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ON THE UNITARY EQUIVALENCE OF N-NORMAL OPERATORS

by

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A THESIS
SUBMITTED TO THE FACULTY
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Houston, Texas
May, 1960
INTRODUCTION

The principal object of this paper is to give a set of unitary invariants for a certain class of operators on a Hilbert space. The operators considered, herein called n-normal, are exactly those operators which generate a ring of type $I_n$ in the terminology of Kaplansky [8]. An n-normal operator is a direct sum of so-called homogeneous k-normal operators, and a homogeneous k-normal operator is very like a continuous function from a totally disconnected topological space to the full ring of $k \times k$ complex matrices. Thus it was believed that if one could find a suitable set of invariants for complex matrices, one could also solve the unitary equivalence problem for homogeneous n-normal operators. In fact, Brown [2] found a set of unitary invariants for $2 \times 2$ matrices and thereby also found a set of invariants for homogeneous binormal operators.

Now a set of unitary invariants for $n \times n$ matrices was furnished by Specht. In [46] he showed that there is an infinite collection of traces attached to every $n \times n$ matrix such that two matrices are unitarily equivalent if and only if the corresponding traces in this collection are equal. Now the generalization of the trace of a matrix to n-normal operators is given by Dixmier [3]. It is thus natural to suppose that the generalized Specht invariants would also serve as unitary invariants for homogeneous n-normal operators, and this is
indeed the case (Theorem 3). Some other interesting related questions are as follows:

1) For \( n \) fixed, is it possible to find some finite subset of the Specht invariants to serve as a collection of unitary invariants for homogeneous \( n \)-normal operators and \( n \times n \) matrices?

2) Will the Specht invariants, or some finite subset thereof, also serve as a set of invariants for not necessarily homogeneous \( n \)-normal operators?

3) If the answer to question 1) is yes, is it possible that the number of traces required is small enough to furnish a practical criterion for determining when two \( n \times n \) matrices are unitarily equivalent?

In what follows, the author shows that the answer to questions 1) and 2) is yes, and the answer to question 3) is yes at least in the case \( n = 3 \). In particular, in connection with 1) it is shown (Theorems 2 and 3) that for a given \( n > 2 \), there is a subset of less than \((n - 1)^4n^2\) Specht traces which is a collection of unitary invariants. With respect to 2), it is shown (Theorem 4) that there is a collection of mutually commuting normal operators (Dixmier traces) attached to an \( n \)-normal operator such that two \( n \)-normal operators are unitarily equivalent if and only if the attached normal operators are simultaneously unitarily equivalent in pairs. With respect to 3), it is shown (Theorem 1) that there is a set of nine traces which form a set of invariants for \( 3 \times 3 \)
matrices. This is a correction and an extension of a result of Murnaghan's [12] on the same subject.
CHAPTER I

A FINITE SET OF UNITARY INVARIANTS FOR $N \times N$ MATRICES

If $A = (a_{ij})$ is an $n \times n$ matrix with entries which are complex numbers, then the adjoint of $A$, denoted by $A^*$, is the matrix $(b_{ij})$ defined by $b_{ij} = \overline{a_{ji}}$ where the bar denotes complex conjugation. A matrix is Hermitian if $A^* = A$, unitary if $UU^* = U^*U$, and normal if $AA^* = A^*A$. Clearly both Hermitian and unitary matrices are normal. Matrices $A$ and $B$ are said to be unitarily equivalent if there exists a unitary matrix $U$ such that $UAU^* = B$. It is clear that this is actually an equivalence relation. A complete set of unitary invariants for a matrix $A$ is a collection of objects $\{O^A_\alpha\}$ associated with every $n \times n$ matrix $A$ such that two matrices $A$ and $B$ are unitarily equivalent if and only if $O^A_\alpha = O^B_\alpha$ for every $\alpha$. If $A = (a_{ij})$, then the trace of $A$, denoted in this chapter by $\sigma(A)$, is the complex number $\sum_{i=1}^{n} a_{ii}$. The reader will recall that $\sigma(A)$ is equal to the sum of the $n$ eigenvalues of $A$.

For normal matrices it has long been known that the set of eigenvalues together with their respective multiplicities is a complete set of unitary invariants. Since the $n$ eigenvalues of a matrix $A$ are determined by the numbers $\sigma(A), \sigma(A^2), \ldots, \sigma(A^n)$ one could equally well take this collection as a complete set of unitary invariants for a normal matrix $A$. 
For arbitrary matrices the problem of unitary equivalence is much more difficult. Many papers have been written on the subject, but it was not until 1940 that Specht [16] discovered a complete set of unitary invariants for arbitrary \( n \times n \) matrices. Let \( A \) be an \( n \times n \) matrix and consider the infinite collection of matrices

\[
\left\{ A^{n_1} A^{n_2} A^{n_3} \ldots A^{n_j} \right\}
\]

where \( n_1, \ldots, n_j \) is any finite sequence of non-negative integers. Specht proved that the collection of complex numbers

\[
\left\{ \sigma(A^{n_1} A^{n_2} \ldots A^{n_j}) \right\}
\]

forms a complete set of unitary invariants for \( A \). This collection has the disadvantage, however, that it is infinite, and for \( n \) fixed it seemed reasonable that some finite subset of the Specht invariants would always suffice. In fact, Murnaghan [12] showed that for \( n = 2 \), the subset

\[
\left\{ \sigma(A), \sigma(A^2), \sigma(A^*) \right\}
\]

is a complete set of unitary invariants, and he also made some progress toward showing that a finite subset of the Specht invariants suffices for \( 3 \times 3 \) matrices, although one of his results is incorrect.

The main result of this chapter is that for arbitrary \( n > 2 \) there is a finite subset of less than \((n - 1)^4 n^2\) Specht invariants which is a complete set of unitary invariants for \( n \times n \) matrices.
Also the case \( n = 3 \) is worked out more or less by straight forward computation, and it is proved that for \( 3 \times 3 \) matrices there is a subset of nine Specht invariants which does the job. (Although the number \((n - 1)^4 n^2\) is only a rough upper bound, the author at present does not see how to improve this estimate enough to approximate the known results for \( n = 2, 3 \).)

We work out the case \( n = 3 \) first. The idea is to treat the different eigenvalue possibilities as separate cases, obtain a canonical form under unitary transformations for each case, and then show that two matrices in canonical form and possessing the same nine traces listed in Theorem 1 are equal. We start with

**Lemma 1.1.** If \( A \) is any \( 3 \times 3 \) complex matrix having one eigenvalue \( a \) of multiplicity 3, then \( A \) is unitarily equivalent to a matrix

\[
\begin{pmatrix}
\ast & d & f \\
 a & g \\
 a & a
\end{pmatrix}
\]

satisfying

1) \( d, g \geq 0 \)

2) \( d = 0 \) implies \( g = 0 \)

3) \( g = 0 \) implies both \( d = 0 \) and \( f \geq 0 \).

**Proof.** If \( A \) is scalar then the theorem is true. So suppose \( A \) is not scalar. It is well known that there exists a unitary matrix \( U \) such that \( A' = UA^* \) is triangular - say
\[ A' = \begin{pmatrix} a & d' & f' \\ g' \\ a \end{pmatrix} \]

The eigenvalues, \(a\), of course must appear on the diagonal.

Transformation of \(A'\) by the diagonal unitary matrix

\[ V = \begin{pmatrix} e^{i\theta_1} \\ e^{i\theta_2} \\ e^{i\theta_3} \end{pmatrix} \]

gives

\[ A'' = V A' V^* = \begin{pmatrix} a & d'e^{i(\theta_1-\theta_2)} & f'e^{i(\theta_1-\theta_3)} \\ a & g'e^{i(\theta_2-\theta_3)} \\ a \end{pmatrix} \]

and \(\theta_1, \theta_2, \theta_3\) can be adjusted so that \(d'' = d'e^{i(\theta_1-\theta_2)}\) and \(g'' = g'e^{i(\theta_2-\theta_3)}\) are non-negative. Now if \(d'' = 0\), further transformation by the unitary matrix

\[ W = \begin{pmatrix} kf'' & \overline{kg''} & 0 \\ -\overline{kg''} & kf'' & 0 \\ 0 & 0 & 1 \end{pmatrix} ; \quad |k|^2(|f''|^2 + g''^2) = 1 \]

yields

\[ \begin{pmatrix} a & 0 & f''' \\ a & 0 \\ a \end{pmatrix} \]

and another application of \(V\) to this matrix ensures \(f > 0\).
If \( g''' = 0 \) in \( A''' \), then transformation of \( A''' \) by the unitary matrix

\[
Y = \begin{pmatrix}
1 & 0 & 0 \\
0 & kf'' & -kd'' \\
0 & kd'' & kf''
\end{pmatrix}
\]

again yields

\[
\begin{pmatrix}
a & 0 & f'''
\end{pmatrix}
\]

\[
\begin{pmatrix}
a \\
0 \\
a
\end{pmatrix}
\]

and \( f'''' \) is rotated to be non-negative as before.

**Lemma 1.2.** If \( A \) has one eigenvalue \( b \) of multiplicity 2, and another \( a \) of multiplicity 1, then \( A \) is unitarily equivalent to a matrix

\[
\begin{pmatrix}
b & d & f \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
b \\
g \\
a
\end{pmatrix}
\]

satisfying

1) \( d, g \geq 0 \)

2) \( d = 0 \) implies \( g = 0 \)

3) \( g = 0 \) implies \( f \geq 0 \).
Proof. It is easy to see that there is a triangular matrix B unitarily equivalent to A which has the eigenvalues arranged on the diagonal in the desired order. Application (if necessary) of the unitary matrices V, W, and V again to B just as was done in Lemma 1.1 gives the desired result.

Lemma 1.3. If A has distinct eigenvalues and \{a, b, c\} is any desired ordering of these eigenvalues, then A is unitarily equivalent to a matrix

\[
\begin{pmatrix}
a & d & f \\
b & g & \\
c & &
\end{pmatrix}
\]

where

1) \( d, g \geq 0 \)
2) \( d = 0 \) implies \( f \geq 0 \)
3) \( g = 0 \) implies \( f \geq 0 \).

Proof. Again it is easy to find a triangular matrix B unitarily equivalent to A which has the desired ordering of the eigenvalues. Then an application of V of Lemma 1.1 ensures that \( d, g \geq 0 \) and if \( d = 0 \) or \( g = 0 \), V can be chosen so that \( f \geq 0 \).

The three lemmas above, together with Lemma 1.4 to follow, were essentially proved in [12], and are included here only for completeness.
Lemma 1.4. Suppose $A_1$ and $A_2$ are the matrices

$$A_1 = \begin{pmatrix}
\lambda_1 & d_1 & f_i \\
& \lambda_2 & g_i \\
& & \lambda_3
\end{pmatrix}$$

with $d_i, g_i > 0$, and suppose the six corresponding traces $\sigma(A_1^2)$, $\sigma(A_1^3)$, $\sigma(A_1^*A_1^*)$, $\sigma(A_1^*A_2^2)$, and $\sigma(A_1^*A_2^2)$ are equal for $i = 1, 2$. Then the three corresponding numbers

1) $d_i^2 + g_i^2 + |f_i|^2$
2) $|\lambda_3 - \lambda_2|^2 d_i^2 + |\lambda_2 - \lambda_1|^2 g_i^2 + d_i g_i f_i$
3) $(\lambda_3 - \lambda_2) d_i^2 + (\lambda_1 - \lambda_2) g_i^2 - d_i g_i f_i$

are equal for $i = 1, 2$.

Proof. There are several properties of the trace function which will be used in the proof of this lemma and subsequently throughout the paper. They are

1) $\sigma(\alpha A + \beta B) = \alpha \sigma(A) + \beta \sigma(B)$
2) $\sigma(AB) = \sigma(BA)$
3) $\sigma(A^*) = \overline{\sigma(A)}$
4) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of $A$ then

$\sigma(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$ and more generally

$\sigma(A^m) = \lambda_1^m + \lambda_2^m + \ldots + \lambda_n^m$. 

Let \( \text{adj } A = A^2 - \sigma(A)A + \frac{1}{2} [(\sigma(A))^2 - \sigma(A^2)] I \), where \( I \) is the identity matrix. Now the matrices \( A^*_1 \text{adj } A_1 \) and \( \text{adj } A_1^* \text{adj } A_1 \) are linear combinations of the six matrices (and adjoints) appearing in the hypothesis. It follows that \( \sigma(A^*_1 \text{adj } A_1) = \sigma(A^*_2 \text{adj } A_2) \) and \( \sigma(\text{adj } A_1^* \text{adj } A_1) = \sigma(\text{adj } A_2^* \text{adj } A_2) \).

Let \( t_1 = d_1^2 + g_1^2 + |f_1|^2, \)

\[
u_i = \lambda_3 d_1^2 + \lambda_1 g_1^2 + \lambda_2 |f_1|^2 - d_1 g_1 \bar{f}_1, \]

and \( v_1 = |\lambda_3|^2 d_1^2 + |\lambda_1|^2 g_1^2 + |f_1|^2 - d_1 g_1 (\lambda_2 f_1 + \bar{\lambda}_2 \bar{f}_1) + d_1^2 g_1^2. \)

Calculations show that

\[
sigma(A^*_1 A_1) = |\lambda_1|^2 + |\lambda_2|^2 + |\lambda_3|^2 + t_1, \]

\[
sigma(A^*_1 \text{adj } A_1) = \overline{\lambda}_1 \lambda_2 \lambda_3 + \lambda_1 \overline{\lambda}_2 \lambda_3 + \lambda_1 \lambda_2 \overline{\lambda}_3 - u_1, \]

and

\[
sigma(\text{adj } A_1^* \text{adj } A_1) = |\lambda_1|^2 |\lambda_2|^2 + |\lambda_1|^2 |\lambda_3|^2 + |\lambda_2|^2 |\lambda_3|^2 + v_1. \]

Thus, \( t_1 = t_2, \ u_1 = u_2, \) and \( v_1 = v_2. \) The equality of the numbers in 1) follows from \( t_1 = t_2, \) that of the numbers in 3) from \( u_1 = \lambda_2 t_1 = u_2 = \lambda_2 t_2, \) and that of the numbers in 2) from \( v_1 = \overline{\lambda}_2 u_1 = \lambda_2 \overline{u}_1 + |\lambda_2|^2 t_1 = v_2 - \overline{\lambda}_2 u_2 = \lambda_2 \overline{u}_2 + |\lambda_2|^2 t_2. \)

We turn now to the lemma which greatly simplifies our future calculations.

**Lemma 1.5.** Suppose \( A = \mathcal{D} + A \) and \( B = \mathcal{D} + B \) are \( n \times n \) matrices, where \( \mathcal{D} \) is a scalar matrix, and suppose \{\( n_1, \ldots, n_j \)\} is a collection of non-negative integers. Suppose that for every collection of non-negative integers \{\( m_1, \ldots, m_j \)\} which
satisfies $m_i \leq n_i$, $i = 1, \ldots, j$, it is true that

$$\sigma(A_{1}^{m_{1}} A_{2}^{m_{2}} \ldots A_{j}^{m_{j}}) = \sigma(B_{1}^{m_{1}} B_{2}^{m_{2}} \ldots B_{j}^{m_{j}}),$$

where $\sigma(A)$ is the trace of $A$. Then

$$\sigma(a_{1}^{n_{1}} a_{2}^{n_{2}} a_{3}^{n_{3}} \ldots a_{j}^{n_{j}}) = \sigma(b_{1}^{n_{1}} b_{2}^{n_{2}} b_{3}^{n_{3}} \ldots b_{j}^{n_{j}}).$$

**Proof.** The proof is by induction on $k = n_{1} + n_{2} + \ldots + n_{j}$.

For $k = 1$ the theorem is trivial. Suppose the theorem is true for $k < p$, and let $k = p$. Now

$$A_{1}^{n_{1}} A_{2}^{n_{2}} \ldots A_{j}^{n_{j}} = (\delta + \alpha)^{n_{1}}(\delta^{*} + \alpha^{*})^{n_{2}} \ldots (\delta^{*} + \alpha^{*})^{n_{j}}$$

which can be expanded into

$$a_{1}^{n_{1}} a_{2}^{n_{2}} a_{3}^{n_{3}} \ldots a_{j}^{n_{j}} + \sum \delta^{t_{1}} \delta^{*} \delta^{t_{2}} \delta^{s_{1}} \delta^{s_{2}} \ldots \delta^{s_{j}},$$

where every term in the above $\Sigma$ sum is such that $s_{1} + s_{2} + \ldots + s_{j} < p$.

Likewise we have

$$B_{1}^{n_{1}} B_{2}^{n_{2}} \ldots B_{j}^{n_{j}} = b_{1}^{n_{1}} b_{2}^{n_{2}} \ldots b_{j}^{n_{j}} + \sum \delta^{t_{1}} \delta^{*} \delta^{t_{2}} b_{1}^{s_{1}} b_{2}^{s_{2}} \ldots b_{j}^{s_{j}}.$$

It follows from the induction hypothesis that for corresponding terms in the above $\Sigma$ sums we have

$$\sigma(a_{1}^{s_{1}} a_{2}^{s_{2}} \ldots a_{j}^{s_{j}}) = \sigma(b_{1}^{s_{1}} b_{2}^{s_{2}} \ldots b_{j}^{s_{j}}),$$

and since $\delta^{t_{1}} \delta^{*} \delta^{t_{2}}$ is scalar and the trace function is linear we have

$$\sigma(\sum \delta^{t_{1}} \delta^{*} \delta^{t_{2}} a_{1} \ldots a_{j}^{s_{j}}) = \sigma(\sum \delta^{t_{1}} \delta^{*} \delta^{t_{2}} b_{1} \ldots b_{j}^{s_{j}}).$$
The fact that
\[ \sigma(A_1^{n_1} A_2^{n_2} \ldots A_j^{n_j}) = \sigma(B_1^{n_1} B_2^{n_2} \ldots B_j^{n_j}) \]
and a subtraction yield the desired results.

**Lemma 1.6.** Suppose \( A_i, i = 1, 2, \) are matrices of the form in Lemma 1.1; i.e., suppose that
\[
A_i = \begin{pmatrix}
a & d_i & f_i \\
a & a & g_i \\
a & & a
\end{pmatrix}
\]
where
1) \( d_i, g_i \geq 0 \)
2) \( d_i = 0 \) implies \( g_i = 0 \)
3) \( g_i = 0 \) implies both \( d_i = 0 \) and \( f_i \geq 0 \).
Suppose in addition that the corresponding numbers \( \sigma(A_i^2), \sigma(A_i^3), \sigma(A_i^2 A_1), \sigma(A_i^2 A_1^2), \sigma(A_i A_1^2 A_1^2), \sigma(A_i A_1^2 A_1 A_1), \sigma(A_i^2 A_i A_1^2 A_1), \) and \( \sigma(A_i A_1^2 A_1^2 A_1) \) are equal for \( i = 1, 2 \). Then \( A_1 = A_2 \).

**Proof.** Write \( A_i = B + A_i \) where \( B \) is the scalar matrix
\[
\begin{pmatrix}
a & a \\
a & a
\end{pmatrix}
\]
From the equality of the nine traces in the hypothesis and the four properties of the trace function mentioned in Lemma 1.4 above, we obtain the equality of a sufficient number of additional
traces to enable us to apply Lemma 1.5 and conclude that the four corresponding numbers $\sigma(a^*_{1} a_{1})$, $\sigma(a^*_{1} a^2_{1})$, $\sigma(a^*_{1} a^2_{1} a^*_{1})$, and $\sigma(a^*_{1} a^2_{1} a^2_{1} a^*_{1})$ are equal for $i = 1, 2$. [Actually the fact that $a^3_{1} = 0$ is also used to show that $\sigma(A^3_{1} A^*_{1}) = \sigma(A^3_{2} A^*_{2})$].

Now calculation shows that

$$\sigma(a^*_{1} a_{1}) = d^2_{1} + g^2_{1} + |f^2_{1}|,$$

$$\sigma(a^*_{1} a^2_{1}) = d^2_{1} g^2_{1},$$

$$\sigma(a^*_{1} a^2_{1} a^*_{1}) = d^2_{1} g^2_{1}, \text{ and}$$

$$\sigma(a^*_{1} a^2_{1} a^2_{1} a^*_{1}) = d^2_{1} g^2_{1} d^2_{1} + |f^4_{1}|),$$

so we have

( I ) $d^2_{1} + g^2_{1} + |f^2_{1}|^2 = d^2_{2} + g^2_{2} + |f^2_{2}|^2$

( II ) $d^2_{1} g^2_{1} = d^2_{2} g^2_{2}$

(III) $d^2_{1} g^2_{1} = d^2_{2} g^2_{2}$

( IV ) $d^2_{1} g^2_{1} (d^2_{1} + |f^2_{1}|^{2}) = d^2_{2} g^2_{2} (d^2_{2} + |f^2_{2}|^{2})$.

Since $d_1, g_1 \geq 0$, it follows from (III) that $d_1 g_1 = d_2 g_2$.

If $d_1 g_1 \neq 0$, then division of (II) yields $f_1 = f_2$ and from (IV) we get $d_1 = d_2$. Then (I) yields $g_1 = g_2$ and $A_1 = A_2$. If $d_1 g_1 = 0$, then by hypothesis $d_1 = g_1 = d_2 = g_2 = 0$ and $f_1, f_2 \geq 0$. Taking square roots in (I) completes the argument.

**Lemma 1.7.** Suppose $A_1$ and $A_2$ are matrices of the form in Lemma 1.2; i.e., suppose that
\[
A_i = \begin{pmatrix}
  b & d_i & f_i \\
  b & g_i \\
  a
\end{pmatrix}
\quad i = 1, 2
\]

where

1) \( a \neq b \)
2) \( d_i, g_i \geq 0 \),
3) \( d_i = 0 \) implies \( g_i = 0 \),
4) \( g_i = 0 \) implies \( f_i \geq 0 \).

Suppose also that the corresponding numbers \( \sigma(A_1), \sigma(A_1^2), \sigma(A_1^3), \sigma(A_1^*A_1), \sigma(A_1^2A_1), \sigma(A_1^3A_1^2), \sigma(A_1^*A_1^2A_1), \sigma(A_1^*A_1^2A_1^2) \)
are the same for \( i = 1, 2 \). Then \( A_1 = A_2 \).

**Proof.** Write \( A_1 = \mathcal{S} + \mathcal{Q}_1 \) where \( \mathcal{S} \) is the scalar matrix

\[
\begin{pmatrix}
  b \\
  b \\
  b
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
  0 & d_i & f_i \\
  0 & g_i \\
  a - b
\end{pmatrix}
\]

Denote \( a - b \) by \( k \). Since we have assumed \( \sigma(A_1) = \sigma(A_2), \sigma(A_1^2) = \sigma(A_2^2) \), and \( \sigma(A_1^3) = \sigma(A_2^3), A_1 \) and \( A_2 \) have the same set of eigenvalues and therefore the same characteristic equations. Thus we
can write $A_1^3 = \alpha A_1^2 + \beta A_1 + \rho$ for $i = 1, 2$, and thus $A_1^3 A_1^2 = 
abla A_1^1 + \beta A_1^1 + \rho A_1^1$ for $i = 1, 2$. It follows from the properties of the trace function and the equality of the traces in the hypothesis that $\sigma(A_1^3 A_1^2) = \sigma(A_2^3 A_2^2)$ and $\sigma(A_1^3 A_1^2) = \sigma(A_2^3 A_2^2)$.

Now by taking adjoints, etc. of the operators whose traces we have equality of, we obtain a sufficient number of equalities to enable us to apply Lemma 1.5 and conclude that the corresponding numbers

$$
\sigma(a_1^* a_1^2) = d_1 g_1 f_1 + k(|f_1|^2 + g_1^2 + |k|^2),
$$

$$
\sigma(a_1^* a_1^2 a_1^* a_1) = d_1^2 |d_1 g_1 + f_1 k|^2 + |\sigma(a_1^* a_1^2)|^2, \text{ and}
$$

$$
\sigma(a_1^* a_1^2 a_1^2) = d_1 g_1 (d_1 g_1 + f_1 k) + \bar{k} \sigma(a_1^* a_1^2)
$$

are equal for $i = 1, 2$. Thus we have

( I ) \quad d_1 g_1 f_1 + k(|f_1|^2 + g_1^2 + |k|^2) = \quad d_2 g_2 f_2 + k(|f_2|^2 + g_2^2 + |k|^2)

( II ) \quad d_1^2 |d_1 g_1 + f_1 k|^2 = d_2^2 |d_2 g_2 + f_2 k|^2

( III ) \quad d_1 g_1 (d_1 g_1 + f_1 k) = d_2 g_2 (d_2 g_2 + f_2 k)

(IIIa) \quad d_1^2 g_1 |d_1 g_1 + f_1 k|^2 = d_2^2 g_2 |d_2 g_2 + f_2 k|^2.

Also, making the necessary changes of notation, we obtain from Lemma 1.4

( IV ) \quad (|k|^2 + g_1^2) d_1^2 = (|k|^2 + g_2^2) d_2^2

( V ) \quad d_1^2 + g_1^2 + |f_1|^2 = d_2^2 + g_2^2 + |f_2|^2

( VI ) \quad k d_1^2 - d_1 g_1 f_1 = k d_2^2 - d_2 g_2 f_2.
Now if \( d_1^2|d_1 g_1 + f_1 k| \neq 0 \), then dividing (IIIa) by (II) gives \( g_1 = g_2 \). Dividing (IV) by \(|k|^2 + g_1^2\), which is non-zero, yields \( d_1 = d_2 \). If \( g_1 \neq 0 \), then (VI) yields \( f_1 = f_2 \) (we have \( d_1 \neq 0 \) from above). If \( g_1 = 0 \), then by hypothesis, \( f_1 \geq 0 \), and (V) yields \( f_1 = f_2 \). Turning to the case \( d_1^2|d_1 g_1 + f_1 k| = 0 \) there are two possibilities. If \( d_1 = 0 \), it follows from (IV) that \( d_2 = 0 \), since \(|k|^2 + g_1^2 > 0\). By hypothesis then \( g_1 = g_2 = 0 \), \( f_1 \geq 0 \), and \( f_1 = f_2 \) follows from (IV). If \( d_1 g_1 + f_1 k = 0 \), \( d_1 \neq 0 \), then \( d_2 \neq 0 \) by (IV), so \( d_2 g_2 + f_2 k = 0 \). Since \( k \neq 0 \), we get

\[
    f_1 = \frac{-d_1 g_1}{k} \quad \text{and} \quad \left| f_1 \right|^2 = \frac{d_1^2 g_1^2}{|k|^2}.
\]

Substituting in (V) we get

\[
    (d_1^2 + g_1^2)|k|^2 + d_1^2 g_1^2 = (d_2^2 + g_2^2)|k|^2 + d_2^2 g_2^2.
\]

Subtracting this equation from (IV) yields

\[
    |k|^2 g_1^2 = |k|^2 g_2^2 \quad \text{and thus} \quad g_1 = g_2.
\]

Then (IV) gives \( d_1 = d_2 \), and \( f_1 = f_2 \) follows as before.

**Lemma 1.8.** Suppose that

\[
    A_i = \begin{pmatrix}
        a & d_i & f_i \\
        b & g_i \\
        c
    \end{pmatrix}, \quad i = 1, 2
\]
where

1) \((a - b)(b - c)(c - a) \neq 0\)
2) \(d_i, g_i \geq 0\)
3) \(d_i = 0\) implies \(f_i \geq 0\)
4) \(g_i = 0\) implies \(f_i \geq 0\)

Suppose also that the corresponding numbers \(\sigma(A_i, 1), \sigma(A_i^2, 1), \sigma(A_i^3, 1),\)
\(\sigma(A_i^* A_i, 1), \sigma(A_i^* A_i^2, 1), \sigma(A_i^* A_i A_i^*, 1), \sigma(A_i^* A_i^2 A_i, 1), \sigma(A_i^* A_i A_i^* A_i)\)
are the same for \(i = 1, 2\). Then \(A_1 = A_2\).

**Proof.** As usual we write \(A_i = \mathcal{L} = A_i\) where

\[
\mathcal{L} = \begin{pmatrix}
  a & \\
  a & \\
  a & \\
\end{pmatrix}
\]

and

\[
A_i = \begin{pmatrix}
  0 & d_i & f_i \\
  k_1 & g_i \\
  k_2 & \\
\end{pmatrix}
\]

with \(b - a = k_1\) and \(c - a = k_2\). As in Lemma 1.7, the \(A_i\) have
the same characteristic equation, and it follows that \(\sigma(A_i^2 A_i^*) = \sigma(A_2^* A_2)\) and \(\sigma(A_i^3 A_i^* A_i) = \sigma(A_i^3 A_i^*)\). Then the usual application of
Lemma 1.5 yields the equality of the corresponding traces of the
\(A_i\). Calculation shows that \(A_i A_i^* - k_1 A_i^* A_i\) is the matrix
$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta_1 \\ 0 & 0 & \rho_1 \end{pmatrix}$$

where $\delta_1 = e_1(d_1^2 + |k_1|^2) + (k_2 - k_1)(f_1 d_1 + \bar{k}_1 g_1)$ and

$$\rho_1 = g_1(d_1 \bar{f}_1 + g_1 k_1) + (k_2 - k_1)(|f_1|^2 + g_1 + |k_2|^2).$$

Now

$$\sigma(B_1 B_1^*) = |\delta_1|^2 + |\rho_1|^2,$$

$$\sigma(B_1 A_1^*) = \delta_1 g_1 + \bar{k}_2 \rho_1,$$

and

$$\sigma(A_1^* A_1 B_1) = \rho_1 (g_1^2 + |f_1|^2 + |k_2|^2) + \delta_1 (d_1 \bar{f}_1 + k_1 g_1).$$

Since $B_1, B_1 B_1^*, B_1 A_1^*$, and $A_1^* A_1 B_1$ are linear combinations of words in $\mathcal{A}_1$ and $\mathcal{A}_1^*$ for which the traces are known to be equal we get

(I) $\rho_1 = \rho_2$

(II) $|\delta_1|^2 + |\rho_1|^2 = |\delta_2|^2 + |\rho_2|^2$

(III) $g_1 \delta_1 + \bar{k}_2 \rho_1 = g_2 \delta_2 + \bar{k}_2 \rho_2$

(IV) $\rho_1 (g_1^2 + |f_1|^2 + |k_2|^2) + \delta_1 (d_1 \bar{f}_1 + k_1 g_1) = \rho_2 (g_2^2 + |f_2|^2 + |k_2|^2) + \delta_2 (d_2 \bar{f}_2 + k_1 g_2).$

Also from Lemma 1.4, by making suitable changes of notation, we get

(V) $d_1^2 + g_1^2 + |f_1|^2 = d_2^2 + g_2^2 + |f_2|^2$

(VI) $|k_1 - k_2|^2 d_1^2 + |k_1|^2 g_1^2 + d_1^2 g_1^2 = |k_1 - k_2|^2 d_2^2 + |k_1|^2 g_2^2 + d_2^2 g_2^2.$
(VII) \( (k_2 - k_1)d_1^2 - k_1g_1^2 - d_1g_1 \bar{f}_1 = \\
(k_2 - k_1)d_2^2 - k_1g_2^2 - d_2g_2 \bar{f}_2 \).

Now (I) and (II) yield \( |\delta_1| = |\delta_2| \) and (I) and (III) yield \( g_1^2|\delta_1|^2 = g_2^2|\delta_2|^2 \). Hence if \( \delta_1 \neq 0 \), it follows that \( g_1 = g_2 \), and subtraction and division in (VI) yield \( d_1 = d_2 \) (remember \( k_1 - k_2 \neq 0 \)). If either \( g_1 \) or \( d_1 \) is zero then \( f_1 = f_2 \) follows from (V), and if \( g_1d_1 \neq 0 \), then \( f_1 = f_2 \) by (VII). Thus in the case \( \delta_1 \neq 0 \) we have \( A_1 = A_2 \). Now if \( \delta_1 = 0 \) and \( \rho_1 \neq 0 \), (I) and (IV) yield

\[ g_1^2 + |f_1|^2 = g_2^2 + |f_2|^2, \]

and this with (V) gives \( d_1 = d_2 \). By subtracting \( |k_1 - k_2|^2d_1^2 \)

from each side in (VI) and dividing the resulting equation by \( |k_1|^2 + d_1^2 \), we get \( g_1 = g_2 \), and \( f_1 = f_2 \) follows as above. Thus if \( \delta_1 \neq 0 \) or \( \rho_1 \neq 0 \), \( A_1 = A_2 \). We complete the proof by showing that \( \delta_1 \) and \( \rho_1 \) can never vanish simultaneously. To do this it suffices to show that \( a^*_1a_1(a_1 - k_1) \neq 0 \). Let \( x \) be an eigenvector for \( A_1 \) corresponding to the eigenvalue \( k_2 \) (\( \neq k_1 \)). Then

\[ a^*_1a_1(a_1 - k_1)x = a^*_1a_1(a_1x - k_1x) = a^*_1a_1(k_2 - k_1)x = a^*_1k_2(k_2 - k_1)x. \]

But the null space of \( A_1^* \) is orthogonal to the range of \( A_1 \) so \( a^*_1k_2(k_2 - k_1)x \neq 0 \). Thus \( a^*_1a_1(a_1 - k_1) \neq 0 \).
Theorem I. If $A_1$ and $A_2$ are $3 \times 3$ complex matrices and the nine corresponding numbers $\sigma(A_1^2)$, $\sigma(A_1^2)$, $\sigma(A_1^3)$, $\sigma(A_1^*A_1^2)$, $\sigma(A_1^*A_1^2)$, $\sigma(A_1^*A_1^3)$, $\sigma(A_1^*A_1^2A_1^*)$, $\sigma(A_1^*A_1^3A_1^*)$, $\sigma(A_1^*A_1^2A_1^3)$ are equal for $i = 1, 2$, then $A_1$ and $A_2$ are unitarily equivalent.

Proof. Let $\lambda_1$, $\lambda_2$, $\lambda_3$, be the eigenvalues of $A_1$ and $\beta_1$, $\beta_2$, $\beta_3$ the eigenvalues of $A_2$. The equality of the first three traces above implies

$$\lambda_1 + \lambda_2 + \lambda_3 = \beta_1 + \beta_2 + \beta_3,$$
$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \beta_1^2 + \beta_2^2 + \beta_3^2,$$
$$\lambda_1^3 + \lambda_2^3 + \lambda_3^3 = \beta_1^3 + \beta_2^3 + \beta_3^3.$$

It follows easily that the collection \{\lambda_1, \lambda_2, \lambda_3\} is the same as the collection \{\beta_1, \beta_2, \beta_3\}. Thus the one of the three lemmas 1.1, 1.2, 1.3 concerning $A_1$ also concerns $A_2$. Applying the appropriate lemma to the $A_i$ yields matrices $B_i$ unitarily equivalent to the respective $A_i$. Now $\sigma(\cdot)$ is a unitary invariant, since unitarily equivalent matrices have the same eigenvalues. Thus the above corresponding nine functions of the $B_i$ are equal. Application of the appropriate lemma 1.6, 1.7, or 1.8 yields $B_1 = B_2$ and the result follows.

In \[12\], it is claimed that the first six of the nine traces in Theorem I are sufficient to ensure the unitary equivalence of matrices $A_1$ and $A_2$ having distinct eigenvalues.
To see that this is incorrect consider the matrices

\[
A_1 = \begin{pmatrix}
0 & 1 & 1 \\
1 & 1 & 2 \\
2 & &
\end{pmatrix}
\quad \text{and} \quad
A_2 = \begin{pmatrix}
0 & \frac{1}{\sqrt{3}} & \frac{-\sqrt{2}}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & 1 & \sqrt{2} \\
\frac{-\sqrt{2}}{\sqrt{3}} & \sqrt{2} & 2
\end{pmatrix}.
\]

Calculations show that for \( i = 1, 2, \sigma(A_i) = 3, \sigma(A_1^2) = 5, \sigma(A_2^3) = 9, \sigma(A_1^*A_1) = 8, \sigma(A_1^{*2}A_1) = 16, \sigma(A_1^{*2}A_1^2) = 36, \) and in fact \( \sigma(A_1^*A_1A_1^*A_1) = 48 \) and \( \sigma(A_1^*A_1A_1^*A_1) = 104. \) But also calculations show that \( \sigma(A_1^{*2}A_1^{*2}A_1) = 240 \) and \( \sigma(A_1^{*2}A_1^{*2}A_1^{*2}) = 232, \) from which it follows that \( A_1 \) and \( A_2 \) are not unitarily equivalent.

We turn now to the result for \( n \times n \) matrices. For technical reasons the result will be obtained in terms of linear operators on an \( n \)-dimensional (complex) Hilbert space, and the result for matrices is, of course, a trivial corollary. We remind the reader that a ring of operators (also called a \( \mathfrak{W}^* \)-algebra) is a self-adjoint algebra of operators on a Hilbert space which is closed in the weak operator topology. Rings of operators do not necessarily contain the identity operator, but every ring does have a unit – namely its maximal central projection.

To simplify the notation we also introduce the following terminology. A word in \( x \) and \( y \), denoted \( \omega(x,y) \), is an expression of one of the four forms

\[
\begin{align*}
x_1y_2x_3y_4 \ldots x_j, & \quad x_1y_2x_3 \ldots \ y_j, \quad \ y_1x_2y_3x_4 \ldots \ x_j, \quad \ x_1y_2x_3 \ldots \ y_j,
\end{align*}
\]
or \[ y^1 x^2 y^3 \ldots y^j \]

where the \( m_i \) are positive integers and \( j \) is a positive integer which is even or odd depending on the form of the word. The length of a word in \( x \) and \( y \) is the number \( j \). A polynomial in \( x \) and \( y \) is a linear combination of words in \( x \) and \( y \). Note that according to this definition polynomials in \( x \) and \( y \) do not contain constant terms. Throughout the remainder of this paper, \( W \) will denote the collection of all words in \( x \) and \( y \), \( W_n \) will denote the subset of \( W \) consisting of words having length less than or equal to \( n \), and \( W^m \) will denote the subset of \( W \) consisting of words no exponent in which exceeds \( m \). We also write \( W^m_n \) for \( W_n \cap W^m \).

**Lemma 1.9.** If \( A \) and \( B \) are operators on an \( n \)-dimensional Hilbert space and \( \sigma(\omega(A,A^*) \cap \omega(B,B^*)) \) for every word \( \omega(x,y) \in W^{n-1}_{4n^2} \cup \{ x^n \} \), then \( \sigma(\omega(A,A^*) \cap \omega(B,B^*)) \) for every word \( \omega(x,y) \in W_{4n^2} \).

**Proof.** It follows from the equations \( \sigma(A_i) = \sigma(B_i) \); \( i = 1, 2, \ldots, n \), that \( A \) and \( B \) have the same eigenvalues and thus the same characteristic equations. It is not hard to see, in view of the fact that \( A \) satisfies its characteristic equation, that any word \( \omega(A,A^*) \) where \( \omega(x,y) \in W_{4n^2} \) can be written as a linear combination of words \( \omega'(A,A^*) \) where each \( \omega'(x,y) \in W^{n-1}_{4n^2} \cup \{ x^n \} \). Furthermore \( \omega(B,B^*) \) is the same linear combination of the \( \omega'(B,B^*) \).
as \( \omega(A,A^*) \) is of the \( \omega'(A,A^*) \). The proof is easily completed by using the linearity of the trace function.

**Lemma 1.10.** If \( A \) is an operator on an \( n \)-dimensional Hilbert space, and \( \mathcal{R} \) is the ring of operators generated by \( A \), then the collection \( K = \{ \omega(A,A^*) / \omega(x,y) \in \mathbb{W}^2 \} \) spans \( \mathcal{R} \) considered as a vector space.

**Proof.** Since \( A \) acts on a finite dimensional space, the weak operator topology coincides with the uniform operator topology, and therefore the ring \( \mathcal{R} \) consists of all polynomials \( p(A,A^*) \) where \( p(x,y) \) is any polynomial in \( x \) and \( y \). Thus the collection of operators \( \{ \omega(A,A^*) / \omega(x,y) \in \mathbb{W} \} \) spans \( \mathcal{R} \). Now since \( A \) satisfies its characteristic equation, it is easy to see that any positive power \( A^j \) of \( A \) can be written as a linear combination of \( A, A^2, \ldots, A^n \). Likewise, any positive power of \( A^* \) can be written as a linear combination of \( A^*, A^{*2}, \ldots, A^{*n} \). It follows easily that the collection of operators
\[
\{ \omega(A,A^*) / \omega(x,y) \in \mathbb{W}^2 \}
\]
spans \( \mathcal{R} \). We now split this collection into two subsets and prove that \( K \) spans each subset. Let \( C_1 = \{ \omega(A,A^*) / \omega(x,y) \in \mathbb{W}^2 \text{ and is of the form } x^{m_1} \ldots \} \). And let \( C_2 = \{ \omega(A,A^*) / \omega(x,y) \in \mathbb{W}^2 \text{ and is of the form } y^{m_1} \ldots \} \). We construct a subset \( K_1 \) of \( K \) which spans \( C_1 \). We begin by putting \( A^{m_1} \) in \( K_1 \) for \( 1 \leq m_1 \leq n \). Next consider the operators of the form \( A^{m_1}A^{m_2} \) where \( 1 \leq m_1, m_2 \leq n \). If every operator of this
form is a linear combination of the $A^{m_1}$ already in $K_1$ then it
can be seen that $K_1$ as it stands spans $C_1$. If some $A^{m_1}A^{m_2}$ is
not a linear combination of the $A^{m_1}$ then put $A^{m_1}A^{m_2}$ in $K_1$ for
$l \leq m_1, m_2 \leq n$ and note that $K_1$ contains at least 2 linearly
independent elements. Now consider the operators $A^{m_1}A^{m_2}A^{m_3}$
where $l \leq m_1, m_2, m_3 \leq n$. If each of these operators is a
linear combination of elements already in $K_1$ then again $K_1$ as
it is spans $C_1$. If some $A^{m_1}A^{m_2}A^{m_3}$ is independent of the $A^{m_1}$'s
and the $A^{m_1}A^{m_2}$'s, put $A^{m_1}A^{m_2}A^{m_3}$ in $K_1$ for $l \leq m_1, m_2, m_3 \leq n$
and note that $K_1$ now contains at least three independent elements.
If we continue by induction and recall that $|R|$ contains at most
$n^2$ linearly independent elements, we see that there is some $j \leq n^2$
such that addition of the operators $A^{m_1}A^{m_2}...A^{m_j}$ (or $A^{m_j}$);
$l \leq m_1, m_2, ... , m_j \leq n$, to those operators already in $K_1$ yields
a collection of operators which spans $C_1$. The construction of a
subset of $K$ which spans $C_2$ proceeds in an analogous fashion and
is therefore omitted.

The reader is reminded that if $A$ is an operator on an
$n$-dimensional Hilbert space and $\sigma(AA^*) = 0$, then $A = 0$. This
fact is used several times in the remainder of this chapter.
Theorem 2. If \( A \) and \( B \) are operators on an \( n \)-dimensional Hilbert space, and \( \sigma(\omega(A, A^*)) = \sigma(\omega(B, B^*)) \) for every word \( \omega(x, y) \in W^{n-1}_{2n^2} \cup \{ x^n \} \), then \( A \) and \( B \) are unitarily equivalent.

Proof. Let \( \mathcal{R}_1 \) be the ring of operators generated by \( A \) and \( \mathcal{R}_2 \) the ring generated by \( B \). If \( A^* = \lambda A \) for some scalar \( \lambda \), then \( \sigma[(B^* - \lambda B)(B - \lambda B^*)] = 0 \), and, as mentioned earlier, this implies that \( B^* = \lambda B \). Thus \( A \) and \( B \) are normal and certainly the traces we have assumed equal are sufficient to ensure that \( A \) and \( B \) are unitarily equivalent. So we can suppose that \( A \) and \( A^* \) are linearly independent. We apply Lemma 1.10 to obtain a basis for \( \mathcal{R}_1 \). The basis consists of words \( \omega_i(A, A^*) \); \( i = 1, 2, \ldots, q \), where the \( \omega_i(x, y) \in W^n_{2n^2} \), and we can arrange it so that \( \omega_1(A, A^*) = A \) and \( \omega_2(A, A^*) = A^* \). Let this set of \( q \) words \( \omega_i(x, y) \) be denoted by \( D \) and define a subset \( C \) of \( \mathcal{R}_2 \) by

\[
C = \{ \omega_i(B, B^*) / \omega_i(x, y) \in D \}.
\]

We prove that \( C \) is a basis for \( \mathcal{R}_2 \). Suppose \( \sum_{i=1}^{q} \alpha_i \omega_i(B, B^*) = 0 \).

Let \( p(A, A^*) = [\sum_{i=1}^{q} \alpha_i \omega_i(A, A^*)][\sum_{i=1}^{q} \beta_i \omega_i(A, A^*)]^* \). Then we observe that \( p(x, y) \) is a linear combination of words every one of which is in \( W^{2n}_{2n^2} \) and also that \( p(B, B^*) = 0 \). Now from Lemma 1.9 it follows that \( \sum_{i=1}^{q} \beta_i \omega_i(A, A^*) = 0 \). Thus \( \alpha_i = 0 \) for \( i = 1, \ldots, q \), and the \( \omega_i(B, B^*) \) are linearly independent. If \( C \) were not a basis it could be extended to one, and an argument similar to the one
above would show that the corresponding extension of the basis of $\mathcal{R}_1$ was a linearly independent set. This, of course, is impossible, and thus $C$ is a basis for $\mathcal{R}_2$. Now by the classical theorem for rings of operators on finite dimensional spaces \[13\], $\mathcal{R}_1$ is a direct sum of factors - say

$$\mathcal{R}_1 = F_1 \oplus F_2 \oplus \cdots \oplus F_t.$$  

Denote by $E_i$ the unit of the ring $F_i$. It is possible to choose in each $F_s$ a set of $n_s^2$ matrix units $M^s_{ij}; 1 \leq i, j \leq n_s$, where the $M^s_{ij}$ have the following properties:

$$M^s_{ij} M^s_{kl} = \delta_{jk} M^s_{il} \quad s = 1, 2, \ldots , t; \quad 0 < i, j, k, l \leq n_s$$

$$\sum_{i=1}^{n_s} M^s_{ii} = E_s \quad s = 1, 2, \ldots , t$$

$$\left\{ M^s_{ij} / 0 < i, j \leq n_s \right\}$$  

is a basis for $F_s; s = 1, 2, \ldots , t$

$$M^s_{ij} M^t_{kl} = 0 \quad t \neq s; \quad i, j, k, l \text{ arbitrary}$$

It clearly follows that the union over $s$ of these collections of matrix units form another basis for $\mathcal{R}_1$. For brevity we denote the elements of this basis by $V_1, V_2, \ldots , V_q$. Let the relationship between these two bases for $\mathcal{R}_1$ be given by

$$V_j = \sum_{i=1}^{q} \alpha_{ij} \omega_i(A, A^*) \quad j = 1, 2, \ldots , q.$$  

Then of course the $q \times q$ matrix $(\alpha_{ij})$ is non-singular.
Now for \( j = 1, 2, \ldots, q \), let
\[
U_j = \sum_{i=1}^{q} a_{ij} u_i(B, B^*).
\]

It follows from the fact that \((a_{ij})\) is non-singular that the collection \(U_j, j = 1, 2, \ldots, q\), is another basis for \(\mathbb{R}_2\).

The \(U_k\) have nice multiplication properties too. Suppose
\[
V_c V_d = \lambda V_t
\]
where \(\lambda\) is a scalar and \(1 \leq c, d, t \leq q\). Then it is also true that
\[
U_c U_d = \lambda U_t.
\]

For, consider the quantity \((U_c U_d - \lambda U_t)(U_c U_d - \lambda U_t)^*\). Since
\[
U_j = \sum_{i=1}^{q} a_{ij} u_i(B, B^*)
\]
where \(u_i(x, y) \in W_{\text{n}^2}\), it can be seen that there exists a polynomial \(p(x, y)\) which is a linear combination of words in \(W_{\text{4n}^2}\) with the properties that
\[
p(B, B^*) = (U_c U_d - \lambda U_t)(U_c U_d - \lambda U_t)^*
\]
and
\[
p(A, A^*) = (V_c V_d - \lambda V_t)(V_c V_d - \lambda V_t)^*.
\]

Now by Lemma 1.9 we have
\[
\sigma[\omega(A, A^*)] = \sigma[\omega(B, B^*)]
\]
for every word \(\omega(x, y) \in W_{\text{4n}^2}\). Thus
\[
\sigma[p(B, B^*)] = \sigma[p(A, A^*)] = 0
\]
since \(V_c V_d - \lambda V_t = 0\), and of course it follows that \(U_c U_d = \lambda U_t\).
We now turn around and write \( \omega_j(A, A^\star) \) in terms of the \( V_i \) - say

\[
\omega_j(A, A^\star) = \sum_{i=1}^{q} \beta_{ij} V_i; \quad j = 1, 2, \ldots, q.
\]

Then of course we also have

\[
\omega_j(B, B^\star) = \sum_{i=1}^{q} \beta_{ij} U_i; \quad j = 1, 2, \ldots, q,
\]

because the matrix \( (\beta_{ij}) \) is just \( (\alpha_{ij})^{-1} \). In particular we get

\[ A = \sum_{i=1}^{q} \beta_{i1} V_i, \quad A^\star = \sum_{i=1}^{q} \beta_{i2} V_i, \quad B = \sum_{i=1}^{q} \beta_{i1} U_i, \quad \text{and} \quad B^\star = \sum_{i=1}^{q} \beta_{i2} U_i. \]

The importance of these equations lies in the fact (already proved) that if \( 1 \leq c, d \leq q \), then there exists a \( t \) satisfying \( 1 \leq t \leq q \) and a scalar \( \lambda \) such that \( V_c V_d = \lambda V_t \) (and this implies \( U_c U_d = \lambda U_t \)). So let \( \omega(x, y) \) be any word in \( W \). Then by using the above expressions for \( A, A^\star, B, \) and \( B^\star \) we can form \( \omega(A, A^\star) \) and \( \omega(B, B^\star) \) by multiplication and after reduction (via the equations \( V_c V_d = \lambda V_t \), etc.) we have

\[
\omega(A, A^\star) = \sum_{i=1}^{q} \delta_i V_i \quad \text{and} \quad \omega(B, B^\star) = \sum_{i=1}^{q} \delta_i U_i.
\]

Since we know that \( \sigma(V_i) = \sigma(U_i) \) for \( i = 1, 2, \ldots, q \), it follows that \( \sigma[\omega(A, A^\star)] = \sigma[\omega(B, B^\star)]. \) Thus by the original theorem of Specht, \( A \) and \( B \) are unitarily equivalent. (Specht's proof is incomplete - a complete unpublished proof has been given by A. Brown.)

Theorem 2 shows that one complete finite set of unitary invariants for \( n \times n \) matrices is the collection of traces.
\[
\{ \sigma[\omega(A_A^*)] \;/ \omega(x,y) \in W_{4n^2}^{n-1} \cup \{x^n\} \}.
\]

It is easy to see, however, that some of the equalities

\[
\sigma[\omega(A_A^*)] = \sigma[\omega(B_B^*)], \omega(x,y) \in W_{4n^2}^{n-1} \cup \{x^n\},
\]

follow from others, and, therefore, one can obtain smaller sets of unitary invariants. For example, it suffices to assume equality for the words in \(W_{4n^2}^{n-1} \cup \{x^n\}\) of the forms \(x^i\) and \(x^i \ldots y^j\) in view of the identities \(\sigma(A^*) = \overline{\sigma(A)}\), \(\sigma(CD) = \sigma(DC)\), and Lemma 1.9. Further use of these identities reduces even more the number of needed equalities, but the question of finding by this technique a "best" function \(f(n)\) to be an upper bound on the number of equalities needed is very difficult. Also the finding of such a function does not appear worthwhile to the author since whatever the function \(f(n)\) is, the number \(f(3)\) certainly is nowhere close to the known result. So for the present we content ourselves with the rough upper bound \(f(n) = (n - 1)^4 n^2\) for \(n > 3\). Corollary 1, below, does show, however, that looking at the isolated case \(n = 2\), Murnaghan’s result can almost be obtained by the general theorem.

**Corollary 1.** For \(n = 2\), the collection of traces

\[
\{ \sigma(A), \sigma(A^2), \sigma(AA^*), \sigma([AA^*]^2) \}
\]

is a complete set of unitary invariants.
Proof. It follows from Theorem 2 and the above remarks that the set
\[
\{ \sigma(A), \sigma(A^2), \sigma(AA^*), \sigma([AA^*]^2), \ldots, \sigma([AA^*]^8) \}
\]
is a complete set of unitary invariants. Since $AA^*$ is a $2 \times 2$ matrix, the last six traces in this list are linear combinations of $\sigma(AA^*)$ and $\sigma([AA^*]^2)$ and thus may be omitted.
CHAPTER II

THE UNITARY EQUIVALENCE OF HOMOGENEOUS N-NORMAL OPERATORS

We begin by reviewing some of the relevant definitions.

For the definitions not given herein, see [16] or [13].

Projections $E$ and $F$ in a ring of operators $\mathcal{R}$ are equivalent if there exists a partial isometry $V$ in $\mathcal{R}$ such that $VV^* = E$ and $V^*V = F$. A projection $E$ is abelian if the ring $E\mathcal{R}E$ is abelian, and homogeneous of order $k$ if there are $k$ orthogonal, equivalent, abelian projections $E_i$ in $\mathcal{R}$ such that $E = \sum_{i=1}^{k} E_i$. If $H_1$ and $H_2$ are Hilbert spaces, $A_1$ is an operator on $H_1$ and $\phi$ is a unitary isomorphism of $H_1$ onto $H_2$, then $\phi A_1 \phi^{-1} = A_2$ is an operator on $H_2$. The mapping thus induced by $\phi$ of operators on $H_1$ to operators on $H_2$ will be denoted by $\phi^\circ$. Thus $\phi^\circ(A_1) = \phi A_1 \phi^{-1}$. If there exists such a $\phi$ with $\phi^\circ(A_1) = A_2$, then $A_1$ and $A_2$ are said to be unitarily equivalent. This, of course, is a generalization of the definition given for matrices in Chapter I. Also if $\mathcal{R}_1$ and $\mathcal{R}_2$ are rings on $H_1$ and $H_2$ respectively, with $\phi^\circ(\mathcal{R}_1) = \mathcal{R}_2$, then $\mathcal{R}_1$ and $\mathcal{R}_2$ are likewise said to be unitarily equivalent.

It is now possible to give the definition of the operators in which we will be interested for the remainder of this paper. The definition is most easily given in terms of the kind of ring the operator generates. A nontrivial ring of operators $\mathcal{R}$ is said to be n-normal if it satisfies the identity
\[(*) \quad \sum_{\sigma} \text{sgn} \sigma \ X_{\sigma(1)}X_{\sigma(2)} \ldots X_{\sigma(2n)} = 0\]

where the \(X_i, i = 1, \ldots, 2n\), are arbitrary elements of \(\mathbb{R}\), and the sum is taken over all permutations \(\sigma\) on \(2n\) objects. An \(n\)-normal ring \(\mathbb{R}\) is **homogeneous \(n\)-normal** if the unit of \(\mathbb{R}\) is homogeneous of order \(n\). An operator is an **\(n\)-normal operator** if the ring it generates (the smallest ring containing it) is \(n\)-normal, and a **homogeneous \(n\)-normal** operator if the ring it generates is homogeneous \(n\)-normal.

It is not hard to see that the effect of imposing (*) on a ring \(\mathbb{R}\) is to limit the number of non-zero, orthogonal, equivalent projections in \(\mathbb{R}\) to a maximum of \(n\). Using this fact, Kaplansky [8] and Brown [2] gave a complete structure theory for \(n\)-normal rings, and solved the unitary equivalence problem for binormal operators. Brown's results are basic to this work and are therefore stated here.

**Theorem A.** Let \(\mathcal{H}\) be a Hilbert space and \(\mathbb{R}\) an \(n\)-normal ring on \(\mathcal{H}\). Then there exist subspaces \(M_i\) where \(\sum_i \Theta M_i = \mathcal{H}\), and homogeneous \(i\)-normal rings \(\mathbb{R}_i\) on \(M_i\) respectively such that \(\mathbb{R} = \sum_i \Theta \mathbb{R}_i\). The sum index \(i\) in these two sums runs over some collection of integers between \(0\) and \(n + 1\).

**Theorem B.** Let \(\mathbb{R}_1\) and \(\mathbb{R}_2\) be homogeneous \(n\)-normal rings on Hilbert spaces \(\mathcal{H}_1\) and \(\mathcal{H}_2\) respectively, and let \(\mathbb{Z}_1\) and \(\mathbb{Z}_2\) be their respective centers. Then \(\mathbb{R}_1\) and \(\mathbb{R}_2\) are unitarily
equivalent if and only if $\mathbb{Z}_1$ and $\mathbb{Z}_2$ are, and furthermore if $\mathcal{O}$ is any unitary isomorphism of $\mathbb{Z}_1$ onto $\mathbb{Z}_2$ then there is a unitary isomorphism $\psi^0$ of $\mathcal{R}_1$ onto $\mathcal{R}_2$ which agrees with $\mathcal{O}$ on the ring $\mathbb{Z}_1$.

**Theorem C.** A homogeneous $n$-normal ring $\mathcal{R}$ is unitarily isomorphic to a matrix ring $\mathcal{T}$ consisting of all $n \times n$ matrices with entries in an abelian ring $\hat{\mathbb{Z}}$ containing 1. The center of $\mathcal{R}$ corresponds under this isomorphism to the $n$-fold copy of $\hat{\mathbb{Z}}$ consisting of all matrices

\[
\begin{pmatrix}
W & & \\
& W & \\
& & \ddots \\
& & & W
\end{pmatrix}
\]

with $W$ in $\hat{\mathbb{Z}}$.

The matrix ring $\mathcal{T}$ of theorem C is a ring of operators acting on a direct sum of $n$ copies of $H_1$, the Hilbert space of the ring $\hat{\mathbb{Z}}$, in the obvious fashion. The various operations are carried out in $\mathcal{T}$ just as if the entries were complex numbers.

For example if $A = (A_{ij})$ is in $\mathcal{T}$, then $A^* = (B_{ij})$ where $B_{ij} = A_{ji}^*$.

**Theorem D.** If $A$ is a homogeneous binormal operator and the functions $f_i(A)$, $i = 1, \ldots, 4$ are defined by

\[
f_1(A) = ([A, A^*])^2
\]

\[
f_2(A) = ([A^2, A^*])^2
\]
\[ f_3(A) = (\{ A, A^* \} + \{ A^2, A^* \})^2 \]
\[ f_4(A) = (A[A, A^*])^2 \]

where \( \{ A, B \} = AB - BA \), then the \( f_i(A) \), \( i = 1, \ldots, 4 \) are mutually commuting normal operators. If \( A_1 \) and \( A_2 \) are homogeneous binormal operators then \( A_1 \) is unitarily equivalent to \( A_2 \) if and only if there exists a unitary isomorphism \( \mathcal{O} \) such that
\[ \mathcal{O}(f_i(A_1)) = f_i(A_2) \quad \text{for} \quad i = 1, \ldots, 4. \]

Now let \( A \) be a homogeneous \( n \)-normal operator generating the ring \( \mathcal{R} \). Then \( \mathcal{R} \) is homogeneous \( n \)-normal, and according to Theorem C we can regard \( \mathcal{R} \) as an \( n \times n \) matrix ring with entries in an abelian ring \( \hat{\mathbb{Z}} \) containing \( 1 \). Now the well known Gelfand representation theory is applicable to the ring \( \hat{\mathbb{Z}} \), and accordingly \( \hat{\mathbb{Z}} \) is isomorphic (as a \( C^* \)-algebra) to the \( C^* \)-algebra \( C(\hat{\mathcal{X}}) \) of all continuous complex valued functions on a compact Hausdorff space \( \hat{\mathcal{X}} \), where of course \( \| f \| \) is defined as \( \sup_{x \in \hat{\mathcal{X}}} \| f(x) \| \), and \( f^* = \overline{f} \).

[Recall that a \( C^* \)-algebra is a Banach \( * \)-algebra with the properties that \( \| x^* x \| = \| x \|^2 \) and that \( x^* x + 1 \) is invertible. It was shown in [5] that the most general \( C^* \)-algebra is, up to isomorphism, the most general uniformly closed \( * \)-algebra of operators on a Hilbert space.] The above is true if \( \hat{\mathbb{Z}} \) is closed only in the uniform operator topology (i.e., is a \( C^* \)-algebra). The fact that \( \hat{\mathbb{Z}} \) is also closed in the weak topology leads to the additional restrictions that \( \hat{\mathcal{X}} \) possess a topological base of compact open sets (i.e., be totally disconnected) and that the closure of every
open set in $\mathcal{X}$ be open \([17]\).

It is easy to extend the above representation for $\hat{\mathbb{Z}}$ to a representation for $\hat{\mathcal{R}}$ itself. Thus consider the collection $M_n(\mathcal{X})$ of all continuous functions from $\mathcal{X}$ to the $C^*$-algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices, where $M_n$ is given the usual uniform topology. Algebraic operations can be defined pointwise in $M_n(\mathcal{X})$, and if $\|A(\cdot)\|$ is defined as $\sup \|A(x)\|$, then $M_n(\mathcal{X})$ becomes a $C^*$-algebra. Now let $A = (A_{ij})$ be in $\hat{\mathcal{R}}$, and suppose $A_{ij} \mapsto a_{ij}(x)$ under the correspondence already obtained between $\hat{\mathbb{Z}}$ and $C(\mathcal{X})$.

Then define $\tilde{\Phi}(A) = \lambda(\cdot)$ in $M_n(\mathcal{X})$ where $\lambda(x) = (a_{ij}(x))$. It is easily verified that $\tilde{\Phi}$ is a $*$-algebra isomorphism of $\hat{\mathcal{R}}$ onto $M_n(\mathcal{X})$, and since the norm in a $C^*$-algebra is unique \([9]\), in addition is norm preserving. The above situation, utilized by Brown, can be stated as

**Theorem F.** Let $\mathcal{R}$ be the homogeneous $n$-normal ring of operators on $\mathcal{H} = M \otimes M \otimes M \otimes \ldots \otimes M$ consisting of matrices with entries in the abelian ring $\hat{\mathbb{Z}}$ on $M$. Let $\mathcal{X}$ be the representation space of $\hat{\mathbb{Z}}$. Then $\mathcal{R}$ is isomorphic, as a $C^*$-algebra, to the $C^*$-algebra $M_n(\mathcal{X})$.

We want to apply the results in Chapter I to homogeneous $n$-normal operators, and therefore a definition of trace for such operators is needed. For this purpose let $A$ be a homogeneous $n$-normal operator generating the ring $\mathcal{R}$. By Theorem C, $\mathcal{R}$ is unitarily isomorphic to a ring of $n \times n$ matrices with entries
in an abelian ring \( \mathbb{Z} \) containing 1. Let \( A \) correspond to the matrix \((A_{ij})\) under this isomorphism, and define \( \bigwedge(A) \) to be the operator in the center of \( \mathcal{R} \) corresponding to the central matrix

\[
\begin{pmatrix}
\frac{1}{n} \sum_{i=1}^{n} A_{ii} \\
\frac{1}{n} \sum_{i=1}^{n} A_{ii} \\
\ddots \\
\frac{1}{n} \sum_{i=1}^{n} A_{ii}
\end{pmatrix}
\]

under the above unitary isomorphism.

It is not at all obvious that this definition is independent of the matrix representation taken for \( \mathcal{R} \), and it is given here only because of its utility. The fact is that the above definition coincides with the more general definition of trace given by Dixmier [3] for operators in rings of type \( \mathfrak{I}_n \). We postpone the proof that this is the case to Chapter III, where consideration of non-homogeneous \( n \)-normal operators occurs. It follows easily from the definition above that \( \bigwedge(\cdot) \) is a linear function, that \( \bigwedge(A^*) = [\bigwedge(A)]^* \), and that \( \bigwedge(A) = A \) for \( A \) belonging to the center of \( \mathcal{R} \).

The main result of this chapter can now be stated. Suppose \( C \) is any collection of words \( \omega(x,y) \) with the property that the equality of the traces \( \sigma[\omega(A,A^*)] \) and \( \sigma[\omega(B,B^*)] \) for all \( \omega(x,y) \in C \)
implies the unitary equivalence of the $n \times n$ complex matrices $A$ and $B$. Let $A$ and $B$ be homogeneous $n$-normal operators each generating the homogeneous $n$-normal ring $\mathcal{R}$. Then if
\[ \cap \omega(A,A^*) = \Omega [\omega(B,B^*)] \]
for all $\omega(x,y) \in C$, $A$ and $B$ are unitarily equivalent via a unitary element $U \in \mathcal{R}$. By virtue of Theorem C and the above definition of trace for the operators in $\mathcal{R}$, we can confine our attention to the case where $\mathcal{R}$ is a ring of $n \times n$ matrices with entries in an abelian ring $\hat{\mathbb{Z}}$ containing $1$, and where $A = (A_{ij}) \in \mathcal{R}$ has
\[
\Omega(A) = \left( \begin{array}{ccc}
\frac{1}{n} \sum_{i=1}^{n} A_{ii} \\
\frac{1}{n} \sum_{i=1}^{n} A_{ii} \\
\vdots \\
\frac{1}{n} \sum_{i=1}^{n} A_{ii}
\end{array} \right)
\]

The proof is based on a theorem of Kaplansky [8] to the effect that a $^{\ast}$-automorphism of an $\mathcal{AW}^{\ast}$-algebra of type I which leaves the center elementwise fixed is inner by a unitary element. This theorem is applied locally and the result follows by a summing process.

**Lemma 2.1.** Suppose $A$ and $B$ are homogeneous $n$-normal operators each of which generates the ring $\mathcal{R}$ of $n \times n$ matrices with entries in the abelian ring $\hat{\mathbb{Z}}$ containing $1$. Suppose that $C$ is a collection of words $\omega(x,y) \in W$ with the property that the
collection of traces \( \{ \sigma[\omega(\mathcal{A}, \mathcal{A}^*)] \} / \omega(x, y) \in C \) is a complete set of unitary invariants for \( n \times n \) complex matrices. Suppose that for every \( \omega(x, y) \in C \), \( \bigwedge \omega(\mathcal{A}, \mathcal{A}^*) = \bigwedge \omega(\mathcal{B}, \mathcal{B}^*) \). Then for every word \( \omega(x, y) \in W, \bigwedge \omega(\mathcal{A}, \mathcal{A}^*) = \bigwedge \omega(\mathcal{B}, \mathcal{B}^*) \). (Recall that \( W \) is the collection of all words in \( x \) and \( y \).)

**Proof.** Let \( \mathcal{A} \leftrightarrow \mathcal{A}() \) and \( \mathcal{B} \leftrightarrow \mathcal{B}() \) via the isomorphism of \( \mathcal{M} \) onto \( M_n(\mathcal{K}) \), where \( \mathcal{A}(x) = (a_{ij}(x)) \) and \( \mathcal{B}(x) = (b_{ij}(x)) \). Also, denote by \( \tau_A(\cdot) \) the image of \( \bigwedge \mathcal{A} \) under the isomorphism between \( \mathcal{M} \) and \( M_n(\mathcal{K}) \). Then

\[
\tau_A(x) = \begin{pmatrix}
\frac{1}{n} \sum_{i=1}^{n} a_{ii}(x) \\
\frac{1}{n} \sum_{i=1}^{n} a_{ii}(x) \\
\vdots \\
\frac{1}{n} \sum_{i=1}^{n} a_{ii}(x)
\end{pmatrix}
\]

By virtue of the equalities of the \( \bigwedge \omega(\mathcal{A}, \mathcal{A}^*) \) with the \( \bigwedge \omega(\mathcal{B}, \mathcal{B}^*) \), we have \( \tau_{\omega(\mathcal{A}, \mathcal{A}^*)}(x) = \tau_{\omega(\mathcal{B}, \mathcal{B}^*)}(x) \) for every \( x \in \mathcal{K} \) and every \( \omega(x, y) \in C \). Also from the various definitions and isomorphisms it follows that

\[
\tau_{\omega(\mathcal{A}, \mathcal{A}^*)}(x) = \begin{pmatrix}
\frac{1}{n} \sigma[\omega(\mathcal{A}(x), \mathcal{A}^*(x))] \\
\frac{1}{n} \sigma[\omega(\mathcal{A}(x), \mathcal{A}^*(x))] \\
\vdots \\
\frac{1}{n} \sigma[\omega(\mathcal{A}(x), \mathcal{A}^*(x))]
\end{pmatrix}
\]
for each \( \omega(x,y) \in C \) and each \( x \in \mathcal{F} \), and similarly for \( B \). Thus it is possible to conclude that \( \sigma[\omega(A(x),A^\ast(x))] = \sigma[\omega(B(x),B^\ast(x))] \) for each \( \omega(x,y) \in C \) and each \( x \in \mathcal{F} \), and by hypothesis, \( A(x) \) is unitarily equivalent to \( B(x) \) for each \( x \in \mathcal{F} \). It follows of course that for any word \( \omega(x,y) \in W \) and for any \( x \in \mathcal{F} \),
\[
\sigma[\omega(A(x),A^\ast(x))] = \sigma[\omega(B(x),B^\ast(x))].
\]
Thus, reversing our direction, we conclude that \( \tau_{\omega(A,A^\ast)}(\cdot) = \tau_{\omega(B,B^\ast)}(\cdot) \) for every \( \omega(x,y) \in W \), and it follows that \( \bigwedge[\omega(A,A^\ast)] = \bigwedge[\omega(B,B^\ast)] \) for every \( \omega(x,y) \in W \).

Now an \( n \)-normal ring of operators is an \( AW^\ast \)-algebra. This fact is the basis for Brown's paper [2], and it enables us to use Kaplansky's theorem mentioned earlier. If \( A \) and \( B \) satisfy the hypothesis of Lemma 2.1, it is easy to see via the conclusion of that lemma, that \( p(A,A^\ast) = 0 \) implies \( p(B,B^\ast) = 0 \) for any polynomial \( p(x,y) \). Thus the correspondence \( p(A,A^\ast) \leftrightarrow p(B,B^\ast) \) between the \( ^\ast \)-algebra (algebraic) generated by \( A \) and the \( ^\ast \)-algebra generated by \( B \) is actually a \( ^\ast \)-automorphism. If this automorphism could be extended to \( \mathcal{R} \) the result would follow immediately. However, the author has found it necessary to return to consideration of the problem "locally" to apply Kaplansky's theorem.

**Lemma 2.2.** Suppose that \( A \) is a homogeneous \( n \)-normal operator generating the ring \( \mathcal{R} \) of \( n \times n \) matrices with entries in the abelian ring \( \mathbb{Z} \). Suppose also that \( A \) corresponds to \( \mathcal{A}(\cdot) \) via the isomorphism between \( \mathcal{R} \) and \( M_n(\mathbb{C}) \). Then the matrix \( \mathcal{A}(x) \) generates the full ring of \( n \times n \) complex matrices for every \( x \in \mathcal{F} \) outside some closed nowhere dense subset.
Proof. Let $\mathcal{A}(x) = (a_{ij}(x))$, and recall that the functions $a_{ij}(x)$ are continuous functions from $\mathcal{X}$ to the complex numbers. We prove that the set $S$ of points $x \in \mathcal{X}$ at which $\mathcal{A}(x)$ does not generate the full ring of complex $n \times n$ matrices is a closed nowhere dense subset of $\mathcal{X}$. Suppose $\mathbf{x}$ is not in $S$. Then there are $n^2$ polynomials $p_i(x,y)$ in $x$ and $y$ such that the $n^2$ matrices $p_i(\mathcal{A}(\mathbf{x}), \mathcal{A}^*(\mathbf{x}))$ are linearly independent. Now the $p_i(\mathcal{A}(\cdot), \mathcal{A}^*(\cdot))$ can be regarded as $n \times n$ matrices with continuous functions as entries, and it is obvious from this interpretation that there is an open set $U$ containing $\mathbf{x}$ such that for $x \in U$, the $p_i(\mathcal{A}(x), \mathcal{A}^*(x))$ are linearly independent. Thus $S$ is closed.

To show that $S$ is nowhere dense, it suffices to show that $S$ contains no open set. So suppose that $S$ contains an open set. Then since $\mathcal{X}$ has a base of compact open sets, $S$ contains a compact open set $\overline{U}$. For $x \in \overline{U}$, let $\mathcal{P}(x)$ be the ring generated by $\mathcal{A}(x)$. From the classical theorem for rings of operators on finite dimensional spaces \cite{13}, and from the facts about polynomial identities \cite{11}, it follows that for each $x \in \overline{U}$, $\mathcal{P}(x)$ satisfies the polynomial identity $(\ast)$ where the summation is taken over all permutations of $2(n - 1)$ objects. Now the characteristic function of the compact open set $\overline{U}$ is continuous on $\mathcal{X}$, and therefore corresponds to a projection $\hat{\mathbf{E}}$ in $\mathcal{L}$. Now let $E$ be the central projection in $\mathcal{R}$ consisting of the matrix with $E$ along the principal diagonal and zeros elsewhere. Consider the $\ast$-algebra (algebraic) $\mathcal{A}$ generated by the operator $E\mathbf{A}$. 
This algebra is weakly dense in the ring $E \mathcal{R}$, and consists of all operators of the form $E p(A, A^*)$ where $p(x, y)$ is any polynomial in $x$ and $y$. Now let $X_i$, $i = 1, 2, \ldots, 2(n-1)$ be arbitrary elements of $\mathcal{A}$, and suppose that $X_i$ corresponds to the function $\chi_i(\cdot)$ in $M_n(\mathcal{H})$. Since for $x \in \overline{U}$ the $\chi_i(x)$ belong to the ring $P(x)$,

$$
\sum_{\sigma} (\text{sgn} \sigma) \chi_{\sigma(1)}(x) \ldots \chi_{\sigma(2n-2)}(x) = 0 \quad \text{for each } x \in \overline{U}.
$$

It follows that the $^*$-algebra $\mathcal{A}$ satisfies this polynomial identity, and by continuity $E \mathcal{R}$ does too. But $E \mathcal{R}$ is a direct summand of $\mathcal{R}$ and is therefore homogeneous n-normal with $\mathcal{R}$.

Since a homogeneous n-normal ring contains a collection of $n^2$ matrix units, and it is known that such a collection does not satisfy (*) when the summation is taken over all permutations on $2(n-1)$ objects [1], we have arrived at a contradiction. The result follows.

**Lemma 2.3.** Suppose $\mathcal{A}(\cdot)$ and $\mathcal{B}(\cdot)$ are elements of $M_n(\mathcal{H})$ such that for every word $\omega(x, y) \in W$ and for every $x \in \mathcal{H}$,

$$
\sigma[\omega(\mathcal{A}(x), \mathcal{A}^*(x))] = \sigma[\omega(\mathcal{B}(x), \mathcal{B}^*(x))].
$$

Suppose further that $\overline{x}$ is such that $\mathcal{A}(\overline{x})$ generates the full ring of $n \times n$ matrices.

Then there exists a compact open set $\overline{U}$ containing $\overline{x}$ and a unitary element $\mathcal{U}(\cdot) \in M_n(\overline{U})$ such that for $x \in \overline{U}$, $\mathcal{B}(x) = \mathcal{U}(x) \mathcal{A}(x) \mathcal{U}^*(x)$.

**Proof.** We have seen that if $\mathcal{A}(\overline{x})$ generates the full ring of $n \times n$ matrices then there is a compact open set $\overline{U}$ containing $\overline{x}$ and $n^2$ polynomials $p_i(x, y)$ such that for each $x \in \overline{U}$, the
matrices $p_1(\mathcal{A}(x), \mathcal{A}^*(x))$, ..., $p_n(\mathcal{A}(x), \mathcal{A}^*(x))$ are linearly independent. It follows, of course, that $\mathcal{A}(x)$ generates the full ring of $n \times n$ matrices for each $x \in \overline{U}$. Furthermore it is clear that $p_1(x, y)$ can be taken to be $x$. Now define the functions $\xi_{ij}(\cdot) \in M_n(\overline{U})$ by letting, for $x \in \overline{U}$, $\xi_{ij}(x)$ be the $n \times n$ matrix consisting of a 1 in the $i$-th row and $j$-th column and zeros elsewhere. Then of course the $\xi_{ij}(\cdot)$ are a collection of matrix units for $M_n(\overline{U})$. Furthermore the collection \{ $\xi_{ij}(x)$ \} for $i, j = 1, 2, ..., n$ forms a basis for the full ring of $n \times n$ matrices for each $x \in \overline{U}$. Thus we can write one basis in terms of the other. Let

$$\xi_{ij}(x) = k_{1}^{ij}(x) p_1(\mathcal{A}(x), \mathcal{A}^*(x)) + ... + k_{n}^{ij}(x) p_n(\mathcal{A}(x), \mathcal{A}^*(x)).$$

Then the $k_{t}^{ij}(x)$ are continuous complex-valued functions on $\overline{U}$ calculable as determinants.

Now, just as in Theorem I, it follows that for $x \in \overline{U}$, the matrices $p_1(\mathcal{B}(x), \mathcal{B}^*(x))$, ..., $p_n(\mathcal{B}(x), \mathcal{B}^*(x))$ are a basis for the full ring of $n \times n$ complex matrices. Consider the functions $\mathcal{F}_{ij}(\cdot) \in M_n(\overline{U})$ defined by

$$\mathcal{F}_{ij}(x) = k_{1}^{ij}(x) p_1(\mathcal{B}(x), \mathcal{B}^*(x)) + ... + k_{n}^{ij}(x) p_n(\mathcal{B}(x), \mathcal{B}^*(x))$$

for $i, j = 1, 2, ..., n$. Now by copying over the proof in Theorem I a point at a time in $\overline{U}$, one sees that the functions $\mathcal{F}_{ij}(\cdot)$ satisfy the same equations as do the $\xi_{ij}(\cdot)$; i.e.,
\[ \mathcal{F}_{ij}(\cdot) \mathcal{F}_{kl}(\cdot) = \delta_{jk} \mathcal{F}_{il}(\cdot) \]
\[ \mathcal{F}_{ij}^*(\cdot) = \mathcal{F}_{ji}(\cdot) \]
\[ \sum_{i=1}^{n} \mathcal{F}_{ii}(\cdot) = 1. \]

We are now in a position to define a mapping of \( M_n(\bar{U}) \) onto itself which will be proved to be a *-algebra automorphism of \( M_n(\bar{U}) \) which leaves the center elementwise fixed and maps \( \mathcal{A}(\cdot) \) to \( \mathcal{B}(\cdot) \).

Let \( \mathcal{J}(\cdot) \) be any element of \( M_n(\bar{U}) \). Then for \( x \in \bar{U} \), \( \mathcal{J}(x) \) can be written as

\[ \mathcal{J}(x) = \sum_{i,j=1}^{n} d_{ij}(x) \mathcal{E}_{ij}(x) \]

where the \( d_{ij}(x) \) are continuous functions on \( \bar{U} \). Let

\[ \mathcal{L}(x) = \sum_{i,j=1}^{n} d_{ij}(x) \mathcal{F}_{ij}(x). \]

Then clearly \( \mathcal{L}(\cdot) \in M_n(\bar{U}) \), and we can map \( \mathcal{J}(\cdot) \) to \( \mathcal{L}(\cdot) \). It is obvious that this mapping is a 1-1 mapping of \( M_n(\bar{U}) \) onto itself.

Furthermore it is obvious from the fact that the \( \{ \mathcal{E}_{ij}(\cdot) \} \) and \( \{ \mathcal{F}_{ij}(\cdot) \} \) satisfy the same equations above that this mapping is in fact a *-algebra automorphism on \( M_n(\bar{U}) \). To see that this mapping leaves the center elementwise fixed, it suffices to observe that if \( \mathcal{J}(\cdot) \) is in the center of \( M_n(\bar{U}) \), then \( \mathcal{J}(x) \) is of the form

\[ \mathcal{J}(x) = \sum_{i=1}^{n} d(x) \mathcal{E}_{ii}(x), \]

and therefore

\[ \mathcal{L}(x) = \sum_{i=1}^{n} d(x) \mathcal{F}_{ii}(x) = d(x) \sum_{i=1}^{n} \mathcal{F}_{ii}(x) = d(x) \cdot 1 \]
\[ = d(x) \sum_{i=1}^{n} \mathcal{E}_{ii}(x) = \mathcal{J}(x). \]
The lemma now follows from Theorem 3 in \([B]\) and the fact that 
\(A(*)\) is mapped to \(B(*)\) by the automorphism.

**Theorem II.** Suppose \(A\) and \(B\) are homogeneous \(n\)-normal operators, each of which generates the ring \(\mathcal{R}\). Also suppose that \(C\) is any collection of words \(\omega(x,y) \in W\) with the property that the collection of traces \(\{ \sigma[\omega(A,A^*)] / \omega(x,y) \in C \}\) is a complete set of unitary invariants for \(n \times n\) complex matrices. Furthermore suppose that \(\bigcap \omega(A,A^*) = \bigcap \omega(B,B^*)\) for each \(\omega(x,y) \in C\). Then there exists a unitary \(U \in \mathcal{R}\) such that \(UAU^* = B\).

**Proof.** As mentioned before it suffices to consider the case where \(\mathcal{R}\) is the ring of \(n \times n\) matrices over the abelian ring \(\mathcal{Z}\) containing \(1\). By Lemma 2.1, for every word \(\omega(x,y) \in W\) we have 
\(\bigcap \omega(A,A^*) = \bigcap \omega(B,B^*)\). Now central projections in \(\mathcal{R}\) correspond in the usual fashion to the characteristic functions of compact open subsets of \(M_n(\mathcal{Z})\). Consider a point \(\bar{x} \in \mathcal{Z}\) such that \(\bar{A}()\) generates \(M_n\). Let \(U\) be a compact open set about \(\bar{x}\) such that for \(x \in U\), \(\bar{B}(x) = U(x) A(x) U^*(x)\) where \(U(*) \in M_n(U)\).

Then of course \(U(*)\) can be extended to a unitary element 
\(U(*) \in M_n(\mathcal{Z})\). Now let \(\mathcal{E}(*)\) be the central projection in \(M_n(\mathcal{Z})\) consisting of the characteristic function of \(U\) on the main diagonal and zeros elsewhere. Then the equation

\[ \mathcal{U}(x) A(x) \mathcal{U}^*(x) \mathcal{E}(x) = A(x) \mathcal{E}(x) \]

is valid for each \(x \in \mathcal{Z}\). The corresponding equation in \(\mathcal{R}\) reads

\[ \mathcal{B}E = \mathcal{VAV}^*E \]  

(1)
where \( E \) is the non-zero central projection in \( \mathcal{R} \) corresponding to \( e(\cdot) \), and \( V \) is unitary in \( \mathcal{R} \). Now consider collections of non-zero, orthogonal, central projections \( E_\lambda \) in \( \mathcal{R} \) for which there exists some unitary element \( V \) such that equation (1) is satisfied. One can apply Zorn's Lemma in standard fashion to obtain a maximal collection \( \{ E_\lambda \} \). Let \( F = \sup_\lambda \{ E_\lambda \} = \sum_\lambda E_\lambda \).

Now \( F \) must be the identity operator of \( \mathcal{R} \), because otherwise the central projection \( 1 - F \) is non-zero, and thus corresponds to a compact open subset of \( \mathcal{K} \). Application of Lemmas 2.2 and 2.3 shows that there is a non-zero central projection in \( \mathcal{R} \) which is orthogonal to \( F \) and for which (1) is satisfied. This contradicts the maximality of the collection \( \{ E_\lambda \} \). Now for each \( \lambda \), let \( U_\lambda \) be a unitary element in \( \mathcal{R} \) for which \( BE_\lambda = U_\lambda AU_\lambda E_\lambda \) holds. (By definition, there must be at least one for each \( \lambda \).) Consider the operator

\[
U = \sum_\lambda E_\lambda U_\lambda
\]

in \( \mathcal{R} \). Then

\[
U^* = \sum_\lambda E_\lambda U_\lambda^*,
\]

and

\[
UU^* = U^*U = \sum_\lambda E_\lambda U_\lambda U_\lambda^* = \sum_\lambda E_\lambda = 1.
\]

Thus \( U \) is unitary, and in addition,

\[
UAU^* = (\sum_\lambda E_\lambda U_\lambda) A (\sum_\lambda E_\lambda U_\lambda^*) = \sum_\lambda E_\lambda A U_\lambda U_\lambda^* E_\lambda = \sum_\lambda E_\lambda B = B.
\]
CHAPTER III

NON-HOMOGENEOUS N-NORMAL OPERATORS

We have thus far confined our attention to homogeneous n-normal operators $A$ and $B$ which generate the same ring. In this chapter the question of the unitary equivalence of arbitrary n-normal operators generating perhaps different rings is attacked. The central result is that $A$ and $B$ are unitarily equivalent provided that a certain finite number of traces of words in $A$ and $A^*$ are simultaneously unitarily equivalent to the corresponding traces of words in $B$ and $B^*$.

We begin by defining a trace function for arbitrary n-normal operators. Let $\mathcal{R}$ be any n-normal ring (i.e., ring of type $I_n$). It follows from Theorems A and C that $\mathcal{R}$ can be taken to be of the form $\mathcal{R} = \mathcal{R}_1 \circ \mathcal{R}_2 \circ \ldots \circ \mathcal{R}_n$, where for $1 \leq i \leq n$, $\mathcal{R}_i$ is either vacuous or an $i \times i$ matrix ring with entries in an abelian ring $\hat{\mathbb{Z}}_i$ containing $l_i$. On each of the homogeneous rings $\mathcal{R}_i$ define the function $\Omega_i(\cdot)$ as in Chapter II. If $A = A_1 \circ A_2 \circ \ldots \circ A_n$ is any operator in $\mathcal{R}$, let $D(A) = \Omega_1(A_1) \circ \Omega_2(A_2) \circ \ldots \circ \Omega_n(A_n)$. This function $D(\cdot)$ is a center valued function on $\mathcal{R}$, and several properties of $D(\cdot)$ follow from the corresponding properties of the functions $\Omega_i(\cdot)$. If $A$, $B$ are arbitrary elements of $\mathcal{R}$, $C$ is in the center of $\mathcal{R}$, and $U = \Sigma \circ U_1$ is a unitary element of $\mathcal{R}$, then one gets immediately from the definition of the $\Omega_i(\cdot)$

1) $D(A + B) = D(A) + D(B)$

2) $D(CA) = CD(A)$
3) \( D(C) = C \)

4) \( D(A^*) = [D(A)]^* \)

5) \( D(AB) = D(BA) \)

6) If \( P \) is a positive operator in \( \mathcal{R} \), so is \( D(P) \).

Also by going to the spaces \( M_i (\mathcal{X}_i) \) and considering the situation pointwise, one gets \( \bigcap_i (\bigcup_i \mathbf{A}_i, \mathbf{U}_i^*) = \bigcap_i (\mathbf{B}_i) \). It follows that

7) \( D(UAU^*) = D(B) \).

Now suppose \( \{ H^\lambda \} \) is a net of homogeneous \( i \)-normal operators in \( \mathcal{R}_i \) converging weakly to the homogeneous \( i \)-normal operator \( H = (H^i_{jk}) \). Let \( H^\lambda = (H^\lambda_{jk}) \) where the \( H^\lambda_{jk} \) are, of course, operators on \( \mathcal{X}_i \). Then it is easy to see that for \( l \leq j, k \leq i \), the net \( \{ H^\lambda_{jk} \} \) must converge weakly to the operator \( H^i_{jk} \). It follows that if \( \{ A^\lambda \} \) is a net of operators in \( \mathcal{R} \) converging weakly to \( A \), then \( \{ D(A^\lambda) \} \) converges weakly to \( D(A) \).

The final property of \( D(A) \) we are interested in requires a little bit of proof. It is:

**Lemma 3.1.** Let \( \{ P^\lambda \} \) be a net of positive operators in \( R \) with \( \sup \sup_{\lambda} (P^\lambda) = P \), another positive operator in \( R \). Then

\[ D(P) = \sup_{\lambda} (D(P^\lambda)). \]

**Proof.** It follows from 6) above that \( D(P) \supseteq D(P^\lambda) \) for every \( \lambda \). Let \( S \) be the set of operators \( P^\lambda \) in the net \( \{ P^\lambda \} \). It suffices to prove that \( P \) is in the weak closure of \( S \), because then there is a net \( \{ P^\beta \} \) in \( S \), converging weakly to \( P \), and by the above \( \{ D(P^\beta) \} \) converges weakly to \( D(P) \). And, it follows easily from this that there can be no \( \overline{P} \) satisfying \( D(P) \supseteq D(P^\lambda) \) for each \( \lambda \).
and $D(P) < D(P)$. To prove $P$ is in the weak closure of $S$, let

$$U = \{3 / \left|([P - B]x_i, y_i)\right| < 1 \text{ for } i = 1, 2, \ldots, n\}$$

be an arbitrary basic open set containing $P$. Then it suffices to find a $P_{\lambda} \in U$. Now clearly if $P_{\lambda_1} \supseteq P_{\lambda_2}$, then

$$0 \leq ([P - P_{\lambda_1}] x, x) \leq ([P - P_{\lambda_2}] x, x) \text{ for all } x.$$  

Also of course if $P_{\lambda_1}, P_{\lambda_2}, \ldots, P_{\lambda_k}$ is any collection of $P_{\lambda}$’s then there is a $P_{\lambda} \in S$ satisfying $P_{\lambda_i} \supseteq P_{\lambda_k}$ for $i = 1, 2, \ldots, k$.

It is easy to see, using these two facts and applying the polarization inequality

$$\left|([P-P_{\lambda}] x, y)\right| \leq \frac{1}{4} \left|([P-P_{\lambda}(x+y), (x+y)])\right| + \frac{1}{4} \left|([P-P_{\lambda}(x-y)), (x-y)]\right|$$

$$+ \frac{1}{4} \left|([P-P_{\lambda}(x+iy), (x+iy)])\right| + \frac{1}{4} \left|([P-P_{\lambda}(x-iy), (x-iy)])\right|$$

to the pairs $x_i, y_i, i = 1, \ldots, n$, that there is some $P_{\lambda} \in U$.

It follows from Theorem 3, page 267, of [3] that $D(\cdot)$ is the unique trace function, as defined by Dixmier, for finite $W^*$-algebras.

**Lemma 3.2.** Suppose $A$ is a $j$-normal operator generating the $j$-normal ring $\mathcal{R}$, and suppose $I$ is any collection of words in $x$ and $y$ such that the collection $\{\sigma(\omega(A, A^*)) / \omega(x, y) \in I\}$ is a complete set of unitary invariants for $j \times j$ complex matrices. Then the collection of traces $\{D(\omega(A, A^*)) / \omega(x, y) \in I\}$ generates the center of $\mathcal{R}$.

**Proof.** Let $\mathcal{R}$ as usual be the direct sum $\bigoplus_{i=1}^{\infty} \mathcal{R}_i$ of the matrix rings $\mathcal{R}_i$. For simplicity, we take each $\mathcal{R}_i$ to be non-vacuous (without compromising the proof). Now by Theorem F, $\mathcal{R}_i$
is isomorphic as a $C^*$-algebra to the $C^*$-algebra $M_i(\mathcal{H}_i)$ of continuous functions from the compact Hausdorff space $\mathcal{H}_i$ to the full ring $M_i$ of $i \times i$ complex matrices. Now let $C$ be in the center of $\mathcal{R}$. Since $A$ generates $\mathcal{R}$, there is a net of polynomials $\{p_\lambda(A,A^*)\}$ which converges weakly to $C$. Furthermore from the properties of the Dixmier trace, it follows that the net $\{D(p_\lambda(A,A^*))\}$ converges weakly to $D(C) = C$. Thus the collection $\{D(\omega(A,A^*)) / \omega(x,y) \in W\}$ generates the center of $\mathcal{R}$ as a $\mathcal{W}^*$-algebra. It follows that it suffices to prove that the collections $\{D(\omega(A,A^*)) / \omega(x,y) \in W\}$ and $\{D(\omega(A,A^*)) / \omega(x,y) \in I\}$ generate the same $\mathcal{W}^*$-algebra. But since collections which generate the same $C^*$-algebra also generate the same $\mathcal{W}^*$-algebra, we content ourselves with showing that the above collections generate the same $C^*$-algebra. Consider the set $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \ldots \cup \mathcal{K}_j$, and introduce a topology on $\mathcal{K}$ by letting the open subsets of $\mathcal{K}$ be the sets $U$ such that $U \cap \mathcal{K}_t$ is open in $\mathcal{K}_t$ for $t = 1, 2, \ldots, j$. The resulting topological space is compact Hausdorff. Consider the collection $\mathcal{F}$ of continuous functions $f$ from $\mathcal{K}$ to the complex numbers. We make $\mathcal{F}$ a $C^*$-algebra by defining the operations pointwise and defining

$$\|f\| = \max_{x \in \mathcal{K}_1} |f(x)| + \max_{x \in \mathcal{K}_2} \frac{1}{2}|f(x)| + \ldots + \max_{x \in \mathcal{K}_j} \frac{1}{j}|f(x)|.$$ 

Now a 1-1 correspondence can be established between $\mathcal{F}$ and the center $\mathcal{Z}$ of $\mathcal{R}$ as follows. Let $f(x) \in \mathcal{F}$, and let $f_{\#}(x)$ be the restriction of $f(x)$ to $\mathcal{K}_i$. Then $f_{\#}(x)$ is continuous on $\mathcal{K}_i$, and
corresponds to the central element $\mathcal{C}_i(\cdot) \in M_i(\hat{X}_i)$ such that

$$
\mathcal{C}_i(x) = \begin{pmatrix}
\frac{1}{i} f_i(x) \\
\frac{1}{i} f_i(x) \\
\ddots \\
\frac{1}{i} f_i(x)
\end{pmatrix}
$$

Thus $f(x)$ can be made to correspond to the element $\mathcal{C}_1(\cdot) \otimes \mathcal{C}_2(\cdot) \otimes \ldots \otimes \mathcal{C}_j(\cdot)$ in $M_1(\hat{X}_1) \otimes M_2(\hat{X}_2) \otimes \ldots \otimes M_j(\hat{X}_j)$, and since this last algebra is $C^*$-isomorphic to $\mathbb{R}$, $f(x)$ can be made to correspond to the element of $\mathbb{Z}$ associated with $\mathcal{C}_1(\cdot) \otimes \ldots \otimes \mathcal{C}_j(\cdot)$. It is clear that the correspondence just established between $\mathcal{F}$ and $\mathbb{Z}$ is in truth a $C^*$-isomorphism, and we use this fact to shift the problem of whether $\{D(\omega(A, A^*)) / \omega(x,y) \in I\}$ and $\{D(\omega(A, A^*)) / \omega(x,y) \in W\}$ generate the same $C^*$-algebra over to $\mathcal{F}$.

Let $A = A_1 \otimes A_2 \otimes \ldots \otimes A_j$ and let $A_i \hookrightarrow \mathcal{A}_i(\cdot)$ under the isomorphism between $\mathbb{R}_i$ and $M_i(\hat{X}_i)$. Then under the above isomorphism between $\mathcal{F}$ and $\mathbb{Z}$, the element $D[\omega(A, A^*)] \in \mathbb{Z}$ corresponds to the function $d_\omega(x) \in \mathcal{F}$ defined by

$$
d_\omega(x) = \sigma[\omega(a_i(x), a_i^*(x))]$$

for each $x \in \hat{X}_i$; $i = 1, 2, \ldots, j$. We have finally got ourselves in a position where it suffices to prove that the collections $\{d_\omega(x) / \omega(x,y) \in W\}$ and $\{d_\omega(x) / \omega(x,y) \in I\}$ generate the same sub $C^*$-algebra of $\mathcal{F}$. We use the Stone-Weierstrass theorem, and it suffices to prove that the sets of constancy of the above
collections are the same. Or, what is the same thing, we prove that if \( x_1, x_2 \in \mathcal{X} \) with

\[
d_{\omega(A,A^*)}(x_1) = d_{\omega(A,A^*)}(x_2)
\]

for every \( \omega(x,y) \in I \), then equation (1) is in fact valid for every \( \omega(x,y) \in W \). Suppose \( x_1 \) and \( x_2 \) are elements of the same \( \mathcal{X} \).

Then the matrices \( A_i(x_1) \) and \( A_i(x_2) \) are of the same size, and equation (1) is the same as

\[
\sigma[\omega(A_i(x_1), A_i^*(x_1))] = \sigma[\omega(A_i(x_2), A_i^*(x_2))]
\]

for every \( \omega(x,y) \in I \).

It follows from the definition of \( I \) that the matrices \( A_i(x_1) \) and \( A_i(x_2) \) are unitarily equivalent, and therefore that (2) is valid for every \( \omega(x,y) \in W \). Hence (1) is valid for every \( \omega(x,y) \in W \).

Now suppose \( x_1 \) and \( x_2 \) are elements of different \( \mathcal{X} \) - say \( x_1 \in \mathcal{X}_{i_1} \) and \( x_2 \in \mathcal{X}_{i_2} \) with \( i_1 < i_2 \). Then the matrices \( A_{i_1}(x_1) \) and \( A_{i_2}(x_2) \) are different sizes but they can both be regarded as \( i_2 \times i_2 \) matrices by considering \( A_{i_1}(x_1) \) to be the direct sum of itself and the \((i_2 - i_1) \times (i_2 - i_1) \) zero matrix. It follows just as before then that \( A_{i_1}(x_1) \) and \( A_{i_2}(x_2) \) are unitarily equivalent, and therefore that equation (1) is valid for every \( \omega(x,y) \in W \).

Thus the sets of constancy of the two collections \( \{d_{\omega(A,A^*)}(x) / \omega(x,y) \in I\} \) and \( \{d_{\omega(A,A^*)}(x) / \omega(x,y) \in W\} \) are the same, and the collections generate the same \( C^* \)-algebra.
Lemma 3.3. Suppose \( A \) generates the \( m \)-homogeneous ring \( \mathcal{R} \) with center \( \mathbb{Z} \), and suppose \( B \) generates the \( n \)-homogeneous ring \( \mathcal{T} \) whose center is also \( \mathbb{Z} \). Furthermore, suppose that
\[
D[\omega(A, A^*)] = D[\omega(B, B^*)]
\]
for each \( \omega(x, y) \in K \), where \( \{\sigma[\omega(\lambda, \lambda^*)] / \omega(x, y) \in K\} \) is a set of unitary invariants for \( mn \times mn \) complex matrices. Then \( m = n \).

Proof. Suppose, without loss of generality, that \( m > n \). Now \( \mathbb{Z} \) is \( C^* \)-isomorphic to an algebra \( C(\mathbf{X}) \) of continuous complex valued functions on a compact Hausdorff space \( \mathbf{X} \). It is easy to see, replacing \( \mathcal{R} \) by \( \mathbb{Z} \) in Theorem F, that \( \mathcal{R} \) is \( C^* \)-isomorphic to \( \mathcal{M}_m(\mathbf{X}) \) and \( \mathcal{T} \) is \( C^* \)-isomorphic to \( \mathcal{M}_n(\mathbf{X}) \). Let \( A \leftrightarrow A(\cdot) \) and \( B \leftrightarrow B(\cdot) \) under these isomorphisms. It follows that if \( \omega(x, y) \) is any word in \( x \) and \( y \), then \( D[\omega(A, A^*)] \) corresponds to the \( m \times m \) matrix
\[
\begin{pmatrix}
\frac{1}{m} \sigma[\omega(\lambda(\cdot), \lambda^*(\cdot))]
& & \\
& \ddots & \\
& & \frac{1}{m} \sigma[\omega(\lambda(\cdot), \lambda^*(\cdot))] \\
\end{pmatrix}
\]
and \( D[\omega(B, B^*)] \) corresponds to the \( n \times n \) matrix
\[
\begin{pmatrix}
\frac{1}{n} \sigma[\omega(\delta(\cdot), \delta^*(\cdot))]
& & \\
& \ddots & \\
& & \frac{1}{n} \sigma[\omega(\delta(\cdot), \delta^*(\cdot))] \\
\end{pmatrix}
\]
It follows from the definition of these isomorphisms that
\[ D[\omega(A,A^*)] \] corresponds to the continuous function \( \frac{1}{n} \sigma[\omega(\Lambda(\cdot),\Lambda^*(\cdot))] \) in \( C(\tilde{\mathcal{F}}) \), and \( D[\omega(B,B^*)] \) corresponds to \( \frac{1}{n} \sigma[\omega(\Theta(\cdot),\Theta^*(\cdot))] \). It also follows from the hypotheses that for \( \omega(x,y) \in K \)
\[ \frac{1}{m} \sigma[\omega(\Lambda(\cdot),\Lambda^*(\cdot))] = \frac{1}{n} \sigma[\omega(\Theta(\cdot),\Theta^*(\cdot))] . \]

Now for any \( x \in X \), consider the \( mn \times mn \) complex matrix \( \Lambda'(x) \) which is the direct sum of \( n \) copies of the matrix \( \Lambda(x) \), and also consider the \( mn \times mn \) matrix \( \Theta'(x) \) which is the direct sum of \( m \) copies of \( \Theta(x) \). For each \( \omega(x,y) \in K \) we have
\[ \sigma[\omega(\Lambda'(x),\Lambda'^*(x))] = n \sigma[\omega(\Lambda(x),\Lambda^*(x))] = \]
\[ m \sigma[\omega(\Theta(x),\Theta^*(x))] = \sigma[\omega(\Theta'(x),\Theta'^*(x))] . \]

It follows that the \( mn \times mn \) matrices \( \Lambda'(x) \) and \( \Theta'(x) \) are unitarily equivalent. Now let \( \overline{x} \) be an element of \( \mathcal{F} \) such that \( \Lambda(\overline{x}) \) generates the full ring \( M_m \). (Lemma 2.2 ensures us of the existence of such an \( \overline{x} \).) It is clear that \( \Lambda'(\overline{x}) \) is not i-normal for any \( i < m \). But \( \Theta'(\overline{x}) \) is n-normal, and since \( \Lambda'(\overline{x}) \) is unitarily equivalent to \( \Theta'(\overline{x}) \), \( \Lambda'(\overline{x}) \) must be n-normal too. This contradicts the assumption \( n < m \).

Now suppose \( A \) is an m-normal operator generating the ring \( \mathcal{R} \) and \( B \) is an n-normal operator generating the ring \( \mathcal{T} \). Suppose also that \( \Theta \) is a unitary isomorphism of \( \mathcal{H} \), the Hilbert space of the ring \( \mathcal{R} \), to \( L \), the Hilbert space of the ring \( \mathcal{T} \), which satisfies
\[ \Theta^*[D(\omega(A,A^*))] = D[\omega(B,B^*)] . \]
for each \( \omega(x,y) \in K \) (as defined in Lemma 3.3). It follows easily from Lemma 3.2 that \( \varnothing^{\circ}(\mathcal{Z}_{1R}) = \mathcal{Z}_{1R} \) where the \( \mathcal{Z} \)'s are the centers of the respective rings. Now \( \varnothing^{\circ}(A) = A' \) will be an operator on the space \( L \), and it is clear that \( A' \) is \( m \)-normal with \( A \). Now for any \( \omega(x,y) \),

\[
D[\omega(A',A'^*)] = D[\omega(\varnothing^{\circ}(A),\varnothing^{\circ}(A^*))] = D[\varnothing^{\circ} \omega(A,A^*)].
\]

To see that it is possible to interchange \( D \) and \( \varnothing^{\circ} \) consider the function \( \overline{D} \) defined on \( \mathcal{R} \) by

\[
\overline{D}(A) = \varnothing^{\circ \ -1}(D[\varnothing^{\circ}(A)]).
\]

It follows from the properties of \( D \) and \( \varnothing^{\circ} \) that \( \overline{D} \) has all of the properties of the Dixmier trace function on \( \mathcal{R} \), and by the uniqueness theorem [ ],

\[
\overline{D}(*) = D(*)\).
\]

It follows that for any \( \omega(x,y) \in K \),

\[
D[\omega(A',A'^*)] = D[\omega(B,B^*)].
\]

Thus, by Lemma 3.2, the center of the ring generated by \( A' \) is the ring \( \mathcal{Z}_{1R} \). We use these facts to prove

**Theorem 4.** Suppose that \( A \) is an \( m \)-normal operator generating the ring \( \mathcal{R} \) and \( B \) is an \( n \)-normal operator generating the ring \( T \). Further suppose that there is a unitary isomorphism \( \varnothing \) of the Hilbert space \( H \) of \( \mathcal{R} \) to the Hilbert space \( L \) of \( T \) such that

\[
\varnothing^{\circ}(D[\omega(A,A^*)]) = D[\omega(B,B^*)]
\]

for each \( \omega(x,y) \in K \), where \( K \) is as in Lemma 3.3. Then there is a unitary isomorphism \( \overline{\varnothing} \) of \( H \) to \( L \) such that \( \overline{\varnothing}(\mathcal{R}) = T \), \( \overline{\varnothing}(A) = B \),
and $\mathcal{O}$ agrees with $\varnothing^o$ on the center of $R$.

**Proof.** It follows from the comments above that it suffices to assume that $H = L$, that the rings $R$ and $\mathcal{T}$ have a common center $\mathcal{Z}$, and that $D[\omega(A,A^*)] = D[\omega(B,B^*)]$ for $\omega(x,y) \in K$.

Now $R = \bigoplus_{i=1}^m R_i$ and $T = \bigoplus_{i=1}^n T_i$ where the $R_i$, $T_i$ are homogeneous i-normal or vacuous. Let $E_i$ be the unit of the non-vacuous ring $R_i$. We show that $E_i$ is also the unit of $T_i$. The operator $E_i$ is at least a central projection in $T$ and thus can be written $E_i = Z \oplus F_k$ where $F_k \in T_k$. Suppose some $F_k$, $k \neq i$, is non-zero. Then the ring $F_k T = F_k T_k$ is homogeneous k-normal, and since $F_k \subseteq E_i$, the ring $F_k R = F_k R_i$ is homogeneous i-normal. Now $F_k A$ generates the ring $F_k R$ and $F_k B$ generates $F_k T$. Also these rings have the common center $F_k Z$, and

$$D[\omega(F_k A,F_k A^*)] = F_k D[\omega(A,A^*)] = F_k D[\omega(B,B^*)] = D[\omega(F_k B,F_k B^*)]$$

for $\omega(x,y) \in K$. These facts are in direct contradiction to Lemma 3.3, and it follows that $F_k = 0$ for $k \neq i$. Now

$$0 < E_i = F_i \leq T_i,$$

the unit of the ring $T_i$, and it follows that $T_i$ is necessarily non-vacuous. Also, the fact $E_i \leq T_i$, coupled with the symmetry of the hypotheses on $R$ and $T$, yields $T_i \leq E_i$, or $T_i = E_i$. It follows that the j-homogeneous summand of $R$ is vacuous if and only if the j-normal summand of $T$ is also, and if $R_j$, $T_j \neq 0$, these rings have the common unit $E_i$ and the common center $E_i Z$. 
Let $I$ denote the collection of integers $i$ such that $\mathcal{R}_i$ is non-vacuous. We denote by $\mathbb{H}_i$ the space of the ring $\mathcal{R}_i$, by $A_i$ the $i$-homogeneous part $E_iA$ of $A$ and by $B_i$ the $i$-homogeneous part $E_iB$ of $B$. Thus $E_i$ is the projection on $\mathbb{H}_i$, and $E_i\mathbb{Z}$ is the center of the $i$-homogeneous rings $\mathcal{R}_i$ and $\mathcal{U}_i$. We now apply Theorem B with the result that for each $i \in I$, there is a unitary isomorphism $\psi_i$ of $\mathbb{H}_i$ onto itself with the properties that $\psi_i^\circ(\mathcal{R}_i) = \mathcal{U}_i$ and $\psi_i^\circ$ leaves $E_i\mathbb{Z}$ elementwise fixed. Denote by $\overline{A}_i$ the operator $\psi_i^\circ(A_i)$. It is clear that $\overline{A}_i$ must generate the ring $\mathcal{U}_i$, and if $\overline{A} = \sum_{i \in I} \overline{A}_i$, then $\overline{A}$ generates the ring $\mathcal{U}$.

Furthermore, if $\psi = \sum_{i \in I} \psi_i$, then $\psi$ is a unitary isomorphism of $\mathbb{H} = \sum_{i \in I} \mathbb{H}_i$ onto itself such that $\psi^\circ(A) = \overline{A}$. It follows as before that $D[\omega(A,A^*\mathbb{)}^\circ] = D[\omega(\overline{A},\overline{A}^\circ\mathbb{)}^\circ]$ for every $\omega(x,y)$, and hence

\[
\bigwedge_i[\omega(\overline{A}_i,\overline{A}_i^\circ\mathbb{)}^\circ] = D[\omega(E_i\overline{A},E_i\overline{A}^\circ\mathbb{)}^\circ] = E_iD[\omega(\overline{A},\overline{A}^\circ\mathbb{)}^\circ] = E_iD[\omega(A,A^\circ\mathbb{)}^\circ] = E_iD[\omega(B,B^\circ\mathbb{)}^\circ] = D[\omega(E_iB,E_iB^\circ\mathbb{)}^\circ] = \bigwedge_i[\omega(B_i,B_i^\circ\mathbb{)}^\circ]
\]

for each $\omega(x,y) \in K$ and each $i \in I$. Hence by Theorem 3, for each $i \in I$, there is a unitary element $U_i \in \mathcal{U}_i$ such that $U_i \overline{A}_i U_i^\circ = B_i$. Let $U = \sum_{i \in I} U_i$, and let $\overline{\psi}$ be the composition $\overline{\psi} = U\psi$ of the isomorphisms $U$ and $\psi$. Then $\overline{\psi}$ is a unitary isomorphism mapping $\mathbb{H}$ onto itself such that $\overline{\psi}^\circ(\mathcal{R}) = \mathcal{U}$, $\overline{\psi}^\circ(A) = B$, and $\overline{\psi}^\circ$ is constant on the center of $\mathcal{R}$. 

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