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FUCHSIAN GROUPS OF GENUS TWO

by

Ernest Carlton Kennedy

Being a major thesis presented to the Faculty of the Rice Institute in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

June 1937
Houston, Texas.
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FUCHSIAN GROUPS OF GENUS TWO

Introduction. — In this thesis we propose to discuss a class of Fuchsian groups having fundamental regions bounded by equal sides. We shall take the unit circle

\[ Q_0 : \quad z \bar{z} = 1 \]

to be the principal circle. The transformations of the group will then have the form

\[ z' = T(z) = \frac{a z + \bar{c}}{c z + \bar{a}}, \quad \Delta = a \bar{a} - c \bar{c} = 1. \]  

We set up the group as follows. Consider 2n \((n > 1)\) equal circles orthogonal to \(Q_0\) and with centers at the vertices of a regular polygon. Let the common distance \(k\) from the centers to the origin be so chosen that adjacent circles intersect at the angle \(\pi/n\). (The reason for this last requirement will appear later). We arrange these circles into \(n\) pairs, letting each pair constitute the isometric circles* of a transformation of type (1) and its inverse.

If these \(n\) transformations be combined into products in all possible ways, they generate a group which has for its fundamental region \(R_0\) that part of the interior of \(Q_0\) which is exterior to the 2n isometric circles.

In this thesis we shall be interested primarily in the case \(n = 4\). We shall study, for this case, the connection

\[ *\text{Ford, L.R., Automorphic Functions, p. 23.} \]
between these groups and the hypergeometric differential
equation. Finally, we shall consider the problem of finding
algebraic functions, which are uniformized by the groups, and
their associated Abelian Integrals.

PART I

1. The Geometry of the Equal Circles.--- Let \( P_0, P_1, \ldots, P_{2n-1} \) be the centers of the \( 2n \) circles arranged in
counter clockwise order as shown in the figure. (It is con-
venient to let \( P_0 \) be on the positive real axis). These points
lie on equally spaced radial lines about the origin. Let
angle \( P_0 O P_1 \) be \( \alpha \), whence
\[ \alpha = \pi / n. \]
Since the angle of intersection of two ad-

cjacent circles is \( \alpha \) it fol-

dows that the value of \( k \)
depends upon \( n \) and is uni-
quely determined when the
latter is given. These
circles, which we shall
call \( C_0, C_1, \ldots, C_{2n-1} \),
have centers at \( k, k e^{\frac{i}{n}}, \ldots, k e^{\frac{(2n-1)i}{n}} \), respectively.
From the figure
\[ \overline{EP}_o = r \csc \alpha, \text{ where } r \text{ is the radius of } C_e. \]
\[ \overline{OE} = \overline{EJ} = r \cot \alpha. \]
Hence
\[ k = \overline{OE} + \overline{EP} = r(\cot \alpha + \csc \alpha), \]
\[ k^2 - 1 = r^2(\cot \alpha + \csc \alpha)^2 - 1, \]
and since
\[ k^2 = r^2 + 1, \]
\[ r^2 = \frac{1}{(\cot \alpha + \csc \alpha)^2 - 1}. \]
We have also
\[ OJ = 2 \overline{OE} \cos \alpha/2. \]
In the case of eight circles \( n = 4 \) we have
\[ r = \sqrt{\frac{\sqrt{2} - 1}{2}} = .45508985, \]
\[ k = \sqrt{\frac{1 + \sqrt{2}}{2}} = 1.0986841, \]
\[ OM = \sqrt{\sqrt{2} - 1} = r \sqrt{2} = .64359425, \]
\[ OJ = \frac{1}{2^{1/4}}. \]
From this we obtain \( J \) as a complex number,
\[ J = r \left[ (1 + \frac{i}{\sqrt{2}}) + \frac{i}{\sqrt{2}} \right] = .77688698 + .32179712 i. \]
It is of interest to note that the real part of
\[ J = \text{real part of } P, \text{ for all values of } n. \]

2. The Transformations. — By a well known theorem*,
the successive performance of an even number of inversions
in circles is equivalent to a linear transformation. Making

---
*Ford, L.R., loc. cit., p. 12.
use of this theorem we can obtain the linear transformation $T_{m,p}$ which carries $C_m$ into $C_p$, in the following manner.

We first invert in $C_m$, carrying $z$ into $z_1$, where

$$(z_1 - P_m)(\bar{z} - \bar{P}_m) = r^2, \quad P_m = k e^{m\alpha i},$$

or

$$z_1 = \frac{P_m \bar{z} - 1}{\bar{z} - P_m},$$

bars indicating conjugate imaginaries.

We then reflect in the real axis, carrying $z_1$ into $z_2$, where

$$z_2 = \bar{z}_1 = \frac{P_m \bar{z} - 1}{\bar{z} - P_m}. $$

Finally, we rotate about the origin thru an angle $(p+m)\alpha$, carrying $z_2$ into $z_3$, where

$$z_3 = e^{(p+m)\alpha i} z_2 = \frac{k e^{p\alpha i} \bar{z} - e^{(p+m)\alpha i}}{\bar{z} - k e^{m\alpha i}}. $$

Each of the preceding inversions and rotations carries the interior of $Q_o$ into itself. The result of the sequence is a linear transformation with $Q_o$ as fixed circle which carries $C_m$ into $C_p$. It is clear that $C_m$ is the isometric circle $I_m$ of this transformation. For, the initial inversion in $C_m$ in creases lengths inside $C_m$ and decreases them outside and the subsequent reflection and rotation do not alter lengths
further. Likewise, it follows that \( G_p \) is the isometric circle \( I_m' \) of the inverse transformation \( T_{m,p} \). On dividing numerator and denominator by \( irC^{(p-m)} \) and replacing \( z_3 \) by \( z' \) we put the linear transformation in the form shown in equation (1):

\[
(2) \quad z' = T_{m,p}(z) = \frac{\frac{ik}{r} C^{\frac{\pi}{2}(p-m)i} z - \frac{i}{r} C^{\frac{\pi}{2}(p+m)i} \left( \frac{\frac{ik}{r} C^{-\frac{\pi}{2}(p+m)i} z - \frac{i}{r} C^{-\frac{\pi}{2}(p-m)i}}{z} \right)}{\frac{\frac{ik}{r} C^{-\frac{\pi}{2}(p+m)i} z - \frac{i}{r} C^{-\frac{\pi}{2}(p-m)i}}{z}}.
\]

From (2) we can read off the isometric circle of the transformation. It is

\[
\left| \frac{\frac{ik}{r} C^{-\frac{\pi}{2}(p+m)i} z - \frac{i}{r} C^{-\frac{\pi}{2}(p-m)i}}{z} \right| = 1,
\]

or

\[
| z - kC^{m\alpha i} | = r,
\]

which is a circle with center \( kC^{m\alpha i} \) and radius \( r \).

Now \( T_{m,p} \) is equivalent to an inversion in \( I_m \) followed by a reflection in the radical axis of \( I_m \) and \( I_m' \). Hence the transformation of \( R_0 \) by \( T_{m,p} \) (represented by \( T_{m,p}(R_0) \)) is the region obtained by inverting \( R_0 \) in \( I_m \) and reflecting in the radical axis. Thus the 2n transforms of \( R_0 \) by (2) are all exactly alike in shape and any one may be obtained from any other by a rotation about the origin. That is, no matter how we pair up our 2n sides, we get the same set of transforms.
of $R_n$. Considering the case where $n = 4$ and referring to figure II, p.4, we see, for example, that $T_{4\alpha}(R_n)$ is exactly like $T_{15}(R_n)$ and either one may be obtained from the other by a rotation about the origin. When we build on the sides of one of these $2n$ transforms, we get exactly the same configuration as we get when we build on the sides of any other one.

3. **Genus of $R_n$** — By the genus of the fundamental region of a group (or, briefly, the genus of the group) we mean the genus of the closed region formed by bringing congruent sides of the region together so that congruent points coincide. By a cycle we mean a complete set of congruent vertices of $R_n$. If $2n$ is the number of sides of $R_n$, $k'$ the number of cycles, then the genus of the fundamental region is

$$p = \frac{1}{2}(n-k'+1).$$

If $n$ is even (odd), then $k'$ is odd (even) since $p$ is a non-negative integer. Also, $k'$ may have any positive integral value which makes $p$ such a number.

Let the sides of $R_n$ be arranged into congruent pairs in two different ways, thus giving rise to two different groups. If by a rotation or a reflection or by a combination of the two we can carry $R_n$ into itself in such a way that one set of congruent pairs goes into the other set, we say
that the two associated groups are essentially the same.
Thus if

\[ T_1, T_2, \ldots, T_n \]

and

\[ T'_1, T'_2, \ldots, T'_n \]

are the transformations of the two groups, suitably arranged, we have

\[ T'_k = S T_k S' \]

where \( S \) is a rotation or reflection or a combination of the two.* One group is thus a simple transform of the other.

If we choose any one of the \( 2n \) sides of \( R_n \), we can pair it off with another side in \( 2n-1 \) different ways. Then we can choose any one of the remaining \( 2n-2 \) sides and pair it off with another side in \( 2n-3 \) different ways, and so on. There are thus \( (2n-1)(2n-3) \ldots 1 \) possible ways of pairing off the sides of \( R_n \). This gives, of course, the total number of groups which can be formed by pairing off the sides regardless of the value of \( p \) or whether or not the groups are essentially different. This number increases rapidly with \( n \). For \( n = 4 \) it has the value of 105.

In the case of eight sides \( k' \) has the value 5, 3, or 1 and so the corresponding values of \( p \) are 0, 1, and 2. These three cases are illustrated in the figures below.

* Make \( S^{-1} \) first, then \( T_k \), then \( S \).
In the first figure the five cycles are: \((1, 3, 5, 7,),(0), (2), (4), \) and \((6)\). That is, the four vertices \(1, 3, 5, 7\) form one cycle; the vertex \(0\) forms a second cycle, etc. In the second figure the three cycles are: \((0, 1, 4, 5), (2, 7), \) and \((3, 6)\). In the third figure all eight vertices form a single cycle. We shall be primarily interested from now on in the case \(p = 2\). In this case the transformations of the groups are hyperbolic*, and so \(|a + \bar{a}| > 2\). By examining the various possibilities (when \(n = 4\)) we find that there are four essentially different groups with genus \(p = 2\). These are shown as cases I, II, III, and IV on pp. 10 and 12. In the future when we refer to \(R_n\), we shall always mean the case where \(n = 4\) unless otherwise stated.

4. Isometric Circles Thru a Vertex of \(R_n\) — Consider a group with genus \(p = 2, n = 4\). Now any vertex of \(R_n\) can be carried into any other vertex by a transformation of the group. Let \(X\) be the transformation which carries vertex \(J\)

*Ford, L.R., loc.cit., p. 23.
Case I.

\[
T = T_{0,2} = R^2 S = R^8 S^R \\
U = T_{1,3} = R^3 S R^T = R^3 S R^T \\
V = T_{4,6} = R^6 S R^T = R^6 S R^T \\
W = T_{5,7} = R^7 S R^{-T} = R^7 S R^T
\]

Case II.

\[
T = T_{0,2} = R^2 S R^T = R^8 S R^T \\
U = T_{1,4} = R^4 S R^{-T} = R^4 S R^{-T} \\
V = T_{3,6} = R^6 S R^{-T} = R^6 S R^{-T} \\
W = T_{5,7} = R^7 S R^{-T} = R^7 S R^T
\]
Case III.

\[ T = T_{0,2} = R^2S = R^2SR^6 \]
\[ U = T_{1,5} = R^5SR^{-1} = R^5SR^7 \]
\[ V = T_{3,4} = R^6SR^{-3} = R^6SR^5 \]
\[ W = T_{4,7} = R^7SR^{-4} = R^7SR^4 \]

Case IV.

\[ T = T_{0,4} = R^4S = R^4SR^8 \]
\[ U = T_{1,5} = R^5SR^{-1} = R^5SR^7 \]
\[ V = T_{2,4} = R^6SR^{-2} = R^6SR^6 \]
\[ W = T_{3,7} = R^7SR^{-3} = R^7SR^5 \]
into vertex P. Then the isometric circle of $X$ passes thru $J$; for if $J$ were outside this isometric circle, then $P$ would be inside the isometric circle of $X^{-1}$ and this is not possible since $P$ is on the boundary of $R_o$.

Hence seven isometric circles of the group pass thru $J$, as shown in figure 3.

These seven circles, along with the line $OJ$, divide the angular space about $J$ into 16 equal parts. That is, two adjacent circles intersect at an angle of $\pi/8$. Alternate circles (shown heavy) form part of the boundary of certain transforms of $R_o$. Starting with $C$ and going around $J$ counter clockwise we let the seven circles be numbered.
1, 2, 3, --- 7. It will cause no confusion if we number the vertices of \( R_0, 0, 1, 2, --- 7 \), starting with \( J = 0 \) and going counter clockwise around the region.

Consider now the isometric circles thru vertex \( J \) (fig. 3) where the transformations involved are those shown in case III, p. 11. Now the transformation \( T \) (\( T_a \) in our earlier notation) takes \( J \) into vertex number 1. Its isometric circle is circle number 1. UT takes \( J \) into vertex 4. Its isometric circle was found to be number 2. WUT takes \( J \) into vertex 6 and the corresponding isometric circle is number 3, and so on. Considering the four cases mentioned previously we obtain Table I below.

<table>
<thead>
<tr>
<th>Circle No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I. ( J ) goes into vertex no.</td>
<td>1</td>
<td>2</td>
<td>7</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>Case II. ( J ) &quot; &quot; &quot; &quot;</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>2</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>Case III. ( J ) &quot; &quot; &quot; &quot;</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>2</td>
<td>7</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>Case IV. ( J ) &quot; &quot; &quot; &quot;</td>
<td>3</td>
<td>6</td>
<td>1</td>
<td>4</td>
<td>7</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

This table tells us, for example, that in case IV the transformation which carries \( J \) into vertex 2 has circle number 6 for its isometric circle.

Let \( X \) be any transformation carrying \( J \) to vertex \( P \). As stated above, the isometric circle of \( X \) passes thru \( J \). If we apply \( X^{-1} \) to \( R_0 \), we get a transform of \( R_0 \), \( X^{-1}(R_0) \).
which has a vertex at J. This transform lies inside the isometric circle of X and abuts on $R_e$. Let Y be a transformation carrying P to vertex Q. That is, the transformation $XY$ carries J to Q. Then the isometric circle of $XY$ passes thru J. Applying $Y^{-1}$ to $R_e$ we get a transform of $R_e$ having a vertex at P and lying inside the isometric circle of Y and abutting on $R_e$. Then applying $X^{-1}$ to $Y^{-1}(R_e)$ we get the transform $X^{-1}Y^{-1}(R_e)$ which has a vertex at J and abuts on $X^{-1}(R_e)$, and so on. In brief, if $S_1, S_2, \ldots, S_7$ are the transformations carrying J to the other seven vertices of $R_e$, then $S_1^{-1}(R_e), \ldots, S_7^{-1}(R_e)$ are the seven transforms of $R_e$ having a vertex at J. Recalling that a linear transformation preserves angles, we see that these seven transforms along with $R_e$ exactly fill out the angular space about J. This is due to the fact that the sum of the angles at the vertices of $R_e$ is $2\pi$. If we had chosen the angle at the vertices of $R_e$ so that their sum was not $2\pi$, then either there would have been an overlapping of $R_e$ and one of the seven transforms, in which case $R_e$ would not be a fundamental region, or else $R_e$ and the seven transforms would not have filled out the space about J. In the latter case $R_e$ is a fundamental region only if the sum of the angles at its vertices is a sub-multiple of $2\pi$, and we are not interested in this case. This explains why we took the angle of intersection of the $2n$ circles to be $\pi/n$.

If there were an eighth isometric circle thru J, then
there would be a transformation $S_3$ such that the region $S_3^{-1}(R_\ast)$ would overlap in part those regions which fill out the space about $J$, and this is impossible.

5. **Transforms of the Sector at a Vertex of $R_\ast$**— Consider the vertex $J$ (fig. 3) and the seven transforms of $R_\ast$ which fit together there. That part of $R_\ast$ or of a transform of $R_\ast$ lying in the neighborhood of $J$ we shall refer to as a sector. Starting with that sector lying inside $C_1$ and abutting on $R_\ast$ and going around $J$ clockwise, we number these sectors $1,2,3,\ldots,7$, as shown in the adjoining figure. The remaining sector we shall call sector $J$. A similar statement is made regarding each of the other vertices of $R_\ast$. Let these vertices be numbered as in the preceding section. If we apply $T_1$ to $R_\ast$, we find that sector $J$ is carried into sector 1 at vertex $J$.

Thus in case I, $T$ takes sector $J$ into sector 1 at vertex 1. (Shown shaded in fig. 5 p. 16). $U$ takes this last sector into sector 2 (shaded) at vertex 2. $T^{-1}$ takes this sector into sector 3 at vertex 7, and so on. In this way we get a shaded sector at each vertex. Thus starting at $J$ and going around $R_\ast$ counter-clockwise we find that sector $J$ goes into sector number 1,2,7,4,5,6,3 at vertices 1,2,3,4,5,6,7, re-
Table 2.

<table>
<thead>
<tr>
<th>At vertex</th>
<th>1 2 3 4 5 6 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sec. J goes into sec.</td>
<td>1 2 7 4 5 6 3</td>
</tr>
</tbody>
</table>

Fig. 5

Table 3.

<table>
<thead>
<tr>
<th>At vertex</th>
<th>1 2 3 4 5 6 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sec. J goes into sec.</td>
<td>1 5 2 7 3 4 6</td>
</tr>
</tbody>
</table>

Fig. 6
Table 4

<table>
<thead>
<tr>
<th>At vertex</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sec. J goes into sec.</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>2</td>
<td>7</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Fig. 7

Table 5

<table>
<thead>
<tr>
<th>At vertex</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sec. J goes into sec.</td>
<td>3</td>
<td>6</td>
<td>1</td>
<td>4</td>
<td>7</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

Fig. 8
pectively. It is of interest to compare the first sequence with the corresponding one listed in Table I. By considering the four cases we obtain the figures and tables listed on pp. 16 and 17.

The first of these tables tells us, for example, that sector J is carried into sector 3 at vertex 7 by the transformations of case I.

6. **The Group of Triangle Function Theory.**— Consider the group generated by R and S, where these transformations are as shown in the figure. Since S is equivalent to an inversion in C, followed by a reflection in the line OP, it is evident that the circular arc triangle CKMJ is a fundamental region for this group.

![Fig. 9](image)

Here

\[ R = C^{\alpha}Z = \frac{C^{\alpha}Z}{C^{-\alpha}Z} \quad \alpha = 1, \quad \nu = \pi. \]

Thus \( R \) is a rotation about the origin thru the angle \( \alpha \); whence

\[ R_m = C^{m \alpha}Z = \frac{C^{m \alpha}Z}{C^{-m \alpha}Z} \quad \Delta = 1, \]

and

\[ R_m^* = 1. \]
To obtain \( S \) we set \( m = 0 \), \( p = 0 \) in (2). Thus

\[
(2'') \quad S = \frac{i \frac{p}{r} z - i}{\frac{p}{r} z - i}, \quad \Delta = 1, \quad \alpha + \overline{\alpha} = 0.
\]

Note that \( S \) is an elliptic transformation of period 2; hence \( S^2 = 1 \). From (2') and (2'') we find for \( R^m S \) the formula

\[
R^m S = \frac{i \frac{p}{r} e^{\frac{m \phi}{r}} z - i e^{\frac{m \phi}{r}}}{i e^{-\frac{m \phi}{r}} z - \frac{i}{r} e^{-\frac{m \phi}{r}}}, \quad \Delta = 1.
\]

Consider one of the transformations \( T_{m,p} \), as described on p. 6. It can be expressed in terms of \( R \) and \( S \). To show this we first make \( R^m \). This takes \( C_m \) back to \( C_0 \). Then we make \( S \), which carries \( C_0 \) into itself; then make \( R^p \) carrying \( C_0 \) to \( C_p \). The result is the transformation \( R^p S R^{-m} \) which carries \( C_m \) to \( C_p \) in the same way, point for point, that \( T_{m,p} \) does. Hence

\[
T_{m,p} = R^p S R^{-m}.
\]

Since the generating transformations of each of our groups are individually combinations of \( R \) and \( S \), it follows that each of the groups is made up wholly of products formed from \( R \) and \( S \). In other words, each of our groups is a subgroup of the group of triangle function theory. The groups I, II, III, and IV contain, however, none of the infinitude of elliptic transformations of the group of the triangle
functions.

7. A Set of Associated Transformations. — Let $E$ be the transformation as shown in fig. (10). Thus $E = T_{o_1} = RS$ is an elliptic transformation of period 8 and so $E^8 = 1$. Since $E$ is equivalent to an inversion in $C_o$ plus a reflection in the radical axes of $C_o$ and $C_t$, it is obvious that the transforms $E(R_o) - - - - E^7(R_o)$ fill out the space about $J$. When $E^7(R_o)$, say, is acted on by $R$ (of the preceding section) it goes into some $T_{m_1}T_{r_1}(R_o)$. Hence it follows that our set of transformations is the same as the set obtained by combining

$$E, E^2, -------- E^7$$

and

$$R, R^2, -------- R^7$$

in any order; that is, the set $R^jE^k, j, k = 1, 2, -------- 7$. Thus in case III, $R^4S = R\cdot RS = RE$ takes $J$ to vertex 1. $J$ is taken into vertex 4 by $R^5SR^{-1}.RE = R^5SE = R^4RSE = R^6E^3$, and so on. By considering the four cases we obtain the table below. Note the sequence of numbers which form the exponents of $E$, e.g., in Case III, 1, 4, 6, 2, 7, 3, 5. It is of interest to
compare this sequence with the corresponding one in tables 1 and 4.

<table>
<thead>
<tr>
<th>J goes into</th>
<th>Vertex</th>
<th>Case I</th>
<th>Case II</th>
<th>Case III</th>
<th>Case IV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>by $R^2E^1$</td>
<td>by $R^2E^5$</td>
<td>by $R^2E^9$</td>
<td>by $R^4E^3$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$R^2E^7$</td>
<td>$R^2E^2$</td>
<td>$R^2E^6$</td>
<td>$R^4E^6$</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>$R^4E^7$</td>
<td>$R^4E^7$</td>
<td>$R^4E^7$</td>
<td>$R^4E^7$</td>
</tr>
<tr>
<td></td>
<td>4</td>
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PART II

1. The Hypergeometric Differential Equation. — We now consider the problem of mapping the upper half of the z-plane (fig. 2) on the circular arc triangle O MJ in such a way that the line segment (0,1) goes into (0,M), (1,∞) into arc MJ, (0,∞) into OJ; with the points 0,1,∞ going into o, M, J, respectively. To this end we consider the hypergeometric differential equation

\[ W(1 - W) \frac{d^2 R}{dw^2} + \left[ 1 - (\alpha + \beta + 1) W \right] \frac{dR}{dw} - \alpha \beta R = 0. \]

In the neighborhood of each of the three regular singular points 0,1,∞ this equation has two well-known linearly independent particular solutions expressed in the form of hypergeometric series. Two such solutions valid in the neighborhood of 0 are

\[ Y_1(w) = F(\alpha, \beta, \gamma; w) = 1 + \frac{\alpha \beta}{\gamma} w + \frac{\alpha(\alpha+1) \beta(\beta+1)}{2\gamma(\gamma+1)} w^2 + \ldots \]

\[ Y_2(w) = w^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; w), \]

where \( \gamma \) is not zero or a negative integer in the first

---

* Klein, F., Vorlesungen Über Die Hypergeometrische Funktion, p. 43.
equation and not a positive integer in the second. Series (4) and (5) converge for \(|w| < 1\) and are analytic there. If \(\gamma > \alpha \gamma \beta\) then these series define \(Y_1\) and \(Y_2\) over the closed circle. The general solution of (3) is, of course, a linear combination of \(Y_1\) and \(Y_2\).

By setting \(t = 1/w\) in (3) we can find a pair of linearly independent solutions \(Y_1^1\) and \(Y_2^1\) valid in the neighborhood of \(w = 1\). Such a pair are

\[
Y_1^1 = F(\alpha, \beta, \alpha + \beta - \gamma + 1: 1-w)
\]

\[
Y_2^1 = (1-w)^{1-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1: 1-w).
\]

Both of these solutions are valid inside the circle \(|w-1| = 1\). If \(1-\gamma > 0\) and \(\gamma > \alpha \gamma \beta\), then the two series converge over the closed circle. We note that (4) and (5) are linear combinations of (5) and (6), since all four solutions have a common domain of existence.

Thus

\[
Y_1 = a Y_1^1 + b Y_2^1
\]

\[
Y_2 = c Y_2^1 + d Y_1^1.
\]

By letting \(t = 1/w\) we find two linearly independent solutions of (3) valid for \(|w| > 1\). Two such solutions are

* Klein, F., doc. cit., p. 43.*
(9) \[ Y_i' = w^{-\alpha} F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1 : \frac{1}{w}) \]

(10) \[ Y_i'' = w^{-\beta} F(\beta, \beta - \alpha + 1, \beta - \beta + 1 : \frac{1}{w}). \]

These series converge outside the circle \(|w| = 1\). If \(\gamma > \alpha + \beta\), the region of convergence includes the boundary of the circle also. Since \(Y_i', Y_i''\) all have a common domain of existence, it follows that each of the first two is a linear combination of the last two. The parts (in the upper half plane) of the domains of definition of these six solutions are shown graphically in fig. 11.

Let \(z = Y_i'' / Y_i'\) be the quotient of the two solutions valid for \(|w| < 1\).

Then

(11) \[ z(\omega) = \frac{Y_i''(\omega)}{Y_i'(\omega)} = w^\lambda \left[ 1 + h_1 w + h_2 w^2 + \cdots \right], \]

where \(\lambda = -\gamma > 0\) and each \(h_i\) is real. Now any branch of \(z(w)\) maps the neighborhood of the origin in the upper half of the \(w\) plane on the interior of an angle of magnitude \(\lambda \pi\) in the \(z\) plane. (In this connection it is convenient to consider \(w\) as being in the \(w\)-plane and \(z\) as being in the \(z\)-plane). Note that when \(w = 0\) then \(z = 0\). That is, (11) maps the segment \((0,1)\) in the \(w\) plane on \((0,z[1])\) in the \(z\)-plane.
Similarly,

\[ z' = \frac{y_2'}{y_1'} = (1-w)^\mu \left[ 1 + a_1(1-w) + a_2(1-w)^2 + \ldots \right], \]

\[ \mu = \gamma - \lambda - \beta \quad \text{and all } a_i \text{ real, maps the neighborhood of } w = 1 \text{ in the upper half plane on the interior of an angle of magnitude } \mu \pi. \]

In the case of \( w = \infty \) we consider

\[ z'' = \frac{y_2''}{y_1''} = \omega^\nu \left[ 1 + b_1w + b_2w^2 + \ldots \right], \quad \nu - \beta = \nu, \]

and a similar statement applies here. Since the quotient \( z(w) \) of any two solutions is a linear transformation of each of the preceding quotients, it may be seen, and, in fact, is well known, that any branch of \( z(w) \) in the upper half plane maps the half plane on a circular arc triangle with angles \( \lambda \pi, \mu \pi, \) and \( \nu \pi. \)

By (8), (11), and (12) we have

\[ z = \frac{y_2}{y_1} = \frac{a}{c} \frac{y_2'}{y_1'} + \frac{b}{c} = \frac{a}{c} \frac{y_2'}{y_1'} + \frac{b}{c} = \frac{a}{c} \frac{z'}{z} + \frac{b}{c}. \]

This shows the connection between the solutions of (3) and the linear transformation. If we evaluate \( a, b, c, d \) in

\[ y_2(\omega) = a y_2'(\omega) + b y_1'(\omega) \]

\[ y_1(\omega) = c y_2'(\omega) + d y_1'(\omega) \]}
we will have continued $z = Y_z(w) / Y_i(w)$ analytically over the interior of the circle $|w-1| = 1$. Then $z(w)$ will be defined over the interior of the two unit circles with centers at $w = 0$ and $w = 1$. To extend $z(w)$ over the whole upper half of the $w$-plane we consider

$$(16) \quad z' = \frac{Y_2'}{Y_i'} = \frac{AY_2'' + BY_i''}{CY_i'' + DY_i'} = \frac{AZ'' + B}{CZ'' + D},$$

where

$$(17) \quad Y_z' = AY_2'' + BY_i'',$$
$$Y_i' = CY_i'' + DY_i''.$$

On evaluating the constants we extend $z(w)$ analytically over the upper half plane exterior to $|w| = 1$. By virtue of $(14)$ we have now extended $z(w)$ over the entire upper half plane.

Thus

$$(18) \quad z = \frac{Y_z}{Y_i} \quad \text{and} \quad z = \frac{Mz'' + N}{Pz'' + Q},$$

where $M, N, P, Q$ are known constants, together map the upper half plane on a certain circular arc triangle.

We now return to our problem as stated at the beginning of this section; namely, the problem of mapping the upper half plane on triangle $O\triangle M$. It is evident that our problem involves merely an application of the above theory.
2. **Mapping on the Triangle** OAB. In our problem \( \lambda = \frac{1}{8} \)

\[
\mu = \frac{1}{2} \quad \text{and} \quad \nu = \frac{1}{8}.
\]

From

\[
\lambda = 1 - \gamma
\]

\[
\mu = \gamma - 2 - \beta
\]

\[
\nu = 2 - \beta
\]

we obtain \( \alpha = \frac{1}{4} \), \( \beta = \frac{1}{8} \), and \( \gamma = \frac{7}{8} \). When these values are substituted in (3) the result is

\[
(w(1-w) \frac{d^2v}{d\omega^2} + \left[ \frac{7}{8} - \frac{11}{8} w \right] \frac{dv}{d\omega} - \frac{v}{32} = 0.
\] (19)

Consider the solutions

\[
\Psi_1(\omega) = F(\alpha, \beta, \gamma: w) = F(\frac{1}{4}, \frac{1}{8}, \frac{7}{8}: w)
\]

\[
= 1 + \frac{w}{28} + \ldots
\]

\[
\Psi_2(\omega) = w^{1-\lambda} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma: w) = w^{\frac{1}{8}} F(\frac{3}{8}, \frac{1}{4}, \frac{7}{8}: w)
\]

\[
= w^{\frac{1}{8}} \left[ 1 + \frac{w}{12} + \ldots \right]
\] (21)

Then \( \Psi_1 \) and \( \Psi_2 \) (they correspond to \( Y_1 \) and \( Y_2 \) of section1) constitute a pair of linearly independent solutions of (19) valid inside the unit circle about \( w = 0 \). In fact, both series converge at \( w = 1 \) since \( \gamma > \alpha + \beta \) in both cases.

We shall consider that branch of \( \Psi_2 \) for which \( w^{\frac{1}{8}} > 0 \) when \( w > 0 \).
We shall study the ratio \( \zeta'(\omega) / \zeta_0'(\omega) \) of these two solutions. This ratio is defined inside the unit circle about the origin and we shall study its analytic continuation throughout the upper half plane. We know that \( \zeta'(\omega) \) and its analytic continuations map the upper half plane on a circular arc triangle with angles \( \frac{\pi}{8}, \frac{\pi}{6}, \) and \( \frac{\pi}{4} \).

We now consider the following pair of linearly independent solutions valid for \( |w-1| < 1 \).

\[
(22) \quad \zeta_1'(\omega) = F(\alpha, \beta, \alpha + \beta - \gamma + 1; 1 - w) = F\left(\frac{1}{4}, \frac{3}{8}, \frac{1}{2}; 1 - w\right)
\]

\[
(23) \quad \zeta_2'(\omega) = (1-w)^{1-\alpha-\beta} F(\gamma - 1 - \gamma - 1 + 1; 1 - w) \\
= \sqrt{1-w} F\left(\frac{5}{8}, \frac{3}{4}, \frac{1}{2}; 1 - w\right).
\]

Here we shall consider that branch of \( \zeta_2' \) such that \( \sqrt{1-w} > 0 \) for \( 0 < w < 1 \). Both of these series converge over \( 0 \leq w \leq 2 \), since \( \frac{1}{4} > \frac{1}{4} + \frac{3}{8} \) and \( \frac{3}{4} > \frac{3}{4} + \frac{3}{8} \).

We now set up the linear relationship

\[
(24) \quad \zeta_2(\omega) = a \zeta_2'(\omega) + b \zeta_1'(\omega)
\]

\[
(25) \quad \zeta_1(\omega) = c \zeta_2'(\omega) + d \zeta_1'(\omega).
\]

To evaluate \( a \) and \( b \) we set \( w = 0 \) and \( w = 1 \), obtaining the following pair of equations.
\[ \mathcal{V}_2(0) = a \mathcal{V}_2(0) + b \mathcal{V}_2'(0) \]
\[ \mathcal{V}_2(1) = a \mathcal{V}_2'(1) + b \mathcal{V}_2''(1). \]

By referring to the proper series expansion we find that

\[ \mathcal{V}_2(0) = 0, \quad \mathcal{V}_2'(1) = 0, \text{ and } \mathcal{V}_2''(1) = 1. \]

To evaluate \( \mathcal{V}_2(1), \mathcal{V}_2'(0) \) and \( \mathcal{V}_2''(0) \) we make use of the relationship

\[ F(a', b', c'; 1) = \frac{\Gamma(c') \Gamma(c'-a'-b')}{\Gamma(c'-a') \Gamma(c'-b')} \]

Thus

\[ \mathcal{V}_2(1) = F(\frac{1}{2}, \frac{1}{4}, \frac{3}{2}; 1) = \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{3}{4})}{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{4})} = 1.2500732 \]

\[ \mathcal{V}_2'(0) = -F(\frac{1}{2}, \frac{1}{4}, \frac{3}{2}; 1) = \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{1}{4})}{\Gamma(\frac{1}{2}) \Gamma(\frac{3}{4})} = -5.0002927 \]

\[ \mathcal{V}_2''(0) = F(\frac{1}{4}, \frac{1}{8}, \frac{5}{4}; 1) = \frac{\Gamma(\frac{5}{4}) \Gamma(\frac{5}{8})}{\Gamma(\frac{3}{4}) \Gamma(\frac{7}{8})} = 1.5537727 \]

On solving (26) we obtain

\[ a = \frac{-\mathcal{V}_2(1) \mathcal{V}_2'(0)}{\mathcal{V}_2''(0)} = 0.38844350. \]

From (25) we obtain, in a similar manner, the system

\[ \mathcal{V}_1(0) = c \mathcal{V}_1(0) + d \mathcal{V}_1'(0) \]
\[ \mathcal{V}_1(1) = c \mathcal{V}_1'(1) + d \mathcal{V}_1''(1). \]

* Whittaker and Watson, Modern Analysis, p. 282.
On solving this system we get

\[ d = \xi_{li}(l) = F(\frac{t}{2}, \frac{k}{2}, \frac{\xi}{2}; 1) = \frac{\Gamma(\frac{t}{2}) \Gamma(\frac{k}{2})}{\Gamma(\frac{\xi}{2}) \Gamma(\frac{\xi}{2})} = 1.0986 \times 10^4 \]

\[ c = \frac{1 - \frac{\xi}{2}(l) \frac{\xi}{2}(0)}{\xi z(0)} = 0.14141308. \]

We have thus extended \( S(w) \) analytically over the upper half of the unit circle \( |w-1| = 1 \) exterior to \( |w| = 1 \) by means of

\[ S = \frac{\xi_1(w)}{\xi_2(w)} = \frac{a \xi_1 + b \xi_2}{c \xi_1 + d \xi_2} = \frac{a \xi_1 + b c}{c \xi_1 + d} = \frac{a S + b c}{c S + d} \]

where \( S' = \xi_1' / \xi_2' \).

Now consider the following two linearly independent solutions valid in the region \( |w| > 1 \).

\[ \eta_{1,ii}(w) = w^{-d} F(\frac{d}{2}, \frac{d}{2}, \frac{d-1}{2}, \frac{1}{w}) = w^{-\frac{d}{2}} F(\frac{d}{2}, \frac{d}{2}, \frac{d-1}{2}, \frac{1}{w}) = w^{-\frac{d}{2}} \left[ 1 + \frac{1}{2w} + \cdots \right] \]

\[ \eta_{1,ii}(w) = w^{-\beta} F(\beta, \beta + 1, \beta, \frac{1}{w}) = w^{-\beta} F(\beta, \beta + 1, \beta, \frac{1}{w}) = w^{-\beta} \left[ 1 + \frac{1}{2\beta w} + \cdots \right] \]

Here we consider those branches of \( \eta_{1,ii} \) and \( \eta_{1,ii} \) such that \( w^{-\frac{1}{2}} \) and \( w^{-\beta} \) are positive when \( w > 1 \). As before, we set up a linear relationship between the solutions. This time we write...
\( (32) \quad \nu_z'(\omega) = A \nu_z''(\omega) + B \nu_z''(\omega) \)

\( (33) \quad \nu_z'''(\omega) = C \nu_z''(\omega) + D \nu_z''(\omega) \).

If we set \( w = 1, \frac{3}{2} \) in (32) and take \( \sqrt{1-w} \) pure imaginary when \( w > 1 \), we obtain the system

\[ \begin{align*}
\nu_z'(1) &= A \nu_z''(1) + B \nu_z''(1) \\
\nu_z''\left(\frac{3}{2}\right) &= A \nu_z''\left(\frac{3}{2}\right) + B \nu_z''\left(\frac{3}{2}\right)
\end{align*} \]

We observe that \( \nu_z'(1) = 0, \nu_z''(1) = \nu_z(1), \nu_z''(1) = \nu_z(1) \).

By actual computation from the power series using the Monroe calculating machine we find \( \nu_z'(\frac{3}{2}), \nu_z''\left(\frac{3}{2}\right) \) and \( \nu_z''\left(\frac{3}{2}\right) \) correct to eight decimal places and on solving (34) we get

\[ A = 5.0002936 \]

\[ B = -4.39473709. \]

In exactly the same manner we obtain a system of equations from (33) which when solved for \( C \) and \( D \) give

\[ C = 1.5537742 \]

\[ D = -0.56565228 \]

Hence \( S(w) \) is extended analytically over the upper half plane exterior to the two circles \(|w-1| = 1\) and \(|w| = 1\) by means of
\[ (35) \quad \zeta' = \frac{V_2(\omega)}{V_1(\omega)} = \frac{A V_2''(\omega) + B V_1''(\omega)}{C V_2''(\omega) + D V_1''(\omega)} = \frac{A \zeta'' + B}{C \zeta'' + D} \]

where \[ \zeta'' = \frac{V_2''(\omega)}{V_1''(\omega)}. \]

By (29) and (35) we have

\[ (36) \quad \zeta = \frac{a \left( \frac{A \zeta'' + B}{C \zeta'' + D} \right) + d}{c \left( \frac{A \zeta'' + B}{C \zeta'' + D} \right) + d} = \frac{\zeta''(Aa + cC) + (cB + dD)}{\zeta''(Aa + dD) + (cB + dD)} = \frac{Ma'' + N}{Pa'' + Q} \]

and this relationship extends \( \zeta(\omega) \) analytically over the upper half plane exterior to \( |w| = 1 \).

Let us now sum up our results in a few paragraphs: That part of the upper half plane for which \( |w| \leq 1 \) is mapped on part of a circular arc triangle with angles \( \pi/6, \pi/2, \) and \( \pi/8 \) by

\[ (37) \quad \zeta(\omega) = \frac{V_2(\omega)}{V_1(\omega)} = \omega^{1/3} \left[ 1 + \frac{w}{2i} + \cdots \right]. \]

Now \( \zeta(0) = 0 \), \( \zeta(1) = \frac{V_2(1)}{V_1(1)} \), and the line segment \((0,1)\) goes into the segment \([0, \zeta(w)]\). The segment \((0,-1)\) goes into a line lying along \(OJ\).

At \( w = 1 \) we have

\[ (38) \quad \zeta_1(\omega) = \frac{V_2(\omega)}{V_1'(\omega)} = -\sqrt{1 - \omega} \left[ 1 + \frac{3}{4}(1 - \omega) + \cdots \right]. \]
For $0 \leq w < 1$, $\zeta'$ is a real negative number, for $w = 1$
$\zeta' = 0$, and for $1 \leq w \leq 2$, $\zeta'$ is pure imaginary. Thus that
part of the real axis in the neighborhood of $w = 1$ is carried
by $\zeta'$ into two perpendicular lines, along one of which $\zeta'$ is
real. Then the linear transformation

\[
\zeta = \frac{a \zeta' + b}{c \zeta' + d}; \quad a, b, c, d \text{ real},
\]

(39)
carries these two lines into two orthogonal circular arcs
(in one case a straight line). It is readily seen that the
segment $(0,1)$ goes into $[0,\zeta(1)]$ and $(1,2)$ goes into a cir-
cular arc thru $\zeta(1)$ and orthogonal to the real axis. It is
obvious that this completely determines the circular arc tri-
angle $OM'J'$ on which the upper half
plane is mapped by (36) and (37).

This triangle is not $OMJ$, but if
we shrink it up by applying properly
the factor

\[
K = \frac{OM}{\zeta(1)} = \sqrt{w^2 - 1} \cdot \frac{\zeta'(w)}{\zeta'(1)} = \frac{w}{\sqrt{w^2 - 1}},
\]

(40)
than the two triangles coincide. Thus

\[
z = K \zeta' = K \cdot \frac{\zeta'(w)}{\zeta'(1)} \quad \text{and} \quad z = K \cdot \frac{MS'' + N}{PS'' + Q}
\]

(41)
are the functions which map the upper half plane on triangle
$OMJ$. We have now solved the problem proposed in section 1.
In the following section we shall point out and prove some
interesting facts concerning the various constants which
appeared above.

3. **The Constants Involved.**— We shall point out the relationship between the various constants by means of the following theorems.

**Theorem 1.**— The distance from the origin to the center of \( C \) is \( \mathcal{K}_1(1) \); i.e. \( \mathcal{K}_1(1) = \ell \) (See p. 3).

We have

\[
\mathcal{K}_1(1) = \frac{\Gamma(\frac{3}{8}) \Gamma(\frac{1}{4})}{\Gamma(\frac{5}{8}) \Gamma(\frac{3}{4})} = \frac{\Gamma(\frac{1}{4}) \Gamma(\frac{3}{2})}{\Gamma(\frac{3}{4}) \Gamma(\frac{5}{8}) \Gamma(\frac{5}{8})}.
\]

From the relation

\[
\Gamma(z) \Gamma(1+z) = \frac{\pi}{\sin \pi z}
\]

we have

\[
\Gamma(\frac{1}{4}) \Gamma(\frac{5}{8}) = \frac{\pi}{\sin \frac{3\pi}{8}}
\]

and

\[
\Gamma(\frac{1}{4}) \Gamma(\frac{3}{4}) = \pi \sqrt{2}.
\]

Also

\[
\Gamma(\frac{1}{4}) \Gamma(\frac{5}{8}) = \frac{1}{\pi} \sqrt{2 \sqrt{2}} \Gamma(\frac{1}{4})^x.
\]

Whence

\[
\mathcal{K}_1(1) = \frac{1}{2^{5/4} \sin \frac{3\pi}{8}} = \frac{1}{2^{5/4} \sqrt{2 - \sqrt{2}}}
\]

\[
\left[\mathcal{K}_1(1)\right]^2 = \frac{1}{2\sqrt{2} - 2} = \frac{1}{2\sqrt{2} - 2} \cdot \frac{\sqrt{2} + 1}{\sqrt{2} + 1} = \frac{\sqrt{2} + 1}{\sqrt{2}} = \ell^2.
\]

Since there can be no doubt regarding signs, this completes our proof.

THEOREM 2.— The coefficients $a, b, c$, and $d$ in the analytic continuation of $V_1(w)$ and $V_2(w)$, as given in (24) and (25), have the values

$$a = \frac{2 + \sqrt{2}}{8 \kappa}$$

$$b = \frac{1}{\sqrt{2} \kappa}$$

$$c = \frac{1}{4 \sqrt{2} \kappa} = \frac{\kappa}{4}$$

$$d = \kappa.$$ 

Making use of the formulas of the preceding proofs, we have

$$a = \frac{V_2(1) V_2'(0)}{-V_2'(0)} = \frac{2}{\sqrt{2}} \cdot \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{8}\right) \Gamma\left(\frac{7}{8}\right)} = \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}{4 \Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{8}\right) \Gamma\left(\frac{7}{8}\right)}$$

$$= \frac{\sqrt{\pi} \cdot \frac{\Gamma\left(\frac{1}{8}\right)}{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{8}\right) \Gamma\left(\frac{7}{8}\right)}}{4 \Gamma\left(\frac{1}{4}\right) \frac{\Gamma\left(\frac{3}{4}\right)}{\pi^2} \frac{\pi^2}{\sin^2 \frac{\pi}{8}}}$$

$$= \left[\frac{\sqrt{2 - \sqrt{2}}}{2} (1 + \sqrt{2})\right] \frac{1}{2^{\frac{3}{8}}} = \frac{2 + \sqrt{2}}{8 \kappa}.$$ 

To get $b$ we observe that $\overline{OM} = \sqrt{2} r$ and that $r \kappa = \frac{1}{2}$ (See p. 6). Whence

$$\kappa = \frac{\overline{OM} \cdot V_2(1)}{V_2(1)} = \frac{1}{\sqrt{2} \kappa (1)} = \frac{1}{\sqrt{2} \kappa}.$$
since, as already shown, \( b = \sqrt{C}(1) \).

By (3.30)

\[
C = \frac{1 - \sqrt{C}(0) \sqrt{C}(0)}{\sqrt{C}(0)}
\]

we first prove that \( \sqrt{C}(0) = -4 \sqrt{C}(1) \) by noting that

\[
\frac{-\sqrt{C}'(0)}{\sqrt{C}(1)} = \frac{F(\frac{3}{8}, \frac{3}{4}, \frac{3}{2}; 1)}{F(\frac{3}{4}, \frac{3}{8}, \frac{3}{2}; 1)} = \frac{\Gamma(\frac{3}{8}) \Gamma(\frac{3}{4}) \Gamma(\frac{3}{2}) \Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4}) \Gamma(\frac{3}{8}) \Gamma(\frac{3}{2}) \Gamma(\frac{1}{2})}
\]

\[
= \frac{\Gamma(\frac{3}{8}) \cdot 8 \cdot \Gamma(\frac{1}{2})}{2 \Gamma(\frac{1}{2}) \Gamma(\frac{3}{8})} = 4
\]

Next we observe that

\[
\sqrt{C}(1) \sqrt{C}(0) = \frac{\Gamma(\frac{3}{8}) \Gamma(\frac{3}{4}) \Gamma(\frac{3}{2}) \Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4}) \Gamma(\frac{3}{8}) \Gamma(\frac{3}{2}) \Gamma(\frac{1}{2})}
\]

By aid of the formulas in the preceding proofs along with the new formula*

\[
\Gamma(\frac{3}{8}) \Gamma(\frac{5}{8}) = \frac{\pi}{\sin \frac{3\pi}{8}}
\]

we have

\[
\sqrt{C}(1) \sqrt{C}(0) = \frac{\pi}{\sin \frac{3\pi}{8}} \cdot 4 \cdot \frac{\sin \frac{3\pi}{8}}{\pi} \cdot \frac{1}{\sqrt{2}} = \frac{\sin \frac{3\pi}{8}}{\sqrt{2} \sin \frac{3\pi}{8}}
\]

\[
= 1 + \frac{1}{\sqrt{2}}
\]

Whence

\[
C = \frac{1 - 1 + \frac{1}{\sqrt{2}}}{4 \sqrt{C}(1)} = \frac{1}{4 \sqrt{C}(1) \sqrt{C}(0)} = \frac{\sqrt{C}(0)}{4}
\]

From p. 30 we have \( d = \sqrt{C}(1) = \frac{\sqrt{2}}{2} \).

* Nielson, N., loc. cit., p. 137.
THEOREM 3. — The coefficients $A, B, C, D$ in the analytic continuation of $\psi_1'(w)$ and $\psi_2'(w)$, as given in (32) and (33), have the values

$$A = \frac{4}{k\sqrt{2}} i$$

$$B = -4k^2 i$$

$$C = \frac{2 + \sqrt{2}}{2k}$$

$$D = -k$$

From (41)

$$z = \frac{A S'' + B}{C S'' + D}$$

and

$$S' = \frac{A S'' + B}{C S'' + D}.$$

When $w = \infty$, $S'' = \infty$, $S' = A/C$ and $z = \infty$. Hence

$$-\frac{d J + k K}{c J - k a} = \frac{A}{C} \quad (42)$$

From (33) we get

$$1 = C \psi_1(1) + D \psi_2(1), \quad (43)$$

since $\psi''_1(1) = \psi_1(1)$ and $\psi''_2(1) = \psi_2(1)$.

From (32) we have

$$\tilde{\psi} = A \psi_1(1) + B \psi_2(1)$$

or
(43') \[ \frac{A}{B} = -\frac{V_2'(1)}{V_1'(1)} \]

From (36) we have
\[ \frac{Z}{K} = S = \frac{S''(A a + c C) + (a B + d D)}{S''(A c + d D) + (c B + d D)} \]

When \( w = \infty \), \( S'' = \infty \) and \( z = j \). Whence

(43'') \[ \frac{J}{K} = \frac{A a + c C}{A c + d C} \]

Now (42), (43), (43'), and (43'') constitute a system of four equations in the four unknowns \( A, B, C, D \). On solving the system and simplifying by the results of the preceding theorem we get the set of values as given in the theorem.

From the results of the last two theorems we obtain the exact values of \( M, N, P, Q \). Thus

\[ M = A a + b C = \frac{1}{\sqrt{2} K \pi} \left[ 2 + \frac{\sqrt{2}}{2} \right] [1 + i] \]

\[ N = a B + b D = -\frac{\sqrt{2}}{2} \left( 1 + \frac{\sqrt{2}}{2} \right) i \]

\[ P = A c + d C = \left( 1 + \frac{\sqrt{2}}{2} \right) + \frac{\sqrt{2}}{2} i \]

\[ Q = c B + d D = -K \pi [1 + i] \]
PART III

1. The Riemann Surface of the Inverse of the Triangle Functions. — Let the branch of

\[ z = z(\omega) = \mathrm{K} \cdot \frac{V_3(\omega)}{V_1(\omega)} \]

used in (41) be extended analytically across the segment 0,1 into and over the lower half of the \( w \)-plane. Then the lower half plane is mapped onto a circular arc triangle which is the inverse of \( \triangle OMQ \) in one of its sides. This is a consequence of the principle of reflection since the lower half plane is the reflection of the upper half in the real axis. The real axis goes into the sides of the triangle. If this process is continued indefinitely, this branch of \( z(w) \) maps the two half planes on an infinite system of non-overlapping triangles fitting together without lacunae. The inverse of any triangle in one of its sides gives another triangle of the system.

If we shade those triangles which are maps of the upper half plane, it follows that any shaded triangle can be carried into any other shaded triangle by a linear transformation \( T_i(z) \).

Consider the inverse \( w = w(z) \) of the preceding function. Then \( w(T_i) = w(z) \) and so \( w(z) \) is automorphic with respect to this group. That is, \( w = w(z) \) is a simple automorphic function with respect to the group generated by \( R \) and \( S \). It maps a shaded triangle on the upper half plane and an unshaded one on the lower half plane. Hence a shaded triangle together with an adjoining unshaded one forms a fundamental region since
in this region the function takes on each value once. Such a function is called a triangle function*. Any triangle function of this group is a rational function of \( w(z) \).

Now the inverse of the preceding [i.e. (44)] is infinitely many valued in the \( w \)-plane. Hence its Riemann surface must have infinitely many sheets. The branch points of this surface are over the points \( w = 0, \ w = 1, \ w = \infty \).

From

\[
Z = K \frac{V_z(w)}{P_z(w)} = K \cdot W^\frac{\theta}{2} \left( 1 + \frac{w}{2^i} + \ldots \right)
\]

it is evident that eight sheets hang together at each of the infinitely many branch points above \( w = 0 \).

If in (38) we set \( w = 1 + re^{\theta i} \), we get

\[
\xi' = \sqrt[1][e^{i(\theta - \pi)}] \left[ 1 + \frac{1}{2^i} (1 - w) \ldots \right].
\]

When \( \theta = \pi, -3 \pi, \ldots \), \( \xi' \) is real and by (41) \( Z \) is real. Hence crossing the axis from the upper half plane into the lower half plane (to the right of \( w = 1 \)) is equivalent to crossing the arc \( \xi \xi' \) in the \( z \)-plane. If we now cross the real axis just to the left of \( w = 1 \) and enter the upper half plane, this is equivalent to crossing the real axis just to the right of \( \xi \) in the \( z \)-plane, and so on. It is now evident that above \( w = 1 \) the sheets hang together in sets of two at each of the infinitude of branch points.

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* Ford, L. R., *Automorphic Functions*, p. 305.
By going around the points \( w = 0, \ w = 1 \) it is readily seen that the sheets hang together in sets of eight at each of the infinitude of branch points above \( w = \infty \).

2. The Eight Sheeted Surfaces Associated with the Groups.

The function \( w = w(z) \) maps \( \mathbb{R}_o \) on an eight sheeted surface. The inverse function \( z = z(w) \) maps a certain eight sheeted surface on \( \mathbb{R}_o \). This surface \( \Sigma \) is closed up about a branch point at the origin where eight sheets intertwine. It is cut by slits in each sheet along the real axis extending from 1 to \( +\infty \) and from 0 to \( -\infty \).

The edges of these slits (1 to \( +\infty \)) are carried by \( z = z(w) \) into sides of \( \mathbb{R}_o \). Hence pairing off the sides of \( \mathbb{R}_o \) is equivalent to pairing off these edges, i.e. connecting up the sheets of the Riemann surface. Thus in case I, p. (10), we see, by referring to fig. 2, that \( \mathcal{T} \) carries \( U_1 \) into a region abutting \( L_1 \) and \( L_1 \) into a region abutting \( U_7 \). We therefore connect upper sheet number 1 with lower sheet number 7 and upper 7 with lower 1. Considering the other three transformations we get the sheets connected as shown.

Since the sides of \( \mathbb{R}_o \) are paired up in a different manner in each of the four cases, it follows that the sheets are connected differently in each case. The Riemann surfaces \( \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4 \)
for the four cases are shown on pp. 10 and 11. In each case the genus of the surface is two.

Consider a simple automorphic function $f(z)$ belonging to one of these groups. Now $f(z)$ is analytic except for poles in $R_0$ and takes on the same values at congruent points along congruent sides. Let $f(z) = f[z(w)] = F(w)$. Now $F(w)$ is analytic on $\Sigma$ except for poles and takes on the same values along the edges we have paired off. Hence if we close up $\Sigma$, say, along the slits the branches of $F(w)$ on the two sides of the slit are the analytic continuations of one another. Thus $F(w)$ is single valued and analytic except for poles on the surface $\Sigma$. Conversely, any function $F(w)$ single valued and analytic except for poles on $\Sigma$, is a simple automorphic function of $z$, $F[w(z)]$. We have a similar situation in the other three cases.

3. An Algebraic Function Belonging to One of the Groups.— In this section we shall find an algebraic function $W(w)$ which has $\Sigma_1$ for its Riemann surface. This problem consists of examining the branch point and finding an algebraic relationship between $W$ and $w$ such that $W$ is single valued on the surface. We have not, as yet, been able to solve the problem for all four cases, but we present below a solution for case IV.

The algebraic function belonging to the two-sheeted
hyperelliptic surface suggests the following form:

\[(45) \quad W^3 = w(w-1)^4\]

The problem now is to find out just how the sheets of the Riemann surface of (45) are connected. To this end we let

\[w-1 = r_1 e^{i\theta_1}\]

\[w = r_1 e^{i\theta_2}\]

Then

\[W = \rho e^{i\left(\frac{4\theta_1 + \theta_2}{3}\right)}, \quad \rho = r_1 r_2\]

Let \(W_1\) be defined by this formula with \(0 \leq \theta_1 \leq 2\pi\), \(-\pi \leq \theta_2 \leq \pi\). Thus \(W_1\) is defined over the whole plane with slits along \(-\infty\) to \(0\) and \(1\) to \(+\infty\). The other branches of \(W\) are got by adding multiples of \(2\pi\) to \(\theta_1\) and \(\theta_2\) in the given formula. Thus we have

\[W_2 = \epsilon W_1\]

\[W_3 = \epsilon W_2 = \epsilon^2 W_1\]

\[W_4 = \epsilon W_3 = \epsilon^3 W_1\]

If we make a circuit about \(w = 0\) in the clockwise direction, angle \(\theta_1\) is decreased by \(2\pi\) and the branch of \(W\) we start with
$W_1$ goes into another branch, as $W_2$. That is

$$W_1 \rightarrow \epsilon W_1 = W_2.$$  

Similarly,

$$W_3 \rightarrow \epsilon W_3 = W_3$$
$$W_7 \rightarrow \epsilon W_7 = W_8.$$  

Now let $w$ go around $w = 1$. Then $\theta$ is decreed by $2\pi$ and we have

$$\downarrow W_1 \rightarrow \epsilon' W_1 = W_5$$
$$W_5 \rightarrow \epsilon' W_5 = W_1.$$  

Similarly, $W_2 \leftrightarrow W_4$; $W_3 \leftrightarrow W_7$; $W_4 \leftrightarrow W_8$.  

On going around both points at the same time we get

$$\downarrow W_1 \rightarrow W_6$$
$$W_6 \rightarrow W_7, \ldots$$
$$W_8 \rightarrow W_5.$$  

The above Riemann surface is exactly $\Sigma_4$. Hence (45) is the desired algebraic relation between $W$ and $w$.

4. The Simple Automorphic Functions Belonging to the Group.—

We can now state our fundamental theorem:

**THEOREM.** — The most general simple automorphic function belonging to the fourth group is a rational function of $W$ and $w$ where $w = w(z)$ is the inverse of the function $z = K \bar{K_z(\omega)}/\psi(\omega)$ and $W = \sqrt[6]{w(\omega-i)^6}$.

We have already proved that any simple automorphic function $f(z)$ belonging to the fourth group is a single valued analytic function of $w$ on $\Sigma_4$ except for poles. It follows* that $f(z)$

is a rational function of \( W \) and \( w \),

\[
f(z) = R(W, w)
\]

Conversely, a rational function of \( W \) and \( w \) is a single valued analytic function of \( w \) on \( \Sigma_{4A} \). In terms of \( z \) it is then a simple automorphic function with respect to the fourth group.

5. **Remarks on the Associated Abelian Integrals.** — Since the algebraic function (45) is of genus 2, it follows by a well known theorem * that there are 2 linearly independent everywhere finite integrals, i.e., Abelian integrals of the first kind. Two such integrals are

\[
I_1 = \int \frac{W^w}{W^{w_0}} \frac{(w-1)^2}{W^5} \, dw \quad ; \quad I_2 = \int \frac{W^w}{W^{w_0}} \frac{(w-1)^3}{W^7} \, dw .
\]

To show this we examine the integrals at the points \( w = 0, w = 1, w = \infty \).

At \( w = 0 \) we have, from (45), \( W^5 = w^{\frac{5}{2}}[1 + a_1 w + \cdots] \)

Thus

\[
I_1 = \int \frac{(w-1)^2}{w^{\frac{5}{2}}[1 + a_1 w + \cdots]} \, dw = \int [w^{-\frac{5}{2}} + \text{higher powers}] \, dw
\]

\[
= C_1 + \frac{5}{3} w^{\frac{3}{2}} + \text{higher powers of } w ,
\]

and this is finite at \( w = 0 \).

At \( w = 1 \) we have
\[
W^5 = (w-1)^{\frac{5}{2}} [1 + b_1(w-1) + \ldots]
\]

and
\[
I_1 = \int \frac{(w-1)^2 \, dw}{(w-1)^{\frac{5}{2}}[1 + c^1(w-1) + \ldots]} = \int [(w-1)^{-\frac{1}{2}} + \text{higher powers of } w-1] \, dw
\]
\[
= C_2 + 2(w-1)^{\frac{1}{2}} + \text{higher powers of } (w-1).
\]
when \( w = 1 \) \( I_1 = C_2 \).

At \( w = \infty \),
\[
W^5 = \omega^{\frac{5}{2}} [1 + \frac{c_1}{\omega} + \ldots] \quad \text{and so}
\]
\[
I_1 = \int \frac{(w-1)^2 \, dw}{\omega^{\frac{5}{2}}[1 + \frac{c_1}{\omega} + \ldots]} = \int [\omega^{-\frac{7}{2}} + \text{lower powers}] \, dw
\]
\[
= C_3 - 8 \omega^{-\frac{1}{2}} + \text{lower powers}.
\]
When \( w = \infty \) this quantity is finite.

In the case of the second integral we have at the origin
\[
I_2 = \int \frac{(w-1)^3 \, dw}{W^\frac{5}{2}[1 + d_1(w-1) + \ldots]} = \int [- \omega^{-\frac{7}{2}} + \text{higher powers}] \, dw
\]
\[
= C_4 - 8 \omega^{\frac{1}{2}} + \text{higher powers}.
\]
At \( w = 0 \) the series is 0 and so \( I_2 = C_4 \).

At \( w = 1 \) we get
\[
I_2 = \int \frac{(w-1)^3 \, dw}{(w-1)^{\frac{5}{2}}[1 + e_1(w-1) + \ldots]} = \int [(w-1)^{-\frac{7}{2}} + \text{higher powers of } (w-1)] \, dw
\]
\[
= C_5 - 8(w-1)^{\frac{1}{2}} + \text{higher powers of } (w-1).
\]
Thus \( I_2 \) is finite at \( w = 1 \).
At \( w = 0 \) we have

\[
I_\omega = \int \frac{(\omega - 1)^3}{\omega^{3/2} [1 + \frac{4}{3} \omega + \cdots]} \delta \omega = \int [\omega^{-3/2} + \text{lower powers}] \delta \omega
\]

\[
= Ce^{-\frac{B}{3} \omega^{-3} + \text{lower powers}}.
\]

Thus \( I_\omega \) and \( I_{\omega} \) when expressed as functions of \( z \) are analytic in the whole interior of the unit circle and are altered by the addition of a constant, at most, when \( z \) moves from one point to a congruent point. (That is, congruent with respect to the fourth group).

The various kinds of Abelian integrals connected with the algebraic relation (45) could be worked out.

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