# ON THE UNIFORMLY BOUNDED TURNING
# OF
# LEVEL CURVES OF THE GREEN'S FUNCTION

by

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# On the Uniformly Bounded Turning of Level Curves of the Green's Function

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## 1. Introduction.

Let C be a simple rectifiable plane curve of arc length s and total length C. Then

(a) C has a tangent almost everywhere; that is, except possibly on a set of zero linear measure.

(b) For any set of rectangular axes the functions,

$$x = x(s), \quad y = y(s),$$

which express the coordinates of points on C in terms of the arc length measured from some arbitrary point on the curve, are absolutely continuous; that is,

$$x(s) - x(\sigma) = \int_{\sigma}^{s} \cos\theta(s)ds, \qquad y(s) - y(\sigma) = \int_{\sigma}^{s} \sin\theta(s)ds,$$

where

$$\frac{dx}{ds} = \cos\theta(s), \qquad \frac{dy}{ds} = \sin\theta(s),$$

at almost all points on C*. $\theta(s)$ is the angle (mod $2\pi$) made with the positive x-axis by the positive (or forward) direction of C at the point s.

Definition.† Let $V_\theta(C)$ be the total variation of $\theta(s)$ over the simple rectifiable curve C. The turning $T(C)$ of the curve is the greatest lower bound of $V_\theta(C)$ for all possible choices of the measure of $\theta(s)$.

Definition. If the turning is finite the curve C is said to be of bounded turning.

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* Lebesgue, Leçons sur l'intégration, Paris (1928), pp. 198-201.
† J. Radon, "Über die Randwertaufgaben beim logarithmischen Potential", Sitzungsberichte der Akademie der Wissenschaften in Wien, Vol. 128 (1919), pp. 1123-1167. See p. 1126.

For curves of bounded turning $\theta(s)$ is a function of bounded variation in the interval $0 \leqslant s \leqslant C$; and for some measure of $\theta(s)$

$$T(C) \equiv \int_C |d\theta| .$$

Thus if C is a curve of bounded turning it has the further properties*:

(c) The curvature $d\theta/ds$ exists at almost all points.

(d) C has a forward and a backward tangent at every point. This follows from the fact that the limits,

$$\lim_{\sigma \to s+} \theta(\sigma) = \theta(s+), \qquad \lim_{\sigma \to s-} \theta(\sigma) = \theta(s-),$$

exist for every value of s in the interval $0 \leqslant s \leqslant C$. By the symbol $\sigma \to s\pm$ is meant that the point $\sigma$ approaches s from the forward (+) direction or from the backward (-) direction on C. The angle $\theta(s-)$ may be termed the direction of arrival at the point s, and $\theta(s+)$ the direction of departure from s. $\theta(s+)$ is the angle of the forward tangent at s.

(e) The set of points of C at which $\theta(s+)-\theta(s-) \neq 0$ is denumerable. For given any arbitrary $k > 0$ there can be only a finite number of values of s at which $|\theta(s+)-\theta(s-)| > k$, since otherwise $\theta(s)$ would fail to be of bounded variation.

We call vertices those points at which C does not have a unique tangent; that is, where $\theta(s+)-\theta(s-) \neq 0$. It is convenient to measure s from a point at which C has a unique tangent.

(f) We may choose the measure of $\theta(s)$ so that $|\theta(s+)-\theta(s-)| \leq \pi$ for all points on C.

(g) At a vertex we may define $\theta(s)$ to have any value between

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*Hobson, Theory of Functions of a Real Variable, 2nd. ed., Cambridge (1921), Vol. 1, pp. 307-318.

$\theta(s+)$ and $\theta(s-)$ without in any manner affecting the properties of the function which are to be used later. However, it is not necessary to do this, so in general the function $\theta(s)$ will not be defined at the vertices of C.

2. Theorems on curves of bounded turning*.

The Radon turning $T(\Gamma_n)$ of a polygonal arc $\Gamma_n$ of n sides is readily found. A positive direction is chosen around $\Gamma_n$, and $\theta_i$ the angle made by the i-th side of the polygonal arc with some arbitrarily fixed direction is measured so that $|\theta_{i+1}-\theta_i| < \pi$. Then

$$T(\Gamma_n) = \sum_{1}^{n-1} {}_i |\theta_{i+1}-\theta_i| .$$

Lemma I†. If $\Gamma$ is a polygonal arc of n sides, and $\Gamma'$ is a second polygonal arc of n sides whose vertices are arbitrarily close to the corresponding vertices of $\Gamma$, then $T(\Gamma')$ is arbitrarily close to $T(\Gamma)$.

Given $\epsilon > 0$. Let the vertices of $\Gamma'$ be inside circles of radius $\epsilon \ell/2n$ described about each vertex of $\Gamma$ as center. $\ell$ is the length of the shortest side of $\Gamma$. Then the angle between corresponding sides of $\Gamma$ and $\Gamma'$ is in absolute value less than $2(\epsilon\ell/2n)\,1/\ell = \epsilon/n$, and therefore

$$|T(\Gamma') - T(\Gamma)| < \epsilon .$$

Lemma II. Let $\Gamma_n$ be a polygon of n sides, and $\Gamma_m$ be a polygon of $m > n$ sides among whose vertices are all of the vertices of $\Gamma_n$. Then $T(\Gamma_m) \geqq T(\Gamma_n)$.

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*Only simple curves are considered in this paper; however it is not difficult to extend the theorems to curves that are not simple.

†Lemma I and Theorems I, II, III, and V are due to R. S. Martin, National Research Council Fellow. They were presented by him in seminar at the Rice Institute during the spring of 1933. The proof to the first three theorems is due to H. E. Bray.

Let $m = n+1$. Consider the polygonal arc $\Gamma_m$ of n sides. Let $\ell_1$, $\ell$, $\ell_2$ be three consecutive sides of $\Gamma_m$, meeting in the vertices $P_1$ and $P_2$, and having the directions $\theta_1$, $\theta$, $\theta_2$, respectively. Let T be the part of the turning of $\Gamma_m$ at $P_1$ and $P_2$,

$$T = |\theta_1 - \theta| + |\theta - \theta_2| \quad ; \quad |\theta_1 - \theta| < \pi, \quad |\theta - \theta_2| < \pi.$$

Between the vertices $P_1$ and $P_2$ insert a new vertex Q to form $\Gamma_{m+1}$. Let $\theta_1'$ and $\theta_2'$ be the directions of the sides $\ell_1'$ and $\ell_2'$ of $\Gamma_{m+1}$ meeting at Q;

$$|\theta_1 - \theta_1'| \leq \pi, \quad |\theta_1' - \theta_2'| \leq \pi.$$

Let $\bar{\theta}_2$ be the new determination of the direction of $\ell_2$; $|\theta_2' - \bar{\theta}_2| \leq \pi$. Of course $\bar{\theta}_2 = \theta_2 + 2j\pi$, j being an integer.
Let T' be the turning of $\Gamma_{m+1}$ at $P_1$, Q, $P_2$,

$$T' = |\theta_1 - \theta_1'| + |\theta_1' - \theta_2'| + |\theta_2' - \bar{\theta}_2| .$$

We need consider only the relation between T and T', since

$$T(\Gamma_m) - T = T(\Gamma_{m+1}) - T'.$$

At Q draw a vector $\bar{\ell}$ with direction $\bar{\theta}$ parallel to the side $\ell$ of $\Gamma_m$. This vector will always lie in or on the boundary of the smaller angular region determined by the vectors $\ell_1'$ and $\ell_2'$, and hence we may determine $\bar{\theta}$ to lie between $\theta_1'$ and $\theta_2'$; that is so

$$|\theta_1' - \bar{\theta}| \leq \pi, \quad |\bar{\theta} - \theta_2'| \leq \pi.$$

As $\bar{\ell}$ is parallel to $\ell$ we have $\bar{\theta} = \theta + 2k\pi$, k being an integer. Thus

$$|\theta_1' - \theta_2'| = |\theta_1' - \bar{\theta}| + |\bar{\theta} - \theta_2'| ,$$

and we have that

$$\begin{aligned} T' &= |\theta_1 - \theta_1'| + |\theta_1' - \bar{\theta}| + |\bar{\theta} - \theta_2'| + |\theta_2' - \bar{\theta}_2| \\ &\geq |\theta_1 - \bar{\theta}| + |\bar{\theta} - \bar{\theta}_2| = |\theta_1 - \theta - 2j\pi| + |\theta - \theta_2 + 2(j-k)\pi| \\ &\geq |\theta_1 - \theta| + |\theta - \theta_2| = T. \end{aligned}$$

Therefore

$$T(\Gamma_{m+1}) \geq T(\Gamma_m).$$

For a general value of m the lemma can now be proved by induction.

Theorem I. Let $\Gamma$ be a polygon whose vertices lie consecutively around a simple curve C of bounded turning. Then $T(C) \geq T(\Gamma)$.

Theorem II. If C is a simple curve of bounded turning then a polygon $\Gamma$ inscribed in C exists for which $T(C) - T(\Gamma) < \epsilon$, where $\epsilon > 0$ is arbitrarily small.

These two theorems may be written as the following.

Theorem III. The turning T(C) of a simple curve C of bounded turning is the least upper bound of the turning of the inscribed polygons whose vertices lie in order on the curve.

The proofs to these statements are contained in the following.

Lemma III. Let C be a curve of bounded turning. Let $\{s_j^n\}$ be a net of points dense on C, such that

$$s_j^n < s_{j+1}^n, \quad \{s_j^n\} \subset \{s_j^{n+1}\}, \quad j = 0,1,2, \ldots, m_n,$$

which includes all of the vertices of C; say that $\{s_j^n\}$ contains all of the vertices for which $|\theta(s+) - \theta(s-)| > 1/n$. Let $\{\Gamma_n\}$ be the sequence of polygons having $\{s_j^n\}$ in order as vertices. Then $T(\Gamma_n) \leq T(C)$ and $\lim_{n\to\infty} T(\Gamma_n) = T(C)$.

Since $\theta(s)$ is of bounded variation it follows that for sufficiently large n the total variation of $\theta(s)$ in the interior of the interval $s_{j-1}^n < s < s_j^n$, $j = 1,2, \ldots, m_n$, can be made arbitrarily small. Consider the interval $[s_0^n, s_1^n]$ and assume, without loss in generality, that the chord $P_0^nP_1^n$ joining these points is parallel to the x-axis. Then x(s) is an absolutely continuous increasing function of s, and conversely, s is a continuous increasing of x. Since the direction of $P_0^nP_1^n$ differs arbitrarily little from the value of $\theta(s)$ in the interval $[s_0^n, s_1^n]$, the function

$$y(s) = \int_0^s \sin\theta(s)ds + y(0)$$

is a continuous single-valued function of x, and a maximum value of y occurs at some interior point $\overline{s_1^n}$ of the interval. At $\overline{s_1^n}$ a ray parallel to $P_0^nP_1^n$ has the angle $\overline{\theta}_1^n = 0$ of the chord, and this angle lies between the angles $\theta(\overline{s_1^n}+)$ and $\theta(\overline{s_1^n}-)$.

This result is general: In the interior of each of the intervals $[s_{j-1}^n, s_j^n]$ there exists a point $\overline{s_j^n}$ such that the direction of the chord $P_{j-1}^nP_j^n$ is determined by an angle $\overline{\theta_j^n}$ which lies between $\theta(\overline{s_j^n}+)$ and $\theta(\overline{s_j^n}-)$.

As the function $\theta(s)$ is of bounded variation

$$T(\Gamma_n) = \sum_{1}^{m_n}{}_j \left|\overline{\theta_{j-1}^n} - \overline{\theta_j^n}\right| \leq \lim_{n\to\infty} \sum_{1}^{m_n}{}_j \left|\overline{\theta_{j-1}^n} - \overline{\theta_j^n}\right| = \lim_{n\to\infty} T(\Gamma_n) = T(C).$$

which is the statement of the lemma.

Theorem IV. If $\{\Gamma_n\}$ is any sequence of polygons which define the length of the simple curve C of bounded turning and whose vertices lie in order around C, then $T(C) = \lim_{n\to\infty} T(\Gamma_n)$.

The proof to this theorem is contained in Lemmas I, II, and III.

Theorem V. Let C be a simple curve of bounded turning. Given $\epsilon > 0$, then $\delta > 0$ exists such that if C' is a simple curve of bounded turning whose Frechet distance from C is less than $\delta$, then $T(C') > T(C) - \epsilon$.

Bu Theorem II we can inscribe in C a polygon $\Gamma$ such that $T(C) - T(\Gamma) < \epsilon/2$. Let $\Gamma'$ be a polygon of the same number of sides inscribed in C', with the corresponding vertices of $\Gamma'$ and $\Gamma$ distant apart by less than $\delta$. Then by Lemma I we have for $\delta$ sufficiently small that

$$|T(\Gamma)-T(\Gamma'')| < \epsilon/2 .$$

Hence we have that

$$T(C') \geqslant T(\Gamma') \geqslant T(\Gamma)-\epsilon/2 > T(C)-\epsilon .$$

Corollary*. If $\{C_n\}$ is a sequence of simple curves of bounded turning approaching a simple curve C of bounded turning in such a manner that the Frechet distance between $C_n$ and C approaches zero, then $\inf.\lim_{n\to\infty} T(C_n) \geqslant T(C)$.

3. The turning of the level curves of the Green's function.

In the complex plane, $z = x+iy$, let the finite region D be bounded by a simple closed rectifiable curve C. We may assume without loss in generality that the origin lies within D.

Let $g(x,y)$ be the Green's function for D, with pole at the origin, and let $h(x,y)$ be the conjugate function†. Then the analytic function

$$w(z) = e^{-g-ih}, \quad w = r\, e^{i\beta} ,$$

maps D upon the unit circle $|w|<1$, with center at the origin in the w-plane. This mapping is such that the level curves $C_r$ of the Green's function, defined by

$$g \equiv \text{constant},$$

are mapped upon the circles $|w| = r < 1$; while the orthogonal curves, given by

$$h \equiv \text{constant},$$

are mapped upon radii of the circle $|w| = 1$.

Let $f(w) = z$ be the inverse function which maps the unit circle

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*For the family of curves to be considered in paragraph 3 this may be established directly from the definition of the turning.

†W. F. Osgood, Lehrbuch der Funktionentheorie, 5th. ed., (1928), Vol. 1, pp. 716-736.

$|w| < 1$ conformally upon D so that the circles $|w| = r < 1$ are mapped upon the level curves $C_r$. Let s denote the arc length of the curves C and $C_r$, measured from the points corresponding to $w = 1$ and $w = r$ (that is where $\beta = 0$), respectively. The point $w = 1$ is chosen to correspond in the mapping to a point at which C has a unique tangent. Thus*

$$s = s(r, \beta) = \int_0^\beta |w f'(w)| \, d\beta, \quad r \leq 1.$$

Let $\theta(r, \beta) \equiv \theta_r(s)$ be the angle made with the positive real axis by the forward direction of the curve $C_r$ at the point z corresponding to $w(z) = r e^{i\beta}$. $\theta(r, \beta)$ is to be measured so that it is a continuous function for all $0 < r < 1$. Let $\theta(s) \equiv \Theta(\beta)$, $s = s(1, \beta)$, be the corresponding angle defined for the boundary curve C.

Now the inverse function f(w) is given by

$$f(w) = z = x + iy = \int_0^s e^{i\theta_r(s)} \, ds + z_0,$$

where the integration is extended over the level curve $C_r$ which passes thru the point z†. Then

$$\frac{df}{ds} = e^{i\theta_r(s)},$$

but also

$$\frac{df}{ds} = f'(w)\frac{dw}{ds} = \frac{i \, w \, f'(w)}{|w \, f'(w)|},$$

from which we obtain the well known result that,

$$\theta(r, \beta) = \theta_r(s) = \pi/2 + \beta + \arg f'(w),$$

for all $0 < |w| < 1$.‡ Thus $\theta(r, \beta)$ is a harmonic function in $0 < |w| < 1$.

As D is a finite region bounded by a rectifiable curve the limit

$$\lim_{r \to 1} f(w) = f(e^{i\beta})$$

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*For an account of the properties of the function $s(r, \beta)$ see W. Seidel, "Über die Ränderzuordnung bei konformen Abbildungen", Mathematische Annalen, Vol. 104 (1931), pp. 182-243.

†Lebesgue, op. cit.

‡G. Pólya und G. Szegö, Aufgaben und Lehrsätze aus der Analysis, Vol. 1, p. 105.

exists uniformly for all $\beta$; and the limit

$$\lim_{r\to 1} f'(w) = f'(e^{i\beta})$$

exists almost everywhere*. Here $f(e^{i\beta})$ is the boundary function and $f'(e^{i\beta})$ is its derivative. From the integral expression for $f(w)$ and the uniform approach to its boundary values we see that

$$\lim_{r\to 1} e^{i\theta_r(s)} = e^{i\theta(s)}, \quad \text{for almost all } s,$$

that is,

$$\lim_{r\to 1} \theta_r(s) = \theta(s) + 2n\pi, \quad n \text{ being an integer},$$

or that

$$\lim_{r\to 1} \theta(r,\beta) = \theta(1,\beta) = \Theta(\beta) + 2n\pi,$$

for almost all $\beta$. We choose the measure of the function $\theta_r(s)$ so that

$$\lim_{r\to 1} \theta_r(0) = \theta(0),$$

which is

$$\lim_{r\to 1} \theta(r,0) = \Theta(0).$$

Theorem VI. For any simple boundary curve $T(C_r)$ is a non-decreasing function of $r$ for all $r < 1$.

For all $0 < r < 1$ the function $\theta(r,\beta)$ is continuous and its total variation over any level curve $C_r$ is given by

$$T(C_r) = \int_0^{2\pi} \left|\frac{\partial}{\partial\beta}\theta(r,\beta)\right| d\beta, \quad 0 < r < 1.$$

Since the function $\theta(r,\beta)$ is harmonic in $0 < |w| < 1$, its derivative is likewise harmonic and the absolute value of the derivative is a sub-harmonic function in $|w| < 1$. Therefore by the well known theorem of F. Riesz the function $T(C_r)$ is a non-decreasing function for all values of $r$ in the interval $0 < r < 1$.†

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*P. Fatou, " Séries Trigonométriques et Séries de Taylor", Acta Mathematica, Vol. 30 (1906), pp. 335-400. See p.345. F. Riesz, " Über die Randwerte einer analytische Funktion", Mathematische Zeitschrift, Vol. 18 (1923), pp. 87-95. See p. 94. For a good presentation see S. Warschawski, " Über einige Konvergenzsätze aus der Theorie der Konformen Abbildung", Göttinger Nachrichten (1930), pp. 344-369.

† F. Riesz, "Sur les fonctions subharmoniques et leur rapport à la théorie du potential", Acta Mathematica, Vol. 48 (1926), pp. 329-343; Vol. 54 (1930), pp. 321-360.

**Theorem VII.*** If C is a simple closed polygon of a finite number of sides, then $\theta(r,\beta)$ is uniformly bounded in $r \leq 1$, $0 \leq \beta \leq 2\pi$, and

$$\lim_{r\to 1} \theta(r,\beta) = \tfrac{1}{2}\left\{\Theta(\beta+) + \Theta(\beta-)\right\}$$

at all points on the curve C.

Let C have n sides. Then the mapping function is†

$$f(w) = A\int_0^w (w-w_1)^{\mu_1}(w-w_2)^{\mu_2}\cdots(w-w_n)^{\mu_n}\,dw$$

where the $w_k = e^{i\beta_k}$ are the maps of the vertices of C, and the turning at the k-th vertex is

$$-\mu_k\pi = \Theta(\beta_k+) - \Theta(\beta_k-).$$

Now

$$\theta(r,\beta) = \tfrac{1}{2}\pi + \beta + \arg f'(w)$$
$$= \tfrac{1}{2}\pi + \beta + \arg A + \sum_{1}^{n}{}_{k}\,\mu_k \arg(w-w_k).$$

Each of the quantities on the right is bounded for all points w within or on the unit circle; that is to say, for $\beta$ in the range $0 \leq \beta \leq 2\pi$ these quantities are uniformly bounded for all values of r in the range $r \leq 1$. For the only other possibility is that some one of the $\arg(w-w_k)$ should become arbitrarily large, and this is impossible since the point w must remain inside or on the unit circle.

In the neighborhood of $w_k$ ‡

$$f'(w) = (w-w_k)^{\mu_k}\varphi_k(w),$$

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* E. Study, Vorlesungen über Ausgewählte Gegenstände der Geometrie, (1913), Vol. 2, pp. 83-99. By making $\lim(r\to 1)\ \theta(r,0) = \Theta(0)$ we have chosen equal to zero the arbitrary constant which enters into Study's result. For curves with a continuously turning tangent at all points a similar theorem is proved by Warschawski, loc. cit. See p. 367.

† E. B. Christoffel, "Ueber die Abbildung einer n-blättrigen, einfach zusammenhängenden Fläche auf einem Kreise", Göttinger Nachrichten, (1870), p. 359.

‡ This is proved for more general boundary curves by W. F. Osgood and E. H. Taylor, "Conformal Transformations on the Boundaries of Their Regions of Definition", Transactions of the American Mathematical Society, Vol. 14 (1913), pp. 277-298.

where $\varphi_k(w)$ is analytic at $w_k$ and $\varphi_k(w_k) \neq 0$. Thus the value of arg $\varphi_k(w_k)$ is independent of the manner of approach of $w$ to $w_k$.

We have

$$\theta(r,\beta) = \tfrac{1}{2}\pi + \beta + \mu_k \arg(w-w_k) + \arg \varphi_k(w).$$

Let $w$ approach $w_k$ along the arc of the circle, then

$$\lim_{\nu\to\beta_k+} \theta(1,\nu) = \tfrac{1}{2}\pi + \beta_k + \mu_k(\tfrac{1}{2}\pi + \beta_k) + \arg \varphi_k(w_k)$$

$$\lim_{\nu\to\beta_k-} \theta(1,\nu) = \tfrac{1}{2}\pi + \beta_k + \mu_k(\tfrac{3}{2}\pi + \beta_k) + \arg \varphi_k(w_k).$$

Hence

$$\lim_{\nu\to\beta_k+} \theta(1,\nu) - \lim_{\nu\to\beta_k-} \theta(1,\nu) = \Theta(\beta_k+) - \Theta(\beta_k-),$$

and since we have chosen the measure of the angles so that

$$\lim_{r\to 1} \theta(r,0) = \Theta(0)$$

we have that

$$\Theta(\beta_k+) = \tfrac{1}{2}\pi + \beta_k + \mu_k(\tfrac{1}{2}\pi+\beta_k) + \arg \varphi_k(w_k)$$

$$\Theta(\beta_k-) = \tfrac{1}{2}\pi + \beta_k + \mu_k(\tfrac{3}{2}\pi+\beta_k) + \arg \varphi_k(w_k)$$

which serve to give the value of arg $\varphi_k(w_k)$.

If $w$ approach $w_k$ along a line making an angle $\alpha$ with the forward direction of the circle at $w_k$ then

$$\lim \arg(w-w_k) = \alpha + \tfrac{1}{2}\pi + \beta_k$$

so that

$$\lim \theta(r,\beta) = \alpha\mu_k + \Theta(\beta_k+)$$
$$= \Theta(\beta_k+) - \frac{\alpha}{\pi}\left\{\Theta(\beta_k+) - \Theta(\beta_k-)\right\}.$$

For radial approach $\alpha = \tfrac{1}{2}\pi$ and

$$\lim_{r\to 1} \theta(r,\beta_k) = \tfrac{1}{2}\left\{\Theta(\beta_k+) + \Theta(\beta_k-)\right\}.$$

Finally, since $f(w)$ is analytic on the unit circle except at the points $w_k$, it follows that

$$\lim_{\substack{r\to 1\\ \nu\to\beta}} \theta(r,\nu) = \Theta(\beta), \quad \beta \neq \beta_k.$$

This completes the proof to the theorem.

Theorem VIII. <u>If C is a simple closed polygon of a finite number of sides, then</u> $\lim_{r\to 1} T(C_r) = T(C)$.

Let $w = r\, e^{i\beta}$, $r < 1$, be a fixed point. Consider the function

$$h(r,\beta;\rho,\nu) = \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2r\rho\cos(\nu-\beta)}, \qquad \begin{array}{l} r < r_0 < \rho \leqslant 1 \\ 0 \leqslant \beta \leqslant 2\pi \\ 0 \leqslant \nu \leqslant 2\pi \end{array}$$

This function and its derivatives are continuous, and

$$\lim_{\rho\to 1} h(r,\beta;\rho,\nu) = h(r,\beta;1,\nu) = h(r,\beta;\nu)$$

$$\lim_{\rho\to 1} \frac{\partial}{\partial\nu} h(r,\beta;\rho,\nu) = \frac{\partial}{\partial\nu} h(r,\beta;\nu)$$

uniformly in $\nu$ for each fixed $r$ and $\beta$. The function $h(r,\beta;\nu)$ is of bounded variation in $\nu$, and $h(r,\beta;\rho,\nu)$ is also of bounded variation in $\nu$, uniformly with respect to $\rho$.

The function $\frac{\partial}{\partial\beta}\theta(r,\beta)$ is harmonic for all $r < 1$ and therefore can be written as a Poisson integral,

$$\frac{\partial}{\partial\beta}\theta(r,\beta) = 1/2\pi\int_0^{2\pi} h(r,\beta;\rho,\nu)\, d_\nu\, \theta(\rho,\nu), \quad r_0 < \rho < 1.$$

An integration by parts yields

$$\frac{\rho - r}{\rho + r} - 1/2\pi\int_0^{2\pi}\theta(\rho,\nu)\, d_\nu\, h(r,\beta;\rho,\nu).$$

Consider

$$\int_0^{2\pi}\theta(\rho,\nu)\, d_\nu\, h(r,\beta;\rho,\nu) - \int_0^{2\pi}\Theta(\nu)\, d_\nu\, h(r,\beta;\nu) =$$

$$\int_0^{2\pi}\left[\theta(\rho,\nu)-\Theta(\nu)\right]\frac{\partial}{\partial\nu}h(r,\beta;\nu)\,d\nu + \int_0^{2\pi}\theta(\rho,\nu)\left[\frac{\partial}{\partial\nu}h(r,\beta;\rho,\nu) - \frac{\partial}{\partial\nu}h(r,\beta;\nu)\right]d\nu\,.$$

The function $\theta(\rho,\nu)$ is uniformly bounded

$$\left|\theta(\rho,\nu)\right| \leqslant M, \qquad \left|\Theta(\nu)\right| \leqslant M,$$

and

$$\lim_{\rho\to 1}\theta(\rho,\nu) = \Theta(\nu) \quad \text{for almost all } \nu,$$

Therefore both the integrals on the right approach zero with $1-\rho$, for every fixed $w = r\,e^{i\beta}$, $0 \leq \beta \leq 2\pi$, $r < 1$.

Thus, allowing $\rho \to 1$ we have that

$$\frac{\partial}{\partial\beta}\theta(r,\beta) = \frac{1-r}{1+r} - 1/2\pi\int_0^{2\pi}\Theta(\nu)\, d_\nu h(r,\beta;\nu)$$

and an integration by parts yields

$$\frac{\partial}{\partial\beta}\theta(r,\beta) = 1/2\pi\int_0^{2\pi} h(r,\beta;\nu)\, d\Theta(\nu).$$

That is, the harmonic function $\frac{\partial}{\partial\beta}\theta(r,\beta)$ is expressed as a Poisson-Stieltjes integral. It has been proved for such functions that the total variation of $\theta(r,\beta)$ in $\beta$ is dominated for all r by the total variation of $\Theta(\beta)$;* or

$$T(C_r) = \int_0^{2\pi}\left|\frac{\partial}{\partial\beta}\theta(r,\beta)\right| d\beta \leq \int_0^{2\pi}|d\Theta|$$

$$\lim_{r\to 1} T(C_r) = \int_0^{2\pi}|d\Theta| = T(C).$$

This completes the proof to the theorem.

We can now prove the corresponding theorem for a more general boundary curve. Let C be a simple closed curve of bounded turning. In C inscribe a sequence $\{\Gamma_m\}$ of simple polygons such that

$$\lim_{m\to\infty} T(\Gamma_m) = T(C).$$

This we can do by Theorem IV. Consider now the sequence $\{f_n(w)\}$ of analytic functions which map the unit circle $|w| < 1$ conformally on the interiors of the polygons $\{\Gamma_m\}$, the origin in the w-plane mapping on the origin in the z-plane. Let f(w) be the corresponding mapping function for the region D bounded by C. By a theorem of T. Rado †

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*G. C. Evans, The Logarithmic Potential, New York (1927), pp. 38-39.

†T. Rado, "Sur la representation conforme de domaines variables", Acta Szeged, (1923), pp. 180-186. See also Warschawski, loc. cit.

$$\lim_{n\to\infty} f_n(w) = f(w)$$

uniformly for all $|w| \leq 1$.

Thus if $\{\Gamma_{n,r}\}$ are the level curves for $\{\Gamma_n\}$ and $\{\theta_n(r,\beta)\}$ are the angles corresponding to $\theta(r,\beta)$,

$$\theta_n(r,\beta) = \tfrac{1}{2}\pi + \beta + \arg f_n'(w),$$

then we have that

$$\lim_{n\to\infty} \theta_n(r,\beta) = \theta(r,\beta), \quad r \text{ fixed},$$

uniformly for $0 < r < 1$. From this follows immediately that

$$\lim_{n\to\infty} T(\Gamma_{n,r}) = T(C_r), \quad r < 1.$$

Since by Theorem VI $T(\Gamma_{n,r})$ is a non-decreasing function of $r$ for each $n$, and by Theorem VIII approaches the limit $T(\Gamma_n)$, we have

$$T(\Gamma_{n,r}) \leq T(\Gamma_n) \leq T(C),$$

and in the limit

$$T(C_r) \leq T(C).$$

This inequality with that of Theorem V shows that

$$\lim_{r\to 1} T(C_r) = T(C).$$

We have then proved the following theorem.

Theorem IX*. <u>If $C$ is a simple closed curve of bounded turning, and $\{C_r\}$ are the level curves of the Green's function for the finite region bounded by $C$, then</u> $T(C_r) \leq T(C)$ <u>and</u> $\lim_{r\to 1} T(C_r) = T(C)$.

Theorem X. <u>If $C$ is a curve satisfying the conditions of Theorem IX, then at every point of the curve</u>

$$\lim_{r\to 1} \theta(r,\beta) = \tfrac{1}{2}\{\Theta(\beta+) + \Theta(\beta-)\}$$

<u>which is</u>

$$\lim_{r\to 1} \theta_r(s) = \tfrac{1}{2}\{\theta(s+) + \theta(s-)\}.$$

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* This theorem was suggested by Radon, <u>loc. cit.</u>, pp. 1154-1155.

## 4. Generalization.

The results obtained in paragraph 2 for curves of bounded turning may be extended to apply to any simple rectifiable curve. In paragraph 2 necessary conditions for curves of bounded turning are found. These are also sufficient.

For let C be a simple rectifiable curve. Consider a dense set of parameter values $\{s_j^n\}$, $s_j^n < s_{j+1}^n$, forming a net on C. If $\theta_m(s)$ represents the step function giving the direction angle of the polygon $\Gamma_m$ whose vertices in order are $s_j^n$, $j=1,2,\ldots,m_n$, then for a point on $\Gamma_m$,

$$x_n(s)-x_n(0) = \int_0^s \cos\theta_m(s)ds, \qquad y_n(s)-y_n(0) = \int_0^s \sin\theta_m(s)ds.$$

If now the functions $\theta_m(s)$ are of uniformly bounded variation there exists a dense denumerable set D(s) of points upon which the functions $\theta_m(s)$, $\cos\theta_m(s)$, and $\sin\theta_m(s)$ converge to functions $\theta(s)$, $\cos\theta(s)$, and $\sin\theta(s)$ which are of bounded variation upon D(s). If we write, by an integration by parts,

$$x_n(s)-x_n(0) = s\cos\theta_m(s) - \int_0^s s\, d\cos\theta_m(s)$$

$$y_n(s)-y_n(0) = s\sin\theta_m(s) - \int_0^s s\, d\sin\theta_m(s)$$

and consider the value s to belong to D(s), and the integrals to be net integrals over D(s), then in the limit*

$$x(s)-x(0) = s\cos\theta(s) - \int_0^s s\, d\cos\theta(s)$$

$$y(s)-y(0) = s\sin\theta(s) - \int_0^s s\, d\sin\theta(s)$$

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*H. E. Bray, "Elementary Properties of the Stieltjes Integral", Annals of Mathematics, Vol. 20 (1919), p. 180. See also Evans, op. cit., p. 15.

which are

$$x(s) - x(0) = \int_0^s \cos\theta(s)\,ds, \qquad y(s) - y(0) = \int_0^s \sin\theta(s)\,ds.$$

Therefore the curve C must be a curve of bounded turning, since the conditions of paragraph 1 are satisfied.

With this result the extension of the results of paragraph 2 is readily carried out. We state explicitly the generalization of Theorem IV.

Theorem XI. _If $\{\Gamma_n\}$ is any sequence of polygons which define the length of the simple rectifiable curve C and whose vertices lie in order around C, then_ $\lim_{n\to\infty} T(\Gamma_n) = T(C)$.

With this theorem and Theorem IX the following is immediate.

Theorem XII. _If C is any simple closed rectifiable curve, and $\{C_r\}$ are the level curves of the Green's function for the finite region bounded by C, then_ $T(C_r) \leqslant T(C)$ _and_ $\lim_{r\to 1} T(C_r) = T(C)$.

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