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SOME PROPERTIES OF A HARMONIC FUNCTION
IN THE NEIGHBORHOOD OF AN ISOLATED SINGULARITY

by

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A THESIS
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Approved
J. P. Whirl
TO MY WIFE
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1. **Introduction**

Let \( \overline{R} \) be a closed and bounded region of the plane, let \( P \) be an interior point of \( \overline{R} \). Let \( u_1(r, \theta) \) be a function single valued and harmonic in \( R = \overline{R} \setminus \{ P \} \) and having an isolated singularity at the point \( P \). For purposes of the work to follow there will be no loss in generality to take \( P \) as the origin \( (r = 0) \) and \( R \) as the region \( \{0 < r \leq 1, 0 \leq \theta \leq 2\pi\} \).

If the function \( u_1(r, \theta) \) is normalized in \( R \)- i.e. if we subtract from \( u_1(r, \theta) \) a function which is single valued and harmonic in \( \overline{R} \) and assumes on \( r = 1 \) the values \( u_1(1, \theta) \)- we obtain a function \( u(r, \theta) \) satisfying the conditions:

a. \( u(r, \theta) \) is single valued and harmonic in \( R \).

b. \( u(1, \theta) = 0 \) in \( \Theta \).

c. \( u(r, \theta) \) has an isolated singularity at \( r = 0 \).

Consider the sets of points of \( R \) defined as follows:

1. \( D_+ \) the set of points of \( R \) for which \( u(r, \theta) > 0 \).

2. \( D_- \) the set of points of \( R \) for which \( u(r, \theta) < 0 \).

3. \( D_0 \) the set of points of \( R \) for which \( u(r, \theta) = 0 \).

The problems to be considered in the following work are:

(1) the characterization of the point sets \( D_+, D_- \), and \( D_0 \).
and (2) the behavior of \( u(r, \theta) \) in these point sets in a neighborhood of \( r = 0 \).

We shall find that the point sets \( D_\phi \), \( D_\theta \) and \( D_0 \) can be resolved into components, i.e. connected subsets, and with at most one exception every component of \( D_\phi \) and \( D_\theta \) is a simply connected domain. Moreover, every component of \( D_\phi \) and \( D_\theta \) will have \( O(r = 0) \) as a boundary point.

Since \( u(r, \theta) \) has an isolated singularity at \( r = 0 \), we know that for \( (r, \theta) \) belonging to \( R_1 \):

\[
1. \quad u(r, \theta) = \sum_{n=0}^{\infty} r^n (c_n \cos n\theta d_n \sin n\theta) \log \frac{1}{r} + \sum_{n=1}^{\infty} \frac{a_n \cos n\theta b_n \sin n\theta}{r^n}
\]

The series \( \sum_{n=1}^{\infty} \frac{a_n \cos n\theta b_n \sin n\theta}{r^n} \) will be called the principal part of \( u(r, \theta) \). The principal parts of \( u_1(r, \theta) \) and the normalized function \( u(r, \theta) \) are identical.

If in some neighborhood of \( r = 0 \), \( u(r, \theta) \) is bounded above or bounded below then \( u(r, \theta) \) reduces to:

\[
u(r, \theta) = \sum_{n=0}^{\infty} r^n (c_n \cos n\theta d_n \sin n\theta) \log \frac{1}{r}\]
and in this case the circumference \( r = 1 \) constitutes the
point set \( D_0 \) and the remaining points of \( R \), i.e. the
interior of \( R \), is a component constituting \( D_0 \) or is a
component constituting \( D_\). Consideration is given to the intersection of the
point sets \( D_0 \), \( D_\) and \( D_\) with the point sets \( r = a \n(0 < a < 1) \). From this consideration we shall obtain
conclusions regarding the behavior of \( u(r, \theta) \) in \( R \) and
in turn conclusions regarding the point sets \( D_0 \), \( D_\), and
\( D_\).
Throughout this work the results have been obtained
without drawing from the properties of an analytic
function of a complex variable of which the harmonic
function under consideration is the real or imaginary
part. This view is adopted with the idea in mind of
using the results here obtained possibly to study the
behavior of an analytic function of a complex variable
in the neighborhood of an isolated essential singularity.
The character of the point set \( D_0 \) in a neighborhood
of a point for which the function is harmonic as well as
the character of the point sets \( D_0 \), \( D_\) and \( D_\) in a
neighborhood of the point $r = 0$ when the series expansion \( l_a \) has but a finite number of terms with negative powers of $r$ is well known. In obtaining the results of this paper, some of these known results may have been obtained again, possibly from somewhat different considerations, but the importance of the paper lies in the results obtained concerning the character of the point sets \( D_+ \), \( D_- \) and \( D_0 \) in a neighborhood of $r = 0$ when there are an infinity of terms in the expansion \( l_a \) with negative powers of $r$.

Last, but not least, the author wishes to express his deep appreciation to Professor Floyd E. Ulrich for his patience, guidance and suggestions in the inception, preparation and completion of this paper.

2. Preliminary Definitions

Defn. 2.1 By a simple arc we shall mean the set of points $(x, y)$ given by the functions $x = x(t)$, $y = y(t)$ continuous for $t$ belonging to the closed interval $(0, 1)$ and such that for $t_1$ and $t_2$ belonging to the open interval $(0, 1)$ $x(t_1) = x(t_2)$, $y(t_1) = y(t_2)$ if and only if $t_1 = t_2$.

Defn. 2.2 The points $(x(0), y(0))$ and $(x(1), y(1))$ are called the endpoints of the arc.
Defn. 2.3 If the arc is such that \( x(o) = x(1) \) and 
\( y(o) = y(1) \), then the arc is called a simple closed arc.

Defn. 2.4 If a finite set of arcs given by \( \{ x_k(t), y_k(t) \} \) \( k = 1 \) to \( n \) 
\( t \) belonging to the closed interval \( (0,1) \), can be so ordered 
that \( x_{k+1}(o) = x_k(1) \) and \( y_{k+1}(o) = y_k(1) \), for \( k = 1, 2, \ldots n-1 \), 
and these are the only points of intersection of these arcs, 
then we say that the set forms a chain of arcs or Jordan Curve.

Defn. 2.5 If the Jordan Curve as defined in Defn. 2.4 is 
such that \( x_n(1) = x_1(o) \) and \( y_n(1) = y_1(o) \) then the Jordan Curve is called a simple closed Jordan Curve.

Defn. 2.6 If a Jordan Curve is such that: for each \( k, \) 
\( k = 1, 2, \ldots n, \) and for each \( t \) in the closed interval 
\( (0,1) \), \( x_k(t) \) and \( y_k(t) \) are not only continuous but also 
have continuous derivatives satisfying \( x_k^2(t) + y_k^2(t) > 0 \) 
(where the dot denotes differentiation with respect to \( t \)) 
then we call each arc a smooth arc, the Jordan Curve a 
contour, and a simple closed Jordan Curve having these 
differentiability properties a simple closed contour.

Defn. 2.7 A set of points is connected if any two points 
of the set can be joined by a Jordan Curve \( J \) every point 
of which belongs to the set.
Defn. 2.8 By a domain we shall mean an open connected set of points.

Defn. 2.9 A finite domain is said to be simply connected if every simple closed Jordan Curve lying in the domain contains in its interior (in the sense of the Jordan Curve Theorem) only points of the domain.

Defn. 2.10 By a Region we shall mean a domain plus some, none, or all of its boundary points. In the context, the sense in which we use the term "region" will, in general, be indicated.

3. Expansion of $u(r,\theta)$

Consider the function $u_1(r,\theta)$ single valued and harmonic throughout the region $0 < r \leq 1$, $0 \leq \theta \leq 2\pi$. The origin ($r = 0$) is an isolated singularity of the function $u_1(r,\theta)$. Under these conditions we know that $u_1(r,\theta)$ has the expansion:

\[ u_1(r,\theta) = \sum_{n=0}^{\infty} r^n (\alpha_n \cos n\theta + \beta_n \sin n\theta) + \operatorname{Log} \frac{1}{r} + \sum_{n=1}^{\infty} \frac{\psi_n(\theta)}{r^n} \]

which converges uniformly for all $0 < \delta \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

$$\psi_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$$

$k$ is a constant.

For simplicity of writing we shall set:

$$v_1(r, \theta) = \sum_{nm \geq 0} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

Moreover, $v_1(r, \theta)$ shall be a function single valued and harmonic in $R + \{0\}$.

**Defn. 5.1** By the symbolism $M \{g(r, \theta)\}$ we shall understand the mean value of $g(r, \theta)$ on the circumference of the circle, center $0$, radius $r$, i.e.

$$M \{g(r, \theta)\} = \frac{1}{2\pi} \int_0^{2\pi} g(r, \theta) d\theta$$

5b. By hypothesis $u_1(r, \theta)$ is continuous (being harmonic in $0 < r \leq 1$) on the circumference $r = 1$. We construct the function $w(r, \theta)$ single valued and harmonic in $r \leq 1$ which takes the values $u_1(1, \theta)$ on $r = 1$. This function is given by the Poisson Integral:
\[ w(r, \theta) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{(1-r^2)u_1(1, \theta)d\theta}{1+r^2-2r\cos(\theta-\theta)} \quad 0 < r \leq 1 \]

Settings:
\[ u(r, \theta) = u_1(r, \theta) - w(r, \theta), \]
then letting
\[ v(r, \theta) = v_1(r, \theta) - w(r, \theta) \]
we have:

\[ u(r, \theta) = v(r, \theta) + k\log \frac{1}{r} + \sum_{n=1}^{\infty} \frac{\psi_n(\theta)}{r^n}. \]

where \( v(r, \theta) \) is harmonic throughout the unit circle.

The function \( u(r, \theta) \) has the properties:

a. it is harmonic and single valued in the region:
\[ 0 < r \leq 1, \quad 0 \leq \theta \leq 2\pi \]

b. it has an isolated singularity at \( r = 0 \).

c. \( u(1, \theta) = 0 \) for all \( 0 \leq \theta \leq 2\pi \).

**Defn. 3.2** When a function \( g(r, \theta) \) satisfies these three conditions a., b., c. we shall say that \( g(r, \theta) \) is **normal** in the region. This normalization is conducive to a certain amount of convenience in the work to follow. However, to realize this same convenience, we shall find that it is enough that the function be of constant sign on the
circumference of some circle with center at the singular point. In the sequel, when referring to the function (3d.) we shall understand it to be normal in the unit circle.

Theorem 3.1* If 
\[ u(r, \theta) = v(r, \theta) + k \log \frac{1}{r} + \sum_{n=1}^{\infty} \frac{\psi_n(\theta)}{r^n} \]

is bounded below (or bounded above) for all \( r \leq 1 \) then

\[ \psi_n(\theta) \equiv 0 \text{ for all } n; \text{ i.e.} \]

3e.

\[ u(r, \theta) = v(r, \theta) + k \log \frac{1}{r} \]

Proof: By hypothesis there is a number \( A \) such that

\[ u(r, \theta) - A > 0 \text{ for all } r \leq 1. \]

Suppose there exist \( a_m \) and \( b_m \) in the expansion of \( u(r, \theta) \) such that \( a_m^2 b_m^2 > 0 \), then

\[ M \{ [u(r, \theta) - A][1 + \cos \theta] \} > 0 \text{ for all } r \leq 1 \text{ and for all } \theta. \]

Therefore:

3f. \[ 0 < M \{ [u(r, \theta) - A][1 + \cos \theta] \} = B k \log \frac{1}{r} + \frac{a_m \cos \theta + b_m \sin \theta}{r^m} \]

\[ = B k \log \frac{1}{r} + \sqrt{a_m^2 b_m^2 \cos (\theta - \phi)} \]

*For a proof of this Theorem in 3-space see O. D. Kellogg "Foundations of Potential Theory" (abbreviated F.P.T. in the sequel) F. Ungar Publishing Co., New York, Page 270 Theorem 12.
where

\[ \phi_0 = \arctan \frac{b_m}{a_m} \]

and

\[ B = M \left\{ \left[ v(r, \theta) - A \right] \left[ i + \cos \theta - \phi_0 \right] \right\} \]

which is bounded for all \( r \leq 1 \).

The inequality \( \exists \phi \) must hold for all \( r \leq 1 \). However it is obvious that \( \phi \) may be chosen so that \( \cos \theta - \phi_0 < 0 \) and by choosing \( r \) sufficiently small the term \( \frac{a_m}{r^m} + \frac{b_m}{r^m} \cos \theta - \phi_0 \) will dominate the expression

\[ B = \frac{1}{r} + \frac{\sqrt{a_m^2 + b_m^2 \cos \theta}}{r^m} \]

since \( m \) is a positive integer. Hence for this choice of \( r \) and \( \phi \) the mean value in \( \exists \phi \) is less than zero. This contradiction assures us that the assumption that there is an integer \( m \) such that \( a_m^2 + b_m^2 > 0 \) must be false and hence \( a_m = b_m = 0 \) for every \( m \). In exactly the same manner it can be shown that if \( u(r, \theta) \) is bounded above for all \( r \leq 1 \) then \( \psi_n(\theta) = 0 \) for all \( n \). With these conclusions the theorem is established.

Since \( u(r, \theta) \) is continuous for \( 0 < r \leq 1 \) and \( 0 < \theta \leq 2\pi \),
if \( \psi_n(\theta) \neq 0 \) for some \( n \), then this theorem tells us that \( u(r, \theta) \) assumes all real values between \( (-\infty, +\infty) \).

The proof of Theorem 3.1 does not depend upon \( u(r, \theta) \) being normal in \( R \) and hence the conclusions hold for \( u_r(r, \theta) \).

4. The Point Sets \( D_\Phi, D_-, D_0 \) and Components of \( D_\Phi \) and \( D_- \).

**Defn. 4.1** Each point \((r, \theta)\) of the region \( R(0 < r \leq 1, 0 < \theta \leq 2\pi) \) is such that \( u(r, \theta) \) is greater than, less than or equal to zero. We shall denote by \( D_\Phi, D_- \) and \( D_0 \) the sets of points of \( R \) for which \( u(r, \theta) > 0 \), \( u(r, \theta) < 0 \) and \( u(r, \theta) = 0 \) respectively.

We note that \( D_\Phi \) and \( D_- \) are open sets, for given any point \((r, \theta)\) belonging to \( D_\Phi \), by the continuity of \( u(r, \theta) \) there exists a neighborhood of this point \((r, \theta)\) such that for all \((r', \theta')\) belonging to this neighborhood \( u(r', \theta') > 0 \). Therefore \((r, \theta)\) is an interior point of \( D_\Phi \) and hence \( D_\Phi \) is open. Similarly for \( D_- \).

**Defn. 4.2** Two points \( P_1 \) and \( P_2 \) belonging to a set \( S \) are said to be **connected in \( S \)** if there exists a connected subset of \( S \) containing \( P_1 \) and \( P_2 \).
**Defn. 4.3** The relation of "connectedness" between points is an equivalence relation for a). $P_1$ is connected to $P_1$ (reflexive), b). If $P_1$ is connected to $P_2$ then $P_2$ is connected to $P_1$ (symmetric). c). If $P_1$ is connected to $P_2$ and $P_2$ is connected to $P_3$ then $P_1$ is connected to $P_3$ (transitive), and this equivalence relation defines a division of a given set $S$ into mutually exclusive connected subsets. Each of these connected subsets is called a component of $S$. Essentially a component of a set $S$ containing a point $P$ of $S$ is the maximal connected set of $S$ containing $P$.

We note that: since $D_+$ and $D_-$ are open sets, then every component of $D_+$ or $D_-$ is a domain, and moreover each point of $D_+$ or $D_-$ belongs to some component.

**Lemma 4.1.** If there exists a component $U_+$ of $D_+$ or $U_-$ of $D_-$ such that this component contains a simple closed Jordan Curve $C$ with the point $r = 0$ in its interior, then there is no other component of $D_+$ or of $D_-$ having this property.

**Proof:** In order to prove this lemma it is enough to show that the annular region bounded by $r = 1$ and $C$, including $C$, is contained in a component $U_+$ of $D_+$ or a component $U_-$.
of $D_\phi$.

Let $C$, Fig. 1, be a simple closed Jordan Curve containing $O(r = 0)$ in its interior and suppose $C$ belongs to $U_\phi$.

Then for $(r, \theta)$ belonging to $C$, $u(r, \theta) > 0$ and on $r = 1$, $u(1, \theta) = 0$. By the principle of the maximum and minimum for a harmonic function we know that for $(r, \theta)$ belonging to the domain bounded by $r = 1$ and $C$, $u(r, \theta) > 0$ and therefore this domain plus $C$ belong to $U_\phi$. Obviously the corresponding fact holds if we assume that $C$ belongs to a component $U_\_\phi$ of $D_\_\phi$.

If we assume the existence of another component containing a simple closed Jordan Curve $J$ with the point $r = 0$ in its interior, then by the first part of the proof we know that the annular region bounded by $r = 1$ and $J$, including $J$, is contained in a component. If $C$ and $J$ belong to components $U_\phi$ and $U_\phi^1$ (components of $D_\phi$) respectively then by definition of the components

$$U_\phi = U_\phi^1.$$
Similarly if \( C \) and \( J \) belong to components \( U_- \) and \( U'_- \) respectively then

\[
U_- = U'_-
\]

Finally we note that the assumption that \( C \) belongs to a component \( U_\phi \) and \( J \) belongs to a component \( U_- \) implies that the common part of the annular regions bounded by \( r = 1 \) and \( C, r = 1 \) and \( J \), is at the same time a subset of a component of \( D_\phi \) and a component of \( D_- \) which is impossible.

Q.E.D.

Lemma 4.2  Each component of \( D_\phi \) or \( D_- \) which does not contain a simple closed Jordan Curve with the point \( r = 0 \) in its interior is simply connected.

Proof:  Let \( U_\phi \) be a component of \( D_\phi \) satisfying the condition that each simple closed Jordan Curve in \( U_\phi \) does not contain \( \text{O}(r = 0) \) in its interior.  Let \( J \) be a simple closed Jordan Curve in \( U_\phi \).  Then on \( J \) and throughout the interior of \( J \), \( u(r,\theta) \) is harmonic and accordingly assumes its maximum and minimum on \( J \).  But for \((r,\theta)\) belonging to \( J \), \( u(r,\theta) > 0 \), and therefore \( u(r,\theta) > 0 \) throughout the interior of \( J \).  Therefore \( U_\phi \) is, by definition, a simply connected domain.
Obviously, considering a component $U_-$ of $D_-$ we obtain
the same results. Q.E.D.

Theorem 4.1. Among all components of $D_+$ and $D_-$ together,
there exists at most one component that is not simply
connected, and this one exceptional component, if it exists,
is characterized by the fact that it contains a simple
closed Jordan Curve $O$ which has the point $r = 0$ in its
interior. Moreover, each simple closed Jordan Curve
contained in the exceptional component and containing the
point $r = 0$ in its interior, also contains in its interior
every remaining component of $D_+$ and $D_-$. Each component of
$D_+$ and $D_-$, without exception, has the origin $O$ as a boundary point.

Proof: Lemma 4.1 and 4.2 give immediately the first two
parts of the theorem. We need show that every component
of $D_+$ and $D_-$ has the origin as a boundary point.

Suppose a component $U_+$ of $D_+$ does not have the origin
as a boundary point, then $U_+$ is a domain lying in the interior
of $R$ and obviously at each boundary point $u(r, \theta)$ is zero or
negative. But this is impossible since $u(r, \theta)$, harmonic in
$U_+$ and zero or negative on the boundary must be zero or
negative throughout the domain $U_+$; i.e. $U_+$ must then be a component of $D_-$. This contradiction assures us that every component of $D_+$ or $D_-$ has the origin as a boundary point. Q.E.D.

Corollary 4.11. If $u(r, \theta)$ is bounded above (or below) for all $r \neq 1$, then the interior of $R$ is a component, and in fact is $D_+$ if $k > 0$ and $D_-$ if $k < 0$, where $k$ is the coefficient of the logarithmic term in the expansion of $u(r, \theta)$.

Proof: By Theorem 3.1

$$u(r, \theta) = v(r, \theta) + k \log \frac{1}{r}$$

Since $v(r, \theta)$ is bounded in $R \setminus \{0\}$, then there exists $r_0 < 1$ such that for all $r \neq r_0$, $u(r, \theta)$ has the same sign as $k \log \frac{1}{r}$, i.e. if $k > 0$ then $u(r, \theta) > 0$ and if $k < 0$ then $u(r, \theta) < 0$. Take any circle, center 0, and radius $r \neq r_0$, then with this circle as the simple closed Jordan Curve $C$ of lemma 4.1 the conditions of this lemma are fulfilled and hence the region bounded by $r = 1$ and $C$, including $C$ belong to a component and hence we conclude the assertions of the corollary.
Corollary 4.12. If in the expansion of \( u(r, \theta) \), \( k = 0 \) then every component of \( D_\phi \) and \( D_- \) is a simply connected domain.

Proof: In order to prove this corollary it is obviously enough to show that there cannot be a component of \( D_\phi \) or \( D_- \) containing a simple closed Jordan Curve which contains in its interior the point \( r = 0 \).

Since \( k = 0 \), by hypothesis, we can write:

\[
    u(r, \theta) = \sum_{n=0}^{\infty} r^n (c_n \cos n \theta + d_n \sin n \theta) + \sum_{n=1}^{\infty} \frac{\psi_n(\theta)}{r^n}
\]

First it is necessary to show that \( c_0 = 0 \). This follows immediately from the fact that for \( r = 1 \), \( u(1, \theta) = 0 \) and hence

\[
    0 = M \{ u(1, \theta) \} = c_0
\]

Now suppose there exist a component \( U_\phi \) of \( D_\phi \) containing a simple closed Jordan Curve \( C \) which contains in its interior the point \( r = 0 \). By Lemma 4.1, the domain bounded by \( C \) and \( r = 1 \) is a subset of \( U_\phi \). Let \( r_0 \) be the least upper bound of the distances from \( C \) to \( r = 0 \) then
in the annulus \( r_0 \leq r < l \), \( u(r, \theta) > 0 \), and for all \( \theta \) and \( \phi \)

\[
u(r, \theta)(1 + \cos(\theta - \phi)) > 0
\]

Hence:

\[
4a. \quad \alpha < \max \{ u(r, \theta)(1 + \cos(\theta - \phi)) \} = \frac{r^m (c_m \cos \phi + d_m \sin \phi) + \Psi_m(\phi)}{r^m}
\]

\[
= (c_m \frac{r^m + \frac{a_m}{r^m}}{r^m}) \cos \phi + (d_m \frac{r^m + \frac{b_m}{r^m}}{r^m}) \sin \phi
\]

\[
= \sqrt{(c_m \frac{r^m + \frac{a_m}{r^m}}{r^m})^2 + (d_m \frac{r^m + \frac{b_m}{r^m}}{r^m})^2} \cos(\phi - \phi_r)
\]

where

\[
m \phi_r = \arctan \frac{d_m \frac{r^m + \frac{a_m}{r^m}}{r^m}}{c_m \frac{r^m + \frac{b_m}{r^m}}{r^m}}
\]

From the right hand member of 4a, it is apparent that an \( r_1 \) can be found in the interval \( r_0 \leq r_1 < l \) such that

\[
c_m r_1^m + \frac{a_m}{r_1^m} \quad \text{and} \quad d_m r_1^m + \frac{b_m}{r_1^m}
\]

are not both zero. With this choice of \( r_1 \) and for a suitably chosen \( \phi \) the mean value would be negative which is a contradiction. If each

\[
\Psi_m(\theta) \equiv 0,
\]

under the present hypotheses \( u(r, \theta) \) is harmonic
at \( r = 0 \) and since it is normal in \( R \) then the corollary is
trivially true; i.e. the only case of any interest is that
in which \( \psi_n(\theta) \neq 0 \) for some \( n \). Q.E.D.

We note in the case of the Laurent expansion of a
function \( f(z) \) analytic in a deleted neighborhood of a
point \( z = a \) and this point is an isolated singularity of
the function \( f(z) \) that the real or imaginary parts of \( f(z) \)
do not contain the logarithmic term. Accordingly, when
these real or imaginary parts are normalized in some region
about the singular point, then each component of the real
or imaginary parts of \( f(z) \) in the region of normalization
will be a simply connected domain.

The essential features of the above conclusions
continue to be valid if the circular region \( \overline{R} \) is replaced
by a simply connected domain \( D \), the singularity an interior
point \( P \) of \( D \), and the function is not normalized in \( D \). The
facts in this more general situation are indicated in:
Theorem 4.2 Let \( f(r, \theta) \) be any function single valued and
harmonic throughout a simply connected domain \( D \) except for
one point \( P \) belonging to \( D \) which is an isolated singularity
of \( f(r, \theta) \). If in \( D \) we define the point sets \( D_+ \), \( D_- \), \( D_0 \) and
the components of \( f(r, \theta) \) as before, then there exists at most one component that is multiply-connected in \( D \) and has the point \( P \) as a boundary point. Moreover, this multiply-connected component contains a simple closed Jordan Curve \( J \) which contains in its interior the point \( P \) and all components of \( D_+ \) and \( D_- \) having \( P \) as a boundary point.

Proof: Suppose \( U_+ \), a component of \( D_+ \), is multiply connected and has \( P \) as a boundary point. Suppose \( U_+ \) does not contain a simple closed Jordan Curve \( C \) having the point \( P \) in its interior. By the same argument as used in Lemma 4.2, it follows that each simple closed Jordan Curve \( K \) belonging to \( U_+ \) contains in its interior only points of \( U_+ \), i.e., \( U_+ \) must then be simply connected in \( D \). By hypothesis \( U_+ \) is multiply connected and hence \( U_+ \) contains a simple closed Jordan Curve \( J \) which has the point \( P \) in its interior.

If we assume the existence of a component, \( U \), distinct from \( U_+ \), interior to \( J \), and satisfying the same conditions as \( U_+ \), then \( U \) must have points in common with \( U_+ \) contrary to the fact that the components are mutually exclusive. That these two components would have to have points in
common is shown by the following: \( U \) is an open connected set having \( P \) as a boundary point. Therefore, we can find a Jordan Curve \( I \) joining a point \( Q \) of \( J \) with the point \( P \) and every point of \( I \), except \( P \), belongs to \( U \). By the first part of the proof we know that \( U \) contains a simple closed Jordan Curve \( I' \) having the point \( P \) in its interior, but \( Q \) being a point exterior to \( I' \) and \( P \) a point interior to \( I' \) then \( I \) must intersect \( I' \) in at least one point \( T \) and this point is then common to \( U \) and \( U \).

By exactly the same argument we can show that there is no component \( U \) having points exterior to \( J \) and satisfying the same conditions as \( U \). Finally: from this last result it follows that every component having \( P \) as a boundary point must be interior to \( J \). Q.E.D.

5. General Properties of \( D_0 \).

5.1 If \( u(r,\theta) \neq 0 \) throughout the region for which it is harmonic, then \( D_0 \) cannot contain a domain (two dimensional) of points.*

*See Kellogg, O.D. F.P.T. Page 259.
5.2 $D_0$ cannot possess an isolated point $(r_0, \theta_0)$ for if so then there exists a neighborhood $N$ of $(r_0, \theta_0)$ for which there are no other points $(r, \theta)$ belonging to this neighborhood such that $u(r, \theta) = 0$. Assuming the existence of such a neighborhood $N$ then either: (a). $u(r, \theta)$ is of constant sign in $N$ in which case if we take a circle center $(r_0, \theta_0)$ lying entirely in $N$, then $u(r_0, \theta_0)$ is the mean value of $u(r, \theta)$ for points on the circumference of this circle which implies $u(r, \theta) = 0$ in the circle contrary to hypothesis that $(r_0, \theta_0)$ is an isolated point of $D_0$. or (b). $u(r, \theta)$ is positive or negative for all points of $N$ except $(r_0, \theta_0)$. But if so take a circle $C_1$ center $(r_0, \theta_0)$ lying entirely in $N$. This circle must contain in its interior two points $(r_1, \theta_1)$ and $(r_2, \theta_2)$ such that $u(r_1, \theta_1) > 0$ and $u(r_2, \theta_2) < 0$. Take a second circle $C_2$ center $(r_0, \theta_0)$ interior to $C_1$ and such that the interior of the annulus thus formed contains $(r_1, \theta_1)$ and $(r_2, \theta_2)$. In this annulus which is closed and bounded $u(r, \theta)$ is continuous and therefore assumes all values intermediate to $u(r_1, \theta_1)$ and $u(r_2, \theta_2)$ and therefore assumes the value zero somewhere in the annulus contrary to the assumption that except for $(r_0, \theta_0)$ $u(r, \theta)$ assumes the value
zero nowhere in the neighborhood \( N \). Thus \( D_0 \) has no isolated points.

**Defn. 7.1** A set \( S \) is said to be **compact** if every sequence of points in \( S \) contains a subsequence that converges to a point in \( S \).

5.3 Given any positive quantity \( \epsilon < 1 \), the intersection of the set \( D_0 \) with the annulus \( A_0 < \mathbb{R} < r \) is a compact set, i.e., \( D_0 \) is a compact set except for a sequence of points of \( D_0 \) converging to \( r = 0 \). In order to show this it is enough to show that the set \( D_0 \cap A \) (the intersection of \( D_0 \) and \( A \)) is closed (for a closed and bounded set is compact). First we note that \( A \) is closed and therefore contains all its limit points. It remains to show that if \( (r_0, \theta_0) \) is a limit point of points \( \{(r_i, \theta_i)\}_{i=1}^{\infty} \) belonging to \( D_0 \cap A \) then \( (r_0, \theta_0) \) belongs to \( D_0 \). But every point of the set \( A \) belongs to either \( D_+ \), \( D_- \) or \( D_0 \). \( (r_0, \theta_0) \) cannot belong to \( D_+ \) or \( D_- \) for if so then \( (r_0, \theta_0) \) is an interior point of one of the two sets \( D_+ \) or \( D_- \) (being open sets) which would make \( (r_0, \theta_0) \) a point such that in some neighborhood of \( (r_0, \theta_0) \) there were no points of \( D_0 \) contrary to hypothesis that \( (r_0, \theta_0) \) is a limit.
point of a sequence of points in \( D_0 \). Therefore \((r_0, \theta_0)\)
belong to \( D_0 \) and hence \( D_0 \cdot A \) is closed and bounded and
therefore compact.

5.4 Defn. 5.2 A set \( S_1 \) is dense in a set \( S_2 \) if every
open subset of \( S_2 \) contains points of \( S_1 \). Obviously, if
\( u(r, \theta) \neq 0 \) in \( R \), then \( D_0 \) is not dense-in-\( R \).

5.5 If \( \psi_n(\theta) \neq 0 \) for some \( n \), then by Theorem 3.1 \( D_0 \) is
not empty in the interior of \( R \).

6. The Boundary of a Simply Connected Component
of \( D_\circ \) or \( D_\circ^* \).

Now consider a simply connected component \( U_\circ \) of \( D_\circ \).
It will be obvious that what we assert regarding the
boundary of \( U_\circ \) can also be asserted regarding the boundary
of a component \( U_\circ^- \) of \( D_\circ^- \).

Defn. 6.1 Let the set of points \( C = C' \cup C'' \) be defined as
follows:

a. \( C' \) consists of all boundary points of \( U_\circ \) except
the origin \( 0 \).

b. If a point \( P \) belonging to \( C' \) has a number (one
or more) of arcs belonging to \( D_0 \) and passing through \( P \)
whether they be points of the boundary of \( U_\circ \) or not, then
all the points belonging to these arcs will constitute the set C^a. (We shall find that C' is a subset of C^a.)

6.1 Every point of C must belong to D_0 since
   a. every point of R belongs either to D_+ D_- or D_0.
   b. D_+ and D_- are open sets and hence no points of these two sets can be boundary points.

6.2 Defn. 6.2 If P is a point of C' such that u_0^2 + u_o^2 > 0 at this point, then we shall call P an ordinary point of C'.

Defn. 6.3 If Q is a point of C' then with Q as the pole of the polar coordinate system (r, φ) we can represent u by a series of the form:

\[ a_0' + \sum_{n=1}^{\infty} r^n (a_n' \cos n\phi + b_n' \sin n\phi) \]

and a_0' = 0 since u is zero at Q. If at the point Q, a_k' = b_k' = 0 for k = 1, 2, ..., m-1 (m > 1) and a_m' + b_m' > 0, then we shall call Q an exceptional point of order m.

It is a well known fact that D_0 consists entirely of analytic arcs.* In particular, we can say of the sets C' and C^a:

---

a. In a neighborhood of a boundary point $P$ of $U_*$ which is an ordinary point, $C'$ consists of a single analytic arc passing through the point $P$.

b. In a neighborhood of a boundary point $Q$ of $U_*$ which is an exceptional point of order $m$, $C''$ consists of $m$ analytic arcs passing through the point $Q$. Each of these arcs has a tangent at the point $Q$ and any two consecutive tangents form an angle $\pi/m$.

c. Since each point of $C'$ is an ordinary point or an exceptional point then in any subregion $R\{0 < \theta < 2\pi\}$ of $R$, $C'$ is a contour.

Defn. 6.4 Two sets $S_1$ and $S_2$ form a partition of a set $S$ if $S_1$ and $S_2$ are closed in $S$, non-null, disjoint, and $S_1 \cup S_2 = S$.

A theorem from Topology states that: The boundary of a bounded simply connected domain in the plane does not admit of a partition. If to the set $D_0$ we add the origin $O$, then each simply connected component of $D_0$ and $D_-$ has a boundary.

that does not admit of a partition. This tells us, for example, that \( C^1 \# 0 \) cannot consist of a set of disjoint arcs or contours.

**Defn. 6.5** A Jordan Curve \( J \) is called an **end-cut** of a domain \( D \) if \( J \) has an end point on the boundary of \( D \) and for each point \( P \) of \( J \), except this boundary point, there exists a neighborhood of \( P \) and every point of this neighborhood is either a point of \( D \) or a point of \( J \).

**Defn. 6.6** A Jordan Curve \( J \) is called a **cross-cut** of a domain \( D \) if \( J \) joins two points of the boundary of \( D \) and except for these two boundary points for each point \( P \) of \( J \) there exists a neighborhood of \( P \) such that every point of this neighborhood is either a point of \( D \) or a point of \( J \).

If \( u(r,\theta) \neq 0 \) in \( R \), then by the mean value principle for harmonic functions we know that: if a point \( P \) belongs to \( C^1 \) then in every neighborhood of \( P \) there are points of \( D_+ \) (in particular \( U_+ \)) and points of \( D_- \). From this we conclude that \( C^1 \) can have no end-cuts, cross-cuts, or "linear combinations" of these cuts in \( U_+ \).

Every point of \( C^1 \) is either an ordinary point or an exceptional point. In a neighborhood of an ordinary point
P, C' consists of two analytic arcs abutting on the point P and these two arcs are parts of one and the same analytic arc. In a neighborhood of an exceptional point Q, C consists of p (the order of the exceptional point) analytic arcs passing through the point Q and parts of two of these arcs abutting on Q constitute the set of points of C' belonging to this neighborhood. From these considerations we conclude that O(r = 0) is the only point of the interior of R plus \{ 0 \} at which C' can terminate.

If to the set C' we add \{ 0 \}, from the above considerations we conclude that this set is a simple closed Jordan Curve and C', from exceptional point to exceptional point, is an analytic arc.

7. The Level Curves of \( u(r, \theta) \)

Suppose now we choose any real number a, then consider \( u(r, \theta) - a \). We know that this function satisfies the following:

1. It is harmonic in R.
2. It has an isolated singularity at \( r = 0 \).
3. It is of constant sign on \( r = 1 \) and in fact is \( -a \) on this circumference.
Therefore the properties which we have attributed to $u(r, \theta)$ can equally well be attributed to this function $u(r, \theta) - a$. In particular we may say that the curves $D_a$, i.e. the points of $R$ for which $u(r, \theta) = a$ are contours and if to $D_a$ we add $\{0\}$ we have that the set $D_a + \{0\}$, except $r = 1$, consists of simple closed Jordan Curves joining at $O$. The components (curves) constituting the set $D_a$ are called Level Curves.

6. The Sign Changes of $u(r, \theta)$ and The Components of $D_+$ and $D_-$

In this section we shall seek information regarding the number and nature of the components of $D_+$ and $D_-$ in the neighborhood of $r = o$ when certain conditions are imposed upon $u(r, \theta)$. In particular we shall be interested in the case in which there is a positive integer $n_0$ such that in the expansion of $u(r, \theta)$, $\psi_{n_0}(\theta) \neq 0$. Under this assumption we know that $u(r, \theta)$ is neither bounded above nor below and therefore there are at least two components one belonging to $D_+$ and one belonging to $D_-$ and each of these components has the origin as a boundary point. Therefore there exists a circle, center $O$, radius $r_0 < 1$, such
that on the circumference of this circle there are intervals for which \( u(r, \theta) > 0 \) and intervals for which \( u(r, \theta) < 0 \), and moreover for every \( r \) such that \( 0 < r \neq r_0 \) similar intervals must exist.

**Defn. 8.1** Suppose on the circumference of the circle \( r = a < r \), there is a \( \theta \) such that in any interval \( \theta_1 - \epsilon < \theta < \theta_1 + \epsilon, \epsilon > 0 \), there are points \((a, \theta)\) for which \( u(a, \theta) > 0 \) and points for which \( u(a, \theta) < 0 \). We then say that \( u(a, \theta) \) has a sign change (or changes sign) at the point \((a, \theta_1)\). By the continuity of \( u(a, \theta) \) we know then that \( u(a, \theta_1) = 0 \). Of course there may be points of the circumference \( r = a \), for which \( u(a, \theta) = 0 \), but \( u(a, \theta) \) does not change sign at these points.

We note that on any circumference \( r = a \) \((0 < a < 1)\) there are at most a finite number of points for which \( u(a, \theta) \) changes sign. If there were infinitely many such points \((a, \theta_i)\), \( i = 1, 2, \ldots \), then on the closed bounded set \( r = a \), this sequence of points would have a limit point \((a, \theta_0)\) belonging to \( D_0 \). \((D_0 \) is compact in the set \( r = a \)) and in every neighborhood of this limit point there would have to be either:
(1) infinitely many arcs belonging to \( D_0 \), or

(2) a single* analytic arc of \( D_0 \) which would oscillate in a neighborhood of the point \((a, \theta_0)\) and intersect the analytic arc \( r = a \) in infinitely many points.

Case (1) is impossible since in some neighborhood of \((a, \theta_0)\) there is only one analytic arc of \( D_0 \) if \((a, \theta_0)\) is an ordinary point, and at most \( p \) arcs of \( D_0 \) if \((a, \theta_0)\) is an exceptional point of order \( p \). In order to show that case (2) is impossible, we assume the existence of an arc of \( D_0 \) that has infinitely many oscillations in a neighborhood of \((a, \theta_0)\). We know that this arc is given by \( u(r, \theta) = 0 \).

Moreover, the point \((a, \theta_0)\) is either an ordinary point or an exceptional point and in either case \( u(r, \theta) = 0 \) is an analytic arc throughout the neighborhood of \((a, \theta_0)\)** and can be written in the form:

\[
(8a') \quad r = a + a_1(\theta - \theta_0) + a_2(\theta - \theta_0)^2 + \ldots
\]

---

*In any closed subset of \( \mathbb{R} \) there are at most a finite number of exceptional points of \( D_0 \) and therefore the assertion "A single analytic arc..." Regarding the number of exceptional points of \( D_0 \) see e.g. Walsh, J.L. "Location of Critical Points" Page 270.

**Cf. Walsh, J.L. loc. cit. Page 269.
This function is continuous throughout a neighborhood of 
\( \theta = \theta_0 \) and it is presumed to have infinitely many zeros, 
\( \theta_i, \ i = 1, 2, \ldots \) in this neighborhood. Suppose \( \theta_i, \ \theta_i \neq \theta_0 \), 
is any one such zero, then writing (5a'):

\[
(r-a) = (\theta-\theta_0) \left[ a_1 \theta a_2 (\theta-\theta_0) + \ldots \right] = (\theta-\theta_0) \phi(\theta)
\]
we have:

\[
c = (\theta_1 - \theta_0) \phi(\theta_1)
\]
but \( \theta_1 - \theta_0 \neq 0 \), hence we conclude that \( \phi(\theta_1) = 0 \). Since
\( \theta_1 \neq \theta_0 \) was presumed to be any zero of (5a') in the neighbor-
hood of \( (a, \theta_0) \) and moreover \( \phi(\theta) \) is a continuous function
of \( \theta \) we conclude, in particular, that \( \phi(\theta_0) = 0 \), i.e. \( a_1 = 0 \).
Proceeding in this manner we have that, in (5a'), \( a_k = 0 \)
for all positive integers \( k \), i.e. in the neighborhood of
\( (a, \theta_0) \) \( u(r, \theta) \equiv r-a = 0 \). In fact these considerations
tell more: if a portion of an analytic arc \( u(r, \theta) = 0 \)
coincides with a part of the circumference \( r = a \), then
these two arcs coincide for all values \( (r, \theta) \), i.e.

\[
u(r, \theta) \equiv r-a = 0 \text{ for all values } \theta.
\]

From the above considerations we conclude the following:
a. If \( u(r, \theta) \) is normal in \( R \) and \( \psi_{\theta_0}(\theta) \neq 0 \), then for all \( a, 0 < a < 1 \), the points of the circumference \( r = a \) for which \( u(a, \theta) = 0 \) are isolated points and hence there are at most a finite number of such points on \( r = a \).

b. If \( u_1(r, \theta) \) is not normal in \( R \) and \( \psi_{\theta_0}(\theta) \neq 0 \), then there exists at most one number \( a, 0 < a < 1 \), such that for \( r = a \), \( u_1(a, \theta) = 0 \) for all \( \theta \). In this case \( u_1(r, \theta) \) is then normal in the region \( R' \{ 0 < r < a, 0 \leq \theta \leq 2\pi \} \). For each \( r, 0 < r < a \), we can then assert that the points of \( D_0 \) are isolated points on this circumference and there are at most a finite number of points belonging to \( D_0 \).

Suppose on \( r = a \), \( u(a, \theta) \) has sign changes at the points \( \{a, \theta_i\}_{i=1}^{n}, 0 \leq \theta_1 < \theta_2 < \cdots < \theta_n \leq 2\pi, \theta_n - \theta_1 < 2\pi \) and these are the only points for which \( u(a, \theta) \) changes sign. By the continuity of \( u(a, \theta) \), we know that in adjacent intervals \( (\theta_{i-1}, \theta_i) \) and \( (\theta_i, \theta_{i+1}) \) \( u(a, \theta) \) has opposite signs.

From these considerations we conclude that the circumference \( r = a \) can be divided into an even number of intervals, in half these intervals \( u(a, \theta) > 0 \) and in the other half \( u(a, \theta) < 0 \); moreover, if \( u(a, \theta) > 0 \) in one interval then \( u(a, \theta) < 0 \) in the adjoining intervals. If \( u(a, \theta) > 0 \) in a
given interval, then there are points of this interval for which \( u(a, \theta) > 0 \), and similarly in intervals for which \( u(a, \theta) \leq 0 \). If there are \( 2p \) such intervals then associated with these intervals are \( 2p \) points at which \( u(a, \theta) \) changes sign. In this case we say that \( u(a, \theta) \) has \( 2p \) sign changes on the circumference \( r = a \).

With these conventions we proceed to:

Lemma 8.1. Given the set \( \{ \theta_i \}_{i=1}^{2n} \),

\[
0 \leq \theta_1 < \theta_2 < \ldots < \theta_{2n} \leq 2\pi, \quad \theta_{2n} - \theta_1 < 2\pi,
\]

there exists a trigonometric polynomial

\[
T_n(\theta) = A \sum_{p=0}^{n} \left( a_p \cos p\theta + b_p \sin p\theta \right)
\]

which is zero and changes sign at each of the points \( \theta_i \). Moreover: \( |a_p|, |b_p| \leq 2 \) for \( p = 0, 1, \ldots, n \), and

\[
\sqrt{a_n^2 + b_n^2} = \frac{1}{2^{n-1}}
\]

Proof: we write:
(8a.) \( T_n(\theta) = A \prod_{i=1}^{2n} \sin \frac{\theta - \theta_i}{2} \)

where \( A \) is arbitrary and accordingly may be assigned in such a way that the signs of \( T_n(\theta) \) alternate in a desired manner. That \( T_n(\theta) \) has the desired zeros (and only these zeros) and that there are \( 2n \) sign changes is obvious from its representation in (8a). That \( T_n(\theta) \) can be written as a trigonometric polynomial of order \( n \) is apparent once we expand the factors of the product, taking them in the order:

(8b.) \( \sin(\theta - \theta_{2p-1})\sin(\theta - \theta_{2p}) = \frac{\cos \left( \frac{\theta_{2p} - \theta_{2p-1}}{2} \right) - \cos \left( \frac{\theta_{2p} + \theta_{2p-1}}{2} \right)}{2} \)

\[ = A_{2p} \cos \theta + B_{2p} \sin \theta \]

There will be \( n \) factors of the form (8b.) and these, when multiplied together and transformed by trigonometric identities, give us the polynomial

(8c.) \( T_n(\theta) = A \sum_{p=0}^{n} (a_p \cos \theta + b_p \sin \theta) \)
This completes the construction of the desired polynomial.

That $T_n(\theta)$ has the additional properties stated in the lemma, we show as follows:

\[ \sum_{i=1}^{2n} \sin^2(\theta - \theta_i) = a_0' + a_1' \cos \theta + b_1' \sin \theta + \ldots + a_n' \cos^n \theta + b_n' \sin^n \theta \]

and

\[ a_p' = \frac{1}{\pi} \int_0^{2\pi} \cos^p \theta \, d\theta = \frac{1}{\pi} \int_0^{2\pi} \left[ \sum_{i=1}^{2n} \sin(\theta - \theta_i) \right] \cos^p \theta \, d\theta \]

or:

\[ |a_p'| \leq \frac{1}{\pi} \int_0^{2\pi} \left| \sum_{i=1}^{2n} \sin(\theta - \theta_i) \right| |\cos^p \theta| \, d\theta \leq 2 \]

Similarly for $|b_p'|$.

\[ a_0' = \frac{1}{2\pi} \int_0^{2\pi} \left[ \sum_{i=1}^{2n} \sin(\theta - \theta_i) \right] \, d\theta \quad \text{so that} \quad |a_0'| \leq 1. \]
If we choose \(|A| \leq \frac{1}{2}\), we can then assert that every coefficient in the polynomial (8c.) is at most 1 in absolute value.

8.2 If \(T(\theta)\) is precisely of order \(n\), i.e. \(a_n^2 + b_n^2 > 0\), then:

\[
a_n' = \frac{(-1)^n \cos(\sum_{i=1}^{2n} \theta_i)}{2^{2n-1}}, \quad b_n' = \frac{(-1)^n \sin(\sum_{i=1}^{2n} \theta_i)}{2^{2n-1}}
\]

and

\[
\sqrt{a_n'^2 + b_n'^2} = \frac{1}{2^{2n-1}}
\]

To obtain this we have from (8b.):

\[
a_n' = \frac{1}{\pi} \int_0^{2\pi} \left[ \prod_{p=1}^{n} \cos \left( \frac{\theta - 2p - 1}{2} \right) - \cos \left( \frac{\theta + 2p - 1}{2} \right) \right] \cos n \theta d\theta
\]

By the orthogonality of the trigonometric functions we obtain:

\[
a_n' = \frac{1}{\pi} \int_0^{2\pi} \left( \prod_{p=1}^{n} \frac{\cos(\theta - \frac{2p + \theta - 2p - 1}{2})}{2^n} \right) \cos n \theta d\theta
\]

If we let \(s_p = \frac{\theta - 2p - 1}{2}\), and set \(\cos(\theta - s_p) = \frac{e^{i(\theta - s_p)} + e^{-i(\theta - s_p)}}{2}\)
\[ a_n^i = \frac{(-1)^n}{\pi \cdot 2^{2n+1}} \int_0^{2\pi} (e^{i(\theta-S_1)} + e^{-i(\theta-S_1)}) \ldots \]

\[ (e^{i(\theta-S_n)} + e^{-i(\theta-S_n)})(e^{i\theta} + e^{-i\theta}) \, d\theta \]

In view of the fact that for \( k \) an integer

\[ \int_0^{2\pi} e^{ik\theta} \, d\theta = \begin{cases} 0 & \text{if } k \neq 0 \\ 2\pi & \text{if } k = 0 \end{cases} \]

we then have:

\[ a_n^i = \frac{(-1)^n}{\pi \cdot 2^{2n+1}} \int_0^{2\pi} (\sum_{p=1}^{n} S_p + e^{-i\sum_{p=1}^{n} S_p}) \, d\theta \]

\[ = \frac{(-1)^n}{2^{2n-1}} \cos(\sum_{p=1}^{n} S_p) = \frac{(-1)^n}{2^{2n-1}} \cos(\frac{2\pi}{2^{2n-1}} \sum_{i=1}^{2n} \theta_i) \]

In exactly the same manner we obtain:

\[ b_n^i = \frac{(-1)^n}{2^{2n-1}} \sin(\frac{2\pi}{2^{2n-1}} \sum_{i=1}^{2n} \theta_i) \]
So that

\[ \sqrt{\frac{2}{n} \cdot \frac{n}{2} \cdot \frac{2}{n}} = \frac{1}{2^{2n-1}} \]

8.3 If \(|A| \leq \frac{1}{2}\) then \(|T_n(\theta)| \leq \frac{1}{2}\).

This follows immediately from:

\[ |T_n(\theta)| = |A| \frac{2n}{n} \prod_{i=1}^{2n} \sin^2(\theta - \theta_i) \leq \frac{1}{2} \]

Theorem 8.1 If \(u(r, \theta) = v(r, \theta) + k \log \frac{1}{r} + \sum_{n=1}^{\infty} \frac{\psi_n(\theta)}{r^n}\) is such that the number of its sign changes is bounded above by \(2N\) for all \(r (0 < r \leq 1)\), then \(\psi_n(\theta) = 0\) for all \(n > N\).

Proof: Suppose \(\psi_g(\theta) \neq 0\), i.e. \(a^2 + b^2 > 0\) and \(g > N\). Suppose moreover that on a circle of radius \(r < 1\), \(u(r, \theta)\) has \(2N_r\) sign changes. We then construct the trigonometric polynomial

\[ T_N^r(\theta) = A_N^r \sum_{p=0}^{N_r} (a_p^r \cos \theta + b_p^r \sin \theta) \]

in which \(N_r \in N\) and \(T_N^r(\theta)\) has, on the circle of radius \(r\), exactly the same sign changes as \(u(r, \theta)\). For each \(r\) and the corresponding \(T_N^r(\theta)\) we now choose \(|A_N^r| = \frac{1}{2}\). The sign of \(A_N^r\) will then be chosen in such a way that, for
the particular \( r \) in question:

\[
T_{N_r}(\theta) u(r, \theta) \geq 0 \quad \text{for all } \theta.
\]

Take \( \phi \) arbitrary and form the product

\[
u(r, \theta)T_{N_r}(\theta) \left[ 1 + \cos(m_r \theta - \phi) \right]
\]

in which \( m_r \in \mathbb{N} \). This product is non-negative for all \( \phi \) and moreover:

\[(\text{d}) \quad 0 < M \left\{ u(r, \theta)T_{N_r}(\theta) \left[ 1 + \cos(m_r \phi) \right] \right\} = \]

\[
M \left\{ v(r, \theta)T_{N_r}(\theta) \left[ 1 + \cos(m_r \phi) \right] \right\} + \int_{N_r} a_i^i \, k \log \frac{1}{r} \, + \]

\[
A_N \sum_{n=1}^{N_r} \left\{ a_i^i \right\} \frac{a_n^i a_n^i + b_i^i b_i^i}{2^n} \int_{N_r} \left\{ T_{N_r}(\theta) \cos(m_r \theta - \phi) \sum_{n=1}^{\infty} \frac{\psi_n(\theta)}{r^n} \right\}
\]

Consider the following terms of the last integral:

\[
T_{N_r}(\theta) \cos(m_r \theta - \phi) = A_N \sum_{n=0}^{N_r} \left\{ (a_i^i \cos \phi + b_i^i \sin \phi) \right\}
\]

\[(\cos m_r \theta \cos \phi + \sin m_r \theta \sin \phi) \}

Let \( c_\phi = \cos \phi \) and \( s_\phi = \sin \phi \), then expanding we obtain:

\[(\text{e}) \quad A_N \sum_{n=0}^{N_r} \frac{a_i^i \cos \phi + b_i^i \sin \phi}{2} \cos(m_r \theta + \phi) \right\} + \frac{a_i^i \cos \phi + b_i^i \sin \phi}{2} \cos(m_r \theta - \phi) \]
\[ \frac{a^r b^r c}{pra} \sin(m_r p)\theta + \frac{a^r b^r c}{pr_b} \sin(m_r p)\theta \}

In (8e.), let:

\[ A_p' = \frac{a^r b^r c}{pra}, \quad B_p' = \frac{a^r b^r c}{pr_b} \]

\[ C_p' = \frac{a^r b^r c}{pra}, \quad D_p' = \frac{a^r b^r c}{pr_b} \]

Also set:

\[ (8f.) \quad I(\theta) = M \left\{ T_N (\theta) \cos(m_r \phi - \psi) \sum_{n=1}^{\infty} \frac{\psi_n(\theta)}{r^d} \right\} \]

Making the above substitutions and replacing the terms of (8f.) that are given by (8e.) we obtain:

\[ I(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \left\{ A_N \sum_{p=0}^{N_r} \left[ A_p' \cos(m_r p)\theta + B_p' \cos(m_r p)\theta + C_p' \sin(m_r p)\theta + D_p' \sin(m_r p)\theta \right] \sum_{n=1}^{\infty} \frac{a_n \cos \theta + b_n \sin \theta}{r^n} \right\} d\theta \]

In performing this integration, we note, by the orthogonality
of the trigonometric functions, that we obtain a series of terms in powers of \( r \), these powers ranging from \(-1\) to 
\[-(N_r \omega_r) = -g, \text{ i.e.:}\]

\[
I(\theta) = A_N r \sum_{p=1}^{N_r \omega_r - 1} \left( \frac{E_p}{r^p} \right) \frac{A_N a g + A_N C_i g}{2r^5}
\]

wherein: \( E_p \) is the appropriate coefficient obtained by the integration and the last term is given by

\[
(\theta_g.) \quad \frac{A_N a \cdot a g + A_N C_i g}{2r^5} = A_N \frac{a^i \cos \theta - b^i \sin \theta}{4r^5} + a g^i
\]

\[
= \frac{a^i \sin \theta + b^i \cos \theta}{4r^5} \quad b g = A_N \frac{a^i a g + b^i \cos \theta (a^i b g - b^i a g) \sin \theta}{4r^5}
\]

We note in \((\theta_g.)\) that the coefficients

\[
a^i \frac{a g + b^i}{g} \quad \text{and} \quad a^{i^2} b^i \frac{a g}{g}
\]

cannot both be zero for squaring

and adding them gives:

\[
\frac{a^i}{g} \frac{a^2}{g} + \frac{a^2}{g} \frac{a^2}{g} + \frac{b^2}{g} \frac{b^2}{g} + \frac{b^i}{g} \frac{a^2}{g} = (\frac{a^i}{g} + \frac{b^i}{g})(a^2 + b^2)
\]
By lemma 8.1, \(\frac{a^2}{N_r} + \frac{b^2}{N_r} = \left(\frac{1}{2N_r - 1}\right)^2\) and by hypothesis \(\psi_g(\theta) \neq 0\) i.e. \(a^2 g + b^2 g > 0\).

If in (8g,) we set:

\[
\cos \theta_0 = \frac{a^2 g + b^2 g}{4 \sqrt{a^2 g^2 + b^2 g^2}} \quad \text{and} \quad \sin \theta_0 = \frac{a b g}{4 \sqrt{a^2 g^2 + b^2 g^2}}
\]

substituting I(\(\theta\)) in (8d,) and collecting corresponding terms, we have:

\((8h,) \quad \forall M \{ u(r, \theta) T_N (\theta) \left[1 + \cos(m_r \theta - g \theta)\right] \} = v_r A_r a^2 + \kappa \log \frac{1}{r} + \sum_{n=1}^{g-1} \left( \frac{F_n}{r^n} \right) A_r \quad \frac{\sqrt{a^2 g^2 + b^2 g^2}}{2N_r - 1} \quad \frac{\cos(g \theta - g \theta)}{r^g}
\]

wherein:

\[V_r = M \{ v(r, \theta) T_N (\theta) \left[1 + \cos(m_r \theta - g \theta)\right] \}\]

\(F_n\) is the coefficient of \(\frac{1}{r^n}\) for \(n = 1, \ldots, g - 1\). In (8h,), we know that as \(r\) varies \(A_r, N_r, T_N (\theta), m_r, V_r\) and \(F_n\) vary. However all these functions are uniformly bounded as \(r \to 0\).
To show this, we have:

(1) \(|A_N| = \frac{1}{r}\) for all \(r\) by definition.

(2) \(N_r \leq N\) by hypothesis.

(3) \(|T_{N_r}(\theta)| \leq \frac{1}{r}\) for all \(r\) by 8.3.

(4) \(m_r \leq g\) and \(g\) is independent of \(r\), since \(m_r \cdot N_r = g\) by definition.

(5) \(V_r\) is uniformly bounded as \(r \to 0\) since \(v(r, \theta)\) is harmonic throughout \(R \cdot \{0\}\) and is therefore bounded for all \(r \leq 1\), \(|T_{N_r}(\theta)| \leq \frac{1}{r}\) and \(1 + \cos(m_r \theta - g\theta)\) is 2.

(6) \(|F_n| \leq F\), a positive constant independent of \(r\), since

\[
|a_{p_r}^1|, |b_{p_r}^1| \leq 2, p_r = 0, 1, \ldots N_r, \text{and } 1 + \cos(m_r \theta - g\theta) \leq 2
\]

for all \(r\).

From items (1)...(6) listed above we have:

\[
\left| \frac{V_r \cdot A_N a_{p_r}^1}{\log \frac{1}{r}} + \sum_{n=1}^{N_r} \frac{F_n}{n} \right| \leq \nu^* \cdot k \cdot \log \frac{1}{r} + \nu^* \cdot \sum_{n=1}^{N_r} \frac{F_n}{n} \frac{1}{r n}
\]

\[
= O\left\{ \frac{1}{r^{g-1}} \right\} \quad \text{as } r \to 0
\]

On the other hand:

\[
\left| \frac{A_N}{r} \frac{\sqrt{a_E^2 + b_E^2}}{2^{2N_r-1}} \cos(g\theta - \varphi_0) \right| \geq \left| \frac{\sqrt{a_E^2 + b_E^2}}{2^{2N_r-1}} \cdot \frac{\cos(g\theta - \varphi_0)}{r g} \right|
\]

\[
= O\left\{ \frac{1}{r^g} \right\} \quad \text{as } r \to 0
\]
Hence for $r$ sufficiently small the right member of \((\text{8h}_r)\) will have the sign of:

\[
A_r \frac{\sqrt{\frac{a^2_r + b^2_r}{2N_r-1}} \cdot \cos(g^2_\theta - \phi_0)}{r^2}
\]

But for a suitable choice of $\phi$ this is negative which contradicts \((\text{8h}_r)\).

This contradiction therefore assures us that in the expansion of $u(r, \theta)$ and under the hypothesis that the number of sign changes of $u(r, \theta)$ is bounded above by $2N$, there cannot be a term \(\psi_n(\theta) \neq 0\) and such that $g > N$, i.e., in the expansion of $u(r, \theta)$, \(\psi_n(\theta) = 0\) for all $n > N$ and Q.E.D.

We note that in the proof of Theorem 8.1, no use was made of the assumption that $u(r, \theta)$ is normal in $R$, hence the theorem holds independent of the normalization.

The following corollary is obvious.

**Corollary 8.11** If $u(r, \theta) = v(r, \theta) + k \log \frac{1}{r} + \sum_{n=1}^{\infty} \frac{\psi_n(\theta)}{r^n}$ is such that $\psi_n(\theta) \neq 0$ for infinitely many $n$, and if on the circle of radius $r$, $u(r, \theta)$ has $2N_r$ sign changes, then $N_r \to \infty$ as $r \to 0$. 
Consider $u(r, \theta)$ on the boundary $r = 1$; since $u(r, \theta)$ is normal in $R$ then:

$$u(1, \theta) = 0 = v(1, \theta) + \sum_{n=1}^{\infty} \psi_n(\theta)$$

Consider the function

$$h(r, \theta) = - \sum_{n=1}^{\infty} r^n \psi_n(\theta)$$

On the boundary $r = 1$,

$$h(1, \theta) = - \sum_{n=1}^{\infty} \psi_n(\theta)$$

and the series $\sum_{n=1}^{\infty} \psi_n(\theta)$ converges uniformly (Par. 3).

From these considerations we conclude that $h(r, \theta)$ is a function harmonic in $R \setminus \{0\}$ (i.e. throughout the unit circle) and assumes on the boundary, $r = 1$, precisely the values of $v(r, \theta)$. But if two functions, harmonic in a region, assume the same values on the boundary then the two functions are identical throughout the region, hence

$$v(r, \theta) = - \sum_{n=1}^{\infty} r^n \psi_n(\theta)$$

and

$$u(r, \theta) = k \log \frac{1}{r} + \sum_{n=1}^{\infty} \left( \frac{1}{r^n} - r^n \right) \psi_n(\theta)$$
Now suppose $u(r, \theta)$ has but a finite number of terms in its expansion, and suppose $\psi_N^2(\theta) = 0$ i.e. $a_N^2 + b_N^2 > 0$, and for all $n > N \geq 1$ $\psi_n(\theta) = 0$, then

$$u(r, \theta) = k \log \frac{1}{r} + \sum_{n=1}^{N} \left( \frac{1}{r^n} - r^n \right) \psi_n(\theta)$$

$$= \frac{1}{r^n} \left\{ kr^N \log \frac{1}{r} + (1-r^{2N}) \psi_N(\theta) + \sum_{n=1}^{N-1} (r^{N-n} - r^{Nn}) \psi_n(\theta) \right\}$$

Let: $S(r, \theta) = kr^N \log \frac{1}{r} + (1-r^{2N}) \psi_N(\theta) + \sum_{n=1}^{N-1} (r^{N-n} - r^{Nn}) \psi_n(\theta)$

If we define $r^N \log \frac{1}{r}$ to be zero at $r = 0$, then $S(r, \theta)$ is continuous in $\Re \{0\}$, moreover $S_0(r, \theta)$ is continuous in $\Re \{0\}$. Suppose $\theta_1$ is a zero of $\psi_N(\theta)$ i.e. $\psi_N(\theta_1) = 0$, then:

a. $S(0, \theta_1) = \psi_N(\theta_1) = 0$ and

b. $S_0(0, \theta_1) \neq 0$

a. is obvious; to show b. we have:

$$S_0(r, \theta) = (1-r^{2N})N(-a_N \sin \theta + b_N \cos \theta) +$$

$$\sum_{n=1}^{N-1} (r^{N-n} - r^{Nn})n(-a_n \sin \theta + b_n \cos \theta)$$
and

\[ S_0(o, \theta_1) = N(-a_N \sin \theta_1 + b_N \cos \theta_1) \]

and if \( S_0(o, \theta_1) = 0 \), then \( N \geq 1 \),

\[ a_N \cos \theta_1 + b_N \sin \theta_1 = 0 \quad \text{and} \quad -a_N \sin \theta_1 + b_N \cos \theta_1 = 0 \]

which imply \( a_N^2 + b_N^2 = 0 \), contrary to hypothesis, and therefore we conclude \( b \). The implicit function theorem then tells us that: there exists a neighborhood \( U \) of \( r = 0 \), \( \theta = \theta_1 \) such that for each \( r \) belonging to \( U \) there exists a \( \theta \) belonging to \( U \) such that for these corresponding values \((r, \theta)\),

\[ S(r, \theta) = 0 \]

The function \( \theta = f(r) \), so determined, is single valued and continuous in \( U \), \( \theta_1 = f(0) \) and \( S(r, f(r)) = 0 \) for \( r \) belonging to \( U \). Moreover if \( N > 1 \), then also \( S_1(r, \theta) \) exists and is continuous in \( \mathbb{R} \setminus \{0\} \), and in this case, not only is the graph of \( \theta = f(r) \) an arc, it is also a smooth arc in some neighborhood of \( r = 0 \), \( \theta = \theta_1 \). In exactly the same manner, it can be shown that for each of the \( 2N \) values, \( \theta_i (i = 1, 2, \ldots, 2N) \)
such that \( \psi_N^{(e_1)} = 0 \), there exists a neighborhood of 
\((0, e_1)\) such that \( S(r, \theta) \) defines an arc (or a smooth arc 
if \( N > 1 \)) passing through \( r = 0 \) and for values of \( r \) and \( \theta \)
belonging to this arc and to the so determined neighborhood, \( S(r, \theta) = 0 \). If \( N > 1 \), then two consecutive tangents 
(i.e. tangent lines to two consecutive arcs) form an angle 
\( \frac{\pi}{N} \) with each other at \( r = 0 \).

Let \( C_1 \) and \( C_2 \) denote two of these \( 2N \) arcs and let 
these two arcs be consecutive arcs, let \( C_3 \) denote another 
of these arcs and in particular the one consecutive with 
\( C_2 \), but not \( C_1 \) if \( N > 1 \). Then in a neighborhood of \( r = 0 \) 
and "between" \( C_1 \) and \( C_2 \), \( S(r, \theta) > 0 \) or \( S(r, \theta) < 0 \). Suppose 
in this region \( S(r, \theta) > 0 \), then "between" \( C_2 \) and \( C_3 \) \( S(r, \theta) < 0 \). 
In fact it is now apparent that in a sufficiently small 
neighborhood of \( r = 0 \), \( S(r, \theta) \) defines exactly \( 2N \) regions 
having the origin as a boundary point, and in \( N \) of these 
regions \( S(r, \theta) > 0 \), alternating with these \( N \) regions for 
which \( S(r, \theta) > 0 \) are \( N \) regions for which \( S(r, \theta) < 0 \), and the 
angular opening at \( r = 0 \) of any such region is \( \frac{\pi}{N} \).

Now since:

\[
u(r, \theta) = \frac{1}{r^N} S(r, \theta)\]
we conclude the following:

Theorem 8.2 If \( u(r, \theta) = k \log \frac{1}{r} + \sum_{n=1}^{N} \left( \frac{1}{r^n} - r^n \right) \psi_n(\theta) \)

and \( \psi_N(\theta) \neq 0 \), then:

a. (Since all components of \( u(r, \theta) \) have the origin as a boundary point.) There are at most \( 2N \) components of \( u(r, \theta) \); at most \( N \) of these components constitute \( D_+ \) and the same for \( D_- \). In any event in some neighborhood of \( r = 0 \), there are exactly \( 2N \) regions having the origin as a boundary point. In exactly \( N \) of these regions \( u(r, \theta) > 0 \) and alternating with these \( N \) regions for which \( u(r, \theta) > 0 \) are exactly \( N \) regions for which \( u(r, \theta) < 0 \).

b. Let \( T \) (the shaded area in Fig. 2) be one of these \( 2N \) regions (in a neighborhood of 0), let \( C_1 \) and \( C_2 \) be the smooth arcs \( (N > 1) \) constituting the designated portion of the boundary of \( T \), and along \( C_1 \) and \( C_2 \) \( u(r, \theta) = 0 \), let \( t_1 \) and \( t_2 \) be the tangents to these arcs at \( r = 0 \), then the angle \( t_1 \hat{t}_2 = \frac{\pi}{N} \).

Fig. 2
c. For any value \( \theta' \) lying interior to the angle \( t_1 \theta t_2 \),

\[
u(r, \theta') = O\left(\frac{1}{r^N}\right)
\]

as \( r \to 0 \).

d. In a sufficiently small neighborhood of \( r = 0 \), the number of sign changes of \( u(r, \theta) \) is bounded above by \( 2N \).

From d, we can obtain immediately the converse of Corollary 8.11, i.e.

Corollary 8.21 If on a circle of radius \( r \), \( u(r, \theta) \) has \( 2N \) sign changes and if \( N \to \infty \) as \( r \to 0 \) then \( \psi_n(\theta) \neq 0 \) for infinitely many \( n \).

From b, we obtain:

Corollary 8.22 If \( \{N_i\}_{i=1}^{\infty} \) is a monotone increasing sequence of positive integers and corresponding to each of these integers we have a function:

\[
u_1(r, \theta) = k_i \log \frac{1}{r} + \sum_{n=1}^{N_i} \left( \frac{1}{r^n} - r^n \right) \psi_{ni}(\theta)
\]

such that:
\[ \psi_{N_i}(\theta) = a_{N_i}^{(i)} \cos N_i \theta + b_{N_i}^{(i)} \sin N_i \theta \neq 0 \]

if \( \alpha_i (= \pi/N_i) \) is the angular opening at \( r = 0 \) of one of the regions \( T_i \) (discussed in \( b_* \)), then \( \alpha_i \to 0 \) as \( i \to \infty \).

**Theorem 8.** If \( u(r, \theta) = O(1/r^N) \) as \( r \to 0 \) then \( \psi_n(\theta) = 0 \) for all \( n > N \). (\( N \) is any positive number).

**Proof:** Suppose \( \psi_k(\theta) \neq 0 \) and \( k > N \) then:

\[ M \{ u(r, \theta) \psi_k(\theta) \} = \frac{1}{r^k} (r - r^k) \left( a_k^2 + b_k^2 \right) \]

By hypothesis, there exists a positive number \( K \) such that

\[ |r^N u(r, \theta)| \leq K \text{ for all } r \text{ (} 0 \leq r \leq 1 \text{), therefore} \]

\[ r^N \{ u(r, \theta) \psi_k(\theta) \} \leq 2M \{ |r^N u(r, \theta)| \} \leq 2 \cdot K \]

and

\[ r^N \frac{(r - r^k)}{2} \left( a_k^2 + b_k^2 \right) \leq 2 \cdot K \]

for all \( r \) (\( 0 \leq r \leq 1 \)). But since \( N < k \), this last inequality is obviously false for \( r \) sufficiently small, and hence we conclude \( \psi_k(\theta) = 0 \) and Q.E.D.

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*For a proof of this theorem in 3-space see Kellogg, O. D. F.P.T., page 270, Theorem XI.*
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