INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps.

ProQuest Information and Learning
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
800-521-0600

UMI®
COLLECTIVE SINGULARITIES OF
FAMILIES OF ANALYTIC FUNCTIONS

by

Alan Wilson

A THESIS
SUBMITTED TO THE FACULTY
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Houston, Texas
April, 1958
## CONTENTS

COLLECTIVE SINGULARITIES OF FAMILIES OF
ANALYTIC FUNCTIONS

<table>
<thead>
<tr>
<th>SECTION</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. The type of a family of entire functions</td>
<td>1</td>
</tr>
<tr>
<td>2. Remarks on family type</td>
<td>1</td>
</tr>
<tr>
<td>3. <strong>THEOREM</strong></td>
<td>1</td>
</tr>
<tr>
<td>4. Remarks on convex sets</td>
<td>4</td>
</tr>
<tr>
<td>5. Some integral relations</td>
<td>5</td>
</tr>
<tr>
<td>6. The conjugate diagram of a family</td>
<td>6</td>
</tr>
<tr>
<td>7. Another integral relation</td>
<td>8</td>
</tr>
<tr>
<td>8. The indicator of a family of entire functions</td>
<td>8</td>
</tr>
<tr>
<td>9. The crucial relation between conjugate diagram and indicator</td>
<td>9</td>
</tr>
<tr>
<td>10. The indicator diagram</td>
<td>12</td>
</tr>
<tr>
<td>11. <strong>THEOREM</strong></td>
<td>12</td>
</tr>
<tr>
<td>12. <strong>THEOREM</strong></td>
<td>13</td>
</tr>
<tr>
<td>13. Interpolated families</td>
<td>14</td>
</tr>
<tr>
<td>14. <strong>THEOREM</strong></td>
<td>14</td>
</tr>
<tr>
<td>15. An example</td>
<td>17</td>
</tr>
<tr>
<td>16. Lemma</td>
<td>17</td>
</tr>
<tr>
<td>17. An example</td>
<td>18</td>
</tr>
<tr>
<td>SECTION</td>
<td>PAGE</td>
</tr>
<tr>
<td>---------</td>
<td>------</td>
</tr>
<tr>
<td>18. A special case of Theorem 14</td>
<td>19</td>
</tr>
<tr>
<td>19. Lemma</td>
<td>20</td>
</tr>
<tr>
<td>21. <strong>THEOREM</strong></td>
<td>21</td>
</tr>
<tr>
<td>22. Remarks on Theorem 21</td>
<td>22</td>
</tr>
<tr>
<td>23. Examples</td>
<td>23</td>
</tr>
<tr>
<td>24. Remarks toward Theorem 25</td>
<td>24</td>
</tr>
<tr>
<td>25. <strong>THEOREM</strong></td>
<td>25</td>
</tr>
<tr>
<td>26. A uniqueness question</td>
<td>28</td>
</tr>
<tr>
<td>27. Almost isolated irregular points</td>
<td>29</td>
</tr>
<tr>
<td>28. <strong>THEOREM</strong></td>
<td>30</td>
</tr>
<tr>
<td>29. <strong>THEOREM</strong></td>
<td>37</td>
</tr>
<tr>
<td>30. <strong>THEOREM</strong></td>
<td>39</td>
</tr>
<tr>
<td>31. A theorem of Polya</td>
<td>39</td>
</tr>
<tr>
<td>32. Lemma</td>
<td>40</td>
</tr>
<tr>
<td>33. Proof of Theorem 30</td>
<td>41</td>
</tr>
</tbody>
</table>
Introduction.

In his thesis \[1\] Johnson gives the following definition of the radius of regularity of a family of holomorphic functions:

Let \( F \) denote a family of functions \( f(z) \) regular at \( z_0 \). \( R \) is called the radius of regularity of \( F \) at \( z_0 \) if \( R \) is the largest number \( r \) such that each function is holomorphic and the family is normal in \[ |z - z_0| < r \] . If the conditions are valid in \[ |z - z_0| < r \] for each \( r > 0 \), then \( R = \infty \). If the conditions are not valid in \[ |z - z_0| < r \] for any \( r > 0 \), then \( R = 0 \). If a function of \( F \) has a singularity at \( z_0 \), then \( R = 0 \).

In the present thesis we will be concerned only with the case in which \( F = \{ f(z) \} = \{ \sum_{n=0}^{\infty} a_n (z - z_0)^n \} \) is uniformly bounded in some neighborhood of \( z_0 \). As Johnson proves, the radius of regularity \( R \) is then given by the formula

\[
\frac{1}{R} = \lim_{n \to \infty} \sup_{t \in F} |a_n^t|^{1/n},
\]

where we take \( R = 0 \) whenever the righthand expression is infinite.
The analogy between the formula (§) and the Cauchy-Hadamard formula for the radius of convergence of a power series was pursued by Johnson [1] in the following ways.

He defines the idea of the regular continuation of a family and obtains a gap theorem analogous to the Hadamard gap theorem for a single function. He obtains a theorem analogous to one of Dienes and Vivanti on a family whose coefficients \( \alpha_n \) are confined to a certain angle. Finally he proves a theorem—analogous to theorems of Wigert, Leroy and Faber—on the irregular points of a family whose coefficients are the values at the integers of a second family of entire functions. That is, on the location of the irregular points of a family \( \mathcal{B} : B(z) = \sum_{n=0}^{\infty} A(n) z^n \), where \( A = \{ A(z) \} \) is a family of entire functions.

The present thesis is a further pursuit of these analogies. We will find that a substantial number of the ideas contained in the two remarkable papers of Polya [1], [3], on the gaps and singularities of power series transplant fruitfully to the study of the irregular points of a family of functions.

We introduce (section 8) an indicator function to measure the "collective growth" of a family of entire
functions along half-rays through the origin. We develop an indicator diagram theory (sections 9 - 12) which relates the "collective growth" of a family of entire functions to the irregular points of the family of their Borel transforms. Our main results of this sort appear as the theorems of sections 11 and 12.

In the theorem of section 14 we generalize a functional transformation due to Pincherele to obtain a transformation of one normal family into another. By regarding an interpolated family (section 13) as the result of a certain "familial transformation" of this kind we obtain two theorems (sections 21 and 25) which complete the theorem of Johnson's mentioned above.

In section 27 we introduce the idea of an "almost isolated" irregular point of a family of functions. We prove, in the theorem of section 28, that a family having just one such irregular point (and no other irregular point) on its circle of regularity can be decomposed into the "sum" of two normal families having certain convenient properties. Indeed the decomposition leads us to a theorem (section 29) which provides a necessary and sufficient condition for the presence of just one irregular point (almost isolated) on the circle of regularity. As the last theorem of the paper (section 30) we prove that when the
only irregular point on the circle of regularity is almost 
isolated the upper density of the sequence \( A_n = \sup_{f \in F} \{|q^n_f|\} \) 
is one.

I am pleased to thank Professor Mandelbrojt for his 
encouragement and vigorous instruction. I am grateful to 
Dr. Guy Johnson for his interest in my work. Professor Polya 
I wish to thank for the pleasure of reading his papers.
1. **Definition.** Let $F = \{ f(z) \}$ be a family of entire functions. We shall say that $F$ is of exponential type $H$ if,

(1) upon setting $\max_{|z|=r} |f(z)| = \mathcal{M}(r)$, the quantity $\sup_{|z|=r} |M^f(r)| = \mathcal{M}(r)$ is finite for every $r \geq 0$, and

(2) \[ \lim_{r \to \infty} \frac{\log \mathcal{M}(r)}{r} = H \].

2. **Remarks.** The type of a single function can be defined only when the function is entire; that is, when the radius of convergence of its Taylor series is infinite. Similarly, the type of a family of entire functions can be defined only when the family has an infinite radius of regularity, for if the radius of regularity is finite, $\mathcal{M}(r)$ will be $+\infty$ for all sufficiently large $r$.

3. **Theorem.** The family $F = \{ f(z) \}$ of entire functions

\[ f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \]

is of finite exponential type if and only if the family $G = \{ g(z) \}$ of functions

\[ g(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} \]
has a finite radius of regularity $R$ and is locally bounded in $|z| > R$. Furthermore, under these circumstances $R$ is the type of $F$. That is,

$$\lim_{n \to \infty} \sup_{f \in F} \left| a_n^f \right|^n = \lim_{r \to \infty} \frac{\log M(r)}{r}$$

Proof: Put $\sup_{f \in F} \left| a_n^f \right| = A_n$.

(1) Suppose that $G$ has a finite radius of regularity and is locally bounded in $|z| > R$. Then [17] the two conditions

(1) $A_n < \infty \quad (n \geq 0)$

(2) $\lim_{n \to \infty} \frac{A_n}{n}^n = R$.

hold. If $\epsilon > 0$ is given, then by (2) there is an integer $N$ such that

$$A_n < (R + \epsilon)^n \quad (n > N).$$

Accordingly, we have for each $f \in F$

$$|f(z)| \leq \sum_{n=0}^{N} \frac{|a_n^f|}{n!} |z|^n + \sum_{n=N+1}^{\infty} \frac{|a_n^f|}{n!} |z|^n$$

$$\leq \sum_{n=0}^{N} \frac{A_n}{n!} |z|^n + \sum_{n=N+1}^{\infty} \frac{(R+\epsilon)|z|^n}{n!}.$$
It follows that
\[ M(r) \leq P(r) + e^{(R+\varepsilon)r} \]
where \( P(r) \) is a polynomial in \( r \) of fixed degree \( N \).
This proves that \( M(r) \) is finite for every \( r \geq 0 \) and that
\[ \lim_{r \to +\infty} \frac{\log M(r)}{r} \leq R + \varepsilon \]
for every \( \varepsilon > 0 \). Consequently, \( F \) is of finite exponential type \( H \leq R \).

(II) Suppose that \( F \) is of finite exponential type \( H \).
By definition of family type there corresponds to \( \varepsilon > 0 \)
an \( R \) such that
\[ |\{r e^{i\phi} \} | \leq M(r) < e^{(H+\varepsilon)r} \quad (r > R) \]
for each \( \{ F \} \). Cauchy's inequality yields
\[ \left| \frac{a_{n}}{n!} r^{n} \right| < e^{(H+\varepsilon)r} \quad (n \geq 0, r > R, F) \]
\[ \frac{A_n}{n!} r^n < e^{(H+\varepsilon)r} \quad (n \geq 0, r > R), \]
which shows that \( A_n \) is finite for every \( n \). Furthermore, if we choose \( r = \frac{n}{H+\varepsilon} \), we can state that for \( n \) sufficiently large
\[ \sqrt[n]{A_n} < (H+\varepsilon) \left( \frac{n/e}{\sqrt[n]{n!}} \right) \]
Therefore, $G$ has radius of regularity

$$R = \lim_{n \to \infty} A_n^{1/n} \leq H + \epsilon$$

and is locally bounded in $|z| > R$. Since $\epsilon > 0$ is arbitrary, we have $R \leq H$.

(III) Parts (I) and (II) combined clearly prove the theorem.

4. It will be convenient to rehearse some results on convex regions. A complete discussion is in Polya [2].

A convex region of the complex $z$-plane ($z = x + iy$) is a closed and bounded set which along with two of its points contains the segment joining them. The sum

$$R_1 + R_2 = \{z_1 + z_2, z_1 \in R_1, z_2 \in R_2\}$$

of two convex regions is a convex region. The difference, similarly defined, is a convex region. In particular the sum $R + \{z : |z| \leq \epsilon\}$ is called the region parallel to $R$ at the distance $\epsilon$ and is denoted by $R_\epsilon$.

If $R$ is a convex region, the function $h(\phi) = \max_{z \in R} \Re \{ze^{-i\phi}\}$ is uniquely defined, periodic and continuous. The line

$$L_\phi : x \cos \phi + y \sin \phi - h(\phi) = 0$$

is called the support line of $R$ of normal direction $\phi$ and always contains at least one point of $R$. Also, every point $x + iy$ of $R$ satisfies

$$x \cos \phi + y \sin \phi - h(\phi) \leq 0$$
for every value of \( \theta \). Indeed, \( R \) is uniquely characterized by \( h(\theta) \) as the set of points which lie on the negative side of every \( L_{\theta} \). Therefore, associated with each convex region \( R \) is a unique function \( h(\theta) \) called the support function of \( R \), which completely characterizes \( R \). We note that if \( h(\theta) \) supports \( R \), then \( h(\theta) + \epsilon \) supports \( R_{\epsilon} \).

The points of a line segment different from its end-points we call the interior points of the segment. A point \( z \) of a convex region \( R \) is an extreme point when it is not an interior point of any line segment wholly contained in \( R \). The points of \( R \) fall into three classes:

1. interior points
2. interior points of line segments entirely contained in the boundary of \( R \)
3. extreme points.

Since every support line shares with \( R \) either a single point or a line segment, any non-void convex region has at least one extreme point. Furthermore, if a sufficiently small neighborhood of an extreme point of \( R \) is deleted from \( R \), the remaining set is contained in a proper convex subset of \( R \).

5. The families \( F \) and \( G \) of Theorem 1 are each generated by the same set of constants \( \{ a_n \} \). Each of the two functions corresponding to one of these sequences is expressible
in terms of the other.

(I) Let $\epsilon > 0$ be arbitrary. Since each $g \in G$ is holomorphic in $|z| > R$, we may write

$$
\frac{1}{2\pi i} \oint g(z) e^{zt} \, dz = \frac{1}{2\pi i} \int \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{a_n^k}{z^n} \frac{(zt)^k}{k!} \, dz.
$$

For any fixed value of $t$, the double series converges absolutely uniformly when $z$ is on the path of integration. Therefore,

(1) $$
\frac{1}{2\pi i} \oint g(z) e^{zt} \, dz = \sum_{n=0}^{\infty} \frac{a_n^t}{n!} t^n = f(t).
$$

(II) Suppose that $z$ is fixed, $z > R$. Then

$$
\int \int f(t) e^{-zt} \, dt = \sum_{n=0}^{\infty} \frac{a_n^t}{n!} \int e^{-zt} \, dt = \sum_{n=0}^{\infty} \frac{a_n^t}{n!} \int t^n e^{-zt} \, dt
$$

(2) $$
= \sum_{n=0}^{\infty} \frac{a_n^t}{z^n} = g(z).
$$

The interchange of summation and integration is justified by the uniform convergence on every finite interval of the series for $f(t)$; the final result is valid because of the convergence of $\sum_{n=0}^{\infty} \frac{|a_n^t|}{|z|^n}$ at the $z$ in question.

6. The reciprocal integral relations which exist between the corresponding members of the families $F$ and $G$ will
enable us to give a precise localization of the irregular points of the family \( G \). To this end we shall use an idea of Pólya [2] and define the conjugate diagram \( \overline{C} \) of the family \( G \) as the smallest convex region which contains the irregular points of \( G \). The convex region \( \overline{R} \) contains the irregular points of \( G \), so we may define \( \overline{C} \) more precisely as the intersection of all such convex regions. We see that outside of \( \overline{C} \) each \( \eta \in G \) is holomorphic and \( G \) itself is locally bounded and normal. No proper subset of \( \overline{C} \) has both these properties. Moreover, each extreme point of \( \overline{C} \) is an irregular point of \( G \).

To see this, we suppose to the contrary that there is an extreme point \( \bar{z}_0 \) of \( \overline{C} \) with the property that \( G \) can be regularly continued [see Johnson [1]] into some neighborhood of \( \bar{z}_0 \) along the half-ray connecting \( \bar{z}_0 \) to the point at infinity and that \( G \) remains locally bounded in that neighborhood. By the statement at the end of section 4., this would imply that there is a proper convex subset of \( \overline{C} \) which contains all the irregular points of \( G \). This contradicts the definition of \( \overline{C} \), so \( \bar{z}_0 \) is an irregular point of \( G \). We note that \( \overline{C} \) is void if and only if every \( \eta(z) \in G \) is free of singularities in the finite plane. Since each \( \eta(z) \) vanishes at \( z = \infty \), the family \( G \) consists then of the single function \( \eta(z) = \infty \).
7. We know of the family $F$ of Theorem 1 that
\[ \lim_{r \to \infty} \frac{\log M(r)}{r} = R. \]
Since $M^f(r) \leq M(r)$ for each $f \in F$, it follows that
\[ \lim_{r \to \infty} \frac{\log M^f(r)}{r} \leq R, \]
which means that each $f \in F$ is an entire function of exponential type not exceeding $R$. Based on this fact alone and without further reference to the properties of $F$ and $G$ as families the following formula is valid for each $g \in G$:
\[ g(t) = \int_{\gamma} f(t) e^{-it} dt \quad \text{for } x \cos \theta + y \sin \theta > R. \]
The symbol $(-\overline{\theta})$ appended to the integral sign means that the path of integration extends from $-\infty$ to $\infty$ along the half-ray making the angle $-\overline{\theta}$ with the positive real $t$-axis. The proof is based on a familiar argument (see, e.g., Polya [2]).

3. We define
\[ h(q) = \lim_{r \to \infty} \frac{\log \sum_{f \in F} |f(re^{i\phi})|}{r} \]
and observe that $h(q) \leq R$ (it could be that $h(q) = -\infty$ for some values of $q$, although we shall see that this is not the case). We name $h(q)$ the indicator function of the family $F$. Clearly, $h(q)$ measures the "collective growth"
of $F$. We also define $k(q)$ to be the support function of the conjugate diagram $\overline{C}$ of $G$. Since $\overline{C}$ is contained in $|z| \leq R$, it follows that $k(q) \leq R$ also.

9. The functions $h(q)$ and $k(q)$ defined in section 8 arise respectively from the growth properties of the family $F$ and the irregular points of the family $G$. We can prove a close connection between them.

(I) The path of integration of the integrals (1) may be changed from $|z| = R + \varepsilon$ to a path which hugs more closely the conjugate diagram $\overline{C}$. Let $\varepsilon > 0$ be arbitrary and denote by $\Gamma_\varepsilon$ the boundary of the parallel region $\overline{C}_\varepsilon$. Since every $q \in G$ is holomorphic outside $\overline{C}$, we may write

$$f(z) = \frac{i}{2\pi\sqrt{1 - 1}} \int_{\Gamma_\varepsilon} f(z) e^{zt} \, dz \quad (q \in G),$$

where the integral is taken in the positive sense around $\Gamma_\varepsilon$. From these representations we can get an estimate on $h(q)$.

If $t = re^{-i\theta}$ and $z \in \Gamma_\varepsilon$, then

$$|e^{zt}| \leq e^{r(k(q) + \varepsilon)}.$$ 

Since $G$ is locally bounded outside $\overline{C}$, there is a constant $M < \infty$ such that for every $q \in G$

$$|q(z)| < M \quad (z \in \Gamma_\varepsilon).$$
Therefore,

\[ |f(re^{-i\theta})| \leq \frac{LM}{2\pi} e^{(k(\phi) + \epsilon)} \quad (\epsilon \in F), \]

where \( L \) is the length of the path \( \Gamma \). It follows that

\[ \sup_{\epsilon} |f(re^{-i\theta})| \leq \frac{LM}{2\pi} e^{(k(\phi) + \epsilon)} \]

\[ h(-\phi) \leq k(\phi) + \epsilon \]

for \( \epsilon > 0 \) arbitrary. Hence \( h(-\phi) \leq k(\phi) \).

(II) Consider a fixed \( \phi = \phi_0 \). Let \( \Delta \) be a compact subset of the open half-plane

\[ \mathcal{H}_{\phi_0} : \quad x \cos \phi_0 + y \sin \phi_0 - h(-\phi_0) > 0. \]

Denote by \( d \) the distance from \( \Delta \) to the line

\[ x \cos \phi_0 + y \sin \phi_0 = h(-\phi_0) \]

and put \( \frac{1}{2}d = \epsilon > 0 \). According to the definition of \( h(\phi) \), the inequality

\[ \sup_{\epsilon \in F} |f(re^{-i\theta})| < e^{r(h(-\phi) + \epsilon)} \]

holds for all sufficiently large \( r \). Consequently, the integrands of (3) permit the inequality
\[
\left| f(re^{-i\theta}) \right| \leq \sup_{t \in \mathbb{R}} |f(re^{-i\theta})| e^{-r(x \cos \phi + y \sin \phi - h(-\theta) - \epsilon)}
\]

for every \( \frac{t}{\epsilon} \in \mathcal{F} \) and for all sufficiently large \( r \).

It follows that the integrals

\[
g(t) = \int_{0}^{\infty} f(x) e^{-xt} \, dx
\]

converge for each \( z \in H_{\varphi} \). Hence, each \( g \in G \) is holomorphic in \( H_{\varphi} \). Indeed, when \( z \in \Delta \), we have

\[
x \cos \phi + y \sin \phi - h(-\theta) \geq d \]

\[
x \cos \phi + y \sin \phi - h(-\theta) - \epsilon \geq d - \epsilon = \frac{1}{2} d
\]

As a result, when \( z \in \Delta \)

\[
|g(z)| \leq \int_{0}^{\infty} e^{-\frac{t}{\epsilon} x d} \, dx = \frac{e^{d}}{d} (g \in G).
\]

This shows that \( G \) is uniformly bounded on \( \Delta \). Since \( \Delta \) was an arbitrary compact subset of \( H_{\varphi} \), we have proved that \( G \) is locally bounded in \( H_{\varphi} \).

The functions \( g \in G \) are holomorphic in \( H_{\varphi} \) and \( G \) is locally bounded there. Therefore, \( H_{\varphi} \) contains no extreme
points of \( \overline{C} \), so that \( \overline{C} \) must be contained in the complementary half-plane

\[
x \cos \varphi + y \sin \varphi \leq h(-\varphi) .
\]

According to this, we may allow \( x + iy \) to run through \( \overline{C} \) and compute \( k(\varphi) \leq h(-\varphi) \).

(III) By parts (I) and (II) we have \( k(\varphi) \leq h(-\varphi) \leq k(\varphi) \) or \( k(\varphi) = h(-\varphi) \). Since \( \varphi \) was arbitrary, we have proved that

\[
k(\varphi) = h(-\varphi) .
\]

10. We have seen that the support function of the conjugate diagram of \( G \) is \( h(-\varphi) \). Let us call the convex region whose support function is \( h(\varphi) \) the indicator diagram of the family \( F \), and denote it by \( C \). Then \( C \) is the reflection of \( \overline{C} \) in the real axis. We may sum up our results in the following way.

11. Theorem. Let \( F = \{ f(\varphi) \} \) be a family of entire functions

\[
f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n
\]

and suppose that \( F \) is of exponential type. Let \( C \) be the indicator diagram of \( F \) and define the conjugate diagram \( \overline{C} \) as the reflection of \( C \) in the real axis. Then the family \( G = \{ g(\varphi) \} \) of functions

\[
g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^{n+1}
\]
is locally bounded and normal outside of $\overline{C}$ and has an irregular point at each extreme point of $\overline{C}$.

A direction $q^*$ defined by

$$\max_{0 \leq \phi \leq \pi} h(q) = h(q^*)$$

is a direction of maximum growth for $F$. From the definition of $h(q)$ and the fact that $F$ is of type $R$ in the whole plane, we see that $h(q^*) = R$. Furthermore, since $h(q)$ is the support function of the indicator diagram $C$, the line

$$L_{q^*} : x \cos q^* + y \sin q^* = h(q^*)$$

has a point in common with $C$. Furthermore, $C$ is contained in the circle $|z| = R$, so that $L_{q^*}$ is tangent to this circle and the point of tangency $Re^{i\theta}$ is an extreme point of $C$. Conversely, any extreme point of $C$ which has the form $Re^{i\theta}$ satisfies an equation of the form $L_q$.

We have, except for an obvious transformation, proved the following:

12. **Theorem.** Let $H : h(z) = \sum_0^\infty a_n z^n$ be a family of exponential type. The type of $H$ along a half-ray attains the type of $H$ in the whole plane if and only if the half-ray cuts the circle of regularity of the family $K : \kappa(z) = \sum_0^\infty a_n z^n$ in an irregular point.
13. If \( A = \{ A(z) \} \) is a family of entire functions, we shall call the family \( B : B(z) = \sum_{n=0}^{\infty} A(-n) z^{-n} \) the family interpolated by \( A \), or just the interpolated family. We call \( A \) the interpolating family.

Johnson \cite{1} has proved a theorem assuming the interpolating family \( A \) to be of exponential type \( \tau < \frac{\pi}{2} \) and concluding that the interpolated family \( B \) has radius of regularity \( \geq e^{-\tau} \) and is locally bounded outside the set \( \{ z \mid 1 + |z| \leq e^{\tau} \} \cap \{ z \mid |z| \geq e^{-\tau} \} \).

Johnson's proof follows certain methods of Wigert, LeRoy, Faber and Mandelbrojt. By approaching the problem through a generalization of a functional transformation introduced by Pincherele, we shall be able to sharpen Johnson's result in a theorem with weaker hypotheses and stronger conclusions. The next eight sections prepare the ground for this "interpolating" theorem.

If \( f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \) is an entire function of exponential type, the function \( \tilde{f}(z) = \frac{e^{-z}}{z} \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \) is customarily called the Borel transform of \( f(z) \). Analogously, if \( F: F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \) is an entire family of exponential type, we shall call \( G : G(z) = \frac{e^{-z}}{z} \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \) the Borel transform family associated with \( F \).

14. **Theorem.** Let \( F = \{ f(z) \} \) be an entire family of
exponential type and \( \mathcal{G}_1 = \{ \phi^{(x)} \} \) be the Borel transform family associated with \( \mathcal{F} \). Let \( C \) denote the indicator diagram of \( \mathcal{F} \).

Suppose that \( \psi(x) \) is a uniform function with only isolated singular points.

Denote by \( S \) the set of singularities of \( \psi(x) \) and put \( S + \epsilon = S^\epsilon \). Let \( D \) be a maximal connected subset of the complement of \( S^\epsilon \).

Denote by \( \Gamma_\epsilon \) the boundary of \( \overline{C_\epsilon} \) (this \( \epsilon > 0 \) can be taken arbitrarily small and is assumed not to exceed a certain upper bound which appears in the course of the proof). Put

\[
\frac{1}{2\pi i} \int_{\Gamma_\epsilon} \psi(x - \omega) \phi(\omega) \, d\omega = \psi^*(x),
\]

where \( \Gamma_\epsilon \) is traversed in the positive sense.

Then,

1) for each \( \psi \in \mathcal{G}_1 \), the corresponding \( \psi^*(x) \)

is holomorphic and uniform in \( D \).

2) the family \( \{ \psi^*(x) \} = \psi^* \) is locally bounded and normal in \( D \).

The notation \( \overline{C_\epsilon} \) is explained in section four.

Proof:

(I) For the proof of conclusion 1) it suffices to apply to each \( \psi^*(x) \) the argument used by Polya [2] to prove
the analogous theorem for a single function and its Borel transform.

(II) Fix a point \( \tilde{z}_0 \) in the complement \( \mathcal{S}^c \). Define \( \mathcal{D} \) as the set of points \( \tilde{z} \) in the complement of \( \mathcal{S}^c \) which can be joined to \( \tilde{z}_0 \) by a continuous curve wholly contained in the complement of \( \mathcal{S}^c \).

Now, \( \mathcal{S}^c = \mathcal{S} + \bar{C} \) has no common point with \( \mathcal{D} \), or, what is the same thing \( \mathcal{D} - \bar{C} \) has no point in common with \( \mathcal{S} \).

Suppose now that \( \Delta \) is a compact subset of \( \mathcal{D} \). As \( \tilde{z} \) runs through \( \Delta \) and \( \bar{u} \) runs through \( \bar{C} \), \( \bar{u} - \bar{u} \) describes a compact subset \( \Delta^c \) of the region of regularity of \( \psi(z) \).

Let \( \delta \) be the distance from \( \Delta^c \) to the boundary of this region. Let \( 0 < \varepsilon < \delta \) and take the integral which defines \( \psi^*(z) \) in the positive sense around the boundary \( \Gamma_\varepsilon \) of \( \bar{C} \).

The integrals (for each \( \eta \in \mathcal{G} \)) are well defined and have the same values for each such \( \varepsilon \).

We can now prove that \( \psi^* \) is uniformly bounded on \( \Delta \).

Let \( M_1 = \max_{z \in \Delta^c} |\psi(z)| \). Observe that the family \( \mathcal{G} \) is locally bounded outside of \( \bar{C} \). Hence, there is an \( M_2 \) such that \( |\psi(z)| \leq M_2 \) for all \( z \) on \( \Gamma_\varepsilon \) and every \( \eta \in \mathcal{G} \).

Accordingly, if \( z \in \Delta \)

\[
|\psi(z)| \leq \frac{M_1 M_2}{\delta^2 \pi} L 
\]

( \( L = \) length of \( \Gamma_\varepsilon \))

for every \( \psi^*(z) \in \psi^* \). Thus \( \psi^* \) is uniformly bounded on \( \Delta \).
Δ is arbitrary and \( \varphi^* \) is locally bounded in \( \mathcal{D} \). This also shows that \( \varphi^* \) is regular in \( \mathcal{D} \). The theorem is proved.

15. **Example.** Suppose in Theorem 14 that \( \varphi(z) = \frac{i}{z - a} \), where \( a \) is a constant. \( S \) is then the point \( a \) and \( \mathcal{S}^* = a + \overline{C} \). Consequently, the family \( \varphi^* \):

\[
\varphi^*(z) = \frac{1}{2\pi i} \int_{C_t} \frac{g(u)}{u - a} \, du = g(z - a)
\]

is locally bounded and normal outside the set \( a + \overline{C} \). In other words with this choice of \( \varphi(z) \) Theorem 14 yields a part of the information contained in Theorem 11. We can apply this special case to prove the following generalization of a lemma due to Pólya [1].

16. **Lemma.** Let the function \( \varphi(z) \) of Theorem 14 be meromorphic with simple poles at the points \( a_1, a_2, \ldots \).

Let \( t \) be a point of precisely one of the regions \( a_i + \overline{C} \ (i = 1, 2, \ldots) \), say \( t \in a_m + \overline{C} \), and suppose that \( t \) is an extreme point of \( a_m + \overline{C} \).

Then \( t \) is an irregular point of the family \( \varphi^* \).

**Proof:** Let \( A_m \) be the residue of \( \varphi(z) \) at \( z = a_m \) and write

\[
\varphi(z) = \frac{A_m}{z - a_m} + \varphi_m(z)
\]

The function \( \varphi_m(z) \) is meromorphic with the same poles.
as \( \psi(z) \), excepting \( a_m \). The integral of Theorem 14 gives, by the lemma,

\[
\psi^*(z) = A_m \psi(z-a_m) + \psi^*(z)
\]

The region \( D \) that corresponds to \( \psi(z) \) results from the plane upon removal of the points of the sets \( a_i + \overline{C} \), \( i = 1, 2, \ldots \). The region \( D_m \) that corresponds to \( \psi_m(z) \) results from the set \( D \) upon replacing the set \( a_m + \overline{C} \).

In particular \( \ell \) is an interior point of \( D_m \) and therefore a regular point of \( \{ \psi_m(z) \} \). On the other hand the family \( \{ \psi(z-a_m) \} \) has an irregular point at \( z = \ell \). Therefore, \( \ell \) is an irregular point of the family \( \psi^* \), as was asserted.

17. Example. If \( \psi(z) = e^{-\lambda z} \), \( \lambda \) constant, Theorem 14 gives for the members of the family \( \psi^* \)

\[
\psi^*(z) = \frac{1}{2\pi i} \int_{\ell} e^{-\lambda(z-u)} \psi(u) \, du
\]

\[
= e^{-\lambda \ell} \frac{1}{2\pi i} \int_{\ell} e^{\lambda u} \psi(u) \, du
\]

\[
= e^{-\lambda \ell} \psi(\lambda) ,
\]
where we have used the results of section five. We are now prepared to discuss in detail the particular choice of \( \psi(z) \) which will produce from Theorem 14 the "interpolating" theorem we are aiming for.

18. Let \( \psi(z) \) be the meromorphic function \( \frac{1}{z - e^{-z}} \).

The set \( S \) then consists of the points \( \pm 2\pi n i \quad (n = 0, 1, 2, \ldots) \) and \( S^* \) of the sets \( \pm 2\pi n i + \bar{c} \quad (n = 0, 1, 2, \ldots) \). As for the domain of regularity \( \Omega \):

(I) If \( \bar{c} \) and \( 2\pi i + \bar{c} \) have a point in common, there are two possibilities: either a) \( \Omega \) contains an infinite segment of the positive real axis, or b) \( \Omega \) contains an infinite segment of the negative real axis. In either case \( \Omega \) is simply connected. Lemma 17 gives no information about the irregular points of \( \psi^n \).

(II) If \( \bar{c} \) and \( 2\pi i + \bar{c} \) are disjoint, the set coincides with the complement of \( S^* \) and is therefore infinitely connected (note that Theorem 14 nonetheless assures us that every \( \psi^n(z) \) is uniform in \( \Omega \)). And, most important, Lemma 17 guarantees that the extreme points of all the sets \( \pm 2\pi n i + \bar{c} \quad (n = 0, 1, 2, \ldots) \) are irregular points of \( \psi^n \).

We remark here that the height of the conjugate diagram is \( h(\pi i) + h(-\pi i) \); when this sum is \( < 2\pi \), and \( 2\pi i + \bar{c} \) are disjoint.
19. Now, according to Polya [2], the integrals

$$\Psi(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{\varphi(u)}{z-u-a} du$$

may be evaluated by a term-by-term integration using

$$\frac{1}{1-e^{-z}} = 1 + e^{-z} + e^{-2z} + \ldots \quad (\Re(w) > 0)$$

$$= e^{-z} + e^{-2z} + e^{-3z} + \ldots \quad (\Re(w) < 0).$$

Using example 17, the results of the evaluations prove:

20. **Lemma.** If in Theorem 14 $\Psi(z) = \frac{1}{1-e^{-z}}$, and if $D$ is chosen to contain an infinite segment of the positive real axis, the functions of the family $\Psi^*$ permit the representations

$$\Psi^*(z) = \sum_{n=1}^{\infty} \varphi_n e^{-nt}$$

in the open half-plane $\Re(z) > \lambda(\sigma)$.

If in addition $D$ contains an infinite segment of the negative real axis (and is therefore infinitely connected), the regular continuation of the family $\Psi^*$ is representable by

$$\Psi^*(z) = \sum_{n=1}^{\infty} -\varphi_n e^{-nt}$$

in the half-plane $\Re(z) < -\lambda(\pi)$. 
21. **Theorem.** Let \( \mathcal{F} = \{ f(z) \} \) be an entire family of exponential type whose indicator satisfies
\[
\lambda(\pi/2) + \lambda(-\pi/2) < \lambda W.
\]
Then the family \( \mathcal{F} \) of functions
\[
\varphi(z) = \sum_{n=0}^{\infty} f(z) z^n
\]
has radius of regularity \( e^{-\lambda(\pi)} \) and is locally bounded in \( |z| < e^{-\lambda(\pi)} \). If \( \mathcal{C} \) is the conjugate diagram associated with \( \mathcal{F} \) and if \( \mathcal{C}^* \) is its image under the mapping \( e^{-z} \), then \( \mathcal{F} \) is locally bounded and regular in the domain of the extended plane exterior to \( \mathcal{C}^* \).

Furthermore, if \( t \) is an extreme point of \( \mathcal{C} \), then \( e^{-t} \) is an irregular point of \( \mathcal{F} \).

The set \( \mathcal{C}^* \) is contained in the annulus
\[
c^{-\lambda(\pi)} < |z| < e^{\lambda(\pi)}
\]
and in the region exterior to this annulus \( \mathcal{F} \) has the representation
\[
\varphi(z) = -\sum_{n=0}^{\infty} \frac{f(-z)}{z^n}.
\]

Finally, the following (not necessarily distinct) sets each contain at least one irregular point of \( \mathcal{F} \):

1. \( |z| = e^{-\lambda(\pi)} \)
2. \( |z| = e^{\lambda(\pi)} \)
3. \( e^{-\lambda(\pi)} < |z| < e^{\lambda(\pi)} \), \( \arg z = \lambda(\pi/2) \)
4. \( e^{-\lambda(\pi)} < |z| < e^{\lambda(\pi)} \), \( \arg z = -\lambda(-\pi/2) \).
Proof: Take \( \psi(z) = (1 - e^{-z})^{-1} \) in Theorem 14. By the remark concluding section 18 and by (II) of the same section, \( D \) is the extended plane with the sets \( \{ \pm n \} + \mathcal{C} \) removed. These sets are contained between the lines \( \Re(z) = -\lambda(z) \) and \( \Re(z) = \lambda(z) \). The image \( \mathcal{D}^* \) of \( D \) under the mapping \( e^{-z} \) is therefore the region of the extended plane exterior to \( \mathcal{C}^* \) (which is clearly contained in the annulus described in the theorem). Hence, the family \( \mathcal{F} : \psi(z) = \psi^*(-\log z) \) is locally bounded and normal in \( \mathcal{D}^* \). Lemma 20 provides the desired representations of the members of the family in the domains \( |z| < e^{-\lambda(z)} \) and \( |z| > e^{-\lambda(z)} \). Finally, the presence of the irregular points of \( \mathcal{F} \) on the sets 1) - 4) is accounted for by section 18, (II) and the fact that each of the lines \( \Re(z) = -\lambda(z) \), \( \Re(z) = \lambda(z) \), \( \Im(z) = \lambda(\arg z) \), \( \Im(z) = -\lambda(\arg z) \) contain at least one extreme point of \( \mathcal{C} \).

22. Remarks: 1) The type of \( F \) is \( \max \lambda(z) = \mathcal{H} \) and \( \mathcal{C} \) is contained in the circle \( |z| = \mathcal{H} \). The theorem allows \( \mathcal{H} \) to be arbitrarily large.

2) We have been able to compute the exact radius of regularity of the interpolated family.

3) We have learned not only where the
irregular points of the interpolated family can be, but where some of them must be.

4) Every function of the interpolated family vanishes at \( z = \infty \).

23. Examples.

1) Let \( F = \{ f_\kappa(z) \} = \{ (z/\kappa)^\kappa \} \). Computing the indicator of \( F \):

\[
h(\kappa) = \lim_{r \to \infty} \frac{\log \sup_{\kappa \geq 1} |f_\kappa(re^{i\theta})|}{r}
= \lim_{r \to \infty} \frac{\log \sup_{\kappa \geq 1} [\log r - \log \kappa]}{r}
= \lim_{r \to \infty} \frac{\log e^{r/e} \log e}{r} = \frac{r}{e}.
\]

Hence, \( h(\kappa_1) + h(-\kappa_2) = \frac{r}{e} + \frac{r}{e} = \frac{2r}{e} < 1\pi \).
This shows that \( \overline{C} \) is the circle \( 1z1 \leq \frac{r}{e} \). By the theorem the family \( F \):

\[
\Phi_\kappa(z) = \sum_{n=0}^{\infty} f_\kappa(n) \frac{z}{n} = \sum_{n=1}^{\infty} \left( \frac{n}{\kappa} \right) \frac{z}{n}
\]
has radius of regularity \( \frac{r}{e} \) and is locally bounded and normal outside the set which is the image of \( \{ \omega \in \frac{r}{e} \text{ under the mapping } z = e^{\frac{r}{e}} \}\).
Furthermore, since each point of \( \{ \omega \in \frac{r}{e} \text{ is an extreme point of } \overline{C} \} \), it follows that every point

\[
z = e^{-i\frac{1}{e}} \quad (0 < \kappa \leq 2\pi)
\]
is irregular for $\bar{f}$.

2) Let $F = \{ f_k(z) \} = \{ e^{i_k} \}$. In this case the indicator of $F$ is

$$h(r) = \lim_{r \to \infty} \frac{\sum_{k=1}^{\infty} \left( \frac{r \cos \theta}{k} \right)}{r}$$

$$= \begin{cases} \cos \theta & \text{for } 0 \leq \theta \leq \pi, \ \frac{\pi}{2} \leq \theta < 2\pi \\ 0 & \text{for } \pi/2 < \theta < 2\pi/2 \end{cases}$$

Consequently, $\bar{C}$ is the segment $0 \leq \theta \leq 1$ of the real axis. The family $\bar{\Gamma}$:

$$\Phi_k(z) = \sum_{n=0}^{\infty} c_n \bar{z}^n$$

is locally bounded and normal off of the segment $e^{-i} \leq \theta \leq 1$ of the real axis and the endpoints $z = e^{-i}, \bar{z} = 1$ are irregular points of $\bar{f}$. (Indeed, since $\Phi_k(z) = (1 - \bar{z}e^{i} \bar{z})^{-1}$, each of the points $z_k = e^{-i} / k$ is an irregular point of $\bar{f}$).

24. In Theorem 22 it is proved that under certain simple conditions a family $\{ \sum a_n z^n \}$ whose coefficients are the values of an entire function $\xi$ at the integers,

$$a_n = \xi(n)$$
is regular off of a certain compact set which neither contains nor surrounds the origin. Indeed it is clear from the theorem that there is always a half-ray from the origin along which the interpolated family
can be regularly continued into the point at infinity.
The question whether every family whose irregular points
are so situated must be an interpolated family is answered
in the next section.

25. **Theorem.** A necessary and sufficient condition that
the family $\mathcal{F}$ of functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be locally bounded and regular in a neighborhood of
the origin and regularly continuable along some
half-ray from $z=0$ to $z=\infty$ and that each $f$
take the value $f(\infty) = 0$ there, is that there exist
an entire family $E = \{f(z)\}$ whose indicator satis-
fies $\ln(n) + \ln(-n) < 2\pi$ and which provides $f(z) =
\sum_{n=0}^{\infty} a_n (n > 0, \quad f \in \mathcal{F})$.

**Proof:** Sufficiency: an immediate consequence of Theorem 22.

Necessity: Let $\mathcal{F}$ be locally bounded and regular in some
neighborhood of the origin. Suppose in particular that $R > 0$
is the radius of regularity of $\mathcal{F}$ at $z=0$. Then every
is holomorphic in $|z| < R$, and if $0 < r < R$, we may write

$$a_n = \frac{i}{2\pi} \int_{|\omega|=r} \frac{f(\omega)}{\omega^{n+1}} \, d\omega \quad (n > 0, \quad f \in \mathcal{F})$$

where the path of integration is positively traversed.
The hypotheses on the regular continuability of $\mathcal{F}$ imply
the existence of three real numbers \( R_1, \theta_1, \varphi_1 \), with the following properties:

(a) \( R_2 > R_1 \), \( 0 < \theta_1 < 2\pi \), \( 0 < \varphi - \theta_1 < 2\pi \)

(b) the branch of each \( \Phi(z) \) defined by \( \sum d_n z^n \) is holomorphic and uniform in the set

\[
S = \{ \left. z \mid 1 < |z| < R_1 \} \cup \{ \left. z \mid R_1 \leq |z| \leq R_2, \theta \leq \text{arg} z \leq \varphi_1 \} \cup \{ \left. z \mid |z| > R_2 \} \}
\]

(c) \( \Phi \) is locally bounded and normal in \( S \).

(d) the regular continuation of \( \Phi \) has the representation

\[
\Phi(z) = \sum_{n=0}^{\infty} d_n z^n / z^n
\]

in \( |z| > R_1 \), where \( d_0 = 0 \).

In particular, the radius of regularity of \( \Phi \) about \( z = \infty \) is \( R_2 \).

If \( r_2 > R_2 \) and the integral (\( \ast \)) is extended in the negative sense about \( \omega = r_2 \), it follows from (d) that the result is always zero. That is,

\[
\frac{i}{2\pi i} \int_{|\omega| = r_2} \frac{\Phi(\omega)}{\omega^{n+1}} \, d\omega = 0 \quad (n \geq 0, \Phi \in \Phi).
\]

Consequently, we may replace the path of integration of (\( \ast \)) with the path \( L \) consisting of the four pieces
\[ |w| = r_1, \quad \phi_2 \leq \arg w \leq \phi_1 + 2\pi \]

\[ |w| = r_2, \quad \phi_2 \leq \arg w \leq \phi_1 + 2\pi \]

\[ r_2 \leq |w| \leq r_1, \quad \arg w = \phi_1 + 2\pi \]

\[ r_1 \leq |w| \leq r_2, \quad \arg w = \phi_2 \]

(traced so that \( |w| = r_1 \) is positively traversed) without changing the values of the integrals. Thus,

\[
\frac{1}{2\pi i} \int_{L} \frac{q(w)}{w^{n+1}} \, dw \quad (n \geq 0, \phi \in \mathbb{R}).
\]

Let \( \mathcal{R} \) be the rectangle in the \( v \)-plane with corners at the points

\[-\log r_1 - i \phi_2, \quad -\log r_2 - i (\phi_1 + 2\pi)\]

\[-\log r_2 - i (\phi_1 + 2\pi), \quad -\log r_1 - i \phi_2.\]

Let \( k(\phi) \) be the support function of \( \mathcal{R} \), so that by (a) above \( k(\pi_1) + k(-\pi_2) = -\pi_1 + (\phi_1 + 2\pi) = 2\pi - (\phi_2 - \phi_1) < 2\pi \).

Now, as \( v \) traces the rectangle \( \mathcal{R} \) in the order the corners are named, \( w = e^{-v} \) traces the path \( L \) in the proper way so that

\[
\frac{1}{2\pi i} \int_{\mathcal{R}} q(e^{-v}) e^{\pi v} \, dv \quad (n \geq 0, \phi \in \mathbb{R}).
\]

Consider the family \( F \) of functions

\[
\int_{\mathcal{R}} q(e^{-v}) e^{\pi v} \, dv \quad (\phi \in \mathbb{R}).
\]
Obviously, each $f^\phi(z)$ is entire. Furthermore, we can compute the indicator $h(\phi)$ of $F$. Since the path $L$ is contained in the set $S$ where (by $(\phi)$) $F$ is locally bounded, we have

$$\frac{\sup_{\phi \in F} |f^\phi(z)|}{M} = \frac{M}{\lambda(R)} < \infty.$$  

Furthermore, when $\phi = \phi_1 + i\phi_2$ is on $R$, $|\phi_1 + \phi_2| \leq k(\phi)$. Therefore,

$$\frac{\sup_{\phi \in F} |f^\phi(z)|}{M} \leq \frac{M}{\lambda(R)} e^{-\frac{M}{\lambda(R)}}$$

where $\lambda(R)$ is the circumference of $R$. Hence, $h(\phi) \leq k(\phi)$ and, from a previous remark, $h(\phi_1) + h(-\phi_1) < 1\pi$. Since

$$f^\phi(z) = a_n z^n \quad (n \geq 0, \phi \in \mathbb{R})$$

the theorem is completely proved.

26. Remarks. Let $F_0 = \{ f_0(z) \}$ be an entire family with an indicator $h_0(\phi)$ which satisfies $h_0(\phi_1) + h_0(-\phi_1) < 1\pi$. The interpolated family $F: f(z) = \sum f(z) e^{i\phi}$ is then uniquely determined by $F_0$. On the other hand, if $F$ is specified in the given form, $F_0$ is not uniquely determined. Indeed, each of the families

$$F_m = \{ c m^{i\phi} f_0(z) \} \quad (m = 0, \pm 1, \pm 2, \ldots)$$

also interpolates for the same family $F$. If we compute the indicator $h_m(\phi)$ of $F_m$, we find that $h_m(\phi) = 2\pi m |\sin\phi| + h_0(\phi)$. Consequently, the conjugate diagram of $F_m$ is $\overline{c}_m = \pm m \pi i + \overline{c}_0$.
which agrees with the fact that $\bar{F}$ is the same for every $w$.

(The remainder of these pages will be devoted principally to theorems concerning a certain type of irregular points whose definition is patterned on the definition given by Pólya [3] for an "almost isolated" singular point of a function as defined by its Taylor series.)

27. The concept of an almost isolated irregular point:
Let $F = \{ f(z) \}$ be a family of functions $f(z) = \frac{1}{r} a_0 \sum (z - z_0)^n$ with radius of regularity $R$ at $z = z_0$ ($0 < R < \infty$).
Let $a$ be a point of the circle $|z - z_0| = R$. We shall say that $a$ is an almost isolated irregular point of $F$ with respect to $z_0$ when the following conditions are satisfied:

There exists an $\epsilon > 0$ such that it is possible to continue $F$ regularly along each ray extending from $z_0$ to a point of the disk $|z - a| < \epsilon$ provided that ray does not pass through the point $a$ itself.

The line joining $z_0$ to $a$ cuts the circumference $|z - a| = \epsilon$ in two points, one of which is exterior to the circle of regularity (at $z_0$) of $F$. Let $a'$ denote that point, and delete the segment joining $a$ to $a'$ from the open disk $|z - a| < \epsilon$. The resulting set, which we shall call $\mathcal{R}$, is a cut disk
in which $F$ is regular. The point $\alpha$ is an irregular point of $F$. The definition states nothing about the points of the segment joining $\alpha$ to $\alpha'$. One of these points may be singular for some $f(z)$, regular for all $f(z)$, regular for $F$, irregular for $F$, etc. Furthermore, such a point may exhibit one kind of behaviour when approached from one side within $R$ and a different behaviour when approached from the other side within $R$.

Certain special cases arise which are not analogous to possibilities for a single function. All the functions of $F$ may be entire, in which case the occurrence of an almost isolated irregular point $\alpha$ shows that the points of the closed segment joining $\alpha$ and $\alpha'$, are irregular points of $F$.

The theorem which we prove about families with an almost isolated irregular point are patterned on theorems of Polya [3] which concern almost isolated singularities of single functions.

28. **Theorem.** Let the family $F$ of functions $f(z) = \sum a_n z^n$ have radius of regularity one and be locally bounded in $|z| < 1$. If $F$ has exactly one irregular point on $|z| = 1$, namely at $z = 1$, and this point is almost isolated, then each $f(z)$ has a decomposition

$$f(z) = \phi(z) + \psi(z) \quad (|z| < 1)$$
where the families \( \mathcal{F} = \{ \phi(z) \}, \mathcal{U} = \{ \psi(z) \} \)

have the following properties:

1. \( \mathcal{U} \) has radius of regularity \( R^* > 1 \)
   and is locally bounded in \( |z| < R^* \)

2. each \( \phi(z) \in \mathcal{F} \) is holomorphic and uniform
   and \( \mathcal{F} \) itself is locally bounded and regular
   off of a finite segment of the real axis.

   The left endpoint of this segment is \( z = -1 \).

   \( \mathcal{F} \) is regular at \( z = \infty \) and each \( \phi(z) \)
   takes the value \( \phi(\infty) = 0 \) there.

Proof: By hypothesis and from the definition of an almost
isolated irregular point, there is a \( \mu > 1 \) such that \( \mathcal{F} \)
is locally bounded and regular in \( |z| < \rho' \) except on the
segment \( 1 < z < \rho' \) of the real axis. \( z = 1 \) is an irregular
point, the remaining points of the segment may be irregular.

(I) We first consider the special case in which there is
a point \( z = \rho \) \( (1 < \rho < \rho') \) such that \( \mathcal{F} \) may be regularly
continued into \( z = \rho \) from both sides of the real axis within
\( |z| < \rho' \). Of course the continuations of a given
\( \phi(z) \in \mathcal{F} \) need not lead to the same value at \( z = \rho \).

Construct the path \( \mathcal{C} = \gamma \cup \Gamma \) consisting of the
two pieces:

\( \mathcal{C}^*: \ |z| = \rho \) positively traversed and
beginning, for definiteness, at \( z = \rho \).

\( \Gamma: \ ) a simple closed path beginning and
ending at \( \bar{z} = \rho \), containing the
segment \( 1 \leq \bar{z} < \rho \) in its bounded component
and the point \( \bar{z} = 0 \) in its unbounded component, entirely contained in \( |\bar{z}| < \rho \)
except for the point \( \bar{z} = \rho \) and traversed in the negative sense.

The path \( C \) is therefore a closed curve with a double
point at \( \bar{z} = \rho \) which is traversed positively (beginning
at \( \bar{z} = \rho \)) around the boundary of a simply connected
domain which contains the origin. Since \( \Phi \) is regular on \( C \) and in its interior, we may write for \( \bar{z} \) in a
neighborhood of the origin

\[
\begin{align*}
\Phi(\bar{z}) & = \frac{i}{2\pi i} \oint_C \frac{f(t)}{t - \bar{z}} \, dt \\
 & = \frac{i}{2\pi i} \oint_{\Gamma} \frac{f(t)}{t - \bar{z}} \, dt + \frac{i}{2\pi i} \oint_{C^*} \frac{f(t)}{t - \bar{z}} \, dt \\
 & = \Phi^f(\bar{z}) + \psi^f(\bar{z}).
\end{align*}
\]

Now the domain \( D \) of the extended plane exterior to the
segment \( 1 \leq \bar{z} < \rho \) is simply connected. Hence, by proper
deformation of \( \Gamma \), it is clear that each \( \Phi^f(\bar{z}) \) is a
uniform function holomorphic in \( D \). Furthermore, from

\[
\Phi^f(\bar{z}) = \frac{i}{2\pi i} \oint_{\Gamma} \frac{f(t)}{t - \bar{z}} \, dt = -\sum_{n=-\infty}^{\infty} \frac{i}{2\pi i} \oint_{\Gamma} \frac{t^n f(t)}{t - \bar{z}} \, dt
\]
it follows that each $\Phi(z)$ is zero at $z = \infty$.

Let $\varepsilon > 0$ be arbitrary and define $D_\varepsilon = \{ z \mid |z - t| < \varepsilon \}$ for all $t \leq \varepsilon < \rho$. For a fixed $\varepsilon$, $\Gamma$ may be deformed so that $\Gamma \cap D_\varepsilon = \emptyset$. Let $d$ be the distance of $\Gamma$ from the boundary of $D_\varepsilon$. Let $M = \sup_{t \in \mathbb{R}, t \in \Gamma} \max_{i \in \mathbb{C}} |\Phi(z)|$ ($F$ is uniformly bounded on $\Gamma$). Then for all $z \in D_\varepsilon$

$$|\Phi(z)| \leq \frac{M}{2\pi d} L \quad (L = \text{length of } \Gamma)$$

for every $t \in F$. Therefore, $\Phi$ is uniformly bounded on every $D_\varepsilon$ and hence locally bounded and regular in $D$ as is asserted in the theorem.

(II) If the segment $1 \leq t < \rho'$ does not contain a point which is regular for $F$ from both approaches, the proof is more involved. First we shall derive the conclusions of the theorem from the assumption that there are paths of finite length which lead into $z = \rho$ from above and below and along which $F$ is uniformly bounded.

Let $S$ be a vertical segment of length $2l$ whose midpoint is $z = \rho$ and which is contained in $|z| < \rho'$.

Suppose that $F$ is uniformly bounded on $S - \{\rho\}$.

Let $R_k$ be the rectangle whose corners are the points

$$\rho \pm i \frac{k}{2} l, (1 - \frac{i}{2k}) : \rho \pm i \frac{k}{2} l, k = 1, 2, 3, \ldots$$

and denote by $\Gamma_k$ the path formed by the three sides of $R_k$ other than the side which is contained in $S$. Let

$$C_k$$

denote the closed arc of the circle $|z| = \sqrt{\rho^2 + (\frac{k}{l})^2}$ which is intercepted by two corners of $R_k$ and which
intersects the negative real axis. Let the curve

\[ C_k = \Gamma_k \cup C_k^\infty \]

be traversed so that the origin lies always to the left. Then for \( z \) in a neighborhood of the origin

\[
\Phi(z) = \frac{1}{2\pi i} \int_{C_k} \frac{f(t)}{t-z} \, dt
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma_k} \frac{f(t)}{t-z} \, dt + \frac{1}{2\pi i} \int_{C_k^\infty} \frac{f(t)}{t-z} \, dt
\]

\[
= \Phi_k^+(z) + \Phi_k^-(z)
\]

for \( k = 1, 2, 3, \ldots \).

Denote by \( S_k \) the side of the rectangle \( R_k \) which is contained in \( S \) (\( = S_r \)). Let \( \epsilon > 0 \) be fixed and consider again the region \( D_\epsilon \) (see (I)).

For sufficiently large \( k \) (\( k > k_0 \), say) \( C_k \) and \( S_k \) lie in the complement of \( D_\epsilon \). Let \( d_1 \) and \( d_2 \) be the distances of \( C_{k_0} \) and \( S_{k_0} \) respectively from the boundary of \( D_\epsilon \) and put \( \min(d_1, d_2) = d \).

It is clear that for every \( k > k_0 \), \( t > 1 \), \( z \in D_\epsilon \) and \( f \in \mathcal{F} \)

\[
\Phi_k^+(z) - \Phi_k^-(z) = \frac{1}{2\pi i} \int_{\sigma_{k,p}} \frac{f(t)}{t-z} \, dt + \frac{1}{2\pi i} \int_{\sigma_{k,p}^\infty} \frac{f(t)}{t-z} \, dt
\]

where \( \sigma_{k,p} \), \( \sigma_{k,p}^\infty \) are appropriate sub-segments of \( S_k \).
properly traversed. The lengths of \( \sigma_{n,p}^\ast \) and \( \sigma_{n,p} \)
are each \( (k^{-1} - (k + b)^{-1}) \). Let \( M = \sup_{f \in F} \max_{t \in S} |f(t)| \).

Then
\[
| \phi_k^f(z) - \phi_{k+1}^f(z) | \leq \frac{M}{\pi d} \frac{1}{k}
\]

for all \( k \geq k_0 \), \( b \geq 1 \), \( z \in D_\varepsilon \) and \( f \in F \). This proves that the limit
\[
(\ast) \quad \lim_{k \to \infty} \phi_k^f(z) = \phi^f(z)
\]
exist uniformly on \( D_\varepsilon \) and uniformly with respect to \( F \).

Since all the \( \phi_k^f(z) \) are uniform functions holomorphic off of the set of points of the paths \( \Gamma_k \), the \( \phi^f(z) \) are uniform functions holomorphic in \( D \).

Now from (\ast) it follows that for \( k \) sufficiently large
( \( k \geq k_0 \), \( z \) fixed), for all \( z \in D_\varepsilon \) and all \( f \in F \)
\[
| \phi_k^f(z) - \phi_k^f(z) | < 1
\]
\[
| \phi_k^f(z) | < 1 + \left| \frac{1}{z \bar{w}} \int_{\Gamma_k} \frac{f(t)}{t-z} \, dt \right|
\]<
\[
< 1 + \frac{M_1}{2\pi d} L
\]

where \( M_1 = \sup_{f \in F} \max_{t \in \Gamma_k} |f(t)| \) and \( L \) is the length of \( \Gamma_k \).

In other words the family \( \bar{f} = \{ \phi^f(z) \} \) is bounded in \( D_\varepsilon \) and therefore locally bounded and regular in \( D \). Finally,
\[ \Phi_k(\infty) = 0 \quad \text{for all } k \text{ and } \Phi, \text{ so that } \Phi(\infty) = 0 \text{ for all } \Phi \in F. \]

By similar reasoning we could show that
\[ \lim_{k \to \infty} \Phi_k(\infty) = \Phi(\infty) \]
exists uniformly on any compact subset \( \Delta \) of \( |z| < \rho \),
where again the uniformity persists with respect to \( \Phi \)
as well as \( \Delta \). Also the family \( \Psi = \{ \Phi_k(\infty) \} \) is locally bounded and regular in \( |z| < \rho \).

To complete this part of the proof we observe that in a neighborhood of the origin
\[ \Phi(\infty) = \Phi_k(\infty) + \Phi_k(\infty) \quad (k \geq 1) \]
\[ = \Phi(\infty) + \Phi(\infty). \]

(III) Suppose that \( z = \rho \) has no special properties with respect to \( F \). By using a device of Poincaré we can still utilize the deliberations of (II).

It is known (Polya and Szego, A. 46, 65, 33) that one can construct a zero-free entire function whose growth along the positive real axis is arbitrarily great. Accordingly, there is a zero-free entire function \( E(z) \) such that
\[ \sup_{\Phi \in F} \frac{|f(z)|}{\sqrt[2]{E[-(z-\rho)^{-2}]}} \]
tends to 0 as \( z \) tends to \( p \) along \( S \) (see II) from either side of the real axis. In particular then the family \( G : \)

\[
q^f(z) = \frac{f(z)}{E[-(z-p)^{-2}]}
\]

is uniformly bounded on \( S - \{p\} \). Furthermore, \( G \) has all the properties of regularity and boundedness of the original family \( F \). Hence, applying part (II), we may write in a neighborhood of the origin

\[
q^f(z) = H^f(z) + K^f(z)
\]

and draw the conclusions about the families \( \{H^f(z)\} \) and \( \{K^f(z)\} \) that were drawn in (II) about \( \Phi \) and \( \Psi \) respectively. This, however, proves the theorem in full with the definitions

\[
\Phi^f(z) = E[-(z-p)^{-2}] H^f(z)
\]
\[
\Psi^f(z) = E[-(z-p)^{-2}] K^f(z)
\]

29. **Theorem.** Let the family \( G = \{g(z)\} = \{ \sum a_n z^n \} \) have radius of regularity one and be locally bounded
in \(|z| < 1\). A necessary and sufficient condition that \(z = 1\) be the only irregular point on the circle of regularity and that this point be almost isolated is that

\[
a_n^q = \{ \gamma^{q}_{\lambda} \} + b_n^q,
\]

where \(\{ \gamma^{q}_{\lambda} \}\) is an entire family of exponential type whose indicator diagram is a finite segment of the real axis whose right endpoint is the origin and

\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}} \left| b_n^q \right|^{1/n} < 1.
\]

Proof: Suppose first that the given decomposition is possible. Then

\[
\gamma^{q}_{\lambda}(z) = \sum_{n=0}^{\infty} \gamma^{q}_{\lambda}(n) z^n + \sum_{n=0}^{\infty} b_n^q z^n.
\]

The first family on the right is locally bounded and regular off of the image under \(e^{-z}\) of the segment described in the theorem. Thus \(\{ \sum \gamma^{q}_{\lambda}(n) z^n \}\) is locally bounded and normal off of a finite segment of the real axis whose left endpoint is one. The family \(\{ \sum b_n^q z^n \}\) has radius of regularity \(> 1\) and is locally bounded in its circle of regularity. This proves that in a disk of radius \(> 1\) the family \(G\) is locally
bounded and regular except at \( z = 1 \) and except possibly at
the points of a finite segment of the real axis whose left
endpoint is \( z = 1 \). Therefore, \( z = 1 \) is an almost isolated
irregular point of \( G \) and is the only irregular point of \( G \)
on \( |z| = 1 \). Hence, the condition of the theorem is
sufficient. The necessity follows from a direct applica-
tion of Theorems 28 and 25.

30. **Theorem.** Let the family \( G \)

\[
q(z) = \sum_{n=0}^{\infty} a_n z^n
\]

have a finite radius of regularity \( R \) and be locally
bounded in \( |z| < R \). Then if there is exactly one
irregular point on \( |z| = R \) and that point is almost
isolated, the upper density of the non-vanishing

\[
A_n = \sup_{q \in G} |a_n|
\]

is one.

31. Theorem 30 generalises the following theorem of Polya[3]:

"**Theorem A.** If on the circle of convergence of
power series only one single singular point
lies, and this singular point is almost iso-
lated for the power series, then the upper
density of the coefficients is one."
We remark that Theorem 30 does not follow from Theorem A, since the functions of \( G \) may well be entire. However, the proof of Theorem A can be extended to prove Theorem 30 without calling on any properties of \( G \) as a normal family. The proof of Theorem A is quite involved and its extension to prove our theorem requires only minor changes. Consequently it seems more economical to describe these changes with respect to Polya's proof than to reproduce it all with the slight alterations.

Polya proves Theorem A by means of a theorem (Satz VI, [3], p. 746) whose generalization we state as:

32. Lemma. Let \( F = \{ f(z) \} \) be a family of functions holomorphic in the half-plane \( \Re(z) \geq 0 \). Let \( M^F_+(r) \) be the maximum of \( |f(z)| \) in the half-circle \( |z| \leq r \), \( \Re(z) \geq 0 \) and put \( \sup_{f \in F} M^F_+(r) = M_+(r) \). Suppose that

\[
\lim_{r \to \infty} \frac{\log M_+(r)}{r} = 0
\]

and

\[
\lim_{r \to \infty} \frac{\log \sup_{f \in F} |f(r)|}{r} = 0.
\]

Then if \( \alpha > 0 \) is arbitrary, the positive integers \( n \) which satisfy

\[
\sup_{f \in F} |f(n)| > e^{-\alpha n}
\]
form a sequence of upper density one.

Similarly, our proof of Theorem 30 rests on Lemma 32.

The proof of Lemma 32 is accomplished by carrying out the deliberations of Polya ([13], pgs. 749-754) with the appropriate substitutions in mind (e.g., for \( |F(z)| \) read \( \sup_{f \in \mathcal{F}} |f(z)| \), etc.).

33. We can now prove Theorem 30: We may assume that \( R = 1 \) and that \( \epsilon = 1 \) is the almost isolated irregularity in question. According to Theorem 29, the coefficients of the family \( G \) allow a decomposition

\[
\alpha_n = \beta_n + b_n,
\]

where the family \( \mathcal{F} = \{ f_n(z) \} \) is of exponential type and has an indicator \( h(\epsilon) \) which supports a finite segment of the real axis with right endpoint at the origin. So \( h(\epsilon) = 0 \) for \( |\epsilon| \leq \sqrt{2} \) and, in particular,

\[
h(\epsilon) = \lim_{\epsilon \to \frac{1}{2^k}} \log \frac{\sup_{z \in \mathbb{C}} |f_n(z)|}{3 + 6} = 0.
\]

Therefore, \( \mathcal{F} \) satisfies the requirements of Lemma 32.

Now \( \lim_{n \to \infty} \sup_{z \in \mathbb{C}} |b_n^3|^{-1} = 1 \), so that there is an \( \delta > 0 \) and an integer \( N(\alpha) \) such that

\[
\sup_{z \in \mathbb{C}} |b_n^3| < e^{-\eta \delta} \quad (n > N(\alpha)).
\]

Furthermore, by Lemma 30 there is a sequence \( \{ \lambda_n \} \) of positive integers having upper density one and satisfying

\[
\sup_{z \in \mathbb{C}} |f_n(\lambda_n)| > e^{-\epsilon \lambda_n} \quad (n = 1).
\]
Now from (4) it follows that
\[ \sup_{g \in G} |a_n^g| \geq \sup_{g \in G} |f_0^g(\lambda)| - \sup_{g \in G} |b_n^g| \]

Consequently, for \( \lambda_n > \max \{ N(\alpha), \log 4^\alpha \} \) we have
\[ \sup_{g \in G} |a_n^g| \geq e^{-\frac{1}{2} \alpha \lambda_n} - e^{-\alpha \lambda_n} \]
\[ = e^{-\alpha \lambda_n} \]

Therefore, the non-zero \( A_n = \sup_{g \in G} |a_n^g| \) have upper density one and Theorem 30 is proved.
REFERENCES

