphi(x) = \sum_n h_n \varphi_{-1,n} = \sqrt{2} \sum_n h_n \varphi(2x - n) \quad n = 0, N - 1
$$

and found that the wavelet function satisfies a similar dilation equation,

$$
\psi(x) = \sqrt{2} \sum_n (-1)^{n-1} h_{N-n} \varphi(2x - n) \quad n = 0, N - 1.
$$

More importantly, a set of $h_n$ coefficients up to $N=20$ (where $N$ has to be even) is constructed. Since $\varphi$ and $\psi$ have finite support (i.e., they only have nonzero values in a finite interval), they can be calculated numerically. Daubechies [18] proved that wavelets construct a set of orthogonal bases for the $L^2$ function space, and gave a detailed construction procedure for these wavelets. Newland [19] gave the wavelet series expansion of the $L^2$ function $f(t)$ as

$$
f(t) = b_0 + \sum_j \sum_k b_{j,k} W(2^j t - k) \quad 0 \leq t < 1, j = 0, \infty \quad k = 0, 2^j - 1
$$
where $W(2^j \cdot k)$ are Daubechies’ wavelets confined in the interval $0 \leq t < 1$ and wrapped around this interval as many times as necessary to ensure that their entire length is included in this interval; therefore, outside of this interval these wrapped around wavelets vanish to zero. The inner product of any single wavelet or any two distinct wavelets from the same family is identically zero. These orthogonality properties are expressed as

\[
\int_0^1 W(2^j t - k)W(2^{j'} t - k')dt = \delta_{jj'}\delta_{kk'} \tag{14a}
\]

\[
\int_0^1 W(2^j t - k)dt = 0 \tag{14b}
\]

where $\delta$ is the Kronecker $\delta$ function. The general coefficients can be calculated by taking the inner product of the function and the wavelet basis as

\[
b_0 = \int_0^1 f(t)dt \tag{15a}
\]

\[
b_{2^j + k} = \int_0^1 f(t)W(2^j t - k)dt \tag{15b}
\]

Newland [19] has developed a very efficient algorithm to compute the discrete wavelet transform Eq. (15) from sampling points of the function. The wavelets $W_m$ are calculated numerically from the inverse discrete wavelet transform. In depth information on wavelets can be found in Appendix B.

1.5 Objectives

Bayazitoglu and Wang [20] introduced wavelet analysis into the solution of radiative transfer problems for nongray media by expanding the spectral intensity function into its wavelet bases in the frequency domain. They chose the $P_N$ approximation in order to deal
with the angular dependency of intensity. Later, Wang and Bayazitoglu [21, 22] replaced the $P_N$ approximation with the discrete ordinates procedure.

In this work, a new numerical scheme is developed, which employs wavelet analysis in the evaluation of radiative intensity in angular domain as

$$I(t,s,\Omega) = \sum_i a_i(t,s)W_i(\Omega)$$

(16)

where $a_i$ are wavelet expansion coefficients and $W_i$ are wavelet bases. This allows the information related to directional distribution of the intensity field to be stored in a wavelet basis leading to the transformation of RTE to a new set of partial differential equations in terms of wavelet expansion coefficients.

The motivational facts behind this work can be extracted from this brief explanation of the wavelet method. First of all, as it is mentioned in Section 1.3, there is no "perfect" numerical solution method of RTE. For most methods, there is a tradeoff between accuracy and computational time. The Discrete Ordinates Method, which is the most popular, has its own disadvantages in ray effects and false scattering. With the goal of formulating a numerical method for the solution of RTE which is accurate, computationally efficient, flexible and easy to implement, wavelets could not be ignored for the following reasons: Wavelets are very useful in approximating a very wide range of functions especially with localized variations. They are apt in picking up edge effects of the functions, performing as a mathematical microscope. The multi-resolution property of wavelets also provides the flexibility of choosing a desired precision level for the approximation. Knowing that the radiative intensity function can and does display sharp
variations over the angular domain, wavelets are used for approximating intensity in this
domain.

The remainder of this thesis is organized as the following: In Chapter 2, the method is
applied to one-dimensional problems involving anisotropically scattering media and
results are compared with exact solution. Chapter 3 includes the extension of the wavelet
method to two-dimensional problems. This requires making use of a two-dimensional
wavelet basis and a different partition of the angular domain. Many of the suggested
numerical methods in the literature present difficulties for multi-dimensional geometries
even though they work well for one-dimensional problems. For instance, the discrete
ordinates method displays the ray effects only for multi-dimensional problems. Comparison of the results of the wavelet and other methods demonstrated the
effectiveness of the method for two-dimensional problems. In Chapter 4, the method is
applied to transient radiative heat transfer problems. An example problem that is
analogous to imaging a tumor in a tissue by using short-time laser is discussed. Finally,
an overall assessment of the method is presented in the concluding chapter.
References


**Nomenclature**

- $a, b =$ wavelet expansion coefficients
- $c =$ speed of light
- $f =$ square integrable function
- $I =$ radiative intensity
- $h =$ convective heat transfer coefficient
- $h_n =$ wavelet coefficients
- $k =$ thermal conductivity
- $q =$ heat flux
- $s =$ spatial coordinate
- $t =$ time
- $T =$ temperature
- $W =$ wrapped around wavelet basis function

**Greek Symbols**

- $\beta =$ extinction coefficient
- $\delta =$ Kronecker $\delta$ function
- $\varepsilon =$ emissivity
- $\Phi =$ scattering phase function
- $\varphi =$ scaling function
- $\kappa =$ absorption coefficient
- $\rho =$ reflectivity
- $\sigma_s =$ scattering coefficient
\( \tau = \text{optical thickness} \)

\( \omega = \text{single scattering albedo} \)

\( \Omega = \text{solid angle} \)

\( \psi = \text{wavelet function} \)

**Subscripts**

\( B = \text{quantity on the boundary} \)

\( i = \text{incoming directions} \)

\( \eta = \text{spectral quantities} \)

**Superscripts**

\( ^\wedge = \text{direction} \)
CHAPTER 2
APPLICATION OF THE WAVELET METHOD TO ONE-DIMENSIONAL PROBLEMS WITH SCATTERING

2.1 Introduction
In this chapter, wavelet analysis [1, 2] is formulated for one-dimensional radiative heat transfer problems involving linear anisotropic scattering media. Anisotropic scattering introduces additional difficulties for the numerical solution techniques of radiative heat transfer. Various researchers [3-5] utilized different solution methods for the solution of this problem.

In the following sections, wavelets [6-8] are introduced into the radiative transfer equation (RTE) in the directional domain by expanding radiative intensity into its wavelet basis. This leads to the conversion of RTE into a set of partial differential equations in terms of wavelet expansion coefficients. It also eliminates the integral term (in-scattering term) in RTE, and therefore makes the resulting equations manageable by finite differencing techniques. Results for a one-dimensional plane-parallel medium are presented and compared with exact solution [10].

2.2 Method of Solution
The equation of transfer for azimuthally symmetric radiation in a scattering gray medium between a one-dimensional, plane-parallel slab (Fig. 1) can be written [9] as
\[
\mu \frac{\partial I(\tau, \mu)}{\partial \tau} = I(\tau, \mu) + (1 - \omega)I_0 + \frac{\omega}{2} \int_{-1}^{1} \Phi(\mu, \mu')I(\tau, \mu')d\mu'
\]

where \(I(\tau, \mu)\) is the total intensity at optical distance \(\tau\) in the direction \(\mu = \cos \theta\). \(I(\tau, \mu) \in L^2(\mathbb{R})\) is bounded in the angular domain. The optical distance \(\tau\) is defined as \(d\tau = (\sigma + \kappa) dx\) where \(\sigma\) is the scattering coefficient, \(\kappa\) is the absorption coefficient and \(x\) is the distance. The parameter \(\Phi(\mu, \mu')\) is the scattering phase function for azimuthally symmetric radiation. Single scattering albedo is defined as \(\omega = \sigma / (\sigma + \kappa)\). In the case of pure scattering the value of \(\omega\) is 1, and for the non-scattering medium \(\omega\) is 0.

\[
\begin{align*}
\mu \frac{\partial I^+(\tau, \mu)}{\partial \tau} &= I^+(\tau, \mu) + (1 - \omega)I_0(\tau) + \frac{\omega}{2} \int_{-1}^{1} \Phi(\mu, \mu')I(\tau, \mu')d\mu' \\
\mu \frac{\partial I^-(\tau, \mu)}{\partial \tau} &= I^-(\tau, \mu) + (1 - \omega)I_0(\tau) + \frac{\omega}{2} \int_{-1}^{1} \Phi(\mu, \mu')I(\tau, \mu')d\mu'
\end{align*}
\]

\textbf{Fig. 1.} 1-D system geometry.

Wavelet analysis is introduced by splitting the angular domain (represented by directional cosine \(\mu\)) into two parts, \(0 \leq \mu < 1\) and \(-1 \leq \mu < 0\). Then Eq.(1) becomes

\[
\begin{align*}
\mu \frac{\partial I^+(\tau, \mu)}{\partial \tau} &= I^+(\tau, \mu) + (1 - \omega)I_0(\tau) + \frac{\omega}{2} \int_{-1}^{1} \Phi(\mu, \mu')I(\tau, \mu')d\mu' \\
\mu \frac{\partial I^-(\tau, \mu)}{\partial \tau} &= I^-(\tau, \mu) + (1 - \omega)I_0(\tau) + \frac{\omega}{2} \int_{-1}^{1} \Phi(\mu, \mu')I(\tau, \mu')d\mu'
\end{align*}
\]
where $0 \leq \mu < 1$. Now, we arrange Eq.'s (2a) and (2b) in order to express the integrations in terms of $I^+$ and $I^-$,

$$
\mu \frac{\partial I^+}{\partial \tau} = I^+ + (1 - \omega)I_b + \frac{\omega}{2} \left[ \int_0^1 \Phi(\mu, -\mu')I^-(x, -\mu')d\mu' + \int_0^1 \Phi(\mu, \mu')I^+(x, \mu')d\mu' \right] 
$$

(3a)

$$
-\mu \frac{\partial I^-}{\partial \tau} = I^- + (1 - \omega)I_b + \frac{\omega}{2} \left[ \int_0^1 \Phi(-\mu, -\mu')I^-(x, -\mu')d\mu' + \int_0^1 \Phi(-\mu, \mu')I^+(x, \mu')d\mu' \right] 
$$

(3b)

The method presented in this study involves approximating positive and negative intensities, $I^+$ and $I^-$, in the angular domain by extending them into their wavelet basis.

$$
I^+(\tau, \mu) = \sum_{i=1}^{N} a_i(\tau)W_i(\mu) 
$$

(4a)

$$
I^-(\tau, \mu) = \sum_{i=1}^{N} b_i(\tau)W_i(\mu) 
$$

(4b)

Introducing above expansions into Eq.'s (4), we obtain

$$
\mu \frac{\partial}{\partial \tau} \sum_i a_i(\tau)W_i(\mu) = \sum_i a_i(\tau)W_i(\mu) + (1 - \omega)I_b + \frac{\omega}{2} \sum_i \left[ b_i(\tau) \int_0^1 \Phi(\mu, -\mu')W_i(\mu')d\mu' + a_i(\tau) \int_0^1 \Phi(\mu, \mu')W_i(\mu')d\mu' \right] 
$$

(5a)

$$
-\mu \frac{\partial}{\partial \tau} \sum_i b_i(\tau)W_i(\mu) = \sum_i b_i(\tau)W_i(\mu) + (1 - \omega)I_b + \frac{\omega}{2} \sum_i \left[ b_i(\tau) \int_0^1 \Phi(-\mu, -\mu')W_i(\mu')d\mu' + a_i(\tau) \int_0^1 \Phi(-\mu, \mu')W_i(\mu')d\mu' \right] 
$$

(5b)

We apply the Galerkin Method to the Eq.'s (5a) and (5b), i.e. integrate Eq.'s (5) in the angular domain after multiplying the individual wavelet on both sides.
\[
\sum_{i}^{1} \int_{0}^{1} \mu W_{j}(\mu) W_{i}(\mu) d\mu \frac{da_{i}}{dt} = \sum_{i} a_{i} \int_{0}^{1} W_{j}(\mu) W_{i}(\mu) d\mu + (1 - \omega) b_{i} \int_{0}^{1} W_{j}(\mu) d\mu
\]

\[
+ \frac{\omega}{2} \sum_{i} \left[ b_{i} \int_{0}^{1} \Phi(\mu, -\mu') W_{j}(\mu) W_{i}(\mu') d\mu' d\mu + a_{i} \int_{0}^{1} \Phi(\mu, \mu') W_{j}(\mu) W_{i}(\mu') d\mu' d\mu \right]
\]

\[
- \sum_{i}^{1} \int_{0}^{1} \mu W_{j}(\mu) W_{i}(\mu) d\mu \frac{db_{i}}{dt} = \sum_{i} b_{i} \int_{0}^{1} W_{j}(\mu) W_{i}(\mu) d\mu + (1 - \omega) b_{i} \int_{0}^{1} W_{j}(\mu) d\mu
\]

\[
+ \frac{\omega}{2} \sum_{i} \left[ b_{i} \int_{0}^{1} \Phi(-\mu, -\mu') W_{j}(\mu) W_{i}(\mu') d\mu' d\mu + a_{i} \int_{0}^{1} \Phi(-\mu, \mu') W_{j}(\mu) W_{i}(\mu') d\mu' d\mu \right]
\]

Utilizing the following orthogonality properties of the Daubechies’ wavelets

\[
\int_{0}^{1} W_{j}(\mu) W_{i}(\mu) d\mu = \delta_{i,j} \quad j = 1,..,N
\]

\[
\int_{0}^{1} W_{j}(\mu) d\mu = \delta_{j,1} \quad j = 1,..,N
\]

and introducing the following bookkeeping notations,

\[
A_{i,j} = \int_{0}^{1} \mu W_{i}(\mu) W_{j}(\mu) d\mu \quad i, j = 1,..,N
\]

\[
S_{i,j}^{1} = \int_{0}^{1} \int_{0}^{1} \Phi(\mu, \mu') W_{j}(\mu) W_{i}(\mu') d\mu' d\mu \quad i, j = 1,..,N
\]

\[
S_{i,j}^{2} = \int_{0}^{1} \int_{0}^{1} \Phi(\mu, -\mu') W_{j}(\mu) W_{i}(\mu') d\mu' d\mu \quad i, j = 1,..,N
\]

\[
S_{i,j}^{3} = \int_{0}^{1} \int_{0}^{1} \Phi(-\mu, \mu') W_{j}(\mu) W_{i}(\mu') d\mu' d\mu \quad i, j = 1,..,N
\]

\[
S_{i,j}^{4} = \int_{0}^{1} \int_{0}^{1} \Phi(-\mu, -\mu') W_{j}(\mu) W_{i}(\mu') d\mu' d\mu \quad i, j = 1,..,N
\]
the Eq.'s (6a) and (6b) can be converted to the following compact forms:

\begin{equation}
\sum_i A_{i,j} \frac{da_i(\tau)}{d\tau} = a_j(\tau) + (1 - \omega)I_b(\tau)S_{j,i} + \frac{\omega}{2} \sum_i [a_i(\tau)S_{i,j}^1 + b_i(\tau)S_{i,j}^2] \tag{10a}
\end{equation}

\begin{equation}
- \sum_i A_{i,j} \frac{db_i(\tau)}{d\tau} = b_j(\tau) + (1 - \omega)I_b(\tau)S_{j,i} + \frac{\omega}{2} \sum_i [a_i(\tau)S_{i,j}^3 + b_i(\tau)S_{i,j}^4]. \tag{10b}
\end{equation}

Eq.'s (10a) and (10b) have 2N unknowns, \(a_i\) and \(b_i\), where \(i, j = 1, N\). The ordinary differential equations system presented in Eq.'s (10a) and (10b) can be solved with proper boundary conditions. To conveniently demonstrate the role of wavelets in the current analysis, we choose the simplest boundary conditions, i.e. the black walls with constant temperatures at \(T_1\) and \(T_2\),

\begin{equation}
\text{at } \tau = 0 \quad \Gamma^+ = I_b(T_1) \tag{11a}
\end{equation}

\begin{equation}
\text{at } \tau = \tau_L \quad \Gamma^- = I_b(T_2). \tag{11b}
\end{equation}

It turns out that the boundary conditions in terms of wavelet expansion coefficients can be obtained by applying the Galerkin method to Eq.'s (11) and taking advantage of orthogonality properties of the Daubechies' wavelets.

\begin{equation}
\text{at } \tau = 0 \quad a_j(0) = I_b(T_1)S_{j,1} \quad j = 1, \ldots, N \tag{12a}
\end{equation}

\begin{equation}
\text{at } \tau = \tau_L \quad b_j(\tau_L) = I_b(T_2)S_{j,1} \quad j = 1, \ldots, N \tag{12b}
\end{equation}

The most commonly used scattering phase function \(\Phi\) is the isotropic phase function, which means that \(\Phi = 1\). It is the simplest possible phase function. For the scattering theories such as Rayleigh scattering and Mie scattering [10], the isotropic phase function model is relatively poor. In this work, we use a linear anisotropic scattering phase function model,

\begin{equation}
\Phi(\mu, \mu') = 1 + \alpha \mu \mu' \tag{13}
\end{equation}
The coefficient $\alpha$ represents different scattering media. Substituting Eq. (13) into Eq.'s (9); the governing bookkeeping matrices for anisotropic scattering become,

$$S_{i,j}^1 = \delta_{i,i} \delta_{j,j} + \alpha B_i B_j$$  \hspace{1cm} (14a)

$$S_{i,j}^2 = \delta_{i,i} \delta_{j,j} - \alpha B_i B_j$$  \hspace{1cm} (14b)

$$S_{i,j}^3 = \delta_{i,i} \delta_{j,j} - \alpha B_i B_j$$  \hspace{1cm} (14c)

$$S_{i,j}^4 = \delta_{i,i} \delta_{j,j} + \alpha B_i B_j$$  \hspace{1cm} (14d)

where

$$B_j = \frac{1}{\mu W_j(\mu)} \int_0^1 \mu W_j(\mu) d\mu.$$  \hspace{1cm} (15)

Notice that Eq.'s (10) can be solved with an arbitrary scattering phase function model provided that it is square integrable. This always holds for real physical situations.

The overall heat flux at optical distance $\tau$ is

$$q(\tau) = \int_4 \hat{\Omega} \hat{I}(\tau, \hat{\Omega}) d\Omega = 2\pi \sum_{i=1}^{N} [a_i(\tau) - b_i(\tau)] B_i.$$  \hspace{1cm} (16)

Eq.'s (10) and (12) close the one-dimensional boundary value problem. The solution algorithm used here to solve this ordinary differential equations system is the method of particular solutions [11].

2.3 Results and Conclusions

To illustrate the accuracy and the effectiveness of the wavelet method, we solve the problem of radiation transfer in a plane-parallel slab with the following assumptions: The medium is emitting, absorbing and scattering. The boundaries are black. The left wall is
hot while the right one is kept cold. In order to compare with other results of prior researchers [3,4,12], various $\alpha$ values including the backward and the forward scattering conditions are considered. The space dependent albedo case results are also presented.

Fig. 2. Angular distribution of the intensity fields at the center of the spatial domain for nonscattering ($\omega = 0$), forward ($\alpha=1.98398, \omega = 0.5$) and backward ($\alpha = -0.56524, \omega = 0.5$) scattering media ($\tau = 1.0$).

The intensity field results produced by the wavelet method are plotted in Fig. 2. The three profiles in this figure represent the wavelet approximations of the intensity fields at the midpoint of the spatial domain, $\tau_L / 2$, for nonscattering ($\omega = 0$), forward scattering ($\omega=0$, $\alpha=1.98398$) and backward scattering ($\omega=0$, $\alpha= -0.56524$) media. The errors in the
approximate intensity field for the nonscattering medium as compared with the exact solution [10] at polar angles $\theta = 0^\circ$, $51.7553^\circ$, $81.7843^\circ$ and $159.4713^\circ$ are 0.53%, 0.72%, 0.32% and 1.79%, respectively. This proves that the present method successfully approximates the angular distribution of the radiative intensity. Fig. 3 shows the nondimensional temperature results produced by the wavelet method in the case of nonscattering medium. As seen from the figure, they compare well with exact results [10].

![Graph showing temperature vs. nondimensional location](image)

**Fig. 3.** Comparison of wavelet method and exact solution [10] nondimensional temperature profiles for a nonscattering medium between isothermal plates.
Fig. 4 presents the heat flux results and their comparison with the exact solutions [3,12] in the case of linear anisotropically scattering “cold” (non-emitting) medium. For the forward scattering medium represented by the scattering phase function coefficient value of $\alpha = 2.319461$, the exact heat flux distribution for the entire domain calculated by using the full phase function is available in the literature [12]. Here, the linear anisotropic function coefficient, $\alpha$, is taken as the first term of the full phase function. The exact and the wavelet method results match very well through the entire spatial domain. For $\alpha = 0.643833$ and $\alpha = 2.602844$, the heat fluxes are plotted in Fig. 4 and the boundary heat flux values are compared with the exact results [3]. The boundary heat flux results of the wavelet method and of other methods from the literature for these three linear anisotropic function coefficients are tabulated in Table 1 as well. Table 2 gives the boundary heat flux results for various albedo values ($\omega = 0.2, 0.5$ and $0.8$) when $\alpha = 0.643833$. In the tables, $F_0$ method represents the exact results.

Finally, the method is applied to a problem where the medium has a spatially varying albedo. Linear and quadratic variations of the albedo are considered. Reflectance, $1-q(0)$, and transmittance, $q(1)$, values for both forward ($\alpha = 1.98398$) and backward ($\alpha = -0.56524$) scattering are given in Table 3.

From the tables and figures, it can be concluded that the current wavelet method can generate accurate results for anisotropic scattering media. As it is stated before, the method has flexibility to handle other scattering phase functions including full phase functions mentioned in [1] as long as they are square integrable. The application of the
method to multi-dimensional geometries is presented in the next chapter. Inclusion of the scattering to the multi-dimensional geometries presents no additional difficulty.

![Graph](image)

**Fig. 4.** Comparison of heat flux distributions for three different linear-anisotropic phase functions with the exact results [3,12]. ($\omega = 0.5$, $\tau = 1.0$)
Table 1. Heat Flux Comparison ($\tau_x=1$, $\omega=0.8$)

<table>
<thead>
<tr>
<th>Methods</th>
<th>I</th>
<th></th>
<th>II</th>
<th></th>
<th>III</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha$ = 0.64383</td>
<td>$\alpha$ = 2.31946</td>
<td>$\alpha$ = 2.60284</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$q(0)$</td>
<td>$q(1)$</td>
<td>$q(0)$</td>
<td>$q(1)$</td>
<td>$q(0)$</td>
<td>$q(1)$</td>
</tr>
<tr>
<td>Schuster-Schwarzchild</td>
<td>0.74265</td>
<td>0.42256</td>
<td>0.92518</td>
<td>0.59763</td>
<td>0.95925</td>
<td>0.63069</td>
</tr>
<tr>
<td>Two-Flux</td>
<td>0.75470</td>
<td>0.47124</td>
<td>0.93134</td>
<td>0.64003</td>
<td>0.96279</td>
<td>0.67078</td>
</tr>
<tr>
<td>$P_1$</td>
<td>0.78743</td>
<td>0.46513</td>
<td>0.94485</td>
<td>0.61671</td>
<td>0.98025</td>
<td>0.65110</td>
</tr>
<tr>
<td>$P_3$</td>
<td>0.76590</td>
<td>0.45636</td>
<td>0.91225</td>
<td>0.59739</td>
<td>0.94477</td>
<td>0.62899</td>
</tr>
<tr>
<td>$P_9$ (9term)</td>
<td>0.76143</td>
<td>0.45637</td>
<td>0.90810</td>
<td>0.60296</td>
<td>0.94297</td>
<td>0.63910</td>
</tr>
<tr>
<td>$F_9$ (Exact)</td>
<td>0.76057</td>
<td>0.45588</td>
<td>0.90706</td>
<td>0.60251</td>
<td>0.94178</td>
<td>0.63874</td>
</tr>
<tr>
<td>DP1</td>
<td>0.7587</td>
<td>0.4543</td>
<td>0.9019</td>
<td>0.5923</td>
<td>0.9336</td>
<td>0.6231</td>
</tr>
<tr>
<td>Wavelet</td>
<td>0.7617</td>
<td>0.4587</td>
<td>0.9081</td>
<td>0.5991</td>
<td>0.9440</td>
<td>0.6344</td>
</tr>
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</table>
**Table 2. Heat Flux Comparison (τ_L=1, α=0.643833)**

<table>
<thead>
<tr>
<th>Methods</th>
<th>( q(0) )</th>
<th>( q(1) )</th>
<th>( q(0) )</th>
<th>( q(1) )</th>
<th>( q(0) )</th>
<th>( q(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Schuster-Schwarzhil</td>
<td>0.96148</td>
<td>0.17657</td>
<td>0.87910</td>
<td>0.26823</td>
<td>0.74265</td>
<td>0.42256</td>
</tr>
<tr>
<td>Two-Flux ( P_1 )</td>
<td>0.96221</td>
<td>0.22271</td>
<td>0.88308</td>
<td>0.31944</td>
<td>0.75740</td>
<td>0.47124</td>
</tr>
<tr>
<td>( P_3 )</td>
<td>-</td>
<td>-</td>
<td>0.93431</td>
<td>0.31287</td>
<td>0.78743</td>
<td>0.46513</td>
</tr>
<tr>
<td>( P_9(9\text{term}) )</td>
<td>0.98130</td>
<td>0.24795</td>
<td>0.90095</td>
<td>0.32450</td>
<td>0.76590</td>
<td>0.45636</td>
</tr>
<tr>
<td>( F_9(\text{Exact}) )</td>
<td>0.96780</td>
<td>0.25435</td>
<td>0.89151</td>
<td>0.32884</td>
<td>0.76143</td>
<td>0.45637</td>
</tr>
<tr>
<td>( D_{P_1} )</td>
<td>0.96513</td>
<td>0.25397</td>
<td>0.88976</td>
<td>0.32843</td>
<td>0.76057</td>
<td>0.45588</td>
</tr>
<tr>
<td><strong>Wavelet</strong></td>
<td><strong>0.9651</strong></td>
<td><strong>0.2513</strong></td>
<td><strong>0.8893</strong></td>
<td><strong>0.3254</strong></td>
<td><strong>0.7588</strong></td>
<td><strong>0.4534</strong></td>
</tr>
</tbody>
</table>
Table 3. Effects of spatial variation of the single-scattering albedo $\omega(\tau)$ on reflectance and transmittance for the wavelet method, DP$_1$ method [3] and the exact method [4].

(a) Wavelet method.

<table>
<thead>
<tr>
<th>$\omega(\tau)$</th>
<th>$\omega_{av}$</th>
<th>Forward Scattering $\alpha=1.98398$</th>
<th>Backward Scattering $\alpha=-0.56524$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear variation of albedo</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5+0.5$\tau$</td>
<td>0.5</td>
<td>0.019061</td>
<td>0.381598</td>
</tr>
<tr>
<td>0.5+0.4$\tau$</td>
<td>0.5</td>
<td>0.024140</td>
<td>0.379662</td>
</tr>
<tr>
<td>0.5+0.3$\tau$</td>
<td>0.5</td>
<td>0.029871</td>
<td>0.378205</td>
</tr>
<tr>
<td>0.5+0.2$\tau$</td>
<td>0.5</td>
<td>0.036300</td>
<td>0.377174</td>
</tr>
<tr>
<td>0.5+0.1$\tau$</td>
<td>0.5</td>
<td>0.043489</td>
<td>0.376559</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.051512</td>
<td>0.376354</td>
</tr>
<tr>
<td>0.5-0.1$\tau$</td>
<td>0.5</td>
<td>0.060449</td>
<td>0.376559</td>
</tr>
<tr>
<td>0.5-0.2$\tau$</td>
<td>0.5</td>
<td>0.070395</td>
<td>0.377174</td>
</tr>
<tr>
<td>0.5-0.3$\tau$</td>
<td>0.5</td>
<td>0.081456</td>
<td>0.378205</td>
</tr>
<tr>
<td>0.5-0.4$\tau$</td>
<td>0.5</td>
<td>0.093751</td>
<td>0.379663</td>
</tr>
<tr>
<td>0.5-0.5$\tau$</td>
<td>0.5</td>
<td>0.107442</td>
<td>0.381598</td>
</tr>
<tr>
<td>Quadratic variation of albedo</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>.45+0.4$\tau$+0.15$\tau^2$</td>
<td>0.5</td>
<td>0.026424</td>
<td>0.380394</td>
</tr>
<tr>
<td>.45+0.2$\tau$+0.15$\tau^2$</td>
<td>0.5</td>
<td>0.038961</td>
<td>0.377870</td>
</tr>
<tr>
<td>.45-0.2$\tau$+0.15$\tau^2$</td>
<td>0.5</td>
<td>0.074053</td>
<td>0.377870</td>
</tr>
<tr>
<td>.45-0.4$\tau$+0.15$\tau^2$</td>
<td>0.5</td>
<td>0.098071</td>
<td>0.380394</td>
</tr>
</tbody>
</table>
Table 3. (Cont') (b) DP₁ Method [3]

<table>
<thead>
<tr>
<th>ω(τ)</th>
<th>M av</th>
<th>Forward Scattering α=1.98398</th>
<th>Backward Scattering α=0.56524</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5+0.5τ</td>
<td>0.5</td>
<td>0.0162</td>
<td>0.3847</td>
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<tr>
<td>0.5+0.4τ</td>
<td>0.5</td>
<td>0.0220</td>
<td>0.3829</td>
</tr>
<tr>
<td>0.5+0.3τ</td>
<td>0.5</td>
<td>0.0284</td>
<td>0.3815</td>
</tr>
<tr>
<td>0.5+0.2τ</td>
<td>0.5</td>
<td>0.0355</td>
<td>0.3805</td>
</tr>
<tr>
<td>0.5+0.1τ</td>
<td>0.5</td>
<td>0.0434</td>
<td>0.3800</td>
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<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.0522</td>
<td>0.3798</td>
</tr>
<tr>
<td>0.5-0.1τ</td>
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<td>0.0618</td>
<td>0.3800</td>
</tr>
<tr>
<td>0.5-0.2τ</td>
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<td>0.0725</td>
<td>0.3805</td>
</tr>
<tr>
<td>0.5-0.3τ</td>
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<td>0.0842</td>
<td>0.3815</td>
</tr>
<tr>
<td>0.5-0.4τ</td>
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<td>0.0972</td>
<td>0.3829</td>
</tr>
<tr>
<td>0.5-0.5τ</td>
<td>0.5</td>
<td>0.1116</td>
<td>0.3847</td>
</tr>
</tbody>
</table>

Linear variation of albedo

Quadratic variation of albedo

| .45+0.4τ+0.15τ² | 0.5  | 0.0247 | 0.3838     | 0.0930  | 0.2961     |
| .45+0.2τ+0.15τ² | 0.5  | 0.0386 | 0.3814     | 0.1255  | 0.2930     |
| .45-0.2τ+0.15τ² | 0.5  | 0.0765 | 0.3814     | 0.2028  | 0.2930     |
| .45-0.4τ+0.15τ² | 0.5  | 0.1019 | 0.3838     | 0.2492  | 0.2961     |
### Table 3 (Con’t) (c) Exact results [4]

<table>
<thead>
<tr>
<th>$\omega(\tau)$</th>
<th>$\omega_{av}$</th>
<th>Forward Scattering</th>
<th>Backward Scattering</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\omega=1.98398$</td>
<td>$\omega=-0.56524$</td>
</tr>
<tr>
<td>0.5+0.5$\tau$</td>
<td>0.5</td>
<td>0.020878</td>
<td>0.386096</td>
</tr>
<tr>
<td>0.5+0.4$\tau$</td>
<td>0.5</td>
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<td>0.384434</td>
</tr>
<tr>
<td>0.5+0.3$\tau$</td>
<td>0.5</td>
<td>0.031273</td>
<td>0.383162</td>
</tr>
<tr>
<td>0.5+0.2$\tau$</td>
<td>0.5</td>
<td>0.037412</td>
<td>0.382265</td>
</tr>
<tr>
<td>0.5+0.1$\tau$</td>
<td>0.5</td>
<td>0.044262</td>
<td>0.381731</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.051899</td>
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<tr>
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<tr>
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<td>0.105416</td>
<td>0.386096</td>
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</tbody>
</table>

**Linear variation of albedo**

<table>
<thead>
<tr>
<th>Quadratic variation of albedo</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.45+0.4$\tau$+0.15$\tau^2$</td>
</tr>
<tr>
<td>0.45+0.2$\tau$+0.15$\tau^2$</td>
</tr>
<tr>
<td>0.45-0.2$\tau$+0.15$\tau^2$</td>
</tr>
<tr>
<td>0.45-0.4$\tau$+0.15$\tau^2$</td>
</tr>
</tbody>
</table>
References


Nomenclature

\( a_i, b_i = \) Wavelet expansion coefficients

\( A_{i,j}, B_{i,j}, S_{i,j} = \) Bookkeeping matrices

\( I = \) Radiative intensity

\( I_b = \) Black body intensity

\( N = \) Number of wavelet expansion coefficients

\( q = \) Heat flux

\( T = \) Temperature

\( W_i = \) Wrapped around wavelet basis

Greek symbols

\( \alpha = \) Linear anisotropic scattering phase function coefficient

\( \delta = \) Kronecker \( \delta \)-function

\( \Phi = \) Scattering phase function

\( \varphi = \) Dilation function

\( \kappa = \) Absorption coefficient

\( \mu = \) Directional cosine

\( \sigma = \) Scattering coefficient

\( \tau = \) Optical thickness

\( \Omega = \) Solid angle

\( \omega = \) Albedo

\( \psi = \) Wavelet function
Subscripts

$L =$ Distance between the plates

$1,2 =$ Left and right walls

Superscripts

$+, \, - =$ Positive and negative directions

$' =$ Incoming direction
CHAPTER 3
APPLICATION OF THE WAVELET METHOD TO TWO-DIMENSIONAL PROBLEMS

3.1 Introduction

The solution of multi-dimensional radiative transfer has been a challenge for researchers for decades. Many numerical solution methods (some of them are mentioned in the first chapter) have been suggested over the years. Among them, the discrete ordinates method stands out with its simplicity and flexibility [1-5]. While most of the methods work well for one-dimensional problems, they suffer various shortcomings for multi-dimensional problems. These shortcomings include ray effects and false scattering in discrete ordinate type of methods [3].

In this chapter, the application of the wavelets method introduced into the two-dimensional radiative heat transfer problems. As it is seen in Chapter 2, one-dimensional radiative transfer equation (RTE) includes only one angular parameter, which is the polar angle $\theta$. However, for two-dimensional radiation problems both polar ($\theta$) and azimuthal ($\phi$) angles have to be taken into account. Therefore, the wavelet expansion of the radiative intensity cannot be achieved by using one-dimensional wavelet basis functions. Moreover, both $\theta$ and $\phi$ need to be accounted for the partitioning of the angular domain. In the following section, the treatment of the two-dimensional RTE by using two-dimensional wavelets is explained. Two test problems are discussed, and issues related to ray effects and false scattering are addressed.
3.2 Mathematical Background

Wavelet expansion can be applied to a two-dimensional function $f(x,y)$ function in a similar fashion to one-dimensional case [6,7]:

$$\mathbf{W}(x, y) = \mathbf{W}(x) \mathbf{C} \mathbf{W}^t(y)$$  (1)

$\mathbf{W}(x)$ and $\mathbf{W}(y)$ are $1 \times N$ wavelet basis matrices, $\mathbf{C}$ is $N \times N$ wavelet coefficient matrix. $N$ is the number of wavelet basis that are used for each dependent variable of the function.

We can rearrange the two-dimensional expansion with a different notation in the following way:

$$f(x, y) = \sum_{m}^{N} \sum_{n}^{N} c_{m,n} W_{m,n}(x, y)$$  (2)

where $W_{m,n}(x,y)$ are the two dimensional wavelet bases.

3.3 Method of Solution

In this section, a radiative heat transfer problem in a rectangular enclosure with an absorbing, emitting medium is considered. The radiative properties are assumed gray and spatially homogeneous. A schematic of the physical model and coordinates is illustrated in Fig. 1. The mathematical description to this problem is [1]

$$\sin \theta \sin \phi \frac{\partial I(\tau_y, \tau_z, \theta, \phi)}{\partial \tau_y} + \cos \theta \frac{\partial I(\tau_y, \tau_z, \theta, \phi)}{\partial \tau_z} + I(\tau_y, \tau_z, \theta, \phi) = I_b(\tau_y, \tau_z)$$  (3)

where $I(\tau_y, \tau_z, \theta, \phi)$ is the total intensity at the position $(\tau_y, \tau_z)$ and in a direction which is expressed in terms of the polar angle $\theta$ and the azimuthal angle $\phi$ (see Fig. 2). $I(\tau_y, \tau_z, \theta, \phi) \in L^2(\mathbb{R})$ is bounded in the angular domain. $I_b$ is the black body intensity

$$I_b(\tau_y, \tau_z) = \sigma \cdot T^4(\tau_y, \tau_z)/\pi$$  (4)
where \( \sigma \) is the Stefan-Boltzmann constant. \( \tau_y = \kappa y \) and \( \tau_z = \kappa z \) are optical thicknesses and \( \kappa \) is the absorption coefficient of the medium. Defining the directional cosines as \( \mu = \cos \theta \) and \( \zeta = \sin \phi \), we will rearrange Eq. (3):

\[
\zeta \sqrt{1 - \mu^2} \frac{\partial I}{\partial \tau_y} + \mu \frac{\partial I}{\partial \tau_z} + I = I_b
\]

(5)

![Diagram of 2-D Enclosure](image)

**Fig. 1.** 2-D Enclosure.

The procedure that we will follow requires the expansion of the intensity function \( I = I(\tau_y, \tau_z, \theta, \phi) \) in angular domain \((\mu, \zeta)\) into the wavelet series. We will use the Daubechies' wavelets and they have only finite support in [0,1]. However, \( \mu \) and \( \zeta \) have values from negative one to positive one. Therefore, we will divide the angular domain into four subdomains (Fig.2), and denote the intensity \( I \) with \( i, j, k, l \) in these subdomains:
\[ i = \mathbf{i}(\tau_y, \tau_z, \mu, \xi) \text{ where } 0 \leq \mu < 1 \text{ and } 0 \leq \xi < 1 \]

\[ j = \mathbf{j}(\tau_y, \tau_z, \mu, \xi) \text{ where } -1 \leq \mu < 0 \text{ and } 0 \leq \xi < 1 \]

\[ k = \mathbf{k}(\tau_y, \tau_z, \mu, \xi) \text{ where } 0 \leq \mu < 1 \text{ and } -1 \leq \xi < 0 \]

\[ l = \mathbf{l}(\tau_y, \tau_z, \mu, \xi) \text{ where } -1 \leq \mu < 0 \text{ and } -1 \leq \xi < 0 \]

**Fig. 2.** Sub-domains for angular variation of the radiative intensity.

If we shift the directional cosines so that \( 0 \leq \mu < 1 \) and \( 0 \leq \xi < 1 \) for each subdomain, we can rewrite RTE in these subdomains as follows:

\[ \xi \sqrt{1-\mu^2} \frac{\partial i}{\partial \tau_y} + \mu \frac{\partial i}{\partial \tau_z} + i = I_b \quad 0 \leq \mu < 1 \text{ and } 0 \leq \xi < 1 \]  
(6a)

\[ \xi \sqrt{1-\mu^2} \frac{\partial j}{\partial \tau_y} - \mu \frac{\partial j}{\partial \tau_z} + j = I_b \quad 0 \leq \mu < 1 \text{ and } 0 \leq \xi < 1 \]  
(6b)

\[ -\xi \sqrt{1-\mu^2} \frac{\partial k}{\partial \tau_y} + \mu \frac{\partial k}{\partial \tau_z} + k = I_b \quad 0 \leq \mu < 1 \text{ and } 0 \leq \xi < 1 \]  
(6c)

\[ -\xi \sqrt{1-\mu^2} \frac{\partial l}{\partial \tau_y} - \mu \frac{\partial l}{\partial \tau_z} + l = I_b \quad 0 \leq \mu < 1 \text{ and } 0 \leq \xi < 1 \]  
(6d)
Boundary conditions for above set of equations are given below:

\[ i(\tau_y, 0, \mu, \xi) = \varepsilon_1 \cdot I_b(\tau_y, 0) \]

\[ + \frac{\rho_1}{\pi} \int_0^1 \left[ j(\tau_y, 0, \mu, \xi) + l(\tau_y, 0, \mu, \xi) \right] \frac{\mu}{\sqrt{1 - \xi^2}} \, d\mu \, d\xi, \quad -\tau_y < \tau_y < \tau_y \tag{7a} \]

\[ i(-\tau_y, \tau_z, \mu, \xi) = \varepsilon_2 \cdot I_b(-\tau_y, \tau_z) \]

\[ + \frac{\rho_2}{\pi} \int_0^1 \left[ k(-\tau_y, \tau_z, \mu, \xi) + l(-\tau_y, \tau_z, \mu, \xi) \right] \frac{\mu}{\sqrt{1 - \xi^2}} \, d\mu \, d\xi, \quad 0 < \tau_z < \tau_z \tag{7b} \]

where \( \varepsilon, \rho \) are emissivity and reflectivity of the boundaries respectively. Similar expressions for \( j, k, l \) could be written. Due to the limited space, we will not include those in this thesis. As it can be seen from above equations, a singularity occurs in the integration. These singularities are on the limits of the integrations. They are treated by excluding a small region surrounding the singularity and considering the limit as the excluding region approaches 0 [8]. Wavelet series expansions of the intensities \( i, j, k, l \) are

\[ i = \sum_m \sum_n a_{m,n}(\tau_y, \tau_z) \cdot W_{m,n}(\mu, \xi) \tag{8a} \]

\[ j = \sum_m \sum_n b_{m,n}(\tau_y, \tau_z) \cdot W_{m,n}(\mu, \xi) \tag{8b} \]

\[ k = \sum_m \sum_n c_{m,n}(\tau_y, \tau_z) \cdot W_{m,n}(\mu, \xi) \tag{8c} \]

\[ l = \sum_m \sum_n d_{m,n}(\tau_y, \tau_z) \cdot W_{m,n}(\mu, \xi) \tag{8d} \]

where \( a_{m,n}, b_{m,n}, c_{m,n} \) and \( d_{m,n} \) are wavelet expansion coefficients, and \( W_{m,n} \) are the two-dimensional wavelets. As it can be seen from Eq. (8), the wavelet coefficients are only the function of position, and all the information related to the directional distribution of the intensities is packed in the wavelets. Now, we will insert the Eq.’s (8) into Eq. (6a).
\[
\xi \sqrt{1 - \mu^2} \frac{\partial}{\partial \tau_y} \left[ \sum_m \sum_n a_{m,n} \langle \tau_y, \tau_z \rangle \cdot W_{m,n} (\mu, \xi) \right] \\
+ \mu \frac{\partial}{\partial \tau_z} \left[ \sum_m \sum_n a_{m,n} \langle \tau_y, \tau_z \rangle \cdot W_{m,n} (\mu, \xi) \right] + \sum_m \sum_n a_{m,n} \langle \tau_y, \tau_z \rangle \cdot W_{m,n} (\mu, \xi) = I_b (\tau_y, \tau_z)
\]

(9)

Similar expressions can be obtained for Eq.'s (6b), (6c) and (6d). We will apply the Galerkin Method to the above equation. The weighting functions are chosen to be the same functions as the wavelet bases. After multiplying an individual wavelet on both sides and integrating in the angular domain, and using the following orthogonality properties of wavelets

\[
\int_{\mu=0}^{1} \int_{\xi=0}^{1} W_{m,n} (\mu, \xi) \cdot W_{m',n'} (\mu, \xi) d\xi d\mu = \begin{cases} 
1 & \text{if } m = m' \text{ and } n = n' \\
0 & \text{otherwise}
\end{cases}
\]

(10a)

\[
\int_{\mu=0}^{1} \int_{\xi=0}^{1} W_{m,n} (\mu, \xi) d\xi d\mu = \delta_{m,1} \cdot \delta_{n,1}.
\]

(10b)

We obtain following equations for four subdomains:

\[
\sum_m \sum_n \left[ A_{m'n,n'} \frac{\partial a_{m,n} (\tau_y, \tau_z)}{\partial \tau_y} + B_{m'n,n'} \frac{\partial a_{m,n} (\tau_y, \tau_z)}{\partial \tau_z} \right] + a_{m',n'} (\tau_y, \tau_z) = I_b (\tau_y, \tau_z) \cdot \delta_{m',1} \cdot \delta_{n',1}
\]

(11a)

\[
\sum_m \sum_n \left[ A_{m'n,n'} \frac{\partial b_{m,n} (\tau_y, \tau_z)}{\partial \tau_y} - B_{m'n,n'} \frac{\partial b_{m,n} (\tau_y, \tau_z)}{\partial \tau_z} \right] + b_{m',n'} (\tau_y, \tau_z) = I_b (\tau_y, \tau_z) \cdot \delta_{m',1} \cdot \delta_{n',1}
\]

(11b)

\[
\sum_m \sum_n \left[ - A_{m'n,n'} \frac{\partial c_{m,n} (\tau_y, \tau_z)}{\partial \tau_y} + B_{m'n,n'} \frac{\partial c_{m,n} (\tau_y, \tau_z)}{\partial \tau_z} \right] + c_{m',n'} (\tau_y, \tau_z) = I_b (\tau_y, \tau_z) \cdot \delta_{m',1} \cdot \delta_{n',1}
\]

(11c)
\[ \sum \sum \left[ -A_{m',n'} \frac{\partial d_{m,n}(\tau_y, \tau_z)}{\partial \tau_y} - B_{m',n'} \frac{\partial d_{m,n}(\tau_y, \tau_z)}{\partial \tau_z} \right] + d_{m',n'}(\tau_y, \tau_z) = I_b(\tau_y, \tau_z) \cdot \delta_{m',1} \cdot \delta_{n',1} \]  

(11d)

where \( A_{m',n'} \) and \( B_{m',n'} \) are defined as

\[ A_{m',n'} = \int_0^1 \int_0^1 \left( \xi \sqrt{1 - \mu^2} \right) \cdot W_{m,n}(\mu, \xi) \cdot W_{m',n'}(\mu, \xi) d\xi d\mu \]  

(12a)

\[ B_{m',n'} = \int_0^1 \int_0^1 \mu \cdot W_{m,n}(\mu, \xi) \cdot W_{m',n'}(\mu, \xi) d\xi d\mu \]  

(12b)

and can be readily calculated. In other words, they are known coefficient matrices of the differential Eq.'s (11).

With the help of the procedure explained above, the RTE has been converted to a new set of partial differential equations (Eq.'s 11) written in terms of wavelet coefficients. There are many types of numerical differencing schemes to solve Eq.'s (11). Here we utilized finite volume method with a step scheme. The details of this procedure are given in Appendix C. After solving Eq.'s (11) for the wavelet expansion coefficients \( a, b, c \) and \( d \), heat fluxes are calculated with the following expression:

\[ q_y(\tau_y, \tau_z) = 2 \sum \sum [a_{m,n}(\tau_y, \tau_z) + b_{m,n}(\tau_y, \tau_z) - c_{m,n}(\tau_y, \tau_z) \left] - d_{m,n}(\tau_y, \tau_z) \right] \int_0^1 \int_0^1 W_{m,n}(\mu, \xi) \frac{\xi \sqrt{1 - \mu^2}}{\sqrt{1 - \xi^2}} d\mu d\xi \]  

(13a)

\[ q_z(\tau_y, \tau_z) = 2 \sum \sum [a_{m,n}(\tau_y, \tau_z) - b_{m,n}(\tau_y, \tau_z) + c_{m,n}(\tau_y, \tau_z) \left] - d_{m,n}(\tau_y, \tau_z) \right] \int_0^1 \int_0^1 W_{m,n}(\mu, \xi) \frac{\mu}{\sqrt{1 - \xi^2}} d\mu d\xi \]  

(13b)
3.4 Test Problems

3.4.1 Test Problem I

The wavelet method is applied to a two-dimensional rectangular enclosure containing a homogenous, absorbing, emitting and non-scattering medium in radiative equilibrium as illustrated in Fig. 1. The bottom wall is hot while the other walls kept cold. All the walls are considered to be black body (\( \varepsilon = 1 \)). For this choice of the boundary conditions, Eq.'s (7) can be written as follows:

\[
\begin{align*}
    i(\tau_y, 0, \mu, \xi) &= 1, \quad -\tau_{yo} < \tau_y < \tau_{yo} \quad (14a) \\
    i(-\tau_{yo}, \tau_z, \mu, \xi) &= 0, \quad 0 < \tau_z < \tau_{zo} \quad (14b) \\
    j(\tau_y, \tau_{zo}, \mu, \xi) &= 0, \quad -\tau_{yo} < \tau_y < \tau_{yo} \quad (14c) \\
    j(-\tau_{yo}, \tau_z, \mu, \xi) &= 0, \quad 0 < \tau_z < \tau_{zo} \quad (14d) \\
    k(\tau_y, 0, \mu, \xi) &= 1, \quad -\tau_{yo} < \tau_y < \tau_{yo} \quad (14e) \\
    k(\tau_{yo}, \tau_z, \mu, \xi) &= 0, \quad 0 < \tau_z < \tau_{zo} \quad (14f) \\
    l(\tau_y, \tau_{zo}, \mu, \xi) &= 0, \quad -\tau_{yo} < \tau_y < \tau_{yo} \quad (14g) \\
    l(\tau_{yo}, \tau_z, \mu, \xi) &= 0, \quad 0 < \tau_z < \tau_{zo} \quad (14h)
\end{align*}
\]

If the procedure used to treat Eq.'s (6) is employed for Eq.'s (14), the boundary conditions in terms of wavelet expansion coefficients can be written as

\[
\begin{align*}
    a_{m,n} &= 1 \text{ at } \tau_z = 0 \quad \text{ and } \quad a_{m,n} = 0 \text{ at } \tau_y = -\tau_{yo} \quad (15a) \\
    b_{m,n} &= 0 \text{ at } \tau_z = \tau_{zo} \quad \text{ and } \quad b_{m,n} = 0 \text{ at } \tau_y = -\tau_{yo} \quad (15b) \\
    c_{m,n} &= 1 \text{ at } \tau_z = 0 \quad \text{ and } \quad c_{m,n} = 0 \text{ at } \tau_y = \tau_{yo} \quad (15c) \\
    d_{m,n} &= 0 \text{ at } \tau_z = \tau_{zo} \quad \text{ and } \quad d_{m,n} = 0 \text{ at } \tau_y = \tau_{yo} \quad (15d)
\end{align*}
\]
This set of the boundary conditions is chosen because the discrete ordinates type of methods is susceptible to ray effects for the specified boundary conditions. The susceptibility arises from the discontinuity of the boundary loading at the corner points where bottom wall meets cold sidewalls. However, the wavelet method is immune to these effects since it fully models the angular distribution of the intensity at every point through the spatial domain. Calculated intensity distribution at the center of the rectangular enclosure is given in Fig. 3. Angular discretization is required to be able to employ the Discrete Wavelet Transform. In this test problem, the angular discretization is accomplished by dividing $\mu = \cos \theta$ and $\xi = \sin \phi$ into 16 even parts, which means uneven discretization in $\theta$ and $\phi$. Therefore, the calculated intensity field displays somewhat a discontinues behavior in the directions adjacent to the Z axis where discretization in $\theta$ is a lot coarser than it is near to the X-Y plane. This can be fixed by making the discretization in $\theta$ rather than in $\mu$.

The nondimensional emissive power at the centerline and the heat flux results at the top wall are presented in Fig.’s 4-8. They are compared with exact results [9], and with those of the Standard Discrete Ordinates (SDO) [5] and the Modified Discrete Ordinates (MDO) [5] methods. The MDO has been developed to mitigate the ray effects in SDO results and it has been successful to some extent. However, it is important to state that it is a semi-analytical method. This analytic nature is a setback in terms of the full automation of the solution method. Besides, it presents difficulties in handling radiation problems that involve anisotropic and other types scattering phase functions [5].
Fig. 3. Angular distribution of intensity at the center point of the rectangular enclosure.

The nondimensional emissive power results for different aspect ratios are plotted in Fig. 4 and compared with the exact results [9]. Emissive power results show good agreement with the exact results. Figures 5-8 present the surface heat flux results at the top wall. For all the aspect ratio values, the surface heat flux results of the SDO are inaccurate. They display oscillations especially for small aspect ratios. The MDO agrees well with the exact results for all the aspect ratios except for $r = 0.1$. Ramankutty and Crosbie [5] attributed the presence of anomalies at the heat flux results in the case of $r = 0.1$ to the inability of the MDO to remove the ray effects completely. The present method exhibits good agreement with exact results for all the aspect ratio cases. The errors of the present method do not show dependence to aspect ratio. This might be explained as follows:
Wavelet bases are very capable of approximating functions with local changes. Therefore, the wavelet method does not suffer from the ray effects. However, the finite differencing scheme that is used to solve the Eq.'s (8) is incapable of completely capturing sharp temperature and intensity gradients through the spatial domain, and causes them to be smoothened out. As it can be seen from Fig.'s 5-8, the surface heat flux profiles are somewhat flattened for all cases. Hence, the utilization of a better spatial discretization method could improve the results.

Fig. 4. Comparison of the centerline non-dimensional emissive power results for different aspect ratios with exact results [9].
Fig. 5. Comparison of the nondimensional surface heat flux results given by Wavelet, the Standard Discrete Ordinates (SDO) [5], the Modified Discrete Ordinates (MDO) [5] methods and exact solution [9] at the top wall for $r=0.1$. 
Fig. 6. Comparison of the nondimensional surface heat flux results given by Wavelet, the Standard Discrete Ordinates (SDO) [5], the Modified Discrete Ordinates (MDO) [5] methods and exact solution [9] at the top wall for $r=0.5$. 
**Fig. 7.** Comparison of nondimensional surface heat flux results given by Wavelet, the Standard Discrete Ordinates (SDO) [5], the Modified Discrete Ordinates (MDO) [5] methods and exact solution [9] at the top wall for r=1.
Fig. 8. Comparison of nondimensional surface heat flux results given by Wavelet, the Standard Discrete Ordinates (SDO) [5], the Modified Discrete Ordinates (MDO) [5] methods and exact solution [9] at the top wall for r=2.

3.4.2 Test Problem II

In this section, the wavelet technique is applied to a two-dimensional rectangular enclosure containing a homogenous, absorbing, emitting and nonscattering medium with uniform heat generation (q'' = 1). All the walls are cold black (ε = 1). Writing out the boundary conditions in terms of the wavelet expansion coefficients is straight forward in this case referring the test problem I;

\[ a_{m,n} = 0 \quad \text{at} \quad \tau_z = 0 \quad \text{and at} \quad \tau_y = -\tau_{yo} \]  \hspace{1cm} (16a)

\[ b_{m,n} = 0 \quad \text{at} \quad \tau_z = \tau_{zo} \quad \text{and at} \quad \tau_y = -\tau_{yo} \]  \hspace{1cm} (16b)
\[ \begin{align*}
    c_{m,n} &= 0 \quad \text{at } \tau_z = 0 \text{ and at } \tau_y = \tau_{yo} \quad (16c) \\
    d_{m,n} &= 0 \quad \text{at } \tau_z = \tau_{zo} \text{ and at } \tau_y = \tau_{yo} \quad (16d)
\end{align*} \]

Cheong and Song [10] handled this case with the Discrete Ordinates Interpolation Method (DOIM) and stated that the DOIM dramatically reduces the false scattering. The nondimensional emissive power results along the centerline with varying optical depth plotted in Fig. 9 and compared with the DOIM and zonal method results. Here, the zonal method is considered to be the benchmark solution.

Fig. 9a
Fig. 9. Comparison of nondimensional z direction centerline emissive power results of wavelet method, DOIM [10] and Zonal Method [10] for a square enclosure containing absorbing-emitting nonscattering medium with a uniform heat source for optical depths (a) 0.1; (b) 1.0; (c) 10.
In Fig. 9a, when optical depth is small, the zonal method predicts a sudden decrease of the medium temperature just near the cold wall. Since the optical thickness is small and boundary is cold, this drop is the indication of jump boundary condition and is considered reasonable. Nondimensional heat flux results for various optical depths are given in Fig. 10 and are again compared with the DOIM and zonal method solutions.
Fig. 10. Comparison of nondimensional surface heat flux results of the wavelet method, DOIM [10] and Zonal Method [10] for a square enclosure containing absorbing-emitting nonscattering medium with a uniform heat source for optical depths (a) 0.1; (b) 1.0; (c) 10.
For all the optical thickness values, the present method produces accurate emissive power and surface heat flux results. However, it cannot be claimed that the wavelet method eliminates false scattering. False scattering or numerical diffusion is a consequence of spatial discretization. Yet the numerical methods that are used for the angular dependency, such as the standard discrete ordinates method, affect the severity of false scattering. In this sense, the wavelet method mitigates the false scattering.

3.5 Concluding Remarks

The present method is fully numerical. It utilizes the wavelet bases to model the intensity field in angular domain. In this work Daubechies’ orthogonal wavelet bases are used. However, any other orthogonal wavelet can be used in the same way. This process converts the RTE into a set of partial differential equations (Eq.’s 12) only in terms of position. This set can be handled with any spatial discretization method. In this work finite differencing with step scheme is used. The results agree fairly well with the exact results. They might be improved by introducing better spatial discretization procedures. The code converges in 15 seconds for $r = 0.1$, 25 seconds for $r = 1$ and 30 seconds for $r = 2$ for the first test problem on a SPARC Ultra-4machine. Extension of the present method to three-dimensional problems is straightforward since the number of the angular parameters, namely polar $\theta$ and azimuthal $\phi$ angles, are the same for two and three-dimensional problems.
References

Nomenclature

\( a_{m,n} , b_{m,n} , c_{m,n} , d_{m,n} = \) Wavelet expansion coefficients of raidative intensity

\( A_{mn',nn'} , B_{mn',nn'} = \) Bookkeeping matrixes

\( E = \) Emissive power \( (= \sigma T^4) \)

\( i, j, k, l = \) Radiative intensity in four subdomains

\( I = \) Radiative intensity

\( N = \) Number of wavelet expansion terms

\( q = \) Radiative heat flux

\( r = \) Aspect ratio of 2-D rectangular enclosure

\( T = \) Temperature

\( W_i = \) Wrapped Daubechies wavelet functions

\( W_{ij} = \) Two dimensional wrapped Daubechies wavelet functions

\( x,y,z = \) Coordinates

Greek Symbols

\( \delta = \) Kronecker \( \delta \)-function

\( \varepsilon = \) Emissivity

\( \phi = \) Azimuthal angle measured from the positive x axis

\( \kappa = \) Absorption coefficient

\( \mu, \xi = \) Directional cosines

\( \theta = \) Polar angle measured from the positive z axis

\( \rho = \) Reflectivity

\( \sigma = \) Stefan-Boltzmann constant

\( \tau_y, \tau_z = \) Optical thicknesses in y and z directions
Subscripts

m, m', n, n' = Wavelet basis
1 = Hot wall
2 = Cold walls

Superscript

t = Transpose
CHAPTER 4

IMPLEMENTATION OF THE WAVELET METHOD IN SOLUTION OF TRANSIENT RADIATIVE TRANSFER PROBLEMS

4.1. Introduction

In this chapter, the wavelet method is implemented in the analysis of transient radiative transfer in an absorbing and scattering turbid media such as tissues. Both one and two-dimensional geometries are considered. As it is demonstrated in the previous chapter, the wavelet method is apt for problems involving abrupt variations in the intensity field. Most of the test cases presented here are of this nature where the intensity and consequently heat flux within the medium display sharp changes. The results of these cases prove the effectiveness of the wavelet method as illustrated in following sections.

In the majority of traditional engineering applications of radiative transfer, the time derivative term is negligible in the radiative transfer equation even if the intensity varies with time [1]. However, the rapid improvement of technology in recent years led to applications of radiative transfer where the time scales are in the order of pico- and femto-seconds, which necessitate the inclusion of transient effects in the transfer equation [2]. One of these applications is noninvasive imaging of tissues i.e., optical tomography [3,4]. In this technique, tissue is exposed to a short-pulse laser and transmittance or heat flux rate is measured at different locations. The goal is to obtain information about the interior structure of the medium by examining time-resolved measurements [4].
Solution of the transient radiative transfer equation gained little attention and only a few studies were reported [5,6], until recently. Increasing numbers of work is reported [4, 6-12] within the past five years. Kumar et al. [4] reported the implementation of the $P_1$ method in the solution of the one-dimensional transient radiative transfer equation, retaining its hyperbolic nature for the first time. Guo and Kumar [7] studied transient radiative transfer in a plane-parallel medium with inhomogeneous properties by using the radiative element method. Mitra and Kumar [8] compared several solution methods of transient radiative transfer for one-dimensional geometries. Mitra et al. [9] by using the $P_1$ method, solved the transient radiative transfer equation in two-dimensional geometry for the first time. Monte Carlo and Discrete Ordinates methods were later applied to the solution of two-dimensional transient radiative transfer equation by Guo et al [10] and Guo and Kumar [11], respectively. Application of the Transient Radiative Element Method to a three dimensional media is explained in [12]. Even with the cited literature above, there is still a considerable amount of uninvestigated transient radiation problems.

The remainder of this chapter is organized as follows: In the next section, the formulations of the wavelet method for one- and two-dimensional transient radiative transfer equations are explained. Then, several test problems are presented and compared with the available literature. Finally, imaging of an inhomogeneous zone within an absorbing, scattering medium is demonstrated.
4.2 Formulation

4.2.1 One-Dimensional Boundary-Driven Problem

The transient radiative transfer equation in a one-dimensional absorbing, emitting and scattering planar gray medium can be written as:

\[
\frac{1}{c} \frac{\partial I}{\partial t} + \mu \frac{\partial I}{\partial x} + \beta I = S(t, x, \mu) \tag{1}
\]

where \( I = I(t, x, \mu) \) is radiative intensity; \( c \) is the speed of the light in the medium; \( x \) is the Cartesian distance; \( t \) is the time; \( \beta \) is the extinction coefficient and \( S \) is the source term. The extinction coefficient is the summation of the absorption (\( \kappa \)) and the scattering coefficients (\( \sigma \)): \( \beta = \kappa + \sigma \).

The wavelet method formulation of Eq. (1) can be accomplished by following steps similar to those explained in Chapter 2, starting with the partitioning of the angular domain (represented by directional cosine \( \mu \)) into two parts, \( 0 \leq \mu < 1 \) and \( -1 \leq \mu < 0 \). The radiative intensities in these subdomains are represented with \( I^+ \) and \( I^- \), respectively. Then, the transient RTE can be written in these subdomains as

\[
\frac{1}{c} \frac{\partial I^+}{\partial t} + \mu \frac{\partial I^+}{\partial x} + \beta I^+ = S(t, x, \mu) \tag{2a}
\]

\[
\frac{1}{c} \frac{\partial I^-}{\partial t} - \mu \frac{\partial I^-}{\partial x} + \beta I^- = S(t, x, \mu) \tag{2b}
\]

where \( 0 \leq \mu < 1 \). The wavelet expansions of \( I^+ \) and \( I^- \) in angular domain are

\[
I^+(t, x, \mu) = \sum_{i=1}^{N} a_i(t, x)W_i(\mu) \tag{3a}
\]
\[ \Gamma(t, x, \mu) = \sum_{i=1}^{N} b_i(t, x)W_i(\mu) \]  

(3b)

where \(a_i\) and \(b_i\) are wavelet expansion coefficients; \(W_i\) are wavelet basis functions. As it can be seen from Eq.'s (3), the information about temporal and spatial variation of the intensity is stored in the wavelet expansion coefficients while the wavelet basis functions embodies the angular distribution of the intensity. Shortcutting the middle steps of the procedure (see Chapter 2 for details), the following transfer equations are obtained:

\[
\frac{1}{c} \frac{\partial a_i(t, x)}{\partial t} + \sum_i A_{i,j} \frac{\partial a_j(t, x)}{\partial x} = -\beta a_j(t, x) + \kappa b_i(t, x) \delta_{j,1} + \frac{\sigma}{2} \sum_i \left[ a_i(t, x) D_{i,j} D_{i,j}^1 + b_i(t, x) D_{i,j}^2 \right],
\]

(3a)

\[
\frac{1}{c} \frac{\partial b_i(t, x)}{\partial t} - \sum_i A_{i,j} \frac{\partial b_j(t, x)}{\partial x} = -\beta b_j(t, x) + \kappa b_i(t, x) \delta_{j,1} + \frac{\sigma}{2} \sum_i \left[ b_i(t, x) D_{i,j} D_{i,j}^3 + b_i(t, x) D_{i,j}^4 \right],
\]

(3b)

with the bookkeeping matrices:

\[
A_{i,j} = \int_{0}^{1} \mu W_i(\mu) W_j(\mu) d\mu, \quad i, j = 1, \ldots, N
\]

(4a)

\[
D_{i,j}^1 = \int_{0}^{1} \int_{0}^{1} \Phi(\mu, \mu') W_i(\mu) W_j(\mu') d\mu' d\mu, \quad i, j = 1, \ldots, N
\]

(4b)

\[
D_{i,j}^2 = \int_{0}^{1} \int_{0}^{1} \Phi(\mu, -\mu') W_i(\mu) W_j(\mu') d\mu' d\mu, \quad i, j = 1, \ldots, N
\]

(4c)

\[
D_{i,j}^3 = \int_{0}^{1} \int_{0}^{1} \Phi(-\mu, \mu') W_i(\mu) W_j(\mu') d\mu' d\mu, \quad i, j = 1, \ldots, N
\]

(4d)

\[
D_{i,j}^4 = \int_{0}^{1} \int_{0}^{1} \Phi(-\mu, -\mu') W_i(\mu) W_j(\mu') d\mu' d\mu, \quad i, j = 1, \ldots, N
\]

(4e)
\[ C_j = \int_0^1 \Phi(\mu, l) W_j(\mu) d\mu, \quad j = 1, \ldots, N \]  

(4f)

The linear-anisotropic scattering assumption is considered:

\[ \Phi(\mu, \mu') = 1 + \alpha \mu \mu'. \]  

(5)

The coefficient \( \alpha \) represents different scattering media.

The problem under consideration is boundary-driven, meaning that time variation of the intensity is initiated by a change in the boundary conditions. In this section, step change of the boundary condition is considered as, illustrated in Fig. 1.

Fig. 1. Sketch of one-dimensional boundary driven problem.

The boundary conditions for Eq.’s (3) can be expressed as

\[ a_j(t, x = 0) = I_b H(t) \delta_{j,1} \]  

(6a)

\[ b_j(t, x = L) = 0 \]  

(6b)

where \( \delta \) is the Kronecker \( \delta \)-function and \( H(t) \) is Heaviside’s unit step function

\[ H(t) = \begin{cases} 
0, & t < 0, \\
1, & t > 0.
\end{cases} \]  

(7)
Hyperbolic partial differential equations (3a) and (3b) with the boundary conditions (6a) and (6b) can be solved with conventional finite differencing methods with an appropriate initial condition (see Appendix C).

4.2.2 Two-Dimensional Medium Exposed to Laser Pulse

For two-dimensional Cartesian coordinates, the transient RTE can be written as (see Chapter 3 for details):

\[
\frac{1}{c} \frac{\partial I}{\partial t} + \xi \sqrt{1 - \mu^2} \frac{\partial I}{\partial y} + \mu \frac{\partial I}{\partial z} + \beta I = S
\]  
\[ \text{(8)} \]

where radiative intensity is \( I = I(t,y,z,\mu,\xi) \); \( \mu \) and \( \xi \) are directional cosines (\( \mu = \cos \theta \) and \( \xi = \sin \phi \)). \( \theta \) and \( \phi \) are polar and azimuthal angles. The system geometry, coordinates and laser pulse, is depicted in Fig. 2. The pulsed laser is normally incident on the medium at the center of the bottom wall.

The radiative heat transfer problems involving a medium exposed to collimated irradiation (laser beam is collimated i.e., all light waves are parallel to one another [13] ) are treated by separating the intensity within the medium into two parts:

\[
I(t,y,z,\mu,\xi) = I_d(t,y,z,\mu,\xi) + I_c(t,y,z,\mu,\xi).
\]  
\[ \text{(9)} \]
In Eq. (9), $I_c$ is the remnant of the collimated beam after partial extinction within the medium and $I_d$ is the diffuse part of intensity due to scattering of the collimated irradiation. This leads to two coupled radiative heat transfer problems in terms of collimated, $I_c$, and diffuse, $I_d$, intensities. Assuming a laser pulse with spatial and temporal square profiles (Fig. 3), the two-dimensional radiation problem of collimated intensity can be reduced to being one-dimensional, i.e., $I_c=I_c(t,z,\mu_c)$ within the range of $y \subset (-d_c/2, d_c/2)$ where $d_c$ is the width of the square laser pulse. In this respect, the transfer equation for collimated intensity can be written as

$$
\frac{1}{c} \frac{\partial I_c}{\partial t} + \mu \frac{\partial I_c}{\partial z} + \beta I_c = 0
$$

(10)
and the solution of this equation is

$$I_c(t, z, \mu) = I_0 e^{-\beta z \{H[t - (z/c)] - H[t - t_p - (z/c)]\} \delta_{\mu, 1}}$$  \hspace{1cm} (11)

where $I_0$ is the incident laser intensity; $t_p$ is the duration of the laser pulse and $\delta$ is the Kronecker $\delta$-function. Finally $H$ is Heaviside’s unit step function.

![Laser Intensity Temporal Profile](image)

**Fig. 3.** Temporal profile of the input laser.

The transfer equation of diffuse intensity, $I_d$, can be written as

$$\frac{1}{c} \frac{\partial I_d}{\partial t} + \xi \sqrt{1 - \mu^2} \frac{\partial I_d}{\partial y} + \mu \frac{\partial I_d}{\partial z} + \beta I_d = S$$  \hspace{1cm} (12)

with a source function $S$:

$$S(t, y, z, \Omega) = k I_b(y, z) + \frac{\sigma}{4\pi} \int_{\Omega'} I_d(t, y, z, \Omega') \Phi(\Omega', \Omega) d\Omega' + S_c.$$  \hspace{1cm} (13)

In Eq. (13), $\Omega$ depicts the total solid angle for outgoing and $\Omega'$ for incoming directions. $S_c$ is the contribution of collimated intensity to the source function and is defined as

$$S_c(t, x, \mu) = \frac{\sigma}{2} \int_{-1}^{1} I_c(t, x, \mu') \Phi(\mu', \mu) d\mu'.$$  \hspace{1cm} (14)

Introducing Eq. (11) into Eq. (14), the following expression is obtained for $S_c$:
\[ S_c(t,z,\mu) = \frac{\sigma}{2} I_0 e^{-\beta z} \left[ H[t - (z/c)] - H[t - \tau_p - (z/c)] \right] \Phi(\mu,1). \]  \hfill (15)

Since the wavelet method for two-dimensional radiative transfer problems is explained in Chapter 3, we will skip the intermediate steps of the method for Eq. (12), and write the resulting hyperbolic differential equations:

\[ \frac{1}{c} \frac{\partial a_{m,n}}{\partial t} + \sum_{m} \sum_{n} \left[ A_{mm',nn'} \frac{\partial a_{m,n}}{\partial y} + B_{mm',nn'} \frac{\partial a_{m,n}}{\partial z} \right] + \beta a_{m',n'} = S \cdot \delta_{m',1} \cdot \delta_{n',1} \]  \hfill (16a)

\[ \frac{1}{c} \frac{\partial b_{m,n}}{\partial t} + \sum_{m} \sum_{n} \left[ A_{mm',nn'} \frac{\partial b_{m,n}}{\partial y} - B_{mm',nn'} \frac{\partial b_{m,n}}{\partial z} \right] + \beta b_{m',n'} = S \cdot \delta_{m',1} \cdot \delta_{n',1} \]  \hfill (16b)

\[ \frac{1}{c} \frac{\partial c_{m,n}}{\partial t} + \sum_{m} \sum_{n} \left[ -A_{mm',nn'} \frac{\partial c_{m,n}}{\partial y} + B_{mm',nn'} \frac{\partial c_{m,n}}{\partial z} \right] + \beta c_{m',n'} = S \cdot \delta_{m',1} \cdot \delta_{n',1} \]  \hfill (16c)

\[ \frac{1}{c} \frac{\partial d_{m,n}}{\partial t} + \sum_{m} \sum_{n} \left[ -A_{mm',nn'} \frac{\partial d_{m,n}}{\partial y} - B_{mm',nn'} \frac{\partial d_{m,n}}{\partial z} \right] + \beta d_{m',n'} = S \cdot \delta_{m',1} \cdot \delta_{n',1} \]  \hfill (16d)

where \( m,m',n,n' = 1,\ldots,N \) and \( N \) indicates the level of wavelet expansion. \( a_{m,n} = a_{m,n}(t,y,z) \), \( b_{m,n} = b_{m,n}(t,y,z) \), \( c_{m,n} = c_{m,n}(t,y,z) \) and \( d_{m,n} = d_{m,n}(t,y,z) \) are wavelet expansion coefficients. \( A_{mm',nn'} \) and \( B_{mm',nn'} \) are defined as

\[ A_{mm',nn'} = \int_{\mu=0}^{1} \int_{\xi=0}^{\mu} \left( \frac{\xi}{\sqrt{1 - \mu^2}} \right) \cdot W_{m,n}(\mu,\xi) \cdot W_{m',n'}(\mu,\xi) d\xi d\mu \]  \hfill (17a)

\[ B_{mm',nn'} = \int_{\mu=0}^{1} \int_{\xi=0}^{\mu} \mu \cdot W_{m,n}(\mu,\xi) \cdot W_{m',n'}(\mu,\xi) d\xi d\mu \]  \hfill (17b)

The boundary conditions for Eq. (16a) in its most general form are (see Chapter 3 for details)
\[ a_{m',n'}(t, y = -H/2, z) = \{ \epsilon_1 \cdot I_b(y = -H/2, z) \]
\[ + \frac{\rho_1}{\pi} \sum_{m} \sum_{n} [c_{m,n}(t, y = -H/2, z) + d_{m,n}(t, y = -H/2, z)] \int_0^1 W_{m,n}(\mu, \xi) \frac{\xi \sqrt{1 - \mu^2}}{\sqrt{1 - \xi^2}} d\mu d\xi \delta_{m',i} \delta_{n',i} \]

\[ (24a) \]

\[ a_{m',n'}(t, y, z = 0) = \{ \epsilon_1 \cdot I_b(y, z = 0) \]
\[ + \frac{\rho_1}{\pi} \sum_{m} \sum_{n} [b_{m,n}(t, y, z = 0) + d_{m,n}(t, y, z = 0)] \int_0^1 W_{m,n}(\mu, \xi) \frac{\mu}{\sqrt{1 - \xi^2}} d\mu d\xi \delta_{m',i} \delta_{n',i} \]  

\[ (24b) \]

Similar expressions can be written for Eq.'s (16b), (16c) and (16d). The set of hyperbolic partial differential equations (16) can be solved by applying first upwind finite differencing scheme, providing a proper set of initial conditions for the intensity field.

Special attention needs to be paid to the choice of time step, \( \Delta t \) to assure the solution method is stable and physically realistic, and also to minimize the numerical diffusion. First of all, the distance that light travels in the time step, \( c\Delta t \), should not exceed spatial mesh size, i.e., \( c\Delta t < \min(\Delta y, \Delta z) \). Stability analysis is carried out by the Von Neuman analysis and numerical experimentation. It has not been possible to find a unique Courant number; however, it is in the range of \((cA_i)(\Delta t/\Delta x)\) for one-dimensional case. To reduce the numerical diffusion, time step should be chosen such a way that value of the Courant number is close to one.

4.3 Results and Discussions

4.3.1 One-Dimensional Boundary-Driven Problem

The one-dimensional absorbing, nonemitting (cold medium assumption [13]) and anisotropically scattering medium is considered. The radiative properties of the medium
are as follows: scattering coefficient $\sigma=6\text{mm}^{-1}$, absorption coefficient $\kappa=0.012\text{mm}^{-1}$ and the coefficient $\alpha$ in Eq. (5) is $2.319461$, which corresponds to the strong forward scattering medium. The bottom plate (Fig. 1) is suddenly heated and maintained at a hot temperature while the top wall is kept cold. By considering only a step change on the boundary and examining the temporal evolution of the scattered intensity field, we can obtain insight into the method and compare it with existing results from the literature. Figure 4 shows the transmittance results of the boundary-driven one-dimensional problem for various optical depths ($\tau=\beta x$). Transmittance for the one-dimensional problem is defined as the heat flux at the top wall in the positive direction, $q^+(x=L)$. As it can be seen from Fig. 4, the higher the optical depth of the medium is, the smaller the magnitude of transmittance is. This is expected since the radiation beam travels across larger distances and is attenuated more for higher optical depths. The times when the transmittance on the top wall is first observed are also significant. They indicate the flight times between two boundaries. As it can be observed from Fig. 4, no transmittance on the top wall is detected in earlier times. As the optical thickness increases, it takes longer for a photon to travel across the medium, consequently flight time values shift right in Fig. 4. The results compare well with the Discrete Ordinates Method solution with 12 discrete directions [8].
Fig. 4. Comparison of the transmittance results from wavelet and discrete ordinates methods [8] as a function of time for various optical depths for a one-dimensional boundary-driven problem.

4.3.2 Two-Dimensional Boundary Driven Problem

Boundary-driven transient radiative transfer in a square enclosure is examined in this section (Fig. 2). The medium is absorbing, nonemitting (cold medium assumption) and isotropically scattering. Medium parameters are $L=H=10\text{mm}$, $\sigma=1\text{mm}^{-1}$, $\kappa=0.001\text{mm}^{-1}$. Initially, medium and medium boundaries are considered to be cold. Then, the bottom wall (see Fig. 2) is suddenly heated and maintained at a constant uniform temperature. There is no laser beam impinging on the boundaries. In this case $S_c$ in Eq. (13) is set to zero. Again, the temporal profile of the transmittance will be examined (Fig. 5). The transmittance for the enclosure given in Fig. 2 is described as the heat flux leaving
medium through the top wall, \( q_x (t, y, z=L, \mu, \xi) \). The temporal transmittance values at three different locations on the top wall; \((y=0, z=L)\), \((y=H/4, z=L)\) and \((y=0.48H, z=L)\), obtained by the wavelet and Monte Carlo [11] methods are plotted in Fig. 5. No transmittance is observed at any of the three locations until the time at which first photons reach the top wall. After this time, the amount of heat flux through the top wall starts increasing as time progresses. Eventually, the transmittance profile levels off and the system reaches the steady-state. The results of the current method are in good agreement with those predicted by the Monte Carlo method. The present method slightly over predicts the transmittance at earlier times. This can be attributed to the first order time differencing used in the upwind differencing scheme.

![Graph showing transmittance comparison](image)

**Fig. 5.** Comparison of temporal transmittance profiles of wavelet and Monte Carlo methods [11] in a square isotropically scattering medium with one hot wall.
4.3.3 Short-Pulse Laser Transport

The physical system under consideration is a square enclosure with an absorbing, nonemitting and scattering medium. A pulsed radiation beam (laser irradiation) is incident externally on the bottom wall at the position \(y=0, z=0\) (Fig. 2). Initially, the medium is cold and no radiative transfer is present. The radiative transfer phenomenon within the medium is triggered by the laser pulse incident on the bottom wall. As the laser pulse travels through the medium, it is attenuated by outscattering and absorption of the medium. The optical and geometrical parameters of the system are as follows: \(L=H=10\text{mm}, \ \sigma=0.997\text{mm}^{-1}, \ \kappa=0.003\text{mm}^{-1}\) and \(n_o=1.33\) (refractive index). The time duration of the pulse is \(t_p=10\text{ps}\) (Fig. 3), and the width of the laser beam is \(d_c=1\text{mm}\) (Fig. 2). The fact that the medium is strongly scattering justifies the assumption of nonemitting medium. The emission of the medium is negligible compared to the intensity generated by laser beam (both the laser beam itself and the intensity scattered away) within the medium.

The temporal transmittance profiles at three different locations \((y=0, z=L), (y=0.2H, z=L)\) and \((y=0.4H, z=L)\) on the top boundary are presented in Fig. 6. The initial response time, which is the minimum time required for a photon to travel from the bottom to top wall following the shortest path, equals to \(t_i = L/c = 44.33\text{ps}\). At earlier times, no transmittance is observed on the top wall. At \(t_i\), a sudden increase in the temporal transmittance profile takes place at location \((y=0, z=L)\), which coincides with the axis of the incident laser beam. Such a peak is produced by the remnant of the laser beam that travels the direct path from bottom to top without any interaction with the medium. Temporal thickness or
duration of the peak is 10ps, which is the pulse duration of the laser pulse, \( t_p \). After the pulse passes through the top boundary completely, the transmittance profile at location \( y=0 \) drops off. From this point on, the transmittance on the top wall is generated by the diffuse intensity, which is an outcome of the outscattering of the laser beam as it travels through the medium. Since the majority of the scattered intensities does not follow the direct path from bottom to top, their effects on transmittance magnify at later times than the arrival of the pulse to the top wall. At locations \( y=0.2H \) and \( y=0.4H \), there are no direct photons from the incident laser, thus the transmittance profiles do not display an abrupt increase. The transmittance at these locations is produced solely by the multiple scattering of photons within the medium.

**Fig. 6.** Temporal transmittance at three different locations on the top wall of square enclosure.
Figure 7 shows the transmittance profiles as a function of dimensionless distance $y^* = y/H$ over the top wall at different time values. Remembering that the flight time of the light from bottom to top wall is 44.33ps, the profile at time $t=44.95$ps corresponds to initial stages of the evolution of transmittance. At this time, the profile consists of a spike with a thickness of 1mm, which is equal to the thickness of the pulse $d_c$. As the system progresses in time, the magnitude of the spike grows, while the more diffuse intensity reaches the top wall creating the skirts of the spike (the profile at $t = 54.56$ps). At time $t=58.9$ps, the effect of the laser beam diminishes and the transmittance profile becomes smooth. For a certain time, the magnitude of the transmittance augments due to the increasing amounts of scattered photons reaching the top wall. Eventually, the magnitude of the profile depletes, approaching steady state.

![Transmittance Profiles](image)

**Fig. 7.** Transmittance profiles at various times on the top wall of a square enclosure.
The reflectance results at the bottom wall at locations \((y = 0, z = L)\), \((y = 0.2H, z = L)\) and \((y = 0.4H, z = 0)\) are plotted in Fig. 8. Reflectance can be described as the heat flux leaving the bottom wall in the outward direction. The initial response time at location \((y = 0, z = L)\) is zero as it seen in Fig. 8. As time elapses, the reflectivity at location \(y=0\) increases rapidly, and reaches a maximum at a time approximately equal to the time duration of incident pulse, \(t_p\). After that, it drops off and eventually reaches the steady state values. As the location at which reflectance is calculated moves away from the center, the initial response time delays and the magnitude of the reflectance peak shrinks substantially. Comparison of the reflectance results of the present method and the Monte Carlo method [10] show good agreement.

**Fig. 8.** Comparison of temporal transmittance of the present and the Monte Carlo methods [10] at three different locations on the bottom wall of square enclosure.
Heat balance of the system can also be investigated. The heat gain to the system is only due to the incident laser beam on the bottom wall. The heat losses from the medium consist of the heat losses from all the boundaries and the absorbed energy by the medium (assuming that the absorbed energy is not reemitted to the medium). Here, we only consider heat losses from the boundaries for the calculation of total heat losses, and neglect the absorption since the amount of absorbed intensity is small ($\kappa=0.003\text{mm}^{-1}$) compared to the scattered intensity ($\sigma=0.997\text{mm}^{-1}$). The total heat gain and heat loss values integrated over time is plotted in Fig. 9. Most of the heat loss occurs at early times. This means that the reflectance at the bottom wall has a significant role in this process. As time evolves, the system approaches the steady state, and consequently total heat loss from the system approaches the total heat gain to the system.

![Graph](image)

**Fig. 9.** Total heat losses from the medium as the system reaches the steady state in time.
4.3.4 Imaging of Inhomogeneous Zones

In this section, the imaging of an inhomogeneous zone in a strongly scattering medium is investigated. This study is analogous to imaging of a tumor inside the tissues. The medium geometry is \( L=H=10\text{mm} \) (Fig. 2). Optical parameters of the medium are as follows: \( \sigma=1\text{mm}^{-1}, \kappa=0.01\text{mm}^{-1} \) and \( n_r=1.4 \). An ultra-short-pulsed laser irradiation (\( t_p=1\text{ps} \) and \( d_c=0.1\text{mm} \)) is incident on the bottom wall. Various rectangular inhomogeneous zones with different dimensions are placed at different locations in the medium. A list of inhomogeneities with their sketches, dimensions (\( H_1, L_1 \)) and locations are listed in Table 1. The optical properties of the inhomogeneous zones are chosen as \( \sigma_1=1.2\text{mm}^{-1}, \kappa_1=0.2\text{mm}^{-1} \) and \( n_{r1}=1.4 \). The refractive indexes of the medium and inhomogeneous zone are chosen to be the same; therefore the light beam does not change direction after entering the inhomogeneous zone.

The transmittance results at the top wall are examined in this section. First of all, they are obtained for the homogeneous medium (no inhomogeneity within the medium) as reference results for inhomogeneous medium cases. The temporal profile of the transmittance and distribution of transmittance on the top wall at various times are given in Fig.'s 10a and Fig 11, respectively. A similar behavior of the transmittance to the one in the previous section can be observed in these figures.
<table>
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<tr>
<th>Inhomogeneity</th>
<th>Sketch</th>
<th>Dimensions $H_1XL_1$</th>
<th>Center Coordinates $(y_1/H, z_1/L)$</th>
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</thead>
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</tr>
<tr>
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<td><img src="image" alt="Sketch D" /></td>
<td>1mmX6mm</td>
<td>$(0, 0.65)$</td>
</tr>
<tr>
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<td><img src="image" alt="Sketch E" /></td>
<td>1mmX5mm</td>
<td>$(0.06, 0.7)$</td>
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<tr>
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<td><img src="image" alt="Sketch G" /></td>
<td>1mmX5mm</td>
<td>$(0.2, 0.7)$</td>
</tr>
</tbody>
</table>

**Table 1.** Shapes, dimensions and locations of the inhomogeneous zones.
Fig. 10. Comparison of logarithmically varied temporal transmittance results of the homogeneous medium ($\sigma=1 \text{mm}^{-1}$, $\kappa=0.01 \text{mm}^{-1}$), (a) inhomogeneous media A, B, C, D and (b) inhomogeneous media C, E, F, G.
Fig. 11. Transmittance distribution at the top wall at various time instances in the case of homogeneous medium.

The effects of inhomogeneity size on transmittance results are examined in Fig.’s 10a and 12. Inhomogeneties with same widths but different lengths (A, B, C, D) are placed on the centerline of the host medium. The temporal behaviors of the transmittance are compared with that of the homogeneous medium in Fig. 10a. As seen from the figure, after the laser pulse completely passes through the top wall (after the spike in the transmittance
profiles), each inhomogeneity exhibit distinct profiles. As the length of the inhomogeneity increases, the transmittance values decreases. This can be better viewed in Fig. 11 where the transmittance distributions over the top wall at time $t = 93$ps are plotted. The longer the inhomogeneous zone gets, the flatter the profile becomes (A and B). Further increase in the length creates valley-like transmittance profiles (D and C).

![Graph showing transmittance at $t = 93$ps]

**Fig. 12.** Comparison of transmittance distributions over the top wall at time $t = 93$ps for inhomogeneities with different sizes (A, B, C, D). Optical parameters of inhomogeneities are $\sigma_1=1\text{mm}^{-1}$, $\kappa_1=0.01\text{mm}^{-1}$.
Change of the location of an inhomogeneity within the medium is also observable in transmittance results. An inhomogeneity with a certain size and shape (inhomogeneity C) is slid on the y axis causing the temporal and spatial transmittance profiles to be altered (Figs. 10b and 13). As the inhomogeneity moves to the right (inhomogeneties E, F, G), spatial transmittance profiles (Fig. 13) slant to the left and the minimal point of the profiles shift to the right. These minimal points coincide with the centerlines of the inhomogeneties. It is also noteworthy that the path of laser does not go through any of the moved inhomogeneties (E, F, and C). As the inhomogeneties are moved further away from the laser, the magnitude of the profiles increases. Since the inhomogeneties considered here have higher scattering coefficients, they create a blocking effect by scattering away more photons. As an inhomogeneity moves away from the laser beam, the blocking effect weakens.

Finally, the optical properties of the inhomogeneity A are changed to $\sigma_1=2\text{mm}^{-1}$, $\kappa_1=0.2\text{mm}^{-1}$ and compared with the results of $\sigma_1=1.2\text{mm}^{-1}$, $\kappa_1=0.2\text{mm}^{-1}$ in Fig. 14. With the increasing scattering coefficient, the transmittance profile shrinks. Similar comparison is also done for inhomogeneity F. Transmittance results for various values of the scattering coefficient of the inhomogeneity, ($\sigma_1=1.1\text{mm}^{-1}$, 1.2 $\text{mm}^{-1}$ and 2 $\text{mm}^{-1}$) are compared with those of the homogeneous medium.
Fig. 13. Comparison of the transmittance distributions of homogeneous medium and inhomogeneous media C, E, F and G at time $t = 93$ps. Optical parameters of inhomogeneties are $\sigma_1=1\text{mm}^{-1}$, $\kappa_1=0.01\text{mm}^{-1}$. 
Fig. 14. Comparison of the transmittance distributions over the top wall at t=93ps for inhomogeneity A with two different scattering coefficients: (i) $\sigma_t=1.2\text{mm}^{-1}$ and (ii) $\sigma_t=2.0\text{mm}^{-1}$. Absorption coefficients and refractive indexes are kept the same ($\kappa_t=0.2\text{mm}^{-1}$ and $n_s=1.4$). The optical properties of the host medium are $\sigma_t=1.2\text{mm}^{-1}$, $\kappa_t=0.2\text{mm}^{-1}$ and $n_s=1.4$. 
Fig. 15. Comparison of the transmittance distributions over the top wall at t=93ps for inhomogeneity F with two different scattering coefficients: (i) $\sigma_1=1.1\text{mm}^{-1}$ (ii) $\sigma_1=1.2\text{mm}^{-1}$ and (iii) $\sigma_1=2.0\text{mm}^{-1}$. Absorption coefficients and refractive indexes are kept the same ($\kappa_1=0.2\text{mm}^{-1}$ and $n_0=1.4$). The optical properties of the host medium are $\sigma_1=1.2\text{mm}^{-1}$, $\kappa_1=0.2\text{mm}^{-1}$ and $n_0=1.4$.

Various inhomogeneous zones are placed in different locations of a host medium. Geometry and optical parameters of the inhomogeneties are altered and transmittance results at the top wall versus time and location are plotted in Figs. 10-15. These results are compared both with those of the homogeneous medium and with each other. As it can
be seen from the figures, valuable information about the locations, shapes and optical properties of the inhomogeneous zones can be obtained from the transmittance profiles.

4.4 Concluding Remarks

The wavelet method is formulated to study transient radiative transfer in absorbing, nonemitting and scattering media. First, boundary-driven problems in one and two-dimensional enclosures are considered. The transmittance results compare well with existent results from the literature [8,10,11]. Then, the radiative transfer problem in a two-dimensional enclosure subjected to a short pulsed laser is studied. Transmittance and reflectance results are presented and compared with the Monte Carlo results from the literature [10]. Results agree well with the Monte Carlo results. In addition, the heat balance of the system is confirmed. Finally, imaging of an inhomogeneity in a turbid medium is studied. It is shown that the geometrical and optical information about the inhomogeneity can be obtained by examining the transmittance results.

The advantages of the wavelet method stand out in laser transport problems where the intensity field has sharp variations. Even for the cases where the medium with inhomogeneous zones, the wavelet method successfully produce heat flux or transmittance profiles without any unrealistic oscillations that could be caused by ray effects.
References


Nomenclature

a, b = wavelet expansion coefficients

c = speed of light

H(t) = Heaviside's unit function

H = width of two-dimensional enclosure

H_i = width of inhomogeneous zone

I = radiative intensity

I_0 = intensity of laser

I_b = blackbody intensity

L = distance between the plates for one-dimensional problem or height of two-dimensional enclosure

n_s = refractive index

S = source function

t = time

W = wavelet functions

x, y, z = Cartesian coordinates

Greek Symbols

\( \alpha \) = linear anisotropic scattering phase function coefficient

\( \beta \) = extinction coefficient

\( \delta \) = Kronecker's \( \delta \) function

\( \varepsilon \) = emissivity of surface

\( \phi \) = azimuthal angle
\( \kappa = \text{absorption coefficient} \)

\( \mu, \xi = \text{directional cosines} \)

\( \theta = \text{polar angle} \)

\( \rho = \text{reflectivity of surface} \)

\( \sigma = \text{scattering coefficient} \)

\( \tau = \text{optical thickness} \)

\( \omega = \text{single scattering albedo} \)

\( \Phi = \text{scattering phase function} \)

\( \Omega = \text{solid angle} \)

**Subscripts**

1 = inhomogeneity

c = collimated

d = diffuse

**Superscripts**

+ , − = positive and negative directions

′ = incoming directions
CHAPTER 5

CONCLUSION

In this study, wavelet analysis is applied to radiative heat transfer problems in the angular domain. Radiative intensity function is expanded in terms of wavelet basis functions allowing the separation of the angular and the other dependencies of the intensity. This expansion is substituted into the Radiative Transfer Equation (RTE). Then, the Galerkin Method is utilized by taking advantage of orthogonality properties of wavelet basis functions in order to convert RTE into a form manageable with finite differencing methods.

The treatment of the in-scattering term in RTE by wavelet analysis is demonstrated for one-dimensional problems in Chapter 2. It is shown that the method can handle anisotropic scattering as long as the scattering phase function is square integrable.

In Chapter 3, the wavelet method is applied to the two-dimensional radiative heat transfer problems in order to prove its accuracy. Two-dimensional geometries require modifications in the formulation procedure of the method. First, the partitioning of the angular domain for two-dimensional geometries is discussed. Then two illustrative examples are presented. The boundary conditions in the first example create a radiative heat transfer problem where the majority of other solution methods suffer from “ray effects”. The comparison of heat flux results with the exact solution shows that the
wavelet method is less susceptible to ray effects. The second example illustrates the
performance of the method in terms of "false scattering". Nondimensional temperature
and heat flux results compare well with those of the Discrete Ordinate Interpolation
Method which is suggested as a remedy to false scattering.

The solution of the transient radiative transfer equation gained more importance in recent
years with the advent of short-pulse lasers. There is relatively limited literature on the
solution methods of the transient RTE. In Chapter 4, the wavelet method is formulated
for the transient radiative heat transfer problems, and imaging of inhomogeneities within
an absorbing, scattering medium exposed to a short-pulse laser beam is studied. In the
case of laser transport problems, the intensity field within the medium displays abrupt
changes, which makes those problems good examples illustrating the effectiveness of the
presented wavelet analysis.

This work is the first to incorporate wavelets in the angular domain of RTE. Therefore,
much more work can be done by formulating the method to three-dimensional and
cylindrical coordinates. The extension of the method to three-dimensional geometries is
straightforward because two- and three-dimensional problems require the same number
of directional parameters, i.e. the directional cosines $\mu$ and $\xi$. However, cylindrical
geometries might present challenges during the implementation of the wavelet method in
RTE where "connection coefficients" of wavelets (inner product of wavelet basis
functions and their derivatives) need to be calculated. The method still needs further
improvement in terms of the choice and incorporation of finite differencing schemes.
APPENDIX A

FORMULATION OF THE DISCRETE ORDINATES METHOD, RAY EFFECTS
AND FALSE SCATTERING

A.1 Discrete Ordinates Method

The Radiative Transfer Equation for absorbing, emitting and anisotropically scattering gray medium is,

\[
\frac{\partial I(s, \hat{s})}{\partial s} = \kappa(s)I_b(s) - \kappa(s)I(s, \hat{s}) - \sigma_s(s)I(s, \hat{s}) + \frac{\sigma_s(s)}{4\pi} \int \int I(s, \hat{s}')\Phi(\hat{s}', \hat{s})d\Omega'
\]  

(1)

subjected to the boundary condition

\[
I(t, s_B, \Omega) = \varepsilon(s_B)I_b(s_B) + \frac{\rho(s_B)}{\pi} \int_{\hat{n} \cdot \hat{s} > 0} I(s_B, \hat{s}')|\hat{n} \cdot \hat{s}'|d\Omega'.
\]

(2)

In the discrete ordinates method, Eq. (1) is solved for a set of n different directions $\hat{s}_i$, $i = 1, 2, \ldots, n$, and integrals over direction are replaced by numerical quadratures:

\[
\int f(\hat{s})d\Omega \equiv \sum w_i f(\hat{s}_i)
\]

(3)

where $w_i$ are quadrature weights associated with the direction $\hat{s}_i$. Therefore, Eq. (1) is approximated by a set of n equations,

\[
\frac{\partial I(s, \hat{s}_i)}{\partial s} = \kappa(s)I_b(s) - \kappa(s)I(s, \hat{s}_i) - \sigma_s(s)I(s, \hat{s}_i) + \frac{\sigma_s(s)}{4\pi} \sum_{j=1}^{n} w_j I(s, \hat{s}_j)\Phi(\hat{s}_j, \hat{s}_i), \ i = 1, 2, \ldots, n
\]

(4)

subjected to the boundary conditions

\[
I(s_B, \hat{s}_i) = \varepsilon(s_B)I_b(s_B) + \frac{\rho(s_B)}{\pi} \sum_{\hat{n} \cdot \hat{s}_j > 0} w_j I(s_B, \hat{s}_j)|\hat{n} \cdot \hat{s}_j|, \ \hat{n} \cdot \hat{s}_i > 0.
\]

(5)
The radiative heat flux is calculated with the following expression:

\[ q(s) = \frac{1}{4\pi} \int I(s, \hat{s}) \hat{s} d\Omega \equiv \sum_{i=1}^{n} w_i I_i(s) \hat{s}_i. \] (6)

A.2 Ray Effect

Ray effect is the fundamental shortcoming of the discrete ordinates method. It is a consequence of the approximation of a continuously varying angular variable by considering a specified set of discrete angular directions. Ray effect is independent of spatial discretization practices. In order to understand the significance of ray effect and the errors caused by it, let us consider the following example.

We have a rectangular enclosure with a nonparticipating medium. All the walls are cold, with an exception of a small heated strip placed at the center of the bottom wall (Fig.1). In this system, an intensity leaving the heated strip does not interact with the medium and reaches the walls without any extinction because the medium is transparent.

Assume that we use a discrete ordinates approximation with six discrete directions (this could be \( S_6 \)). Physically, heat flux on the top and side walls must be continuous. However, with discrete ordinates approximation; the heat flux has finite values at the locations where chosen intensity strikes the walls while other locations on the walls do not see any radiation resulting in zero heat flux values. This is clearly physically unrealistic. This effect is known as ray effect.
Fig. 1: Illustration of discrete ordinates directions for a rectangular enclosure with a transparent medium.

In this example, no spatial approximation has been made. It can therefore concluded that the ray effect is solely the outcome of angular discretization practice.

The problems with a participating medium are susceptible to ray effect as well. While the ray effect is noticeable in the heat flux and other indicative results of discrete ordinate solution of radiative heat transfer problems involving an optically thick medium, it becomes more eminent for problems with optically thin medium.

A.3 False Scattering

In contrast with ray effect, false scattering is a consequence of spatial discretization practice and independent of the angular discretization. False scattering is known as
numerical diffusion in the CFD community. It could be explained better with the following example:

Suppose we have a rectangular enclosure containing a transparent medium. It consists of three cold walls, and the upper half of the left wall is subjected to collimated incidence as illustrated in Fig. 2.

![Diagram](image)

**Fig. 2:** Illustration of false scattering.

Since there is no interaction of intensity with the medium, the intensity field within the medium must be a step function which has the magnitude of the collimated intensity incident on the left wall in the upper half of and zero magnitude in the lower half of the medium. However, the numerical diffusion introduced by spatial discretization schemes causes smoothing of the profile and creating finite intensity values in the lower part of the medium, which is obviously unrealistic, as illustrated in Fig. 2. This phenomena is called false scattering by radiative heat transfer community.
APPENDIX B

BASICS OF WAVELETS

B.1. Theory of Wavelets

The goal of the theory of wavelets is to provide a coherent set of concepts, methods, and algorithms that are well adapted to a variety of non-stationary signals and that are also suitable for numerical analysis. Historical perspectives of application of wavelets in engineering are discussed first by showing the generalization of the Haar basis [1] in the modern theory of the multiresolution analysis [2]. The general framework of the multiresolution analysis motivated Ingrid Daubechies [3] to construct wavelet bases which are orthogonal, have local support, and some other useful properties. Mallat [2] and Newland [4] constructed a very efficient algorithm for Discrete Wavelet Transform to make the theory of wavelets applicable for numerical computation.

B.1.1 Concept of Wavelets

Historically, the elementary concept of wavelets appeared in several branches of engineering. Nevertheless, neither of these separate intuitive efforts was finalized as a part of a coherent theory.

The first orthonormal system of localized functions \( \{ \psi_{j,k}(x), j,k = \pm 1, \pm 2, \ldots \} \) defined on [0,1] resembling wavelets was constructed by [1]. In this case, for any function \( f(x) \) continuous on [0,1], the Haar series
\[ \hat{f}_n(x) = \langle f, \psi_0 \rangle \psi_0(x) + \langle f, \psi_1 \rangle \psi_1(x) + \ldots + \langle f, \psi_n \rangle \psi_n(x) \]  

converge to \( f(x) \) uniformly on the interval \([0,1]\). For \( n = 2^{-j} + k \geq 1, j = 0, -1, -2, \ldots \) and \( 0 \leq k < 2^{-j} \), the basis functions \( \psi_n(x) \) are defined as

\[ \psi_n(x) = \psi_{j,k}(x) = 2^{-j/2} \psi(2^{-j}x - k) \]  

where

\[ \psi(x) = \begin{cases} 
1 & 0 \leq x < \frac{1}{2} \\
-1 & \frac{1}{2} \leq x < 1 \\
0 & \text{otherwise}
\end{cases} \]  

Also, the function

\[ \phi(x) = \begin{cases} 
1 & 0 \leq x < 1 \\
0 & \text{otherwise}
\end{cases} \]  

is augmented to complete the Haar basis. It is straightforward to show (Fig. 1) that this basis is, indeed, orthogonal. The support of \( (\psi_{j,k}) = [2^{-j}k, 2^{-j}(k+1)] \) and, therefore, any two Haar wavelets of the same scale (same value of \( j \)) never overlap. Overlapping of two basis functions can occur in such a way that the wavelet with the smaller support lies entirely within the region where the other wavelet is constant. Then the inner product of any two wavelets from different scales is also zero.
Fig. 1 Two Haar Wavelets from Different Scales.

Any function \( f(x) \) of \( L^2([0, 1]) \) can be arbitrarily closely approximated by a function \( f_0 \) that has a constant value on the intervals \([2^{-j_0} k, 2^{-j_0} (k + 1)]\). It is just necessary to select a sufficiently large value for \( J_0 \). This fact pointed out by Daubechies [5] provides a proof that the Haar wavelets \( \{ \psi_{j,k}(x) \} \) do constitute a basis of \( L^2([0, 1]) \). Since,

\[
f(x) \approx f_0(x) = \sum_k C_{-J_0,k} \phi_{-J_0,k}
\]

(5)

where

\[
\phi_{j,k}(x) = 2^{-j/2} \phi(2^{-j} x - k)
\]

(6)

\[
C_{-J_0,k} = \int_{[0,1]} f_0(x) \phi_{-J_0,k}(x) dx
\]

(7)
Now the function $f_0$ can be split into two components $f_1$ and $d_1$, where $f_1$ is a piecewise constant function over the intervals $[2^{-J_0} + 1, k, 2^{-J_0} + 1(k + 1)]$ and is defined similarly to $f_0$ (see Fig. 2). Namely,

$$f_1(x) = \sum_{k} C_{-J_0+1,k} \phi_{-J_0+1,k}$$

(8)

where

$$C_{-J_0+1,k} = \frac{(C_{-J_0,2k-1} + C_{-J_0,2k})}{\sqrt{2}}$$

(9)

Alternatively, $d_1$ accumulates information lost in the transition from a fine scale to the coarser scale; it is a piece-wise constant function with the same step width as $f_0$

$$d_1(x) = \sum_{k} d_{-J_0+1,k} \psi_{-J_0+1,k}$$

(10)

where

$$d_{-J_0+1,k} = \frac{(C_{-J_0,2k-1} + C_{-J_0,2k})}{\sqrt{2}}$$

(11)
Fig. 2 Decomposition of a piece-wise constant function using the Haar Wavelets.

The procedure above is shown in Fig. 2. Further, the same decomposition can be applied to $f_1$ by increasing the step width twice. Again
\[ f_1(x) = f_2(x) + \sum_k d_{-J_0+2,k} \psi_{-J_0+2,k} \]  \hspace{1cm} (12)

Finally after \( J_0 \) steps of averaging an expansion of \( f_0 \) by using the Haar wavelets can be found. Since \( f_0 \) is an arbitrary piece-wise constant function, the Haar wavelets comprise an orthogonal basis for \( L^2([0, 1]) \). This proof of the completeness of the Haar basis was proposed by Daubechies [5] as an illustration of the multiresolution analysis. It implicitly uses the multiresolution approach by projecting the function \( f(x) \) on coarser and less detailed scales. One "paradox" about Haar function is that the Haar basis which is used to represent a continuous function \( f(x) \) is not continuous itself.

To generalize the Haar wavelets concept, Calderon [6] developed a principally new approach to the so-called "atomic decomposition" introducing the "Calderon's identity". A double-indexed family of wavelets can be generated from a single function \( \psi \in L^2(\mathbb{R}) \) by dilation and translation

\[ \psi_{a,b}(x) = |a|^{-1/2} \psi\left(\frac{x-b}{a}\right) \quad a, b \in \mathbb{R}, a \neq 0 \]  \hspace{1cm} (13)

Daubechies [3] described that depending on the type of application, different families of wavelets may be chosen. One way is to let the parameters \( a, b \) in (13) vary continuously on their range \( \mathbb{R}^* \times \mathbb{R} \) (where \( \mathbb{R}^* = \mathbb{R} \setminus \{0\} \)). For instance, represent functions \( f \in L^2(\mathbb{R}) \) by the functions \( Uf, \)

\[ Uf(a, b) = \left< \psi_{a,b}, f \right> = |a|^{-1/2} \int \overline{\psi\left(\frac{x-b}{a}\right)} f(x) dx \]  \hspace{1cm} (14)

If \( \psi \) satisfies the condition
\[ \int d\xi |\xi|^{-1} |\hat{\psi}(\xi)|^2 < \infty \] (15)

where \(^\wedge\) denotes the Fourier transform,

\[ \hat{\psi}(\xi) = \frac{1}{\sqrt{2\pi}} \int dx e^{ix\xi} \psi(x) \] (16)

Then \( U \) (as defined by (14)) is an isometry (up to a constant) from \( L^2(R) \) into \( L^2(R^* \times R, a^{-2} da \) \( db \)). The map \( U \) is called the "continuous wavelet transform".

Note that the "admissibility condition" eq.(15) implies, if \( \psi \) has sufficient decay which we shall always assume in practice, that \( \psi \) has mean zero,

\[ \int dx \psi(x) = 0 \] (17)

Typically, the function \( \psi \) will therefore have at least some oscillations. A standard example is

\[ \psi(x) = \left( \frac{2}{\sqrt{3}} \right) \pi^{-1/4} (1 - x^2) e^{-x^2/2} \] (18)

For other applications, including those in numerical computation, one may choose to restrict the values of the parameters \( a, b \) in (13) to a discrete sublattice, which means that one may fix a dilation step \( a_0 > 1 \), and a translation step \( b_0 \neq 0 \). The family of wavelets of interest becomes then, for \( m, n \in Z \),

\[ \psi_{m,n}(x) = a_0^{-m/2} \psi(a_0^{-m} x - nb_0) \] (19)

Note that this corresponds to the choices \( a = a_0^m \), \( b = nb_0 a_0^m \), indicating that the translation parameter \( b \) depends on the chosen dilation rate. For \( m \) large and positive, the
oscillating function $\Psi_{m,0}$ is very much spread out, and the large translation steps $b_0a_0^m$ are adapted to this wide width. For large but negative $m$ the opposite happens; the function $\Psi_{m,0}$ is very much concentrated, and the small translation steps $b_0a_0^m$ are necessary to still cover the whole range.

A corresponding "discrete wavelet transform" $T$ is associated with the discrete wavelets (19). It maps functions $f$ to sequences indexed by $\mathbb{Z}^2$,

$$(Tf)_{mn} = \langle \psi_{m,n}, f \rangle = a_0^{-m/2} \int \overline{\psi(a_0^{-m}x - nb_0)} f(x) dx$$  \hspace{1cm} (20)

If $\gamma$ satisfies the "admissibility condition" (15), and if $\gamma$ has sufficient decay, the $T$ maps $L^2(\mathbb{R})$ into $l^2(\mathbb{Z}^2)$. If $T$ has a bounded inverse on its range, i.e., if for some $A > 0, B < \infty, A\|f\|^2 < \sum_{m,n \in \mathbb{Z}} |\langle \psi_{m,n}, f \rangle|^2 < B\|f\|^2$

for all $f$ in $L^2(\mathbb{R})$, then the set $\{\psi_{m,n}; m, n \in \mathbb{Z}\}$ is called a "frame". In this case one can construct numerically stable algorithms to reconstruct $f$ from its wavelet coefficients $\langle \psi_{m,n}, f \rangle$. For some applications, one can turn to choices of $\gamma$ and $a_0, b_0$ (typically $a_0 = 2$) for which the $\gamma_{m,n}$ constitute an orthonormal basis. This is the case to which we shall be restricting ourselves in the remainder of this work.
B.1.2 Multiresolution Analysis

The idea of a multiresolution analysis is to write $L^2$-functions $f$ as a limit of successive approximations, each of which is a smoothed version of $f$, with more and more concentrated smoothing functions. The successive approximations thus use a different resolution, whence the name multiresolution analysis. The successive approximation schemes are required to have following properties:

1. A family of embedded closed subspaces $V_m \in L^2(R), m \in \mathbb{Z}$,

   $\ldots \subset V_2 \subset V_1 \subset V_0 \subset V_1 \subset V_2 \subset \ldots$;

   such that

2. $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \bigcup_{j \in \mathbb{Z}} V_j = L^2(R)$;

3. $f(x) \in V_j \iff f(2^j \cdot x) \in V_0$;

4. $f(x) \in V_0 \rightarrow f(x - k) \in V_0$ for all $k \in \mathbb{Z}$.

5. $\exists \phi(x) \in V_0$ such that $\{ \phi_{0,k}(x) = \phi(x - k), k \in \mathbb{Z} \}$ constitute an orthonormal basis for $V_0$. Where, for all $j, k \in \mathbb{Z}$,

   $\phi_{j,k}(x) = 2^{-j/2} \phi(2^{-j}x - k)$

   is an orthonormal basis for $V_j$ for all $j \in \mathbb{Z}$.

The basic idea of multiresolution analysis is that whenever a collection of closed subspaces satisfies (1)-(5), then there exists an orthonormal basis $\{\psi_{j,k}: j, k \in \mathbb{Z}\}$ of $L^2(R)$, $\psi_{j,k}(x) = 2^{-j/2} \psi(2^{-j}x - k)$, such that, for all $f$ in $L^2(R)$,
\[ P_{V_{j-1}} f = \text{Proj}_{V_j} f + \sum_{k \in \mathbb{Z}} f, \psi_{j,k} \langle \psi_{j,k} \]  

(21)

where \( P_{V_{j-1}} \) is the orthogonal projection of \( f \) onto \( V_j \). Function \( f \) is called scaling function and \( \psi \) is called wavelet.

According to eq.(21), the space \( W_j \) is defined as the orthogonal compliment of \( V_j \) in \( V_{j-1} \),

\[ V_{j-1} = V_j \oplus W_j \]  

(22)

Equation (22) and the definition of the multiresolution analysis show that the spaces \( W_j \), \( j = \ldots, -1, 0, 1, \ldots \) must satisfy the following equations:

\[ W_j \perp W_{j'}, \text{ if } j \neq j' \]  

(23)

\[ V_j = V_j \oplus \bigoplus_{k=0}^{j-1} W_{j-k} \]  

(24)

\[ L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j \]  

(25)

The space \( W_j \) contains information lost in the projection of a function into the space \( V_j \) in comparison with its projection into \( V_{j-1} \). The useful principle of the multiresolution analysis is the fact that eq.(25) yields that the set \( \{\psi_{j,k} ; j, k \in \mathbb{Z}\} \) forms an orthogonal wavelet basis of \( L^2(\mathbb{R}) \). Note that the Haar basis satisfies all of the preceding conditions and represents the simplest orthogonal wavelets with a multiresolution structure.
Since $\phi \subset V_0 \subset V_{-1}, \psi \subset W_0 \subset V_{-1}$, and $\{\phi_{-1,n}\}$ form an orthogonal basis of $V_{-1}$, the following multi-scale representation of the functions $f(x)$ and $\gamma(x)$ can be readily established

$$\phi(x) = \sum_k h_k \phi_{-1,k} = \sqrt{2} \sum_k h_k \phi(2x - k) \quad (26)$$

$$\psi(x) = \sum_k g_k \phi_{-1,k} = \sqrt{2} \sum_k g_k \phi(2x - k) \quad (27)$$

The relationship between $h_k$ and $g_k$ is the fundamental for the construction of orthogonal wavelets.

The well-known notation for the Fourier transform is

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)e^{-ix\xi}dx \quad (28)$$

equation (26) and (27) can be rewritten using the eq.(28)

$$\hat{\phi}(\xi) = m_0(\frac{\xi}{2})\hat{\phi}(\frac{\xi}{2}) \quad (29)$$

and

$$\hat{\psi}(\xi) = m_1(\frac{\xi}{2})\hat{\phi}(\frac{\xi}{2}) \quad (30)$$

where the 2p periodic function $m_0(x)$ and $m_1(x)$ are defined as

$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_k h_k e^{-ik\xi} \quad (31)$$
and
\[ m_1(\xi) = \frac{1}{\sqrt{2}} \sum_k g_k e^{-i\xi k} \] (32)

The scaling function \( f(x) \) is normalized so that
\[ \int_{-\infty}^{+\infty} \phi(x) dx = 1 \] (33)

\[ \hat{\phi}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(x) dx = \frac{1}{\sqrt{2\pi}} \] (34)

Therefore
\[ m_0(0) = \frac{1}{\sqrt{2}} \sum_k h_k = 1 \] (35)

The orthogonality of \( \{\phi_{0,n}; n \in \mathbb{Z}\} \) requires
\[ \int_{-\infty}^{+\infty} \phi(x) \phi(x-k) dx = \int_{-\infty}^{+\infty} \left| \phi(\xi) \right|^2 e^{i\xi k} d\xi = \int_{-\infty}^{+\infty} \left( \sum_l \left| \phi(\xi + 2\pi l) \right|^2 e^{i\xi k} d\xi = \delta_{k,0} \] (36)

where \( \delta_{k,0} \) is the kronecker delta, yields
\[ \sum_l \left| \hat{\phi}(\xi + 2\pi l) \right|^2 = \frac{1}{2\pi} \] (37)

Substituting eq.(29) leads to (let \( z=x/2 \))
\[ \sum_l \left| m_0(\xi + \pi l) \right|^2 \left| \hat{\phi}(\xi + \pi l) \right|^2 = (2\pi)^{-1} \] (38)

Splitting the sum into even and odd \( l \), using the periodicity of \( m_0 \),
\[
\sum_{l} (|m_{0}(\zeta + 2\pi l)|^2 |\hat{\phi}(\zeta + 2\pi l)|^2 + |m_{0}(\zeta + (2l + 1)\pi)|^2 |\hat{\phi}(\zeta + (2l + 1)\pi)|^2 )
\]
\[
= \sum_{l} (|m_{0}(\zeta)|^2 + |m_{0}(\zeta + \pi)|^2 |\hat{\phi}(\zeta + 2\pi l)|^2 ) = (2\pi)^{-1}
\]

Equations (37) and (39) yield
\[
|m_{0}(\zeta)|^2 + |m_{0}(\zeta + \pi)|^2 = 1
\]

Substituting eq. (31) into eq. (40) yields

\[
\sum_{k} h_{k} h_{n+2k} = \delta_{k,0}
\]

The orthogonal condition \(\psi \perp V_{0}\) implies \(\psi \perp \phi_{0,k}\), in Fourier domain, which means,

\[
\int \hat{\psi}(\zeta) \hat{\phi}(\zeta) e^{ik\xi} d\xi = 0 \quad \text{or} \quad \int_{0}^{2\pi} d\xi e^{ik\xi} \sum_{l} \hat{\psi}(\zeta + 2\pi l) \hat{\phi}(\zeta + 2\pi l) = 0
\]

hence

\[
\sum_{l} \hat{\psi}(\zeta + 2\pi l) \hat{\phi}(\zeta + 2\pi l) = 0
\]

Substituting (29) and (30), regrouping the sums for odd and even \(l\), and using (37) leads to

\[
m_{1}(\zeta)m_{0}(\zeta) + m_{1}(\zeta + \pi)m_{0}(\zeta + \pi) = 0
\]

Since \(m_{0}(\zeta)\) and \(m_{0}(\zeta + \pi)\) cannot vanish at the same time because of eq. (40), this implies the existence of a \(2\pi\)-periodic function \(l(z)\) so that

\[
m_{1}(\zeta) = \lambda(\zeta)m_{0}(\zeta + \pi)
\]

and
\[ \lambda(\zeta) + \lambda(\zeta + \pi) = 0 \]  

(45)

If we choose

\[ \lambda(\zeta) = e^{i\xi} \]  

(46)

then

\[ \hat{\psi}(\xi) = e^{i\xi/2} \frac{m_0(\xi/2 + \pi)}{e^{i\xi/2} \hat{\phi}(\xi/2)} \]  

(47)

On the other hand if function \( f(x) \in W_0 \), therefore \( f(x) \in V_{-1} \), we have

\[ f = \sum_k f_k \phi_{-1,k} \]  

(48)

This implies

\[ \hat{f}(\xi) = \frac{1}{\sqrt{2}} \sum_k f_k e^{-ik\xi/2} \hat{\phi}(\xi/2) = m_f(\xi/2) \hat{\psi}(\xi/2) \]  

(49)

where

\[ m_f(\xi) = \frac{1}{\sqrt{2}} \sum_k f_k e^{-ik\xi} \]  

(50)

Substituting (47) into (49) for \( \hat{\phi}(\xi/2) \)

\[ \hat{f}(\xi) = \frac{m_f(\xi/2)}{e^{i\xi/2} m_0(\xi/2 + \pi)} \hat{\psi}(\xi) \]  

(51)

Since \( \hat{f}(\xi) \) and \( \hat{\psi}(\xi) \) are 2p periodic functions, \( \frac{m_f(\xi/2)}{e^{i\xi/2} m_0(\xi/2 + \pi)} \) is also 2p periodic.

Therefore eq.(51) can be written as
\[ \hat{f}(\xi) = \left( \sum_k \gamma_k e^{-ik\xi} \right) \hat{\psi}(\xi) \]  \hspace{1cm} (52)

or

\[ f = \sum_k \gamma_k \psi(\xi - k) \]  \hspace{1cm} (53)

which means \( \psi(\xi - k) \) is a set of basis of \( W_0 \).

Next we need to verify that the \( \gamma_{0,k} \) are an orthonormal basis for \( W_0 \).

\[ \int_{\mathbb{R}^2} \psi(x) \overline{\psi(x-k)} \, dx = \int_0^{2\pi} d\xi e^{ik\xi} |\hat{\psi}(\xi)|^2 = \int_0^{2\pi} \left( \sum_l |\hat{\psi}(\xi + 2\pi l)|^2 \right) e^{ik\xi} d\xi \]  \hspace{1cm} (54)

Substituting eq.(47),

\[ \sum_l |\hat{\psi}(\xi + 2\pi l)|^2 = \sum_l \left| m_0(\xi/2 + \pi l + \pi) \right|^2 \left| \phi(\xi/2 + \pi l) \right|^2 \]

grouping \( l = 2n \) (even), \( l = 2n + 1 \) (odd) terms,

\[ \sum_l |\hat{\psi}(\xi + 2\pi l)|^2 = \left| m_0(\xi/2 + \pi) \right|^2 \sum_n \left| \phi(\xi/2 + 2\pi n) \right|^2 \]

\[ + \left| m_0(\xi/2) \right|^2 \sum_n \left| \phi(\xi/2 + \pi + 2\pi n) \right|^2 \]

\[ = \left| m_0(\xi/2 + \pi) \right|^2 (2\pi)^{-1} + \left| m_0(\xi/2) \right|^2 (2\pi)^{-1} \text{ (by (1.37))} \]

\[ = (2\pi)^{-1} \text{ (by (1.40))} \]

Hence

\[ \int_{\mathbb{R}^2} \psi(x) \overline{\psi(x-k)} \, dx = (2\pi)^{-1} \int_0^{2\pi} e^{ik\xi} d\xi = \delta_{k,0} \]  \hspace{1cm} (55)

Equation(47) implies,
\[
\psi = \sum_k (-1)^{k-1} h_{-k-1} \phi_{-1,k} \quad \text{i.e.,} \quad \psi = \sqrt{2} \sum_k (-1)^{k-1} h_{-k-1} \phi(2x - k)
\]

compare with Eq.(27)

\[
g_k = (-1)^{k-1} h_{-k-1}
\]

(56)

As a summary, a multiresolution analysis consists of a ladder of spaces \((V_j)_{j \in \mathbb{Z}}\) and a special function \(\phi \in V_0\) such that properties (1)-(5) are satisfied. One may try to start the construction from an appropriate choice for the scaling function \(f\). After all, \(V_0\) can be constructed from the \(\phi(\cdot - k)\), and from there, all the other \(V_j\) can be generated. On the other hand, wavelet basis \((W_j)_{j \in \mathbb{Z}}\) is the orthogonal complement of \(V_j\) in \(V_{j-1}\), which means wavelets \(\psi_{j,k}\) describes the difference between \(V_j\) and \(V_{j-1}\). Therefore \(Y\) can be constructed from \(f\).

### B.1.3 Compactly Supported Orthogonal Wavelets

If we build the multiresolution structure from \(f\) or the \(V_j\), wavelet bases will be, normally, infinitely supported functions. To construct compactly supported wavelets bases, it pays to start from \(m_0\), i.e., eq.(31).

Daubieches (1992) proved for compactly supported \(f\) the 2p-periodic function \(m_0\),

\[
m_0(\xi) = \frac{1}{\sqrt{2}} \sum_k h_k e^{-ik\xi}
\]

(31)
becomes a trigonometric polynomial. Besides the orthonormality of the \( f_{0,k} \), i.e.,

\[
|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1
\]  
(40)

\( m_0 \) should be of the form

\[
m_0(\xi) = \left( \frac{1 + e^{i\xi}}{2} \right)^N \eta(\xi)
\]  
(57)

with \( N \geq 1 \) and \( h \) a trigonometric polynomial. Combining (40) and (57) with notation

\[
M_0(\xi) = |m_0(\xi)|^2, \quad L(\xi) = |\eta(\xi)|^2
\]

\[
M_0(\xi) = \left( \cos^2 \frac{\xi}{2} \right)^N L(\xi)
\]  
(58)

\( L(x) \) is also a polynomial in \( \cos x \), so rewrite \( L(x) \) as a polynomial in

\[
\sin^2 \frac{\xi}{2} = (1 - \cos \xi) / 2,
\]

\[
M_0(\xi) = \left( \cos^2 \frac{\xi}{2} \right)^N P\left( \sin^2 \frac{\xi}{2} \right)
\]  
(59)

\( P \) has the form

\[
P(x) = P_N + x^N R \left( \frac{1}{2} - x \right)
\]  
(60)

where

\[
P_N = \sum_{k=0}^{N-1} \binom{N-1+k}{k} x^k
\]  
(61)

\( R \) is an odd polynomial, chosen such that \( P(x) \geq 0 \) for \( x \in [0,1] \). Daubieches proved that with eq.(40), eq.(57) and condition \( m_0(0)=1 \), one can define \( f, y \) by
\[ \hat{\phi}(\xi) = (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\xi) \]

\[ \hat{\psi}(\xi) = e^{-i\xi/2} m_0(\xi/2 + \pi) \hat{\phi}(\xi/2) \]

Then \( f, y \) are compactly supported \( L^2 \)-functions, satisfying

\[ \phi(x) = \sqrt{2} \sum_n h_n \phi(2x - n) \]

\[ \psi(x) = \sqrt{2} \sum_n (-1)^n h_{-n} \phi(2x - n) \]

Where \( h_n \) is determined by \( m_0 \) via \( m_0(\xi) = \frac{1}{\sqrt{2}} \sum_k h_k e^{-i k \xi} \). Moreover, the

\[ \psi_{j,k}(x) = 2^{-j/2} \psi(2^{-j}x - k), \quad j, k \in \mathbb{Z} \]

constitute an orthonormal basis if and only if the following condition is satisfied [5]:

The eigenvalue 1 of the \([2(N_2 - N_1) - 1] \times [2(N_2 - N_1) - 1]\) dimensional matrix \( A \) defined by

\[ A_{1,k} = \sum_{n=N_1}^{N_2} h_n h_{k-2l+n} \quad -(N_2 - N_1) + 1 \leq 1, k \leq (N_2 - N_1) + 1 \] (62)

(where we assume \( h_n = 0 \) for \( n < N_1; n > N_2 \)) is nondegenerate.

The well-known Daubechies’ wavelets were constructed by selection of \( R = 0 \) in eq.(60). Therefore \( L(x) = P_N \sin^2 x/2 \). The zeros of \( P_N \) are within the unit circle. For each \( N \), \( m_0 \) has \( 2N \) nonvanishing coefficients,
\[ m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{n=0}^{2N-1} h_n e^{-i \xi n} \]  

(63)

Both \( f \) and \( y \) have support width \( 2N-1 \). If the support of \( f(x) \) is \([0, 2N-1]\), from eq.(61) \( h_n \) goes from \( N_1 \) to \( N_2 \), the basic scaling function recursion equation becomes

\[ \phi(x) = \sqrt{2} \sum_{n=N_1}^{N_2} h_n \phi(2x - n) \]

where the support of the right hand side is \([N_1/2, (2N-1+N_2)/2]\). Since the support of both sides must be the same, the limits on the sum; or the limits on the indices of the nonzero \( h_n \) are such that \( N_1=0 \), and \( N_2=2N-1 \). In the wavelet recursion equation \( h_{-n-1} \) can be modulated by the support length \( 2N \). Therefore the basic recursion equations for compactly supported wavelets are,

\[ \phi(x) = \sqrt{2} \sum_{n=0}^{2N-1} h_n \phi(2x - n) \]  

(64)

\[ \psi(x) = \sqrt{2} \sum_{n=0}^{2N-1} h_{2N-n-1} \phi(2x - n) \]  

(65)

Notice that eq.(41) and \( m_0(0)=1 \), i.e.,

\[ \sum_{n=0}^{2N-1} h_n = \sqrt{2} \]  

(66)

construct only \( N+1 \) equations for \( 2N \) coefficient \( h_n \). More equations are needed if \( N > 1 \). Regularity conditions come to help us.

Burrus (1996) described that regularity and moments are related to the smoothness or differentiability of \( f \) and \( y \). Define the \( k \) th moments of \( f \) and \( y \) as
\[ M(k) = \int x^k \phi(x) \, dx \]  
(67)

\[ M_1(k) = \int x^k \psi(x) \, dx \]  
(68)

and the discrete \( k \)th moments of \( h_n \) and \( h_{2N-n-1} \) as

\[ \mu(k) = \sum_n n^k h_n \]  
(69)

\[ \mu_1(k) = \sum_n n^k (-1)^n h_{2N-1-n} \]  
(70)

To achieve higher order differentiability, which is required to solve differential equations, more vanishing wavelet moments and the discrete moments are required. Therefore the rest \( N-1 \) conditions are,

\[ \sum_n n^k (-1)^n h_{2N-1-n} = 0 \quad k = 0, \ldots, N-1 \]  
(71)

Table 1 shows the \( h_n \) coefficients calculated by Daubechies up to \( N = 10 \). Next we will focus on the numerical algorithm on calculation of wavelets.

\[ \text{B.1 4 Approximation Properties of Wavelet Bases and Fast Wavelet Transform} \]

The multiresolution analysis yields that any \( L^2(R) \) function can be uniquely represented by its wavelet series

\[ f(x) = \sum_{k=-\infty}^{\infty} c_k \phi_k(x) + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} d_{j,k} \psi_{j,k}(x) \]  
(72)
Where the infinite summations converge in $L^2(R)$. However, since computational resources are limited. Equation (72) must be truncated, and information from only several scales can be retained.

In this regard, it is pointed out that the approximation properties of wavelets depend on the difference between the function and its projection on the $j$-th scale. Here we will follow closely Newland’s [4] description of circular wavelet expansion of $L^2$ functions.

First for any function $f(x) \in L^2(R)$, we limit the range of the independent variable $x$ to one unit interval so that $f(x)$ is assumed to be defined only for $0 \leq x < 1$. It is also convenient to assume that $f(x)$, $0 \leq x < 1$, is one period of a periodic function so that the function $f(x)$ is exactly repeated in the adjacent unit intervals to give

$$F(k) = \sum_k f(x - k)$$  \hspace{1cm} (73)

Suppose that we are using D4 wavelet (D means Daubechies, which means we use the $h$ coefficients in table 1; 4 means $2N = 4$), which occupies three unit intervals $0 \leq x < 3$. In the interval $0 \leq x < 1$, $f(x)$ will receive contributions from the first third of $y(x)$, the middle third of $y(x + 1)$, and the last third of $y(x + 2)$. This is the same as if $y(x)$ is “wrapped around” the unit interval twice. Therefore, when any wavelet which starts in the interval $0 \leq x < 1$ runs off the end at $x = 1$, it may be assumed to be wrapped around the interval as many times as their entire length is within the unit interval. So that eq. (72) can be written as
\[ f(x) = a_0 \phi(x) + a_1 W(x) + \begin{bmatrix} a_2 & a_3 \\ W(2x) & W(2x - 1) \end{bmatrix} + \begin{bmatrix} a_4 & a_5 & a_6 & a_7 \\ W(4x) & W(4x - 1) & W(4x - 2) & W(4x - 3) \end{bmatrix} + \cdots + a_{2^j+k} W(2^j x - k) \]  

(74)

where \( W(x) \) is the wrapped around wavelets. Wrapped around scaling function \( f(x) \) will become a constant. Actually since the vanishing wavelet moments, eq.(68),

\[
\int \psi(x) dx = 0
\]

(75)

or mean zero eq.(17), if we integrate both sides of eq.(72), the integral of first term is equal to the integral of function \( f \), which is a constant for any given \( f \). One obvious way is to set \( \phi(x) = 1 \). The coefficients \( a_1, a_2, a_3, \ldots \) give the amplitudes of each of the contributing wavelets (after wrapping) to one cycle of the periodic function (73) in the interval \( 0 \leq x < 1 \). It also turns out that at scale zero \((j=0)\), there is \( 2^0 = 1 \) wavelet, at scale one there are \( 2^1 = 2 \) wavelets, therefore at level \( j \), there are \( 2^j \) wavelets each spaced \( \Delta x = 2^{-j} \) apart along the \( x \)-axis.

By the orthogonality conditions eq.(55) and (75),

\[ a_0 = \int_0^1 f(x) \phi(x) dx \]  

(76)

\[ a_{2^j+k} = \int_0^1 f(x) W(2^j x - k) dx \]  

(77)

The discrete wavelet transform (DWT) is an algorithm for computing eq.(76) and (77) when \( f(x) \) is sampled at equally spaced intervals over \( 0 \leq x < 1 \). Obviously the integrals in eq.(76) and (77) can be computed to whatever accuracy is required after first
generating $\mathcal{W}(2^j x - k)$. However, a remarkable feature of the DWT algorithm is that this is not necessary.

The DWT algorithm introduced here is called Mallat’s tree algorithm. Mathematically, it is a matrix multiplication algorithm. We will approach the algorithm by considering first its inverse, suppose that the DWT has been computed to generate the sequence

$$a = [a_0 \ a_1 \ a_2 \ a_3 \ ... \ a_{2^j+k} \ ...]$$ \hspace{1cm} (78)

In order to include all the wavelets at any particular scale, the total number of terms in the transform must always be a power of 2.

As an example, consider the case there are only eight terms,

$$a = [a_0 \ a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \ a_7]$$ \hspace{1cm} (79)

The first element $a_0$ is the amplitude of the scaling function term $f(x)$. $f(x)$ can be generated by iteration from a unit box function over $0 \leq x < 1 \ [4]$

$$\phi^{(1)}(x) = \sum_{n=0}^{2N-1} c_n \phi^{(0)}(2x - n)$$ \hspace{1cm} (80)

where $N = 2$ (D4 wavelets), $c_n = \sqrt{2} h_n$, $h_n$ are Daubechies coefficients and $\phi^{(0)}(x) = 1$. Therefore $a_0 \phi(x)$ can be generated by iteration a box of height $a_0$. The first step in the iteration is
\[
\phi^{(1)} = \begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3 \\
\end{bmatrix}
\begin{bmatrix}
a_0 \\
\end{bmatrix}
\]
(81)

If the part outside the interval is ped round to fall back into the unit interval, it becomes,

\[
\phi^{(1)} = \begin{bmatrix}
c_0 + c_2 \\
c_1 + c_3 \\
\end{bmatrix}
\begin{bmatrix}
a_0 \\
\end{bmatrix}
\]
(82)

Taking the second iteration, including wrap-around,

\[
\phi^{(2)} = \begin{bmatrix}
c_0 & c_2 \\
c_1 & c_3 \\
c_2 & c_0 \\
c_3 & c_1 \\
\end{bmatrix}
\begin{bmatrix}
c_0 + c_2 \\
c_1 + c_3 \\
\end{bmatrix}
\begin{bmatrix}
a_0 \\
\end{bmatrix}
\]
(83)

For the third iteration step,

\[
\phi^{(3)} = \begin{bmatrix}
c_0 & c_2 \\
c_1 & c_3 \\
c_2 & c_0 \\
c_3 & c_1 \\
\end{bmatrix}
\begin{bmatrix}
c_0 & c_2 \\
c_1 & c_3 \\
c_2 & c_0 \\
c_3 & c_1 \\
\end{bmatrix}
\begin{bmatrix}
c_0 + c_2 \\
c_1 + c_3 \\
\end{bmatrix}
\begin{bmatrix}
a_0 \\
\end{bmatrix}
\]
(84)

This generates the eight ordinates in interval \(0 \leq x < 1\). Let \(M_i\) be the matrices, the algorithm for generating the contribution of \(a_0 \phi(x)\) to \(f(x)\) is

\[
f^\phi(1:8) = M_3M_2M_1a_0
\]
(85)

or, in diagrammatic form,
\[
a_0 = f^{\phi}(1) \xrightarrow{M_1} f^{\phi}(1:2) \xrightarrow{M_2} f^{\phi}(1:4) \xrightarrow{M_3} f^{\phi}(1:8)
\]

where \(f^{\phi}(1:8)\) means an array of eight elements that represents the contribution of \(a_0^\phi(x)\) to \(f(x)\) at \(x = 0, \frac{1}{8}, \frac{1}{4}, \ldots, \frac{7}{8}\).

Next consider the second term \(a_1\), the amplitude of the wavelet function \(W(x)\) which is also generated from the unit box by iteration [4]. The matrix operations for doing this are the same as for generating the scaling function \(f(x)\) except that the first step involves replacing

\[
\begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3
\end{bmatrix}
\]

by

\[
\begin{bmatrix}
-c_3 \\
c_2 \\
-c_1 \\
c_0
\end{bmatrix}
\]

Due to wrapping around, the algorithm for generating the contribution of \(a_1W(x)\) to \(f(x)\) is

\[
f^{(0)}(1:8) = M_3M_2G_1a_1
\]

where

\[
G_1 = \begin{bmatrix}
-c_3 - c_1 \\
c_2 + c_0
\end{bmatrix}
\]

In diagrammatic form,

\[
a_1 = f^{(0)}(1) \xrightarrow{G_1} f^{(0)}(1:2) \xrightarrow{M_2} f^{(0)}(1:4) \xrightarrow{M_3} f^{(0)}(1:8)
\]

The third and fourth terms, \(a_2\) and \(a_3\), are the amplitudes of \(W(2x), W(2x-1)\). They are at the same level.
\[ f^{(1)}(1:8) = M_3 G_2 \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} \] (90)

where

\[ G_2 = \begin{bmatrix} -c_3 & -c_1 \\ c_2 & c_0 \\ -c_1 & -c_3 \\ c_0 & c_2 \end{bmatrix} \] (91)

and, in a diagram,

\[ \begin{array}{c}
\begin{bmatrix} a_2 \\ a_3 \end{bmatrix} \\
\xrightarrow{G_2} \\
\xrightarrow{M_3} \\
\xrightarrow{f^{(1)}(1:4)} \\
\xrightarrow{f^{(1)}(1:8)}
\end{array} \] (92)

The remaining four elements of eq.(79), \( a_4, a_5, a_6, a_7 \) are at level 2. It turns out the algorithm is,

\[ f^{(2)}(1:8) = G_3 \begin{bmatrix} a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \] (93)

\[ G_3 = \begin{bmatrix} -c_3 & -c_1 \\ c_2 & c_0 \\ -c_1 & -c_3 \\ c_0 & c_2 \\ -c_1 & -c_3 \\ c_0 & c_2 \end{bmatrix} \] (94)

Diagrammatically
Finally,

\[
f(1:8) = f^{(\phi)}(1:8) + f^{(0)}(1:8) + f^{(1)}(1:8) + f^{(2)}(1:8)
\]

(96)

\[
\begin{align*}
    f^{\phi}(1) & \xrightarrow{M_1} f^*(1:2) \xrightarrow{M_2} f^*(1:4) \xrightarrow{M_3} f^*(1:8) = f \\
    \uparrow & \quad G_1 \uparrow \quad G_2 \uparrow \quad G_3 \uparrow \\
    a &= [a(1) \quad a(2) \quad a(3:4) \quad a(5:8)]
\end{align*}
\]

(97)

This is the complete inverse of DWT.

Now consider how to break down an arbitrary function \(f(1:2^3)\) into its wavelet transform \(a(1:2^3)\). Because of the orthogonality condition, eq.(41), we have

\[
\begin{align*}
    \frac{1}{2} M_r^t M_r &= I \\
    M_r^t G_r &= 0 \\
    G_r^t M_r &= 0 \\
    \frac{1}{2} G_r^t G_r &= I
\end{align*}
\]

(98)

where \(t\) means matrix transpose.

Define \(H\) and \(L\) for the transposes of \(G\) and \(M\), so that

\[
H = G^t \quad L = M^t
\]

(99)

Reversing each of the steps that make up the diagram (97). For example,

\[
a_0 = \frac{1}{2} L_1 \frac{1}{2} L_2 \frac{1}{2} L_3 f^\phi
\]

(100)

Replacing \(f^\phi\) by \(f\), according to eq.(98),
\[ \frac{1}{2} L_1 \frac{1}{2} L_2 \frac{1}{2} L_3 f = \frac{1}{2} L_1 \frac{1}{2} L_2 \frac{1}{2} L_3 f^\phi \]  

(101)

Therefore,

\[ a_0 = \frac{1}{2} L_1 \frac{1}{2} L_2 \frac{1}{2} L_3 f \]  

(102)

and, similarly

\[ a_1 = \frac{1}{2} H_1 \frac{1}{2} L_2 \frac{1}{2} L_3 f \]  

(103)

\[ \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \frac{1}{2} H_2 \frac{1}{2} L_3 f \]  

(104)

\[ \begin{bmatrix} a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \frac{1}{2} H_3 f \]  

(105)

The diagram (97) can be reversed,

\[
\begin{array}{cccc}
\frac{1}{2} L_1 & \frac{1}{2} L_2 & \frac{1}{2} L_3 & f^\phi(1:8) \\
\uparrow & \frac{1}{2} H_1 & \uparrow & \frac{1}{2} H_2 & \uparrow & \frac{1}{2} H_3 \\
& a(1) & a(2) & a(3:4) & a(5:8) \\
\end{array}
\]

(106)

Rewrite eq.(96),

\[ f(1:8) = M_3M_2M_1a_0 + M_3M_2G_1a_1 + M_3G_2 \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} + G_3 \begin{bmatrix} a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \]  

(107)

If any \( a_i \) (in this case \( i = 0, \ldots, 7 \)) = 1, \( a_j = 0 \) (\( j \neq i \)), the corresponding wavelet function is numerically calculated.
References


APPENDIX C

SPATIAL DISCRETIZATION

C.1 Numerical Solution to One-dimensional Radiative Transfer

Equation After Application of Wavelets Method

The details of the finite differencing applied to the one-dimensional radiative transfer equation (RTE) with scattering terms are given in this section. Equation (10a) of the Chapter 2, which represents the set of ordinary differential equations written in terms of wavelet expansion coefficients $a_i(\tau)$, is discretized by using the first-order forward differencing. Recall the Eq. (10a) of the Chapter 2:

$$\sum_i A_{ij} \frac{d a_i(\tau)}{d\tau} = a_j(\tau) + (1 - \omega)I_h(\tau)\delta_{j,1} + \frac{\omega}{2} \sum_i [a_i(\tau)S_{i,j} + b_i(\tau)S_{i,j}^2]$$

(1)

Applying the first-order forward differencing to this equation,

$$\sum_i A_{ij} \frac{a_i^{m+1} - a_i^m}{\Delta\tau} = a_j^m + (1 - \omega)I_h^m\delta_{j,1} + \frac{\omega}{2} \sum_i [a_i^m S_{i,j}^1 + b_i^m S_{i,j}^2]$$

(2)

and rearranging we obtain the following set of equations:

$$\sum_i A_{ij} a_i^{m+1} = \sum_i A_{ij} a_i^m + (\Delta\tau) b_j^m + (\Delta\tau)(1 - \omega)I_h^m\delta_{j,1} + (\Delta\tau) \frac{\omega}{2} \sum_i [a_i^m S_{i,j}^1 + b_i^m S_{i,j}^2]$$

(3)

where $m=1,\ldots,M$ and $i,j=1,\ldots,N$. $M$ is the number spatial discretization points. $N$ indicates the level of the wavelet approximation. Boundary conditions for the above set are

$$a_j^1 = I_b(T^1)\delta_{j,1}$$

(4)
If $I_b^m$, $S_{i,j}^1$ and $S_{i,j}^2$ are known, Eq. (3) is consist of MxN linear independent equations can be easily solved. However, $I_b^m$, $S_{i,j}^1$ and $S_{i,j}^2$ are not known, therefore an iterative method is needed to solve the equation system. First assuming a temperature profile, initial values for $I_b^m$, $S_{i,j}^1$ and $S_{i,j}^2$ calculated. To move the next step, temperature profile is updated based on the energy equation (in the case of radiative equilibrium, $\nabla \cdot \mathbf{q} = 0$). This iteration is continued until the energy equation is satisfied.

### C.2 Numerical Solution to Two-dimensional Radiative Transfer

#### Equation After Application of Wavelets Method

In the Chapter 3, RTE takes the following form after the application of the wavelet method:

\[
\sum_m \sum_n \left[ A_{m,m',nn'} \frac{\partial a_{m,n}(\tau_y, \tau_z)}{\partial \tau_y} + B_{m,m',nn'} \frac{\partial a_{m,n}(\tau_y, \tau_z)}{\partial \tau_z} \right] + a_{m',n'}(\tau_y, \tau_z) = I_b(\tau_y, \tau_z) \cdot \delta_{m',1} \cdot \delta_{n',1}
\]

(5a)

\[
\sum_m \sum_n \left[ A_{m,m',nn'} \frac{\partial b_{m,n}(\tau_y, \tau_z)}{\partial \tau_y} - B_{m,m',nn'} \frac{\partial b_{m,n}(\tau_y, \tau_z)}{\partial \tau_z} \right] + b_{m',n'}(\tau_y, \tau_z) = I_b(\tau_y, \tau_z) \cdot \delta_{m',1} \cdot \delta_{n',1}
\]

(5b)

\[
\sum_m \sum_n \left[ -A_{m,m',nn'} \frac{\partial c_{m,n}(\tau_y, \tau_z)}{\partial \tau_y} + B_{m,m',nn'} \frac{\partial c_{m,n}(\tau_y, \tau_z)}{\partial \tau_z} \right] + c_{m',n'}(\tau_y, \tau_z) = I_b(\tau_y, \tau_z) \cdot \delta_{m',1} \cdot \delta_{n',1}
\]

(5c)

\[
\sum_m \sum_n \left[ -A_{m,m',nn'} \frac{\partial d_{m,n}(\tau_y, \tau_z)}{\partial \tau_y} - B_{m,m',nn'} \frac{\partial d_{m,n}(\tau_y, \tau_z)}{\partial \tau_z} \right] + d_{m',n'}(\tau_y, \tau_z) = I_b(\tau_y, \tau_z) \cdot \delta_{m',1} \cdot \delta_{n',1}
\]

(5d)

Finite Volume Method is applied to solve Eq. (5). Choosing the finite volume as shown in Fig. 1, Eq. (5a) can be discretized as:
\[
\sum_{m} \sum_{n} \left[ A_{mn',nn'} \frac{a_{m,n}' - a_{m,n}}{\Delta \tau_y} + B_{mn',nn'} \frac{a_{m,n}' - a_{m,n}}{\Delta \tau_y} \right] + a_{m,n}' = [I_b \cdot \delta_{m',1} \cdot \delta_{n',1}]^p \tag{6}
\]

Value of wavelet expansion coefficients at the center of the finite volume, \(a_{m,n}'\), is related to their values on the surfaces of the finite volume (\(a_{m,n}^e\), \(a_{m,n}^n\), \(a_{m,n}^w\) and \(a_{m,n}^s\)) with the following equation (Step Scheme):

\[
a_{m,n}' = a_{m,n}^e = a_{m,n}^n
\tag{7}
\]

Substituting this into Eq. (6):

\[
\sum_{m} \sum_{n} \left[ A_{mn',nn'} \frac{a_{m,n}' - a_{m,n}}{\Delta \tau_y} + B_{mn',nn'} \frac{a_{m,n}' - a_{m,n}}{\Delta \tau_y} \right] + a_{m',n'}' = [I_b \cdot \delta_{m',1} \cdot \delta_{n',1}]^p \tag{8}
\]

which can be reorganized as

\[
\sum_{m} \sum_{n} \left[ A_{mn',nn'} + \left( \frac{\Delta \tau_y}{\Delta \tau_z} \right) B_{mn',nn'} \right] a_{m,n}' + \Delta \tau_y a_{m',n'}' = \Delta \tau_y [I_b \cdot \delta_{m',1} \cdot \delta_{n',1}]^p
\]

\[
+ \sum_{m} \sum_{n} \left[ A_{mn',nn'} a_{m,n}' + \left( \frac{\Delta \tau_y}{\Delta \tau_z} \right) B_{mn',nn'} a_{m,n}^s \right].
\tag{9}
\]

Fig. 1. Two-dimensional finite volume.
Boundary conditions for Eq. (9) in the case where all the walls of the rectangular enclosure are black body and the bottom wall is hot while the others are cold can be formulated as,

\[ a^{\delta}_{m,n} = \delta_{m,n} \quad \text{and} \quad a^{\omega}_{m,n} = 0. \]  

(10)

Now, it is possibly to sweep through the spatial domain from SW to NE direction with Eq. (9) provided that \( I_b^{\mu} \) is known. Therefore, a similar iterative method explained in previous section is utilized; namely: after making an initial temperature profile assumption, Eq. (9) is solved starting from the boundary conditions (Eq. 9) and then the temperature profile is updated with help of the energy equation before moving to second iteration.

C.3 Numerical Solution to Two-dimensional Transient Radiative

Transfer Equation After Application of Wavelets Method

After the application of wavelets, the transient RTE can be expressed as (see Chapter 4 for details),

\[ \frac{1}{c} \frac{\partial a_{m,n}}{\partial t} + \sum_m \sum_n \left[ A_{mnm',nm'} \frac{\partial a_{m,n}}{\partial y} + B_{mnm',nm'} \frac{\partial a_{m,n}}{\partial z} \right] + \beta a_{m',n'} = S \cdot \delta_{m',1} \cdot \delta_{n',1} \]  

(11a)

\[ \frac{1}{c} \frac{\partial b_{m,n}}{\partial t} + \sum_m \sum_n \left[ A_{mm',nn'} \frac{\partial b_{m,n}}{\partial y} - B_{mm',nn'} \frac{\partial b_{m,n}}{\partial z} \right] + \beta b_{m',n'} = S \cdot \delta_{m',1} \cdot \delta_{n',1} \]  

(11b)

\[ \frac{1}{c} \frac{\partial c_{m,n}}{\partial t} + \sum_m \sum_n \left[ -A_{mm',nn'} \frac{\partial c_{m,n}}{\partial y} + B_{mm',nn'} \frac{\partial c_{m,n}}{\partial z} \right] + \beta c_{m',n'} = S \cdot \delta_{m',1} \cdot \delta_{n',1} \]  

(11c)

\[ \frac{1}{c} \frac{\partial d_{m,n}}{\partial t} + \sum_m \sum_n \left[ -A_{mm',nn'} \frac{\partial d_{m,n}}{\partial y} - B_{mm',nn'} \frac{\partial d_{m,n}}{\partial z} \right] + \beta d_{m',n'} = S \cdot \delta_{m',1} \cdot \delta_{n',1} \]  

(11d)
where \( m,n,m',n' = 1,\ldots,M \) and \( M \) represents the level of the wavelet approximation.

Equation (11) is very similar to Eq. (5) with the addition of transient term. With an implicit differencing in time, the following can be written for Eq. (11a):

\[
\frac{1}{c} \left( \frac{(a_{m',n'})_p - (a_{m',n'})_w}{\Delta t} + \sum_{m} \sum_{n} \left[ A_{mn',nn'} \frac{(a_{m,n})_p - (a_{m,n})_w}{\Delta y} + B_{mn',nn'} \frac{(a_{m,n})_p - (a_{m,n})_w}{\Delta z} \right] + \beta(a_{m',n'})_p = S_p \cdot \delta_{m',1} \cdot \delta_{n',1} \right) 
\]  \hspace{1cm} (12)

After rearranging this equation,

\[
(a_{m',n'})_p (\beta \Delta y + \Delta y / \Delta t) + \sum_{m} \sum_{n} \left[ A_{mn',nn'} + B_{mn',nn'} (\Delta y / \Delta z) \right] (a_{m,n})_p = \Delta y S_p \cdot \delta_{m',1} \cdot \delta_{n',1} 
\] 
\[
+ (\Delta y / \Delta t) (a_{m',n'})_p + \sum_{m} \sum_{n} [A_{mn',nn'} (a_{m,n})_w + B_{mn',nn'} (\Delta y / \Delta z) (a_{m,n})_w] \right) 
\]  \hspace{1cm} (13)

Boundary conditions (with the assumption of walls being black body) are as follows:

\[
(a_{m',n'})_s = (I_b)_s \cdot \delta_{m',1} \cdot \delta_{n',1} \quad \text{and} \quad (a_{m',n'})_w = (I_b)_w \cdot \delta_{m',1} \cdot \delta_{n',1} \]  \hspace{1cm} (14)

This system of equations then can be solved for a set of given initial conditions.