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CONCERNING THE UNIFORMIZATION
OF CERTAIN RIEMANN SURFACES
ALLIED TO THE INVERSE COSINE
AND INVERSE GAMMA SURFACES

by

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A Thesis presented to the faculty
of The Rice Institute in
partial fulfilment of the requirements
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Chapter I

Introduction

1. By the fundamental mapping theorem of Poincaré and Koebe any simply-connected open Riemann surface $\mathcal{F}$ may be mapped (1-1) and conformally (except at the algebraic branch points) onto the Schlicht region of the $z$-plane: $|z| < R \leq \infty$. The well known type problem is to determine whether a given surface $\mathcal{F}$ is parabolic ($R = \infty$), or hyperbolic ($R < \infty$). There is then the further question as to the properties of the mapping function—e.g., if $\mathcal{F}$ is parabolic it is the image of the $z$-plane by a meromorphic function $f(z)$—what can be said about the order of $f(z)$?

2. Beyond those surfaces actually constructed by commencing with a meromorphic function certain simple classes of surfaces (which are to be thought of as covering the Riemann sphere) are known to be parabolic. R. Nevanlinna [1] showed that all $\mathcal{F}$ with no algebraic branch points and only a finite number of logarithmic singularities are parabolic. This was extended by Elfving [1] to include $\mathcal{F}$ with a finite number of algebraic branch points. This in turn was extended by Ullrich [1], [2] to $\mathcal{F}$ with a finite number of "periodic ends" (e.g., half of a cosine surface). Numerous papers have been published on the type of a surface having only logarithmic branch points projecting into a finite number of points on the sphere—e.g. R. Nevanlinna [2], Kakutani [1], Blanc [1].
More generally, results are known for surfaces whose singularities and algebraic branch points project into a finite number of disjoint closed sets on the sphere and such that no two of these points of the surface projecting into the same closed set are connected in any portion of the surface over that closed set. These are the surfaces which may be represented by a line-complex (tree)-Elfving [1], R. Nevanlinna [5], Chapter XI. The strongest result here is that of Wittich [1]: if \( \lambda_n \) denote the number of boundary-knots in the \( n \)-th generation of the line-complex and if \( \sum \frac{1}{\lambda_n} \) diverges, \( \mathcal{F} \) is parabolic.

3. By way of general sufficient conditions for parabolicity there is very little which may actually be applied effectively to a given surface. There is the metric condition of Ahlfors [1]: a necessary and sufficient condition that \( \mathcal{F} \) be parabolic is that there exist a metric on \( \mathcal{F} \) such that if \( L(\rho) \) is the circumference of the circle of radius \( \rho \) with a fixed center then \( \int \frac{d\rho}{L(\rho)} \) diverges; (both \( \rho \) and \( L(\rho) \) are measured in the given metric). This condition is particularly effective when applied to surfaces with no singularities (cf. Ahlfors [3]), e.g. the doubly periodic surfaces. It may also be used to derive Kobayashi's condition for surfaces whose singularities and branch points are isolated (cf. Kobayashi [1]; R. Nevanlinna [5], p. 310).

This criterion has also been used by F. E. Ulrich [1] to study a class of surfaces modelled after that of \( w=1/\Gamma (z) \). We shall obtain wider results on this class of surfaces in Chapters III and VI.
Another useful method which was introduced into the type problem by Ahlfors [2] and since extensively used by several writers is the idea of maps of bounded eccentricity. A discussion and one application of this method will be found in Chapters V and VI.

4. It is enlightening to examine the results mentioned in §§ 2 and 3 in terms of the singularities exhibited by the surface. We recall the following classification of Iversen [1] of the isolated singularities of a Riemann surface defined by the inverse, \( z = \phi(w) \) of a meromorphic function \( w = f(z) \). Let \( w = w_o \) be an isolated singularity of \( F \), \( F_o \) the largest connected piece of \( F \) lying over \( |w - w_o| < r \) and abutting on the singularity. For \( r \) sufficiently small \( F_o \) contains no other singularities. Then we say

I) \( w_o \) is a **directly critical singularity** if along every ray,
\[
\text{arg}(w - w_o) = \text{const}, \text{ of } F_o, \; z = \phi(w) \to \infty .
\]
A directly critical point is said to be of the **first species** if \( F_o \) contains no algebraic branch points for \( r \) sufficiently small; otherwise it is of the **second species**.

II) \( w_o \) is an **indirectly critical singularity** if along no ray,
\[
\text{arg}(w - w_o) = \text{const}, \text{ of } F_o, \; z = \phi(w) \to \infty .
\]

III) If \( w_o \) falls in neither of these categories it is said to be **directly and indirectly critical**.

All the surfaces mentioned in § 2 have isolated singularities and branch points, so their singularities are all directly critical of the first species. To the best of the author's knowledge, all surfaces in the literature which have been proved parabolic possess only this one simplest type of singularity, with the single exception
of the gammic surfaces treated by F. E. Ulrich which possess two directly critical points, one of the first, and one of the second species.

The present paper will prove three different classes of surfaces parabolic, each class containing members exhibiting an indirectly critical point.

5. In chapter II it is shown that all symmetric surfaces "like" that defined by \( w = \cos \sqrt{z} \) are parabolic. The method employed also yields the precise form of the corresponding entire function. The converse is then proved: i.e., all entire functions of this form map the punched plane onto a symmetric semi-cosinice surface.

Chapters III and IV follow an analogous scheme for the symmetric surfaces modelled after those defined by \( w = 1/\Gamma'(z) \) and \( w = \cos z \).

Chapter V contains a brief discussion of the fundamental theorem connecting the type problem with functions of bounded eccentricity.

In Chapter VI this method is used to show that all surfaces of the above three classes are parabolic, whether symmetric or not. The question of the form of the corresponding entire function in the unsymmetric cases is as yet unanswered.

The method employed in Chapters II, III, IV is that of approximation by elliptic surfaces. How far it is capable of extension to other surfaces depends, roughly speaking, on how much may be determined about the numerical nature of rational functions with prescribed surfaces.

6. At this point it is convenient to gather certain well-known theorems to be employed subsequently.
The usual statement of Koebe's distortion theorem is:

Lemma I. Let \( w = g(z) \) be holomorphic, schlicht, in \( |z| < R, \ g(o) = 0, \ g'(a) = l \). Let \( B \) be the image of \( |z| < \alpha \) in the \( w \)-plane. Then the distance of \( w = 0 \) from the boundary of \( B \) is \( \geq \frac{1}{\alpha}R \), the equality prevailing if and only if \( B \) is a "Koebe Schlitzbereich", i.e. the \( w \)-plane slit along a straight line thru \( w = 0 \) from \( w = \frac{1}{\alpha}Re^{i\alpha} \) to \( \infty \) (\( \alpha \) real).

Proofs of this are to be found in various places, e.g. R. Nevanlinna [3] (pp. 81-85), or Bieberbach [1] (pp. 71-75). A useful variant is obtained by replacing \( |z| < R \) by an equivalent Schlitzbereich:

Lemma II. Let \( w = g(z) \) be schlicht, holomorphic, in the \( z \)-plane cut from \( z = R > 0 \) to \( +\infty \) along the real axis; \( g(o) = 0, \ g'(0) = 1 \). Let the image of this Schlitzbereich in the \( w \)-plane be \( B \). Then the distance from \( w = 0 \) to the boundary of \( B \) is \( \geq R \), the equality obtaining if and only if \( B \) is a congruent Schlitzbereich.

The following two versions are also useful. \( K_x (R, \infty) \) denotes the Schlitzbereich obtained by slitting the \( z \)-plane along the real axis from \( z = R > 0 \) to \( z = +\infty \).

Lemma III. Let \( B_1, B_2 \) be two simply-connected schlicht domains in the \( \mathbb{S} \)-plane, \( B_1 \triangleleft B_2 \), \( B_2 \triangleleft B_1 \), containing \( \mathbb{S} = 0 \). Let \( \zeta = g_1(z), \ g_1(o) = 0, \ g_1'(0) = 1, \) map \( K_x (R_1, \infty) \) schlichtly onto \( B_1 \), and \( \zeta = g_2(z), \ g_2(o) = 0, \ g_2'(0) = 1 \) map \( K_x (R_2, \infty) \) schlichtly onto \( B_2 \). Then \( R_1 < R_2 \).

This follows immediately by applying Lemma II to the function \( w = g_2^{-1} [g_1(z)] \) which maps \( K_x (R_1, \infty) \) schlichtly onto a subset of \( K_w (R_2, \infty) \).

Lemma IV. Let \( B \) be a schlicht domain in the \( \mathbb{S} \)-plane containing
the circle $|z| < R$ with the exception of the segment $\rho \leq \Im z \leq R$.

Let $\Im z = g(z)$ map $K_{\rho}(r, \infty)$ schlichtly onto $B$, $g(0) = 0$, $g'(0) = 1$.

Then $r \geq \rho/(1 + \rho/R)^{\gamma}$, the equality obtaining if and only if

$B$ consists exactly of the slit circle mentioned.

This follows immediately from Lemma III and the fact that

$$w = \frac{\sqrt{R}}{4} \left( 1 - \left( \frac{R - r}{R + r} \right)^{\gamma} \right)$$

maps the above slit circle onto the domain $K_w(\rho/(1 + \rho/R)^{\gamma}, \infty)$ with $w(0) = 0$, $w'(0) = 1$.

A proof of the following theorem on families of schlicht functions may be found in Bieberbach [1] (pp. 13-15).

**Lemma V.** Let $A_n$ be a sequence of schlicht domains in the $z$-plane, all containing the origin. Let $B_n$ be a sequence of schlicht domains in the $\Im z$-plane, all containing the origin. Let $F_n(z) = \Im z$ map $A_n$ schlichtly onto $B_n$, $F_n(0) = 0$, $F_n'(0) = 1$. If the sequence $A_n$ converges to its kernel $A$ then a necessary and sufficient condition that $F_n(z)$ converge uniformly in any closed subset of $A$ is that the $B_n$'s converge to their kernel $B$. The limit function $F(z)$ maps $A$ schlichtly onto $B$.

**Remarks.** It is important to note that neither $A$ nor $B$ need be bounded. Also, it is a trivial matter to replace the conditions $F_n(0) = 0$, $F_n'(0) = 1$ by $F_n(0) \to 0$, $F_n'(0) \to 1$.

In case $A$ is the punched $z$-plane, $|z| < \infty$, we have the slightly stronger form:

**Lemma VI.** Let $A_n$, $B_n$, $F_n$ be as in lemma V. Let $A_n$ converge to its kernel $A$: $|z| < \infty$. Then $B_n$ converges to $|\Im z| < \infty$ and $F_n(z) \to z$ uniformly in $|z| \leq R$ for any finite $R$.

The following general form of Darboux's theorem is found in Osgood [1] (p. 400).
Lemma VII. Let \( S \) be a domain of the \( z \)-plane which may be mapped conformally onto a domain \( S' \), bounded by a Jordan curve on the \( w \)-sphere, the mapping remaining continuous on the boundary.

Let \( f(z) \) be meromorphic in \( S \) and assume a boundary value at every boundary point of \( S \), with the exception of at most a single point where it may become infinite, the boundary values at two distinct points being different.

Let there be a value \( w = A \) (finite or infinite) which is not assumed by \( f(z) \) at any interior or boundary point of \( S \). The image of the boundary of \( S \) divides the \( w \)-plane into two domains---let that one not containing \( A \) be denoted by \( \Sigma' \).

Then \( w \cdot f(z) \) maps \( S \) (1-1) and conformally onto \( \Sigma' \), the correspondence remaining continuous on the boundary.

Finally, the following will prove useful (cf. R. Nevanlinna [3] pp. 61-62).

Lemma VIII. Let \( f(z) \) be a single valued analytic function in the region between to Jordan arcs ending in a common point \( P \). If \( \lim_{z \to f} f(z) \) exists along each of these arcs and these two limits are different, then \( f(z) \) assumes every value but two (at most) infinitely often in the given region.
Chapter II

The Symmetric Semi-cosinic Surface

1. This surface $F_w$, over the $w$-plane, is defined by the sequence of real numbers, $\{a_k\}_{k=1}^{\infty}$ with $a_1 > 0$, $a_{2n+1} > a_{2n}$. $F_w$ consists of the sheets $S_1$, $S_2$, ..., $S_k$, ....; $S_1$ is a replica of the $w$-plane cut along the positive real-axis from $w = a_1$ to $w = \infty$; $S_k$ $(k > 1)$ is a replica of the $w$-plane cut along all the real-axis except the segment $(a_{k-1}, a_k)$. $S_1$ and $S_2$ are joined along their cuts from $a_1$ to $+\infty$, forming a first order branch point over $w = a_1$; $S_k$ and $S_{k+1}$ are joined along their cuts from $a_k$ to $(-)^{k-1}\infty$.

2. In the particular case: $a_k = (-)^{k-1}$, $F_w$ is the Riemann surface of the function $z = (\cos^{-1} w)^2$, as is readily seen. Any symmetric semi-cosinic surface $F_w$ is obviously topologically equivalent to this prototype, which is the parabolic surface of the inverse of an entire function; hence $F_w$ is simply-connected.

$F_w$ is open and hence is either of parabolic or hyperbolic type.

3. The nature of the singularities of $F_w$ depends on the sequence $\{a_k\}$. There are the following possibilities:

I) $|a_k| < \infty$, $(k \ n, 2, \ldots)$ and $\lim_{k \to \infty} a_k$ does not exist. Then $F_w$ has a single logarithmic branch point over $w = \infty$ — i.e. a single directly critical point of the first species.

II) There exists a subsequence of the $a_k$ tending to
infinity. Then $F_w$ has a single directly critical point of the second species.

III) There exists $\lim_{w \to \infty} a_\kappa = a \neq 0$. Then $F_w$ has two critical points: a directly critical point of the first species over $w = \infty$ and an indirectly critical point over $w = a$.

4. Let $F_w$ be mapped onto the circle $\mathbb{H} |\gamma| < R \leq \infty$ by the normalized uniformizing function

(1) $\mathcal{S} = \phi(w)$, $w = \phi(\mathcal{S})$

(2) $f(0) = 0 \in S_1$, $f'(0) = 1$

It follows that $f(\mathcal{S})$ is real for real $\mathcal{S}$ and the image of the branch point of $F_w$ over $w = a_\kappa$ is a point $\mathcal{S} = b_\kappa$ of the positive real axis with $0 < b_1 < b_2 < \ldots < b_\kappa < \ldots$. For let $F_w$ be cut into two symmetric halves by slicing $S_\kappa$ $(k > 1)$ from $a_{\kappa-1}$ to $a_\kappa$ and $S_1$ from $-\infty$ to $a_1$. The piece containing the upper half of $S_1$ may be mapped into a semicircle $|\gamma| < R \leq \infty$, $f'(S) > 0$ in such a way that the edges of the slices correspond to the diameter $-R < \mathcal{S} < R$, $w = 0 \in S_1$ corresponds to $\mathcal{S} = 0$. This mapping carries the branch points over $\{a_\kappa\}$ into a monotonic sequence of points $\{b_\kappa\}$ of the positive real axis. Applying the Schwarz reflection principle to the inverse of this mapping function and adjusting to make $f'(0) = 1$ we have exactly the function (1).

5. The fundamental regions in the $\mathcal{S}$-plane will be as follows: $S_\kappa$ $(k > 1)$ is mapped into a portion of $|\gamma| < R$ bounded by two curves $C_{\kappa-1}$, $C_\kappa$ symmetric about the real axis and intersecting the real axis at $b_{\kappa-1}$ and $b_\kappa$ respectively. The uncut segment $(a_{\kappa-1}, a_\kappa)$ of $S_\kappa$ corresponds to the segment $(b_{\kappa-1}, b_\kappa)$;
the two shores of the cut commencing at $a_{\kappa-1}$ correspond to the
two halves of $C_{\kappa-1}$, and the shores of the cut from $a_{\kappa}$ correspond
to the two halves of $C_{\kappa}$. Finally, the sheet $S_1$ is mapped into
the connected portion of $|\gamma|<R$ to the left of $C_1$, the shores of
$S_1$, corresponding to $C_1$ and the segment $(-\infty, a_1)$, to $(-\infty, b_1)$.

The curves $C_{\kappa}$ and the real axis constitute the real-paths
of $w=f(\gamma)$. These real-paths have no intersections except at
the points $b_{\kappa}$ of the real axis. Each $C_{\kappa}$ is a Querschnitt of
the domain $|\gamma|<R$ which it divides into two simply-connected
portions.

All the facts mentioned in this paragraph are essentially
topological and are the same in either of the possible cases
$R<\infty$, $R=\infty$.

6. Elliptic Approximating Surface $\Omega^{(n)}_W$. We shall utilize
the $(n+1)-$-sheeted surface consisting of $S_1$, $S_2$, $\ldots$, $S_n$, $S'_{n+1}$,
being the first $(n+1)$ sheets of $\Omega^{(n)}_W$ with the hanging cut from
$A_{n+1}$ in $S_{n+1}$ deleted—i.e. $S'_{n+1}$ is a replica of the $w$-plane
cut from $A_n$ to $(-)^n\infty$ along the real axis.

$\Omega^{(n)}_W$ is a simply-connected closed surface with $n$ first-order
branch points over $w=a_1, \ldots, a_n$ and a branch point of order
$n$ over $w=\infty$. Hence it is the Riemann surface of the inverse
of a rational function of degree $n+1$ which we may take to be a
polynomial since there is only one point of $\Omega^{(n)}_W$ over $w=\infty$:
\begin{equation}
(3) \quad w = p_\eta(z).
\end{equation}

We normalize this so that
\begin{equation}
(4) \quad p_\eta(0) = 0 \in S_1, \quad p_\eta'(0) = 1.
\end{equation}
Arguing as in §4 we see that the images of the branch points over \( a_1, \ldots, a_n \) are points of the real axis \( b_{\eta_1}, \ldots, b_{\eta_n} \) with \( 0 < b_{\eta_1} < b_{\eta_2} < \ldots < b_{\eta_n} < \).

The fundamental regions in the z-plane are bounded by curves \( C_{\eta_1}, \ldots, C_{\eta_n} \) symmetric about the real axis. \( C_{\eta_1} \) intersects the real axis at \( b_{\eta_1} \). The curves \( C_{\eta_1}, \ldots, C_{\eta_n} \) together with the real axis constitute the real-paths of \( w = P_n(z) \). \( S_k (1 < k < n) \) corresponds to the region between \( C_{\eta_1} \) and \( C_{\eta_k} \); \( S'_1 \) to the region to the left of \( C_{\eta_1} \); and \( S'_{n-1} \) to the region to the right of \( C_{\eta_n} \).

Observing (4) and the fact that the zeros of \( P_n'(z) \) are the points \( b_{\eta_1}, \ldots, b_{\eta_n} \) we have

\[
(5) \quad P_n(z) = \prod_{k=1}^{n} \left( 1 - \frac{z}{b_{\eta_k}} \right)
\]

\[
(6) \quad P_n(z) = \int_0^\infty P_n'(-t) \, dt
\]

I. The domain \( D_n \), being the z-plane cut along the real axis from \( b_{\eta_1} \) to \( +\infty \) is mapped by \( w = P_n(z) \) onto \( \mathbb{C}^{(n)} \) supplied with a cut in \( S'_{n-1} \) extending from \( a_n \) to \( (-)^n \infty \). But the function \( \zeta = \phi(w) \) maps this cut surface \( (1-1) \) onto a region \( \Delta_n \) of the \( \zeta \)-plane bounded by \( |\zeta| = R \), the segment \( (b_n, b_{n+1}) \), the curve \( C_{n+1} \), and containing \( \zeta = 0 \).

Thus \( D_n \) is mapped schlichtly onto \( \Delta_n \) by the composite function

\[
(7) \quad \zeta = \phi(P_n(z)) = \psi_n(z)
\]

By (2) and (4)

\[
(8) \quad \psi_n(0) = 0, \quad \psi_n'(0) = 1.
\]
By Lemma II (p. 5) we have:

(9) \[ b_{n,n} < b_n \ ; \ b_{n,n} < R \ ; \ b_{n,n} < \text{dist} \ 0, C_{n+1} \].

Let \( \tilde{b}_n \) be defined by

(10) \[ \frac{1}{b_n} = \frac{1}{n} \left( \frac{1}{b_{n,1}} + \cdots + \frac{1}{b_{n,n}} \right) \]

(11) \[ b_{n,1} < \tilde{b}_n < b_{n,n} \]

For \( 0 \leq z \leq b_{n,1} \) each term in the product (5) is positive.

From the relation between the geometric and arithmetic means

\[ 0 < P_n^r (z) < \left[ \frac{1}{n} \sum_{k=1}^{n} \left( 1 - \frac{z}{b_{k,n}} \right) \right]^n = \left( 1 - \frac{z}{b_{n}} \right)^n \]

By (6)

\[ 0 < a_1 = \int_{0}^{b_{n,1}} P_n^1 (z) \, dz < \int_{0}^{b_{n,1}} \left( 1 - \frac{z}{b_{n}} \right)^n \, dz < \int_{0}^{\tilde{b}_n} \left( 1 - \frac{z}{b_{n}} \right)^n \, dz \]

\[ a_1 < \frac{\tilde{b}_n}{n+1} \]

\[ \frac{1}{\tilde{b}_n} < \frac{a_1}{n+1} \]

(12) \[ \frac{1}{b_{n,1}} + \cdots + \frac{1}{b_{n,n}} < \frac{n}{(n+1) a_1} < \frac{1}{a_1} \]

For any \( r \leq r' \leq n \)

\[ \frac{r}{b_{n,r'}} < \frac{1}{b_{n,1}} + \cdots + \frac{1}{b_{n,r'}} < \frac{1}{a_1} \]

(13) \[ b_{n,r'} > a_1 r' \quad r'=1, \ldots, n \quad n=1,2, \ldots \]

From (9) and (13): \( R > a_1 n \) for all \( n \); hence \( R = \infty \) and \( \mathbb{F}_n \) is parabolic.

8. Since \( b_{n,n} \rightarrow \infty \) Lemma VI (p. 6) applies to the family of schlicht maps (7). Hence

\[ \phi (P_n (z)) \rightarrow \mathbb{F}\, , \text{ or} \]

(14) \[ P_n (z) \rightarrow \phi^{-1} (z) = f(z) \]
uniformly for $|z| \leq R$. Also of course $P_n'(z) \to f'(z)$, again uniformly for $|z| \leq$ any $R$.

From the theorem of Hurwitz

$$\lim_{n \to \infty} b_n, \nu = b_\nu, \nu = 1, 2, \ldots$$

It is important to note that (15) is not uniform for all $\nu$.

For a fixed $N$, using (12) and (15),

$$\sum_{\nu=1}^{N} \frac{1}{b_\nu} = \lim_{n \to \infty} \sum_{\nu=1}^{N} \frac{1}{b_n, \nu} \leq \lim_{n \to \infty} \sum_{\nu=1}^{N} \frac{1}{b_n, \nu} \leq \frac{1}{a_1}$$

This being valid for all $N$,

$$\sigma_1 = \sum_{\nu=1}^{\infty} \frac{1}{b_\nu} \leq \lim_{n \to \infty} \sum_{\nu=1}^{N} \frac{1}{b_n, \nu} \leq \frac{1}{a_1}$$

Now for $n > n_o$ and $|z| < \frac{1}{2}b_1$, $f'(z) \neq 0$, $P_n'(z) \neq 0$. Hence, taking that determination of the logarithm which is zero for $z = 0$:

$$\log P_n'(z) \to \log f'(z), \quad |z| < \frac{1}{2}b_1$$

By (5):

$$\log P_n'(z) = -z \sum_{\nu=1}^{\infty} \frac{1}{b_n, \nu} - \frac{z^2}{2} \sum_{\nu=1}^{\infty} \frac{1}{b_n, \nu}$$

From (17) and (18) we see that the limit

$$\sigma_2 = \lim_{n \to \infty} \sum_{\nu=1}^{N} \frac{1}{b_n, \nu} \leq \frac{1}{a_1}$$

exists.

Because of (16) it is natural to consider the function

$$\frac{1}{\Pi(z)} = \Pi_{\nu=1}^{\infty} \left(1 - \frac{z}{b_\nu}\right)$$

Still for $|z| < \frac{1}{2}b_1$:

$$\lim_{n \to \infty} \frac{P_n'(z)}{\Pi(z)} = \log \frac{f'(z)}{\Pi(z)}$$

By (18) and (20):

$$\log \frac{P_n'(z)}{\Pi(z)} = -z \left\{ \sum_{\nu=1}^{\infty} \frac{1}{b_n, \nu} - \frac{z^2}{2} \sum_{\nu=1}^{\infty} \frac{1}{b_n, \nu} \right\} - \sum_{\nu=2}^{\infty} \frac{z^2}{p_\nu} \left\{ \sum_{\nu=1}^{N} \frac{1}{b_n, \nu} - \frac{z^2}{2} \sum_{\nu=1}^{N} \frac{1}{b_n, \nu} \right\}.$$
From (13) and (15): $b_v > a_v$, and for $p > 2$ we obtain the limit (which exists by virtue of (21)):

$$\lim_{n \to \infty} \left\{ \sum_{\nu=1}^{n} \frac{1}{b_{n,v} - \sum_{\nu=1}^{\infty} \frac{1}{b_{n,v}}} \right\} = \lim_{n \to \infty} \left\{ \sum_{\nu=1}^{n} \frac{1}{b_{n,v}} - \sum_{\nu=1}^{\infty} \frac{1}{b_{n,v}} \right\} = 2 \sum_{\nu=1}^{\infty} (a_v - a_{\nu+1}) < \frac{C}{n^p}$$

This being true for all $n$, this limit is zero, and by (21),

(22):

$$\log \frac{f'(z)}{\Pi(z)} = -z \lim_{n \to \infty} \left\{ \sum_{\nu=1}^{n} \frac{1}{b_{n,v}} - \sum_{\nu=1}^{\infty} \frac{1}{b_{n,v}} \right\} = -(\sigma_2 - \sigma_1)z$$

Let $\sigma_2 - \sigma_1 = \delta > 0$ by (16), (19). Then

(23) $f'(z) = e^{-\delta z} \Pi(z) = e^{-\delta z} \prod_{\nu=1}^{\infty} \left( 1 - \frac{z}{b_{n,v}} \right)$.

Thus we have proved:

**Theorem 1.** The symmetric semi-cosinic surface $\widetilde{T}_w$ corresponding to any sequence of real numbers $\{\alpha_\nu\}_{\nu=1}^{\infty}$, $\alpha_1 > 0$, $\alpha_{2n+1} > \alpha_{2n}$ is parabolic. There exists a positive monotone sequence of real numbers $\{b_{n,v}\}$ and a non-negative real number $\delta$ such that $\widetilde{T}_w$ is the image of the $z$-plane by the function

$$w = f(z) = \int_{0}^{z} f'(t) \, dt$$

(23) $f'(z) = e^{-\delta z} \prod_{\nu=1}^{\infty} \left( 1 - \frac{z}{b_{n,v}} \right)$,

the branch point of $\widetilde{T}_w$ over $a_\nu$ corresponding to $z = b_{n,v}$.

**Remark** If the normalization $w(0) = 0 \in S$, $w'(0) = 1$ is dropped we obtain any entire function $w = g(z)$ which maps the $z$-plane onto $\widetilde{T}_w$ is of the form

$$g(z) = f(\alpha z + \beta)$$

$\alpha$, $\beta$ constants, $\alpha \neq 0$. 
9. Let \( 0 < x_{\eta, \nu} \leq b_{\eta, \nu} \) ; \( \phi(P_n(x_{\eta, \nu})) = \xi_o \). Let
\[ \Delta_n(\xi_o) \] be that region of the \( \xi \)-plane to the left of \( C_{\eta+1} \)
exclusive of the interval \((\xi_o, b_{\eta+1})\). Then (cf. §7) \( \psi_n(z) = \phi(P_n(z)) \)
maps \( \mathcal{K}(x_{\eta, \nu}, \infty) \) onto \( \Delta_n(\xi_o) \). By lemma II, \( x_{\eta, \nu} < \xi_o \).
In particular if \( x_{\eta, \nu} = b_{\eta, \nu} \), then \( \xi_o = b_{\eta, \nu} \) and \( b_{\eta, \nu} < b_{\eta, \nu} \).
Furthermore, for a given \( \xi_o \) : \( \Delta_n(\xi_o) \subset \Delta_{\eta+1}(\xi_o) \) and by
Lemma III (p. 5), \( x_{\eta, \nu} < x_{\eta+1, \nu} \). Setting \( \xi_o = b_{\nu} \):
\[ b_{\eta, \nu} < b_{\eta+1, \nu} < b_{\nu} \, . \] (24)

Now \( \Delta_n(\xi_o) \) contains the circle \( |\xi| < b_{\eta, \nu} \) by (9)
except for the slit \((\xi_o, b_{\eta, \nu})\), where we assume \( \xi_o < b_{\eta, \nu} \).
Then by Lemma IV (p. 5)
\[ x_{\eta, \nu} > \frac{\xi_o}{(1 + \xi/o/b_{\eta, \nu})^L} . \]
Again setting \( \xi_o = b_{\nu} \):
\[ b_{\eta, \nu} > \frac{b_{\nu}}{(1 + b_{\nu}/b_{\eta, \nu})^L} \quad \text{for} \, b_{\nu} < b_{\eta, \nu} . \] (25)

The problem of determining the sets of numbers \( b_{\eta, \nu}, \ldots, b_{\eta, \nu} \)
may be regarded as a simple problem in polynomial interpolation
where the extreme values of a polynomial together with the order
in which they occur are specified, but not the abscissae; i.e.
given \( a_1, a_2, \ldots, a_n \) with \( a_1 > 0, a_{2k+1} > a_{2k} \), find \( 0 < b_{\eta, \nu} < \ldots < b_{\eta, \nu} \)
such that
\[ a_k = \int_0^{b_{\eta, \nu}} \frac{\prod_{\nu=1}^n (1 - \frac{x}{b_{\eta, \nu}})}{b_{\eta, \nu}} \, dx \quad k = 1, 2, \ldots, n. \]
Regarded from this standpoint there seem to be no simple proofs
of (24) and (25).
10. It is interesting now to consider the converse problem to Theorem I: given an entire function of the form

\[
\begin{align*}
f'(z) &= e^{-\xi} \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{b_\nu}\right) \\
w &= f(z) = \int_{0}^{z} f'(t) dt
\end{align*}
\]

(26)

\[S \geq 0 \quad 0 < b_1 < b_2 < \ldots \]

\[
\sum_{\nu=1}^{\infty} \frac{1}{b_\nu} = \sigma < \infty
\]

construct the Riemann surface \(\mathcal{R}_w\) of the inverse function \(z = \phi(w)\).

This may be done by finding the real-paths and using Darboux's theorem. The real axis is a real path and there will be one other real path \(C_\nu\), symmetric about the real axis, passing thru \(z = b_\nu\) and extending to \(z = \infty\). No two of these real paths intersect except at the points \(b_\nu\), since any such intersection is a zero of \(f'(z)\). Furthermore, these are the only real paths, and each \(C_\nu\) eventually remains in the left half-plane. This last fact is suggested by consideration of specific examples such as \(f(z) = \cos \sqrt{z}\) or \(f(z) = \int_{0}^{z} e^{-\xi} z^{-\frac{1}{2}} \sin \sqrt{z} \, dz\). The simplest method of proof again seems to be by polynomial approximation; and indeed this yields much more information about the real paths than is actually necessary to prove the converse to Theorem I.

11. Now

\[
\lim_{n \to \infty} \frac{1}{\nu=1} \left(1 - \frac{\xi}{b_\nu}\right) = \lim_{n \to \infty} \frac{1}{\nu=1} \left(1 - \frac{\xi}{b_\nu}\right)
\]

\[
\lim_{n \to \infty} \left(1 - \frac{\xi}{n}\right)^n = e^{-\xi}
\]

both uniformly for \(|z|\) bounded. We shall consider the polynomials
(27) \[ Q'_n(z) = (1 - \frac{s z}{\lambda_n})^{\frac{n}{\nu}} \left( 1 - \frac{z}{b_\nu} \right) \]

(28) \[ Q_n(z) = \int_{-\infty}^{x} Q'_n(\tau) \, d\tau \]

where \( \{\lambda_n\} \) is a sequence of positive integers increasing to infinity. We have

(29) \[ Q_n(z) \to f(z), \quad Q'_n(z) \to f'(z) \]

uniformly for \( |z| \) bounded. The procedure is to consider the real-paths of \( Q_n(z) \).

12. Case I. \( \delta = \alpha \). Here \( Q_n(z) \) is similar to the approximating polynomial \( P_n(z) \) used previously. \( Q_n(z) \) is of degree \( n+1 \) and has as its real-paths the real axis and \( n \) curves \( C_{n,1}^+, \ldots, C_{n,1}^- \), \( C_{n,1}^+ \) symmetric about the real axis, \( C_{n,1}^- \) passing thru \( b_\nu \).

Let \( C_{n,1}^\pm \) denote the upper half of \( C_{n,1}^+ \) and \( C_{n,1}^- \) the lower half.

The point \( z = \infty \) is a branch point of order \( n \) for \( Q_n(z) \) so:

\[
\begin{align*}
\begin{cases}
\bar{z} \in C_{n,1}^+ & \arg z \sim \frac{\pi}{2} - \frac{\nu\pi}{n+1} & (z \to \infty) \\
\bar{z} \in C_{n,1}^- & \arg z \sim \frac{\pi}{2} + \frac{\nu\pi}{n+1} & (z \to \infty)
\end{cases}
\end{align*}
\]

The equation of the real-paths is

(31) \[ I_n(x,y) = \int Q_n(x + i y) = \frac{i}{2\pi} \left\{ Q_n(z) - Q_n(\bar{z}) \right\} = 0 \]

which is of degree

\[ n+1 \quad \text{in } x \text{ and } y \quad \text{simultaneously} \]

(32) \[ n \quad \text{in } x \]

\[ 2 \left[ \frac{n}{2} \right] + 1 \quad \text{in } y \]

From this we see that any vertical line \( \Re(z) = x < b_i \), intersects each of the curves \( C_{n,1}^+, \ldots, C_{n,\left[\frac{n}{2}\right]}^- \) exactly twice in conjugate points, and the real axis once; for by (30) there are at least two
intersections with each of these C's which together with the
one intersection with the real axis makes up the precise degree
2 \left[ \frac{n}{2} \right] + 1 of the equation I_N (x_o , y) = 0.

Again, any horizontal line \( \gamma(z) = y_o > 0 \) intersects \( C_{n, \nu}^+ \)
( \( \nu = 1, \ldots, n \) ) at least once and, by (52), exactly once.

Also by considerations of degree, the line \( \arg(z-z_o) = \theta_o \),
z_o real, \( z_o < b_1 \) or \( z_o > b_n \) intersects each half of \( C_{n, \nu} \) once
or not at all:

for \( z_o < b_1 \), \( \frac{\nu \pi}{n+1} < \theta_o < \frac{(\nu+1) \pi}{n+1} \), \( 0 \leq \nu \leq n \)
the intersections are with \( C_{n, \nu}^+ \), \( \ldots, C_{n, n-\nu}^+ \) and \( C_{n, \nu}^- \), \( \ldots, C_{n, n-\nu}^- \).

Similarly for \( z_o > b_n \).

Collecting the information obtained so far:

A) The section of \( C_{n, \nu}^+ \), \( \nu \leq \left\lfloor \frac{n}{2} \right\rfloor \), contained in
\( x < b_1 \) has a negative slope.

B) As the point P travels along \( C_{n, \nu}^+ \) from \( b_\nu \) to \( \infty \) its
ordinate increases monotonically.

C) \( C_{n, \nu} \) is convex with respect to any point of the real
axis exterior to \( (b_1 , b_n) \).

In general A) is not true at all points of \( C_{n, k}^+ \), e.g.
\( C_{n, k} \), \( k > 1 \), may be concave to the right at \( z = b_k \). This may
be seen as follows: set \( z = x + iy \), \( x = b_k = \xi \). From (27), (28),
(51) it is readily seen that in the neighborhood of \( z = b_k \):

\[ I_N (x, y) = y \left[ A_k \xi + B_k (\xi - \frac{y}{3}) + O(|\xi|^2, |y|^3) \right] \]

where

\[ A_k = -\frac{\partial^4 (b_\nu - b_k)}{\partial b_\nu} ; B_k = -\frac{\alpha_k}{\mu \neq k \sum_{\mu k} \frac{1}{b_\nu - b_k}} \]

Dropping the factor \( y \),

\[ \frac{d^2 \xi}{dy^2} \mid_{x, y} = \frac{2}{3} \frac{B_k}{A_k} = -\frac{2}{3} \sum_{\nu=1}^{n} \frac{1}{b_\nu - b_k} \]

\[ \frac{1}{y} \]
whence: $C_{\eta, \kappa}$ is concave to the right (left) at $b_\kappa$ if 

$$\sum' \frac{1}{b_\nu - b_\kappa}$$

is negative (positive).

There are limitations however upon how far $C_{\eta, \kappa}$ may wander to the right etc. heading for home in the left half-plane. Let the extreme abscissa on $C_{\eta, \kappa}$ be $x_\sigma$, with $b_\nu - 1 \leq x_\sigma < b_\nu$.

Assume $\left\lceil \frac{n}{2} \right\rceil \geq \nu - 1$. Then the vertical $R(z) = x_\sigma$ intersects each of $C_{\eta, \kappa}, C_{\eta, \kappa+1}, \ldots, C_{\eta, \nu - 1}$ in at least four points, each of $C_{\eta, \nu}, \ldots, C_{\eta, \left\lceil \frac{n}{2} \right\rceil}$ in at least two points, and the real axis once. Hence by (32)

$$2 \left\lceil \frac{n}{2} \right\rceil + 1 \geq 4 \left[ (\nu - 1) - (k - 1) \right] + 2 \left[ \left\lceil \frac{n}{2} \right\rceil - (\nu - 1) \right] + 1$$

or \[ \nu \leq 2k - 1 \] when $\left\lceil \frac{n}{2} \right\rceil \geq 2k - 2$.

Finally, the roots of $I_\eta (x_\sigma, y) = 0, x_\sigma < b_1$, are all real, distinct, and hence alternate with the (real, distinct) roots of

$$\frac{\partial}{\partial y} I_\eta (x_\sigma, y) = 0.$$

Using (31), (35) is equivalent to

$$Q'_\eta (x_\sigma + i y) + Q'_\eta (x_\sigma - i y) = 0$$

or \[ R Q'_\eta (x_\sigma + i y) = 0 \]

or $\arg Q'_\eta (x_\sigma - i y) = (2k - 1) \frac{\pi}{2}$

If we take $\arg Q'_\eta (x_\sigma) = 0$, the positive roots of (35) in increasing order of magnitude correspond to $k = 1, k = 2; \ldots, k = \left\lceil \frac{n}{2} \right\rceil$.

13. Using (29) we have the following information concerning the real paths of $w = f(z)$ in case $\delta = 0$.

1° The real paths of $w = f(z)$ consist of the real axis and one curve $C_\nu$ thru each critical point $b_\nu$, symmetric about the real axis.

2° As the point $P$ travels along the upper half of $C_\nu$ from $b_\nu$ to $\infty$ its ordinate increases monotonically.
3° Each $C_{\nu}$ eventually enters the half-plane $R(z) < b_1$.

The upper half of $C_{\nu}$ has a finite negative slope at every point of this half-plane. $C_{\nu}$ has precisely one simple intersection with the line $R(z) = x_0 < b_1$. Let this point be $(x_\nu, y_\nu, x_0)$.  

4° The values $y_\nu(x_\nu)$ alternate with the simple roots of $\arg f'(x_\nu - iy) = (2k-1)\frac{\pi}{2}$. If $\arg f'(x_\nu) = 0$ and $\arg f'(x_\nu - i\eta_\nu) = (2k-1)\frac{\pi}{2}$ then $0 < \eta_1 < \eta_2 < \eta_3 < \cdots$.

5° Each $C_{\nu}$ is convex with respect to any point $z < b_1$ of the real axis.

6° $C_{\nu}$ is concave left (right) at $b_\nu$ according as $\sum_{j=1}^{\infty} \frac{1}{2j+1} b_{\nu-j} - b_\nu$ is positive (negative).

7° $C_{\nu}$ lies entirely in the half-plane $R(z) < b_{2\kappa-1}$.

14. To show that the image of the $z$-plane by $w = f(z)$ is a symmetric semi-cosinoidal surface we demonstrate that the strip in the upper half plane bounded by $C_{\nu}^+, C_{\nu+1}^+$ and the segment $(b_{\nu}, b_{\nu+1})$ is mapped schlichtly by $w = f(z)$ on the half plane $(-)^\nu \mathcal{J}(w) > 0$.

This follows from Lemma VII (p. 7). $\mathcal{J}(z)$ has the constant sign $(-)^\nu$ in this region since there are no more real paths "between" $C_{\nu}$ and $C_{\nu+1}$.

Hence we may take $A = (-)^{\nu+1} i$ for the exceptional values. $R(w)$ varies monotonically on the boundary and tends to $(-)^{\nu+1}$ on $C_{\nu}$ and $(-)^{\nu}$ on $C_{\nu+1}$. Hence Lemma VII applies and $\mathcal{J}(w)$ is semi-cosinoidal.

That the asymptotic value of $f(z)$ along $C_{\nu}$ is infinity and not finite follows from 3° and $\left| \int_{C_{\nu}} f'(z)dz \right| = \int_{C_{\nu}} |f'(z)dz|$. 


with \(|f'(z)| > 1\) in the left half-plane.

Notice that here we have used only \(1^\circ\) and \(3^\circ\) of \(\S\,13\).

15. Case II. \(\delta > 0\) Here, see (27), we take

\[
Q_n(z) = (1 - \sum_{\nu=1}^{n} z/\lambda_n)^{\lambda_n} \cdot \prod_{\nu=1}^{n} \left(1 - \frac{z}{b_{\nu}}\right)
\]

where \(\lambda_n\) is a sequence of positive integers tending to infinity. We assume \(\lambda_n > b_{\nu}\) \(\forall\). Without more ado we state: \(Q_n(z)\) has the same general characteristics as \(Q_{n+\lambda_n}(z)\) has in case \(\delta = 0\), except that the real paths \(C_{n, n+1}, \ldots, C_{n, n+\lambda_n}\) now all pass thru a single point, \(z = \lambda_n/\delta\) of the real axis. But this point and all points of \(C_{n, \nu}\) \((\nu > n)\) pass out of the picture as \(n \to \infty\) and hence are of only passing interest. For if \(C_{n, n+1}\) had points interior to \(|z| < R\) for infinitely many \(n\) there would be points of \(C_{n, \nu}\) for all \(\nu < n\) interior to \(|z| < R\).

None of these paths intersect and by (29) there would be a point \(z_0\) \(|z_0| < R\) in the neighborhood of which \(f(z)\) would have segments of infinitely many non-intersecting real paths. This is impossible since \(f(z)\) is entire, and hence holomorphic at \(z = z_0\).

Having shown that these real paths are extraneous it immediately follows that the seven properties of \(\S\,13\) are valid in the general case, with the single exception of \(6^\circ\) which becomes:

\(6^\circ\) \(C_\nu\) is concave left (right) at \(b_\nu\) according as \(S + \sum_{\nu=1}^{\infty} \frac{1}{b_{\nu} - b_\nu}\) is positive (negative).

The proof that the general case gives a semi-cosinusoidal surface now follows exactly as in \(\S\,14\). Thus we have proved

**Theorem II** The image of the \(z\)-plane by any entire function of the form (26) is a symmetric semi-cosinusoidal surface.

16. It is possible to obtain fairly accurate information on the fundamental regions of a particular function of this type.
by using $4^o$ of §13 and an analogous method on the intersections of the real-paths with the line $\gamma(z) = \text{constant}.$

17. The interesting question now arises: what distinguishes the two cases, $\delta = 0$ and $\delta > 0,$ in terms of the branch points $a_\nu?$ This may be partially answered by commencing with

$$(26) \quad f'(z) = e^{-\delta z} \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{b_\nu} \right) = e^{-\delta z} \prod(z)$$

and obtaining estimates on

$$(35) \quad a_\nu = \int_{0}^{b_\nu} f'(z) \, dz.$$  

Let $\lambda < 1$ be the order of $f(z)$ and let $n(\zeta)$ be the number of zeros of $\prod(z)$ which are $\leq \zeta : b_\nu \leq \zeta < b_{\nu+1}, \quad n(\zeta) = \nu$.

Then $\lambda$ is the order of $n(\zeta)$ and the convergence-exponent of

$\{b_\nu\}$ —cf. Nevanlinna [8] (p. 216). Also

$$(36) \quad \sum_{\nu=1}^{\infty} \frac{1}{b_\nu} = \int_{0}^{\infty} \frac{dn(\zeta)}{\zeta} = \int_{0}^{\infty} \frac{n(\zeta)}{\zeta^2} \, d\zeta$$

is convergent.

Let $M(r) = \max_{|z|=r} \prod(z) = \prod(-r)$. If $0 < \lambda < \lambda \log M(r)$ and $n(r)$ are of the same order, type, class (loc. cit. p. 217).

Now

$$\log M(r) = \sum_{\nu=1}^{\infty} \log(1 + \frac{r}{b_\nu}) = \int_{0}^{\infty} \log(1 + \frac{r}{\zeta}) \, dn(\zeta)$$

and

$$\log M(r) = r \int_{0}^{\infty} \frac{n(\zeta)}{\zeta} \, d\zeta \quad \text{whence} \quad$$

$$(37) \quad \log M(r) < \int_{0}^{r} \frac{n(\zeta)}{\zeta} \, d\zeta + r \int_{r}^{\infty} \frac{n(\zeta)}{\zeta^2} \, d\zeta$$

Using (36): \log $M(r) < A(\varepsilon) + \varepsilon \cdot r$, or

$$(38) \quad M(r) < A(\varepsilon) e^{\varepsilon \cdot r}$$

for any $\varepsilon > 0$ and any $\lambda$. If $\lambda < 1$, $n(\zeta) < A(\varepsilon) \zeta^{\lambda+\varepsilon}$.
and using \( r^*_0 = r \) in (37)

\[
M(r) < A(\varepsilon) e^{\lambda r + \varepsilon}
\]

18. Suppose now \( \delta > 0 \). Then by (38)

\[
\sum_{k=0}^{\infty} |a_{k+1} - a_k| = \int_{b_\varepsilon}^{\infty} e^{-\delta r} |\Pi(z)| \, dz 
\leq \int_{b_\varepsilon}^{\infty} e^{-\delta r} M(r) \, dr < A(\varepsilon) e^{-(\delta - \varepsilon)b_\varepsilon}
\]

Or since \( \nu = o(b_\varepsilon) \) and if \( \lambda < 1 \)

\( b_\varepsilon > \frac{C(\varepsilon)}{\nu^{1/(\lambda + \varepsilon)}} \)
we obtain

\[
\sum_{k=0}^{\infty} |a_{k+1} - a_k| = o\left( e^{-\delta \nu} \right) \quad \text{any fixed } \nu
\]

\[
\sum_{k=0}^{\infty} |a_{k+1} - a_k| = o\left[ e^{-A(\varepsilon) \nu^{1/(\lambda + \varepsilon)}} \right] \quad \text{for } \lambda < 1.
\]

We see that \( \delta > 0 \) implies the existence of \( \lim_{k \to \infty} a_k = a \neq \infty \)
and since the \( a_k \)'s are the extreme values of \( f(z) \) on the positive real axis

\[
\lim_{x \to +\infty} f(x) = a \neq \infty
\]

and

\[
|f(x) - a| < \int_{x}^{\infty} e^{-\delta r} M(r) \, dr < A(\varepsilon) \varepsilon^{-(\delta - \varepsilon)x}
\]

Conversely, (42) and (43) imply \( \delta > 0 \), for suppose (42) and \( \delta = 0 \).
Consider the function \( F(z) = (f(z) - a) \cdot (f(-z) - a) \). Let \( \mu(r) \) be its maximum modulus. By (38)

\[
|f(z)| \leq \int_{0}^{1|z|} M(r) \, dr < A(\varepsilon) \varepsilon^{|z|}
\]

and

\[
\mu(r) < A(\varepsilon) \varepsilon^{r}, \quad \text{any } \varepsilon > 0.
\]

If \( F(z) \) were bounded along the real axis, then by the Phragmen-Lindelof principle (Nevanlinna [5] p. 43) either

\[
\lim_{r \to \infty} \frac{\log \mu(r)}{r} > 0 \quad \text{or } |F(z)| \text{ bounded.}
\]
The first is nonsense by (44) and the second since \( F(z) \) is not constant. Thus \( F(z) \) is not bounded on the real axis; i.e. for any constant \( K \) there exists a sequence \( x_{n} \rightarrow +\infty \) such that

\[
|f(x_{n}) - a| > \frac{K}{|f(-x_{n}) - a|} > \frac{K}{2|f(-x_{n})|}
\]

or

\[
(45) \quad |f(x_{n}) - a| > A(\varepsilon) \varepsilon^{-x_{n}} \quad \text{for any } \varepsilon > 0,
\]

which is impossible simultaneously with (45).

Suppose \( b_{n} < x_{n} \), then \( f(x_{n}) \) lies between \( a_{n} \) and \( a_{n+1} \), so one of the two quantities \( |a_{n} - a| \) and \( |a_{n+1} - a| \) is \( > K(\varepsilon) \varepsilon^{-b_{n+1}} \).

Thus we have proved the

**Theorem** If \( \delta > 0 \), then

\[
\lim_{x \to +\infty} f(x) = a \neq \infty \quad \text{and}
\]

\[
\sum_{n=1}^{\infty} \left| a_{n+1} - a_{n} \right| < A(\varepsilon) \varepsilon^{-\delta - \varepsilon b_{n+1}}
\]

If \( \delta = 0 \) and \( \lim_{x \to +\infty} f(x) = a \neq \infty \), then there exists an infinite subsequence of indices \( \nu \) for which

\[
|a_{\nu} - a| > K(\varepsilon) \varepsilon^{-b_{\nu+1}}
\]

19. This leaves hanging the more complicated problem of the connection between the order \( \lambda \) of \( f(z) \) and the branch points.

Roughly speaking, for a fixed \( \delta \), the larger \( \lambda \) the more rapidly does \( \sum_{\nu=1}^{N} |a_{\nu+1} - a_{\nu}| \) increase with \( N \). The only result actually obtained here, and that by laborious chopping, is that in the very particular case \( \delta = 0 \), \( b_{\nu} = \nu^{\alpha} \), we have \( |a_{\nu+1} - a_{\nu}| < M \) for \( 1 < \alpha < 1.95 \). Undoubtedly the correct upper limit here would be \( \alpha = 2 \) for then we have the prototype surface for

\[
w = -\frac{2}{\pi^{\nu}} \cos \nu \sqrt{x} \quad \text{with} \quad |a_{\nu+1} - a_{\nu}| = \frac{4}{\nu^{\nu}}.
\]
Chapter III

The Symmetric Gamnic Surface

1. This surface, \( \mathcal{T}_w \), is obtained by adding a pair of logarithmic ends to the symmetric semi-cosinic surface of the previous chapter. Let the sequence \( a_k (k=1,2,\ldots) \) and the sheets \( S_1, S_2, \ldots \) be as in chapter II §1. The sheet \( S_1 \) is opened up along the negative real axis from \(-\infty\) to \(-a_o\), \( a_o > 0 \). A positive logarithmic end is attached to the upper shore. It is convenient to regard this as being built up of half-sheets \( Q_1, Q_2, \ldots \) as follows: \( \mathcal{W}_{2\nu-1} \) is a replica of \( \mathcal{W} < 0 \), \( \mathcal{W}_{2\nu} \) is a replica of \( \mathcal{W} > 0 \). \( Q_1 \) is attached to the upper shore of \( S_1 \) along \(-\infty, -a_o\). \( Q_1 \) and \( Q_2 \) are joined on \(-a_o, \infty\). In general \( \mathcal{W}_{2\nu-1} \) and \( \mathcal{W}_{2\nu} \) are joined on \(-a_o, \infty\), \( \mathcal{W}_{2\nu} \) and \( \mathcal{W}_{2\nu+1} \) are joined on \(-\infty, -a_o\). The negative logarithmic end, attached to the lower shore of \( S_1 \), is similarly built up of half-sheets \( Q_{-1}, Q_{-2}, \ldots; Q_{-(2\nu+1)} \) being an image of \( \mathcal{W} > 0 \), \( Q_{-2\nu} \) an image of \( \mathcal{W} < 0 \), and so on.

2. This surface is based on that defined by \( \mathcal{W}=1/\Gamma(-z) \) — cf. Lense [1], Ginzel[1]. In this prototype the logarithmic branch point lies over \( w=0 \) rather than \(-a_o\); the algebraic branch points spread out in a uniform fashion: \( \frac{\Gamma(z-h\nu)}{\log \nu} \), \( \lim \nu h
\nu = 0 \). The type of the symmetric gammic surface has been considered by F. E. Ulrich [1] who proves, using Ahlfors' metric condition, that \( \mathcal{T}_w \) is parabolic when \( \operatorname{sgn} a_\nu = (-)^{\nu-1} \) and

\[
|a_\nu| > \frac{m}{\log (\nu+1)} \exp \left[ \frac{\nu^2+1}{\log (\nu+1)} \right]
\]
for some positive constant m. This condition is satisfied by the prototype surface.

3. All these surfaces \( \sigma^T_w \) are open and simply-connected.

The nature of the singularities depends on the sequence \( a_n \).

I) \( |a_n| < M, \lim a_n < \lim a_K \). In this case \( \sigma^T_w \) will have one logarithmic branch point over \( w = -a_\nu \) and two over \( w = \infty \) — all these are directly critical of the first species.

II) \( \lim |a_K| = \infty \). \( \sigma^T_w \) will have one directly critical singularity of the first species over \( w = -a_\nu \) and one of the second species over \( w = \infty \).

III) \( \lim a_K = a \neq \infty \). Then \( \sigma^T_w \) has the three singularities of case I and also an indirectly critical point over \( w = a \).

4. Let \( \sigma^T_w \) be mapped onto the disc \( |\zeta| < R < \infty \) by the normalized function

\[
\begin{align*}
(1) \quad w &= f(\zeta) \\
(2) \quad f(0) &= 0 \in S_1 \\
f'(0) &= 1
\end{align*}
\]

By the same argument as used in chapter II § 4 we see that the image of the branch point over \( w = a_\nu \) is a point \( 1 = b_\nu \) of the real axis with \( 0 \leq b_1 < b_2 < \ldots < b_\nu \rightarrow \infty \).

5. Again the fundamental regions are bounded by the real-paths of \( f(\zeta) \). We have the semi-infinite herringbone of Chapter II § 5 consisting of the real axis and curves \( C_\nu \) thru \( b_\nu \) and symmetric about the real axis. In addition we have a sequence of curves \( \Gamma_{\pm \nu}^i \) \( (\nu = 1, 2, \ldots) \) as follows: \( \Gamma_1^i \) lies in the upper half-plane to the left of \( C_1 \); \( \Gamma_{\nu+1}^i \) lies above \( \Gamma_\nu^i \); \( \Gamma_{-\nu}^i \) is the image of \( \Gamma_\nu^i \) in the real axis. Each \( \Gamma_\nu^i \) is a Querschnitt
of the disc \(|S| < R \leq \infty\) and intersects no other real-path. The region containing \(J = 0\) and bounded by \(\mathcal{P}_r, \mathcal{P}_l, \mathcal{P}_{-l}\) is the image of \(S_1\); the image of \(Q_v\) is the strip between \(\mathcal{P}_r\) and \(\mathcal{P}_{r+1}\).

This essentially topological description is indifferent to 
\(R < \infty\) or \(R = \infty\).

6. The elliptic approximating surface \(\mathcal{F}^{(m)}_W\) is assembled from the \(n+1\) sheets \(S_1, S_2, \ldots, S_n, S_{n+1}\) and the \(2n\) half-sheets \(Q_{\pm 1}, Q_{\pm 2}, \ldots, Q_{\pm (n-1)}, Q_{\pm n}\). The unprimed sheets are exactly as in \(\mathcal{F}_W\); \(S_{n+1}\) is obtained by closing \(S_{n+1}\) schlichtly across the cut \((a_{n+1}, (-)^n a_n)\); \(Q_{1n}, Q_{2n}\) are obtained by connecting \(Q_n, Q_{-n}\) together along their free shores rather than passing on to \(Q_{n+1}\) and \(Q_{-(n+1)}\).

Thus \(\mathcal{F}^{(m)}_W\) is a \(2n+1\) sheeted simply-connected closed surface with \(n\) first order branch points over \(a_1, \ldots, a_n\), an \(n\)th order branch point over \(-a_0\), and a branch point of order \(2n\) over \(w = \infty\). This is the Riemann surface defined by a rational function, which we may take to be a polynomial since there is only one point of the surface over \(w = 0\). We normalize this map to correspond with that of the complete surface.

\[(3) \quad w = P_n(z) \text{ of degree } 2n + 1\]
\[(4) \quad P_n(0) = 0 \in S_1, \quad P'_n(0) = 1.\]

The image of the branch point over \(w = a_v\) is \(z = b_{n_v} (v = 1, 2, \ldots, n)\), \(0 < b_{n_1} < \ldots, < b_{n_n}\). The image of the \(n\)th order branch point is \(z = -C_n, C_n > 0\). The fundamental regions in this case are bounded by curves \(C_{n_v} (v = 1, \ldots, n)\) thru \(b_{n_v}, v\) and by \(2n\) curved rays emanating from \(-C_n\).
7. Let $D_n$ be the $z$-plane cut along the real axis except for the segment $(-c_n, b_{n+1})$. Let $\Delta_n$ be that region of the $\mathfrak{R}$-plane containing the origin and bounded by $|\mathfrak{R}| = R$, $\Gamma_{n+1}^r$, $\Gamma_{-n+1}^l$, $C_{n+1}$, and the interval $(b_n, b_{n+1})$.

As is readily seen $D_n$ and $\Delta_n$ correspond to the same piece of $\mathfrak{R}_w$ by the maps $w = P_n(z)$ and $w = f(\mathfrak{R})$. The composite function

$$
\mathfrak{R} = \psi_n(z) = \phi(P_n(z))
$$

maps $D_n$ simply onto $\Delta_n$, and by (2) and (4)

$$
\psi_n(0) = 0, \quad \psi'_n(0) = 1.
$$

The domain $D_n$ contains the disc $|z| < \min(c_n, b_{n+1})$.

Applying Lemma I (p. 5) to the map of this circle by $\psi_n(z)$ we obtain

$$
R > \frac{1}{4} \min(c_n, b_{n+1}).
$$

8. Now $P_n^r(z)$ has simple zeros at $z = b_{n,\nu}$ ($\nu = 1, \ldots, n$) and an $n$th order zero at $z = -c_n$. Consulting (4),

$$
P_n^r(z) = (1 + \frac{z}{c_n})^n \prod_{\nu=1}^{n} \left(1 - \frac{z}{b_{n,\nu}}\right)
$$

$$
P_n(z) = \int_0^z P_n^r(t)dt.
$$

In particular, from (9)

$$
-Q_{\nu} = \int_0^{-c_n} P_n^r(z)dz \quad Q_1 = \int_0^{b_{n,1}} P_n^r(z)dz.
$$

Let $\tilde{b}_n$ be defined by

$$
\frac{1}{\tilde{b}_n} = \frac{1}{n} \sum_{\nu=1}^{n} \frac{1}{b_{n,\nu}}; \quad \tilde{b}_{n,1} < \tilde{b}_n < b_{n,1}.
$$

For $-c_n < z < b_{n,1}$, all the terms in the product (8) are positive and by the relation between geometric and arithmetic means

$$
P_n^r(z) < (1 + \frac{z}{c_n})^n \left\{ \frac{1}{n} \sum_{\nu=1}^{n} (1 - \frac{z}{b_{n,\nu}}) \right\}^n = (1 + \frac{z}{c_n})^n (1 - \frac{z}{\tilde{b}_n})^n.
$$

We consider two possibilities:

1) $\tilde{b}_n \gg c_n$. Then for $-c_n < z < 0$
\( p_n (z) < 1 \left(1 + \frac{\pi}{C_n} \right)^n \left(1 - \frac{\pi}{C_n} \right)^n \) = \( 1 - \frac{\pi}{C_n} \) \n
and by (10)
\[
q_\omega = \int_{-C_n}^{C_n} (1 - \frac{\pi}{C_n})^n \, dz = c_n \int_{\cos^2 \theta = \sqrt{\pi} / \sqrt{n}}^{\pi / 2} \frac{\Gamma(\eta+1)}{\Gamma(n+\eta/2)} \frac{\sqrt{\pi}}{2} \frac{c_n}{\sqrt{\pi}} \, d\theta
\]
Thus \( \tilde{b}_n \geq c_n \geq A \sqrt{n} \), \( A \) a positive constant.

II) \( c_n \gg \tilde{b}_n \). Then for \( 0 < z < b_{n,1} \)
\[
p_n (z) < 1 \left(1 + \frac{\pi}{C_n} \right)^n \left(1 - \frac{\pi}{C_n} \right)^n \) = \( 1 - \frac{\pi}{C_n} \) \n
and by (10)
\[
a_1 = \int_{\tilde{b}_n}^{1} (1 - \frac{\pi}{C_n})^n \, dz < \int_{\tilde{b}_n}^{1} \left(1 - \frac{\pi}{C_n} \right)^n \, dz \approx \tilde{b}_n \int_{\cos^2 \theta = \sqrt{\pi} / \sqrt{n}}^{\pi / 2} \frac{\sqrt{\pi}}{2} \frac{\tilde{b}_n}{\sqrt{\pi}} \, d\theta
\]
Thus \( c_n \gg \tilde{b}_n \geq A \sqrt{n} \), \( A \) a positive constant.

In either case, by (11):
\[
(12) \quad c_n > A \sqrt{n} \, , \, b_{n,n} > \tilde{b}_n > A \sqrt{n}
\]
with some positive constant \( A \). By (7) then \( R \) must be infinite, i.e., \( f \) is parabolic.

2. We have just shown that the sequence of domains \( D_n \)
converges to the punched \( z \)-plane, and \( \Delta_n \) converges to the punched \( \tilde{z} \)-plane. Hence Lemma VI (p. 6) applies to the sequence of mappings, and \( \psi_n (z) = \phi (p_n (z)) \rightarrow \mathcal{H} \), uniformly for \( |z| \) bounded. Thus
\[
(13) \quad P_n (z) \rightarrow f(z) \, , \, P_n (z) \rightarrow f'(z)
\]
uniformly for \( |z| \) bounded. By Hurwitz' theorem the zeros of
\( P_n (z) \) approach those of \( f'(z) \),
\[
(14) \quad \lim_{n \rightarrow \infty} b_{n,\nu} = b_{\nu} \quad \nu = 1, 2, \ldots
\]
and the multiple zero at \( -c_n \) passes out of the picture by (12).

Note that (14) is of course not uniform for all \( \nu \).
For \(|z| < \frac{1}{2} b_1\) and \(n > n_0\), \(P_{n}^{(1)}(z)\) and \(f'(z)\) do not vanish, and \(\log P_{n}^{(1)}(z) \rightarrow \log f'(z)\). From (8)
\[
\log P_{n}^{(1)}(z) = \left\{ \frac{n}{c_n} - \sum_{\nu=1}^{n-1} \frac{i}{b_{n,\nu}} \right\} z + \left\{ \frac{n}{c_n} + \sum_{\nu=1}^{n-1} \frac{i}{b_{n,\nu}} \right\} z^2 + \ldots
\]
and therefore the following limits exist:
\[
\lim_{n \rightarrow \infty} \left( \frac{n}{c_n} - \sum_{\nu=1}^{n-1} \frac{i}{b_{n,\nu}} \right) = \sigma_1^n
\]
\[
\lim_{n \rightarrow \infty} \left( \frac{n}{c_n} + \sum_{\nu=1}^{n-1} \frac{i}{b_{n,\nu}} \right) = \sigma_2^n > 0.
\]
All terms in the parenthesis of (17) are positive, hence bounded for all \(n\):
\[
\frac{n}{c_n} < M, \quad \sum_{\nu=1}^{n-1} \frac{i}{b_{n,\nu}} < M.
\]
Now for any fixed \(N\), by (14)
\[
\frac{1}{2} \sum_{\nu=1}^{N} \frac{i}{b_{\nu}} = \lim_{n \rightarrow \infty} \frac{1}{2} \sum_{\nu=1}^{n} \frac{i}{b_{n,\nu}} \leq \lim_{n \rightarrow \infty} \frac{1}{2} \sum_{\nu=1}^{n-1} \frac{i}{b_{n,\nu}} \leq \sigma_2^n
\]
and allowing \(N\) to become infinite
\[
\sigma_3 = \frac{1}{2} \sum_{\nu=1}^{\infty} \frac{i}{b_{\nu}} \leq \sigma_2^n
\]
Now from (18), (19) and the monotone character of the sequences \(b_\nu, b_{n,\nu}\)
\[
c_n > A \sqrt{n}, \quad b_{n,\nu} > A \sqrt{\nu}, \quad b_\nu > A \sqrt{\nu}
\]
for some positive constant \(A\). For \(p \geq 3\) and any \(N\)
\[
\lim_{n \rightarrow \infty} \left| \sum_{\nu=1}^{n} \frac{i}{b_{\nu}^p} - \sum_{\nu=1}^{n-1} \frac{i}{b_{n,\nu}^p} \right| \leq \lim_{n \rightarrow \infty} \sum_{\nu=1}^{N} \left| \frac{i}{b_{\nu}^p} - \frac{i}{b_{n,\nu}^p} \right| + \sum_{\nu=N+1}^{\infty} \frac{i}{b_{\nu}^p} + \lim_{n \rightarrow \infty} \sum_{\nu=N+1}^{n-1} \frac{i}{b_{n,\nu}^p}
\]
Using (14) and (20)
\[
\lim_{n \rightarrow \infty} \left| \sum_{\nu=1}^{n} \frac{i}{b_{\nu}^p} - \sum_{\nu=1}^{n-1} \frac{i}{b_{n,\nu}^p} \right| < C \sum_{\nu=N+1}^{\infty} \left( \frac{i}{b_{\nu}^p} \right) < C N^{-\frac{1}{p} - \frac{1}{2}}
\]
where \(C\) is a constant independent of \(N\).
Allowing $N$ to become infinite,

\[
(21) \quad \lim_{n \to \infty} \sum_{\nu=1}^{n} \frac{1}{b_{\nu}^{n}} = \sum_{\nu=1}^{\infty} \frac{1}{b_{\nu}} \quad p > 3.
\]

So, cf. (15), for $|z| < \frac{3b}{2}$,

\[
\log f'(z) = \lim_{n \to \infty} \log P_{n}(z) = \lim_{n \to \infty} \left\{ \frac{n}{c_{n}} - \frac{1}{b_{\nu}} \right\} z - \lim_{n \to \infty} \frac{1}{b_{\nu}} \left\{ \frac{n}{c_{n}} + \sum_{\nu=1}^{n} \frac{1}{b_{\nu}} \right\} z^2 \nonumber
\]

\[
+ \lim_{n \to \infty} \frac{\sum_{\nu=1}^{n} \frac{1}{b_{\nu}^{n}}}{n} - \frac{1}{p} \sum_{\nu=1}^{n} \frac{1}{b_{\nu}^{p}} \right\} z^p.
\]

By (16), (17), (20), (21)

\[
(22) \quad \log f'(z) = \sigma_1 z - \sigma_3 z^2 - \sum_{\nu=1}^{\infty} \frac{z^p}{p} \sum_{\nu=1}^{\infty} \frac{1}{b_{\nu}^p}.
\]

By (19) the canonical product

\[
(23) \quad P(z) = \prod_{\nu=1}^{\infty} \left\{ (1 - \frac{z}{b_{\nu}}) e^\frac{z}{b_{\nu}} \right\}
\]

converges, and for $|z| < \frac{3b}{2}$

\[
(24) \quad \log P(z) = -\frac{z}{2} \sum_{\nu=1}^{\infty} \frac{1}{b_{\nu}} - \sum_{\nu=1}^{\infty} \frac{z^p}{p} \sum_{\nu=1}^{\infty} \frac{1}{b_{\nu}^p}.
\]

Comparing (22) and (24) and using (19):

\[
\log f'(z) = \sigma_1 z - (\sigma_2 - \sigma_3) z^2 + \log P(z)
\]

with $\sigma_2 - \sigma_3 > 0$, or writing $\sigma_2 - \sigma_3 = \alpha$, $\sigma_1 = \beta$

\[
(25) \quad f'(z) = e^{-\alpha z^2 + \beta z} \prod_{\nu=1}^{\infty} \left\{ (1 - \frac{z}{b_{\nu}}) e^\frac{z}{b_{\nu}} \right\}
\]

$\alpha > 0$, $\beta$ real.

Now it is conceivable (cf. § II) that in certain cases $\sum_{\nu=1}^{\infty} \frac{1}{b_{\nu}}$ would be convergent, allowing the exponential terms in the product to be removed and combined with the term $\beta z$, giving the simpler representation

\[
(26) \quad f'(z) = e^{-\alpha z^2 + \beta' z} \prod_{\nu=1}^{\infty} \left( 1 - \frac{z}{b_{\nu}} \right)
\]

in which case either $\alpha > 0$ or $\alpha = 0$ and $\beta' > 0$ since (cf. Theorem II, p 21) $\sigma_{\nu}$ is not a semi-cosine surface. This
restriction may be written, for the form (25), as: either $\alpha > 0$
or $\beta + \sum_{\nu=1}^{\infty} \frac{1}{b_\nu} > 0$. Summarizing these results we have

Theorem III. The symmetric gammaic surface, $\psi_\nu$ (described in §1) is always parabolic. Furthermore $\psi_\nu$ is the image of the $z$-plane by an entire function $w = f(z) = \int_0^z f'(t)dt$ with

$$f'(z) = e^{-\alpha z^2 + \beta z} \prod_{\nu=1}^{\infty} \left\{(1 - \frac{z}{b_\nu}) e^{\frac{z}{b_\nu}}\right\}$$

with $0 < b_1 < b_2 < \ldots,$ $\sum_{\nu=1}^{\infty} \frac{1}{b_\nu}$ convergent, and $\alpha > 0,$ $\beta$ real. In case $\sum_{\nu=1}^{\infty} \frac{1}{b_\nu}$ converges, then either $\alpha > 0$ or $\alpha = 0$ and $\beta + \sum_{\nu=1}^{\infty} \frac{1}{b_\nu} > 0$.

10. We now consider the converse to Theorem III: to show that any function $w = f(z)$, with $f'(z)$ of the form (25), $\alpha, \beta, b_\nu$ satisfying the results of Theorem III, maps the $z$-plane onto a symmetric gammaic surface.

Let

$$\beta_n = \beta + \sum_{\nu=1}^{\infty} \frac{1}{b_\nu}$$

Then for any sequence of positive integers, $\lambda_n$, which increase rapidly enough the polynomials

$$Q_n(z) = \int_0^z Q_0(t)dt$$

$$Q_n'(z) = (1 - \frac{\alpha z}{\lambda_n})^\lambda_n (1 + \frac{\beta z}{\lambda_n})^\lambda_n \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{b_\nu}\right)$$

approximate $f(z)$:

$$\lim_{n \to \infty} Q_n'(z) = f'(z), \lim_{n \to \infty} Q_n(z) = f(z)$$

uniformly for $|z|$ bounded. This depends on

$$\lim_{N \to \infty} \left(1 - \frac{z}{N}\right)^N = e^{-z}, \text{ uniformly for } |z| \leq R.$$ 

So

$$\lim_{n \to \infty} \left(1 - \frac{\alpha z}{\lambda_n}\right)^\lambda_n = e^{-\alpha z^{\lambda_n}}.$$
Also

\[ \left( 1 + \frac{\beta_n z}{\lambda_n} \right)^{\lambda_n} \prod_{\nu=1}^{\eta} \left( 1 - \frac{z}{b_{\nu}} \right) = e^{\beta_n z} \prod_{\nu=1}^{\eta} \left\{ \left( 1 - \frac{z}{b_{\nu}} \right) e^{z/b_{\nu}} \right\} \]

\[ \leq \left| \left( 1 + \frac{\beta_n z}{\lambda_n} \right)^{\lambda_n} - e^{\beta_n z} \prod_{\nu=1}^{\eta} \left( 1 - \frac{z}{b_{\nu}} \right) \right| \]

\[ + \left| e^{\beta_n z} \prod_{\nu=1}^{\eta} \left( 1 - \frac{z}{b_{\nu}} \right) - e^{\beta_n z} \prod_{\nu=1}^{\eta} \left\{ \left( 1 - \frac{z}{b_{\nu}} \right) e^{\alpha / b_{\nu}} \right\} \right| \]

The last term on the right of (33) tends to zero, uniformly for \( |z| \leq R \), by the very definition of an infinite product. If we choose \( \lambda_n \) so that, by virtue of (31),

\[ \left( 1 + \frac{\beta_n z}{\lambda_n} \right)^{\lambda_n} - e^{\beta_n z} \leq \left[ \eta \frac{\lambda_n}{\prod_{\nu=1}^{\eta} \left( 1 + \frac{z}{b_{\nu}} \right)} \right]^{-1} \]

for \( |z| \leq n \) then the left hand member of (33) tends to zero uniformly for \( |z| \leq \) any \( R \), or

(34) \[ \lim_{\eta \to \infty} \left( 1 + \frac{\beta_n z}{\lambda_n} \right)^{\lambda_n} \prod_{\nu=1}^{\eta} \left( 1 - \frac{z}{b_{\nu}} \right) = e^{\beta_n z} \prod_{\nu=1}^{\eta} \left\{ \left( 1 - \frac{z}{b_{\nu}} \right) e^{\alpha / b_{\nu}} \right\} \]

Finally, combining (32) and (34) we have the first part of (30). The second follows immediately.

11. We shall now construct the Riemann surface for the function \( z = \phi(w) \), inverse to \( w = f(z) \), for the case \( \alpha > 0 \). The general idea is to use the fact that the real-paths of \( Q_n(z) \) tend to those of \( f(z) \).

Increasing the terms in the sequence \( \lambda_n \) does not affect (30) so we shall assume that \( \lambda_n / |\beta_n| > \sqrt{\lambda_n / \alpha} \) \( > b_n \). By (29) it is readily seen that the real-paths of \( Q_n(z) \) are as follows:
1) the real axis, 2) curves \( C_{n, \nu} \) ( \( \nu = 1, \ldots, n \) thru \( b_{\nu} \),
3) 2 \( \lambda_{n} \) curved rays \( \bar{l}_{n, \nu} \) ( \( \nu = 1, \ldots, \lambda_{n} \) ) emanating from \( z = -\sqrt{\lambda_{n}/\alpha} \),
4) 2 \( \lambda_{n} \) curved rays \( \bar{l}' \) from \( z = \sqrt{\lambda_{n}/\alpha} \),
and 5) 2 \( \lambda_{n} \) curved rays \( \bar{l}'' \) from \( z = -\lambda_{n}/\lambda'_{n} \). This whole scheme is symmetric about the real axis.

The rays \( \bar{l}' \) and \( \bar{l}'' \) disappear in the limit, for if not some circle \( |z| \leq R \) would contain sections of the limiting curves of all the \( C_{n, \nu} \) and \( \bar{l}_{n, \nu} \), and \( f(z) \) would possess sections of infinitely many different real-paths in \( |z| \leq R \), which is impossible for an entire function.

The curves \( C_{n, \nu} \) will tend to limiting curves \( C_{\nu} \). They cannot pass out of the picture since \( C_{n, \nu} \) contains the fixed point \( z = b_{\nu} \) for all \( n \), but it is conceivable that might consist of several pieces, all ends being at \( z = \infty \). The strip, \( D_{n, \nu} \), bounded by \( C_{n, \nu} \) and \( C_{n, \nu+1} \), is mapped by \( Q_{n}(z) \) onto a \( w \)-plane, \( \Delta_{n, \nu} \) slit along the real axis except for the segment \( (Q_{n}(b_{\nu}), Q_{n}(b_{\nu+1})) \). The sequence \( \Delta_{n, \nu} \) converges to its kernel \( \Delta_{\nu} \) the \( w \)-plane slit along the real axis except for the segment \( (a_{\nu}, a_{\nu+1}) = (f(b_{\nu}), f(b_{\nu+1})) \). Since these maps are schlicht and converge, and since the curves \( C_{n, \nu} \) and \( C_{n, \nu+1} \) converge to \( C_{\nu} \) and \( C_{\nu+1} \), we may apply Lemma V (p, b), replacing the origin by \( \frac{1}{2}(b_{\nu} + b_{\nu+1}) \) and noting the remark following the Lemma. Thus \( w = f(z) \) maps \( D_{\nu} \), the domain "between" \( C_{\nu} \) and \( C_{\nu+1} \) onto \( \Delta_{\nu} \). Thus \( D_{\nu} \) is simply connected. Furthermore \( C_{\nu} \) is all in one piece, for applying similar considerations to the map of \( D_{n, \nu-1} + D_{n, \nu} \) onto a two-sheeted
domain (which may be unwrapped to form a schlicht domain) we see that \( f(z) \) is analytic at every point of \( C_\nu \) except one at infinity—i.e. \( C_\nu \) can’t have any ends at \( \infty \) where \( f(z) \) would have a finite value.

Thus the curves \( C_{n,\nu} \) approach curves \( C_\nu \) of the character described in §5, and these fundamental regions map into sheets forming a semi-cosinic end.

The logarithmic ends are obtained in a similar fashion: the region of the upper half-plane bounded by \( C_{n,1} \), \( \Gamma_{n,2} \), the real axis, and containing \( \Gamma_{n,1} \) is mapped by \( C_{n,1}(z) \) onto a plane slit along the real axis from \( +\infty \) to \( C_\nu(-\sqrt{\lambda_n/\kappa^2}) \), \( \Gamma_{n,1} \), mapping into the remainder of the real axis. Applying Lemma V again there are two possibilities: 1) \( \Gamma_{n,1} \) (and hence all \( \Gamma_{n,\nu} \)) disappear in the limit and the (appropriate) region bounded by the real axis and \( C_1 \) is mapped into an upper half-plane. This is impossible since we would then have a semi-cosinic surface, which won’t go with (25) by Theorem I. This leaves 2) the curves \( \Gamma_{1,1} \), \( \Gamma_{2,2} \), the limits of \( \Gamma_{n,1} \) and \( \Gamma_{n,2} \) actually exist, the region bounded by \( \Gamma_{n,2} \), the real axis, and \( C_1 \) is mapped by \( w = f(z) \) onto a plane slit along \( (-\infty, +\infty) \). The curve \( \Gamma_1 \) is mapped onto \( (-\infty, -a_\nu) \) and hence \( \Gamma_1 \) is all in one piece. Applying a similar argument to the strip between \( \Gamma_{n,1} \) and \( \Gamma_{n,3} \) we see that \( \Gamma_{\nu,2} \), the limit-curve of \( \Gamma_{n,2} \), is actually present, for along \( \Gamma_{\nu,2} \) \( f(z) \) must assume the values \( (-a_\nu, +\infty) \). Repeating this argument we see that we have for \( f(z) \) an infinite sequence of real paths \( \Gamma_\gamma \) as described in §5 and that the
associated strips map into a logarithmic end with singularities
over $w = -a_0$ and $w = \infty$. Using the relation $f(\bar{z}) = \bar{f}(z)$ we
obtain the other logarithmic end, which completes the proof of

**Theorem IV** Let $w = f(z) = \int_0^z f'(t) \, dt$ be an entire function
with

$$f'(z) = e^{-\alpha \frac{z^\gamma}{\beta}} \prod_{\nu=1}^\infty \left\{ \left(1 - \frac{z}{b_{\nu}} \right) e^{\frac{z}{b_{\nu}}} \right\}$$

with $0 < b_1 < b_2 < \ldots$, $\sum_{\nu=1}^\infty \frac{1}{b_{\nu}}$ convergent, $\alpha > 0$,
$\beta$ real, and such that $\sum_{\nu=1}^\infty \frac{1}{b_{\nu}}$ converges either $\alpha > 0$ or $\alpha = 0$
and $\beta + \sum_{\nu=1}^\infty \frac{1}{b_{\nu}} > 0$.

The image of the $z$-plane by $w = f(z)$ is a symmetric gænnic

**surface.**

12. In the proof just given we assumed $\alpha > 0$. The proof
when $\alpha = 0$ differs only in that the paths $\Gamma_{n}$ come from the
multiple zero $z = -\frac{\lambda_n}{\beta_n}$ ($< 0$ for $n$ sufficiently large, by hypothesis).
In this case none of the real paths of $\mathcal{C}_n(z)$ disappear.

Also we appealed to Theorem I to show that $\Gamma_{1}$ actually exists;
but this may be done directly by showing that

$$a_0 = \lim_{x \to -\infty} f(x), \quad x \text{ real}$$

is finite. By (25),

$$a_0 = -\int_0^\infty f'(z) \, dz = \int_0^\infty e^{-\alpha t^\gamma/\beta} \prod_{\nu=1}^\infty \left\{ \left(1 + \frac{t}{b_{\nu}} \right) e^{-t/b_{\nu}} \right\} \, dt$$

For $t > 0$, $(1 + t/b_{\nu}) e^{-t/b_{\nu}} \leq 1$,

$$0 < \prod_{\nu=1}^n \left\{ \left(1 + \frac{t}{b_{\nu}} \right) e^{-t/b_{\nu}} \right\} \leq \prod_{\nu=1}^n \left\{ \left(1 + \frac{t}{b_{\nu}} \right) e^{-t/b_{\nu}} \right\} < A (1+t)^n e^{-t \sum_{\nu=1}^n \frac{1}{b_{\nu}}}$$

where $A$ depends on $n$ only. Thus the integrand in (36) is

$$< A (1+t)^n \exp \left\{ -\alpha t^\gamma - (\beta + \sum_{\nu=1}^n \frac{1}{b_{\nu}}) t \right\}$$
For \( \alpha > 0 \) we may use \( n = 0 \) and for \( \alpha = 0 \) we may take, by hypothesis, \( n \) large enough to make \( \beta + \sum_{\nu=1}^{n} \frac{1}{b_{\nu}} > 0 \). Therefore (36) always converges to a finite \( A_\nu > 0 \).

15. It may be shown that the line \( y = y_\circ > 0 \) intersects all the real paths of \( Q_\nu(z) \) which lie in the upper half-plane exactly once. These intersections correspond to the roots of the equation

\[
\sum Q_\nu(x + iy_\circ) = 0
\]

which, for \( \alpha > 0 \), is of degree \( 3 \lambda_n + n \), in \( x \). Considering the fact that the real paths of a polynomial divide the angle at \( z = \infty \) equally, the line \( y = y_\circ \) must intersect each of the following at least once: \( C_{\nu_0}, \ldots, C_{\nu_n}, \Gamma_{\nu_1}, \ldots, \Gamma_{\nu_n}, \lambda_n \), \( \lambda_n \) of the \( \Gamma_1 \), and \( \lambda_n \) of the \( \Gamma_2 \). But this makes up the precise degree of (37), so each of these is a simple intersection. The roots of (37) are all real, simple. Hence the roots of

\[
\frac{\partial}{\partial x} \sum Q_\nu(x + iy_\circ) = 0
\]

are all real, simple, and alternate with those of (37). The following lemma is easily established by elementary means.

**Lemma** Let \( P_n(x) \) be a sequence of real polynomials, each with all real distinct zeros. Let \( g(x) \neq 0 \), \( g'(x) \), \( g''(x) \) be defined for all \( x \). Let \( P_n^{(k)}(x) \rightarrow g^{(k)}(x) \) \((k = 0, 1, 2)\) for all \( x \), uniformly in any bounded interval. Let the distance between any two zeros of \( P_n(x) \) contained in a finite interval possess a positive lower bound, dependent on the interval but not on \( n \).

Then the zeros of \( g(x) \) are all simple and alternate with those of \( g'(x) \) which are also all simple.
This lemma applies to the case in hand: the roots of
\[(39) \quad \nabla f(x + iy_0) = 0 \quad y_0 \neq 0\]
are simple roots and alternate with the (simple) roots of
\[(40) \quad \frac{\partial}{\partial x} \nabla f(x + iy_0) = 0 \quad y_0 \neq 0.\]
That this result holds also for \(\lambda = 0\) is proved in a similar manner. This result does not state that the line \(y = y_0 > 0\) actually intersects \(\Gamma^\nu\) and \(\Gamma^\nu\) for all values of \(y_0\); some of these intersections might disappear in the limit, e.g. \(\Gamma^\nu\) might have a horizontal asymptote \(y = y_1 > 0\) etc. We shall pursue this further now.

Equation (40) is equivalent to \(\nabla f'(x + iy_0) = 0\), or
\[(40') \quad \Theta = \arg f'(x + iy_0) = -k\pi \quad k = 0, \pm 1, \pm 2, \ldots\]
Consulting (25),
\[(41) \quad \Theta = -2\alpha y + \beta y + \sum_{\nu=1}^{\infty} \left\{ \frac{y}{b_\nu} - \frac{1}{b_\nu^2 - x} \right\}\]
where we choose \(0 < \tan^{-1} \frac{y}{b_\nu - x} < \pi\). With this determination the sum in (41) is convergent, and \(\Theta\) is a continuous function of both \(x\) and \(y\) for all \(x\) and for \(y > 0\).

Differentiating (41),
\[(42) \quad \frac{\partial \Theta}{\partial x} = -2\alpha y - \sum_{\nu=1}^{\infty} \frac{y}{y^2 + (b_\nu - x)^2} < -2\alpha y \quad (y > 0)\]
Thus \(\Theta\) decreases as we move to the right along \(y = y_0\). Since \(\Theta\) is continuous, the value of \(k\) associated with the root of (40') which lies between two given \(C\) or \(\Gamma\) curves is a constant.

The particular value of this constant may be determined by setting
\[x = \frac{1}{x} (b_\nu + b_\nu)\] in (41) and letting \(y \to 0:\)
\[\lim_{y \to 0} \Theta = -\nu \pi\]
Let the root of $\Theta(x, y) = -k\pi$ be $x = \xi_k(y)$. Let the equation of the curve $C_\nu$ be $x = \mathcal{K}_\nu(y)$ and the equation of $\Gamma_\nu$ be $x = \mathcal{K}_{\nu-1}(y)$. Then we have

$$\begin{cases} \xi_{\nu-1}(y) < \mathcal{K}_\nu(y) < \xi_\nu(y) & \nu = 1, 2, \ldots \\ \xi_{\nu-1}(y) < \mathcal{K}_{\nu-1}(y) < \xi_{\nu+1}(y) & \nu = 1, 2, \ldots \end{cases}$$

There is one immediately obvious fact to be drawn from (42):

if $\alpha > 0$, for then

$$\Theta(x_0, y_0) - \Theta(x_1, y_1) = \int_{x_0}^{x_1} \frac{\partial \Theta}{\partial x} \, dx > 2\pi y \cdot |x_1 - x_0|.$$ 

Therefore $0 < \xi_{\nu+1}(y) - \xi_\nu(y) < \pi / \alpha y$, and by (45),

$$\begin{cases} 0 < \mathcal{K}_{\nu+1}(y) - \mathcal{K}_\nu(y) < \pi / \alpha y & \nu = 1, 2, \ldots \\ 0 < \mathcal{K}_\nu(y) - \mathcal{K}_{\nu-1}(y) < \pi / \alpha y & \nu = 1, 2, \ldots \\ 0 < \mathcal{K}_{\nu-1}(y) - \mathcal{K}_\nu(y) < \pi / \alpha y & \nu = 1, 2, \ldots \end{cases}$$

In this case we see that the real paths have no horizontal asymptotes, and the horizontal width of all strips decreases uniformly as $1/y$.

Now for $x < 0$, $\tan^{-1} \frac{y}{b_{\nu} - x} < \frac{y}{b_{\nu} - x} < \frac{y}{b_{\nu}}$.

$$\sum_{\nu = 1}^{\infty} \left\{ \frac{y}{b_{\nu}} - \tan^{-1} \frac{y}{b_{\nu} - x} \right\} \text{ is a monotone function of } x \text{ (see (42)) and for any } N,$$

$$\lim_{x \to -\infty} \sum_{\nu = 1}^{\infty} \left\{ \frac{y}{b_{\nu}} - \tan^{-1} \frac{y}{b_{\nu} - x} \right\} \geq \lim_{x \to -\infty} \sum_{\nu = 1}^{N} \left\{ \frac{y}{b_{\nu}} - \tan^{-1} \frac{y}{b_{\nu} - x} \right\} = \sum_{\nu = 1}^{N} \frac{y}{b_{\nu}}.$$

Also

$$\lim_{x \to -\infty} \sum_{\nu = 1}^{\infty} \left\{ \frac{y}{b_{\nu}} - \tan^{-1} \frac{y}{b_{\nu} - x} \right\} \leq \sum_{\nu = 1}^{\infty} \frac{y}{b_{\nu}}.$$

Thus this limit is $y \cdot \sum_{\nu = 1}^{\infty} \frac{1}{b_{\nu}}$.

If $\alpha = 0$, then

$$\lim_{x \to -\infty} \Theta(x, y) = y \left( \beta + \sum_{\nu = 1}^{\infty} \frac{1}{b_{\nu}} \right)$$

Thus if $\sum_{\nu = 1}^{\infty} \frac{1}{b_{\nu}}$ diverges all curves $\Gamma_\nu$ are intersected by any
horizontal line, but if \( \sum_{\nu=1}^{\infty} \frac{1}{b_{\nu}} = \tau < \infty \) then \( y = y_0 \) intersects \( \Gamma_{\nu} \) for \( y_0 > \frac{y_0 \pi}{\beta + \nu} \) but does not intersect \( \Gamma_{\nu} \) for \( y_0 < \frac{(\nu - 1) \pi}{\beta + \nu} \).

On the other side of the picture, \( \Theta \) is a monotone function of \( x \) and so \( \lim_{x \to +\infty} \Theta(x, y) \) exists and the value determines roughly which curves \( C_{\nu} \) are intersected by \( y = y_0 \). Here we consider only \( \alpha = 0 \), since we know, by (44), the answer when \( \alpha > 0 \).

For \( b_{n} < x < b_{n+1} \), \( b_{n} > y \) we have from (42)

\[
\frac{\partial \Theta}{\partial x} < - \sum_{\nu=1}^{n} \frac{y}{y_x^x - y} - \sum_{\nu=n+1}^{\infty} \frac{y}{y_{\nu}^x - y}
\]

\[
< - \sum_{\nu=1}^{n} \frac{y}{y_x^x - y} = - \frac{ny}{2} \cdot \frac{1}{x^x}
\]

and

\[
\Theta(b_n, y) - \Theta(b_{n+1}, y) = \int_{b_n}^{b_{n+1}} \frac{\partial \Theta}{\partial x} dx > \frac{ny}{2} \left( \frac{1}{b_n} - \frac{1}{b_{n+1}} \right)
\]

Adding these inequalities for all \( n > m \), where \( b_m > y \):

\[
\Theta(b_m, y) - \lim_{x \to \infty} \Theta(x, y) > \frac{y}{2} \sum_{n=m}^{\infty} \left( \frac{n}{b_n} - \frac{n}{b_{n+1}} \right)
\]

\[
= \frac{y}{2} \left( \frac{m}{b_m} + \sum_{n=m+1}^{\infty} \frac{1}{b_n} \right)
\]

Therefore if \( \sum_{\nu} \frac{1}{b_{\nu}} \) diverges, \( \lim_{x \to \infty} \Theta(x, y) = -\infty \) and \( C_{\nu} \) is intersected by every horizontal line.

If \( \sum_{\nu} \frac{1}{b_{\nu}} \) converges we proceed as follows: from (41) \( \alpha = 0 \)

(46) \[ \Theta = \beta y + y \sum_{\nu=1}^{\infty} \frac{1}{b_{\nu}} - \sum_{\nu=1}^{\infty} \int_{b_{\nu}^x}^{y} \frac{y}{b_{\nu}^x - y} \]

Then since \( \int_{b_{\nu}^x}^{y} \frac{y}{b_{\nu}^x - y} > 0 \) we have for \( x > b_n \),

\[ \Theta < \beta y + y \sum_{\nu=1}^{n} \frac{1}{b_{\nu}} - \sum_{\nu=1}^{n} \frac{\pi}{2} = A y - n \cdot \frac{\pi}{2} \]

and \( \lim_{x \to +\infty} \Theta(x, y) = -\infty \), and again \( C_{\nu} \) is intersected by every horizontal line.
Sharper results on the course of the real-paths may obviously be obtained using the relation between (39) and (40), especially when a reasonably regular behaviour is assumed for the sequence $b_\nu$.

14. The problem of the relation between the numbers $\alpha, \beta, b_\nu$ and the branch points, $a_\nu$, of the Riemann surface is not simple. The following however may easily be established: if $\alpha > 0$ then

$$
\sum_{\nu=1}^{\infty} |a_{\nu+1} - a_\nu| = O \left( e^{-\frac{\alpha}{2} b_\nu} \right) = O \left( e^{-\beta r^n} \right)
$$

for any positive $r$. Let $\Pi(z)$ be the canonical product in (25) and $M(r)$ its maximum modulus:

$$
f'(z) = e^{-\alpha z^2 + \beta z \Pi(z)} \quad M(r) \overset{\text{Max}}{=} |\Pi(z)|
$$

Let $n(t)$ be the number of $b_\nu$'s which are less than $t$. Then


$$
\log M(r) < kr \left\{ \int_0^r \frac{n(t)}{t^2} \, dt + r \int_r^\infty \frac{n(t)}{t^3} \, dt \right\}
$$

Now

$$
\sum_{i=1}^{\infty} \frac{1}{b_i} = \int_0^\infty \frac{d n(t)}{t^2} = \int_0^\infty \frac{n(t)}{t^3} \, dt
$$

these integrals being convergent, and $n(t)/t^2 \to 0$. Let $\varepsilon > 0$ and $r_o$ be fixed such that $n(t)/t^2 < \varepsilon$ for $t > r_o$.

Then

$$
\int_0^r \frac{n(t)}{t^2} \, dt = \int_0^{r_o} \frac{n(t)}{t^2} \, dt + \int_{r_o}^r \frac{n(t)}{t^2} \, dt
$$

$$
< A(\varepsilon) + (r-r_o) \varepsilon \quad < A(\varepsilon) + \varepsilon r
$$

and for $r$ sufficiently large

$$
\int_r^\infty \frac{n(t)}{t^3} \, dt < \varepsilon
$$

Thus by (49)

$$
\log M(r) < A(\varepsilon) r + 2k \varepsilon r^2
$$

or

$$
M(r) = O \left( e^{\varepsilon r^2} \right) \quad (r \to \infty).
$$
Then

\[ \sum_{\nu=n}^{\infty} |a_{\nu+1} - a_{\nu}| = \int_{b_{n}}^{\infty} |f'(z)| \, dz = O \int_{b_{n}}^{\infty} e^{-\frac{2}{\delta} r^2 + \frac{3}{\delta} r^2} M(r) \, dr \]

\[ = O\int_{b_{n}}^{\infty} r e^{-\frac{2}{\delta} r^2} \, dr = O \left( e^{-\frac{2}{\delta} b_{n}^2} \right) \]

which is the first part of (47). The rest follows since \( b_{n}^2 / n \to \infty \).
Chapter IV

The Symmetric Cosinic Surface

1. This class of surfaces $\sigma_{w}$ bears the same relation to that of $z = \cos^{-1} w$ as the class of symmetric semi-cosinic surfaces bears to the surface of $z = (\cos^{-1} w)^2$. $\sigma_{w}$ consists of the sheets $S_{\nu}$ $(\nu = 0, \pm 1, \pm 2, \ldots)$; $S_{\nu}$ is a replica of the $w$-plane slit open along the real axis except for the segment $(a_{\nu}, a_{\nu+1})$ and $S_{\nu}$ and $S_{\nu+1}$ are joined along their slits extending from $a_{\nu+1}$, forming a first order branch point over $a_{\nu+1}$. $\sigma_{w}$ is determined by the sequence $a_{\nu}$ $(\nu = 0, \pm 1, \pm 2, \ldots)$ where $a_{z \nu} > a_{2 \nu}$. We shall assume $a_{1} > 0$, $a_{0} < 0$ so that $w = 0$ is in the unslit portion of $S_{0}$.

2. This surface, $\sigma_{w}$, is open and simply-connected. The nature of its singularities depends on the sequence $\{a_{\nu}\}$.

I) If $|a_{\nu}| < \infty$ for all $\nu$, $\sigma_{w}$ has two logarithmic branch points over $w = \infty$. In addition $\sigma_{w}$ will have no, one, or two indirectly critical singularities according as neither, one, or two of the limits, $\lim_{\nu \to +\infty} a_{\nu}$, $\lim_{\nu \to -\infty} a_{\nu}$ exists (and is finite).

II) If $\lim_{\nu \to +\infty} |a_{\nu}| = \infty$, $\sigma_{w}$ has a directly critical singularity of the second species over $w = \infty$. If in addition $\lim_{\nu \to -\infty} a_{\nu}$ exists (finite), there will be an indirectly critical singularity. The same results hold if we switch $\nu \to +\infty$ and $\nu \to -\infty$. 
3. Let \( \sigma_{W}^{\nu} \) be mapped onto the disc \( |z| < R \leq \infty \) by

\[
\begin{align*}
\begin{cases}
  z = \phi(w), & w = f(z) \\
  f(0) = 0 \in S_{\nu}, & f'(0) = 1.
\end{cases}
\end{align*}
\]

As in the two previous cases the symmetry of \( \sigma_{W}^{\nu} \) implies that if \( z = b_{\nu} \) is the image of the branch point over \( w = a_{\nu} \), then

\[
R \downarrow \quad \ldots < b_{-1} < b_{0} < 0 < b_{1} < b_{1} < \ldots \uparrow R
\]

The fundamental regions in \( |z| < R \) are topologically equivalent to those for the inverse-cosine surface: the fundamental region corresponding to \( S_{\nu} \) is bounded by a curve \( C_{\nu} \) thru \( b_{\nu} \) and a curve \( C_{\nu+1} \) thru \( b_{\nu+1} \). The two shores of the cut in \( S_{\nu} \) extending from \( a_{\nu} \) to \( \pm \infty \) correspond to the two halves of \( C_{\nu} \). The uncut segment of the real axis in \( S_{\nu} \) maps onto \( (b_{\nu}, b_{\nu+1}) \).

The curves \( C_{\nu} \) extend to \( |z| = R \), are symmetric about the real axis, and no two of them have any point in common. The curves \( C_{\nu} \) and the real axis constitute the real-paths of \( w = f(z) \). The points \( b_{\nu} \) are the (simple) zeros of \( f'(z) \).

4. The approximating elliptic surface, \( \sigma_{W}^{(n)} \), to be used here consists of the \( 2n + 1 \) sheets \( S_{-n} \), \( S_{-n+1} \), \ldots, \( S_{0} \), \ldots, \( S_{n} \) exactly as in \( \sigma_{W} \), with the cuts from \( a_{-n} \) and \( a_{n+1} \) (in \( S_{-n} \) and \( S_{n} \)) closed smoothly. \( \sigma_{W}^{(n)} \) is closed, simply-connected, with a branch point of order \( 2n \) over \( w = \infty \); it is the image of the \( z \)-plane by a polynomial of degree \( 2n + 1 \):

\[
\begin{align*}
\begin{cases}
  w = P_{n}(z) \\
  P_{n}(0) = 0 \in S_{0}, \quad P'_{n}(0) = 1.
\end{cases}
\end{align*}
\]

The fundamental regions in the \( z \)-plane are topologically the same as those for the complete surface, except that now there are only
a finite number of them. The first order branch point of \( \varphi^{(n)}_w \)
over \( w = a, -(n-1) \leq \nu \leq n \) will correspond to \( z = b_{\nu}, \nu \) with
\begin{align}
& b_{\nu}, -n+1 \leq \ldots < b_{\nu}, 0 < b_{\nu}, 1 < \ldots < b_{\nu}, n, \\
\end{align}
and we have the representation
\begin{align}
P'_n(z) &= \frac{n}{z^{n+1}} \left( 1 - \frac{z}{b_{\nu}, \nu} \right) \\
P_n(z) &= \int_0^z P'_n(t) \, dt
\end{align}

5. Let \( D_n \) be the \( z \)-plane slit along the real axis except for the interval \( (b_{\nu}, -n+1, b_{\nu}, n) \). Let \( \Delta_n \) the domain in the \( \Im \)-plane containing \( \Im = 0 \) and bounded by \( |\Im| = R, b_{-\nu}, \nu, b_{n+1} \), and the intervals \( (b_{-\nu}, b_{-n+1}), (b_{\nu}, b_{n+1}) \).
The parts of \( \varphi^{(n)}_w \) and \( \varphi^{(n)}_w \) corresponding to \( D_n \) and \( \Delta_n \) are identical. Hence the function
\begin{align}
\psi = \varphi(P_n(z)) = \psi_n(z)
\end{align}
maps \( D_n \) onto \( \Delta_n \) schlichtly. By (3), (1) \( \psi_n(0) = 0 \), \( \psi'_n(0) = 1 \). Also \( D_n \) contains the disc \( |z| < \min (b_{n}, -n+1, b_{\nu}, n) = \mu_n \). Therefore by the Koebe distortion theorem \( \Delta_n \) will contain the disc \( |\Im| < \frac{1}{4} \mu_n \), which implies
\begin{align}
R > \frac{1}{4} \mu_n, \quad \text{all } n.
\end{align}

6. Now if we use \( A_i = \int_a^{b_{\nu}, i} P'_n(z) \, dz \) and \( a_o = \int_a^{b_{\nu}, o} P'_n(z) \, dz \)
together with (5) we obtain
\begin{align}
\sum_{\nu=1}^n \frac{1}{b_{\nu}, \nu} < R \sqrt{n} \quad \text{and} \quad \sum_{\nu=-n+1}^{n-1} \frac{1}{b_{\nu}, \nu} < R \sqrt{n}
\end{align}
The procedure is virtually identical with that of Chapter III \( \S 8 \)
if we replace \( 1/C_n \) by \( -\frac{1}{n} \sum_{\nu=-n+1}^{n-1} \frac{1}{b_{\nu}, \nu} \). Then
\begin{align}
\frac{n}{b_{\nu}, n} \quad < \sum_{\nu=1}^n \frac{1}{b_{\nu}, \nu} \quad < \quad R \sqrt{n}
\end{align}
or
\begin{align}
b_{\nu}, n \quad > \quad \frac{\sqrt{n}}{R}
\end{align}
and similarly \( b_{n,-n+1} > \frac{\sqrt{n}}{n^2} \).

Hence \( \mu_\eta = \min (b_{n,\eta}, b_{n,-n+1}) \to \infty \) as \( n \to \infty \) and by (8) \( R = \infty \) and \( \sigma _f \) is parabolic.

7. We have just shown that the sequence of domains \( D_\eta \) converges to the punched \( z \)-plane, and \( \Delta_\eta \) converges to the punched \( \Sigma \)-plane. Hence by Lemma VI (p. 6), \( \phi (z) = \phi (P_\eta (z)) \to z \).

Thus
\[
(10) \quad P_\eta (z) \to f(z) \quad P'_\eta (z) \to f'(z)
\]
uniformly for \(|z| \) bounded. By Hurwitz' theorem

\[
(11) \quad \lim \frac{b_{n,v}}{n} = b_v \quad v = 0, \pm 1, \pm 2, \ldots
\]
but not uniformly for all \( v \).

Proceeding as in Chapter III §9 we see, for \(|z| < \frac{1}{2} (b_1 + |b_\eta|)\):

\[
(12) \quad \log P'_\eta (z) = -\frac{x}{2} \sum_{\nu = -n+1}^{n} \frac{1}{b_{\eta,\nu}} - \frac{x}{2} \sum_{\nu = -n+1}^{n} \frac{1}{b_{\eta,\nu}} - \ldots
\]

and by (10) there exist

\[
(13) \quad \lim_{n \to \infty} \frac{1}{b_{\eta,\nu}} = \sigma _1 \quad \lim_{n \to \infty} \frac{1}{b_{\eta,\nu}} = \sigma _2
\]

\[
(14) \quad \sigma _3 = \frac{1}{2} \sum_{-\infty}^{\infty} \frac{1}{b_{\eta,\nu}} \leq \sigma _2
\]

Also

\[
(15) \quad |b_{\eta,\nu}| > A \sqrt{\nu} \quad |b_{\eta,\nu}| > A \sqrt{\nu}
\]

and

\[
(16) \quad \lim_{n \to \infty} \left( \sum_{-n+1}^{n} \frac{1}{b_{\eta,\nu}} - \sum_{-\infty}^{\infty} \frac{1}{b_{\eta,\nu}} \right) = 0 \quad p \geq 3.
\]

Hence if

\[
(17) \quad T(z) = \prod_{\nu = 1}^{\infty} \left\{ \left( 1 - \frac{x}{b_{\eta,\nu}} \right) e^{\frac{x}{b_{\eta,\nu}}} \right\}
\]

we see that

\[
\log \frac{f'(z)}{T(z)} = \lim_{n \to \infty} \log \frac{P'_\eta (z)}{T(z)} = -\sigma _1 z - (\sigma _2 - \sigma _3) z \nu,
\]
and by (14) \( \sigma_2^* - \sigma_1^* > 0 \). Thus we have proved

**Theorem V**  The symmetric conic surface \( \sigma_{w_{*}} \) is always parabolic. Furthermore \( \sigma_{w_{*}} \) is the image of the \( z \)-plane by an entire function \( w = f(z) \) with \( f'(z) \) of the form

\[
(18) \quad f'(z) = e^{-\alpha z + \beta z^*} \prod_{\nu=-\infty}^{\infty} \left\{ (1 - \frac{z}{b_{\nu}}) e^{\frac{z}{b_{\nu}}} \right\}
\]

where \( \alpha > 0, \beta \) real, \( b_i < b_{i+1}, b_0 < 0 < b_1 \), and \( \sum_{\nu} \frac{1}{b_{\nu}} \) is convergent.

8. The converse to theorem V goes thru in the same fashion as theorem IV. Starting with (18) as given we construct the sequence of polynomials

\[
(19) \left\{ \begin{array}{l}
Q_n(z) = \int_{-\rho_n}^{\rho_n} Q_{n-1}(t) dt \\
Q'_{n}(z) = \left(1 - \frac{\alpha z + \beta z^*}{\rho_n}\right)^{\lambda_n} \prod_{\nu=-n}^{n} \left(1 - \frac{z}{b_{\nu}}\right)
\end{array} \right.
\]

where \( \beta_n = \beta + \sum_{\nu=-n}^{n} \frac{1}{b_{\nu}} \)

As in Chapter III 8.10, if the sequence of positive integers \( \lambda_n \) increases sufficiently rapidly

\( (20) \quad Q_n(z) \rightarrow f(z), \quad Q'_{n}(z) \rightarrow f'(z) \)

uniformly for \( |z| \) bounded.

The Riemann surface of \( Q_n(z) \) is readily constructed by finding the real-paths of \( Q_n(z) \), which are: 1) the real axis, 2) one curve \( C_{n, \nu} \), symmetric about the real axis, thru \( b_{\nu}, \nu = 0, \pm 1, \ldots, \pm n \), and 3) \( \lambda_n \) rays emanating from each of the points \( \sqrt{\lambda_n/\alpha} \), \(-\sqrt{\lambda_n/\alpha}'\), \( \lambda_n/\beta_n \). These last three sets all disappear since \( f(z) \) has only a finite number of distinct real paths in any bounded region.

The paths \( C_{n, \nu} \) converge to the real-paths of \( f(z) \). Let \( D_{n, \nu} \)
be the region in the upper half-plane between \( C_{n, \nu-1} \) and \( C_{n, \nu+1} \).


\[ D_{n, \nu} \] converges to its kernel \( D_{\nu} \) bounded by the limiting curves \( C_{\nu-1}, C_{\nu+1} \) of \( C_{n, \nu-1} \) and \( C_{n, \nu+1} \). The polynomial \( w = Q_{n}(z) \) maps \( D_{n, \nu} \) onto \( \Delta_{n, \nu} \), being the w-plane slit along the real axis from \( Q_{n}(b_{\nu}) \) to \( (-)^{\nu \infty} \). By (20) \( \Delta_{n, \nu} \) converges to its kernel \( \Delta_{\nu} \), the w-plane slit from \( a_{\nu} = f(b_{\nu}) \) to \( (-)^{\nu \infty} \). Then by Lemma V (p. 6), \( D_{\nu} \) is a simply-connected region, which is mapped by \( f(z) \) onto \( \Delta_{\nu} \). \( C_{\nu} \), which maps into \( (a_{\nu}, (-)^{\nu+1 \infty}) \) is a simple curve in one piece---i.e. \( C_{n, \nu} \) cannot break up into "scallop" around infinity.

If we piece these results together for all \( \nu \) we have

**Theorem VI** Let \( w = f(z) = \int_{0}^{z} f'(t) dt \) be an entire function with

\[
(18) \quad f'(z) = e^{-\alpha z^2 + \beta \sum_{\nu=\infty}^{\infty} \left( \frac{z}{b_{\nu}} \right)^{2\nu} \left( 1 - \frac{z}{b_{\nu}} \right)^{2\nu-b_{\nu}}} \]

where \( \alpha > 0, \beta \) real, \( b_{\nu} < b_{\nu+1} \), \( b_{\nu} < 0 < b_{i} \), and

\[
\sum_{\nu=\infty}^{\infty} \frac{1}{b_{\nu}}
\]

is convergent. Then the image of the z-plane by \( w = f(z) \) is a symmetric cosine surface.

9. As in the case of the symmetric gammaic surface it is shown that the line \( y = y_{c} > 0 \) intersects the upper half of each \( C_{\nu} \) once (or conceivably not at all).

Let the equation of \( C_{\nu} \) be \( x = \chi_{\nu}(y) \) where \( \chi_{\nu} \) is an even single valued function of \( y \), a solution of

\[
(20) \quad \gamma f(x + iy) = 0.
\]

For \( y \) fixed the (simple) roots of (20) alternate with the (simple)

roots of

\[
(21) \quad \Theta = \arg f'(x + iy) = -k\pi , \quad k = 0, \pm 1, \pm 2, \ldots
\]

where, referring to (18) we take, for \( y > 0 \)
\[ (22) \quad \Theta = (\beta - 2\alpha x)y + \sum_{\nu = 1}^{\infty} \left\{ \frac{y}{b_{\nu}} - \tan^{-1} \frac{y}{b_{\nu} - x} \right\} + \sum_{\nu = 0}^{\infty} \left\{ \frac{y}{b_{\nu}} + \pi - \tan^{-1} \frac{y}{b_{\nu} - x} \right\} \]

with \( 0 < \tan^{-1} \frac{y}{b_{\nu} - x} < \pi \)

which gives us a continuous determination of \( \Theta \) for all \( x \) and \( y > 0 \). Also

\[ (25) \quad \frac{\partial \Theta}{\partial x} = -2\alpha y - \sum_{\nu = 1}^{\infty} \frac{y}{(b_{\nu} - x)^{2} + y^{2}} < 0 \]

Thus for a given value of \( k \) there is just one root of (21), \( x = \xi_{k}(y) \). Allowing \( y \to 0 \) in (22) we locate \( \xi_{k} \) with respect to \( \kappa'_{k} \) as:

\[ (24) \quad \xi_{k-1}(y) < \kappa_{k}(y) < \xi_{k}(y) \quad k = 0, \pm 1, \pm 2, ... \]

From (25) we see that if \( \alpha > 0 \)

\[ (26) \quad 0 < \kappa_{\nu+1}(y) - \kappa_{\nu}(y) < \frac{\pi}{\alpha y} \]

To show, if \( \alpha = 0 \), that \( C_{\nu} \) is intersected by every horizontal line it is sufficient to show that \( \lim_{x \to +\infty} \Theta(x, y) = -\infty \) and \( \lim_{x \to -\infty} \Theta(x, y) = +\infty \) for then \( \xi_{k}(y) \) and hence \( \kappa_{k}(y) \) will be finite for all \( k \) and \( y \).

Now as \( x \to +\infty \),

\[ \sum_{\nu = 0}^{\infty} \left\{ \frac{y}{b_{\nu}} + \pi - \tan^{-1} \frac{y}{b_{\nu} - x} \right\} \]

tends to a finite limit, since the terms are all positive and decrease as \( x \) increases. Then proceeding with the other sum in (22) exactly as in Chapter III \( \S 13 \) we see that

\[ \lim_{x \to +\infty} \Theta(x, y) = -\infty . \]

Similarly, \( \lim_{x \to -\infty} \Theta(x, y) = +\infty \), and \( C_{\nu} \) is intersected by every horizontal line.
10. Finally, cf. Chapter III §14, if \( \alpha > 0 \)

\[
\sum_{\nu=n}^{\infty} |a_{\nu+1} - a_{\nu}| = O(e^{-\frac{\alpha}{2} b_n}) = O(e^{-\gamma n})
\]

\[
\sum_{\nu=-\infty}^{-n} |a_{\nu+1} - a_{\nu}| = O(e^{-\frac{\alpha}{2} b_{-n}}) = O(e^{-\gamma n})
\]

for any \( \gamma > 0. \)

11. The striking similarity between the results of the last two chapters leads one to speculation concerning a general equivalence of a pair of logarithmic ends to a semi-cosinic end.

As a matter of fact, nowhere in Chapters II, III, IV has it been essential that the branch points of the surface considered be all distinct. The proof of the parabolicity for instance depends only on a mean value of the zeros of \( \Pi_n'(\xi) \).

This includes then, e.g., for the symmetric cosinic surfaces, all functions (18) with \( b_i \leq b_{i+1} \) rather than \( b_i < b_{i+1} \). The symmetric enemic surface might be considered as an extreme case of the cosinic, with \( a_{\nu} = -a_0, \; \nu \leq 0. \)
Chapter V

**Distortion of Riemann Surfaces**

1. The present chapter is concerned with the fact that a quasi-conformal map (map of bounded eccentricity) preserves the type of a Riemann surface. This result is well-known, e.g. Ahlfors [2], Kakutani [1], but the following development is simple and in certain applications permits the condition that the eccentricity be bounded to be weakened.

2. The point of departure is the well known **Metric Theorem of Ahlfors**. Let \( \mathcal{F}_\omega \) be a simply-connected open Riemann surface over the \( \omega \)-plane. Let \( U(\omega) = U(\xi + i\eta) \) be a real-values function, continuous and single-valued on \( \mathcal{F}_\omega \), such that

1) \( U(\omega) \to \infty \) as \( \omega \) tends to any boundary point of \( \mathcal{F}_\omega \).

2) \( \frac{\partial U}{\partial \xi}, \frac{\partial U}{\partial \eta} \) are piecewise continuous.

3) \( |\operatorname{grad} U| = \sqrt{\left(\frac{\partial U}{\partial \xi}\right)^2 + \left(\frac{\partial U}{\partial \eta}\right)^2} > 0 \) except at isolated points of \( \mathcal{F}_\omega \).

Let \( \Gamma_\rho \) denote the level curve where \( U = \rho \), and

\[
(1) \quad L(\rho) = \int_{\Gamma_\rho} |\operatorname{grad} U| \, d|\omega|
\]

Then if

\[
(2) \quad \int_0^\infty \frac{d\rho}{L(\rho)}
\]

is divergent, \( \mathcal{F}_\omega \) is parabolic. Also if \( \mathcal{F}_\omega \) is parabolic, there exists a level function \( U \) on \( \mathcal{F}_\omega \) for which (2) is divergent, namely \( U = |z| \) when \( \mathcal{F}_\omega \) is mapped conformally.

onto the $z$-plane, for (1) is invariant under a conformal map.

[Equation]

\[ w = u + i v = T'(\omega) = T(\frac{\xi}{\xi} + i \eta) \]

where $T$ is single valued and continuous on $\mathcal{F}_\omega$ with

\[ U_5, U_\eta, V_5, V_\eta \]

continuous except along isolated curves of $\mathcal{F}_\omega$. Let $E, F, G, J$ have their usual meaning.

\[ E = U_5 + V_5 \]
\[ F = U_\eta + V_\eta \]
\[ G = U_\eta + V_\eta \]
\[ J = U_5 V_\eta - U_\eta V_5 \]

Now the level function $U$ defined on $\mathcal{F}_\omega$ is carried by $T$ over to $\mathcal{T}_\omega$. It may be shown that

\[ |\text{grad}_w U|^2 = \frac{1}{j^2} \left[ G \cdot U_5 + E \cdot U_\eta - 2F U_5 U_\eta \right] \]

Also since $EG-F^2 \geq 0$,

\[ 0 \leq \frac{1}{j^2} \left[ E \cdot U_5 + G U_\eta + 2F U_5 U_\eta \right] \]

Adding (5) and (6)

\[ |\text{grad}_w U| \leq \frac{\sqrt{E+G}}{j} |\text{grad}_\omega U| \]

Along the curve $\Gamma_\rho$

\[ |dw|^2 = (U_5 d\xi + U_\eta d\eta)^2 + (V_5 d\xi + V_\eta d\eta)^2 \]

and by Cauchy's inequality

\[ |dw|^2 \leq (U_5^2 + U_\eta^2) |d\omega|^2 + (V_5^2 + V_\eta^2) |d\omega|^2 \]

or

\[ |dw| \leq \sqrt{E+G} \cdot |d\omega| \]

Combining (7) and (8),
\[ L_w(\rho) = \int_{\Gamma_\rho} \nabla \cdot \mathbf{U} | \frac{d\mathbf{w}}{d\rho} | \leq \int_{\Gamma_\rho} M(\omega) \left| \nabla \omega \cdot \mathbf{U} \right| |d\omega| \]

where

\[ M(\omega) = \frac{E + G}{J} \geq 2 \]

The equality in (10) holds only at points where \( \Gamma \) is conformal. \( \Gamma \) is said to be of bounded eccentricity if \( M(\omega) \) is bounded; the eccentricity of an infinitesimal ellipse on \( \mathcal{F}_w \), image of an infinitesimal circle on \( \mathcal{F}_\omega \), will be bounded away from 1 if and only if \( M(\omega) \) is bounded.

From (9) we see that if \( \Gamma \) is of bounded eccentricity

\[ L_w(\rho) < M(\rho) \]

and hence

\[ \int_{\Gamma_\rho} \frac{d\rho}{L_w(\rho)} \]

diverges if \( \int_{\Gamma_\rho} \frac{d\rho}{L_\omega(\rho)} \) diverges.

Since \( M(\omega) \) and \( M(\rho) \) are equal at corresponding points we have

**Theorem VII** If \( \Gamma \) is of bounded eccentricity, \( \mathcal{F}_\omega \) and \( \mathcal{F}_w \) are of the same type.

If we know to start with that \( \mathcal{F}_\omega \) is parabolic, the image of the \( z \)-plane by a meromorphic function \( \omega = f(z) \), we may take \( \mathbf{U} = |z| \) on \( \mathcal{F}_\omega \) and obtain

**Theorem VIII** Let \( \mu(\rho) = \max_{|z|=\rho} M(\omega) \). If \( \int_{\rho \mu(\rho)} \frac{d\rho}{\rho} \) diverges, \( \mathcal{F}_w \) is parabolic.

And so on; there are for example advantages to be gained in using (9) directly rather than \( \mu(\rho) \) in certain instances.

4. It is convenient to have (4) and (10) in polar co-ordinates:

\[
\begin{align*}
\mathbf{w} &= u + i v = R e^{i\phi}, \\
\omega &= \xi + i \eta = r e^{i\theta} \\
E + G &= R_r \phi_r + R_\theta \phi_\theta + \frac{1}{r^2} R_r^2 + \frac{1}{r^2} R_\theta^2 \\
J &= \frac{R}{r} (R_r \phi_\theta - R_\theta \phi_r)
\end{align*}
\]

In the next chapter we shall use one fundamental transformation.
of a half-plane which is as follows. Let $D_\omega$ be the half-plane $\gamma \omega = \eta > 0$ and $D_\nu$ be the half-plane $\gamma \nu = \nu > 0$.

Let $\alpha$ and $\beta$ be two fixed angles, $0 < \alpha < \frac{\pi}{2}$, $0 < \beta < \frac{\pi}{2}$.

The transformation $T$:

$$
\begin{align*}
R &= r \\
\phi &= \frac{\beta}{\alpha} \theta \quad \text{for} \quad 0 \leq \theta \leq \alpha \\
\phi &= \frac{\pi - \beta}{\pi - \alpha} \theta + \frac{\pi (\beta - \alpha)}{\pi - \alpha} \quad \text{for} \quad \alpha \leq \theta \leq \pi
\end{align*}
$$

maps $D_\omega$ topologically on $D_\nu$. $T$ is the identity along the real axis, and the line $\arg \omega = \alpha$ is rotated into $\arg \nu = \beta$ with $|\omega| = \nu$. Also, by (11)

$$
\begin{align*}
\{ 0 \leq \theta \leq \alpha : \hat{M}(\omega) &= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} \\
\{ \alpha \leq \theta \leq \pi : \hat{M}(\omega) &= \frac{\pi - \alpha}{\pi - \beta} + \frac{\pi - \beta}{\pi - \alpha}
\end{align*}
$$

For any given $\alpha$ and $\beta$, $T$ is of bounded eccentricity.

If

$$
0 < h \leq \alpha \leq \pi - h, \quad 0 < h \leq \beta \leq \pi - h
$$

then

$$
\hat{M}(\omega) < \frac{2\pi}{h}
$$

5. It is easily seen that if $T$ is a product of two transformations,

$$
w = T(\omega) = T_1(T_2(\omega))
$$

with $T_2(\omega)$ analytic that

$$
\hat{M}(\omega) = \hat{M}_1(T_2(\omega)).
$$

If $T_1(\omega)$ is analytic,

$$
\hat{M}(\omega) = \hat{M}_2(\omega).
$$
Chapter VI

The General Semi-cosinic, Gammic, and Cosinic Surfaces

1. The object now is to show that all surfaces of these three classes remain parabolic when the symmetry of the surface is destroyed by moving the branch points off the real axis. It is first necessary to consider just what is meant by this.

The general semi-cosinic surface \( \mathcal{F}_w \) is characterized by a sequence of complex numbers \( \{a_k\}_{k=1}^{\infty} \) with \( a_k \neq \infty \) and \( a_k \neq a_{k+1} \) for all \( k \). \( \mathcal{F}_w \) is to consist of sheets \( S_1, S_2, S_3, \ldots \) as follows. \( S_1 \) is a replica of the \( w \)-plane cut along \( \gamma_1 \), a simple curve extending from \( w = a_1 \) to \( w = \infty \). \( S_k \) is a replica of the \( w \)-plane with two cuts: along \( \gamma_{k-1} \) from \( w = a_{k-1} \) to \( w = \infty \) and along \( \gamma_k \) from \( w = a_k \) to \( w = \infty \). \( \gamma_{k-1} \) and \( \gamma_k \) have no points in common. Of course \( \gamma_k \) in \( S_k \) and \( \gamma_k \) in \( S_{k+1} \) are to lie over the same curve in the \( w \)-plane. \( \mathcal{F}_w \) is formed by connecting \( S_k \) and \( S_{k+1} \) together along \( \gamma_k \), \( k = 1, 2, \ldots \), thus forming a first order branch point over each \( a_k \). Just how we choose the curves \( \gamma_k \) is immaterial so long as

\[
\begin{align*}
\text{(A)} & \quad \gamma_k \text{ is a simple curve extending from } a_k \text{ to } \infty \\
& \quad \gamma_k \text{ and } \gamma_{k+1} \text{ have no points in common.}
\end{align*}
\]

Let \( \Pi \) be the polygonal line with successive vertices \( a_1, a_2, a_3, \ldots \). Let \( \ell_n \) be the segment of \( \Pi \) joining \( a_n \) and \( a_{n+1} \), and let \( \theta_n \) be the non-reflex angle of \( \Pi \) at \( a_n \):

\[ 0 \leq \theta_n \leq \Pi \]
A simple choice of cuts is to take for $\gamma_n$ the ray from $a_n$ to $\infty$ which bisects the reflex angle at $a_n$. Obviously this choice satisfies (A). In case the $a_K$'s are all real and $a_{2K+1} > a_{2K}$ we have an example of the symmetric surface of Chapter II with the same system of cuts.

Note that the characters of the singularities of $\mathcal{F}_w$ are determined by precisely the same conditions as in the symmetric case—pp. 3·9.

2. The general gnnmic surface is obtained from the general semi-cosinic surface by opening up the initial sheet from $a_o$ to $\infty$ and attaching a pair of logarithmic ends. The general cosinic surface is obtained by opening the initial sheets of two semi-cosinic surfaces and joining them.

3. The system of cuts mentioned in §1 is not quite suitable for our purposes. We shall alter it as follows:

Let $\delta$, $0 < \delta < \frac{\pi}{2}$, be fixed. Then

1) If $\theta_n < \pi - \delta$, $\gamma_n$ is the ray bisecting the reflex angle $2\pi - \theta_n$.

2) If $\theta_n > \pi - \delta$, but $\theta_{n+1} < \pi - \delta$ and $\theta_{n-1} < \pi - \delta$, $\gamma_n$ is the ray bisecting the reflex angle $2\pi - \theta_n$.

3) There remains the possibility of a block: $\theta_K > \pi - \delta$, for $k = n + 1, n + 2, \ldots, n + m$, $m > 1$. $\theta_n < \pi - \delta$, $\theta_{n+m+1} < \pi - \delta$.

In this case let $\gamma_K$ be perpendicular to $l_K$ ($k = n+1, \ldots, n + m$) with $\gamma_{n+1}$, $\gamma_{n+2}$, $\ldots$ lying on one side of $l_K$, $\gamma_{n+2}$, $\gamma_{n+4}$, $\ldots$ lying on the other, and $\gamma_{n+1}$ lying in the reflex angle at $a_{n+1}$. 
Note in 2), if \( \theta_n = \pi \), either ray will go.

in 3), the block may be infinite.

This system of cutting applies to all three classes of surfaces considered.

4. We distort \( \sigma_{w}\) into a symmetric surface of the same class as follows. \( \sigma_{w}\) is cut into sheets as in \( \S \, 3 \). Each sheet \( S_n \) is then distorted into a sheet \( S'_n \) with cuts \( \gamma_{n-1}' \) and \( \gamma'_n \) from \( a_{n-1}' \) to \( \infty \) and from \( a_n' \) to \( \infty \) so that 1) \( \gamma_{n-1}' \) and \( \gamma'_n \) are the two ends of the line joining \( a_{n-1}' \) and \( a_n' \), and 2) \( \gamma_n' \) is carried, point by point, into \( \gamma_n' \) by a rigid motion, and similarly for \( \gamma_{n-1} \), \( \delta_k \) and \( \gamma_{n-1}' \). The sheets \( S'_n \) are then joined to form a symmetric surface \( \sigma_{w}' \) of the same class as \( \sigma_{w} \). This last step will involve rigid motions of the sheets \( S'_n \) to align the cuts before rejoining, but this (cf. p. 54) will not affect the eccentricity of the map. Also the map of \( \sigma_{w} \) onto \( \sigma_{w}' \) is one to one, and by 2) continuous.

It is known that \( \sigma_{w}' \) is parabolic. To show, by Theorem VII, that \( \sigma_{w} \) is parabolic it is sufficient to show that \( S_n \) may be mapped onto \( S'_n \) (satisfying conditions 1) and 2)) by a map of uniformly bounded eccentricity for all \( n \).

5. Now consider the nature of an individual sheet \( S_{n+1} \), supplied with its cuts \( \gamma_n \) and \( \gamma_{n+1} \). In what follows the term "obtuse angle" is to include right angles.
Case I. Neither $a_n$ nor $a_{n+1}$ in a block, $y_n$ and $y_{n+1}$ lying on the same side of $\Pi$.

Then $y_n$, $y_{n+1}$ form obtuse angles $\Pi - \frac{\theta_n}{2}$, $\Pi - \frac{\theta_{n+1}}{2}$ with $l_n$, one of which, at least, is greater than $(\Pi - \frac{\phi}{2})/2$.

Since these angles are obtuse, $y_n$ and $y_{n+1}$ do not intersect.

Let $L_n$, $L_{n+1}$ be the pair of parallel lines thru $a_n$ and $a_{n+1}$ making equal angles $\phi$ with $y_n$, $y_{n+1}$:

\[
\begin{align*}
\alpha &\leq \frac{\theta_n + \theta_{n+1}}{4} \leq \frac{\Pi}{2} - \frac{\phi}{4} \\
\phi &\leq \frac{\Pi}{2} - \frac{\theta_n + \theta_{n+1}}{4} \\
\frac{\pi}{4} &\leq \phi + \frac{\theta_n}{2} = \frac{\Pi}{2} + \frac{\theta_n - \theta_{n+1}}{4} \leq \frac{3\pi}{4} \\
\frac{\pi}{4} &\leq \phi + \frac{\theta_{n+1}}{2} = \frac{\Pi}{2} + \frac{\theta_{n+1} - \theta_n}{4} \leq \frac{3\pi}{4} \\

\end{align*}
\]

We distort $S_{n+1}$ in three parts: the strip between $L_n$ and $L_{n+1}$ is left fixed; the half-plane bounded by $L_n$ (and not containing $a_{n+1}$) is distorted by the fundamental transformation (p. 53) so as to rotate $y_n$ into the extension of $l_n$ — i.e., either $\alpha = \phi$,

$\beta = \phi + \frac{\theta_n}{2}$, or $\alpha = \pi - \phi$, $\beta = \pi - (\phi + \frac{\theta_n}{2})$

according as $y_n$ must be rotated in a positive or negative sense.

The remaining half-plane is transformed similarly to rotate $y_{n+1}$.
into the extension of \( \nu_n \) — either \( \alpha = \phi + \frac{\theta_{n+1}}{2} - \phi \), or \( \alpha = \pi - \phi \), \( \beta = \pi - \left( \phi + \frac{\theta_{n+1}}{2} \right) \).

Since the fundamental transformation of a half-plane leaves the boundary point-wise fixed, we have a one to one continuous map of \( S'_{n+1} \) onto the desired sheet \( S'_{n+1} \); also (which is not essential) \(|a'_{n+1} - a'_n| = |a_{n+1} - a_n| \).

The first partials of this map are continuous except on \( L_n \), \( L_{n+1} \), \( J_n \), and \( J_{n+1} \). By (1) and p. 54 (13), in the strip \( \Pi(w) \equiv 2 \), and in the half-plane bounded by \( L_n \),

\[
\Pi(w) \leq \max \left\{ \frac{\phi}{\phi + \frac{\theta_n}{2}}, \frac{\phi + \frac{\theta_{n+1}}{2}}{\phi - \frac{\theta_n}{2}}, \frac{\pi - \phi + \frac{\theta_n}{2}}{\pi - \phi - \frac{\theta_n}{2}}, \frac{\pi - \phi - \frac{\theta_n}{2}}{\pi - \phi + \frac{\theta_n}{2}} \right\}
\]

\[
\leq \max \left\{ 1 + \frac{1}{\pi}, \frac{\pi}{\sqrt{2}}, 1 \right\}
\]

\[
\leq \max \left\{ 2 + \frac{\pi}{\sqrt{2}}, 5 \right\} \leq 2 + \frac{2\pi}{\sqrt{2}}
\]

Since \( S < \frac{\pi}{4} \). The other half-plane yields the same result, and so for all \( w \)

\( \Pi(w) \leq 2 + \frac{2\pi}{\sqrt{2}} \).

**Case II** Neither \( a_n \) nor \( a_{n+1} \) in a block, \( J_n \) and \( J_{n+1} \) lying on opposite sides of \( \Pi \). For the same reason as in case I

\( J_n \), \( J_{n+1} \) do not intersect. Parallels \( L_n \), \( L_{n+1} \) forming a fixed acute angle \( \frac{\pi}{2} - \eta \) with \( \nu_n \) and forming angles with \( J_n \) and \( J_{n+1} \):

\[
\begin{align*}
\theta &= \frac{\pi}{2} + \eta - \frac{\theta_n}{2} \\
\psi &= \frac{\pi}{2} + \eta - \frac{\theta_{n+1}}{2}
\end{align*}
\]

\( \eta \leq \phi \leq \frac{\pi}{2} + \eta \)

\( \eta \leq \psi \leq \frac{\pi}{2} + \eta \)
As in case I we leave the strip alone and distort the two half-planes with \( \alpha = \phi \), \( \beta = \phi + \frac{\theta_n}{2} \) in one case and \( \alpha = \psi \), \( \beta = \psi + \frac{\theta_{n+1}}{2} \) in the other, obtaining the symmetric sheet desired:

\[
\gamma_n' - \frac{l_n'}{a_{n+1}' - a_n'} = \frac{l_n}{a_{n+1} - a_n} = |\gamma_n' - \gamma_n|
\]

As is readily verified,

\[
(4) \quad \tilde{h}(w) \leq \max \left\{ 2 + \frac{\pi}{2\eta}, 1 + \frac{2\pi}{\pi - 2\eta} \right\}
\]

**Case III** \( a_n, a_{n+1} \) both members of a block. Then \( \gamma_n, \gamma_{n+1} \) lie on opposite sides of \( l_n, \gamma_n \) perpendicular to \( l_n, \gamma_{n+1} \) making an angle \( \tau \) with \( l_n \), \( \frac{\pi}{2} - \delta < \tau < \frac{\pi}{2} + \delta \). Let \( L_n \) be the line bisecting the angle formed by \( \gamma_n \) and \( l_n \), and let \( L_{n+1} \), thru \( a_{n+1} \), be parallel to \( L_n \).
Again we leave the strip unaltered and distort the two half planes by the fundamental transformation with \( \alpha = \pi/4 \), \( \beta = 3\pi/4 \) in one case and \( \alpha = \pi - \pi/4 \), \( \beta = 3\pi/4 \) in the other, obtaining

\[
\begin{array}{c}
\gamma_n' \\
a_n' \\
a_{n+1}' \\
\gamma_{n+1}'
\end{array}
\]

\[|a_{n+1}' - a_n'| = |a_{n+1} - a_n|\]

It is readily verified that

\[(S) \quad M(w) \leq 1 + \frac{3\pi/4}{\pi/4 - \delta}\]

**Case IV**  
\( a_{n+1} \) the initial member of a block: the procedure here is identical with either case I or case II where we used only the fact that one of the angles formed by \( \gamma_n \) and \( \beta_n \), and \( \gamma_{n+1} \) and \( \beta_n \) was \( > \frac{\pi + \delta}{2} \). This is the case here, since \( \alpha_n \) is not in the block.
Note This case makes evident why the cut issuing from the initial vertex of a block was placed in the reflex angle.

Case V \( a_n \) the terminal member of a block: this reduces to either I or II exactly as case IV did.

This exhausts all possibilities and we have distorted \( \mathcal{F}_w' \) into \( \mathcal{F}_w' \) by a transformation of bounded eccentricity, for by (2), (4), (5)

\[
\mathfrak{m}(w) \leq \max \left\{ 2 + \frac{2\pi}{\delta}, 2 + \frac{\pi}{2\eta}, 1 + \frac{2\pi}{\pi-2\eta}, 1 + \frac{3\pi}{\pi-4\delta} \right\}
\]

where \( 0 < \eta < \frac{\pi}{2} \), \( 0 < \delta < \frac{\pi}{4} \).

If we set \( \eta = \frac{\pi}{4} \), \( \delta = \frac{2\pi}{11} \), we obtain

\[
(6) \quad \mathfrak{m}(w) \leq 13
\]

Theorem IX Any semi-cosinic (gammic, cosinic) surface \( \mathcal{F}_w' \) is parabolic. More precisely, \( \mathcal{F}_w \) may be mapped onto a symmetric semi-cosinic (gammic, cosinic) surface \( \mathcal{F}_w' \), the eccentricity of the map satisfying (6), and the distance between two consecutive branch points being preserved. In the case of the gammic surface the logarithmic ends undergo only a rigid motion.

8. The question of the form of the entire function corresponding to one of these non-symmetric surfaces is left hanging. The elliptic approximating surfaces may be constructed as before, but it is not obvious where the zeros of the derivative of the corresponding polynomial will be.
REFERENCES


