ON CONTINUED FRACTIONS AND INFINITE PRODUCTS

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Table of Contents

I

Introduction. A continued fraction is an expression of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots \; b_{n-1} + \cfrac{a_n}{b_n + \cfrac{a_{n+1}}{\ddots}}}}} \qquad (1 \cdot 1)$$

where the elements a_n and b_n may be finite or infinite in number. The elements a_n and b_n are real or complex quantities and are called the n^{th} partial numerator and the n^{th} partial denominator, respectively. For convenience in the work to follow the continued fraction (1 · 1) shall be designated by

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} + \cdots \qquad (1 \cdot 2)$$

A finite continued fraction is one which has only a finite number of elements, and an infinite continued fraction has infinitely many elements. An infinite continued fraction having an element $a_n = 0$ is equivalent to a finite continued fraction and shall be classed with finite continued fractions.

The n^{th} convergent of the continued fraction (1 · 2) is the quotient A_n / B_n where A_n and B_n are given by the recurrence formulas (Perron 1, p. 5)

$$\begin{aligned} &A_0 = 1, && B_0 = 1, \\ &A_1 = b_0 b_1 + a_1, && B_1 = b_1, \\ &A_n = b_n A_{n-1} + a_n A_{n-2}, && \\ &B_n = b_n B_{n-1} + a_n B_{n-2}, && (n = 2, 3, \cdots). \end{aligned} \qquad (1 \cdot 3)$$

A continued fraction is said to converge if the sequence of convergents tends to a limit as n becomes infinite. The continued fraction is said to be convergent at least in the wider sense if the convergents tend to a limit as n becomes infinite or ~~if the reciprocals of the convergents tend to a limit as~~

~~n becomes infinite or~~ if the reciprocals of the convergents tend to a limit as n becomes infinite.

Worpitzky* convergence criterion. If the elements a_n and b_n of the continued fraction (1 · 2) are functions of any variables, real or complex, the continued fraction converges uniformly in the region characterized by the inequalities

$$(1 \cdot 4) \qquad \left|\frac{a_1}{b_1}\right| < C, \qquad \left|\frac{a_n}{b_{n-1} b_n}\right| \leq \frac{1}{4} \quad (n = 2, 3, \cdots),$$

where C is any positive number. In particular, this convergence criterion is valid when the elements a_n and b_n are independent real or complex variables.

A continued fraction which fails to converge at least in the wider sense is said to diverge by oscillation. An important sufficient condition for this type of divergence has been obtained by Stern and by Stolz. (Perron 1, p. 235).

Theorem. The infinite continued fraction

$$(1 \cdot 5) \qquad b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} + \cdots$$

where the elements b_n are any complex constants diverges by oscillation if the series $\sum_{i=1}^{\infty} |b_i|$ converges.

von Koch's theorem. If the series $\sum_{i=1}^{\infty} |b_i|$ converges,

$$\lim_{n=\infty} A_{2n} = A^0, \qquad \lim_{n=\infty} B_{2n} = B^0,$$

$$\lim_{n=\infty} A_{2n-1} = A^1, \qquad \lim_{n=\infty} B_{2n-1} = B^1,$$

$$A^0 B^1 - A^1 B^0 = -1,$$

where A_n / B_n is the n^{th} convergent of (1 · 5). (Perron 1, p. 235).

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* See Szász (1). This criterion was proved independently by van Vleck (1) and by Pringsheim (1).

The two preceding theorems have been stated for continued fractions of the form (1 · 5), but according to a theorem of Seidel (Perron 1, p. 196), every infinite continued fraction of the form (1 · 2) ($a_n \neq 0$) has a unique equivalent continued fraction of the form (1 · 5). The two continued fractions are equivalent in the sense that their n^{th} convergents are formally identical.

A convergence theorem due to Leighton (1) will be given next. The proof of this theorem does not appear in print at the time this paper is written and so will be included here.

Root-ratio test. The infinite continued fraction

(1 · 6) $$1 + \frac{a_1}{1} + \frac{a_2}{1} + \cdots + \frac{a_n}{1} + \cdots$$

where the elements a_n are arbitrary complex constants converges at least in the wider sense if

(1 · 7) $$\overline{\lim_{n=\infty}} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

and diverges by oscillation if

(1 · 8) $$\underline{\lim_{n=\infty}} \left| \frac{a_{n+1}}{a_n} \right| < 1.$$

Further, if

(1 · 9) $$\lim_{n=\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1,$$

the continued fraction may converge at least in the wider sense or may diverge by oscillation.

If condition (1 · 7) holds, there can be found an integer n_o large enough so that for all $n \geq n_o$

$$\left| \frac{a_{n+1}}{a_n} \right| < (1 - k)$$

where k is fixed, $0 < k < 1$. Then

$$|a_{n+1}| < (1-k)\,|a_n|$$

for all $n \geq n_0$, and for all m

$$|a_{n_0+m}| < (1-k)^m |a_{n_0}| \quad .$$

For m_0 sufficiently large

$$(1-k)^m\,|a_{n_0}| \leq$$

for all $m \geq m_0$, and hence the continued fraction

$$1 + \frac{a_{n_0+m_0}}{1} + \frac{a_{n_0+m_0+1}}{1} + \cdots$$

converges by the Worpitsky criterion. It follows that (1 • 6) converges at least in the wider sense.

If condition (1 • 1) holds, there can be found an even integer $2n_0$ large enough so that for all $n \geq 2n_0$

$$\left|\frac{a_{n+1}}{a_n}\right| > (1+k)$$

where $k > 0$ is fixed. The continued fraction (1 • 6) is now put in the equivalent form (1 • 5) where

$$b_1 = \frac{1}{a_1}, \qquad b_{2n} = \frac{a_1 a_3 \cdots a_{2n-1}}{a_2 a_4 \cdots a_{2n}},$$
$$b_{2n+1} = \frac{a_2 a_4 \cdots a_{2n}}{a_1 a_3 \cdots a_{2n+1}} \quad (n = 1,2,\cdots).$$

These relations are used to show that $\sum_{i=1}^{\infty} |b_i|$ converges.

$$\sum_{i=1}^{\infty} |b_i| = \sum_{i=1}^{\infty} \left(|b_{2i-1}| + |b_{2i}| \right)$$
$$= \sum_{i=1}^{n_0} \left(|b_{2i-1}| + |b_{2i}| \right) + \sum_{i=n_0+1}^{\infty} \left(|b_{2i-1}| + |b_{2i}| \right),$$

The first sum of the right member is finite. For the second sum

$$\sum_{i=n_0+1}^{\infty} \left(|b_{2i-1}| + |b_{2i}| \right) \leq \sum_{i=n_0+1}^{\infty} \left[\frac{|b_{2n_0-1}|}{(1+k)^{2(i-n_0)-1}} + \frac{|b_{2n_0}|}{(1+k)^{2(i-n_0)}} \right]$$
$$= \sum_{i=1}^{\infty} \left[\frac{|b_{2n_0-1}|}{(1+k)^{2i-1}} + \frac{|b_{2n_0}|}{(1+k)^{2i}} \right]$$
$$\leq \; \sum_{i=1}^{\infty} \left(\frac{1}{1+k} \right)^i$$

where k is the greater of $|b_{2n_0-1}|$ and $|b_{2n_0}|$. Hence $\sum_{n=1}^{\infty} |b_n|$ converges and the continued fraction diverges by oscillation.

The following examples demonstrate the truth of the final statement of the theorem. The continued fraction

$$1 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \cdots$$

converges at least in the wider sense, and in fact, converges in the strict sense. The continued fraction

$$\frac{-\frac{1}{4} - e}{1} + \frac{-\frac{1}{4} - e}{1} + \frac{-\frac{1}{4} - e}{1} + \cdots \quad (e > 0)$$

diverges by oscillation (Szász 1).

The test-ratio test as given above applies to continued fractions of the form (1 · 6), but any infinite continued fraction (1 · 2) for which the elements b_n are not zero has a unique equivalent continued fraction of the form (1 · 6) and the test then applies.

Another theorem of Leighton (2) is given below. This theorem, together with the test-ratio test, forms the basis for this paper.

Expansion theorem. Every power series

(1 · 10) $$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

determines uniquely a continued fraction of the form

(1 · 11) $$a_0 + \frac{a_1 x^{\alpha_1}}{1} + \frac{a_2 x^{\alpha_2}}{1} + \cdots + \frac{a_n x^{\alpha_n}}{1} + \cdots$$

where the exponents α_n are positive integers and the coefficients a_n are complex constants. Thus any infinite set of numbers which may be regarded as the differential coefficients of a function at the origin determines uniquely a continued fraction of the form (1 · 10). This is true, in particular, of every function analytic at the origin.

The continued fraction (1 • 11) is said to correspond to the power series (1 • 10) and is called a <u>corresponding continued fraction</u>. The power series (1 • 10) is called the <u>corresponding power series</u> of the continued fraction (1 • 11).

There is no loss of generality in assuming that $a_0 = 1$. The corresponding continued fraction is then

$$(1 \cdot 12) \qquad 1 + \frac{a_1 x^{\alpha_1}}{1} + \frac{a_2 x^{\alpha_2}}{1} + \cdots + \frac{a_n x^{\alpha_n}}{1} + \cdots .$$

The continued fractions to be discussed henceforth in this paper will be of the form (1 • 12), where all of the coefficients a_n are different from zero. If $a_n = 0$ for some finite index n, the continued fraction is equivalent to a terminating one and represents a rational function of x.

II

<u>Convergence theorems</u>. In this section some general convergence results shall be given.

The <u>nth</u> convergent of the continued fraction

(2·1) $$1 + \frac{a_1 x^{\alpha_1}}{1} + \frac{a_2 x^{\alpha_2}}{1} + \cdots + \frac{a_n x^{\alpha_n}}{1} + \cdots$$

is $A_n(x)/B_n(x)$ where

(2·2) $$\begin{aligned} A_0(x) &= 1 & B_0(x) &= 1 \\ A_1(x) &= 1 + a_1 x^{\alpha_1} & B_1(x) &= 1 \\ A_n(x) &= A_{n-1}(x) + a_n x^{\alpha_n} A_{n-2}(x) \\ B_n(x) &= B_{n-1}(x) + a_n x^{\alpha_n} B_{n-2}(x) \end{aligned}$$

It can easily be established by induction from (2·2) that the polynomials $A_n(x)$ and $B_n(x)$ satisfy for all n and for all x,

(2·3) $$A_n(x)B_{n-1}(x) - A_{n-1}(x)B_n(x) \equiv (-1)^{n-1} \prod_{i=1}^{n} a_i x^{\alpha_i}.$$

Then

(2·4) $$\frac{A_n(x)}{B_n(x)} - \frac{A_{n-1}(x)}{B_{n-1}(x)} \equiv (-1)^{n-1} \frac{\prod_{i=1}^{n} a_i x^{\alpha_i}}{B_{n-1}(x)B_n(x)}$$

If both sides of this identity are expanded in power series,

(2·5) $$\frac{A_n(x)}{B_n(x)} - \frac{A_{n-1}(x)}{B_{n-1}(x)} = \sum_{i=\sigma_n}^{\infty} e_i^{(n-1)} x^i$$

where

(2·6) $$e_{\sigma_n}^{(n-1)} = (-1)^{n-1} \prod_{i=1}^{n} a_i \qquad \sigma_n = \sum_{i=1}^{n} \alpha_i .$$

Then the power series expansion of $A_{n-1}(x)/B_{n-1}(x)$ agrees with the power series expansion of $A_n(x)/B_n(x)$ up to but not including the σ_n <u>th</u> power of x.

If the formal power series corresponding to (2·1) is

(2·7) $$P(x) \equiv 1 + \sum_{i=1}^{\infty} c_i x^i$$

it is known [Leighton 2] that when $A_n(x)/B_n(x)$ is expanded in formal power series, the two power series are identical up to but not including the terms in $x^{\sigma_{n+1}}$. Then formally at least,

$$F(x) - \frac{A_n(x)}{B_n(x)} \equiv \sum_{i=\sigma_{n+1}}^{\infty} h_i^{(n)} x^i. \tag{2·8}$$

If the power series (2·7) converges uniformly in some neighborhood of the origin, <u>f(x) is defined as the analytic function whose power series is F(x).</u>

<u>Theorem:</u> <u>Every infinite subsequence of convergents of the corresponding continued fraction which converges uniformly in a neighborhood T of the origin has the same limit function in T.</u>

Let $\{A_{n_k}(x)/B_{n_k}(x)\}$ and $\{A_{m_k}(x)/B_{m_k}(x)\}$ be uniformly convergent subsequences in T. That is,

$$\lim_{k=\infty} \frac{A_{n_k}(x)}{B_{n_k}(x)} \equiv f_n(x),$$

$$\lim_{k=\infty} \frac{A_{m_k}(x)}{B_{m_k}(x)} \equiv f_m(x),$$

where $f_n(x)$ and $f_m(x)$ are analytic in T. Now [Perron 1, p. 17]

$$f_m(x) - f_n(x) \equiv \lim_{k\to\infty} \left[\frac{A_{m_k}(x)}{B_{m_k}(x)} - \frac{A_{n_k}(x)}{B_{n_k}(x)} \right]$$

$$\equiv \lim \left[\frac{(-1)^{\nu_k - 1} \prod^{\nu_k} [illegible]}{B_{m_k}(x) B_{n_k}(x)} B^{(\nu_k)}_{\mu_k - \nu_k}(x) \right]$$

where μ_k is the larger and ν_k is the smaller of m_k and n_k, and where $B^{(\nu_k)}_{\mu_k-\nu_k}(x)$ is the denominator of the $(\mu_k - \nu_k)$<u>th</u> convergent of

$$1 + \frac{a_{\nu_k+1} x^{[illegible]}}{1} + \frac{a_{\nu_k+2} x^{[illegible]}}{1} + \ldots\ldots .$$

Since both subsequences converge uniformly to analytic functions in T there is an integer k_0 such that for $k \geq k_0$, $A_{m_k}(x)/B_{m_k}(x)$ and $A_{n_k}(x)/B_{n_k}(x)$ are uniformly bounded in T. The same is true in any circle K about the origin and lying within T. In K the power series expansions of all functions involved are valid. Thus, in K,

$$\phi_m(x) - \phi_n(x) \equiv \lim_{k \to \infty} \left(\sum_{i=\sigma_{\nu_k}} p_i^{(k)} x^i \right)$$

and the right member is an analytic function. Further, the kth approximating function, $\sum_{i=\sigma_{\nu_k}} p_i^{(k)} x^i$, is analytic, uniformly bounded, and has at least σ_{ν_k} zeros within K. From this.

$$\phi_m(x) - \phi_n(x) \equiv 0$$

in K since it is an analytic function in K with an interior limit point of zeros. By the principle of analytic continuation this identity must also hold throughout T.

This completes the proof of the theorem.

Theorem. If the convergents $A_n(x)/B_n(x)$ are uniformly bounded in T for n sufficiently large, then

$$\lim_{n \to \infty} \frac{A_n(x)}{B_n(x)} = \phi(x)$$

uniformly in T, and $\phi(x)$ is an analytic function in T.

It is sufficient to show that

$$\lim_{n \to \infty} \frac{A_n(x)}{B_n(x)} = \phi(x)$$

in T, where $\phi(x)$ is analytic in T since it will follow from this [Montel 1, p.177] that the convergence is uniform throughout T.

Since the convergents are uniformly bounded in T for n sufficiently large they constitute a normal family of analytic functions in T [Montel 1, p.] and there can be extracted a subsequence of convergents which converges uniformly in T to an analytic function $\phi(x)$. By the preceding theorem every other subsequence of convergents which converges uniformly in T must also converge to $\phi(x)$. Suppose now that the set of convergents does not converge to $\phi(x)$ at some point x_0 of T. Then there must exist a number $\mathcal{E} > 0$ and a sequence of indices

$$n_1, n_2, n_3, \quad \cdots$$

such that

$$\left| \phi(x_0) - \frac{A_{n_i}(x_0)}{B_{n_i}(x_0)} \right| > \mathcal{E} \quad .$$

However, the convergents $A_{n_i}(x)/B_{n_i}(x)$ are uniformly bounded in T for i sufficiently large and constitute a normal family in T. Hence a uniformly convergent subsequence of the $A_{n_i}(x)/B_{n_i}(x)$ can be extracted and this subsequence must have the same analytic limit function $\phi(x)$ in T. This gives a contradiction of the inequality immediately above, and the contradiction completes the proof of the theorem.

Theorem. A necessary and sufficient condition that the corresponding continued fraction converge uniformly to f(x) in a neighborhood T of the origin is that the convergents of the continued fraction be uniformly bounded in T for n sufficiently large.

In order to establish the sufficiency of the condition of the theorem one needs only to show that f(x) is identical with the function $\phi(x)$ of the preceding theorem.

The function $\phi(x)$ is analytic in T and consequently has a power series expansion

$$\phi(x) = 1 + \sum_{i=1}^{\infty} d_i x^i$$

which is valid at least in the largest circle K about the origin and lying in T. Also,

$$\frac{A_n(x)}{B_n(x)} = 1 + \sum_{i=1}^{J_{n+1}-1} c_i x^i + \sum_{i=J_{n+1}}^{\infty} g_i^{(n)} x^i$$

is valid in k, where the c_n are the coefficients of the power series $P(x)$. Then

$$\mathfrak{f}(x) - \frac{A_n(x)}{B_n(x)} \equiv \sum_{i=1}^{J_{n+1}-1} (d_i - c_i)x^i + \sum_{i=J_{n+1}}^{\infty} (d_i - g_i^{(n)})x^i$$

is valid in K and the limit function is identically zero in K and hence in T. It follows that, for any index i_o, n can be taken large enough so that

$$d_i - c_i = 0$$

for all $i \leq i_o$ and hence

$$\mathfrak{f}(x) = 1 + \sum_{i=1}^{\infty} c_i x^i$$

is valid in K, or

$$\mathfrak{f}(x) = P(x)$$

in K. Hence $P(x)$ is the power series of an analytic function $f(x)$, and it follows that in K and in T as well

$$\mathfrak{f}(x) \equiv f(x).$$

This completes the proof of the sufficiency of the condition for the convergence of the continued fraction to $f(x)$.

The necessity of the condition is easily established. If

$$\lim_{n=\infty} \frac{A_n(x)}{B_n(x)} \equiv f(x)$$

uniformly for all x in T, the limit function is analytic in T and bounded in T. For any given $\epsilon > 0$ there exists an integer n_o such that for $n \geq n_o$,

$$\left| f(x) - \frac{A_n(x)}{B_n(x)} \right| < \epsilon$$

uniformly for all x in . It follows that

$$\left| \frac{A_n(x)}{B_n(x)} \right| < M + \epsilon$$

uniformly in T for all $n \geq n_o$, and hence the convergents are uniformly bounded in T for $n \geq n_o$.

<u>Corollary</u>. *If the continued fraction*

$$1 + \frac{a_1 x^{\alpha_1}}{1} + \frac{a_2 x^{\alpha_2}}{1} - \cdots$$

converges uniformly in some neighborhood T of the origin, it converges to the analytic function represented by the corresponding power series, and this power series converges uniformly at least in the largest circle about the origin and lying in T.

III

Some results on semi-normal power series and continued fraction. A power series

$$(3 \cdot 1) \qquad 1 + \sum_{i=1}^{\infty} c_i x^i$$

is called semi-normal if all of the determinants

$$(3 \cdot 2) \qquad \phi_n = \begin{vmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \cdots & & & \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{vmatrix} \quad (n = 1, 2, 3, \cdots)$$

$$\psi_n = \begin{vmatrix} c_2 & c_3 & \cdots & c_n \\ c_3 & c_4 & \cdots & c_{n+1} \\ \cdots & & & \\ c_n & c_{n+1} & \cdots & c_{2n-2} \end{vmatrix} \quad (n = 2, 3, 4, \cdots)$$

are different from zero. The corresponding continued fraction is then (Perron 1, p. 304)

$$(3 \cdot 3) \qquad 1 + \frac{a_1 x}{1} + \frac{a_2 x}{1} + \cdots + \frac{a_n x}{1} + \cdots ,$$

where

$$(3 \cdot 4) \qquad \begin{aligned} a_1 &= \phi_1, \quad \phi_0 = 1, \quad \psi_1 = 1, \\ a_{2n} &= - \text{[illegible]}, \\ a_{2n+1} &= - \text{[illegible]} \qquad (n = 1, 2, \cdots) \end{aligned}$$

The continued fraction (3 · 3) is called the corresponding semi-normal continued fraction of the power series (3 · 1).

A power series

$$c_0 + \sum_{i=1}^{\infty} c_i x^{K_i}$$

is said to have Hadamdrd gaps, if for infinitely many indices i,

$$(3 \cdot 5) \qquad K_{i+1} > (1 + \theta) K_i$$

where θ is positive and fixed.

Theorem. A semi-normal power series may have Hadamdrd gaps.

This result will be established by exhibiting an example.

Consider the power series

(3 · 6) $$1 + \sum_{i=0}^{\infty} c_{2^i} x^{2^i}$$

where $c_{2^i} \neq 0$, $i = 0, 1, 2, \cdots$. In order to show that the power series (3 · 6) is semi-normal, it must be shown that the determinants (3 · 2) are all different from zero. The elements c_K where $K \neq 2^i$ ($i = 0, 1, 2, \cdots$) are all zero.

If $n = 2^i$, both determinants are essentially diagonal determinants and

$$\Phi_{2^i} = (-1)^{2^{(i-1)}} [c_{2^i}]^{2^i},$$

(3 · 7) $$\Psi_{2^i} = (-1)^{2^{(i-1)}-1} [c_{2^i}]^{2^i-1} \quad (i = 1, 2, \cdots).$$

Neither of these determinants is zero.

If $n = 2^i + j$ where $0 < j < 2^i$, the largest index of any c_K appearing in Φ_{2^i+j} is $2^{(i+1)} + 2j - 1$. Thus $c_{2^{(i+1)}}$ appears in the last $2j$ rows and the last $2j$ columns of Φ_{2^i+j}. Also, no non-zero elements other than $c_{2^{(i+1)}}$ appear in the last j rows or the last j columns. Φ_{2^i+j} is now expanded by Laplace's method in terms of the $2j$ rowed minors in the last $2j$ columns, together with their complementary minors. (Bôcher 1, p. 26). The value of the minor formed from the last $2j$ rows is $(-1)^j [c_{2^{(i+1)}}]^{2j}$, and its complementary minor is Φ_{2^i-j}. Any other $2j$ rowed minor which is not zero must have at least one non zero element in each row and in each column, and hence at least one of the last j rows of Φ_{2^i+j} is not used in such a minor. Then the complementary minor must have at least one row whose elements are all zero. Finally

(3 · 8) $$\Phi_{2^i+j} = (-1)^j [c_{2^{(i+1)}}]^{2j} \Phi_{2^i-j} \quad \begin{pmatrix} i = 1, 2, \cdots \\ j = 1, 2, \cdots, 2^i-1 \end{pmatrix}.$$

A similar expansion of ψ_{2^i+j} by $2j-1$ rowed minors in the last $2j-1$ columns gives

(3 · 9) $$\psi_{2^i+j} = (-1)^{j-1}\left[c_{2^{(i+1)}}\right]^{2j-1}\psi_{2^i-j+1} \qquad \begin{pmatrix} i = 1, 2, \cdots \\ j = 1, 2, \cdots, 2^i - 1 \end{pmatrix}.$$

Since all of the coefficients c_{2^i} are different from zero, it follows that all of the ϕ and ψ determinants are different from zero, and the power series (3 · 6) is semi-normal.

It is easily seen that the power series has Hadamard gaps since

$$2^{(i+1)} > (1+\theta)\,2^i$$

for any positive and fixed value of $\theta < 1$.

It is clear that the power series (3 · 6) is in a sense a limiting form for semi-normal power series. In fact, <u>no semi-normal power series has more zero coefficients, with indices not exceeding a given positive integer, than has the power series</u> (3 · 6). This fact is implied in the following theorem.

<u>Theorem</u>. <u>A necessary condition that the power series</u>

$$c_0 + \sum_{i=0}^{\infty} c_{n_i} x^{n_i},$$

<u>where</u> $c_{n_i} \neq 0$ (i = 0, 1, 2, ⋯), <u>be semi-normal is that</u>

$$n_0 = 1,$$
$$n_i < n_{i+1} \leq 2n_i .$$

For the power series (3 · 6) the exponents satisfy $n_{i+1} = 2n_i$ for all i (= 0, 1, 2, ⋯)

The continued fraction of the form (3 · 3) corresponding to the power series (3 · 6) is to be discussed next. The formulas (3 · 4), (3 · 7), (3 · 8) and (3 · 9) are used to find the coefficients of the continued fraction.

$$a_{2(2^i+j)} = -\frac{[\psi_{2^i+j+1}]\cdot[\phi_{2^i+j-1}]}{[\phi_{2^i+1}]\cdot[\psi_{2^i+1}]}$$

$$= a_{2(2^i-j)+1} \qquad (j = 1, 2, \cdots, 2^i - 1),$$

$$a_{2(2^i+j)+1} = -\frac{[\Phi\ 2^i+j+1]\cdot[\Psi\ 2^i+j]}{[\Psi\ 2^i+j+1]\cdot[\Phi\ 2^i+j]}$$

$$= a_{2(2^i-j)} \qquad (j = 1, 2, \cdots 2^i - 1),$$

$$a_{2^{(i+1)}} = -\frac{[\Psi\ 2^i+1]\cdot[\Psi\ 2^i-1]}{[\Phi\ 2^i]\cdot[\Phi\ 2^i]}$$

$$= \frac{c_1\ c_{2^{(i+1)}}}{[c_{2^i}]^2} \qquad (i = 1, 2, \cdots),$$

$$a_{2^{(i+1)}+1} = -\frac{[\Phi\ 2^i+1]\cdot[\Phi\ 2^i]}{[\Psi\ 2^i+1]\cdot[\Phi\ 2^i]}$$

$$= -a_{2^{(i+1)}} \qquad (i = 1, 2, \cdots).$$

A summary of these results is

$$(3 \cdot 10) \qquad \begin{aligned} a_{2^{(i+1)}+2j} &= a_{2^{(i+1)}+1-2j}, \\ a_{2^{(i+1)}+2j+1} &= a_{2^{(i+1)}-2j} \end{aligned} \quad \left(\begin{aligned} j &= 1, 2, \cdots 2^i - 1 \\ i &= 1, 2, \cdots \end{aligned}\right),$$

$$a_{2^{(i+1)}} = \frac{c_1\ c_{2^{(i+1)}}}{[c_{2^i}]^2},$$

$$a_{2^{(i+1)}+1} = -\frac{c_1\ c_{2^{(i+1)}}}{[c_{2^i}]^2} \qquad (i = 1, 2, \cdots).$$

These recurrence formulas together with the initial values

$$(3 \cdot 11) \qquad a_1 = c_1, \quad a_2 = -\frac{c_2}{c_1}, \quad a_3 = \frac{c_2}{c_1}$$

completely determine the coefficients of the continued fraction. A useful relation which can easily be established is

$$(3 \cdot 12) \qquad a_{2n} = -a_{2n+1} \qquad (n = 1, 2, \cdots).$$

It is interesting to note that the condition (3 · 12) is, according to a theorem of Perron (Perron 1, p. 336), the necessary and sufficient condition that the corresponding semi-normal power series of the continued fraction have no odd powers of x except the first power.

In the case where the coefficients of the power series (3 · 8) are all the same, the coefficients of the corresponding continued fraction are 1 and -1, except for the first, which is equal to the coefficient of the power series. The continued fraction converges uniformly for $|x| \leq$ [illegible] by the Worpitsky convergence criterion.

The power series is

$$1 + \sum_{i=0}^{\infty} x^{2^i}$$

and this has $|x| = 1$ as its circle of convergence. Thus

$$f(x) = 1 + \sum_{i=0}^{\infty} x^{2^i}$$

is analytic for $|x| < 1$, and according to the Hadamard gap theorem (Titchmarsh 1, p. 223), $f(x)$ has the circle $|x| = 1$ as a natural boundary.

The corresponding semi-normal continued fraction converges to $f(x)$ for $|x| \leq$ [illegible] (p. 9), but existing convergence criteria are not sharp enough to determine the behavior of the continued fraction in the remainder of the circle $|x| = 1$. It is evident that the continued fraction cannot converge uniformly in any region lying partly or wholly outside $|x| = 1$.

IV

Recursion systems and functional relations. A corresponding type continued fraction is uniquely determined formally by the recursion system

(4 · 1) $$P_0 = P_1 + a_1 x P_2, \qquad P_n = P_{n+1} + a_{n+1} x^{n+1} P_{n+2} \quad (n = 1, 2, \cdots).$$

The recursion system can be written as

$$\frac{P_n}{P_{n+1}} = 1 + a_{n+1} x^{n+1} \frac{P_{n+2}}{P_{n+1}} \quad (n = 0, 1, 2, \cdots),$$

and when the change of notation

(4 · 2) $$\frac{P_n}{P_{n+1}} = p_n \quad (n = 0, 1, 2, \cdots)$$

is made, the recursion system takes the new form

(4 · 3) $$p_n = 1 + \frac{a_{n+1} x^{n+1}}{p_{n+1}} \quad (n = 0, 1, 2, \cdots).$$

But (4 · 3) is precisely the set of relations which defines the continued fraction corresponding to a power series $p_0(x)$ (Section 2.).

(4 · 4) $$p_0 \sim 1 + \frac{a_1 x}{1} + \frac{a_2 x^2}{1} + \cdots + \frac{a_n x^n}{1} + \cdots.$$

This continued fraction is one to which the test-ratio test may be applicable. The ratio of the $(n+1)^{st}$ partial numerator to the n^{th} is

$$\left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = |x| \cdot \left| \frac{a_{n+1}}{a_n} \right| .$$

If the condition

(4 · 5) $$\lim_{n=\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

is imposed on the coefficients, the continued fraction will converge uniformly in the wider sense at least for $|x| < 1/L$ and will diverge by oscillation for $|x| > 1/L$. If $L = 0$ the continued fraction converges to a function meromorphic in the finite plane, and if $L = \infty$ it diverges by oscillation everywhere except at the origin. If L is finite and not zero, the change of variable

$x = 1/L$ will make the new continued fraction convergent at least in the wider sense for $|x| < 1$, and hence we can replace condition (4 · 5) by

$$(4 \cdot 6) \qquad \lim_{n=\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

without loss of generality.

Subject to condition (4 · 6) the continued fraction (4 · 4) converges uniformly at least in the wider sense for $|x| < 1$. Let a positive number M be chosen so that

$$|a_1| < M, \qquad \left| \frac{a_{n+1}}{a_n} \right| < M, \qquad (n = 1, 2, \cdots).$$

Then in the circle $|x| \leq 1/4M$,

$$|a_n x^n| \leq \qquad (n = 1, 2, \cdots).$$

This fact is immediate since

$$|a_n x^n| = |a_1| \cdot \left| \frac{a_2}{a_1} \right| \cdots \left| \frac{a_n}{a_{n-1}} \right| \cdot |x^n| \leq (\ ) \qquad (n = 1, 2, \cdots).$$

Thus the continued fraction converges uniformly in the strict sense for $|x| \leq 1/4M$, and by an earlier result (p. 9), the continued fraction converges for $|x| \leq 1/4M$ to an analytic function whose power series is $\phi_0(x)$. Let $\phi_0(x)$ now represent this analytic function. By the principle of analytic continuation

$$\phi_0(x) \equiv 1 + \frac{a_1 x}{1} + \frac{a_2 x^2}{1} + \cdots$$

throughout the interior of the unit circle except possibly at isolated points which are poles of $\phi_0(x)$.

The behavior of the denominators $B_n(x)$ of the convergents $A_n(x)/B_n(x)$ of the continued fraction (4 · 4) will be discussed for the case where condition (4 · 6) holds.

Consider the circular neighborhood of the origin, $|x| \leq \delta$, where $\delta > 0$ is to be chosen later. Let M_i be the maximum value of $|B_i(x)|$ for $|x| \leq \delta$ and let $C_i = |a_i \delta^i|$. From the recurrence relation for $B_i(x)$,

$$|B_i(x)| = |B_{i-1}(x)| + |a_i x^i| \cdot |B_{i-2}(x)|$$

and for $|x| \leq \delta$

$$M_i = M_{i-1} + C_i M_{i-2} .$$

Let M^*_{i-1} be the larger of M_{i-1} and M_{i-2}.

$$M_i \leq M^*_{i-1} (1 + C_i) .$$

Consider the continued fraction

$$1 + \frac{C_i}{1} + \frac{C_{i+1}}{1} + \cdots .$$

The convergents of this are P_k / Q_k where

$$P_0 = 1, \quad Q_0 = 1,$$
$$P_1 = 1 + C_i, \quad Q_1 = 1,$$
$$P_k = P_{k-1} + C_{k+i-1} P_{k-2},$$
$$Q_k = Q_{k-1} + C_{k+i-1} Q_{k-2}.$$

Then

$$M_{i+k} \leq M^*_{i-1} P_k ,$$

and (Perron 1, p. 235)

$$|P_k| < (1 + C_i)(1 + C_{i+1}) \cdots (1 + C_{i+k-1}).$$

As a first choice, $\delta > 0$ is taken small enough so that for $j \geq i$

(4.7) $$|a_j \delta^j| \leq \left(\tfrac{1}{2}\right)^j .$$

This condition is satisfied if

$$\delta \leq \frac{1}{2|a_1|}, \qquad \delta \leq \frac{1}{2}\left|\frac{a_n}{a_{n+1}}\right| \quad (n = 1, 2, \cdots).$$

The series $\sum_{k=i}^{\infty} C_k$ is convergent since

$$\sum_{k=i}^{\infty} C_k \leq \sum_{k=i}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{2^{k-1}} ,$$

and the infinite product dominating $|P_k|$ converges. Thus for any given $\epsilon' > 0$ there can be found an integer i_0, sufficiently large, so that

$$\prod_{k=i_0}^{\infty} (1 + c_k) = 1 + \epsilon' .$$

Since $P_{i_0-1}(x)$ and $P_{i_0-2}(x)$ are polynomials having the value 1 at $x = 0$, $\delta > 0$ can be chosen small enough so that for any given $\epsilon'' > 0$,

$$M_{i_0-1} \leq 1 + \epsilon'' .$$

For all k it follows that

$$M_{i_0+k} \leq (1 + \epsilon') \cdot (1 + \epsilon'')$$

or

(4 · 8) $$M_{i_0+k} \leq 1 + \epsilon .$$

Let m_i be the minimum value of $|P_i(x)|$ for $|x| \leq \delta$ where $\delta > 0$ is to be chosen later. From the recurrence formula for $P_i(x)$,

$$|P_i(x)| \geq |P_{i-1}(x)| - |a_i x^i| \cdot |P_{i-2}(x)| .$$

It is clear that i can be taken so large that the right member will be positive. In particular, take i large enough and $\delta > 0$ small enough so that

$$M_{j-2} \leq 1 + \epsilon_1 , \quad |a_j \delta^j| \leq \frac{1}{2^j} , \quad j \geq i ,$$

where $\epsilon_1 > 0$ is any number, however small. It has already been shown that such a choice of i and δ is possible.

$$m_i \geq m_{i-1} - \frac{1}{2^i}(1 + \epsilon_1)$$

$$m_{i+1} \geq m_{i-1} - \frac{1}{2^i}(1 + \epsilon_1) \cdot \left(1 + \frac{1}{2}\right) .$$

$$m_{i+k} \geq m_{i-1} - \frac{1}{2^i}(1 + \epsilon_1)\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k}\right).$$

The geometric series is summed and there results

$$m_{i+k} \geq m_{i-1} - \frac{1 + \epsilon_1}{2^{i-1}} .$$

This inequality holds for $i \geq i_0$ sufficiently large and $\delta > 0$ sufficiently small. A new choice of δ is made so that

$$m_{i_0-1} \geq 1 - \epsilon_2$$

where $\epsilon_2 > 0$. Then

$$m_{i_0+k} \geq 1 - \left(\epsilon_2 + \frac{1 + \epsilon_1}{2 i_0 - 1} \right)$$

or

(4 · 9) $$m_{i_0+k} \geq 1 - \epsilon .$$

The results obtained above are summarized in the following statement.

Theorem. If the continued fraction

$$1 + \frac{a_1 x}{1} + \frac{a_2 x^2}{1} + \cdots + \frac{a_n x^n}{1} + \cdots$$

is subject to the condition

$$\lim_{n=\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 ,$$

then for any given $\epsilon > 0$ there can be found a $\delta > 0$ and an integer n_0 such that

$$1 - \epsilon \leq | B_n(x) | \leq 1 + \epsilon$$

for $n \geq n_0$ and $|x| \leq \delta$.

It has already been shown that the continued fraction (4 · 4) subject to the condition (4 · 6) converges uniformly in the wide sense, at least, to the function $\phi_0(x)$ for $|x| < 1$ and that this function is meromorphic for $|x| < 1$. A proof of this fact, independent of the original proof, can now be given by using the theorem just proved.

It can easily be verified by induction from (4 · 1) that

$$F_0 = A_{n-1}(x) \; F_n + a_n \; x^n \; A_{n-2}(x) \; F_{n+1}$$

$$F_1 = B_{n-1}(x) \; F_n + a_n \; x^n \; B_{n-2}(x) \; F_{n+1}$$

holds for $n = 1, 2, \cdots$. Also

$$\frac{F_0}{F_1} = \frac{A_{n-1}(x)\,\phi_n + a_n x^n A_{n-2}(x)}{B_{n-1}(x)\,\phi_n + a_n x^n B_{n-2}(x)}$$

and

$$\frac{F_0}{F_1} - \frac{A_{n-1}(x)}{B_{n-1}(x)} = \frac{(-1)^{n-1}\prod_{i=1}^{n} a_i x^i}{B_{n-1}(x)\left[B_{n-1}(x)\phi_n + a_n x^n B_{n-2}(x)\right]}$$

This last relation is obtained from the preceding one after using the identity

$$A_{n-1}(x)\,B_{n-2}(x) - A_{n-2}(x)\,B_{n-1}(x) \equiv (-1)^{n-2}\prod_{i=1}^{n-1} a_i x^i .$$

For n_0 sufficiently large and $\delta > 0$ sufficiently small

$$1 + \epsilon_1 \geq |B_{n-2}(x)| \geq 1 - \epsilon_1 \quad (n \geq n_0) .$$

Also (Perron 1, p. 288)

$$\lim_{n=\infty} \phi_n = 1 , \quad (|x| \leq \delta < 1) ,$$

and n_0 can be taken so large that

$$|\phi_n| \geq 1 - \epsilon_2 \quad (n \geq n_0).$$

Further, for n_0 large enough and $\delta > 0$ small enough

$$|a_n x^n| \cdot |B_{n-2}(x)| \leq \epsilon_3 , \quad (|x| \leq \delta, n \geq n_0) .$$

Then

$$|B_{n-1}(x)\phi_n + a_n x^n B_{n-2}(x)| \geq (1-\epsilon_1)(1-\epsilon_2) - \epsilon_3(1-\epsilon_1)$$

for $n \geq n_0$ and $|x| \leq \delta$. Thus

$$\left| \frac{F_0}{F_1} - \frac{A_{n-1}(x)}{B_{n-1}(x)} \right| \leq \frac{\prod_{i=1}^{n} |a_i \delta^i|}{(1-\epsilon_1)\left[(1-\epsilon_1)(1-\epsilon_2) - \epsilon_3\right]}$$

for $n \geq n_0$ and for $|x| \leq \delta$. The denominator of the right member is bounded away from zero for $n \geq n_0$ and $|x| \leq \delta$, where suitable ϵ_1, ϵ_2, δ and ϵ_3 have been chosen, and the numerator tends to zero as n becomes infinite since $\delta < 1$.

$$\lim_{n=\infty} \left| \frac{F_0}{F_1} - \frac{A_{n-1}(x)}{B_{n-1}(x)} \right| = 0$$

uniformly for all $|x| \leq \delta$ and

$$\frac{F_0}{F_1} \equiv \phi_0 \equiv \lim_{n=\infty} \frac{A_{n-1}(x)}{B_{n-1}(x)}$$

for $|x| \leq \delta$. Since the right member is a meromorphic function for $|x| < 1$, the identity must hold for all $|x| < 1$.

Theorem. The convergents $A_n(x) / B_n(x)$ of the continued fraction (4 · 4) subject to (4 · 6) are such that

$$\lim_{n \to \infty} A_n(x) \equiv A(x)$$

$$\lim_{n \to \infty} B_n(x) \equiv B(x)$$

where $A(x)$ and $B(x)$ are analytic for $|x| < 1$.

From the recurrence relation for $A_n(x)$

$$|A_n(x) - A_{n-1}(x)| \leq |a_n x^n A_{n-2}(x)| .$$

For $|x| \leq r < 1$

$$|A_{n-2}(x)| \leq \prod_{i=1}^{\infty} (1 + |a_i| \cdot r^i)$$

uniformly for all n. The infinite product converges since the series

$$\sum_{i=1}^{\infty} |a_i| \cdot r^i$$

converges by the test-ratio test. Then $A_{n-2}(x)$ is uniformly bounded for all $|x| \leq r$ and

$$|A_n(x) - A_{n-1}(x)| \leq L \cdot |a_n| \cdot r^n$$

uniformly for all n. Thus the general term of the telescopic series

$$A_0(x) + (A_1(x) - A_0(x)) + \cdots + (A_n(x) - A_{n-1}(x)) + \cdots$$

does not exceed the corresponding term of a convergent positive term series, and the telescopic series converges absolutely and uniformly for $|x| \leq r < 1$. That is

$$\lim_{n \to \infty} A_n(x) \equiv A(x) ,$$

where $A(x)$ is analytic for $|x| < 1$.

A similar treatment will suffice to show that

$$\lim_{n \to \infty} B_n(x) \equiv B(x) .$$

Before the discussion of recursion systems of the form (4 · 1) is concluded, one special case will be considered, namely

$$F_0 = F_1 + axF_2$$

(4 · 10) $$F_n = F_{n+1} + ax^{n+1} F_{n+2} \qquad (n = 0, 1, 2, \dots) .$$

This leads to the continued fraction

(4 · 11) $$\frac{F_0}{F_1} \equiv 1 + \frac{ax}{1} + \frac{ax^2}{1} + \cdots + \frac{ax^n}{1} + \cdots$$

and all of the results obtained for the continued fraction (4 · 4) subject to (4 · 6) are valid here.

__Theorem.__ _There is a function_ $A(a, x)$, _analytic in_ x _for each value of_ $a \neq 0$, _such that_

(4 · 12) $$\frac{A(a, x)}{A(ax, x)} \equiv 1 + \frac{ax}{1} + \frac{ax^2}{1} + \cdots + \frac{ax^n}{1} + \cdots$$

for $|x| < 1$ _except possibly at isolated points which are poles of_ $A(a, x)/A(ax, x)$.
~~$A(ax, x)$~~.

Let the n^{th} convergent of the continued fraction be $A_n(a, x) / B_n(a, x)$. Now (Perron 1, p. 15)

$$B_n(a, x) \overset{a,x}{\equiv} A^{(1)}_{n-1}(a, x)$$

where $A^{(1)}_{n-1}(a, x)$ is the numerator of the $(n - 1)^{st}$ convergent of

$$1 + \frac{ax^2}{1} + \frac{ax^3}{1} + \cdots + \frac{ax^n}{1} + \cdots .$$

This continued fraction is the same as (4 · 11) if the a in (4 · 11) is replaced by ax. It follows that

$$A^{(1)}_{n-1}(a, x) \overset{a,x}{\equiv} A_{n-1}(ax, x) .$$

From the theorem of page 24

$$\lim_{n=\infty} A_n(a, x) \equiv A(a, x)$$

uniformly for $|x| < 1$ for each $a \neq 0$.

Also,

$$\lim A_{n-1}(ax, x) \equiv A(ax, x)$$

uniformly for $|x| < 1$ for each $a \neq 0$. That is,

$$\lim_{n=\infty} B_n(x) \equiv A(ax, x).$$

The theorem follows from this.

If the notation

$$F(a, x) \equiv \frac{A(a, x)}{A(ax, x)}$$

is adopted, it is seen that $F(a, x)$ is a meromorphic function of x for $|x| < 1$ for each value of $a \neq 0$. Also, for each $|x| < 1$, $F(a, x)$ is a meromorphic function of a for all finite a (Perron 1, p. 345).

It should be noted that the recursion system (4 · 10) has the solution

$$F_n \equiv A(ax^n, x) \quad (n = 0, 1, 2, \cdots).$$

That is, the continued fraction is defined by the system of <u>functional relations</u>

(4 · 13) $A(ax^n, x) \equiv A(ax^{n+1}, x) + ax^{n+1} A(ax^{n+2}, x)$

for $n = 0, 1, 2, \cdots$.

Another remarkable property of the function represented by (4 · 11) will now be derived.

Let

(4 · 14)
$$g_1(x) \equiv 1 + \frac{ax}{1} + \frac{ax}{1} + \cdots + \frac{ax}{1} + \cdots,$$
$$g_2(x) \equiv 1 + \frac{ax}{1} + \frac{ax^2}{1} + \frac{ax^2}{1} + \cdots,$$
$$- \quad - \quad - \quad - \quad - \quad -$$
$$g_n(x) \equiv 1 + \frac{ax}{1} + \frac{ax^2}{1} + \cdots + \frac{ax^n}{1} + \frac{ax^n}{1} + \cdots,$$

and let

(4 · 15) $\qquad f_n(x) \equiv 1 + \dfrac{ax^{n+1}}{1} + \dfrac{ax^{n+2}}{1} + \cdots\;.$

These identities hold wherever the continued fractions converge uniformly. Let $A_n(x)/B_n(x)$ be the n^{th} convergent of (4 · 11). The following identities can easily be verified by induction.

(4 · 16) $$g_n(x) \equiv \frac{A_{n-1}(x)\, g_1(x^n) + ax^n A_{n-2}(x)}{B_{n-1}(x)\, g_1(x^n) + ax^n B_{n-2}(x)}\,,$$

(4 · 17) $$f(x) \equiv \frac{A_{n-1}(x)\, f_n(x) + ax^n A_{n-2}(x)}{B_{n-1}(x)\, f_n(x) + ax^n B_{n-2}(x)}$$

From these it is found that

(4 · 18) $$f(x) - g_1(x) \equiv \frac{(-1)^{n-1}\,[f_n(x) - g_1(x^n)]\,\prod_{i=1}^{n} a\, x^i}{[B_{n-1}(x)\, g_1(x^n) + a\, x^n B_{n-2}(x)]\cdot[B_{n-1}(x)\, f_n(x) + ax^n B_{n-2}(x)]}\,.$$

The continued fraction $f(x)$ is limitar - periodic (Perron 1, p. 286) and for $|x| < 1$

$$\lim_{n=\infty} f_n(x) \equiv 1.$$

The continued fraction $g_1(x)$ is periodic and

$$g_1(x^n) \equiv \frac{1 + \sqrt{1 + 4\, ax^n}}{2}$$

is valid for all x except the points for which

$$1 + 4\, ax^n < 0.$$

These points lie outside the circle

(4 · 19) $\qquad |x| = |-4a|^{-1/n}$

and along the lines drawn through the origin and the points $1 + 4\, a\, x^n = 0$. That is, the lines are branch lines of the algebraic function

$$y^2 - y - 4\, ax^n = 0\,,$$

and $g_1(x^n)$ is a branch of this function. It is clear that for $|x| < 1$

$$\lim_{n=\infty} g_1(x^n) \equiv 1\;;$$

and from the theorem of page 22 it follows that

$$\lim_{n=\infty} |f_0(x) - g_n(x)| \equiv 0$$

for $|x| \leq \delta$, where $\delta > 0$ is sufficiently small. By analytic continuation

$$f_0(x) \equiv \lim_{n=\infty} g_n(x)$$

for $|x| < 1$. A summary of this result is given below.

__Theorem.__ __For $|x| < 1$ the function defined by the continued fraction__ (4 · 11) __is uniformly approximated by the sequence__ (4 · 16) __of algebraic functions, and the branch points of the approximating functions cluster in every arc of__ $|x| = 1$. This last fact follows from (4 · 19), since the branch points of $g_1(x^n)$ are equally spaced on the circle (4 · 19).

It has been shown that the recursion system (4 · 1) is actually equivalent to the set of functional relations (4 ·) when the complex constants a_i $(i = 1, 2, \cdots)$ are equal, and further, that the analytic function defined by the continued fraction (4 · 11) is the limit of a particular sequence of algebraic functions with the property that the branch points of these functions cluster in every arc of $|x| = 1$. This latter property is also possessed by the analytic function $f(x)$ which is defined for $|x| < 1$ by the continued fraction

$$1 + \frac{ax}{1} \frac{ax^{K}}{+ 1} + \frac{ax^{K^2}}{1} + \cdots + \frac{ax^{K^n}}{1} + \cdots ,$$

where K is a positive integer. It can easily be shown that $f(x)$ satisfies the functional relation

$$f(x) \equiv 1 + \frac{ax}{f(x^K)} .$$

V

<u>A natural boundary theorem for infinite products.</u> It has been shown by Ritt (1) that every power series

(5 · 1) $$1+\sum_{i=1}^{\infty} c_i x^i$$

determines uniquely an infinite product of the form

(5 · 2) $$\prod_{i=1}^{\infty} (1+a_i x^i)$$

In fact, Ritt has shown that (5 · 2) converges uniformly at least for $|x| < 1/6r$ where r is the least upper bound of

$$|c_1| \ , \quad \sqrt{|c_2|} \ , \quad \cdots \quad , \quad \sqrt[n]{|c_n|} \cdots .$$

The following theorem is believed to be the first result relating to natural boundaries of functions given in infinite product representation.

<u>Theorem. If the complex constants a_n are such that</u>

(5 · 3) $$\lim_{n\to\infty} |a_n| = a > 1,$$

<u>the infinite product</u>

(5 · 2) $$\prod_{i=1}^{\infty} (1+a_i x^i)$$

<u>converges uniformly to a function analytic for $|x| < 1$ and having the circle $|x| = 1$ as a natural boundary.</u>

The infinite product converges uniformly for $|x| \le r < 1$ since the series

$$\sum_{i=1}^{\infty} |a_i| \cdot r^i$$

converges (Titchmarsh 1, p. 13 ff.). Hence the infinite product is an analytic function $f(x)$ for $|x| < 1$. For i_0 sufficiently large $|a_i| > 1$ for $i \ge i_0$, and the zeros of the i^{th} factor are equally spaced on the circle $|x| = |a_i|^{-1/i}$. Each of these zeros is a zero of the analytic function represented by the product since they lie within $|x| < 1$. Now

$$\lim_{i\to\infty} |a_i|^{-1/i} = 1,$$

and the zeros of the i^{th} factor of the product cluster in every arc of the circle $|x| = 1$.

Suppose $f(x)$ is analytic in the neighborhood of some point on $|x| = 1$. Then it is certainly continuous in this neighborhood, and hence $f(x) = 0$ at a dense set of points on $|x| = 1$ and in the neighborhood of analyticity. It follows that $f(x) \equiv 0$ for $|x| < 1$, and this is a contradiction since $f(0) = 1$. This completes the proof of the theorem.

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In several cases it has been more convenient to refer to Perron rather than to original papers. In such cases the reader will find references to the original papers in Perron.