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Some Applications
of Adherent Series

by

Henry Curt Lefkovits

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The author wishes to express his appreciation and indebtedness to Professor S. Mandelbrojt who supervised the research presented in this thesis. Through the generous contribution of his ideas and time, Professor Mandelbrojt has provided invaluable guidance and encouragement.
TO MY WIFE
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CHAPTER I
INTRODUCTION

The notion of "adherence", or "adherent series" has been created by Mandelbrojt and applied to a number of diverse problems, such as generalized quasianalyticity, the theory of closure of certain sequences of functions, unicity problems for generalized moment problems, etc. [11] (1). The theory of adherent series is a most valuable and fundamental concept that serves to show the allied nature of these varied problems and provides a method by which the relations between them can be explored. The present work represents on one hand an extension of some results of Mandelbrojt on composition theorems to a more specialized problem, and the application of composition theorems to the study of a new quasianalytic class of functions, and on the other hand, an application of the concept of adherent series to the problem of characterizing an analytic almost periodic function.

The concept of adherent series generally arises in a situation where a function holomorphic in a strip containing a horizontal band is represented in a certain

(1) Numbers in brackets refer to the bibliography at the end of the paper.
asymptotic manner by a Dirichlet series. An adherence hypothesis is a relation between the shape and size of the band, the exponents of the Dirichlet series, and the "precision" of the asymptotic representation. Mandelbrojt has shown that if such a hypothesis is satisfied it is possible to majorize the moduli of the coefficients by a product of quantities depending only on the series of exponents of the Dirichlet series, and the modulus and domain of holomorphism of the function.

It is in this sense that the concept of adherence furnishes a very wide extension of the Cauchy evaluation of the magnitude of the coefficients of a Taylor series. Whereas there we have a series that converges to the function, in the case of adherent series it is not necessary to presuppose convergence of the Dirichlet series, and in fact, in most cases of interest the series does not converge. In that case, even though the series diverges, an adherence hypothesis is sufficient to give a majorization of the magnitude of the coefficients.

The results on adherent series are generally obtained through the use of Mandelbrojt's Fundamental Inequality [8], which will be stated later. Some definitions will be needed first.
1.1 Definitions concerning an increasing positive sequence.

Let \{\lambda_n\} (n \geq 1) be an increasing sequence of positive numbers. For \lambda > 0, let \( N(\lambda) \) denote the number of \( \lambda_n \) smaller than \( \lambda \). Then let

\[
D(\lambda) = \frac{N(\lambda)}{\lambda}
\]

\[
D'(\lambda) = \sup_{x \leq \lambda} D(x)
\]

\[
\overline{D}(\lambda) = \frac{1}{\lambda} \int_0^\lambda D(x) \, dx
\]

\[
\overline{D}'(\lambda) = \sup_{x \leq \lambda} \overline{D}(x)
\]

these functions being called the density function, upper density function, mean density function, and upper mean density function, respectively, of the sequence \{\lambda_n\}.

Also of interest are the following constants:

\[
D' = \lim_{\lambda \to \infty} D'(\lambda)
\]

\[
\overline{D}' = \lim_{\lambda \to \infty} \overline{D}'(\lambda)
\]

called the upper density and upper mean density, respectively, of the sequence \{\lambda_n\}. We will assume in the following that we always have \( D' < \infty \).

The excess function of the sequence \{\lambda_n\} is defined to be

\[
u(D) = \sup_{\lambda \geq 0} \lambda [\overline{D}(\lambda) - D]
\]
This function is non-negative and nonincreasing, having the property that for $D < \overline{D}$, $\mathcal{V}(D) = \infty$, and that $\mathcal{V}(D)$ is finite for $D > \overline{D}$.

We shall need a certain sequence of numbers which are derived from the sequence $\{\lambda_n\}$. It has been shown by Mandelbrojt [11] that the numbers

$$\Delta_n^* = \prod_{m \neq n} \frac{\lambda_m,}{1 - \lambda_m - \lambda_n}$$

are finite and non-zero. The sequence $\{\Delta_n^*\}$ is said to be the sequence associated with the sequence $\{\lambda_n\}$.

1.2 Logarithmic precision.

It was mentioned earlier that the concept of adherence involves a certain asymptotic representation of a holomorphic function by a Dirichlet series. This asymptotic behavior is expressed by a function called the logarithmic precision, defined in the following manner:

Let $\Delta$ be a domain in the $s = \sigma + it$ plane such that the intersection of $\Delta$ with any half-plane $\sigma > \sigma_0$ is not empty. Let $\{d_n\}$ be a sequence of complex numbers and let $\{\lambda_n\}$ be an increasing sequence of positive numbers. Let $F(s)$ be a holomorphic and uniform function in $\Delta$.

If $k$ is a positive integer and $P_k(x)$ is a non-decreasing function tending toward $+\infty$ ($P_k(x)$ may be equal
to \( +\infty \) for \( \kappa \) sufficiently large), such that for \( \kappa \) sufficiently large,

\[
\inf_{m_k^2} \sup_{\sum_{n=1}^{\infty} d_n e^{-\lambda_n s}} |F(s) - \sum_{n=1}^{\infty} d_n e^{-\lambda_n s}| \leq e^{-p_k(\kappa)},
\]

we say that the series \( \sum d_n e^{-\lambda_n s} \), with \( m_k \), represents \( F(s) \) in \( \Delta \) with the logarithmic precision \( p_k(\kappa) \).

This definition clearly does not assume the convergence of the series \( \sum d_n e^{-\lambda_n s} \). If the series converges for \( \sigma > \sigma_0 \), then \( p_k(\kappa) \) is equal to \( +\infty \) for \( \kappa > \sigma_0 \) for any domain contained in this half-plane with \( |t| < a \). The converse of this proposition is not true; if \( p_k(\kappa) = +\infty \) for \( \kappa > \sigma_0 \), then one can merely assert that the series \( \sum d_n e^{-\lambda_n s} \) overconverges to \( F(s) \) in the intersection of \( \Delta \) with \( \sigma > \sigma_0 \).

1.3 Adherence hypotheses.

A condition which allows the coefficients of the Dirichlet series to be majorized by the Fundamental Inequality is called an adherence hypothesis, and if such a condition is satisfied the series is said to "adhere" to the function. The strongest adherence hypothesis given by Mandelbrojt in [11] is the condition \( A(g(\sigma), p(\sigma), \lambda_n) \), given in the following:

Let \( g(\sigma) \) be a continuous positive function defined for \( \sigma \) sufficiently large such that \( g(\sigma) \) is of bounded
variation for \( \sigma \geq \sigma_0 \). Let \( p(\sigma) \) be a positive non-decreasing function tending to \(+\infty\) (where \( p(\sigma) \) may be equal to \(+\infty\) for \( \sigma \) sufficiently large). Let \( \{\lambda_n\} \) be a positive increasing sequence, and \( \gamma(\Omega) \) its excess function. Then we have:

**Hypothesis** \( A\{g(\sigma), p(\sigma), \{\lambda_n\}\} \). If \( g = \lim_{\sigma \to \infty} g(\sigma) \), there exists a continuous non-increasing function \( h(\sigma) \), with \( \lim_{\sigma \to \infty} h(\sigma) = h \), such that

\[
\begin{align*}
\int_0^\infty \left\{ p(\sigma) - 2 \gamma[h(\sigma)] \right\} \exp \left\{ -\frac{1}{2} \int_0^\sigma \frac{du}{g(u) - h(u)} \right\} d\sigma &= \infty.
\end{align*}
\]

A somewhat simpler and more restricted adherence hypothesis is given by the following [11]:

**Hypothesis** \( A_3\{g(\sigma), p(\sigma), \{\lambda_n\}\} \). \( g > \gamma^* \), and there exists a positive constant \( \delta \) such that

\[
\int_0^\infty p(\sigma) \exp \left[ -\frac{1}{2} \int_0^\sigma \frac{du}{g(u) - \delta \gamma[p(\sigma)u]} \right] d\sigma = \infty.
\]

The general character of adherence hypotheses is to state that if a holomorphic function \( F(z) \) can be represented in a domain \( \Delta \) by \( \sum d_n e^{-\lambda n z} \) with a logarithmic precision \( p(\sigma) \), then if the domain is sufficiently large, the numbers \( \lambda_n \) are sufficiently rare and well-distributed, and the function \( p(\sigma) \) is sufficiently large, then the series adheres to the function in \( \Delta \).
1.4 Fundamental inequality.

Mandelbrojt has given the following fundamental theorem [11]:

**IF:**

1° \( \{\lambda_n\} \) is an increasing sequence of positive numbers with finite upper density; \( \Delta \) is a domain in the \( s = \sigma + it \) plane given by \( \sigma > \alpha \), \( |t| < \pi g(\sigma) \), where \( g(\sigma) \) is a continuous function of bounded variation such that \( g(\sigma) > \overline{\beta} \), where \( \overline{\beta} \) is the mean upper density of the sequence \( \{\lambda_n\} \);

2° \( F(s) \) is a holomorphic function in \( \Delta \) which can be continued analytically from \( \Delta \) to the circle \( C(s_0, \pi R) \) of center \( s_0 \) and radius \( \pi R \) through a channel of diameter greater than \( 2\pi \overline{\beta} \);

3° The sequence of numbers \( \{d_n\} \) and the positive integer \( k \) are such that the sums \( \sum_{n=1}^{\infty} d_n e^{-\lambda_n s} \), with \( m < k \), represent \( F(s) \) in \( \Delta \) with the logarithmic precision \( p_k(x) \);

**THEN**

if the hypothesis \( A(g(\sigma), p_k(\sigma), \{\lambda_n\}) \) is satisfied,

\[
|d_k| \leq \frac{1}{2} \pi^2 \lambda_k \text{Re}^{2\gamma(R)} \Delta_k^* e^{\lambda_k R(s_0)} M(s_0, \pi R),
\]

where

\[
M(s_0, \pi R) = \sup_{s \in C(s_0, \pi R)} |F(s)|.
\]

1.5 Associated functions.

The concept of associated functions has been intro-
duced by Mandelbrojt in [10] and in his lectures at Rice Institute in 1956. The theorems obtained on these functions can be considered as consequences of theorems on Watson's problem and adherent series, respectively.

We consider first the definition given in [10]. Let $E$ denote a closed set on the real axis $R$ in the complex $z=x+iy$ plane. If $E=R$ we denote by $\Delta E$ the upper half-plane: $y>0$. If $E \neq R$ we let $\Delta E$ denote the complement of the set $E$ with respect to the entire $z$-plane.

Let $M(r)$ be a positive function defined for $r>0$. $M(r)$ is said to be a function associated with the set $E$ if every function $\phi(z)$ which is single-valued and holomorphic in $\Delta E$ and satisfies there the inequality

$$|\phi(z)| \leq \frac{M(|z|)}{|y|}$$

is identically zero.

Before proceeding to a more general case we will make the following definitions:

If $E$ is a closed set on $R$, and $\alpha$ is a positive number, we say that the set

$$E_\alpha = \bigcup_{x_0 \in E} [x_0-\alpha, x_0+\alpha]$$

is an $\alpha$-covering of $E$.

If $E$ is a closed set on $R$, and $\alpha$ is a positive number, by a two-dimensional $\alpha$-covering of $E$ we mean the set
\[ E_{a,2} = \bigcup_{x_0 \in \mathbb{E}} J(x_0, a) \]

where \( J(x_0, a) \) is the closure of the bounded region determined by

\[
\begin{align*}
y &= a \\
y &= -a \\
x^2 + y^2 &= (x_0 - a)^2 \quad \text{if } x_0 + a, \quad x = a \quad \text{if } x_0 = a \\
x^2 + y^2 &= (x_0 + a)^2 \quad \text{if } x_0 - a, \quad x = -a \quad \text{if } x_0 = -a.
\end{align*}
\]

Suppose now that \( \mathbb{E} \) is a closed set on \( \mathbb{R} \), \( \Phi(z) \) is holomorphic and uniform in \( \Delta_\mathbb{E} \) and that there

\[ |\Phi(z)| \leq \frac{M(|z|)}{|y|} \]

\( M(r) \) being as before. It can then easily be shown that if \( a \) is a positive number and \( E_{a,2} \) is a two-dimensional \( a \)-covering of \( \mathbb{E} \), that there exists a constant \( A \) depending only on \( a \) such that

\[ |\Phi(z)| \leq A M(|z|) \]

in the complement of \( E_{a,2} \) with respect to the \( z \)-plane.

1.6 Associated sequences.

We now come to a more extended definition; in this case, however, we will speak of an associated sequence of numbers.

Let \( \lambda_n \) be a positive increasing sequence of integers with \( D_\lambda < \frac{1}{2} \), \( |M_n| (n \geq 0) \) a sequence of positive numbers, and \( \mathbb{E} \) a closed set on \( \mathbb{R} \), \( \Delta_\mathbb{E} \) having the same
meaning as before.

We say that the sequence \( \{M_n\} \) is associated with the closed set \( E \) and the sequence \( \{\lambda_n\} \) if the following conditions are satisfied:

If \( \phi(z) \) is a holomorphic and uniform function in \( \Delta_E \), and if there exists a sequence \( \{d_n\} \) of complex numbers and a constant \( A \) such that for every \( a > 0 \),

\[
|\phi(z) - \sum_{k=1}^{m} \frac{d_k}{z - \lambda_n} | \leq \frac{AM_{\lambda_n + 1}}{a |z - \lambda_n|^{m+1}}, \quad \lambda_m < \eta < \lambda_{m+1}
\]

\[
|\phi(z)| \leq \frac{A}{a}
\]

for \( z \) in the complement of \( E_{a,2} \), then \( \phi(z) \equiv 0 \).

Whereas before we had a function associated with a set, we now have a sequence associated with a set and a sequence of integers. The generalization of the second definition over the first rests on the sequence \( \{\lambda_n\} \).

This can be seen from the following considerations: Let \( E \) be a closed set on the real axis of the \( \mathbb{C} = \mathbb{R} + i\mathbb{R} \) plane, and let \( \phi(z) \) be a holomorphic and uniform function in \( \Delta_E \) such that

\[
|\phi(z)| \leq \frac{M_n}{|y| |z|^{m}}, \quad (n \geq 0)
\]

where \( \{M_n\} \) is a sequence of positive numbers. It then follows that

\[
|\phi(z)| \leq \frac{1}{|y|} \inf_{n} \frac{M_n}{|z|^{m}}
\]
so that if the function

\[ T(r) = \sum_{n} \frac{M_n}{r^n} \]

is associated with the set \( E \) it follows that \( \phi(z) = 0 \).

From the preceding it also follows that if \( E_{a_2} \)

is a two-dimensional \( \alpha \)-covering of \( E \), then in the com-

plement of \( E_{a_2} \)

\[ |\phi(z)| \leq A \frac{M_n}{|z|^n} . \]

Hence, if \( \{M_n\} \) is associated with \( E \) and the empty

sequence, the function \( T(r) \) defined before is associated

(in the sense given before) with \( E \).

Mandelbrojt has given relationships between adherence

hypotheses and the fact that a sequence \( \{M_n\} \) is associated

with a set \( E \) and a sequence \( \{\lambda_n\} \).

1.7 The Fourier-Carleman transform.

We will present in this section some well-known

results concerning the Fourier-Carleman transform. These

results will be used in subsequent chapters.

Suppose \( f(x) \in L_1 \) on \( (-\infty, \infty) \); then we can define \( \mathcal{F}(f) \),

its Fourier transform as

\[ \mathcal{F}(f) = \phi(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ixu} dx , \]

and the function \( \phi(u) \) is a bounded, continuous function.

The spectrum of \( f(x) \) is defined as the set

\[ \sigma(f) = \{ u | \phi(u) \neq 0 \} . \]
The integral defining $\mathcal{G}(t)$ can be split into the difference of two integrals, e.g.,

$$
\mathcal{G}(t) = \frac{1}{2\pi i} \left[ \int_{-\infty}^{\infty} f(x) e^{-ix} dx - (-\int_{0}^{\infty} f(x) e^{-ix} dx) \right]
$$

so that if $u \notin \sigma(t)$, then these two integrals are equal, i.e.,

$$
\int_{-\infty}^{0} f(x) e^{-ix} dx = -\int_{0}^{\infty} f(x) e^{-ix} dx.
$$

We now proceed to a class of functions for which in general it is not possible to define a Fourier transform. Suppose that $f(x) \in L^p$ ($p > 1$); we define the Fourier-Carleman ($FC$) transform of $f(x)$ to be the pair of functions $(F^+(y), F^-(y))$ given by

$$
F^+(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ixy} dx, \quad (y = \xi + i\eta)
$$

$$
F^-(y) = -\frac{1}{2\pi} \int_{0}^{\infty} f(x) e^{ixy} dx
$$

These functions are holomorphic and uniform in the upper half-plane $\Theta(y) > 0$ and lower half-plane $\Theta(y) < 0$ respectively. Consider now the difference

$$
F^+(\xi + i\eta) - F^-(\xi - i\eta), \quad (\eta > 0).
$$

From the preceding it follows that if $f(x) \in L^1$, then this difference tends to $\phi(\xi) = \mathcal{G}(t)$ as $\eta \to 0$. In particular, this difference tends to zero if $\xi \notin \sigma(t)$; moreover, if $[a, b]$ is an interval such that $[a, b] \cap \sigma(t) = \emptyset$, then this difference tends to zero uniformly for $\xi \in [a, b]$. 
Suppose now that \( f(x) \in L^p (p > 1) \) and that
\[
\lim_{n \to 0} \left[ F^+ (t + i \eta) - F^- (t - i \omega) \right] = 0
\]
uniformly for \( \eta \in [a, b] \). We then say that the interval
\( [a, b] \) is a concordance interval of \( f(x) \), or that \( [a, b] \subset c(f) \),
the concordance set of \( f(x) \). We define the spectrum of
\( f(x) \), \( \sigma(f) \), to be the complement of \( c(f) \) with respect
to \( \mathbb{R} \). This definition is consistent with the one given
for the spectrum of a function \( f(x) \in L_1 \).

A further consequence is of particular interest.
If \( [a, b] \subset c(f) \), then the functions \( F^+(y) \) and \( F^-(y) \)
are analytic continuations of each other across the interval \( [a, b] \).
The converse of this proposition is also true: if \( F^+(y) \) and \( F^-(y) \)
are analytic continuations of each other across the interval \( [a, b] \),
then \( [a, b] \subset c(f) \).

Consider now a function \( f(x) \in L^p (p \geq 1) \), and suppose
that \( \sigma(f) \neq \mathbb{R} \). Let the function \( F(y) \) be defined to be
equal to \( F^+(y) \) for \( \sigma(y) > 0 \), \( F^-(y) \) for \( \sigma(y) < 0 \), and the common
value of \( F^+(y) \) and \( F^-(y) \) along any interval \( [a, b] \subset c(f) \).
If \( P[\sigma(f)] \) denotes the domain obtained by removing the
points of \( \sigma(f) \) from the \( \gamma \)-plane, then \( F(y) \) is holomorphic
and uniform in \( P[\sigma(f)] \). Moreover, if \( \overline{\sigma}(f) \) is a two-
dimensional \( \alpha \)-covering of \( \sigma(f) \), where \( \alpha \) is sufficiently
small so that \( \overline{\sigma}(f) \neq \mathbb{R} \), then it can be shown (through the
use of Schwarz's inequality if \( p \neq \infty \)), similarly to the
result of 1.5, that
\[ |F(t)| \leq A \| f \|_p \]

in the domain \( P[\alpha, \lambda(t)] \), where \( A \) is a constant depending only on \( \alpha \), and \( \| f \|_p \) is the norm of \( f(x) \) in \( L^p \).

**1.8 A problem of generalized quasianalyticity.**

Consider the following definition of a class of functions:

Given a sequence \( \{M_n\} (n \geq 0) \) of positive numbers, a function \( f(x) \), infinitely differentiable on \( (-\infty, \infty) \), is said to belong to the class \( L^p[M_n] \), \( (p \geq 1) \) provided that

1. \( f^{(m)}(x) \in L^p \) \( (n \geq 0) \)
2. \( \| f^{(m)} \|_p \leq M_n \) \( (n \geq 0) \).

We will now apply the previous considerations to a problem of generalized quasianalyticity.

Given a positive sequence \( \{M_n\} (n \geq 0) \), an increasing sequence \( \{\lambda_n\} \) of positive integers such that if \( \{\lambda_n\} \) is the sequence complementary to \( \{\lambda_n\} \) with respect to the sequence of positive integers, then \( \hat{D}_\mu < \frac{1}{2} \), and a closed set \( E \) on \( R \) such that \( E \neq R \), suppose that \( f(x) \) is a function such that

1. \( f(x) \in L^p[M_n] \), \( (p \geq 1) \)
2. \( f(0) = f^{(\lambda_n)}(0) = 0 \) \( (n \geq 1) \)
3. \( \sigma(f) \subset E \).

The problem is to find conditions on the sequence \( \{M_n\} \), the set \( E \), and the sequence \( \{\lambda_n\} \), so that it is possible
to conclude from 1°, 2°, and 3° that \( f(x) \equiv 0 \).

Consider the \( \mathcal{F} \) transform (\( F^+(s), F^-(s) \)) of \( f(x) \).

By repeated integration by parts we obtain for \( \mathcal{D}(s) > 0 \),
\[
\sqrt{2\pi} F^+(s) = \frac{f_0(s)}{i s} + \frac{f_1(s)}{(i s)^2} + \ldots + \frac{f_m(s)}{(i s)^m} + \frac{1}{(i s)^{m+1}} \int_{-\infty}^{\infty} f^{(m+1)}(x) e^{-isx} dx,
\]
and a similar equation for \( F^-(s) \) in \( \mathcal{D}(s) < 0 \). Since
\( f(x) = f_{\mathcal{D}(s)} = 0 \), and \( \{\lambda_n\} \) is the sequence complementary to \( \{\lambda_n\} \), the above equation reduces to
\[
\sqrt{2\pi} F^+(s) = \sum_{n=0}^{m} \frac{f^{(n+1)}(0)}{(i s)^{n+1}} + \frac{1}{(i s)^{m+1}} \int_{-\infty}^{\infty} f^{(m+1)}(x) e^{-isx} dx,
\]
where \( q \) is any integer such that \( \lambda_m \leq q < \lambda_{m+1} \), and \( s \) is such that \( \mathcal{D}(s) > 0 \); a similar equation existing for \( F^-(s) \) in \( \mathcal{D}(s) < 0 \).

Consider now a two-dimensional \( a \)-covering \( E_{a,2} \) of \( E \), where \( a \) is small enough so that \( E_a \neq \emptyset \). Then we notice that \( E_{a,2}(s) \subset E_{a,2} \). Let \( P[E_{a,2}] \) denote the domain complementary to \( E_{a,2} \) and consider the function \( F(s) \) as defined before. Then \( F(s) \) is holomorphic in \( P[E_{a,2}] \) and evaluating the integral in the last equation by the same method as mentioned in 1.7, it follows that in \( P[E_{a,2}] \)
\[
\left| F^+(s) - \sum_{n=0}^{m} \frac{f^{(n+1)}(0)}{(i s)^{n+1}} \right| \leq A \frac{\lambda_{m+1}}{\lambda_{m+1}} \frac{\lambda_{m+1}}{\lambda_{m+1}}
\]
where \( \lambda_m \leq q < \lambda_{m+1} \).

Consequently, from the definition of associated sequences, if the sequence \( \{M_n\} \) is associated with the set \( E \) and the sequence \( \{\lambda_n\} \), it follows that \( F(s) \equiv 0 \), which then implies that \( f(x) \equiv 0 \).
The solution to the problem we have stated before is then given in the following theorem:

IF

1. \( \{ \lambda_n \} \) (n \geq 0) is a sequence of positive numbers, \( E \) is a closed set on \( R \) such that \( E + R \), \( \{ \mu_n \} \) is a sequence of positive integers;

2. \( \{ \lambda_n \} \) is the sequence of integers complementary to \( \{ \mu_n \} \) with respect to the sequence of positive integers;

3. The sequence \( \{ \lambda_n \} \) is associated with the set \( E \) and the sequence \( \{ \mu_n \} \);

THEN

every function \( f(x) \in L^p(\{ \lambda_n \}, (p>1)) \) which is such that
\[
    f(0) = f(\{ \lambda_n \}(0)) = 0 \quad (n \geq 1),
\]
\[
    \sigma(f) \subseteq E
\]
is identically zero.
CHAPTER II
COMPOSITION THEOREMS

The present chapter is devoted to the presentation of some composition theorems. These theorems are specializations of theorems given by Mandelbrojt in [11], the specialization occurring mainly in the sense that the spectra of the functions involved are introduced. In doing so, the striking similarity of these theorems to the famous theorem of Hadamard on the composition of singularities of Taylor series is further emphasized.

We will require the following lemma [11]:

2.1 Lemma.

Let \( g(x) \) be an even function, infinitely differentiable on \((-\infty, \infty)\), such that
\[
\int_{0}^{\infty} |g^{(n)}(t)|^2 dt \leq M_n^2 \quad (n \geq 0),
\]
where \( \{M_n\} (n \geq 0) \) is a non-decreasing positive sequence, \( \log M_n \) being a convex function of \( n \). Let \( \log M(x) \) denote the ordinate of the polygonal line \((x \geq 0)\) with vertices \((n, \log M_n)\). If
\[
H(\omega) = \lim_{A \to \infty} \frac{1}{A} \int_{-A}^{A} g(t) e^{-i\omega t} dt,
\]

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then $H(\omega)$ is even, $|H(\omega)| e^x$ is integrable on $[0, \infty)$ for all $x > 0$, and for $n \geq 0$

$$g^{(n)}(t) = \frac{(i)^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(\omega) \omega^n e^{i\omega t} d\omega$$

and

$$\int_{-\infty}^{\infty} |H(\omega)| \omega^x d\omega \leq 2M(x+1) \quad (x > 0).$$

2.2 Composition theorem.

We first make the following definitions:

If $A$ and $B$ are closed sets on $\mathbb{R}$, the set

$$A \oplus B = \{ \gamma | \gamma = \alpha + \beta, \alpha \in A, \beta \in B \}$$

is called the composite sum of the sets $A$ and $B$; the set

$$A \otimes B = \{ \gamma | \gamma = \alpha \beta, \alpha \in A, \beta \in B \}$$

is called the composite product of the sets $A$ and $B$.

The following notation will also be used: If $E^\omega$ and $E^\omega$ are closed symmetric sets on $\mathbb{R}$, $a$ is a positive number, the domain $D[E^\omega, E^\omega, a]$ is defined in the following manner: Let

$$E = C \{ E^\omega \oplus E^\omega \}$$

the complement being taken with respect to $\mathbb{R}$, and then let

$$E = \{ x | e^x \in E \}.$$ 

Consider then the $s = \sigma + it$ plane, where we define the domain $D_a = D[E^\omega, E^\omega, a]$ by
\[ D_a = \{ s = \sigma + it \mid \sigma > \log a, \ \lvert t \rvert < \frac{\pi}{2} - \arcsin (ae^{-\sigma}) \}, \text{ if } \sigma \notin E \}. \]

We now present a theorem which is not in the symmetrical form of a composition theorem, but contains as a special case a composition theorem for functions in \( L^2 \).

**Theorem 2A**

**IF**

1. \( \{ M_n \} \) and \( \{ M'_n \} \) \((n \geq 2)\) are positive sequences, \( \log M'_n \) being a convex function of \( n \);

2. \( \{ \lambda_n \} \) and \( \{ \lambda'_n \} \) \((n \geq 1)\) are two increasing sequences of even positive integers, such that if \( \{ \lambda_n \} \) is the sequence complementary to \( \{ \lambda_n \} \cup \{ \lambda'_n \} \) with respect to the sequence of even positive integers, then \( \overline{D}_n < \frac{1}{2} \);

3. \( E^{(n)} \) and \( E^{(\alpha)} \) are closed symmetric sets on \( R \) such that \( \emptyset \notin E^{(n)}, E^{(\alpha)} \);

4. The sequence \( \{ M_n, M'_n \} \) is associated with the set \( E^{(n)} \otimes E^{(\alpha)} \) and the sequence \( \{ \lambda_n \} \);

5. \( f(x) \) and \( g(x) \) are even functions, infinitely differentiable on \((-\infty, \infty)\), such that

\[
\begin{align*}
    f(x) \in L^p \{ M_n \} \ (p \geq 1), \ g(x) \in L^2 \{ M'_n \} \\
    f(0) = f^{(\lambda_n)}(0) = 0 \ (n \geq 1) \\
    g(0) = g^{(\lambda'_n)}(0) = 0 \ (n \geq 1) \\
    \sigma_a(f) \subset E^{(n)}, \ \sigma_a(g) \subset E^{(\alpha)};
\end{align*}
\]

**THEN**

if \( g(x) \neq 0 \) it follows that \( f(x) = 0 \).
Proof:
It can be assumed without loss of generality that \( \{M'_n\} \) is an increasing sequence. Then let
\[
H(\omega) = \lim_{A \to \infty} \int_{-A}^{A} g(t)e^{-i\omega t} dt
\]
and
\[
K(x) = H(e^x)e^x
\]
We note that the spectrum of \( g(x) \) is a well-defined set and that
\[
H(\omega) = 0 \text{ pp if } \omega \in \sigma(g)
\]
and hence that if we let
\[
\mathcal{S}_a(g) = \{ x \mid e^x \in \mathcal{S}_a(g) \}
\]
then
\[
K(x) = 0 \text{ pp if } x \in \mathcal{S}_a(g).
\]
By the lemma of 2.1 we then have that
\[
\int_{-\infty}^{\infty} K(x)e^{\mu x} dx = \pi(i)^{\mu m} g^{(\mu m)}(0) = 0
\]
and that
\[
\int_{-\infty}^{\infty} |K(x)| e^{\xi x} dx \leq 2M'(1+1), \quad (\xi > 0).
\]
Now let \( \{\gamma_n\} \) be the sequence complementary to \( \{\lambda_n\} \) with respect to the sequence of even positive integers. If \( \mathcal{C}_{a,2}(f) \) is a two-dimensional \( a \)-covering of \( \sigma(f) \), let \( P[\mathcal{C}_{a,2}(f)] \) denote the complement of \( \mathcal{C}_{a,2}(f) \) with respect to the \( z = x + iy \) plane. Consider the \( \mathcal{F} \mathcal{C} \) transform \((F^t(\mathfrak{z}), F^r(\mathfrak{z}))\) of \( \{f(x) \) and the function \( F(z) \) equal to \( F^t(z) \) in \( \mathcal{D}(\mathfrak{z}) \).
$F^{-1}(x)$ in $O(x)<0$, and the common value on the concordance set. From the considerations of Chapter I it then follows that in $P[\sigma_{a,2}(f)]$,

$$\left| \sqrt{2\pi} F(x) - \sum_{n=1}^{m} \frac{f(x_n) (x)}{(i^2)^{\frac{n+1}{2}}} \right| \leq A \frac{M_{q+1}}{|x|^{\frac{q+1}{2}}}, \quad p_m \leq q < p_{m+1},$$

where $A$ is a constant depending only on $a$. Now let

$$\delta_a(f) = \{ x \in \mathbb{C} | x \in \sigma_a(f) \}$$

and consider the transformation $z = -ie^{y}$. Since $f(x)$ is an even function, $\sigma(f)$ is symmetric, and so is $\sigma_a(f)$; this transformation maps $\sigma_a(f)$ into the set

$$\{ s + (2k+1)\pi i | s \in \delta_a(f), k = 0, \pm 1, \ldots \}.$$ 

Then the image of $P[\sigma_{a,2}(f)]$ is the domain $B[\sigma_{a,2}(f)]$ given by

$$B[\sigma_{a,2}(f)] = \{ s = \sigma + it | if \sigma \in \delta_a(f), \sigma > \log a, \,(2k-1)\pi + \arcsin(ae^{-\sigma}) < t < (2k+1)\pi - \arcsin(ae^{-\sigma}), k = 0, \pm 1, \ldots \}.$$ 

Let $B_c[\sigma_{a,2}(f)]$ denote the following simply-connected subdomain of $B[\sigma_{a,2}(f)]$:

$$B_c[\sigma_{a,2}(f)] = \{ s = \sigma + it | if \sigma \in \delta_a(f), -\infty < t < \infty, \sigma > \log a, 1t < \pi - \arcsin(ae^{-\sigma}) \}.$$ 

Then if we let $\Phi(s) = \sqrt{2\pi} F(-ie^{y})$, in the domain $B[\sigma_{a,2}(f)]$ and ad fortiori in $B_c[\sigma_{a,2}(f)]$,

$$\left| \Phi(s) - \sum_{n=1}^{m} f(x_n) (x) e^{-(p_{n+1})s} \right| \leq A M_{q+1} e^{-\frac{p_{n+1}}{2}},$$
where $p_m \leq q < p_{m+1}$.

If we let $\Psi(s) = e^s \Phi(s)$, then for $s \in B_c[\alpha, 2\alpha(f)]$,

$$|\Psi(s) - \sum_{n=1}^{m} f(p_n)(s) e^{-p_n s}| \leq AM_{q+1} e^{-(q+1)s} e^s,$$

where again $p_m \leq q < p_{m+1}$. Also, since $f(0) = 0$, we have that for $s \in B_c[\alpha, 2\alpha(f)]$

$$|\Psi(s)| \leq AM_2 e^{-s}.$$

We now consider the integral

$$\chi(s) = \int_{-\infty}^{\infty} \Psi(s-x) K(x) dx,$$

this integral clearly converging for $s$ real. We will show that $\chi(s)$ is a holomorphic function in a suitably defined domain. We first note that we may write

$$\chi(s) = \int_{\Delta_a(t_0)} \Psi(s-x) K(x) dx + \int_{C_{\alpha}(g)} \Psi(s-x) K(x) dx.$$

But $K(x) = 0$ pp in $C_{\alpha}(g)$, so that we have

$$\chi(s) = \int_{\Delta_a(t_0)} \Psi(s-x) K(x) dx.$$

Consider now the set

$$E_\alpha = \{ x \mid e^\ast \in E_{f,g} \},$$

where

$$E_{f,g} = C_\alpha(f) \otimes C_{\alpha}(g),$$

the complement being taken with respect to $R$.

Then

$$E_\alpha = C_\alpha(f) \otimes C_{\alpha}(g),$$
and we note that $\mathcal{E}_a \supset \mathcal{E}$, where $\mathcal{E}$ has been defined previously.

Let $\sigma_a$ be such that for $\sigma < \sigma_a$, $\sigma \in \mathcal{E}_a$ (since $0 < \sigma(t)$), for $\alpha$ such that $\log \alpha < \sigma_a$ (which is satisfied if $\alpha$ is sufficiently small). We let

$$D_{N,a} = \left\{ s = \sigma + it \mid \left| t \right| < \pi_2 - \arcsin \left( \alpha e^{-\sigma} \right) + N \Psi(\mathcal{E}_a), \text{ if } \sigma > \sigma_a \right\}$$

$$= \left\{ s = \sigma + it \mid \left| t \right| < \pi_2 + N \Psi(\mathcal{E}_a), \text{ if } \sigma < \sigma_a \right\}$$

where $\Psi(\mathcal{E}_a)$ is the characteristic function of the set $\mathcal{E}_a$ and $N$ is an arbitrary positive constant. Denote also by

$$D_{N,a}^* = \left\{ s = \sigma + it \mid \left| t \right| < \pi_2 + N \Psi(\mathcal{E}_a) \right\},$$

we will show that if $\alpha$ is sufficiently small and $N$ is an arbitrary positive constant, then $\gamma(s)$ is a holomorphic function in $D_{N,a}^*$.

Consider the intersection of $D_{N,a}^*$ with the strip $\sigma_1 < \sigma < \sigma_2$ and a point $s_0 = \sigma_0 + i\xi_0$ in this domain. Then

- either $1^0 \sigma_0 \notin \mathcal{E}_a$,
- or $2^0 \sigma_0 \in \mathcal{E}_a$.

We consider case $1^0$. Let $\alpha$ be sufficiently small in order to have $\left| \xi_0 \right| < \pi_2 - \arcsin(\alpha e^{-\sigma_0})$. Whereas it is not true in general that $s_0 - x \in \mathcal{G}[\mathcal{S}_{\alpha,2}(t)]$, there exists, however, an $\alpha$ sufficiently small such that $s_0 - x \in \mathcal{G}[\mathcal{S}_{\alpha,2}(t)]$ for all $x$.

We then have
\[
|\chi(s_0)| \leq \int_{-\infty}^{\infty} |\bar{\chi}(s_0-x)||k(x)| \, dx
\]

\[
\leq A M_2 e^{-\sigma_0} \int_{-\infty}^{\infty} |k(x)|e^x \, dx
\]

\[
\leq A M_2 e^{-\sigma_1} \int_{-\infty}^{\infty} |k(x)|e^x \, dx
\]

\[
\leq c_1 e^{-\sigma_1} \quad (c_1 \text{ is constant}).
\]

We now consider case 2°. If \(s_0 \in \mathcal{E}_a\), then \(l|l < \frac{\pi}{2} - \arcsin(a e^{-\sigma_0}) + \mathbb{N}\) for \(a\) sufficiently small. We will show that when \(x \in \mathcal{L}_a(g)\), which is the range of integration, then \(s_0 - x \notin \mathcal{C}[s_0, 2l(f)]\). Suppose this is not true. Then there exists an \(x' \in \mathcal{L}_a(g)\) such that \(s_0 - x' = s' \notin \mathcal{L}_a(f)\). Then \(s' \in \mathcal{L}_a(f)\).

But then \(s_0 = s' + x'\), where \(s' \in \mathcal{L}_a(f)\) and \(x' \in \mathcal{L}_a(g)\), and from the definition of \(\mathcal{E}_a\) we have that \(s_0 \notin \mathcal{E}_a\). This, however, contradicts our initial hypothesis that \(s_0 \in \mathcal{E}_a\). Hence, if \(x \in \mathcal{L}_a(g)\), and \(s_0 \in \mathcal{E}_a\), then \(s_0 - x \notin \mathcal{C}[s_0, 2l(f)]\). Then as before

\[
|\chi(s_0)| \leq A M_2 e^{-\sigma_0} \int_{\mathcal{L}_a(g)} |k(x)|e^x \, dx \leq A M_2 e^{-\sigma_1} \int_{\mathcal{L}_a(g)} |k(x)|e^x \, dx
\]

\[
\leq c_1 e^{-\sigma_1}
\]

Thus, for every point \(s \in \mathbb{D}_{N,a}\), and \(a\) for the integral defining the function \(\chi(s)\) has a meaning. It can be seen from the preceding that not only has the integral a meaning, but it converges uniformly in every closed bounded compact subset, this being true for every \(a > 0\).
We now proceed with the proof. From before, if
\[ \sigma \in \mathcal{O}_2 \{ \mathcal{E}_a (t) \} , \]
then
\[ | \overline{f}(x) - \sum_{n=1}^{\infty} f(\eta_n) e^{-\eta_n s} | \leq AM_{q+1} e^{-(q+1)\sigma} e^\sigma , \]
where \( s_m = q < s_{m+1} \). Then by the preceding
\[ \left| \int_{-\infty}^{\infty} \left[ \overline{f}(s-x) - \sum_{n=1}^{\infty} f(\eta_n) e^{-\eta_n (s-x)} \right] K(x) dx \right| \leq AM_{q+1} e^{-(q+1)\sigma} e^\sigma \int_{-\infty}^{\infty} |K(x)| e^{\sigma x} dx \]
for \( s \in \mathcal{D}_a \), defined by
\[ \mathcal{D}_a = \left\{ \sigma + it : -\infty < t < \infty , \forall t \in \mathcal{E}_a , \sigma > \log , \right\} \]
this inequality being true a fortiori in the domain \( D[\mathcal{E}_a, \mathcal{E}_a, a] \) defined before.

Now, for each \( m \), let the integers \( l_m \) and \( n_m \) be defined by
\[ \{ \eta_k \}_m = \{ \mu_k \}_m \cup \{ \nu_k \}_m ; \quad \{ \mu_k \}_m \cap \{ \nu_k \}_m = \emptyset . \]
Then
\[ \sum_{n=1}^{\infty} f(\eta_n) e^{-\eta_n (s-x)} K(x) = \sum_{n=1}^{l_m} f(\mu_n) e^{-\mu_n s} K(x) e^{\mu_n x} + \sum_{n=1}^{n_m} f(\nu_n) e^{-\nu_n s} K(x) e^{\nu_n x} . \]
But since
\[ \int_{-\infty}^{\infty} \sum_{n=1}^{l_m} f(\mu_n) e^{-\mu_n s} K(x) e^{\mu_n x} dx = \sum_{n=1}^{l_m} f(\mu_n) e^{-\mu_n s} \int_{-\infty}^{\infty} K(x) e^{\mu_n x} dx = 0 \]
if we let
\[ a_n = f^{(\nu_n)}(0) \int_{-\infty}^{\infty} k(x) e^{\nu_n x} \, dx, \]
then
\[ |\chi(s) - \sum_{n=1}^{m_n} a_n e^{-\nu_n s}| \leq A M_{\nu+1} e^{-(\nu+1)\sigma} e^{\int_{-\infty}^{\infty} |k(x)| e^{\nu_1 x} \, dx} \leq B M_{\nu+1} M_{\nu+1}' e^{-(\nu+1)\sigma} e^{\sigma} \]
for \( \nu_m < q < \nu_{m+1} \), \( B = \text{constant} \), for \( s \in D[E^{(0)}, E^{(s)}, a] \). Now, if we let
\[ \Theta(s) = e^{s \gamma} \chi(s) \]
then the preceding inequality becomes
\[ |\Theta(s) - \sum_{n=1}^{m_n} a_n e^{-(\nu_n+1)s}| \leq B M_{\nu+1} M_{\nu+1}' e^{-(\nu+1)\sigma}, \]
where it is interesting to note that, in similarity to the Hadamard theorem, \( a_n \) is proportional to \( f^{(\nu_n)}(0) g^{(\nu_n+1)}(0) \).

By a change in notation the last inequality can be rewritten as
\[ |\Theta(s) - \sum_{n=1}^{m_n} a_n e^{-(\nu_n+1)s}| \leq B M_{\nu+1} M_{\nu+1}' e^{-(\nu+1)\sigma} \]
for \( \nu_m \leq q < \nu_{m+1} \), and \( s \in D[E^{(0)}, E^{(s)}, a] \).

By the change of variable \( z = -e^{s} \) we see that from the fact that the sequence \( \{M_n M_n'\} \) is associated with the set \( E^{(0)} \otimes E^{(s)} \) and the sequence \( \{\gamma_n\} \) it follows that \( \Theta(s) = 0 \), i.e.
\[ \Theta(s) = e^{s} \int_{-\infty}^{\infty} Y(s-x) K(x) \, dx \equiv 0. \]
Consider now the function \( e^{\lambda x} \Phi(x) \), with \( 1 < \lambda < 2 \), \( x \) real. Then

\[
| e^{\lambda x} \Phi(x) | \leq AM_2 e^{(\lambda-1)x}
\]

\[
| e^{\lambda x} \Phi(x) - f_0(x) e^{-f_0(x-\lambda)x} | \leq AM_3 e^{-(f_0-\lambda)x}
\]

which implies that \( e^{\lambda x} \Phi(x) \in L_1 \). Also \( e^{\lambda x} \Phi(x) \in L_1 \). Then let \( \Phi^*(\xi) \) and \( K^*(\xi) \) denote the Fourier transforms of \( e^{\lambda x} \Phi(x) \) and \( e^{\lambda x} \Phi(x) \), respectively; then

\[
\sqrt{2\pi} \Phi^*(\xi) K^*(\xi) = \mathcal{G} \left( \int_{-\infty}^{\infty} e^{\lambda x} \Phi(x) K(x) dx \right)
\]

\[
= \mathcal{G} \left( e^{\lambda x} \chi(x) \right)
\]

\[
= 0
\]

Now let, for \( \xi = \xi + i\eta \),

\[
K^*(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(x) e^{(x-i\xi)} dx
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(\omega) e^{(\omega-i\xi)} d\omega
\]

so that \( K^*(\xi) \) is holomorphic for \( \eta > -\lambda \). But \( K(x) = g(x) e^x \) and since \( g(x) \neq 0 \) it follows that \( K^*(\xi) \neq 0 \). Then \( \Phi^*(\xi) \equiv 0 \), which implies that \( \Phi^*(\xi) \), \( \Phi^*(\xi) \), and \( F(\xi) \) are identically zero, so that \( \Phi(\xi) \equiv 0 \). This completes the proof of the theorem.

The composition theorem for functions of \( L^2 \) is a special case of the preceding theorem and is given in the following:
Corollary.

IF

1° \{M_n\} and \{M'_n\} \text{ (n \geq 0)} are positive sequences,
\log M_n, \log M'_n being convex functions of \( n \);
2°, 3°, 4° as in theorem 2A;
5° as in theorem 2A with \( p = 2 \), i.e., \( \{f(x)\} \in L^2[M_n] \);

THEN

one of the functions \( f(x) \) or \( g(x) \) is identically zero.

2.4 \text{ N-function composition theorem.}

The preceding theorem is concerned with the composition of two functions, this idea having been generalized to \( N \) functions by Mandelbrojt [9], and Mandelbrojt and Brunk [13]. Using the same method, the corollary to theorem 2A can also easily be generalized to \( N \) functions, the proof consisting mainly in a repetition of a certain part of the proof of theorem 2A.

We first need to extend some earlier definitions.

If \( A_1, A_2, \ldots, A_N \) are closed sets belonging to \( \mathbb{R} \), let

\[
A_1 \odot A_2 = A_{12} \\
A_{12} \odot A_3 = A_{123} \\
\vdots \\
A_{1 \ldots j} \odot A_{j+1} = A_{1 \ldots (j+1)} \\
\vdots
\]
This procedure defines the composite product of 
\(A_1, A_2, \ldots, A_N\) as the set 
\[A_{12\ldots N} = A_1 \circ A_2 \circ \cdots \circ A_N = \frac{\prod_{n=1}^{N} A_n}{n!}.\]
We note that since this "multiplication" is commutative and associative it is sufficient to speak of the composite product of the collection of sets \(\{A_n\}, \ n = 1, \ldots, N.\)

**Theorem 2B.**

1. \(\mathcal{M}_m^{(k)}(n>0) \ (k = 1, 2, \ldots, N)\) are positive sequences, such that for each \(k\), \(\log M_m^{(k)}\) is a convex function of \(n\);
2. \(\mathcal{N}_m^{(k)}(n \geq 1) \ (k = 1, 2, \ldots, N)\) are increasing sequences of even positive integers such that if \(\{\nu_n\}\) is the sequence complementary to \(\{m_1^{(k)} \cup \cdots \cup m_{(q)}^{(k)}\}\) with respect to the sequence of even positive integers, then \(\bar{\nu} < \frac{1}{2};\)
3. \(E^{(k)} \ (k = 1, 2, \ldots, N)\) are closed symmetric sets on \(R\) such that for each \(k\), \(0 \notin E^{(k)};\)
4. The sequence \(\{M_m^{(k)} \circ M_m^{(k)} \circ \cdots \circ M_m^{(k)}\}\) is associated with the set \(E^{(k)} \circ E^{(k)} \circ \cdots \circ E^{(k)}\) and the sequence \(\{\nu_n\} ;\)
5. For each \(k\), \(1 \leq k \leq N, f_k(x)\) is an even function, infinitely differentiable on \(R\), such that 
   \[f(0) = f^{(n\geq 1)}(0) = 0, \ (n>1)\]
   \[\sigma_a(f_k) \subset E^{(k)};\]
THEN

one of the functions $f_k(x)$ is identically zero.

Proof:

Our proof follows the method of [13]. Consider the function $\Theta(s)$ of theorem 2A and the argument given there as applied to the functions $f_1(x)$ and $f_2(x)$. Let $\Theta_2(s) = \Theta(f_1, f_2)$ and note that $\Theta_2(s)$ has similar properties as the function $\Phi(s)$ at the beginning of the proof. Then let $\Theta_3(s) = \Theta(\Theta_2, f_3)$ in the obvious fashion; repeating this we finally arrive at the function $\Theta_N(s) = \Theta(\Theta_{N-1}, f_N)$ which satisfies the inequality

$$|\Theta_N(s) - \sum_{n=1}^{m} b_n e^{-(\nu_{n+1})s}| \leq C \left( \prod_{n=1}^{m} M_{q,n}^{(n)} \right) e^{-(\eta_1 + \eta_2 + \cdots + \eta_n)S},$$

where $\nu_m < \eta_m < \eta_{m+1}$, in the domain $D[E^{(n)}, E^{(n)}, \cdots, E^{(m)}, a]$ defined by the following: if

$$E = \left\{ \prod_{n=1}^{N} E^{(n)} \right\}$$

let

$$E = \left\{ x \in \mathbb{R}^m | x \in E \right\}$$

and then the domain $D_a = D[E^{(n)}, E^{(n)}, \cdots, E^{(m)}, a]$ is given by

$$D_a = \left\{ s = \sigma + it | |t| < \frac{1}{2\pi} - \arcsin(\text{arc} \sigma), \sigma > \log a, \text{if } \sigma \notin E \right\}.$$

As before, with the aid of $4^0$, the preceding inequality implies that $\Theta_N(s) \equiv 0$, which by the argument of theorem 2A implies that either $\Theta_{N-1}(s) \equiv 0$ or $f_N(x) \equiv 0$. If $f_N(x) \equiv 0$, then $\Theta_{N-1}(s) \equiv 0$, this implying that either $\Theta_{N-2}(s) \equiv 0$.
or $f_{n-1}(x) \equiv 0$. Hence one of the functions $f_k(x)$ is identically zero. This concludes the proof of the theorem.
CHAPTER III

N-POINT QUASIANALYTICITY

We will present in this chapter an extension of the notion of quasianalyticity, this extension being possible for both ordinary and generalized quasianalyticity. In speaking of ordinary quasianalyticity we say that a function belonging to a quasianalytic class is completely determined by the value of the function and the values of all the derivatives at one point, whereas in generalized quasianalyticity the function is determined by its value and the values of some of the derivatives at one point. The extension we present here consists in determining \( N \)-point quasianalytic classes, for which a function is determined by its value at \( N \) points and the values of its derivatives at any one of \( N \) points. A similar extension exists for generalized \( N \)-point quasianalytic classes. We will also present an \( N \)-point composition theorem.

The theorems to be obtained will be easy consequences of a composition theorem. We will require the following theorem which is an extension of a composition theorem of Mandelbrojt [11], this extension being achieved
in the same manner as in theorem 2B of the preceding chapter.

1. \( \{ M^{(k)}_m \} \) (\( m/2 \)) (k = 1, 2, ..., N) are positive sequences such that for each \( k \), \( \log M^{(k)}_m \) is a convex function of \( n \);

2. \( \{ \lambda^{(k)}_m \} \) (\( n/2 \)) (k = 1, 2, ..., N) are sequences of even integers such that \( \{ \lambda^{(k)}_m \} \cup \{ \lambda^{(k)}_m \} \cup \{ \lambda^{(k)}_m \} = \{ 2n \} \);

3. \[ \sum \frac{M^{(1)}_m M^{(2)}_m \cdots M^{(N)}_m}{M^{(1)}_m M^{(2)}_m \cdots M^{(N)}_m} = \infty \] ;

4. \( \{ f_k(x) \} \) (k = 1, 2, ..., N) are even functions, infinitely differentiable on \( \mathbb{R} \), such that for each \( k \)

   \[ |f^{(n)}_k(x)| \leq M^{(k)}_m \] (\( m/2 \))

   \[ f(0) = f^{(\lambda)}_m (0) = 0 \] (\( n/2 \)) ;

Then

one of the functions \( f_k(x) \) is identically zero.

3.1 Definitions and preliminary lemmas.

We first make the following definition:

A function \( f(x) \), finitevalued on \(( -\infty, \infty )\) is said to be **non-odd** if

either 1. \( f(x) = 0 \)

or 2. there exists an \( x \) such that

\[ f(x) \neq -f(-x) \] .
The set of non-odd functions then is the union of the complement of the set of odd functions and the set containing only the function \( f(x) \equiv 0 \). In particular, the set of even functions is a proper subset of the set of non-odd functions.

The following statement then is obvious:

If \( f(x) \) is a non-odd function, then

\[
F(x) = \frac{1}{2} [f(x) + f(-x)]
\]

is an even function and \( F(x) \equiv 0 \) implies that \( f(x) \equiv 0 \).

We are now in a position to establish the following lemma which will be of use later:

**Lemma 1.**

The composition theorem where the spectrum is not involved is true if the functions in question are assumed to be non-odd instead of even.

The proof is immediate. If \( f(x) \) is non-odd we have only to consider the function

\[
F(x) = \frac{1}{2} [f(x) + f(-x)]
\]

and we have:

1° if \( f(x) \in L^\infty \{ M_n \} \), then \( F(x) \in L^\infty \{ M_n \} ; \)

2° if \( f^{(n)}(0) = 0 \), then \( F^{(n)}(0) = 0 \).

Since \( F(x) \) is even it satisfies the hypotheses of the theorem. Since \( F(x) \equiv 0 \) implies that \( f(x) \equiv 0 \) the lemma is proved.
We now make the following definition of a class of functions which will be of importance in later theorems:

A function \( f(x) \), finitevalued on \( (-\infty, \infty) \) is said to belong to the class \( Q \) if and only if:

1° \( f(x) \) is even;

2° \( f(x) \) is not periodic unless \( f(x) \equiv \) constant, i.e. \( f(x) \) is not periodic with a non-zero primitive period.

The following simple lemma will be of use:

Lemma 2.

If \( f(x) \in Q \), then for any real value of \( a \), the function \( f_{a}(x) = f(x+a) \) is non-odd.

Proof:

The lemma is obviously true if \( f(x) \equiv 0 \). Assume then that \( f(x) \neq 0 \) and that there exists an \( a \) for which the lemma is not true. Since \( f(x) \neq 0 \), \( a \neq 0 \) (since \( f(x) \) is even), and \( f_{a}(x) \) is odd, and we know that \( f(x) \equiv \) constant. We will show that \( f(x) \) is periodic with period \( 4a \), and that then \( f(x) \) has a non-zero primitive period.

Since \( f(x) \) is even,

\[
f(x) = f(-x),
\]
and since \( f_{a}(x) \) is odd,

\[
f(a+x) = -f(a-x).
\]

Since \( x = a + (x-a) \), then

\[
f(x) = f(a + \lceil x-a \rceil)
\]
so that by (2),
\[ f(x) = -f(2a-x) \]
and by (1),
\[ f(x) = -f(x-2a). \]  
(3)

But again, \( x-2a = a+(x-3a) \), so that
\[ f(x-2a) = f(a+[x-3a]), \]
so that by (2),
\[ f(x-2a) = -f(4a-x), \]
and by (1),
\[ f(x-2a) = -f(x-4a). \]  
(4)

Combining (3) and (4) then gives
\[ f(x) = f(x-4a) \]
which, since \( f(x) \) is not a constant, contradicts the hypothesis that \( f(x) \in \mathbb{Q} \).

3.2 N-point quasianalyticity.

We formulate the following problem:

Given a function \( f(x) \), infinitely differentiable on \( \mathbb{R} \) such that \( f^{(m)}(x) \in L^\infty (n \geq 0) \), and \( N \) points \( a_1, a_2, \ldots, a_N \) (\( N > 1 \)) on the real axis such that \( f(a_1) = f(a_2) = \ldots = f(a_N) = 0 \), we furthermore suppose that for each positive integer \( h \), the function \( f^{(am)}(x) \) vanishes at (at least) one of the points \( a_k, k = 1, 2, \ldots, N \).

What conditions must be satisfied by the function \( f(x) \) so that the above implies that \( f(x) \equiv 0 \) ?
The solution to this problem is given by the following theorem:

**Theorem 3A.**

IF

1° \( \{M_n\} (n \geq 0) \) is a positive sequence such that \( \log M_n \) is a convex function of \( n \), and for some positive integer \( N \)

\[
\sum \left( \frac{M_n}{M_{n+1}} \right)^N = \infty;
\]

2° \( a_1, a_2, \ldots, a_N \) denote \( N \) points on \( R \);

3° \( f(x) \) is a function, infinitely differentiable on \( R \), belonging to the class \( Q \), and such that

\[
f(x) \in L^\infty \{M_n\},
\]

\[
f(a_k) = 0 \quad k = 1, 2, \ldots, N
\]

\[
\prod_{k=1}^{N} f^{(n)}(a_k) = 0 \quad \text{, } n \geq 1;
\]

THEN

\[f(x) \equiv 0.\]

**Proof:**

Consider the functions

\[F_k(x) = f(x + a_k), \quad k = 1, 2, \ldots, N.\]

Since \( f(x) \in Q \) it follows that \( F_k(x) \) is non-odd from lemma 2 of 3.1. Clearly \( F_k(x) \in L^\infty \{M_n\} \) for each \( k \). Then the conditions of Mandelbrojt's \( N \)-function composition theorem, as
modified by lemma 1 of 3.1, are satisfied, and it follows that one of the functions \( f(x) \) is identically zero. This implies that \( f(x) \equiv 0 \) and proves the theorem.

3.3 N-point quasianalytic classes.

Given a positive sequence \( \{M_n\} (n \geq 0) \) such that \( \log M_n \) is a convex function of \( n \), we define the class \( C_{\alpha}(M_n) \) to be the class of functions \( f(x) \) which are infinitely differentiable on \( \mathbb{R} \), belong to the class \( \mathcal{Q} \), and are such that

\[
|f^{(m)}(x)| \leq M_n \quad (n \geq 0).
\]

The class \( C_{\alpha}(M_n) \) is said to be an \emph{N-point quasianalytic class} if every function \( f(x) \in C_{\alpha}(M_n) \) which is such that for a set \( \{a_k\} \) of \( N \) points on \( \mathbb{R} \)

\[
f(a_k) = 0 \quad k = 1, 2, ..., N
\]

\[
\prod_{k=1}^{N} f^{(m)}(a_k) = 0 \quad (n \geq 1)
\]

is identically zero.

The preceding theorem can then be restated as:

A sufficient condition that \( C_{\alpha}(M_n) \) be \( N \)-point quasianalytic is that

\[
\sum \left( \frac{M_n}{M_m} \right)^N = \infty.
\]

An immediate corollary to the preceding is:

If the class \( C_{\alpha}(M_n) \) is \( N \)-point quasianalytic, and \( N' \) is a positive integer smaller than \( N \), then the class \( C_{\alpha}(M_n) \) is \( N' \)-point quasianalytic.
3.4 N-point composition theorem.

Theorem 3B

IF

1° \{M_n\}, \{M'_n\} (n>0) are positive sequences such that logM_n, logM'_n are convex functions of n;

2° \{λ_n\} and \{μ_n\} are sequences of even positive integers which are complementary with respect to the sequence of even positive integers;

3° \[ \sum \left( \frac{M_nM'_n}{M_{n+1}M_{n+1}} \right)^N = ∞ \];

4° N is a positive integer and a_1, a_2, ..., a_N are N points on R;

5° f(x) and g(x) are functions, infinitely differentiable on R, belonging to the class Q, such that

\[ f(a_k) = g(a_k) = 0 \quad k=1,2,...,N \]

\[ \prod_{k=1}^N f^{(λ_n)}(a_k) = 0 \quad (n≥1) \]

\[ \prod_{k=1}^N g^{(μ_n)}(a_k) = 0 \quad (n≥1) \];

THEN

either f(x) ≡ 0 or g(x) ≡ 0.

Proof:

Consider the 2N functions
\[ F_k(x) = f(x+\alpha k), \quad k=1,2,\ldots,N \]
\[ F_{k+N}(x) = g(x+\alpha k). \]

Since \( f(x) \) and \( g(x) \) belong to \( A \), each function \( F_k(x) \), \( k = 1, 2, \ldots, 2N \), is non-odd. Furthermore, \( F_k(x) \in L^\infty(M_n) \) for \( k = 1, 2, \ldots, N \) and \( F_k(x) \in L^\infty(M_n') \) for \( k = N+1, N+2, \ldots, 2N \). Then again the conditions of the composition theorem as modified by lemma 1 of 3.1 are satisfied, and it follows that one of the functions \( F_k(x) \) is identically zero. Hence either \( f(x) \equiv 0 \) or \( g(x) \equiv 0 \), and the theorem is proved.

We note here that this theorem can be extended without difficulty to a composition theorem for more than two functions.
CHAPTER IV

AN APPLICATION TO THE THEORY OF
ANALYTIC ALMOST PERIODIC FUNCTIONS

The work presented in this chapter is an application of the theory of adherent series to the study of analytic almost periodic functions. It will be shown that given a function $F(s)$, holomorphic in a vertical strip, if this function is represented in the strip by Dirichlet polynomials with a suitable precision, and provided a certain adherence hypothesis is satisfied, one can then conclude that the function $F(s)$ is an analytic almost periodic function in the strip. It will be shown that an analogous result can be obtained by the use of a modified form of asymptotic behavior introduced by Sunyer Balaguer [15]. The work presented here is an extension of a paper of Genys [7].

4.1 Definition of almost periodic functions of a real variable.

Bohr [1] has defined almost periodic functions of a real variable in the following manner:

Let $f(x)$ be a continuous function on $(-\infty, \infty)$. If
for every positive number \( \varepsilon \) there exists a positive number \( l = l(\varepsilon) \) such that every interval of length \( l \) contains a number \( \tau = \tau(\varepsilon) \) with the property that

\[
|f(x + \tau) - f(x)| \leq \varepsilon \quad \text{for} \quad -\infty < x < \infty,
\]

then \( f(x) \) is said to be an almost periodic function. A number \( \tau(\varepsilon) \) with the above property is called a translation number of \( f(x) \) corresponding to \( \varepsilon \).

It follows from this definition that if \( f(x) \) is an almost periodic function, \( f(x) \) is bounded and uniformly continuous. Also, if \( \{f_n(x)\} \) is a sequence of almost periodic functions converging uniformly to \( f(x) \) on \( (-\infty, \infty) \), then \( f(x) \) is almost periodic. If \( f_1(x) \) and \( f_2(x) \) are almost periodic functions, then \( f_1(x) + f_2(x) \) is an almost periodic function. It can be seen from the definition that every continuous periodic function is an almost periodic function; hence, in particular, any finite sum of the form

\[
S(x) = \sum_{n=1}^{N} a_n e^{i\lambda_n x}
\]

is an almost periodic function.

We will now state Bohr's fundamental theorem of the theory of almost periodic functions. Consider the set of all finite sums

\[
S(x) = \sum_{n=1}^{N} a_n e^{i\lambda_n x}, \quad -\infty < x < \infty,
\]

where the coefficients \( a_n \) are arbitrary complex numbers.
and the exponents $\lambda_n$ are real. We will say that such a sum is a Dirichlet polynomial. Let $H(\{\lambda_n\})$ denote the closure of this set, the close being taken in the sense of uniform convergence on $(-\infty, \infty)$; i.e., $f(\omega) \in H(\{\lambda_n\})$ provided that for each positive number $\varepsilon$ there exists a Dirichlet polynomial $S(\omega)$ such that $|f(\omega) - S(\omega)| \leq \varepsilon$ for all $\omega$ in $(-\infty, \infty)$.

The fundamental theorem states that the class is identical with the class of almost periodic functions.

The above theorem bears a resemblance to the Weierstrass theorem on the approximation of continuous functions on a bounded interval by means of polynomials. The theorem here provided uniform approximation to a class of functions, more restricted than the class of continuous functions, over the entire real axis, the approximating functions being Dirichlet polynomials.

### 4.2 Analytic almost periodic functions.

The concept of almost periodicity has been extended by Bohr [2] to analytic functions in a strip.

Let $f(\omega)$ be holomorphic and uniform in the strip $\alpha < \sigma < \beta$ ($-\infty < \alpha < \beta < \infty$) of the $s = \sigma + it$ plane. If for every positive number $\varepsilon$ there exists a length $L = L(\varepsilon)$ such that every interval of length $L$ contains a number $t = t(\varepsilon)$ with the property that

$$|f(s + it) - f(s)| \leq \varepsilon$$
for all $s$ in the strip $\alpha < \sigma < \beta$, then $f(s)$ is said to be an almost periodic function in the strip $(\alpha, \beta)$.

The preceding definition implies that if $f(s)$ is an almost periodic function in the strip $(\alpha, \beta)$, then for every $\sigma_0$ such that $\alpha < \sigma_0 < \beta$, the function $f_{\sigma_0}(t) = f(\sigma_0 + it)$ is an almost periodic function of the real variable $t$.

The definition however implies more than that; it implies that the almost periodicity of the set of functions $\{f_{\sigma_0}(t)\}$ is "uniform". It can be shown [5] that there exist analytic functions $f(s)$, almost periodic on every line of a vertical strip, but lacking this "uniformity" in the almost periodicity on the different straight lines.

The fundamental theorem of almost periodic functions of a real variable can be extended to the case of analytic almost periodic functions, which then says that if $f(s)$ is almost periodic in $(\alpha, \beta)$, $f(s)$ can be approximated uniformly in each strip $\alpha' \leq \sigma \leq \beta'$, with $\alpha < \alpha' < \beta < \beta'$ by Dirichlet polynomials $\sum a_n e^{\lambda_n s}$. The converse of this last proposition is also true, namely, if $\{f_m(s)\}$ is a sequence of Dirichlet polynomials converging uniformly on each strip $\alpha' \leq \sigma \leq \beta'$, with $\alpha < \alpha' < \beta < \beta'$, to a function $f(s)$, then $f(s)$ is almost periodic in $(\alpha, \beta)$.

4.3 Characterization of analytic almost periodic functions.

We adopt the following notation:
A function \( f(s) \) defined in the strip \( (a, b) \) is said to be \textit{bounded in} \( (a, b) \) provided that for every pair of numbers \( a, b \), such that \( a < s < b \), there exists a constant \( K = K(a, b) \) such that

\[
|f(s)| < K \quad \text{in} \quad (a, b).
\]

Similarly, a function \( f(s) \) is said to be \textit{almost periodic in} \( (a, b) \) if \( f(s) \) is almost periodic in every sub-strip \( (a_i, b_i) \), \( a < a_i < b_i < b \).

Suppose now that \( f(s) \) is an almost periodic function in the strip \( (a, b) \). It can then be shown as in the case of almost periodic functions of a real variable that \( f(s) \) is bounded in \( (a, b) \). A kind of converse of the preceding is also true, namely the following theorem of Bohr [3]:

If \( f(s) \) is holomorphic in \( (a, b) \), bounded in \( (a, b) \), and for some \( s_0 \), \( a < s_0 < b \), \( f(s_0 + it) \) is an almost periodic function of the real variable \( t \), then \( f(s) \) is an almost periodic function in \( (a, b) \).

This theorem provides a characterization for analytic almost periodic functions since it says that if a function \( f(s) \), holomorphic in a strip \( (a, b) \), is known to be almost periodic on a single vertical line of the strip, the almost periodicity is carried over into any strip (in the sense of analytic almost periodicity) containing this line in which the function is bounded.
This characterization has been refined to some extent by Bohr [4]. Consider the following definition:

A continuous function $\phi(t)$ defined for $0 < t < \infty$ is said to be **almost periodic on** $0 < t < \infty$ if for each $\varepsilon > 0$ there exists an $l = l(\varepsilon)$ such that in each interval $0 \leq a < b$ of length $l$ there is a translation number $\tau = \tau(\varepsilon)$; i.e.,

$$|\phi(t + \tau) - \phi(t)| \leq \varepsilon \text{ for } 0 < t < \infty.$$ 

The following theorem can then be proved:

If $\phi(t)$ is almost periodic on $0 < t < \infty$ there exists a unique almost periodic function $F(t)$ such that for $0 < t < \infty$,

$$\phi(t) = F(t),$$

i.e., $\phi(t)$ has a unique extension $F(t)$.

Using this theorem Bohr obtains the following characterization theorem for analytic almost periodic functions [4]:

Let $f(s)$ be holomorphic and bounded in the half-strip $\alpha < \sigma < \beta$, $0 < t < \infty$, i.e., $|f(s)| \leq K$ for $\alpha < \sigma < \beta$, $0 < t < \infty$. Suppose that there exists a $\sigma_0$, $\alpha < \sigma_0 < \beta$ such that

$$\phi(t) = f(\sigma_0 + it) \quad (0 < t < \infty)$$

is almost periodic on $0 < t < \infty$. Then $f(s)$ can be continued analytically through the entire strip $\alpha < \sigma < \beta$ to a holomorphic function $F(s)$ which is bounded over the entire strip by the bound of $f(s)$ on the half-strip, and such that $F(s)$ is almost periodic in $\langle \alpha, \beta \rangle$. 
In the following sections we will give some theorems that also serve to characterize analytic almost periodic functions. These theorems are along the lines of some work of Genuys [7].

We will consider certain Dirichlet polynomials,

\[ y_m(s) = \sum [a_y e^{-\lambda y s} + b_y e^{\lambda y s}] \] ; if \( F(s) \) is a holomorphic function which is represented asymptotically by these polynomials in a sufficiently large strip with a logarithmic precision \(^{(1)}\), then we will show that in order for \( F(s) \) to be almost periodic in the strip it is sufficient that an adherence hypothesis be satisfied.

4.4 Auxiliary theorem.

We will assume, without loss of generality, that the strip is symmetric about the imaginary axis of the \( s = \sigma + \text{i}t \) plane.

Theorem 4A.

1. If

1° \( \{ \lambda_n \} \) \((\mathbb{N}_1)\) is a positive increasing sequence

with \( D < \infty \);

2° \( F(s) \) is a holomorphic function in the domain

\[ \Delta : |\sigma| < \pi R, \text{ where } R > D ; \]

3° \( f(m) > m \) is a positive increasing function taking on integral values for all positive integers \( m \), and the

\(^{(1)}\) We are dealing here, since the strip is vertical, with what Mandelbrojt in [11] calls the case of a complex sequence \( \{ \lambda_n \} \).
\[ y_m(s) = \sum_{\nu=1}^{f(m)} \left[ a^{(m)}_{\nu} e^{\lambda_{\nu} s} + b^{(m)}_{\nu} e^{\lambda_{\nu} s} \right] \quad (m \geq 1) \]

are such that for some given integer \( k \) the \( a^{(m)}_{\nu} \) and \( b^{(m)}_{\nu} \) do not depend on \( m \) for \( \nu = k, m \geq k \); i.e., \( a^{(m)}_{k} = a_{k} \), \( b^{(m)}_{k} = b_{k} \) for \( m \geq k \).

4° The \( y_m(s) \) represent \( F(s) \) in \( \Delta \), with \( m \geq k \), with a logarithmic precision \( p_{k}(x) \), and the hypothesis \( A(R, p_{k}(x), \lambda_{R}) \) is satisfied;

THEN

\( F(s) \) is almost periodic (and hence bounded) in \( (-\pi R, \pi R) \).

**Proof:**

Let

\[ R_m(s) = F(s) - y_m(s) \]

and

\[ Q_m(x) = \sup_{s \in \Delta} |R_m(s)|. \]

Consider then an \( R_i \) such that \( \overline{0} < R_i < R \). Then \( R_m(s) \) is holomorphic and bounded on \( C(it_0, \pi R_i) \), where \( t_0 > -\infty \), and there

\[ |R_m(s)| \leq Q_m(t_0 - R). \]

Now let

\[ \Lambda_{k}(z) = \prod_{\nu \neq k} \left( 1 + \frac{z^2}{\lambda_{\nu}^2} \right) = \sum_{j=0}^{\infty} c^{(k)}_{j} z^{2j} \]
and let

$$R_{m,k}(s) = \sum_{j=0}^{\infty} (-1)^j C_j^{(k)} R_m^{(2j)}(s).$$

By a lemma of Mandelbrojt [11], this series converges uniformly on \(C(i\pi_0, \pi \gamma)\), where \(0 < \gamma < R_1 - \delta\), and there represents a holomorphic function satisfying

$$|R_{m,k}(s)| \leq \prod (R_1 - \gamma) L_k \left[\prod (R_1 - \gamma)\right] Q_m \left(i\pi_0 - \pi R\right)$$

where

$$L_k(u) = \int_0^\infty e^{-ur} \Lambda_k(r) \, dr.$$

But

$$R_{m,k}(s) = \sum_{j=0}^{\infty} (-1)^j C_j^{(k)} F^{(2j)}(s) - \sum_{j=0}^{\infty} (-1)^j C_j^{(k)} \sum_{n=1}^{f(m)} [a_n^{(m)} \lambda_n^{2j} e^{-\lambda ns} + b_n^{(m)} \lambda_n^{2j} e^{\lambda ns}]$$

$$= F_k(s) - \sum_{n=1}^{f(m)} \left[ a_n^{(m)} e^{-\lambda ns} + b_n^{(m)} e^{\lambda ns} \right] \sum_{j=0}^{\infty} (-1)^j C_j^{(k)} \lambda_n^{2j}$$

$$= F_k(s) - \sum_{n=1}^{f(m)} \left[ a_n^{(m)} e^{-\lambda ns} + b_n^{(m)} e^{\lambda ns} \right] \Lambda_k(i\lambda_n)$$

where we have set

$$F_k(s) = \sum_{j=0}^{\infty} (-1)^j C_j^{(k)} F^{(2j)}(s).$$

But

$$\Lambda_k(i\lambda_n) = 0 \quad \text{if} \quad k \neq n,$$

so that

$$R_{m,k}(s) = F_k(s) - \left[ a_k^{(m)} e^{-\lambda ks} + b_k^{(m)} e^{\lambda ks} \right] \Lambda_k(i\lambda_k).$$

Since \(f(m) > m\), if \(m \neq k\), then \(R_{m,k}(s) = \Phi_k(s)\) is independent of \(m\). Then in \(C(i\pi_0, \pi \gamma)\)
$|\Phi_k(s)| \leq C L_k [\tau(R_1, \varphi)] Q_m (t_0 - \tau R), \quad m \geq k$

$\leq C L_k [\tau(R_1, \varphi)] \inf_{m \geq k} Q_m (t_0 - \tau R)$,

and if $t_0$ is sufficiently large, then in $\Delta_{t_0}^*$ defined by

$\Delta_{t_0}^* : \quad 10^{-1} < \tau \varphi \ , \ t > t_0$

we have that

$|\Phi_k(s)| \leq C L_k [\tau(R_1, \varphi)] e^{-P_k(t_0 - \tau R)}$.

But the adherence hypothesis $A(R, P_k(x), \{\lambda_n\})$ implies that in $10^{-1} < \tau \varphi$

$\Phi_k(s) \equiv 0$.

Hence

$F_k(s) = \sum_{j=0}^\infty (-1)^j c^{(k)}_j F^{(2j)}(s) = [a_k e^{-j \lambda_k s} + b_k e^{j \lambda_k s}] \Delta_k (i \lambda_k)$

where the $c^{(k)}_j$ have previously been defined by

$\prod_{y \neq k} (1 + \frac{z^2}{\lambda_y}) = \sum_{j=0}^\infty c^{(k)}_j z^{2j}$. 

Now consider this product split up into the products

$\prod_{y \neq k, y \neq m} (1 + \frac{z^2}{\lambda_y}) = \sum_{j=0}^\infty \alpha^{(k)}_{m, j} z^{2j}$

$\prod_{y \neq k} (1 + \frac{z^2}{\lambda_y}) = \sum_{j=0}^\infty \beta^{(k)}_{m, j} z^{2j}$

which defines the $\alpha^{(k)}_{m, j}$ and $\beta^{(k)}_{m, j}$, and where then

$\left(\sum_{j=0}^\infty \alpha^{(k)}_{m, j} z^{2j}\right) \left(\sum_{j=0}^\infty \beta^{(k)}_{m, j} z^{2j}\right) = \sum_{j=0}^\infty c^{(k)}_j z^{2j}$. 
Now we have that
\[ 0 < \alpha_{m,j}^{(k)} \leq c_j^{(k)} \]
and it follows that
\[ \alpha_{m,0}^{(k)} = c_0^{(k)} = 1 \]
\[ \lim_{m \to \infty} \alpha_{m,j}^{(k)} = 0 \quad \text{decreasingly.} \]

Consider now the function
\[ \psi_m(s) = \sum_{j=0}^{\infty} (-1)^j \alpha_{m,j}^{(k)} F^{(2j)}(s). \]

Since for some \( t_0 > -\infty \) \( F(s) \) is bounded in \( K = \{\Re s > t_0 \} \),
if we evaluate \( \psi_m(s) \) in \( \Delta_{t_0}^* \), by Cauchy's theorem we have that
\[ \alpha_{m,j}^{(k)} |F^{(2j)}(s)| \leq c_j^{(k)} (2j)! (\pi k)^{-2j} M(t_0) \]
where
\[ M(t_0) = \sup_{s \in \Delta_{t_0}^*} |F(s)| \]
and
\[ |\psi_m(s)| \leq M(t_0) \pi k \int_0^{\infty} e^{-\pi k r} \left( \sum_{j=0}^{\infty} c_j^{(k)} r^{2j} \right) dr \]
\[ \leq \pi k L_k(\pi k) M(t_0). \]

We then have that the \( \psi_m(s) \) are holomorphic and bounded, \( m \geq k \), in \( \Delta_{t_0}^* \) and there converge uniformly to \( F(s) \). We will now show that these functions are Dirichlet polynomials.

Following a method introduced by Brunk in [6], we let \( D \) represent the differential operator and then have
\[ \psi_m(s) = \prod_{\nu \neq k, \nu \leq m} \left( 1 - \frac{\nu^s}{\lambda_k^s} \right) F(s) \]

so that
\[
\prod_{\nu \neq k, \nu \leq m} \left( 1 - \frac{\nu^s}{\lambda_k^s} \right) \psi_m(s) = \prod_{\nu \neq k} \left( 1 - \frac{\nu^s}{\lambda_k^s} \right) F(s) = \sum_{j=0}^{\infty} (-i)^j c_j^{(k)} F^{(j)}(s) = F_k(s)
\]

and hence
\[
\prod_{\nu \neq k, \nu \leq m} \left( 1 - \frac{\nu^s}{\lambda_k^s} \right) \psi_m(s) = (a_\nu e^{-\lambda_k s} + b_\nu e^{\lambda_k s}) \Delta_k(i \lambda_k)
\]

so that there exist constants \( A_j^{(m)} \) and \( B_j^{(m)} \) such that
\[
\psi_m(s) = \sum_{j=1}^{\infty} \left[ A_j^{(m)} e^{-\lambda_j s} + B_j^{(m)} e^{\lambda_j s} \right] + \alpha_k^{(m)} \left[ a_\nu e^{-\lambda_k s} + b_\nu e^{\lambda_k s} \right]
\]

where
\[
\alpha_k^{(m)} = \frac{\Delta_k(i \lambda_k)}{\prod_{\nu \neq k, \nu \leq m} \left( 1 - \frac{\nu^s}{\lambda_k^s} \right)}
\]

and hence \( \lim_{m \to \infty} \alpha_k^{(m)} = 1. \)

Now let \( \sigma_0 \) be some number such that \( \left| \sigma_0 \right| < \pi \) and consider the functions \( \psi_m(\sigma_0 + it), t_0 < t < \infty \). Since each one of these functions is almost periodic on \( t_0 < t < \infty \) and the sequence converges uniformly on the interval to the function \( F(\sigma_0 + it) = F_1(t) \) it follows that \( F_1(t) \) is almost periodic on \( t_0 < t < \infty \).

Now the function \( F(s) \) satisfies all the hypotheses of Bohr's theorem on analytic continuation of almost
periodic functions in a half-strip, and it follows from that theorem that \( F(s) \) is almost periodic in \( (-\pi R, \pi R) \). This completes the proof of the theorem.

4.5 Characterization theorem.

We can now relax some of the conditions imposed on the coefficients of the \( \Phi_m(s) \). We first make the following definition:

Let \( A_\varepsilon(x) \), \( \varepsilon \in I \), be an asymptotic family of functions [11], and let \( A(x) \) be the lower envelope of this family. Let \( \Delta \) be as before, let \( \{\lambda_n\} \) be an increasing positive sequence, and \( \{d_n\} \) a sequence of complex numbers. Let \( F(s) \) be a function holomorphic in \( \Delta \).

If for any positive integer \( k \) there exists a number \( \varepsilon_k \) such that for any \( \varepsilon \in I \), \( \varepsilon > \varepsilon_k \), there exists an \( m > k \) for which

\[
|F(s) - \sum_{n=1}^{m} d_n e^{-\lambda n s}| \leq A_\varepsilon(\sigma) \quad (s \in \Delta, \sigma > \sigma_0)
\]

we then say that \( \sum d_n e^{-\lambda n s} \) represents \( F(s) \) asymptotically in \( \Delta \) with respect to the function \( A(\sigma) \).

We then have the following characterization theorem for analytic almost periodic functions:

**Theorem 4B.**

**IF**

1° \( \{\lambda_n\} (n \geq 1) \) is a positive increasing sequence with \( D < \infty \);
2° \( F(s) \) is a holomorphic function in the domain 
\[ \Delta: |\sigma| \leq \pi R, \quad \text{where } R \geq \bar{R} ; \]

3° \( f(m) > m \) is a positive increasing function 

taking on integral values for all positive integers \( m \), and 
\[ \psi_m(s) = \sum_{\nu=1}^{(m)} \left[ a_{\nu}^{(m)} e^{-\lambda \nu s} + b_{\nu}^{(m)} e^{\lambda \nu s} \right] ; \quad (m \geq 1) \]

4° The \( \psi_m(s) \) represent \( F(s) \) in \( \Delta \) asymptotically 

with respect to the function \( A(t) \), and the hypothesis 

\[ A(R, -\log A(t), \{\lambda_m\}) \]

is satisfied;

THEN

\( F(s) \) is almost periodic in \( (-\pi R, \pi R) \).

Proof:

Let \( \lambda_0 \) be a number such that \( 0 < \lambda_0 < \lambda_1 \). Let 
\[ a_0^{(m)} = b_0^{(m)} = 0 \quad \text{for } m \geq 1. \]

Then let 
\[ \psi_m^*(s) = \sum_{\nu=1}^{q(m)} \left[ a_{\nu}^{(m)} e^{-\mu \nu s} + b_{\nu}^{(m)} e^{\mu \nu s} \right] \quad (m \geq 1) \]

where, for \( \nu \geq 1, \)

\[ \mu_{\nu} = \lambda_{\nu-1}; \quad a_{\nu}^{(m)} = a_{\nu-1}^{(m)}, \quad b_{\nu}^{(m)} = b_{\nu-1}^{(m)}; \quad q(m) = f(m+1) \quad (m \geq 1). \]

Then for \( m \geq 1 \)

\[ \psi_m^*(s) = \psi_m(s) \]

and hence the \( \psi_m^*(s) \) represent \( F(s) \) in \( \Delta \) asymptotically 

with respect to \( A(t) \). Also \( \overline{D}_\mu = \overline{D}_\lambda \) and the hypothesis 

\[ A(R, -\log A(t), \{\mu_m\}) \]

implies the hypothesis \( A(R, -\log A(t), \{\lambda_m\}) \).
By a theorem of Mandelbrojt [11] there exists a constant $p$, such that the $\psi_m^*(s)$ represent $F(s)$ in $\Delta$, with $m \geq 1$, with a logarithmic precision $-\log A(t) + p$. However, the hypothesis $A(R, -\log A(t), \{\mu_n\})$ implies the hypothesis $A(R, -\log A(t) + p, \{\mu_n\})$. Now, the $\psi^*_m$ and $\psi^*_m$ do not depend on $m$ for $m \geq 1$ by definition. Then all the hypotheses of theorem 4A are satisfied and the conclusion of the present theorem follows.

4.6 The logarithmic $l$-precision.

We will present a characterization theorem using a definition of precision that has been introduced by Sunyer Balaguer [15]. Consider the following definition:

If in the definition of logarithmic precision we replace

$$\inf_{m \geq 1} \sup_{s \in \Delta} \left| F(s) - \sum_{\nu=1}^{m} a_{\nu} e^{-\lambda \nu s} \right| \leq e^{-p_k(x)},$$

by

$$\inf_{m \geq 1} \sup_{s \in \Delta} \left| F(s) - \sum_{\nu \geq 1} a_{\nu} e^{-\lambda \nu s} \right| \leq e^{-p_k(x)},$$

we say that the $\sum a_{\nu} e^{-\lambda \nu s}$ represent $F(s)$, with $m \geq 1$, in $\Delta$ with the logarithmic $l$-precision $p_k(x)$.

We will obtain a characterization theorem which is similar to theorem 4B, and where it is necessary to assume the boundedness of $F(s)$ in a half-strip. In the case of theorem 4B the boundedness of $F(s)$ could be established.
from the logarithmic precision; this can however not be done in general if the \( L \)-precision is used.

We first prove an auxiliary theorem similar to theorem 4A.

**Theorem 4C.**

**IF**

The conditions 1\(^{\text{o}}\), 2\(^{\text{o}}\), 3\(^{\text{o}}\), and 4\(^{\text{o}}\) of theorem 4A are satisfied with the following modifications:

We assume that \( F(s) \) is bounded in the half-strip:

\[ s \in \Delta, t > t_1; \]

We replace logarithmic precision by logarithmic \( L \)-precision, with \( L > 2\pi R \);

**THEN**

\( F(s) \) is almost periodic in \( (-\pi R, \pi R) \).

**Proof:**

We use the same notation as in theorem 4A except for the following:

If \( L < 2\pi R \), let \( R_1 \) be such that \( 0 < R_1 < R \);

if \( L < 2\pi R \), let \( R_1 = L/2\pi \).

Then if

\[ Q_m(x) = \sup_{x \in \mathbb{Z} \times \mathbb{Z}^2} |R_m(s)| \]

\[ s \in \Delta \]

we have

\[ |R_{m, k}(s)| \leq \pi (R_1 - \delta) L_k [\pi (R_1 - \delta)] Q_m(t_0 - \varepsilon/2) \]
so that then

\[ |\Phi_k(s)| \leq \pi(R_1 - R) L_k \left[ \pi(R_1 - R) \right] \int_{m^2 k} Q_m(t_0 - \theta/2) \]

so that in \(|\sigma| < \pi R, \, t > t_0\) sufficiently large,

\[ |\Phi_k(s)| \leq \pi(R_1 - R) L_k \left[ \pi(R_1 - R) \right] e^{-p_k(t_0 - \theta/2)} \]

and it follows as in 4A that

\[ \Phi_k(s) \equiv 0. \]

Since \(F(s)\) is assumed to be bounded in the half-strip the rest of the proof proceeds as in theorem 4A. This completes the proof.

4.7 Characterization theorem with the \(\lambda\)-precision.

**Theorem 4D.**

If

1. \(\{\lambda_m\} (m \geq 1)\) is an increasing positive sequence such that \(D < \infty\);

2. \(F(s)\) is a function holomorphic in \(\Delta : |\sigma| < \pi R, R > D\), and bounded in a half-strip \(s \in \Delta, t > t_1\);

3. \(f(m) > m\) is a positive increasing function taking on integral values for all positive integers \(m\), and

\[ g_m(s) = \sum_{v=1}^{f(m)} \left[ a_v^{(m)} e^{-\lambda vs} + b_v^{(m)} e^{\lambda vs} \right] \quad (m \geq 1); \]

4. The \(g_m(s)\) represent \(F(s)\) in \(\Delta\), with \(m \geq 1\), with a logarithmic \(\lambda\)-precision \(p(x)\), where \(L > 2\pi D\), and the hypothesis \(A(R, p(x), \{\lambda_m\})\) is satisfied;
THEN

\( F(s) \) is almost periodic in \(-\pi R, \pi R\).

**Proof:**

The proof is essentially the same as for proving 4B from 4A except that of course this theorem is based on 4C. With the same notation as in 4B we have that the hypothesis \( A(R, p(x), \lambda_\mu \gamma) \) implies \( A(R, p(x), \lambda_\mu \gamma) \) and the \( \psi_m(s) \) represent \( F(s) \) in \( \Delta \) with a logarithmic \( \ell \)-precision \( p(x) \). Then the conditions of theorem 4C are satisfied and the theorem follows.
BIBLIOGRAPHY


