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THE BEHAVIOR OF FUNCTIONS
HARMONIC AND POSITIVE IN AN
ANGULAR DOMAIN

by

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A THESIS
SUBMITTED TO THE FACULTY
IN PARTIAL FULFILLMENT OF THE
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TO MY WIFE
INTRODUCTION

The principal question to be investigated is the character of a function harmonic and positive in a simply connected domain $G$ and zero on the boundary of $G$ except possibly at a finite number of points. It is well known that if a function is harmonic and regular in a simply connected domain and zero everywhere on the boundary, then the function is identically zero. In fact, if the harmonic function is zero except possibly at a finite number of boundary points and if in a neighborhood of each of these points the function is bounded, then again the function is identically zero (3)*. Thus if a function is harmonic in $G$ and zero on the boundary except at $w_o$, a boundary point of $G$, and if the function is not identically zero, it must become infinite as $w$ approaches $w_o$. The purpose of this paper is to relate the growth of the function as $w$ approaches $w_o$ to the character of the boundary of $G$ in a neighborhood of $w_o$. It is shown that the behavior or the growth of the function as $w$ approaches $w_o$ depends on the size of the angular opening of $G$ at $w_o$. The precision with which the growth can be expressed depends on the

*Numbers in parentheses refer to the bibliography at the end of the paper.
regularity of the boundary of $G$ in a neighborhood of $w_0$.

If $u(w)$ is a positive harmonic function in a simply connected domain $G$ and zero on the boundary of $G$ except at $w_0$, then the boundary of $G$ consists of analytic arcs. Even though the arcs are analytic, there may not exist tangents to the arcs at $w_0$. However, it may be that in a neighborhood of $w_0$ the domain contains an angle of opening $\alpha$ with vertex at $w_0$, or is contained in an angle of opening $\beta$ with vertex at $w_0$, or both. (See figures 2.4.1, 2.7.3, and 2.7.4.) In these cases something can be said about the growth of $u(w)$, as $w$ approaches $w_0$, in terms of the angles $\alpha$ and $\beta$. The precision with which the growth is described depends upon the difference between $\alpha$ and $\beta$. If the difference between the angles $\beta$ and $\alpha$ can be made arbitrarily small by sufficiently restricting the neighborhood of $w_0$, then at $w_0$ there exist tangents to the boundary of $G$. The growth of $u(w)$ then depends on the angle $\alpha = \beta$ between the tangent lines. If the boundary of $G$, in a neighborhood of $w_0$, can be mapped onto an analytic arc by a function $(w - w_0)^{\gamma}$ which maps a neighborhood of $w_0$ in $G$ conformally onto a schlicht neighborhood, then the boundary of $G$ is said to be very
regular in a neighborhood of \( w_0 \). If the boundary of \( G \) is very regular in a neighborhood of \( w_0 \), then a very precise determination of the growth of \( u(w) \) can be obtained.

The results concerning the growth of \( u(w) \) as \( w \) approaches \( w_0 \) are obtained in the following way. The upper half of the \( z \)-plane is mapped conformally onto \( G \), mapping \( z = 0 \) into \( w = w_0 \), by the function \( w = F(z) \). The function \( u^*(z) = u[F(z)] \) which is defined and harmonic on the upper half plane and is zero on the real axis except at \( w_0 \), can be continued in the lower half plane. Stewart (G) has proved a theorem concerning the growth of a harmonic function in a neighborhood of an isolated singularity in terms of the number of sign changes of the function on a circle with center at \( w_0 \) and radius \( r \). This theorem, along with a knowledge of the mapping function \( w = F(z) \) is used to determine the growth of \( u(w) \) as \( w \) approaches \( w_0 \).

In Section 1 the case in which the domain \( G \) has a very regular boundary in the neighborhood of \( w_0 \) is considered, and the character of the function mapping the upper half plane conformally onto \( G \), in the neighborhood of \( w_0 \), is determined. In Section 2 the character of the function \( w = F(z) \) which maps the
upper half plane conformally onto \( G \) in such a way that \( z = 0 \) corresponds to \( w = w_0 \) is determined. This is done for a very general boundary in a neighborhood of \( w_0 \). The results concerning \( w = F(z) \) when the boundary of \( G \) is very general are contained in the proofs of some theorems of Ostrowski (5). These results are stated as Theorems 2.1, 2.3, and 2.4. The proofs of Ostrowski are replaced by simpler proofs, since all the results of his work are not needed. Section 2 describes the character of \( F(z) \) when the boundary of \( G \) has certain special forms. These results are interpreted geometrically in Note 2.7. In Section 3 the mapping theorems of Sections 1 and 2 are used to prove the principal results of this work. In Section 4 a result of Section 3 is used to study the growth of an analytic function \( f(z) \) as \( z \) approaches a singularity of \( f(z) \) between two extremal curves which abut on this singularity.

Most important of all, the author wishes to thank Professor Floyd E. Ulrich for his guidance throughout the course of this work, from the initial proposal of the problem to the completion of this paper.
SECTION 1

The first section consists of a description of a function which maps the upper half plane onto a simply connected domain $G$ with a very regular boundary. The proof of Theorem 1.1, which describes this mapping function, is in two parts. In the first part a function accomplishing the desired mapping is obtained, and in the second part it is shown that any other function performing the same mapping must have the same form.

**Theorem 1.1:** Consider a simply connected domain $G$ in the $w$-plane and suppose that in a neighborhood of $w_0$, a boundary point of $G$, the boundary of $G$ consists of two Jordan arcs $C_-$ and $C_+$ abutting on $w_0$. If there exists a number $\delta$ such that $u = (w - w_0)^\delta$ maps a neighborhood of $w = w_0$ onto a schlicht neighborhood of $u = 0$ and the boundary of $G$ onto an arc analytic in a neighborhood of $u = 0$, then any conformal map of $\mathfrak{J} \{z\} > 0$ onto $G$, such that $w = w_0$ corresponds to $z = 0$, has the form

$$w - w_0 = z^{\frac{\gamma}{\pi}} \phi(z),$$

where $\lim_{z \to 0} \phi(z)$ exists and is not zero.
**Proof:** Let \( H \) be the domain in the \( u \)-plane corresponding to \( G \) under the map \( u = (w - w_0)^{\frac{1}{n}} \), and let \( u = f(z) \) be a conformal map of the half plane \( \mathcal{J} \{ z \} > 0 \) onto \( H \) sending \( z = 0 \) into \( u = 0 \). If \( u = f(z) \), an analytic function, maps a domain \( D_z \) onto a domain \( D_u \) and if in this mapping an analytic boundary arc \( A \) of \( D_z \) corresponds to an analytic boundary arc \( B \) of \( D_u \), then \( f(z) \) is regular at the points of \( A \) (4).

Hence, in a neighborhood of \( z = 0 \), \( f(z) \) has a Taylor expansion \( u = a_1 z + a_2 z^2 + \ldots \), where \( a_1 \neq 0 \) since the arcs are analytic. Thus, \( (w - w_0)^{\frac{1}{n}} = z(a_1 + a_2 z + \ldots) \).

Since \( (w - w_0)^{\frac{1}{n}} \) is a schlicht map in a neighborhood of \( w_0 \) contained in \( G \), a determination of \( z^{\frac{1}{n}}(a_1 + a_2 z + \ldots)^{\frac{1}{n}} = w - w_0 \) can be chosen so that \( w - w_0 \) is single valued for \( z \) in the upper half plane. Set \( \phi(z) = (a_1 + a_2 z + \ldots)^{\frac{1}{n}} \).

Then \( w - w_0 = z^{\frac{1}{n}} \phi(z) \), where \( \lim_{z \to 0} \phi(z) = a_1^{\frac{1}{n}} \) is not zero, is a conformal map of \( \mathcal{J} \{ z \} > 0 \) onto \( G \) such that \( w = w_0 \) corresponds to \( z = 0 \). This completes the proof of the first part.

Suppose \( w - w_0 = \psi(z) \) defines a conformal map of \( \mathcal{J} \{ z \} > 0 \) onto \( G \), sending \( z = 0 \) into \( w = w_0 \). It will be shown that \( \psi(z) = z^{\frac{1}{n}} \phi_2(z) \) where \( \lim_{z \to 0} \phi_2(z) \) exists and is not zero. If we define \( w_1(z) = z^{\frac{1}{n}} \phi(z) \) and \( w_2(z) = \psi(z) \), then \( w = w_1(z) \) and \( w = w_2(z) \) map
the upper half plane \( \mathbb{H} \) onto \( \mathbb{G} \) so that \( w = 0 \)
corresponds to \( z = 0 \). It is convenient to think of the
inverse functions \( z = w_1^{-1}(w) \) and \( z = w_2^{-1}(w) \)
mapping \( \mathbb{G} \) respectively on copies \( \mathbb{H}_1 \) and \( \mathbb{H}_2 \) of \( \mathbb{H} \).
Let \( f \) be any particular linear transformation which
maps \( \mathbb{H}_1 \) and \( \mathbb{H}_2 \) respectively on the interiors of \( \mathbb{C}_1 \)
and \( \mathbb{C}_2 \), copies of the unit circle, and let \( g \) be that
map of \( \mathbb{C}_1 \) onto \( \mathbb{C}_2 \) implied by the above correspondence.
Then \( g = f w_2^{-1} w_1 f^{-1} \). (See figure 1.1.2.) Thus \( g \)
is an analytic function. Now any analytic function which
maps the interior of a circle onto the interior of a
circle is a linear transformation. Hence \( g \) and thus
\( g^{-1} \) and \( f^{-1} g^{-1} f \) are linear transformations. Therefore,
\[
w_2 = w_1 f^{-1} g^{-1} f = w_1 \left( \frac{Az + B}{Cz + D} \right),
\]
where \( B = 0 \) and \( A, D, C \) are not zero, since \( z = 0 \)
maps into \( w = 0 \). Hence
\[
w_2 = w_1 \left( z \left[ \frac{A}{C} - \frac{AD}{C^2} z - \frac{A}{C^3} z^2 - \ldots \right] \right) = z^\frac{1}{N} \left[ \frac{A}{C} - \ldots \right]^\frac{1}{N} \Phi \left( z \left[ \frac{A}{C} - \ldots \right] \right).
\]
Define \( \Phi_2(z) = \left[ \frac{A}{C} - \ldots \right]^\frac{1}{N} \Phi \left( z \left[ \frac{A}{C} - \ldots \right] \right) \).
Then \( w_2(z) = \psi(z) = z^\frac{1}{N} \Phi_2(z) \), where \( \lim_{z \to 0} \Phi_2(z) \) exists
and is not zero. This completes the proof of the theorem.

Corollary 1.2: Any conformal map of \( \mathbb{J} \{ z \} > 0 \)
Figure 1.1.2
onto \( G \), a simply connected domain in the \( w \)-plane with part of its boundary consisting of two straight line segments meeting at \( w_0 \) in the angle \( \gamma \), such that \( w = w_0 \) corresponds to \( z = 0 \), has the form

\[
    w - w_0 = z^{\frac{\gamma}{2\pi}} \phi(z),
\]

where \( \lim_{z \to 0} \phi(z) \) exists and is not zero.
SECTION 2

In this section a description of the function which maps the upper half plane onto a simply connected domain \( G \) bounded by Jordan arcs is given (Theorem 2.5). The result is obtained by transforming a similar result concerning a function which maps the interior of the unit circle onto \( G \) (Theorem 2.4). Several preliminary theorems are needed to prove Theorem 2.4. Theorem 2.1, the first of these, expresses an inequality involving a function \( f(z) \) expressed in terms of the Poisson integral defined by a set of real boundary values \( \chi(\Theta) \).

**Theorem 2.1:** If \( f(z) \) is a complex valued function and \( \chi(\Theta) \) is a real valued function such that

\[
2.1.1 \quad f(z) = \frac{i}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\Theta} + z}{e^{i\Theta} - z} \chi(\Theta) d\Theta, \quad \text{and if}
\]

\[
2.1.2 \quad \lim_{\Theta \to 0} \frac{\pi(\Theta)}{\Theta} = k_+, \quad k_+ \geq 0, \quad \text{and}
\]

\[
2.1.2 \quad \lim_{\Theta \to 0} \frac{\pi(\Theta)}{\Theta} = k_-, \quad k_- \geq 0, \quad \text{where}
\]

\[\Phi(\Theta) = \int_0^\Theta \chi(\tau) d\tau, \quad \text{then}
\]

\[
2.1.3 \quad \lim_{z \to 0} \frac{\pi |f(z)|}{-\log|1-z|} \leq k_+ + k_-
\]
11.

**Definition:** The symbol $z \rightarrow 1$ means that $z$ approaches 1 in an angular opening $\arg(1 - z) < \alpha < \frac{\pi}{2}$.

See figure 2.1.4.

![Figure 2.1.4](image)

**Proof:** Integrate 2.1.1 by parts implying

$$f(z) = \frac{1}{2\pi} \int_0^{\pi/2} \left[ \frac{e^{i\phi} + z}{e^{i\phi} - z} \right] \frac{\Phi(\phi)}{z - \phi} d\phi$$

$$= \frac{i}{2\pi} \int_0^{\pi/2} \left[ \frac{e^{i\phi} - (e^{i\phi} - z)ie^{i\phi} - (e^{i\phi} + z)ie^{i\phi}}{(e^{i\phi} - z)^2} \right] \Phi(\phi) d\phi$$

$$= \frac{i}{2\pi} \int_0^{\pi/2} \left[ \frac{e^{i\phi} + z}{e^{i\phi} - z} \right] \frac{\Phi(\phi)}{z - \phi} d\phi$$

If $z = r e^{i\phi}$, then for $|\phi| < \frac{\pi}{4}$, $|z - 1| > \frac{\sqrt{2}}{2} > \frac{1}{2}$.

See figure 2.1.5.
Thus, for $|\Phi| < \frac{\pi}{4}$ and $|z| \leq 1$,

$$ |f(z)| \leq \frac{|z|}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{|e^{i\phi} - z|^2} d\phi + \frac{4}{\pi} \left[ |\Phi(\pi/2)| + |\Phi(-\pi/2)| \right], $$
or

$$ \frac{\pi}{r} |f(z)| \leq \int_{-\pi/2}^{\pi/2} \frac{1}{|e^{i\phi} - z|^2} d\phi + \frac{4}{r} M, \text{ where } M \text{ is a }
positive constant. Define \( \psi(\theta) \) so that \( \psi(\theta) \sin \theta = \Phi(\theta) \).

Since \( \frac{\sin \theta}{\theta} \) approaches 1 as \( \theta \) goes to 0, it follows from 2.1.2 that given any \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( |\psi(\theta)| \leq k_- + \epsilon \) for \( -\delta \leq \theta < 0 \), and \( |\psi(\theta)| \leq k_+ + \epsilon \) for \( 0 < \theta \leq \delta \). Then

\[
2.1.6 \quad \frac{\nabla}{r} |f(z)| \leq \left( \int_{-\delta}^{0} + \int_{0}^{\delta} \right) \frac{|\Phi(\theta)|}{|e^{i\theta - z}|^2} d\theta + \frac{4M}{r} + (k_+ + \epsilon) \int_{-\delta}^{0} \frac{|\sin \theta|}{|e^{i\theta - z}|^2} d\theta + (k_- + \epsilon) \int_{0}^{\delta} \frac{|\sin \theta|}{|e^{i\theta - z}|^2} d\theta.
\]

Let \( \theta = \phi = x \). Then \( \theta = x + \phi \) and

\[
\int_{0}^{\delta} \frac{\sin \theta}{|e^{i\theta - z}|^2} d\theta = \int_{0}^{\delta} \frac{\sin \theta}{(1 - 2r \cos(\theta - \phi) + r^2)} = \int_{-\phi}^{\phi} \frac{\sin(x + \phi)}{(1 - 2r \cos x + r^2)} dx = \int_{-\phi}^{\phi} \frac{\sin x}{1 + r^2 - 2r \cos x} dx + \int_{-\phi}^{\phi} \frac{\cos x}{1 + r^2 - 2r \cos x} dx.
\]

Hence,

\[
2.1.7 \quad \int_{0}^{\delta} \frac{\sin \theta}{|e^{i\theta - z}|^2} = \frac{\cos \phi}{2r} \log(1 + r^2 - 2r \cos x) \bigg|_{-\phi}^{\delta - \phi} + \sin \phi I(\phi, r), \text{ where}
\]

\[
I(\phi, r) = \int_{-\phi}^{\phi} \frac{\cos x}{1 + r^2 - 2r \cos x} dx.
\]
\[
\frac{\cos \phi}{2r} \log(1 + r^2 - 2r \cos x) \bigg|^{\frac{\delta - \phi}{-\phi}}
\]

and \((\delta - \phi)\) is bounded away from zero as \(\phi\) goes to zero. Thus \(1 + r^2 - 2r \cos(\delta - \phi)\) is bounded away from zero, which implies that

\[
\frac{\cos \phi}{2r} \log(1 + r^2 - 2r \cos(\delta - \phi)) \leq (1 + \varepsilon) \log \frac{1}{|1 - z|},
\]

for \(|\phi| < \delta\) and \(|1 - r| < \delta\).

Hence, given an \(\varepsilon_i\), there is a \(\delta_i\), so that

\[
2 \frac{\cos \phi}{2r} \log(1 + r^2 - 2r \cos x) \bigg|_{-\phi}^{\delta - \phi} \leq (1 + \varepsilon) \log \frac{1}{|1 - z|},
\]

for \(|\phi| < \delta\) and \(|1 - r| < \delta\).

The integral \(I(\phi, r)\) can be evaluated and the result is

\[
I(\phi, r) = \frac{-x}{2r} + \frac{1 + r^2}{(1 - r^2)^2} \frac{1}{r} \tan^{-1} \frac{1 + r}{1 - r} \tan \frac{x}{2} \bigg|_{-\phi}^{\delta - \phi}
\]

where \(|\phi| < \frac{\pi}{4}\). Thus,

\[
\sin \phi I(\phi, r) = \sin \phi \left\{ -\frac{\delta}{2r} + \frac{1 + r^2}{(1 - r^2)^2} \left[ \tan^{-1} \frac{1 + r}{1 - r} \tan \frac{\delta - \phi}{2} 
\right.ight.
\]

- \left. \tan^{-1} \frac{1 + r}{1 - r} \tan(-\frac{\phi}{2}) \right\}.

The quantity in the square brackets is bounded by \(\gamma\); therefore, in order to show that \(\sin \phi I(\phi, r)\) is bounded as \(z\) approaches \(1\) inside some angular opening, it is sufficient to show that \(\frac{\sin \phi}{1 - r^2}\) or \(\frac{\sin \phi}{1 - r}\) is bounded.

If \(z\) is in the angular opening \(2 \alpha\) (see figure
2.1.9.) \( \frac{|\sin \phi|}{1-r} \) is bounded by \( \tan \alpha \). For, if \( z = re^{i\phi} \), then \( 1 - r \geq 1 - r' \), where \( r'e^{i\phi} \) lies on \( L \), a side of the angular opening. This implies that \( \frac{\sin \phi}{1-r} \leq \frac{\sin \phi}{1-r'} \). Now \( r' = \sqrt{x^2 + y^2} \), where \((x, y)\) is the intersection of \( L \) and \( L_1 \), the line through the origin and \( z \).

Figure 2.1.9
The line $L$ has the equation $y = (1 - x)\tan \alpha$, and the
line $L'$ is given by $y = x \tan \phi$. Set $K = \tan \alpha$.

Then $x = \frac{K}{K + \tan \phi}$ and $y = \frac{K}{K + \tan \phi} \tan \phi$, which implies

$$r' = \frac{K}{K + \tan \phi} \sqrt{1 + \tan^2 \phi} = \frac{K \sec \phi}{K + \tan \phi}.$$

Then,

$$1 - r' = 1 - \frac{K \sec \phi}{K + \tan \phi} = \frac{K + \tan \phi - K \sec \phi}{K + \tan \phi},$$

from which it follows that

$$\frac{\sin \phi}{1 - r'} = \frac{K + \tan \phi}{K (1 - \sec \phi) + \sec \phi}.$$

Now

$$\frac{1 - \sec \phi}{\sin \phi} = \frac{1 - \cos \phi}{\sin \phi \cos \phi} = \frac{-\tan \frac{\phi}{2}}{\cos \phi}.$$

Hence,

$$\lim_{\phi \to 0} \frac{\sin \phi}{1 - r'} = \frac{K + 0}{K (0) + 1} = K = \tan \alpha.$$

Thus

$$\frac{\sin \phi}{1 - r'^2}$$

is bounded; which implies that

$\sin \phi I(\phi, r)$ is bounded in some angular neighborhood of

$z = 1$.

Hence in this neighborhood,

$$\int_0^\phi \frac{\sin \theta d\theta}{|e^{i\theta} - z|^2} = (1 + \epsilon_2) \log \frac{1}{|1 - z|},$$

from 2.1.7 and 2.1.8. Let $\theta = - \theta'$, then

$$\int_0^\phi \frac{|\sin \theta| d\theta}{|e^{i\theta} - z|^2} = - \int_\phi^0 \frac{|\sin \theta'| d\theta'}{|e^{i\theta'} - z|^2} = \int_0^\phi \frac{\sin \theta' d\theta'}{|e^{i\theta'} - z|^2}.$$
But \(|e^{i\theta'} - z| = |e^{i\theta'} - \overline{z}|\). Thus,

\[
2.1.11 \quad \int_{\frac{\delta}{2}}^{\frac{\delta}{2}} \frac{|\sin \theta| d\theta}{|e^{i\theta} - z|^2} = \int_{0}^{\delta} \frac{|\sin \theta|}{|e^{i\theta} - \overline{z}|^2} d\theta
\]

\[
\approx (1 + \varepsilon_2) \log \frac{1}{|1 - \overline{z}|} = (1 + \varepsilon_2) \log \frac{1}{|1 - z|}
\]

in some angular neighborhood of \(z = 1\). If the results of 2.1.10 and 2.1.11 are applied to 2.1.6, then

\[
\frac{\pi}{r} |f(z)| \leq (k_+ + k_- + \varepsilon) \log \frac{l}{|1 - z|}
\]

in a suitably restricted angular neighborhood of \(z = 1\), where \(\varepsilon > 0\) is arbitrary. Thus,

\[
\lim_{z \to 1} \frac{\pi |f(z)|}{-\log |1 - z|} \leq k_+ + k_-.
\]

The following corollary demonstrates that result 2.1.3 depends only on the value of \(\chi(\theta)\) in an arbitrary neighborhood of \(\Theta = 0\).

**Corollary 2.2:** If \(\chi^*(\theta)\) and \(f^*(z)\) are such that

\[
f^*(z) = \frac{i}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \chi^*(\theta) d\theta,
\]

and
\( \chi^*(\theta) = \chi(\theta) \) for \( |\theta| < \delta > 0 \), then

\[
\lim_{z \to \infty} \frac{\pi}{-\log |1-z|} |f^*(z)| \leq k_+ + k_-
\]

**Proof:**

\[
f^*(z) = \frac{1}{z \pi} \left( \int_{-\pi}^{\pi} e^{i \theta + z} \chi^*(\theta) \, d\theta + \frac{1}{z} \int_{-\pi}^{\pi} \frac{e^{i \theta + z}}{e^{i \theta} - z} \chi^*(\theta) \, d\theta \right)
\]

\[
= f(z) + \frac{1}{z \pi} \left( \int_{-\pi}^{\pi} e^{i \theta + z} \left[ \chi^*(\theta) - \chi(\theta) \right] \, d\theta \right)
\]

where \( M(z, \delta) \) is bounded in the angular opening. Hence, it follows from 2.1.3 that

\[
\lim_{z \to \infty} \frac{\pi}{-\log |1-z|} |f^*(z)| = \lim_{z \to \infty} \left[ \frac{\pi |f(z)|}{-\log |1-z|} + \frac{|M(z, \delta)|}{-\log |1-z|} \right] \leq k_+ + k_-
\]

The next theorem generalizes Theorem 2.1.

**Theorem 2.2:** If \( f(z) \) and \( \chi(\theta) \) are such that

\[
f(z) = \frac{1}{z \pi} \int_{-\pi}^{\pi} \frac{e^{i \theta} + z}{e^{i \theta} - z} \chi(\theta) \, d\theta
\]

2.3.1 \( \lim_{\theta \to 0} \frac{\Phi(\theta)}{\Theta} - g_+ \leq k_+ (\leq 0) \),

2.3.2 \( \lim_{\theta \to 0} \frac{\Phi(\theta)}{\Theta} - g_- \leq k_- (\leq 0) \),

where \( \Phi(\theta) = \int_{0}^{\theta} \chi(z) \, dz \), then

2.3.3 \( \lim_{z \to \infty} \left| \frac{\pi f(z)}{-\log |1-z| + i \delta} \right| \leq k_+ + k_- \).
where \( \Delta = \varepsilon_+ - \varepsilon_- \).

**Proof:** Let \( g(z) \) be that determination of \( \log \frac{1+z}{1-z} \), in \( |z| < 1 \), which is zero at \( z = 0 \). Denote by \( \Psi(\theta) \),

\[
\int \{ g(e^{i\theta}) \} = \arg \frac{1+e^{i\theta}}{1-e^{i\theta}} = \arg(1 + e^{i\theta}) - \arg(1 - e^{i\theta}).
\]

Then

\[
\Psi(\theta) = \begin{cases} \frac{\pi}{2}, & 0 < \theta < \pi \\
-\frac{\pi}{2}, & 0 > \theta > -\pi, \end{cases}
\]

since the diagonals of a rhombus intersect in an angle \( \frac{\pi}{2} \).

See figure 2.3.3.
Define $\chi^*(\theta) = \chi(\theta) - \frac{g_+ + g_-}{2} - \frac{\Delta}{4\pi} \psi(\theta)$. Then,

$$f^*(z) = \frac{i}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \chi^*(\theta) d\theta = \frac{i}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \chi(\theta) d\theta - \frac{i}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \left( \frac{g_+ + g_-}{2} \right) d\theta$$

$$- \frac{\Delta}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \psi(\theta) d\theta$$

$$= f(z) + C + \frac{i}{4\pi} g(z),$$

since the real part of

$$i g(e^{i\theta}) = -\psi(\theta),$$

and $C = \frac{i}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \left( \frac{g_+ + g_-}{2} \right) d\theta$

is a constant. Also,

$$\lim_{\theta \to 0} \left| \int_{0}^{\theta} \chi^*(\xi) d\xi \right|$$

$$= \lim_{\theta \to 0} \left| \frac{i}{\theta} \int_{0}^{\theta} \chi(\xi) d\xi - \frac{g_+ + g_-}{2} - \frac{\Delta}{4\pi} \frac{\pi}{2} \right|$$

$$= \lim_{\theta \to 0} \left| \frac{i}{\theta} \int_{0}^{\theta} \chi(\xi) d\xi - g_+ \right| \leq k_+ \quad \text{from hypothesis 2.3.1.}$$

Likewise

$$\lim_{\theta \to 0} \left| \int_{0}^{\theta} \chi^*(\xi) d\xi \right|$$

$$= \lim_{\theta \to 0} \left| \frac{i}{\theta} \int_{0}^{\theta} \chi(\xi) d\xi - \frac{g_+ + g_-}{2} + \frac{\Delta}{4\pi} \frac{\pi}{2} \right|$$

$$= \lim_{\theta \to 0} \left| \frac{i}{\theta} \int_{0}^{\theta} \chi(\xi) d\xi - g_- \right| \leq k_-.$$ Hence, from Corollary 2.2,

$$\lim_{z \to 1} \pi \left| \frac{f^*(z)}{z - 1} \right| \leq k_+ + k_-.$$ Now
21. \[
\frac{\Re f^*(z)}{-\log |1-z|} = \frac{1}{-\log |1-z|} |\Re f(z) + \pi C + i\Delta \log(1 + z) - \log(1 - z)|.
\]

Here \(|1 - z| < 1\), and consequently \(-\log |1 - z| > 0\). Hence,

\[
\frac{\Re f^*(z)}{-\log |1-z|} = \frac{\Re f(z)}{-\log |1-z|} + \frac{i\alpha \log(1 - z)}{-\log |1-z|} + \frac{\pi C + i\Delta \log(1 + z)}{-\log |1-z|} \geq \frac{\Re f(z)}{-\log |1-z|} + i\Delta - \epsilon
\]

for any \(\epsilon > 0\) and for \(z\) in some angular neighborhood of \(z = 1\). Thus

\[
\lim_{z \to 1} \left| \frac{\Re f(z)}{-\log |1-z|} + i\Delta \right| \leq k_+ + k_-.
\]

Theorem 2.3 is used to prove the following mapping theorem.

**Theorem 2.4:** Consider a simply connected domain \(G\) in the \(w\)-plane with a boundary composed of Jordan arcs, and suppose that in a neighborhood of \(w_0\), a boundary point of \(G\), the boundary of \(G\) consists of the two Jordan arcs \(C_-\) and \(C_+\) abutting on \(w_0\). Suppose in a neighborhood of \(w_0, C_-\) lies in the angle \(|\arg(w - w_0) - h_-| \leq k_-\) and \(C_+\) lies in the angle \(|\arg(w - w_0) - h_+| \leq k_+\), where \(h_-, h_+, k_-, k_+\) are finite and \(k_+, k_- \geq 0\).

(See figure 2.4.1.) If \(w = f(z)\) maps \(|z| < 1\) conformally onto \(G\) in such a way that \(w = w_0\) corresponds
to $\gamma = h_+ - h_-$.

Proof: If one Jordan domain is transformed conformally into another, then the transformation is one to one and continuous in the closed domain (1). Let $f^*(z) = \frac{1}{\zeta} \log [f(z) - w_0]$ where any particular determination of the logarithm is used. The real part of $f^*(z)$ is the argument of $f(z) - w_0$. Then $f^*(z)$ maps a neighborhood of $z = 1$ contained in $|z| < 1$ into a Jordan domain. Thus $\chi^*(\theta) = \arg \left[ f(e^{i\theta}) - w_0 \right]$ is defined, and $f^*(z)$ has the Poisson integral representation

$$f^*(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \chi^*(\theta) d\theta.$$ From the hypotheses,

$$|\chi^*(\theta) - h_-| \leq k_-$$

for $0 < \theta < \theta_-$. This follows that $\gamma = h_+ - h_-$. It follows that

$$\lim_{Z \to 1} \left| \gamma \frac{f^*(z)}{-\log |z|} \right| \leq k_+ + k_-,$$

which implies

$$\lim_{Z \to 1} \left| \frac{\log [f(w) - w_0]}{-i\log |1 - z|} + i\gamma \right| \leq k_+ + k_-,$$
Figure 2.4.1
and
\[ \lim_{z \to 1} \frac{\log |f(z) - w_0|}{-\log |z - 1|} - \frac{\gamma}{\pi} \leq \frac{k_+ + k_-}{\pi}. \]

Finally, since
\[ \left| \frac{\log |f(z) - w_0|}{-\log |z - 1|} - \frac{\gamma}{\pi} \right| \geq \left| \frac{\log |f(z) - w_0|}{\log |z - 1|} - \frac{\gamma}{\pi} \right|, \]
we have that
\[ \lim_{z \to 1} \left| \frac{\log |f(z) - w_0|}{\log |z - 1|} - \frac{\gamma}{\pi} \right| \leq \frac{k_+ + k_-}{\pi}. \]

Note: It is not necessary that the complete boundary of \( G \) consist of Jordan arcs. One need only assume that \( C_+ \) and \( C_- \) are free Jordan arcs on the boundary \( \partial \).

The next theorem transforms the result of Theorem 2.4 from the interior of a unit circle to the upper half plane.

**Theorem 2.5:** If \( G \) is defined as in Theorem 2.4, and if \( w = F(z) \) maps the upper half plane \( H \) \( \mathcal{Q} \{z\} > 0 \) conformally onto \( G \) in such a way that \( w = w_0 \) corresponds to \( z = 0 \), then

\[ \lim_{z \to 0} \left| \frac{\log |F(z) - w_0|}{\log |z|} - \frac{\gamma}{\pi} \right| \leq \frac{k_+ + k_-}{\pi}. \]

**Proof:** Let \( t = g(z) \) be any conformal map of \( H \) onto \( |t| < 1 \) sending \( z = 0 \) into \( t = 1 \). Let \( w = f(t) \) map \( |t| < 1 \) onto \( G \) in such a way that \( w = w_0 \) corresponds
to $t = 1$. Then $F = fg$. See figure 2.5.2.

Figure 2.5.2

Hence $f = Fg^{-1}$ is a conformal map of $|t| < 1$ onto the interior of $G$ sending $t = 1$ into $w = w_0$. It then follows from Theorem 2.4 that

$$\lim_{t \to 1} \left| \frac{\log |f(t) - w_0|}{\log |1-t|} - \frac{\gamma}{\pi} \right| \leq \frac{k_+ + k_-}{\pi}.$$ 

Now $t = g(z)$ is a linear transformation, implying $z = g^{-1}(t)$ is a linear transformation of the form $z = \frac{1-t}{at + b}$. Thus $\log|z| = \log|1-t| - \log|at + b|$. Also, $t \to 1$
if and only if \( z \to 0 \). Therefore,

\[
\lim_{z \to 0} \left| \frac{\log |F(z) - w_0|}{\log |z|} - \frac{\gamma}{\pi} \right| = \lim_{t \to 1} \left| \frac{\log |f(t) - w_0|}{\log |1-t| - \log |at + b|} - \frac{\gamma}{\pi} \right|
\]

\[
= \lim_{t \to 1} \left| \frac{\log |f(t) - w_0|}{\log |1-t|} \cdot \frac{1}{1 - \frac{\log |at + b|}{\log |1-t|}} - \frac{\gamma}{\pi} \right|
\]

\[
= \lim_{t \to 1} \left| \frac{\log |f(t) - w_0|}{\log |1-t|} - \frac{\gamma}{\pi} \right| \leq \frac{k_+ + k_-}{\pi},
\]

which proves the theorem.

**Corollary 2.6:** If the conditions of Theorem 2.5 are satisfied, then given any \( a, \varepsilon > 0 \), there exists a \( \delta \) such that

\[
|z| \left( \frac{\gamma + k_+ k_-}{\pi} + \varepsilon \right) \leq |F(z) - w_0| \leq |z| \left( \frac{\gamma - (k_+ + k_-)}{\pi} - \varepsilon \right)
\]

for \( |z| < \delta \) and \( 0 < a \leq \arg z \leq \pi - a \).

This follows immediately from the definition of the superior limit.

It is interesting to notice the form which 2.6 takes if \( \gamma - (k_+ + k_-) > 0 \), \( \gamma - (k_+ + k_-) < 0 \), or
\[ \gamma + k_+ + k_- = 0. \] These results are stated as

**Corollary 2.6.1:** If \( \gamma - (k_+ + k_-) > 0 \), then

\[
\left| w - w_0 \right| \left( \frac{\pi}{\gamma - (k_+ + k_-) + \varepsilon} \right) \leq |z| \leq \left| w - w_0 \right| \left( \frac{\pi}{\gamma - (k_+ + k_-) - \varepsilon} \right),
\]

if \( \gamma - (k_+ + k_-) \leq 0 \), then

\[
|z| \leq \left| w - w_0 \right| \left( \frac{\pi}{\gamma + k_+ + k_-} - \varepsilon \right);
\]

and if \( \gamma + (k_+ + k_-) = 0 \), then

\[
|z| \leq \left| w - w_0 \right| \frac{1}{\varepsilon},
\]

for any arbitrary \( \varepsilon > 0 \), in a suitably restricted angular neighborhood of \( w_0 \).

**Note 2.7:** The above results may be interpreted in the following way. The general case is that in which there is no tangent to \( C_+ \) or \( C_- \) at \( w_0 \), but in a sufficiently restricted neighborhood of \( w_0 \) the domain \( G \) contains the angle with vertex at \( w_0 \) and opening \( \alpha = h_+ - k_+ - (h_- + k_-) \), and is contained in the angle with vertex at \( w_0 \) and opening \( \beta = h_+ + k_+ - (h_- + k_-) \).

(See figure 2.4.1.) The results in this case are given by 2.6.2. If the two interior rays coincide, that is,
if $\alpha = 0$ (see figure 2.7.3) or if $h_+ - k_+ < h_- + k_-$ (see figure 2.7.4), then $h_+ - k_+ \leq h_- + k_-$. This implies that $(h_- - h_+) - (k_+ - k_-) \leq 0$, or $\gamma - (k_+ + k_-) \leq 0$.

In this case the results are given by 2.6.3. If 
\[ \gamma + (k_+ + k_-) = 0, \]
that is, if $\beta = 0$ (case 2.6.4), then the boundary of $G$ has a cusp at $w_0$; for,
\[ \gamma + k_+ + k_- = 0 \]
if and only if $\gamma = k_+ = k_- = 0$.

Also, if the boundary of $G$ has a cusp at $w_0$, then $\gamma$, $k_+$ and $k_-$ can be made arbitrarily small in a sufficiently restricted neighborhood of $w_0$. Under these circumstances the result is expressed by 2.6.4.

It should also be noted that if tangents exist at $w_0$, $k_+$ and $k_-$ can be made arbitrarily small, and 2.6 and 2.6.2 become

2.7.1 \[ |z|^{\frac{\gamma}{\beta}} + \varepsilon \leq |w - w_0| \leq |z|^{\frac{\gamma}{\beta}} - \varepsilon \]
and, for $\gamma \neq 0$

2.7.2 \[ |w - w_0|^{\frac{\gamma}{\beta}} + \varepsilon' \leq |z| \leq |w - w_0|^{\frac{\gamma}{\beta}} - \varepsilon'. \]
Figure 2.7.3

Figure 2.7.4.
SECTION 3

This section contains the main result of this work. It describes the growth, as \( w \) approaches \( w_0 \), of a function \( u(w) \) harmonic and positive in a domain \( G \) and zero on the boundary of \( G \) except at \( w_0 \), a boundary point of \( G \). The character of \( u(w) \) in the neighborhood of \( w_0 \) depends on the character of the boundary of \( G \) in the neighborhood of \( w_0 \). The most general theorem, Theorem 3.1, is proved by using a result of Stewart (G) along with Corollary 2.6.1. In the case in which \( G \) has a very regular boundary, a more precise result can be obtained by using Theorem 1.1 instead of Corollary 2.6.1. This result is stated in Theorem 3.2.

In what follows, it is convenient to indicate by \( \mathcal{N}(w_0) \) a closed region contained in \( G \), abutting on \( w_0 \), and such that its image by a conformal mapping of \( G \) on the upper half plane, sending \( w = w_0 \) to \( z = 0 \), is contained in an angle of opening \( 2\alpha \) which may be arbitrarily close to \( \Pi \).

**Theorem 3.1:** Consider a simply connected domain \( G \) in the \( w \)-plane, and suppose that in a neighborhood of \( w_0 \), a boundary point of \( G \), the boundary of \( G \) consists
of two Jordan arcs $C_-$ and $C_+$ abutting on $w_0$. Suppose in some neighborhood of $w_0$, $C_-$ is contained in an angle $|\arg(w - w_0) - h_-| \leq k_-$ and $C_+$ is contained in an angle $|\arg(w - w_0) - h_+| \leq k_+$, where $h_+, h_-, k_+$, and $k_-$ are finite and $k_+$ and $k_-$ are non-negative. (See figure 2.4.1.) Define $\gamma = h_+ - h_-$. Let $u(w)$ be a function harmonic and positive in $G$ and zero on the boundary except at $w = w_0$, where $u(w)$ becomes infinite. Then, given any $\varepsilon > 0$ and a neighborhood $N(w_0)$ there exists a $\delta$ such that the following inequalities are true for all $|w - w_0| < \delta$ and $w$ in $N(w_0)$.

If $\gamma - (k_+ + k_-) > 0$, then

3.1.1 $M|w - w_0| \left(\frac{-\pi}{\gamma + (k_+ + k_-)} + \varepsilon\right) \leq u(w) \leq M|w - w_0| \left(\frac{-\pi}{\gamma - (k_+ + k_-)} - \varepsilon\right)$

where $M$ is a positive constant.

If $\gamma - (k_+ + k_-) \leq 0$, then

3.1.2 $u(w) \geq M|w - w_0| \left(\frac{-\pi}{\gamma + (k_+ + k_-)} + \varepsilon\right)$.

If $\gamma + (k_+ + k_-) = 0$, or if the boundary of $G$ has a cusp at $w_0$, then

3.1.3 $u(w) \geq \frac{1}{|w - w_0|} K$. 
where $K$ is any positive number.

**Proof:** Let $w = F(z)$ map $H : \{z\} > 0$ conformally onto $G$ and $z = 0$ on $w = w_0$. Then $z = F^{-1}(w)$ maps $G$ conformally onto $H$. Define $u^*(z) = u[F(z)]$. Then $u^*(z)$ is harmonic and positive for $\{z\} > 0$ and zero for $\{z\} = 0$ except at $z = 0$, where $u^*(z)$ is infinite. From the Schwarz reflection principle for harmonic functions $u^*(z)$ can be continued harmonically throughout $\{z\} < 0$, so that $u^*(z)$ is harmonic in the entire plane except for $z = 0$, and $u^*(z) < 0$ for $\{z\} < 0$. Stewart has proved the following theorem (6). Suppose $u(r, \theta)$ is harmonic in $r = |z| \notin R$ except at $r = 0$, an isolated singularity of $u(r, \theta)$. Then $u(r, \theta)$ has a representation

$$u(r, \theta) = a(r, \theta) + k \log \frac{1}{r} + \sum_{n=1}^{\infty} \frac{a_n \cos n\theta + b_n \sin n\theta}{r^n}$$

for $r < r_0$, where $a(r, \theta)$ is harmonic throughout $r \leq 1$. Let $m(r)$ be the number of sign changes of $u(r, \theta)$ on the circle of radius $r$ and center at the origin. If $m(r)$ is bounded by $2N$ as $r \to 0$, then $a_n = b_n = 0$ for $n > N$, and thus $u(r, \theta)$ has the representation
\[ u(r, \Theta) = a(r, \Theta) + k \log \frac{1}{r} + \sum_{n=1}^{N} \frac{a_n \cos n\Theta + b_n \sin n\Theta}{r^n}. \]

In the present situation \( u^*(z) \) has exactly two sign changes on every circle \( |z| = r \). Thus \( u^*(z) \) has the form

\[ u^*(z) = a(r, \Theta) + k \log \frac{1}{r} + \frac{m(\Theta)}{r}, \]

where \( m(\Theta) = a \sin \Theta + b \cos \Theta \). Hence, given any \( \epsilon_1 > 0 \) there is a \( \delta \) such that

\[ 3.1.4 \quad \frac{|m(\Theta)|}{|z|} |1 - \epsilon_1| \leq |u^*(z)| \leq \frac{|m(\Theta)|}{|z|} |1 + \epsilon_1|, \]

for all \( |z| < \delta \). From 2.6.2 for \( \gamma - (k_+ + k_-) > 0 \),

\[ \frac{|m(\Theta)|}{|F(z) - w_0|} \frac{|1 - \epsilon_1|}{r} \leq u^*(z) \leq \frac{|m(\Theta)|}{|F(z) - w_0|} \frac{|1 + \epsilon_1|}{r}. \]

Thus, given an \( \epsilon \) and a closed neighborhood \( N(w_0) \) there exists a \( \delta \) such that

\[ \frac{M}{|w - w_0|} \frac{|1 - \epsilon_1|}{r} \leq u(w) \leq \frac{M}{|w - w_0|} \frac{|1 + \epsilon_1|}{r}, \]
or
\[ M|w - w_0| \leq \frac{\eta}{\gamma + (k_+ + k_-) + \varepsilon} \leq u(w) \leq \frac{\eta}{\gamma - (k_+ + k_-) - \varepsilon} |w - w_0| \]

for \( w \) in \( N(w_0) \) and \( |w - w_0| < \delta \). If
\( \gamma + (k_+ + k_-) = 0 \), or if the boundary of \( G \) has a
cusp at \( w_0 \), then from 2.6.4
\[ u^*(z) \geq \frac{|m(\Theta)| [1 - \varepsilon]}{|F(z) - w_0|^{\varepsilon}}. \]
Thus, given any \( K \) and \( N(w_0) \) there exists a \( \delta \)
such that \( u(w) \geq \frac{1}{|w - w_0|} K \) for all \( w \) in \( N(w_0) \)
and \( |w - w_0| < \delta \).

A more precise result is possible when the boundary
is very regular. This more precise result is given in
the following theorem.

**Theorem 3.2:** Consider a simply connected domain \( G \)
in the \( w \)-plane, and suppose that in a neighborhood of
\( w_0 \), a boundary point of \( G \), the boundary of \( G \) is very
regular. Let \( u(w) \) be a function harmonic and positive
in \( G \) and zero on the boundary except at \( w = w_0 \), where
\( u(w) \) becomes infinite. Given any \( \varepsilon > 0 \) and a neighbor-
hood \( N(w_0) \) there exists a \( \delta > 0 \) such that
\[ |w - w_0| - \frac{\eta}{\delta}(M - \varepsilon) \leq u(w) \leq |w - w_0| - \frac{\eta}{\delta}(M + \varepsilon), \]
for all \( w \) in \( M(w_0) \) and \( |w - w_0| < \delta \).

**Proof:** Construct \( u^*(z) \) as in Theorem 3.1 and get, as in Theorem 3.1,

\[
3.1.4 \quad \frac{|\text{Im}(\Theta)|}{|z|} [1 - \varepsilon_i] \leq |u^*(z)| \leq \frac{|\text{Im}(\Theta)|}{|z|} [1 + \varepsilon_i].
\]

From 1.1.1,

\[
\frac{|\text{Im}(\Theta)|}{|z|} \left[ \phi(z) \right]^{\frac{\pi}{\delta}} [1 - \varepsilon_i] \leq u^*(z) \leq \frac{|\text{Im}(\Theta)|}{|z|} \left[ \phi(z) \right]^{\frac{\pi}{\delta}} [1 + \varepsilon_i],
\]

\[
\frac{|F(z) - W_0|}{|F(z) - W_0|^{\frac{\pi}{\delta}}}
\]

where \( \lim_{z \to 0} \phi(z) \) exists and is not zero. Hence,

\[
M [1 - \varepsilon] |w - w_0|^{-\frac{\pi}{\delta}} \leq u(w) \leq M [1 + \varepsilon] |w - w_0|^{-\frac{\pi}{\delta}},
\]

which implies the result.
SECTION 4

The last section contains an application of Theorem 3.1 to extremal curves. Hunsaker has determined the extremal curves for certain classes of functions (2). Theorem 4.2 is in the nature of a converse theorem which describes the function, the extremal curves of which have certain properties in the neighborhood of a singular point of the function.

Definition 4.1: Suppose \( f(z) \) is a function meromorphic in some domain \( D \). Let \( C(r, D) \) denote, for a fixed \( r \), the set of points on the circle \( |z| = r \) which belong to \( D \). Consider \( w(r, \Theta) = |f(re^{i\Theta})| \) as a function of \( \Theta \) on \( C(r, D) \), and denote by \( M(r, D) \) the points of \( C(r, D) \) which are points of relative maxima or relative minima of \( w(r, \Theta) \) or poles of \( f(z) \). Then let \( r \) vary so that the sets \( C(r, D) \) exhaust all of \( D \). The union of all the sets \( M(r, D) \) arising by allowing \( R \) to vary in this way is called \( M \). The set of extremal curves of the function \( f(z) \) in the domain \( D \), Hunsaker has shown (2) that this set \( M \) consists of the zeros and poles of \( f(z) \) and certain curves along which the harmonic function \( \int \left[ \frac{e^{i\Theta}f'(z)}{f(z)} \right] \) is zero.
Theorem 4.2: Suppose $f(z)$ has a singularity at $z = a$, $G$ is the domain bounded by two extremal curves which abut on a in a cusp, and there is no extremal curve lying in $G$. If there exists a ray $\arg(z - a) = \phi_0$ through $a$ such that in a neighborhood of $a$ the ray is contained in $G$, then

$$\lim_{z \to a} \frac{\log |f(z)|}{\log |z - a|} = +\infty.$$ 

Proof: The function $f(z)$ is analytic and not zero in $G$, since any zeros or poles of $f(z)$ will lie on the extremal curves. Since $\int \left\{ \frac{z f'(z)}{f(z)} \right\}$ maintains the same sign throughout $G$, $\int \left\{ \frac{z f'(z)}{f(z)} \right\}$ is a positive harmonic function in $G$ and zero on the boundary of $G$ except at $z = a$. Define $N(a)$ to be a closed region contained in $G$, abutting on $z = a$ and such that its image by a conformal mapping of $G$ on the upper half plane, sending $z = a$ to $t = 0$, is contained in an angle of opening $2\alpha$ which may be arbitrarily close to $\pi$. From Theorem 3.1, given any $M$, and any neighborhood $N(a)$, there exists a $\delta > 0$ such that

$$\frac{1}{|z - a|^M} < \left| \int \left\{ \frac{z f'(z)}{f(z)} \right\} \right| \leq \left| z \frac{f'(z)}{f(z)} \right|,$$
for \(|z - a| < \delta\) and \(z\) in \(N(a)\). Thus,

\[
\left| \frac{d}{dz} \log f(z) \right| = \left| \frac{f'(z)}{f(z)} \right| > \frac{1}{|z| |z - a|^{M_1}};
\]

that is, for \(z = re^{i\theta}\),

\[
4.2.1 \quad \left| \frac{\partial}{\partial r} \log |f(z)| + i \frac{\partial}{\partial r} \arg f(z) \right| > \frac{1}{|z - a|^{M_1}},
\]

where \(M\) may be taken arbitrarily large. Throughout the discussion to follow it is assumed that a particular argument of \(f(z)\) at some point \(z_0\) on the ray \(\arg(z - a) = \phi_0\) is determined; and thus, the argument of \(f(z)\) for \(z\) in \(G\) is defined by continuity. It should be recalled in this connection that \(f(z) \neq 0\) in \(G\).

Suppose there exists an \(M_0\) so that

\[
\left| \frac{\partial}{\partial r} \log |f(z)| \right| \leq \frac{1}{|z - a|^{M_0}},
\]

and

\[
\left| \frac{\partial}{\partial r} \arg f(z) \right| \leq \frac{1}{|z - a|^{M_0}}
\]

for all \(z\) in \(N(a)\). Then

\[
\left| \frac{\partial}{\partial r} \log |f(z)| + i \frac{\partial}{\partial r} \arg f(z) \right| \leq \frac{2}{|z - a|^{M_0}} < \frac{1}{|z - a|^{M}}
\]

for some \(M\), which contradicts 4.2.1. Thus, for any \(M\), either
\[
\left| \frac{\partial}{\partial r} \log |f(z)| \right| > \frac{1}{|z-a|^M}, \quad \text{or}
\]
\[
\left| \frac{\partial}{\partial r} \arg f(z) \right| > \frac{1}{|z-a|^M}, \quad \text{or}
\]
both inequalities hold for \( z \) in \( N(a) \) and \(|z - a|\) sufficiently small. Define \( \rho e^{i\phi} = z - a \), and choose an \( N(a) \) so that in a neighborhood of \( a \), \( N(a) \) contains the ray through \( a \) specified in the hypothesis. Then

4.2.2
\[
\left| \frac{\partial}{\partial \rho} \arg f(z) \right| > \frac{1}{\rho^M}, \quad \text{or}
\]
4.2.3
\[
\left| \frac{\partial}{\partial \rho} \log |f(z)| \right| > \frac{1}{\rho^M},
\]
for all \( \rho \) sufficiently small and \( z \) in \( N(a) \), in particular, for all \( z \) on the ray \( \arg(z - a) = \phi_0 \).

If 4.2.2 holds, then either

4.2.4
\[
\frac{\partial}{\partial \rho} \arg f(z) > \frac{1}{\rho^M}, \quad \text{or}
\]
4.2.5
\[
\frac{\partial}{\partial \rho} \arg f(z) < -\frac{1}{\rho^M}.
\]

If 4.2.4 holds, then for \( \rho < \rho_0 \) and \( z - a = \rho e^{i\phi_0} \)
\[
\int_{\gamma} \left[ \frac{\partial}{\partial \rho} \arg f(z) \right] d\rho > \int_{\rho}^{\rho_0} \frac{d\rho}{\rho^M}; \quad \text{which implies}
\]
that
\[
\arg f(\rho e^{i\phi_0} + a) - \arg f(\rho e^{i\phi_0} + a) > \frac{\rho_0^{-M+1}}{-M+1} + \frac{1}{M-1} \cdot \frac{1}{\rho^{M-1}}.
\]
Hence,
4.2.6 \[ |\arg f(z)| > \frac{1}{M-1} \frac{\rho_0}{|z-a|^{M-1}} - K, \]
where \( K \) is a constant larger than \( |\arg f(a + \rho_0 e^{i \phi_0})| \)
\[ + \frac{\rho_0^{-M+1}}{M-1}. \]

If 4.2.5 holds, then
\[ \left| -\frac{3}{2\rho} \arg f(z) \right| = \left| \frac{3}{2\rho} [-\arg f(z)] \right| > \frac{1}{\rho} M. \]

As in the above, this again implies that
\[ | -\arg f(z) | = | \arg f(z) | > \frac{1}{M-1} \cdot \frac{1}{|z-a|^{M-1}} - K. \]

If 4.2.3 holds, it is shown in a similar manner that

4.2.7 \[ |\log |f(z)|| > \frac{1}{M-1} \cdot \frac{1}{|z-a|^{M-1}} - K. \]

Now \( |\log f(z)| \geq |\log |f(z)|| \), and
\[ |\log f(z)| \geq |\oint \log f(z)| = |\arg f(z)|. \]
Since either 4.2.6 or 4.2.7 holds, given any arbitrary positive \( M \),
there exists a \( \delta > 0 \) such that
\[ |\log f(z)| > \frac{1}{M-1} \frac{1}{|z-a|^{M-1}} - K, \]
for \( \arg (z-a) = \phi_0 \) and \( |z-a| < \delta \). Thus for such \( z \)
such that \( \arg (z-a) = \phi_0 \),
\[ \lim_{z \to a} \frac{|\log |f(z)||}{-\log |z-a|} = +\infty. \]
Therefore, for $z$ in $G$,

\[ \lim_{{z \to a}} \frac{\log |\log f(z)|}{-\log |z - a|} = +\infty. \]
BIBLIOGRAPHY


