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Thesis
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I. A Mathematical Theory of Competition.
II. Generalized Lagrange Problems.

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A Mathematical Theory of Competition.
By C. F. Roos.

1. INTRODUCTION. With reference to the fact that the demand for a commodity often depends not merely on the price at any given time, but also on whether the price is increasing or decreasing at this time,—or, more exactly, assuming that the demand is a linear function both of the price and of the rate of change of price as well,—G. C. Evans has developed a theory of monopoly. The following is an account of a corresponding theory of competition, which seems to have new elements of mathematical interest. In order to save space the author limits himself, as far as possible, to these latter considerations.

Let us assume there are two producers, each manufacturing subject to the same cost function.

\[ q(u) = Au^2 + Bu + C \]

amounts \( u_1 \) and \( u_2 \), respectively in unit time, and each trying to make his profit \( \Pi_i \), \( i = 1, 2 \), a maximum.

If we assume that the demand is a linear function of the price \( p \) and the rate of change of price, \( dp/dt \), and further that as many units are sold as are produced then,

\[ y(t) = u_1 + u_2 = ap(t) + h \frac{dp}{dt} + b \]

where as in Evans' earlier paper † we take \( a < 0 \), \( b > 0 \), \( A > 0 \), \( B > 0 \), \( C > 0 \).

In the theory of competition the following two problems are of interest.

Problem (1). Given \( p = p_1 \) at \( t = t_1 \) (\( h \) being negative) choose \( u_1 \) so that \( \Pi_1 \) is a maximum when \( u_2 \) is regarded as not subject to variation, and at the same time choose \( u_2 \) so that \( \Pi_2 \) is a maximum when \( u_1 \) is regarded as not subject to variation.

In this case the restriction on \( p \) at \( t = t_1 \) involves no restriction on \( u_1 \) and \( u_2 \) since (2) involves an arbitrary constant.


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Problem (2). Given \( p = p_2 \) at \( t = t_2 \) and \( p = p_1 \) at \( t = t_1 \), choose \( p \) as a function of \( t \) so that \( \Pi_1 \) will be a maximum when the function \( u_1 \) alone varies with \( p \) and so that \( \Pi_2 \) will be a maximum when the function \( u_2 \) alone varies with \( p \).

In this case the variation of \( u_1 \) (or \( u_2 \)) is not arbitrary on account of (2) and the boundary conditions.

In both cases the total profits during the interval of time \( t_1 \) to \( t_2 \) will be given respectively by,

\[
\Pi_1 = \int_{t_1}^{t_2} [pu_1 - q(u_1)] dt
\]

\[
\Pi_2 = \int_{t_1}^{t_2} [pu_2 - q(u_2)] dt.
\]

2. Conditions for Maximum. These problems do not seem to reduce strictly to problems in the calculus of variations but solutions, however, are given below. Let us discuss them in the order already given.

In order to obtain a solution of problem (1) let \( \bar{u}_1(t), \bar{u}_2(t), \) and \( \bar{p}(t) \) be the values of \( u_1, u_2, \) and \( p \) giving the desired solution if such values exist. Replace \( u_1, u_2, \) and \( p \) by,

\[
\begin{align*}
u_1 &= \bar{u}_1 + \omega_1 \psi_1(t) \\
u_2 &= \bar{u}_2 + \omega_2 \psi_2(t) \\
p &= \bar{p} + \omega \theta(t)
\end{align*}
\]

where \( \psi_1(t) \) and \( \psi_2(t) \) are continuous functions of \( t \) admitting continuous derivatives in the interval \( t_1 \) to \( t_2 \) and \( \theta(t) \) is a continuous function of \( t \) admitting continuous derivatives in this interval and vanishing at \( t_1 \). By (2) it is easily seen that \( \theta(t) \) satisfies the differential equation

\[
a_0 \theta + h_0 \theta' = \omega_1 \psi_1 + \omega_2 \psi_2
\]

and is therefore determined by its value at one point say \( t_1 \).

It is evident that \( \omega_1 \psi_1(t), \omega_2 \psi_2(t), \) and \( \omega \theta(t) \) will be less in absolute value than a fixed quantity \( \epsilon \) in all the intervals provided \( |\omega_1| \) and \( |\omega_2| \) are small enough.

Obviously, \( u_1 = \bar{u}_1, u_2 = \bar{u}_2, \) and \( p = \bar{p} \) when \( \omega_1 = \omega_2 = 0 \), because of (2).

Now if we replace \( u_1, u_2, \) and \( p \) in (3) and (4) by their values as given by (5) we get \( \Pi_1 \) and \( \Pi_2 \) as functions of \( \omega_1 \) and \( \omega_2 \); hence the necessary conditions for a solution of our problems are,

\[
\frac{\partial \Pi_1}{\partial \omega_1} = \frac{\partial \Pi_2}{\partial \omega_2} = 0 \quad \text{when} \quad \omega_1 = \omega_2 = 0.
\]
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By the ordinary rules of differentiation applied to (3),

$$
\frac{\partial \Pi_1}{\partial \omega_1} = \int_{t_1}^{t_2} \left[ \frac{\partial u_1}{\partial \omega_1} \rho + u_1 \frac{\partial p}{\partial \omega_1} - q'(u_1) \frac{\partial u_1}{\partial \omega_1} \right] \, dt.
$$

By (5) this expression becomes,

$$
\frac{\partial \Pi_1}{\partial \omega_1} = \int_{t_1}^{t_2} \left( \psi_1(t) \left[ \rho(t) + \omega(t) \right] + \left[ u_1 + \omega \psi_1(t) \right] \frac{\partial p}{\partial \omega_1} - 2A \left( u_1 + \omega \psi_1(t) \right) \psi_1(t) \right) dt.
$$

From the initial conditions and (2) and (5),

$$
\omega(t) = e^{-\lambda \rho \tau} \int_{t_1}^{t} \frac{\omega \psi_1(\tau) + \omega \psi_2(\tau)}{h} e^{(\lambda \rho) \tau} \, d\tau, \text{ so that}
$$

$$
\frac{\partial p}{\partial \omega_1} = \frac{1}{h} e^{-\lambda \rho \tau} \int_{t_1}^{t} \psi_1(\tau) e^{(\lambda \rho) \tau} \, d\tau.
$$

Substituting this in (6) gives,

$$
\frac{\partial \Pi_1}{\partial \omega_1} = \int_{t_1}^{t_2} \psi_1(t) \left[ \rho(t) + \omega(t) - 2A \left( u_1 + \omega \psi_1(t) \right) \right] dt + \int_{t_1}^{t_2} dt \int_{t_1}^{t} \frac{u_1(t)}{h} e^{(\lambda \rho) \tau} \psi_1(\tau) \, d\tau
$$

Before we can draw any definite conclusions from the above expression it is necessary to write the iterated integral in a different form by applying Dirichlet's formula and then interchanging $t$ and $\tau$, the parameters of integration. This gives,

$$
\frac{\partial \Pi_1}{\partial \omega_1} = \int_{t_1}^{t_2} \left[ \rho - \omega(t) - 2A \left( u_1 + \omega \psi_1(t) \right) \right] + \int_{t_1}^{t_2} \frac{u_1(t)}{h} e^{(\lambda \rho) \tau} \psi_1(\tau) \, d\tau
$$

and for $\omega_1 = \omega_2 = 0$ this becomes

$$
\left( \frac{\partial \Pi_1}{\partial \omega_1} \right) = \int_{t_1}^{t_2} \left[ \rho - 2Au_1 \right] - B + \int_{t_1}^{t_2} \frac{u_1(t)}{h} e^{(\lambda \rho) \tau} \psi_1(\tau) \, d\tau = 0.
$$

Since $\psi_1(t)$ is arbitrary, we can conclude that the integrand is zero and therefore write as a necessary condition,

$$
p - 2Au_1 - B + 1/h \int_{t}^{t_2} u_1(\tau) e^{(\lambda \rho) \tau} \, d\tau = 0.
$$
In an entirely similar manner we derive from (4)

\[(8) \quad p - 2A u_2 - B + 1 \cdot h e^{\alpha_2 \cdot t} \int_{t}^{t_1} u_2(\tau) e^{-\alpha_2 \cdot \tau} d\tau = 0.\]

These equations are proved to be sufficient by showing that \(\partial \Pi_1 / \partial \omega_1 < 0\) when \(\omega_2 = 0\) and \(0 < \omega_1 < 1\), and \(\partial \Pi_2 / \partial \omega_2 < 0\) when \(\omega_1 = 0\) and \(0 < \omega_2 < 1\), no matter what the functions \(\psi_1(t)\) and \(\psi_2(t)\) are. The reasoning follows:

When (7) is true, (6) becomes

\[
\frac{\partial \Pi_1}{\partial \omega_1} = \int_{t_1}^{t} \psi_1(t) \left[ \omega_1(t) - 2A \omega_1 \psi_1(t) + \omega_1 \cdot h \int_{t_1}^{t} e^{-\alpha_1 \cdot \tau} \psi_1(\tau) d\tau \right] dt.
\]

Let \(\theta_1(t)\) be the value of \(\theta\) when \(\omega_2 = 0\). Then from (3) and (5)

\[\omega_1 \psi_1(t) = \alpha \omega_1 (t) + h \omega \theta_1'(t)\]

or solving this differential equation for \(\omega_1(t)\) we can write

\[\omega_1(t) = (\omega_1, h) e^{-\alpha_1 \cdot t} \int_{t_1}^{t} e^{\alpha_1 \cdot \tau} \psi_1(\tau) d\tau\]

If we substitute for \(\psi_1(t), \omega_1(t)\) and \(\omega_1(h) \int_{t_1}^{t} e^{\alpha_1 \cdot \tau} \psi_1(\tau) d\tau\) their values as given by the relations above, then

\[
\frac{\partial \Pi_1}{\partial \omega_1} = \int_{t_1}^{t_1} \frac{\omega_1 \theta_1(t) + h \omega \theta_1'(t)}{\omega_1} dt
\]

\[= 2A \int_{t_1}^{t_1} \omega_1 \theta_1^2(\tau) d\tau - 2A \int_{t_1}^{t} \left[ \alpha \omega_1 (t) + h \omega_1 \theta_1'(t) \right] dt
\]

\[+ 2h \int_{t_1}^{t_1} \frac{\omega_1 \theta_1(t) \theta_1'(t)}{\omega_1} dt\]

Now by an integration performed on the last term of this expression we can write

\[
\frac{\partial \Pi_1}{\partial \omega_1} = 2A / \omega_1 \int_{t_1}^{t_1} \omega_1^2 \theta_1^2(\tau) d\tau - 2A / \omega_1 \int_{t_1}^{t_1} \left[ \alpha \omega_1 (t) + h \omega_1 \theta_1'(t) \right] dt
\]

\[+ [h \omega^2 / \omega_1 \theta_1^2(\tau)]_{t_1}^{t_1}\]

The first two terms of this expression are negative (see (2)), and the term outside the integral sign vanishes at \(t_1\), so that it is negative when \(h\) is
negative. It is evident that the expression is negative independently of the sign of \( h \) if it happens that,

\[
\left| 2a \omega_1 \int_{t_1}^{t_2} \omega \delta \tilde{\beta}^2 dt - 2A \omega_1 \int_{t_1}^{t_2} [a \omega \theta_1 + h \omega \theta']^2 \right| > h \omega^2 \omega_1 \theta_1^2(t_2)
\]

but we shall not attempt to discuss this phase of the problem. It is perhaps worth mentioning that if we be given \( p = p_1 \) at \( t = t_2 \) instead of \( p = p_1 \) at \( t = t_1 \) we can find \( u \) as a function of \( t \) so that \( \Pi_1 \) during the interval of time \( t_1 \) to \( t_2 \) will be a maximum and \( u \) as a function of \( t \) so that \( \Pi_2 \) will be a maximum provided \( h \) is positive.

3. Solution of Integral Equations (7) and (8). We wish to find \( p \) as an explicit function of \( t \) such that it will satisfy equations (7) and (8) and (2), or in other words solve these three equations for the three unknowns \( u_1 \), \( u_2 \), and \( p \). Adding (7) and (8) and remembering that \( u_1 + u_2 = ap + b + hp' \) we get,

\[
he^{-a \lambda t} [a p - 2A (ap + b + hp') - 2B] + \int_t^{t_2} (ap + b + hp')e^{-a \lambda t} dt = 0.
\]

Before proceeding further let us notice that this expression conditions the value of \( p \) at \( t = t_2 \), i.e. \( p \) must satisfy the equation,

\[
2p - 2A (ap + b + hp') - 2B = 0 \quad \text{when} \quad t = t_2.
\]

The integral equation (9) can readily be reduced to a second order differential equation with constant coefficients by a differentiation with respect to \( t \). We get on collecting terms,

\[
-2A^2 p'' + hp' - a(3 - 2A)p = b - 2Ab - 2Ba.
\]

The general solution of this equation is,

\[
p = \frac{b - 2Ab - 2Ba}{-a(3 - 2A)} + C_1 e^{m_1 t} + C_2 e^{m_2 t} \quad \text{where}
\]

\[
m_1 = \frac{1 + \sqrt{1 - 8A(3 - 2A)}}{4Ah} \quad \text{and} \quad m_2 = \frac{1 - \sqrt{1 - 8A(3 - 2A)}}{4Ah}
\]

and \( C_1 \) and \( C_2 \) are determined from the initial condition \( p = p_1 \) at \( t = t_1 \), and the condition (10). In fact the determinant of the linear equations in \( C_1 \) and \( C_2 \neq 0 \). There always exists therefore a unique solution of the problem proposed.
Since \( u_1 \) and \( u_2 \) are solutions of the same integral equation [see (7) and (8)] we can conclude that \( u_1 = u_2 \) and proceed to determine the amount of goods produced by means of (2).

4. Solution of Problem 2. Given \( p = p_1 \) at \( t = t_1 \), and \( p = p_2 \) at \( t = t_2 \), we wish to find \( u_1 \) as a function of \( t \) so that \( \Pi_1 \) is a maximum, and \( u_2 \) as a function of \( t \) so that \( \Pi_2 \) is a maximum no matter what the variation of \( p \).

Let \( \psi_1(t) \), \( \psi_2(t) \), and \( \theta(t) \) be continuous functions of \( t \) with their derivatives and write

\[
\begin{align*}
    u_1 &= \bar{u}_1 + \omega_1 \psi_1(t) \\
    u_2 &= \bar{u}_2 + \omega_2 \psi_2(t) \\
    p &= \bar{p} + \omega \theta(t)
\end{align*}
\]

where \( \bar{u}_1, \bar{u}_2, \) and \( \bar{p} \) are the values of \( u_1, u_2, \) and \( p \) giving the desired solution. The three families \( \omega_1 \psi_1, \omega_2 \psi_2, \omega \theta \) are therefore arbitrary except that they are always related by \( \omega_1 \psi_1 + \omega_2 \psi_2 = a \omega \theta + h \omega \theta' \). Let us now impose the further condition that \( \theta(t) \) vanish at \( t = t_1 \) and at \( t = t_2 \).

As before we get \( u_1 = \bar{u}_1, u_2 = \bar{u}_2 \) and \( p = \bar{p} \) when \( \omega_1 = \omega_2 = 0 \). The necessary conditions for \( \Pi_1 \) to be a maximum as a function of \( u_1 \) and \( \Pi_2 \) a maximum as a function of \( u_2 \) are \( \delta \Pi_1 / \delta \omega_1 = \delta \Pi_2 / \delta \omega_2 = 0 \) when \( \omega_1 = \omega_2 = 0 \).

For the determination of \( u_1 \), the parameter \( \omega_2 \) will be zero, and therefore the work is simplified by considering the family \( \omega \theta_1(t) \) for which \( \omega_2 = 0 \).

In fact, we can write \( \omega_1 \psi_1(t) = a \omega \theta_1(t) + h \omega \theta_1'(t) \).

As in article (2) we obtain,

\[
\frac{\partial \Pi}{\partial \omega_1} = \int_{t_1}^{t_2} \left( \frac{\partial u_1}{\partial \omega_1} p + \frac{\partial p}{\partial \omega_1} \psi_1(t) \right) + q'(u_1) \frac{\partial u_1}{\partial \omega_1} \right) dt
\]

where \( \frac{\partial p}{\partial \omega_1} = \frac{1}{h} e^{-a \omega_1 t} \int_{t_1}^{t} \psi_1(t) e^{a \omega_1 t} dt = \omega_1 \omega_1 \theta_1(t) \).

By means of the relation \( \omega_1 \psi_1 = a \omega \theta_1 + h \omega \theta_1' \) this expression becomes,

\[
(A) \quad \frac{\partial \Pi}{\partial \omega_1} = \int_{t_1}^{t_2} \left\{ \frac{a \omega \theta_1 + h \omega \theta_1'}{\omega_1} (\bar{p} + \omega \theta_1) + \bar{u}_1 \omega_1 / \omega_1 \theta_1 \\
- [2 A (\bar{u}_1 + a \omega \theta_1 + h \omega \theta_1') + B] \frac{a \omega \theta_1 + h \omega \theta_1'}{\omega_1} \right\} dt
\]

\[
- \int_{t_1}^{t_2} \left\{ \bar{p} \omega_1 + a \omega \theta_1 + h \omega \theta_1' + \bar{u}_1 - 2 A \omega \theta_1 - 2 A \omega^2 \theta_1 - 2 A h \omega \theta_1' \\
- B \bar{\theta} - a h \omega \theta_1' \right\} \omega / \omega_1 \theta_1 dt
\]

\[
+ \int_{t_1}^{t_2} (h \omega \theta_1' - 2 A h \bar{u}_1 \theta_1' - 2 A h^2 \omega \theta_1'' - h B \theta_1') \omega / \omega_1 dt.
\]
By integrating by parts the first three terms under the second integral sign and remembering \( \theta(t_i) = \theta(t_j) = 0 \) this can be written as,

\[
\frac{\partial \Pi}{\partial \omega_i} = \int_{t_i}^{t_j} \left[ a\ddot{p} + a\omega \dot{\theta}_i + h\omega \dot{\theta}_i' + \dddot{u}_i - 2.1a\ddot{u}_i - 2.1a^2 \omega \theta_i - 2.1a \omega \dot{\theta}_i' - B\ddot{u}_i - a\omega \dot{\theta}_i' - \dddot{u}_i + 2.1 \omega \dot{\theta}_i' + 2.1 \omega \theta_i' \right] \omega / \omega_i \theta_i \, dt
\]

and for \( \omega_i = \omega_j = 0 \) this becomes,

\[
\frac{\partial \Pi}{\partial \omega_i} = \int_{t_i}^{t_j} \left[ \ddot{p} + a\ddot{p} + (1 - 2.1a) \dddot{u}_i - B\ddot{u}_i + 2.1 \omega \theta_i' \right] \theta_i(t) \omega / \omega_i \, dt = 0.
\]

If we let the value \( \omega_i = 1 \) correspond to the value \( \omega = 1 \), i.e., let \( \psi_i(t) = u(t) - \dddot{u}(t) \) when \( \theta_i(t) = \dddot{p}(t) - \dddot{p}(t) \), the relation between \( \omega_i \psi_i(t) \) and \( \omega \theta_i(t) \) yields the identity \( \omega = \omega_i \). But \( \theta_i(t) \) is arbitrary except that it vanishes at \( t_i \) and \( t_j \). Hence we can conclude that the above integrand is zero and write

\[(13) \quad \ddot{p} + a\ddot{p} + (1 - 2.1a) \dddot{u}_i - B\ddot{u}_i + 2.1 \omega \theta_i' = 0.\]

In an entirely similar manner we obtain from the condition \( \partial \Pi / \partial \omega_i = 0 \)

\[(14) \quad \ddot{p} + a\ddot{p} + (1 - 2.1a) \dddot{u}_i - B\ddot{u}_i + 2.1 \omega \theta_i' = 0.\]

It happens that these conditions are also sufficient, for when (13) is true the expression (4) for \( \partial \Pi / \partial \omega_i \) becomes

\[
\frac{\partial \Pi}{\partial \omega_i} = \int_{t_i}^{t_j} \left[ (a\omega \theta_i + h\omega \theta_i') \omega / \omega_i \theta_i + (a\omega \theta_i + h\omega \theta_i') \omega / \omega_i \theta_i + (a\omega \theta_i + h\omega \theta_i') \omega / \omega_i \theta_i \right]
\]

\[
- 2.1(a\omega \theta_i + h\omega \theta_i') \left( \frac{a\omega \theta_i + h\omega \theta_i'}{\omega_i} \right)
\]

\[
+ B \omega / \omega_i \theta_i' \right] \, dt.
\]

If we integrate the terms in \( \theta_i' \) and remember that \( \theta_i \) vanishes at \( t_i \) and at \( t_j \), this can be written as,

\[
\frac{\partial \Pi}{\partial \omega_i} = \frac{2a}{\omega_i} \int_{t_i}^{t_j} \omega \theta_i^2 \, dt - 2.1 \int_{t_i}^{t_j} (a\omega \theta_i + h\omega \theta_i')^2 \, dt
\]

and for \( \omega_i = 0, 0 < \omega_i < 1 \) this is evidently negatived by the inequalities (2). The conditions (13) and (14) and equation (2) yield the solution of our problem. If we add (13) and (14) we obtain on dropping dashes

\[
2a\ddot{p} - 2\ddot{p} + (1 - 2.1a)(u_i + u_j) - 2B\ddot{u}_i + 2.1h(u'_i + u'_j) = 0
\]

and by (2) this becomes

\[(14A) \quad 2Ah^2 p'' + h\ddot{p} - a(3 - 2.1a)p = b - 2.1ab - 2B a.\]
This differential equation is the same as (11) but the constants in the solution are determined differently. The general solution is,

\[ p = p_0 + C_1 e^{m_1 t} + C_2 e^{m_2 t}, \]

where \( m_1 \) and \( m_2 \) have the values previously given, and

\[ p_0 = \frac{b - 2Aab - 2Ba}{a(3 - 2Aa)}. \]

For simplicity suppose \( t_1 = 0 \), then if we let \( r_1 = p_1 - p_0 \) and \( r_2 = p_2 - p_0 \), substitution in (15) gives us

\[ C_1 = \frac{r_1 - r_2 e^{m_1 t_2}}{1 - e^{m_1 r_1 - r_2 t_2}}; \quad C_2 = \frac{r_1 - r_2 e^{m_2 t_2}}{1 - e^{m_1 r_1 - r_2 t_2}}. \]

We can therefore say that a unique solution of the problem proposed always exists.

5. Discussion of Solution of Problem 2. It is interesting to note that although (15) is a solution of a differential equation containing a term in \( p' \) while Evans' corresponding result for monopoly is a solution of a differential not involving \( p' \) similar conclusions regarding the solution can be made.

A particular solution of (14A) is the constant \( p = p_0 \), which is the Cournot competition price obtained when the equation of demand does not involve the rate of change of price.*

If we choose the end values such that \( p = p_0 \) etc., it continues to be a solution of our problem. Also, as in the case of monopoly, no solution not identically equal to \( p_0 \) can take on the value \( p_0 \) more than once; in fact, the real non-negative value of \( t \leq t_2 \) for which \( p = p_0 \) is given by

\[ e^{(m_1 - m_2)t} = \frac{C_2}{C_1}. \]

The derivative \( dp/dt \) will be zero for the single real value of \( t \) in the interval \( 0 \leq t \leq t_2 \) for which

\[ e^{(m_1 - m_2)t} = \frac{m_2 C_2}{m_1 C_1}. \]

We can show by using the values of \( C_1 \) and \( C_2 \) in terms of \( r_1 \) and \( r_2 \) that, if \( r_1 \) and \( r_2 \) have opposite signs the graph of price against time cuts the line \( p = p_0 \) once and has no horizontal tangent in the interval, the price continuously increasing or decreasing as the case may be, from \( p_1 \) to \( p_2 \); if \( r_1 \) and \( r_2 \)

have the same sign the graph fails to cross the line $p = p_0$ at all and has one and only one horizontal tangent provided the interval of time is large enough. As is seen by direct computation we must have

$$\frac{m_1 - m_2}{m_1 e^{-m_1 t} - m_2 e^{-m_2 t}} \leq r_2 \leq \frac{m_1 e^{m_1 t} - m_2 e^{m_2 t}}{m_1 - m_2},$$

otherwise there is no horizontal tangent. Since the second derivative does not vanish where $dp/dt$ does we can conclude that $p$ has a maximum in the interval if $dp/dt$ is positive at $t = 0$, and a minimum if $dp/dt$ is negative at $t = 0$. This slope $dp/dt$ has the same algebraic sign as the quantity

$$r_2 = \frac{m_1 e^{m_1 t} - m_2 e^{m_2 t}}{m_1 - m_2}.$$

A comparison of this quantity with the previous inequality shows that when $r_1$ and $r_2$ are both positive $p(t)$ has only a minimum and when $r_1$ and $r_2$ are both negative it has only a maximum. All of the above results are independent of the sign of $h$.

Since (13) and (14) are necessary and sufficient conditions for the solution of the problem, $u_1$ and $u_2$ are not necessarily identical, as functions of $t$, but may have initial values conditioned merely by the fact that their sum is given by (2).

6. **Extension to $n$ Producers.** It is instructive to investigate the phenomenon of competition when $n$ producers are involved, inasmuch as we can derive a formula which includes (15) as well as the monopoly formula of Evans.

Altho each of the problems already discussed can be extended to the case of $n$ producers I shall only give the extension of problem (2).

If we let $u_1, u_2, \ldots, u_n$ represent the respective amounts produced by each competitor then,

$$(2)' \quad u_1 + u_2 + \cdots + u_n = ap + b + kp'.$$

If $\Pi_1, \Pi_2, \ldots, \Pi_n$ denote the respective profits we have,

$$\Pi_i = \int_{t_i}^{t_{i+1}} [pu_i - q(u_i)] dt, \quad i = 1, 2, 3, \ldots, n.$$

where each competitor assumes that the production of the other or others is independent of his and each tries to make his profit a maximum.
Let $\psi_1(t), \psi_2(t), \ldots, \psi_n(t)$ and $\theta(t)$ be continuous functions of $t$ with their derivatives and write,

$$u_i = \bar{u}_i + \omega_i \psi_i(t) \quad \text{where } i = 1, 2, 3, \ldots, n$$

$$p = \bar{p} + \omega \theta(t)$$

where $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n$, and $\bar{p}$ are the values of $u_1, u_2, \ldots, u_n, p$ when $p$ is chosen as a function of $t$ so that $\Pi_i$ will be a maximum when the function $u_i$ alone varies with $p$. Because of (2)' the $n + 1$ families $\omega_1 \psi_1, \omega_2 \psi_2, \ldots, \omega_n \psi_n, \omega \theta(t)$ are always related by

$$\omega_1 \psi_1 + \omega_2 \psi_2 + \cdots + \omega_n \psi_n = \alpha \omega + \beta \omega \theta.$$

The necessary conditions for $\Pi_i$ to be a maximum when $u_i$ alone varies with $p$ are therefore

$$\frac{\partial \Pi_i}{\partial u_1} = \frac{\partial \Pi_i}{\partial u_2} = \cdots = \frac{\partial \Pi_i}{\partial u_n} = 0$$

when $\omega_1 = \omega_2 = \cdots = \omega_n = 0$.

By an analysis similar to that of article (3) we obtain the $n$ necessary and sufficient conditions,

$$-\beta p' + ap + (1 - 2Aa)u_i - Ba + 2Ah u_i' = 0, \quad i = 1, 2, 3, \ldots, n.$$

If we add these $n$ equations and remember that $u_1 + u_2 + \cdots + u_n = ap + b + \beta p'$ we obtain the second order differential equation,

$$-2Ahp' + (n - 1)hp' - a(n + 1 - 2Aa)p = b - 2Aab - nBa.$$

The general solution of this differential equation is,

$$p = p_0 + C_1 e^{m_1 t} + C_2 e^{m_2 t}$$

where

$$p_0 = \frac{b - 2Aab - nBa}{-a(n + 1 - 2Aa)},$$

$$m_1' = \frac{(n - 1) + \sqrt{(n - 1)^2 - 8Aa(n + 1 - 2Aa)}}{4Ah},$$

$$m_2' = \frac{(n - 1) - \sqrt{(n - 1)^2 - 8Aa(n + 1 - 2Aa)}}{4Ah},$$

and $C_1'$ and $C_2'$ are determined as before in terms of $r_1$ and $r_2$.

If $n = 2$ our expression (15)' reduces to the formula discussed for two competitors, and again if $n = 1$ we get the monopoly price already referred to. Again if $n = \infty$, we find that $C_1 \rightarrow 0, C_2 \rightarrow r,$ and $p \rightarrow B$ for all $t \neq t$, —that is, the larger the number of competitors the more rapid is the adjustment to new price levels.
7. Application of Integral Equations. In the present paper we have assumed that the demand is a linear function of the price and the rate of change of price. Another likely hypothesis is to consider the demand as depending not only on the present price but on all previous prices as well. This new hypothesis will lead into the theory of integral equations. Such an assumption is of the form,

$$y(t) = ap(t) + b + \int_{-\infty}^{t} \Phi(t - \tau)p(\tau)d\tau.$$  

where we assume $p(-\infty)$ is finite and $\Phi(t - \tau)$ is negligible if $t - \tau$ is large and negative. This is the usual restriction in treating the phenomena of hysteresis. Consider the problem in which the interval of time extends from $t = -\infty$ to $t = t$, and each $\Pi_t$ is to be a maximum considered as a functional of the corresponding $u_t$. We have then,

$$\Pi_t = \int_{-\infty}^{t} [pu_t - q(u_t)] dt.$$  

Using the methods of this paper as given in article (2) we get,

$$\omega \phi(t) = \frac{u_1 \psi_1 + u_2 \psi_2}{a} - \int_{-\infty}^{t} \omega \theta(t) \frac{\Phi(t - \tau)}{a} d\tau,$$

$$= \frac{u_1 \psi_1 + u_2 \psi_2}{a} + \int_{-\infty}^{t} \frac{u_1 \psi_1(\tau) + u_2 \psi_2(\tau)}{a} \Phi(t - \tau) d\tau,$$

where $\phi(t - \tau)$ is the resolving kernel of $\frac{\Phi(t - \tau)}{a}$.

Then

$$\frac{\partial p}{\partial u_1} = \frac{\psi_1(t)}{a} + \int_{-\infty}^{t} \frac{\psi_1(\tau)}{a} \Phi(t - \tau)d\tau,$$

so that after applying Dirichlet's formula, we get,

$$\frac{\partial \Pi_t}{\partial u_1} = \int_{-\infty}^{t} \psi_1(t)[p + \omega \theta(t) + \frac{u_1}{a} + \frac{u_2 \psi_2}{a} - 2.1(u_1 + u_2 \psi_1) - B$$

$$+ \frac{u_1}{a} \int_{-\infty}^{t} \phi(t - \tau) \psi_1(\tau) d\tau + \int_{-\infty}^{t} \frac{u_1(\tau)}{a} \phi(\tau - t) d\tau] d\tau,$$

and for $u_1 = u_2 = 0$ this becomes

$$\left(\frac{\partial \Pi_t}{\partial u_1}\right)_{u_1 = u_2 = 0} = \int_{-\infty}^{t} [p + u_1/a - 2Au_1 - B + \int_{t}^{t} \frac{u_1(\tau)}{a} \phi(\tau - t) d\tau] \psi_1(t) dt.$$
In the same manner as before we have as necessary conditions for a solution of the problem proposed,
\[ p + u_i/a - 2Bu_i - B + 1/a \int_{t}^{t'} u_i(\tau) \phi(\tau - t) d\tau = 0, \]
\[ p + u_i/a - 2Bu_i - B + 1/a \int_{t}^{t'} u_i(\tau) \phi(\tau - t) d\tau = 0. \]

Adding these two equations gives
\[ 2p + y(t), a - 2Ay(t) - B + 1/a \int_{t}^{t'} y(\tau) \phi(\tau - t) d\tau = 0. \]

Now from (16),
\[ p = \frac{y - b}{a} - \int_{-\infty}^{t'} \frac{\Phi(t - \tau)}{a} p(\tau) d\tau. \]
\[ = \frac{y - b}{a} + \int_{-\infty}^{t'} \frac{y(\tau)}{a} \phi(t - \tau) d\tau. \]

We can therefore write (17) as
\[ y(t) = \frac{B}{3 - 2Aa} + \frac{2B}{3 - 2Aa} \int_{-\infty}^{t'} \phi(t - \tau) d\tau - \frac{2}{3 - 2Aa} \int_{-\infty}^{t} \phi(t - \tau) y(\tau) d\tau. \]

If we define:
\[ f(t) = \frac{2b}{3 - 2Aa} \int_{-\infty}^{t} \phi(t - \tau) d\tau + \frac{B}{3 - 2Aa} \]
\[ K(\tau) = -2\phi(t - \tau) \text{ when } \tau < t \text{ and} \]
\[ = -\phi(t - \tau) \text{ when } \tau > t \text{ and } \lambda = \frac{1}{3 - 2Aa} \]
\[ = 0 \text{ when } \tau = t \]

we can write this Fredholm Integral Equation as,
\[ y(t) = f(t) + \lambda \int_{-\infty}^{t'} K(t, \tau) y(\tau) d\tau. \]
\[ = f(t) - \lambda \int_{-\infty}^{t} K(t, \tau) f(\tau) d\tau, \]

where \( k(t, \tau) \) is the resolving kernel of \( K(t, \tau) \) unless \( \lambda \) happens to be a characteristic value. It is perhaps worth while to mention that \( \lambda \) is a positive fraction less than one (see inequalities given in (2)).

Returning to the earlier problems, if we add (7) and (8) [the necessary conditions for a solution of problem 1, hypothesis (2)] and substitute for \( p \)
its value obtained as a solution of the differential equation (2), the following integral equation for the determination of \( y \) results.

\[
y(t) = f(t) + \int_{t_1}^{t} K(t - \tau) y(\tau) d\tau, \quad \text{where}
\]

\[
f(t) = -\frac{b + aB}{aA} + \frac{ap_2 + aB}{aA} e^{-(a/h)(t-t_1)} \quad \text{and}
\]

\[
K(t - \tau) = \frac{2e^{-a/h}(t-\tau) e^{(a/h)(t-\tau)}}{2Ah}
\]

and \( p_2 \) is the value of \( p \) at \( t = t_2 \), to be determined later. This equation can be solved without appealing to the results already obtained in the discussion of problem 1.

In fact, since \( K(t - \tau) \) satisfies a simple second order differential equation we can obtain the revolving kernel from the fundamental reciprocal formula. Differentiating twice and combining by \( K''(\tau) - a^2 \cdot h^2 K(\tau) = 0 \) gives a second order differential equation whose solution is,

\[
k(t - \tau) = g_1 e^{m_1(t-\tau)} + g_2 e^{m_2(t-\tau)}
\]

where \( m_1 \) and \( m_2 \) are the quantities previously called \( m_1 \) and \( m_2 \), and

\[
g_1 = \frac{2A\lambda m_2 + 6.4a - 1}{4A\lambda^2 h^2 (m_1 - m_2)}; \quad g_2 = \frac{2A\lambda m_1 + 6.4a - 1}{4A\lambda^2 h^2 (m_1 - m_2)}
\]

We can therefore write the solution of (19) as

(20) \[ y(t) = f(t) - \int_{t_1}^{t} k(t - \tau)f(\tau) d\tau \]

where the value of \( p_2 \) is determined as follows:

From the initial conditions and the linear differential equation (2) we obtain

\[
p_2 = p_1 e^{-(a/h)(t-t_1)} + (1/h) e^{-(a/h)t_1} \int_{t_1}^{t_1} e^{(a/h)\tau} y(\tau) d\tau
\]

and by (20) this becomes

\[
p_2 = p_1 e^{-(a/h)(t-t_1)} + (1/h) e^{-(a/h)t_1} \int_{t_1}^{t_1} e^{(a/h)\tau} f(\tau) d\tau - 1/h e^{-(a/h)t_1} \int_{t_1}^{t_2} e^{(a/h)\tau} d\tau \int_{t_1}^{t} k(\tau - z)f(z) dz
\]

an implicit equation for the determination of \( p_2 \).

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**GENERALIZED LA GRANGE PROBLEM**

1. **Introduction.** In the theory of dynamical economics there arises a problem which is a generalization of the LaGrange problem in the Calculus of Variations. Briefly, this problem is that of determining a curve \( f \) in the space \((u_1, u_2, u_3, x)\) satisfying a differential equation

\[ G(u_1, u_1', u_2, u_2', u_3, u_3', x) = 0 \]

such that an integral

\[ \Pi' = \int_{x_0}^{x_1} F_1(u_1, u_1', u_2, u_2', u_3, u_3', x) \, dx \]

is a maximum when \( u_2 \) is not allowed to vary, and such that a second integral

\[ \Pi_2 = \int_{x_0}^{x_1} F_2(u_1, u_1', u_2, u_2', u_3, u_3', x) \, dx \]

is a maximum when \( u_1 \) is not allowed to vary. The function \( u_3 \) and its variations are considered to be determined by the differential equation except for an arbitrary constant. The end point \( x_0 \) and the corresponding end values of \( u_1, (i = 1, 2, 3) \) are considered to be fixed. The end point \( x_1 \) and the corresponding end values of the \( u_1 \) can be regarded as fixed or not, depending upon the nature of the problem under consideration. Both cases will be discussed at some length in the following paragraphs. The functions \( F_1, F_2, \) and \( G \) are assumed to possess continuous derivatives of the third order with respect to \((u_1, u_1', x)\).

2. **Interpretation of Problem.** Let \( u_2 = u_2(x) \) be any function continuous with its first derivative, in the hyperspace \((u_1, u_2, u_3, x)\), and substitute this value of \( u_2 \) in \( F \) and the differential equation \( G = 0 \). The function \( F \) becomes a function \( F'(u_1, u_1', u_2(x), u_2'(x), u_3, u_3', x) \), and
\[ G(u, u', u, u', u, u', u^2, z) = 0 \] becomes the differential equation

\[ G(u, u', u(z), u'(z), u, u', z) = 0 \]

which determines \( u_3 \) in terms of \( u_1 \) and the function \( u_2 \) except for an arbitrary constant. This constant is determined by the initial condition \( u_3(x_0) = u_{30} \).

The problem of finding \( u_1 = y_1(x) \) which maximizes the integral \( \Pi' \) is thus reduced to the problem of finding a function \( y_1(x) \) which maximizes the integral

\[
\Pi' = \int_{x_0}^{x_1} P(u, u', u(z), u'(z), u, u', u', z) \, dz
\]

where \( u_3 \) is determined by \( G = 0 \) and the initial condition in terms of \( y_1(x) \) and the function \( u_2(x) \).

If \( u_1(x) = y_1(x) \) be substituted in \( P \) and \( G = 0 \) they become respectively, \( P_1(y_1(x), y_1'(x), u_2, u_2', u, u', z) \) and \( G(y_1(x), y_1'(x), u_2, u_2', u_3, u_3', x) = 0 \). Choosing the function \( u_2(x) = y_2(x) \) so that it maximizes the integral

\[
\Pi_2 = \int_{x_0}^{x_1} P_2(y_1(x), y_1'(x), u_2, u_2', u_3, u_3', z) \, dz
\]

where \( u_3 \) is determined by \( G = 0 \) and the initial condition, completes the solution of the problem. The curve \( \Gamma \) is the common intersection of the three hyper cylinders, \( u_1 = y_1(x) \); \( u_2 = y_2(x) \); \( u_3 = y_3(x) \), where \( y_1(x) = u_1(x) \) is the solution of the differential equation

\[ G(y_1, y_1', y_2, y_2', u, u', z) = 0 \]

satisfying the initial condition \( u_3(x_0) = u_{30} \).

3. **Admissible Arcs and Variations.** An admissible arc \( u_1 = u_1(x) \) (\( i = 1, 2, 3 \)) is an arc which is continuous on the interval \( x_0 \leq x \leq x_1 \) and is such that the interval can be divided into a finite number of sub intervals on each of which the functions \( u_1(x) \) have continuous derivatives up to those of the third order. All of the elements of the arc must lie in a connected region of the hyperspace \( (u_1, u_2, u_3, z) \) and
satisfy the differential equation

\[ G(u, u', u'', u_1, u_2, u_3, u_4, x) = 0. \]

In the following paragraphs all admissible arcs are to be regarded as fixed at \( x_0 \), i.e., \( u_i(x_0) = u_{i0} \); \( u_2(x_0) = u_{20} \); \( u_3(x_0) = u_{30} \). The behavior of the arcs at \( x_1 \) will be pointed out as the work progresses.

If a two parameter family of admissible arcs

\[ u \approx u_1(x, a, b) \]

containing a particular admissible arc \( \Gamma \) for the parametric values \( a = b = 0 \) be given, the functions

\[ \psi_i(x) = \frac{\partial u_1(x, a, b)}{\partial a} \]
\[ \psi_j(x) = \frac{\partial u_1(x, a, b)}{\partial b} \]

are by definition the partial variations of the family along \( \Gamma \).

To obtain necessary conditions for a solution write

\[ u = y + \psi_1(x, a, b) \]
\[ u = y + \psi_2(x, a, b) \]
\[ u = y + \Theta(x, a, b) \]

where \( \psi \) and \( \Theta \) are functions continuous with continuous derivatives of the first order and vanish when \( a = b = 0 \). The function \( \Theta(x, a, b) \) must satisfy the partial differential equations:

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial a} + \frac{\partial^2 \psi}{\partial x \partial b} + \frac{\partial^2 \psi}{\partial a^2} + \frac{\partial^2 \psi}{\partial b^2} = 0 \]
\[ \frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial x \partial a} + \frac{\partial^2 \Theta}{\partial x \partial b} + \frac{\partial^2 \Theta}{\partial a^2} + \frac{\partial^2 \Theta}{\partial b^2} = 0 \]

and must therefore be continuous and have continuous partial derivatives of the first order because of the continuity requirements on \( G \). The \( \psi_i \) and \( \Theta \) are further defined by the following equations, holding when \( a = b = 0 \):

\[ \frac{\partial \psi}{\partial a} = \psi_i(x) \]
\[ \frac{\partial \psi}{\partial b} = 0 \]
\[ \frac{\partial \Theta}{\partial a} = \Theta(x) \]
\[ \frac{\partial \Theta}{\partial b} = \Theta_b(x) \]
If the end values of \( u_1 \) and \( u_2 \) are fixed at \( x_2 \) as well as at \( x_0 \), it follows that
\[
\psi_1'(x_0) = \psi_1(x_1) = \psi_2(x_0) = \psi_2(x_1) = 0
\]

For the parametric values \( a = b = 0 \) the function \( \Theta(x, a, b) = \Theta(x) \) must satisfy the partial differential equations
\[
\begin{align*}
\frac{\partial}{\partial u_1} \psi_1' + \frac{\partial}{\partial u_1'} \psi_1' + \frac{\partial}{\partial u_2} \Theta_a + \frac{\partial}{\partial u_3} \Theta_b' &= 0 \\
\frac{\partial}{\partial u_1} \psi_2' + \frac{\partial}{\partial u_2} \psi_2' + \frac{\partial}{\partial u_3} \Theta_b + \frac{\partial}{\partial u_3'} \Theta_b' &= 0
\end{align*}
\]

(1)

The first of these determines \( \Theta_a \) in terms of \( \psi_1 \) and the partial derivatives of \( G \) with respect to \( u_1 \) and \( u_1' \), whereas the second determines \( \Theta_b \) in terms of \( \psi_2 \) and the partial derivatives of \( G \) with respect to \( u_2 \) and \( u_2' \). Choosing each of the partial variations \( \Theta_a(x_0) \) and \( \Theta_b(x_0) \) equal to zero implies that the total variation of the function \( \Theta \) be zero at \( x_0 \), i.e.
\[
\delta \Theta = \Theta_a \delta a + \Theta_b \delta b = 0 \quad \text{at} \quad x = x_0.
\]
The equations (1) and the initial conditions therefore completely determine the variations of \( u_3 \).

The functions \( u_1 \) and \( u_2 \) have thus been classified as independent functions as is done in the ordinary theory of maxima and minima of functions. The choice of different independent functions, unlike the ordinary case seems to lead to different problems, for certainly the end values are different.

If the functions \( u_1(x, a, b) \) defining a two parameter family of admissible arcs, containing \( \Gamma \) for the parametric values \( a = b = 0 \), are substituted in \( \Pi \), this integral becomes the function of \( a \) and \( b \) defined by the formula
\[
\Pi = \int_{x_0}^{x_1} \left( \psi_1(x, a, b) u_1'(x, a, b) + \psi_2(x, a, b) u_2'(x, a, b) + \psi_3(x, a, b) u_3'(x, a, b) \right) dx
\]
The partial variation of this integral with respect to
a reduces for \( a = b = 0 \) to the expression

\[
(2) \quad \frac{\partial \Pi}{\partial a} \Delta a = \int_{x_0}^{x_1} \left( \frac{\partial P}{\partial y_1} \psi + \frac{\partial P}{\partial y_3} \phi + \frac{\partial P}{\partial y_3} \phi' + \frac{\partial P}{\partial y_3} \phi'' \right) dx \Delta a
\]

By solving the first differential equation of (1) for
\( \theta_a \), a necessary condition for a maximum value of the integral follows
without the use of the LaGrange multipliers. This procedure avoids the
numerous definitions employed in the classical treatment of the LaGrange
employed in the following analysis problem. The elimination method seems to be more directly an extension
of the theory of maxima of functions, and yields sufficient conditions
for the classical problem as well as for the problem proposed in this
paper.

If \( \frac{\partial G}{\partial y_3} \) is not zero in the interval \( x_0 \leq x \leq x_1 \)
the solution of (1) for \( \theta_a \) is

\[
(3) \quad \theta_a = \int_{x_0}^{x_1} \frac{2 G \cos \psi}{y_3} \left( \frac{\partial G}{\partial y_1} \psi + \frac{\partial G}{\partial y_3} \psi' \right) dx
\]

where as notation
\[
\frac{\partial G}{\partial y_1} = \frac{\partial G}{\partial y_1} \quad \frac{\partial G}{\partial y_3} = \frac{\partial G}{\partial y_3} \quad \frac{\partial G}{\partial y_3} = \frac{\partial G}{\partial y_3}
\]

A differentiation of (3) with respect to \( x \) determines \( \theta' \). By the formula

\[
(4) \quad \theta_a' = \frac{\partial G}{\partial y_1} \sin \psi + \frac{\partial G}{\partial y_3} \sin \psi' + \frac{\partial G}{\partial y_3} \int_{x_0}^{x_1} \left( \frac{\partial G}{\partial y_1} \psi + \frac{\partial G}{\partial y_3} \psi' \right) dx
\]

If these values of \( \theta_a \) and \( \theta_a' \) are substituted in the
expression (2) defining the partial variation of \( \Pi \) with respect to
\( a \), it becomes for \( a = b = 0 \)

...
\[ \frac{\partial \Pi}{\partial a} = \int_{x_0}^{x_1} \left[ \left( \frac{\partial F_1}{\partial y_1} + \frac{\partial F_1}{\partial y_2} + \frac{\partial G_1}{\partial y_1} \right) \psi + \left( \frac{\partial F_1}{\partial y_2} + \frac{\partial F_1}{\partial y_3} + \frac{\partial G_1}{\partial y_2} \right) \psi' \right] \, dx \]

An application of Dirichlet's formula for changing the order of integration followed by an interchange of the parameters of integration, t and x, yields the equation

\[ \frac{\partial \Pi}{\partial a} = \int_{x_0}^{x_1} \left[ \left( \frac{\partial F_1}{\partial y_1} + \frac{\partial F_1}{\partial y_2} + \frac{\partial G_1}{\partial y_1} \right) \psi + \left( \frac{\partial F_1}{\partial y_2} + \frac{\partial F_1}{\partial y_3} + \frac{\partial G_1}{\partial y_2} \right) \psi' \right] \, dx \]

where as notation,

\[ \Psi_1 = \int_{x_0}^{x_1} \left( \frac{\partial F_1}{\partial y_3} + \frac{\partial F_1}{\partial y_2} + \frac{\partial G_1}{\partial y_3} \right) \, dt \]

Now since \( \psi_1(x) \) vanishes at \( x_0 \) and \( x_1 \) by hypothesis, an integration by parts performed on the last term of the partial variation of \( \Pi_1 \), with respect to \( a \), furnishes the expression

\[ \frac{\partial \Pi_1}{\partial a} = \int_{x_0}^{x_1} \left( \frac{\partial F_1}{\partial y_1} + \frac{\partial F_1}{\partial y_2} + \frac{\partial G_1}{\partial y_1} \right) \psi + \left( \frac{\partial F_1}{\partial y_2} + \frac{\partial F_1}{\partial y_3} + \frac{\partial G_1}{\partial y_2} \right) \psi' \, dx \]

If \( \Pi_1 \) is to be a maximum it is necessary that \( \frac{\partial \Pi_1}{\partial a} \) be zero for all values of the function \( \psi_1(x) \). It follows then that the integrand must vanish, or that is,

\[ \left( \frac{\partial F_1}{\partial y_1} + \frac{\partial F_1}{\partial y_2} + \frac{\partial G_1}{\partial y_1} \right) \Psi_1 + \left( \frac{\partial F_1}{\partial y_2} + \frac{\partial F_1}{\partial y_3} + \frac{\partial G_1}{\partial y_2} \right) \Psi_1 = 0 \]

In an entirely analogous manner \( \Pi_2 \) yields as a necessary condition for the solution of the problem.
The differential equations (5) and (6) of the second order and the differential equation \( G(y, y', y''; y_1, y_2, y_3, x) = 0 \) determine the solution if such a solution exists, for the end values \( u_1(x_0) = u_{10} \), \( u_1(x_1) = u_{11} \), \( u_2(x_1) = u_{21} \). The end value \( u_3(x_1) \) is determined by \( G = 0 \) in terms of the end values of \( u_1 \) and \( u_2 \).

If in particular \( F_1 = F_2 \), this problem reduces to a Lagrange problem in the Calculus of Variations. Since no assumption which would prevent this has been made, the equations resulting from (5) and (6) by letting \( F_1 = F_2 \) are necessary in order that a curve satisfying a differential equation \( G(u_1, u_1', u_2, u_2', u_3, u_3', x) = 0 \) and the initial conditions \( u_1(x_0) = u_{10} \), \( u_1(x_1) = u_{11} \), \( u_2(x_1) = u_{21} \), furnish a maximum of an integral

\[
I = \int_{x_0}^{x_1} F(u, u_1, u_2, u_3, u_1', u_2', u_3') \, dx
\]

in which both \( u_1 \) and \( u_2 \) vary independently.

The methods of this paragraph can be extended immediately to the case of \( n \) integrals

\[
\Pi_i = \int_{x_0}^{x_1} F_i(u_1, u_1', \ldots, u_i, u_i', \ldots, u_n, u_n', u_{n+1}, u_{n+1}') \, dx
\]

\((i = 1, 2, \ldots, n)\) and one differential equation \( G(u_1, u_1', \ldots, u_n, u_{n+1}, u_{n+1}', x) = 0 \), in which case \( n \) equations of the type (5) result.

4. **Variable End Points.** In the preceding paragraphs a problem in simultaneous maxima for fixed end points has been discussed. The problem is still possible when one end point, say \( x_2 \), and the corresponding end values are allowed to vary.
Consider the problem of determining a curve \( \Gamma \) in the space \((u_1, u_2, u_3, u_4, x)\) satisfying a differential equation
\[
G(u_1, u_2', u_3, u_2', u_3', u_4', u_4', x) = 0
\]
such that an integral
\[
\Pi_1 = \int_{x_0}^{x_1} F_1(u_1, u_2, u_3, u_2', u_3', u_4', u_4', x) \, dx
\]
is a maximum when \(u_1\) and \(u_2\) are allowed to vary but not \(u_3\) and such that a second integral
\[
\Pi_2 = \int_{x_0}^{x_1} F_2(u_1, u_2, u_3, u_2', u_3', u_4, u_4', x) \, dx
\]
is a maximum when \(u_3\) is allowed to vary but not \(u_1\) and \(u_2\). The function \(u_4\) is considered to be determined by the differential equation \(G = 0\) and the initial condition \(u_4(x_0) = u_{40}\), in terms of \(u_1, u_2, u_3\), and \(u_4\). The end point \(x_1\) and the corresponding end values are variable but not the end point \(x_0\). As already stated the end values \(u_1(x_0) = u_{10}\) are assumed to be given.

Replace the functions \(u_i(x)\), \((i = 1, 2, 3, 4)\) by the set \(u_i = f_i(x, a, b, c)\) where the \(f_i(x, a, b, c)\) are functions of \(x\) and the parameters \(a, b,\) and \(c\), continuous and admitting continuous partial derivatives up to the third order with respect to \(x\) and these parameters in the domain \(0 \leq a < b < c < x_1\), \(x_0 \leq x_1\). \(x_1 = \phi_1(a, b, c)\)

By the ordinary rules of differentiation, the total differential of the integral \(\Pi_1\) which is also a function of \(a, b,\) and \(c\) is

\[
\frac{d}{dx} \left( \int_{x_0}^{x_1} F_1(u_1, u_2', \ldots, u_4', x) \, dx \right) + \int_{x_0}^{x_1} \left( \frac{\partial F_1}{\partial u_1} f_1 + \frac{\partial F_1}{\partial u_2} f_2 + \ldots + \frac{\partial F_1}{\partial u_4} f_4 \right) \, dx
\]
where \( \delta f'_4 = \frac{\partial G}{\partial u_4} \delta a + \frac{\partial G}{\partial u_k} \delta b + \frac{\partial G}{\partial u_k} \delta c \) and \( i \) is an umbral symbol for the values 1, 2, 4 but not 3. The variation of \( u_3 \) in \( F_1 \) is by hypothesis equal to zero, hence \( \delta f_3 = 0 \).

Now the variations \( \delta f'_4 \) and \( \delta f'_2 \) are to be arbitrary but the variation \( \delta f'_4 \) is to be determined by the differential equation

\[
\frac{\partial G}{\partial u_4} \delta f'_4 + \frac{\partial G}{\partial u_k} \delta f'_2 + \frac{\partial G}{\partial u_k} \delta f'_k = 0
\]

where \( k \) is an umbral symbol for the values 1 and 2 only. Assume as is usually done that \( \frac{\partial G}{\partial u'_4} \neq 0 \) in the interval \( x_0 < x < x_1 \).

Since \( \frac{\delta f}{dx} = \frac{d}{dx} \delta f \) this expression can be regarded as a first order differential equation for the determination of \( \delta f'_4 \) in terms of \( \delta f'_1 \) and \( \delta f'_2 \). The variation \( \delta f'_4 \) is therefore determined except for a constant, hence if the variation \( \delta f'_4 \) at \( x_0 \) is zero, then

\[
\delta f'_4 = \int_{x_0}^{x} e^{\int_{x_0}^{t} \frac{\partial G}{\partial u_4} \delta f'_k + \frac{\partial G}{\partial u_k} \delta f'_k} dt
\]

where \( \frac{\partial G}{\partial u'_4} \) has the same meaning as in the first part of this article.

By a differentiation with respect to \( x \)

\[
\delta f'_4 = \frac{\partial G}{\partial u_4} \delta f'_k + \frac{\partial G}{\partial u_k} \delta f'_k + \frac{\partial G}{\partial u_k} \int_{x_0}^{x} e^{\int_{x_0}^{t} \frac{\partial G}{\partial u_4} \delta f'_k + \frac{\partial G}{\partial u_k} \delta f'_k} dt
\]

If these values of \( \delta f'_4 \) and \( \delta f'_2 \) are substituted in (7) it becomes,
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\[ \int_{x_0}^{x_1} \int_{x_0}^x \left[ \left( \frac{\partial F}{\partial u_k} + \frac{\partial F}{\partial u_4} \frac{\partial G}{\partial u_k} \right) \delta f_k + \left( \frac{\partial F}{\partial u_k} + \frac{\partial F}{\partial u_4} \frac{\partial G}{\partial u_k} \right) \delta f'_k \right] dx \]

\[-\left( \frac{\partial F}{\partial u_4} + \frac{\partial F}{\partial u_4} \frac{\partial G}{\partial u_4} \right) \int_{x_0}^x \left[ \int_{t}^{x} \frac{\partial G}{\partial u_4} ds \right] \left( \frac{\partial G}{\partial u_k} \delta f_k + \frac{\partial G}{\partial u_k} \delta f'_k \right) dt \right] dx.\]

An application of Dirichlet's formula to the iterated integral followed by an interchange of the parameters \( x \) and \( t \) reduces this integral to the expression

\[ \int_{x_0}^{x_1} \int_{x_0}^x \left[ \left( \frac{\partial F}{\partial u_k} + \frac{\partial F}{\partial u_4} \frac{\partial G}{\partial u_k} + \frac{\partial G}{\partial u_4} \frac{\partial G}{\partial u_k} \right) \delta f_k + \left( \frac{\partial F}{\partial u_k} + \frac{\partial F}{\partial u_4} \frac{\partial G}{\partial u_k} + \frac{\partial G}{\partial u_4} \frac{\partial G}{\partial u_k} \right) \delta f'_k \right] dx \]

where \( W_1 = \int_{x_0}^x \int_{t}^{x} \frac{\partial G}{\partial u_4} ds \left( \frac{\partial F}{\partial u_4} + \frac{\partial F}{\partial u_4} \frac{\partial G}{\partial u_4} \right) dt. \)

Now if the \( f_k(x,a,b,c) \) have continuous second derivatives with respect to \( x \), the formula for integration by parts can be applied to the second member of the above. It follows then, that

\[ \int_{x_0}^{x_1} \left[ \frac{\partial F}{\partial u_k} + \frac{\partial F}{\partial u_4} \frac{\partial G}{\partial u_k} + \frac{\partial G}{\partial u_4} W_1 \right] \delta f_k \left. \right|_{x_0}^{x_1} + \]

\[ \int_{x_0}^{x_1} \left[ \frac{\partial F}{\partial u_k} + \frac{\partial F}{\partial u_4} \frac{\partial G}{\partial u_k} + \frac{\partial G}{\partial u_4} W_1 - \frac{d}{dx} \left( \frac{\partial F}{\partial u_k} + \frac{\partial F}{\partial u_4} \frac{\partial G}{\partial u_k} + \frac{\partial G}{\partial u_4} W_1 \right) \right] \delta f_k dx. \]

In the preceding analysis the end point \( x_1 \) has been taken as a function of the parameters \( a,b \) and \( c \) so that the end point \( x_1 \) of (4k) the space \( (u_1,u_2,u_3,u_4,x) \) describes a curve \( C \) given the extremals in
by the equations

\[ \begin{align*}
  a &= a(t) \quad ; \quad b = b(t) \quad ; \quad c = c(t) \quad ; \quad x = x(t) \\
  u_1 &= f_1(x_1(t), a(t), b(t), c(t)) = f_1(t) \\
  u_3 &= u_3(x_1) 
\end{align*} \]

Now the variation of \( u_k \) along \( C \) is given by

\[ \delta u_k = u_k' \delta x + \delta f_k \]

for by definition

\[ \delta f_k = \frac{\partial f_k}{\partial a} \delta a + \frac{\partial f_k}{\partial b} \delta b + \frac{\partial f_k}{\partial c} \delta c \]

If this value of \( \delta f_k \) is substituted in \( \delta \Pi \), it becomes

\[ \delta \Pi = \left[ F \delta x + \left( \frac{\partial F}{\partial u_k} u_k' \delta x + \frac{\partial F}{\partial u_k} \delta u_k \right) \right] \bigg|_{x_0}^{x_1} + \int_{x_0}^{x_1} \left( \delta f_k \frac{\partial f_k}{\partial u_k} + \frac{\partial f_k}{\partial u_k} \delta u_k + \frac{\partial f_k}{\partial u_k} \delta u_k \right) \, dx. \]

Now along an extremal arc the expression

\[ \frac{\partial F}{\partial u_k} u_k' + \frac{\partial F}{\partial u_k} \frac{\partial G}{\partial u_k} \delta u_k - \frac{d}{dx} \left( \frac{\partial F}{\partial u_k} + \frac{\partial F}{\partial u_k} \frac{\partial G}{\partial u_k} \right) \delta u_k = 0 \]

\( k = 1 \), and \( \delta \) by the definition of an extremal, for

\[ \delta \Pi = 0 \]

An arc \( u_k = u_k(x) \) is called an extremal if it has continuous derivatives \( u_k' \) and \( u_k'' \) on the interval \( x_0 = x \leq x_1 \) and if furthermore it satisfies the set of equations (8) and the differential equation

\[ G(u_k, u_k', u_k'', u_k', u_k', u_k', \delta u_k) = 0. \]

If it is assumed that \( \delta u_k \) is zero at the fixed end value \( x_0 \), all extremals issue from a fixed \( x_0 \) since \( \delta u_k(x_0) \) has already been assumed to be zero. The following theorem therefore results.

The value of the integral \( \Pi \), taken along a three parameter family of extremal arcs \( E_{01} \), one of whose end points is fixed while
the other describes a curve $C$ has the differential
\[ d\Pi = F_1(u_1, p_1, u_2, p_2, u_3, u_3', u_4, p_4, x_1)dx_1 + (2\frac{\partial F}{\partial u_1} + 2\frac{\partial F}{\partial u_2} \frac{\partial \Phi}{\partial u_3} + 2\frac{\partial F}{\partial u_4} \frac{\partial \Phi}{\partial u_3'}) (du_1 - p_2 dx_1) \]
where at the point $1$ the differentials $dx_1$ and $du_1$ are those belonging to $C$, whereas the $u_1$, $u_3$, $p_1$ and $u_3'$ refer to the extremal $E_{01}$. The functions $F_1$ and $G$ have arguments $(u_1, p_1, u_2, p_2, u_3, u_3', u_4, p_4, x_1)$.

The integral of $d\Pi_1$ corresponds to the Hilbert integral and possesses similar properties. In a way analogous to the method of the Calculus of Variations it is possible to obtain the analogues of the necessary conditions of Weierstrass and Legendre and to obtain sufficient conditions for strong and weak maxima.

By an entirely analogous treatment it follows that:

The value of the integral $d\Pi_1$ taken along a three-parameter family of extremal arcs $E_{01}$, one of whose endpoints is fixed while the other describes a curve $C$ has the differential
\[ d\Pi_2 = F_2 dx_1 + (2\frac{\partial F}{\partial u_3} + 2\frac{\partial F}{\partial u_4} \frac{\partial \Phi}{\partial u_3} + 2\frac{\partial F}{\partial u_3'} \frac{\partial \Phi}{\partial u_3}) (du_3 - p_3 dx_1) \]
where at the point $1$ the differentials $dx_1$ and $du_3$ are those belonging to $C$, whereas the $u_1$, $u_3'$, $u_3$ and $p_3$ refer to the extremal $E_{01}$. The functions $F_2$ and $G$ have arguments $(u_1, u_3', u_2, u_3, p_3, u_4, p_4, x_1)$.

5. **Necessary Conditions of Weierstrass.** A connected region $R$ of the space $(u_1, u_2, u_3, u_4, x)$ is a simply covered extremal field if there exist a family of extremals dependent on two parameters such that one and only one extremal of this field pass thru every point of $R$ and if the angular coefficients $p_h(u_1, u_2, u_3, u_4, x)$, $h=1, 2, 3, 4$, of the tangent to the extremal which passes thru the point $(u_1, u_2, u_3, u_4, x)$, are continuous functions admitting continuous partial derivatives in $R$ up to the second order.
It is quite evident that along an extremal arc of a field
the integral \( \Pi_i \) has the same value as \( \Pi_i^{(p)} \) for \( 6u_i = p_i \delta x \)
along an extremal and the integrand of \( \Pi_i \) thus reduces to the inte-
TEGRAND of \( \Pi_i^{(p)} \).

To obtain the analogue of the Weierstrass\( \text{e} \) conditions select
a point \( l \) so near to 0 that there is no corner of \( E_{01}^{(p)} \) between them
and thru this point \( l \) pass an arbitrary curve \( C_{12} \) with equations
\[
\begin{align*}
\mathbf{u}_1 &= \mathbf{u}_1(x) ; \\
\mathbf{u}_2 &= \mathbf{u}_2(x) ; \\
\mathbf{u}_3 &= \mathbf{u}_3(x) ; \\
\mathbf{u}_4 &= \mathbf{u}_4(x)
\end{align*}
\]
and join the fixed point \( C \) to a movable point \( 2 \) on \( C \) by a three-parameter
family of arcs \( E_{20} \) containing \( E_{01} \) as a member when the point \( 2 \) is in
the position \( l \).

Now if \( E_{01} \) is to give a maximum, it is clear
that the inequality \( \Pi_i^{(p)} (E_{02} + C_{21}) \geq \Pi_i^{(p)} (E_{01}) \) must
hold, hence \( d\Pi_i^{(p)} (E_{02} - C_{21}) \geq 0 \) or, that is
\( d\Pi_i^{(p)} (C_{12} - E_{02}) \leq 0 \) must hold.

This differential is given by the value at the point \( l \)
of the quantity
\[
\begin{align*}
P_l (\mathbf{u}_1, \mathbf{u}_1', \mathbf{u}_2, \mathbf{u}_2', \mathbf{u}_3, \mathbf{u}_3', \mathbf{u}_4, \mathbf{u}_4', \mathbf{x}) 6x &= P_l (\mathbf{u}_1, \mathbf{u}_1', \mathbf{u}_2, \mathbf{u}_2', \mathbf{u}_3, \mathbf{u}_3', \mathbf{u}_4, \mathbf{u}_4', \mathbf{x}) 6x \\
&= -(\frac{\partial P_l}{\partial \mathbf{u}_k} + \frac{\partial P_l}{\partial \mathbf{u}_4} \frac{\partial \mathbf{G}^i}{\partial \mathbf{u}_k} + \frac{\partial P_l}{\partial \mathbf{u}_4} \frac{\partial \mathbf{G}^j}{\partial \mathbf{u}_k} \mathbf{u}_k - \mathbf{u}_k' 6x)
\end{align*}
\]
the differentials in this expression belonging to the arc \( C \) and,
therefore, satisfying the equation \( 6u_k = \mathbf{u}_k' 6x \). At the point \( l \) the
coordinates of \( C \) and \( E \) are equal, and \( \Pi_i \) is zero so that this expres-
sion becomes
\[
\begin{align*}
\int P_l (\mathbf{u}_1, \mathbf{u}_1', \mathbf{u}_2, \mathbf{u}_2', \mathbf{u}_3, \mathbf{u}_3', \mathbf{u}_4, \mathbf{u}_4', \mathbf{x}) - P_l (\mathbf{u}_1, \mathbf{u}_1', \mathbf{u}_2, \mathbf{u}_2', \mathbf{u}_3, \mathbf{u}_3', \mathbf{u}_4, \mathbf{u}_4', \mathbf{x}) \\
= (\mathbf{u}_k' - \mathbf{u}_k) (\frac{\partial P_l}{\partial \mathbf{u}_k} + \frac{\partial P_l}{\partial \mathbf{u}_4} \frac{\partial \mathbf{G}^i}{\partial \mathbf{u}_k} ) 6x.
\end{align*}
\]
This function corresponds to the Weierstrasse E function. Since this differential must be negative or zero for an arbitrarily selected point \( l \) and arc \( C \) thru it, it follows that,

\begin{equation}
(11) \text{At every element } (u_1, u'_1, u_2, u'_2, u_3, u'_3, u_4, u'_4, x) \text{ of an arc } E_0 \text{ which}
\end{equation}

maximizes an integral

\[
\int_{x_0}^{X_1} F_1(u_1, u'_1, u_2, u'_2, u_3, u'_3, u_4, u'_4, x) \, dx
\]

when \( u_3 \) is not allowed to vary and satisfies a differential equation

\[
G(u_1, u'_1, u_2, u'_2, u_3, u'_3, u_4, u'_4, x) = 0
\]

the condition

\[
E(u_1, u'_1, u_2, u'_2, u_3, u'_3, u_4, u'_4, x) = 0
\]

must be satisfied for every admissible set \((u_1, u'_1, u_2, u'_2, u_3, u'_3, u_4, u'_4, x)\) different from \((u_1, u'_1, u_2, u'_2, u_3, u'_3, u_4, u'_4, x)\) for all values of the coordinates \((u_1, u_2, u_3, u_4, x)\) in the region \( R \).

An analysis similar to the preceding shows that,

\[
E(u_1, u'_1, u_2, u'_2, u_3, u'_3, u_4, u'_4, x) = 0
\]

must be satisfied in order that an arc \( f \) satisfy a differential equation

\[
G(u_1, u'_1, u_2, u'_2, u_3, u'_3, u_4, u'_4, x) = 0 \quad \text{and maximize}
\]

an integral \( \mathcal{J} = \int_{x_0}^{X_1} F_1(u_1, u'_1, u_2, u'_2, u_3, u'_3, u_4, u'_4, x) \, dx \)

in which all of the functions \( u_h \) \((h=1, 2, 3, 4)\) are allowed to vary. This condition is therefore a necessary condition for the problem of Lagrange.

Return now to the problem of simultaneous maxima and consider the second integral of the problem proposed. A similar analysis to that employed in the treatment of the first integral shows that the inequality

\[
F_2(u_1, u'_1, u_2, u'_2, u_3, u'_3, u_4, u'_4, x) - F_2(u_1, u'_1, u_2, u'_2, u_3, u'_3, u_4, u'_4, x)
\]

\[
(12) \quad (u'_3 - u'_4) (\frac{\partial F_2}{\partial u'_3} + \frac{\partial F_2}{\partial u'_4} \frac{G'_4}{u'_3}) \leq 0
\]

must be satisfied for every admissible set \((u_1, u'_1, u_2, u'_2, u_3, u'_3, u_4, u'_4, x)\)
different from \((u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x)\) at every element \((u_1, u_2, u_3, u_4, x)\) of a maximizing arc in the region \(R\).

The conditions (8), (11) and (12) are necessary conditions for a solution of the problem proposed in this paper. In the following paragraph a fourth necessary condition will be obtained.

6. Necessary Condition of Legendre. In order to save space consider only the first integral. If the function \(F_1(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x)\) be expanded by means of Taylor's formula the following expression is obtained,

\[
F_1(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x) = F_1(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x) + \\
\left( u_k' - u_k^{1}\right) \left( \frac{\partial F_1}{\partial u_k^1} + \frac{\partial F_1}{\partial u_1^1} \frac{\partial u_1^1}{\partial u_k^1} \right) + \\
\frac{1}{2} \left( u_k' - u_k^{1}\right) \left( u_j' - u_j^{1}\right) \frac{\partial^2 F_1}{\partial u_k^1 \partial u_j^1} A_{u_j^1}^{u_k^1}
\]

where \(A_{u_j^1}^{u_k^1} = \frac{\partial^2 F_1}{\partial u_j^1 \partial u_k^1} + \frac{\partial^2 F_1}{\partial u_1^1 \partial u_j^1} \frac{\partial u_1^1}{\partial u_k^1}\) and \(j\) and \(k\) are umbral symbols of index 2.

The arguments of \(A_{u_j^1}^{u_k^1}\) are \((u_1, u_1', u_2, u_2', \ldots, x)\) where \(0 < \theta < 1\).

Now in as much as the partial derivative of \(G\) with respect to \(u_k\) determines \(\frac{\partial u_k^1}{\partial u_k^1}\), the function \(E\) is given by the formula

\[
E(u_1, u_1', u_2, u_2', \ldots, u_4, u_4', x) = \frac{1}{2} \left( u_k' - u_k^{1}\right) \left( u_j' - u_j^{1}\right) \frac{\partial^2 F_1}{\partial u_k^1 \partial u_j^1} A_{u_j^1}^{u_k^1}.
\]

If the extremal \(E_{01}\) makes \(T_1\) a maximum when \(u_4\) is not allowed to vary and at the same time makes \(T_2\) a maximum when \(u_2\) and \(u_3\) are not allowed to vary, it is necessary that the quadratic forms

\[
(13) \quad T_2 \left( \frac{\partial^2}{\partial u_2^1} A_{u_2^1}, T_2 \left( \frac{\partial^2}{\partial u_1^1} A_{u_1^1} \right) + \frac{\partial^2}{\partial u_2^1} A_{u_2^1} \right) + \frac{\partial^2}{\partial u_3^1} A_{u_3^1} \right) + \frac{\partial^2}{\partial u_2^1} A_{u_2^1} \right)
\]

\[
(14) \quad T_2 \left( \frac{\partial^2}{\partial u_3^1} A_{u_3^1} \right)
\]

be definite negative forms for all systems of finite values of \(u_1', u_2', u_3', \) and
u^4 where the point \((u_1^4, u_2^4, u_3^4, u_4^4, x)\) remains in the domain \(R\).

Now, if \(U^4\) is allowed to approach \(u^4\) in the above quadratic forms, expressions corresponding to the Legendre condition result.

7. **Sufficient Conditions For Simultaneous Maxima.** By definition an extremal curve \(E^4\), furnishes a weak maximum for the integral \(\Pi^4\) when \(u^4\) is not allowed to vary, if there exist a positive number \(\varepsilon\) such that the integral

\[
\Pi^4 = \sum_{x_0}^{x_1} F_1(u_1, u_1^4, u_2^4, u_3^4, u_4, u_4^4, x) dx
\]

is less than the integral

\[
\sum_{x_0}^{x_1} F_1(u_1 + \varepsilon(x), x, u_3^4, u_4 + \varepsilon(x), x) dx
\]

for all possible forms of the functions \(w_j(x)\) of class \(I\) in the interval \((x_0, x_1)\) and satisfying the conditions

\[(15) \quad w_j(x_0) = 0 ; \quad |w_j(x)| < \varepsilon ; \quad |w_j'(x)| < \varepsilon \quad \text{for} \quad x_0 \leq x \leq x_1\]

When the functions \(w_j(x)\) satisfy the conditions

\[
w_j(x_0) = 0 ; \quad |w_j(x)| < \varepsilon \quad \text{but not} \quad |w_j'(x)| < \varepsilon \quad \text{for} \quad x_0 \leq x \leq x_1\]

and the other conditions of the above are satisfied the maximizing arc furnishes a strong maximum for the integral \(\Pi^4\).

By means of the above definitions it is possible to write sufficient conditions for both Weak Relative Maxima and Strong Relative Maxima, in fact,

If \(E_{01}\) is an extremal arc without vertices containing no point 2 conjugate to 1 (in the ordinary sense) and if the conditions (11) and (12) without the equality sign are satisfied at every element \((u_1^4, u_1^4, \ldots, u_4^4, u_4^4, x)\) in a neighborhood \(R^4\) of those on \(E_{01}\) for every admissible set \((u_1^4, U_1^4, \ldots, u_4^4, U_4^4, x)\) such that in (11)

\[
U_1^4 \neq u_1^4 ; \quad U_2^4 \neq u_2^4 ; \quad U_3^4 = u_3^4 ; \quad \text{and} \quad U_4^4 \neq u_4^4
\]
and in \((13)\), \(u'_1 = u'_2 \neq u'_3 \neq u'_4\),

then \(F_{(1)}\) is a Strong Relative Maximum when \(u_3\) is not allowed to vary, and \(F_{(2)}\) is a Strong Relative Maximum when \(u_1\) and \(u_2\) are not allowed to vary.

If \(F_{(0)}\) is an extremal arc without vertices containing no point \(z\) conjugate to 1 and if the Legendre conditions \((13)\) and \((14)\) without the equality sign are satisfied at every set of values \((u'_1, u'_2 \ldots u'_4, x)\) on this arc, then \(F_{(1)}\) is a Weak Relative Maximum when \(u_3\) is not allowed to vary, and \(F_{(2)}\) is a Weak Relative Maximum when \(u_1\) and \(u_2\) are not allowed to vary.

Although the above sufficient conditions apply strictly to the Generalized Lagrange Problem, they can be made to apply to the classic problem by slight modification. In fact, allowing \(u_3\) to vary in \(F_{(1)}\) requires that the subscript \(k\) in \((11)\) and \((13)\) take on the values 1, 2, 3 instead of 1 and 2 only. The arguments of \(F_{(1)}\) and \(G\) in these relations must, of course, be changed so that \(U'\) is accorded its proper place.

3. Problem of Lagrange for More Than One Differential Equation. The analysis of the preceding paragraphs applies to the problem of Lagrange for one differential equation. By introducing the theory of Volterra integral equations this analysis can be modified to the Lagrange problem for more than one differential relation, and to a problem in which the differential relation is replaced by an integral relation. In as much as the method employed in solving the problem for two differential equations is perfectly general, this problem will be discussed in order to save notation.

Briefly this problem is that of determining a curve \(F_{(2)}\) in the space \((u'_1, u'_2, u'_3, x)\) which satisfies two differential equations.
\[ G_k(u_1, u_1', u_2, u_2', u_3, u_3', x) = 0 \quad (k = 1, 2, \ldots) \]

and which furnishes a maximum for an integral

\[ I = \int_{x_0}^{x_1} F(u_1, u_1', u_2, u_2', u_3, u_3', x) \, dx. \]

The \( G_k \) are assumed to be functionally independent and to possess continuous second order partial derivatives with respect to \( u_1, u_1', u_2, u_2', u_3, u_3', x \).

Let \( E_\alpha \) be the curve defined by the equations

\[
\begin{align*}
  u_1 &= y_1(x) \\
  u_p &= z_p(x) & (p = 2, 3, \ldots)
\end{align*}
\]

if such curves exist and write

\[
\begin{align*}
  u_1 &= y_1 + \theta(x, a) \\
  u_p &= z_p + f_p(x, a) & (p = 2, 3, \ldots)
\end{align*}
\]

where \( \theta \) and \( f_p \) are functions continuous with their second derivatives with respect to \( u_1, u_1', \ldots, x \) and which vanish when \( a \) vanishes. This notation for the \( u_1 = (1, 2, 3) \) is used to indicate that \( u_1 \) is to be regarded as the independent function and that the \( u_p \) are to be regarded as determined by the \( G_k \) and the initial conditions \( u_p(x_0) = u_{po} \).

If these values of \( u_1 \) are substituted in \( F \), the integral \( I \) becomes a function of the parameter \( a \) and yields on differentiation with respect to this parameter

\[
\frac{\partial I}{\partial a} = \int_{x_0}^{x_1} \left[ \frac{\partial F}{\partial y} \theta_x + \frac{\partial F}{\partial y'} \theta_x' + \sum \left( \frac{\partial F}{\partial z_p} \theta_{z_p} + \frac{\partial F}{\partial z_p'} \theta_{z_p}' \right) \right] \, dx
\]

where \( \theta_x = \frac{\partial \theta}{\partial x} \theta_a \) and \( \theta_{z_p} = \frac{\partial \theta}{\partial z_p} \theta_a \) at \( a = 0 \).

Now if the variations of \( u_1 \) are considered to be independent, the variations of \( u_2 \) and \( u_3 \) are determined by the differential equations \( G_k = 0 \) in terms of \( \theta \) and the initial conditions.
3. **Dependent Variations by Theory of Volterra Integral Equations.** If the $G_k$ be differentiated parametrically the following set of equations result, for $a \geq 0$

$$
(15) \quad \frac{\partial G_k}{\partial y} \delta \phi + \frac{\partial G_k}{\partial y} \delta \psi + \frac{1}{2} \sum_{i=1}^{2} \frac{\partial G_k}{\partial x_i} \delta f_x + \frac{\partial G_k}{\partial x_i} \delta f'_x = 0
$$

In the classical treatment of this problem, Lagrangian multipliers are introduced at this stage but they can be advantageously avoided by integrating equation (15) with respect to $x$. If $x$ is replaced by $s$ under the integral sign

$$
\sum_{i=1}^{2} \frac{\partial G_k}{\partial x_i} \delta f_x + \frac{\partial G_k}{\partial y} \delta \phi + \int_{x_0}^{x_1} \left( \frac{\partial G_k}{\partial y} - \frac{\partial G_k}{\partial s} \frac{\partial G_k}{\partial s} \right) \delta \psi ds + \int_{x_0}^{x_1} \frac{\partial G_k}{\partial x_i} \delta f'_x ds,
$$

for the $\delta f_x$ must vanish at $x_0$ if the determinant $\Delta G = \left| \frac{\partial G_k}{\partial x_i} \right|$ is not zero in the interval $x_0 \leq x \leq x_1$.

The variations $\delta f_x$ are then determined by the system of Volterra integral equations

$$
\delta f_x = \hat{\phi}_x(x) + \sum_{p=1}^{2} \int_{x_0}^{x_1} K_{xp}(x,s) \delta f_p(s) ds
$$

where

$$
\hat{\phi}_x(x) = \sum_{h=1}^{N} A_{hr} \frac{\partial G_h}{\partial y} \delta \phi + \int_{x_0}^{x_1} \left( \frac{\partial G_h}{\partial y} - \frac{\partial G_h}{\partial s} \frac{\partial G_h}{\partial s} \right) \delta \psi ds,
$$

$$
K_{xp}(x,s) = \sum_{h=1}^{N} A_{hr} \left( \frac{\partial G_h(s)}{\partial s} - \frac{3}{2} \frac{\partial G_h(s)}{\partial s} \right)
$$

and $A_{hr}$ is the cofactor of the $hr$th element of $G$ divided by the determinant $\Delta G$.

These integral equations form a Volterra system of the second type for the determination of the variations $\delta f_x$, uniquely if the ker-
nals \( K_{rp}(x,s) \) are finite and integrable in the interval \( x_0 < s < x < x_1 \).

Now if the determinant \( \Delta G \neq 0 \) in this interval, the \( K_{rp}(x,s) \) will be finite and integrable in the interval on account of the continuity requirements on the \( G_i \). The unique solution of the system is therefore

\[
\delta f_r(x) = \phi_r(x) + \sum_{p=1}^{\infty} \int_{x_0}^{x_1} S_{rp}(x,s) Q_p(s) \, ds
\]

where \( S_{rp}(x,s) \) is the resolvent kernel of \( K_{rp}(x,s) \) defined by the equations

\[
S_{rp}(x,s) = \int_{x_0}^{x_1} K_{rh}(x,t) S_{rp}(t,s) \, dt
\]

\[
S_{rp}(x,s) = \sum_{k=0}^{\infty} K_{rp}(x,s)
\]

It follows then that

\[
\delta f_r(x) = \sum_{h=1}^{n} A_{hr} \frac{\partial G_h}{\partial y} \delta \Theta + A_{hr}(x) \int_{x_0}^{x_1} \frac{\partial G_h(s)}{\partial y} - \frac{d}{ds} \frac{\partial G_h(s)}{\partial y} \, ds
\]

\[
\int_{x_0}^{x_1} A_{hr}(s) S_{rp}(x,s) \left( \frac{\partial G_h(s)}{\partial y} - \frac{d}{ds} \frac{\partial G_h(s)}{\partial y} \right) \delta \Theta \, ds
\]

If Dirichlet’s formula be applied to the iterated integral and the parameters of integration interchanged, the variation of \( f_r \) becomes

\[
\delta f_r(x) = \Phi_r(x) \delta \Theta + \int_{x_0}^{x_1} V_r(x,s) \delta \Theta \, ds
\]

where

\[
\Phi_r(x) = \sum_{h=1}^{n} A_{hr} \frac{\partial G_h}{\partial y}
\]
\[ V(x,a) = \sum_{i=1}^{n} A_{i} \frac{\partial G_{i}}{\partial y} \left( \frac{\partial G_{i}}{\partial y} - \frac{d}{ds} \frac{\partial G_{i}}{\partial y} \right) + A_{i} \frac{\partial G_{i}}{\partial y} \frac{\partial G_{i}}{\partial y} - \frac{d}{ds} \frac{\partial G_{i}}{\partial y} \right) \]

By differentiation with respect to \( x \)

\[ g_{r}(x) = \frac{\partial}{\partial x} \left( W_{r} \phi \right) + V_{r}(x,a) \phi + \int_{0}^{x} \frac{\partial V_{r}(x,a)}{\partial x} \phi \, dx. \]

A substitution of \( g_{r}(x) \) and \( \frac{\partial V}{\partial x} \) in the first variation of \( I \) follows by an application of Dirichlet's formula as before yields

\[ \frac{\partial I}{\partial a} \phi = \sum_{r=1}^{n} \int_{x_{0}}^{x_{1}} \left( \frac{\partial F_{r}}{\partial y} + \frac{\partial F_{r}}{\partial z} \frac{V_{r}}{z_{r}} + V_{r} \frac{\partial F_{r}}{\partial z_{r}} + T_{r}(x) \right) \phi \, dx. \]

\[ + \int_{x_{0}}^{x_{1}} \left[ \frac{\partial F_{r}}{\partial y}, \phi, \frac{\partial F_{r}}{\partial z}, \frac{d}{dx} \left( W_{r} \phi \right) \right] \, dx \quad \text{where} \]

\[ T_{r}(x) = \int_{x_{0}}^{x_{1}} \left[ \frac{\partial F_{r}}{\partial z_{r}} V_{r}(x,a) + \frac{\partial F_{r}}{\partial z_{r}} \frac{\partial V_{r}}{\partial z_{r}} \right] \, ds. \]

An integration by parts on the \( \phi \)ed terms gives

\[ \frac{\partial I}{\partial a} \phi = \sum_{r=1}^{n} \int_{x_{0}}^{x_{1}} \left[ \frac{\partial F_{r}}{\partial y} + \frac{\partial F_{r}}{\partial z} \frac{V_{r}}{z_{r}} + V_{r} \frac{\partial F_{r}}{\partial z_{r}} + T_{r}(x) + \frac{d}{dx} \frac{\partial F_{r}}{\partial y} \cdot W_{r} \frac{\partial F_{r}}{\partial z_{r}} \right] \phi \, dx. \]

It is necessary that the integrand vanish for a maximum hence, if the values of \( x \) and \( V \) be substituted in the integrand, it follows that

\[ \frac{\partial F_{r}}{\partial y} - \frac{d}{dx} \frac{\partial F_{r}}{\partial y} + \int_{A_{r}} \left[ \frac{\partial G_{r} \frac{\partial F_{r}}{\partial y}}{y} + \frac{\partial G_{r} \frac{\partial F_{r}}{\partial z}}{y} - \frac{d}{dx} \frac{\partial G_{r} \frac{\partial F_{r}}{\partial z}}{y} \right] + T_{r}(x) = 0. \]
The function $T_q(x)$ is an integral involving the resolvent kernel of the system of integral equations defining the variations.

If some variable other than $u_1$ had been chosen to be independent a different set of conditions of the type (18) would hold. Presumably this is a different problem from the above, for certainly the $n$ points would be different. Such a problem has already been investigated for special functions, but no attempt to consider the general problem of this type has been made.

10. Problem for Integral Relations. A special problem in which a linear integral equation replaces the first order differential equation of the type $\frac{d}{dx}G(u_1, u_1', \ldots, u_n, u_n') = 0$ has already been discussed. A more general problem is that of finding a curve $E$, in the space $(u_1, u_2, u_3, x)$ satisfying the integral relations

$$
G_k(u_1, u_1', u_2, u_2', u_3, u_3', x) = \int_{x_0}^{x} R_k(u_1, u_1', u_2, u_2', u_3, u_3', x, s)ds
$$

$(k = 1, 2)$ such that an integral

$$
I = \int_{x_0}^{x_1} F(u_1, u_1', u_2, u_2', u_3, u_3', x)dx
$$

is a maximum. The end point $x_1$ and the corresponding end values $u_1(x_1)$ are fixed. The problem is possible when the end point $x_0$ and the end value $u_1(x_0)$ are fixed or variable, but for the sake of brevity suppose $x_1$ and $u_1(x_1)$ to be fixed.

If the $u_i (i = 1, 2, 3)$ are replaced by functions satisfying the same conditions as the corresponding functions of paragraph (6), the integral equations become functions of these parameters and yield by a parametric differentiation
\[ \frac{\partial G_k}{\partial y} \dot{\theta} + \frac{\partial G_k}{\partial y'} \dot{\theta}' + \sum_{r=1}^{L} \frac{\partial G_k}{\partial z_r} \dot{f}_r + \frac{\partial G_k}{\partial z_r'} \dot{f}_r' \]

\[ = \int_{x_0}^{x} \left[ \frac{\partial P_k}{\partial y} \dot{\theta} + \frac{\partial P_k}{\partial y'} \dot{\theta}' + \sum_{r=1}^{L} \frac{\partial P_k}{\partial z_r} \dot{f}_r + \frac{\partial P_k}{\partial z_r'} \dot{f}_r' \right] ds. \]

An integration with respect to \( x \), followed by an integration by parts on the primed variations yields,

\[ \frac{\partial G_k}{\partial y} \dot{\theta} + \int_{x_0}^{x} \left( \frac{\partial G_k}{\partial y} - \frac{1}{\alpha} \frac{\partial G_k}{\partial y} \right) \dot{\theta} ds + \sum_{r=1}^{L} \int_{x_0}^{x} \left( \frac{\partial G_k}{\partial z_r} \dot{f}_r + \int_{x_0}^{x} \left( \frac{\partial G_k}{\partial z_r} - \frac{1}{\alpha} \frac{\partial G_k}{\partial z_r} \right) \dot{f}_r ds \right) \]

\[ = \int_{x_0}^{x} \left( \frac{\partial P_k}{\partial y} - \frac{1}{\alpha} \frac{\partial P_k}{\partial y} \right) \dot{\theta} ds + \int_{x_0}^{x} \left( \frac{\partial P_k}{\partial z_r} - \frac{1}{\alpha} \frac{\partial P_k}{\partial z_r} \right) \dot{f}_r ds. \]

By Dirichlet's formula this expression becomes,

\[ \sum_{r=1}^{L} \int_{x_0}^{x} \frac{\partial G_k}{\partial z_r} \dot{f}_r = - \left[ \frac{\partial G_k}{\partial y} \dot{\theta} + \int_{x_0}^{x} \left( \frac{\partial G_k}{\partial y} - \frac{1}{\alpha} \frac{\partial G_k}{\partial y} \right) \dot{\theta} ds \right] \]

\[ + \int_{x_0}^{x} \left( \frac{\partial G_k}{\partial z_r} - \frac{1}{\alpha} \frac{\partial G_k}{\partial z_r} \right) \dot{f}_r ds. \]

Now as far as the variations \( \dot{f}_r \) and \( \dot{\theta} \) are concerned, this expression is of the same form as (15), and, therefore, the analysis of the preceding paragraph applies from this point on.
11. Further Extensions. The extension to the case of more than one independent variable is obtained by placing a subscript on the $y$ and an
regarding this subscript as umbral symbol of the proper index.

The problem of Simultaneous $x_1$ima for more than one differential or integral relation can be solved by this same method; in fact, if there are no integral terms two independent variables $y_1$ and $y_4$, equation (17) with the proper arguments for $F_1$, $F$ and the $G_k$ substituted is a necessary condition that a curve $E_0$ in the space $(u_1, u_2, u_3, u_4, x)$ satisfy the differential equations

$$ G_k(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x) = 0 \quad (k = 1, 2) $$

and make an integral

$$ \int_{x_0}^{x_1} F_k(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x) \, dx $$

a maximum when $u_4$ is not allowed to vary.

The corresponding problems for variable end points lead to the Neumann and Legendre necessary conditions and to sufficient conditions for strong and weak relative maxima. The reasoning of the preceding paragraphs is sufficient to obtain these conditions.

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