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DOUBLY ORTHOGONAL FUNCTIONS
ON RIEMANN SURFACES

by

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Gerald R. Mac Lean
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INTRODUCTION

An analytic function has the remarkable property that a single function element in the Weierstrass sense determines completely the behavior of the function in the large. That is, the values of an analytic function in some neighborhood of a point determine a Riemann surface upon which the function is single valued (the so called analytic form, c.f. Weyl [10] for example) and also the value of the function at each point of this surface. Thus, if we are given a function defined by a Taylor's series in the unit circle, in theory it is possible to write down conditions upon the coefficients of this series in order that the function have any desired properties. Some theorems along these lines have been given. For example, the well known formula for the radius of convergence of the series gives us a criterion as to whether or not the function has no singularities in the circle \(|z| < R, R \leq \infty\). Many other results of this type could be quoted, many of them quite remarkable.

However, many questions along these lines have remained unanswered. What are the conditions that a Taylor's series, convergent in the unit circle, represent a function having a branch point of order \(p\) at \(z = 1\)? Such conditions can be found -- see theorem 9 below for example.
A powerful tool for studies of this kind is available in the concept of doubly orthogonal functions discovered by Bergman, see [3] p. 14. These ideas have been used for a study of an approximation problem by Davis [4] with considerable success.

It is our purpose in this work to look more deeply into the theory of such doubly orthogonal functions. In particular, we wish to study doubly orthogonal sets of functions on Riemann surfaces rather than to restrict ourselves to the plane case.

For the sake of completeness, the first paragraph of this work is a review of some of the fundamental properties of Hilbert spaces. The information given is essentially the same as that in the first few pages of Stone [9], which may be referred to for a much more complete account of the subject. Paragraph 1 includes a proof of theorem 1, the Riesz-Fischer theorem.

In paragraph 2, the existence proof for the doubly orthogonal sets is given. This is based on the proof found in Bergman [3], p. 15, but applies to a considerably more general case. The major change in the proof follows closely the method given by Scholz [8] in a similar situation.

Bergman, in his proof mentioned above, makes the erroneous statement that the set so constructed is
unique. Scholz [8] shows that the doubly orthogonal set he is considering will be essentially unique in one particular case, and states that the proof is invalid and no unique system exists in the remaining cases. However, by means of an extension of his proof, we are able to find the exact degree of difference which may exist between two doubly orthogonal sets. This is carried out in paragraph 3.

Paragraph 4 reviews some of the needed information about the kernel function, following Bergman [3]. For a more detailed account of this subject, the paper by Aronszajn [1] may be consulted.

In paragraph 5 we consider a constructive method for obtaining doubly orthogonal sets. A modification of a transformation given by Davis [4] is shown to lead to a constructive process. The results of this paragraph are of considerable interest, for the minimum problem from which the existence is proved is not suitable for the actual construction of doubly orthogonal sets. The method described here seems to be the first constructive process given for this problem.

Paragraphs 6 and 7 are concerned with the application of these results to spaces of analytic functions on Riemann surfaces. The principal difficulty to be overcome in paragraph 7 is the approximation of
a function, regular and uniform in an open region $G$, by a function regular and uniform in a larger domain $B$. For this purpose we make use of a theorem of Behnke and Stein [2] which is essentially Runge's theorem on Riemann surfaces. We also make use of a device given by Schiffer and Spencer, [7] p. 137, to approximate a function regular and uniform in an open region $G$ by a function regular and uniform in $G$ and on the boundary. For this process to work, we are forced to assume that the boundary of $G$ is analytic. It would seem that this restriction should not be needed, for in the plane case it can be avoided (see the paper by Farrell [5]), but no method of relaxing this restriction on a Riemann surface in general has been found as yet.

In paragraph 8 we make very brief mention of how these results may be carried over to spaces of harmonic functions and in paragraph 9 we point out some of the consequences of the theory.

In the last two paragraphs of this work, some applications are considered. No effort has been made to be complete here. Rather, these applications are given merely to illustrate some of the possibilities.

Finally, let me here express my appreciation of the help and guidance given to me by Professor G. R. MacLane during the preparation of this work.
1. Let $H$ be a Hilbert space, that is, a complete normed linear space having a complex valued inner product $(f, g)$ and a norm $\|f\| = (f, f)^{\frac{1}{2}}$ satisfying

\[
\begin{align*}
(f, g) &= (g, f), \\
(af + bg, h) &= a(f, h) + b(g, h) & \text{a,b scalars,} \\
\|f\| &\geq 0 & \text{all } f \text{ in } H, \\
\|f\| &= 0 & \text{if and only if } f = 0, \\
\|af\| &= |a|\|f\| & \text{a scalar,} \\
\|f_n - f\| \rightarrow 0 & \implies \|f_n\| \rightarrow \|f\|, \\
|\langle f, g \rangle| &\leq \|f\| \cdot \|g\|, \\
\|f + g\| &\leq \|f\| + \|g\|. 
\end{align*}
\]

The last two are Schwarz's inequality and the triangle inequality respectively. No assumptions are made at present about the separability of the space.

Two elements of $H$ are called orthogonal if $(f, g) = 0$. A set of elements $\{\varphi_n\}$ is called an orthogonal set if every pair of elements are orthogonal. If in addition $\|\varphi_n\| = 1$ for all $n$, the set is called orthonormal.

Any finite set of non-zero orthogonal elements is linearly independent. For if $f = \sum_{n=1}^{k} c_n \varphi_n = 0$, then $\langle f, \varphi_n \rangle = c_n \|\varphi_n\|^2 = 0$, $n = 1, \ldots, k$ which implies that all of the $c_n$ are zero.

If $\{\varphi_n\}$ is an orthonormal set, then

\[
\|\sum_{j=1}^{n} c_j \varphi_j\|^2 = \sum_{j=1}^{n} |c_j|^2
\]

as can be seen by an elementary calculation.
If \( f \) is an element of \( H \), we define the Fourier coefficients of \( f \) with respect to a given orthonormal set \( \{ \varphi_n \} \) to be

\[
(1.3) \quad a_n = (f, \varphi_n).
\]

Then

\[
\| f - \sum_{j=1}^{n} c_j \varphi_j \|^2 = \| f - \sum_{j=1}^{n} a_j \varphi_j + \sum_{j=1}^{n} (a_j - c_j) \varphi_j \|^2
\]

\[
= \| f - \sum_{j=1}^{n} a_j \varphi_j \|^2 + \sum_{j=1}^{n} |a_j - c_j|^2
\]

\[+ 2\Re \langle f - \sum_{j=1}^{n} a_j \varphi_j, \sum_{j=1}^{n} (a_j - c_j) \varphi_j \rangle \]

\[
= \| f \|^2 - 2\Re \langle f, \sum_{j=1}^{n} a_j \varphi_j \rangle + \sum_{j=1}^{n} |a_j|^2
\]

\[+ \sum_{j=1}^{n} |a_j - c_j|^2.
\]

From this, we see that \( \| f - \sum_{j=1}^{n} c_j \varphi_j \| \) is minimized by setting each \( c_j = a_j \). Further, since the norm is always non-negative, letting \( n \to \infty \), we have Bessel's inequality:

\[
(1.4) \quad \sum_{j=1}^{n} |a_j|^2 \leq \| f \|^2.
\]

We say that an orthonormal set \( \{ \varphi_n \} \) is closed or complete in \( H \) if and only if

\[
(1.5) \quad \sum_{j=1}^{n} |a_j|^2 = \| f \|^2
\]

for all \( f \) in \( H \). This is equivalent to

\[
(1.6) \quad \| f - \sum_{j=1}^{n} a_j \varphi_j \| \to 0 \text{ as } n \to \infty.
\]

**Theorem 1.** If \( \{ \varphi_n \} \) is an orthonormal set and if the sequence \( \{a_n\} \) of complex numbers satisfies

\[
\sum_{j=1}^{\infty} |a_j|^2 < \infty
\]

then
\[ f = \sum_{j=1}^{\infty} a_j \varphi_j \]

is in \( H \) (the series converging in the norm) and is such that

\[ (f, \varphi_j) = a_j \]

and

\[ \| f \|^2 = \sum_{j=1}^{\infty} |a_j|^2. \]

**PROOF:** If we define

\[ f_n = \sum_{j=1}^{n} a_j \varphi_j \]

then

\[ \| f_{n+p} - f_n \|^2 = \sum_{j=n+1}^{\infty} |a_j|^2 \]

and hence the \( f_n \) form a Cauchy sequence in \( H \). The completeness of \( H \) shows that the limit \( f \) is in \( H \). We have

\[ \| f_n \|^2 = \sum_{j=1}^{n} |a_j|^2 \]

and hence \( \| f \|^2 = \sum_{j=1}^{\infty} |a_j|^2 \). Finally, for any \( m \), if \( n > m \) then

\[ |(f, \varphi_m) - a_m| = |(f - f_n, \varphi_m)| \leq \| f - f_n \| \]

by Schwarz's inequality. Letting \( n \to \infty \), the theorem is proved.

We note that up to this point, the separability of the space \( H \) has not been assumed. It is easily seen that the space \( H \) is separable if and only if a complete orthonormal sequence exists in \( H \). For if \( \{x_n\} \) is a countable dense subset of \( H \), then the set \( \{y_n\} \)
obtained from \( \{x_n\} \) by the Gram-Schmidt process is a complete orthonormal set. If \( \{y_n\} \) is a complete orthonormal set, then the collection of all finite linear combinations \( a_1y_1 + \ldots + a_ny_n \), where the \( a_j \) run over all complex numbers with rational real and imaginary parts, is a countable dense subset of \( H \).

**Lemma 1.1.** Let the Hilbert space \( H \) be a function space, the functions being complex valued. If for \( Z_0 \) in the domain of these functions

\[
|f(Z_0)| \leq m(Z_0) \| f \|
\]

for all \( f \) in \( H \), and if \( \{\varphi_n\} \) is an orthonormal sequence in \( H \), then

\[
\sum_{n=1}^{\infty} |\varphi_n(Z_0)|^2 \leq [m(Z_0)]^2.
\]

**Proof:** Consider the function

\[
f_N = \sum_{n=1}^{N} \overline{\varphi_n(Z_0)} \varphi_n.
\]

Here,

\[
|f_N(Z_0)|^2 = \sum_{n=1}^{N} |\varphi_n(Z_0)|^2
\]

and

\[
\|f_N\|^2 = \sum_{n=1}^{N} |\varphi_n(Z_0)|^2.
\]

From (1.7), these two give

\[
\sum_{n=1}^{N} |\varphi_n(Z_0)|^2 \leq [m(Z_0)]^2.
\]

and letting \( N \to \infty \), we have (1.8).
2. In the remainder of this work we shall be considering a space equipped with two distinct inner products and associated norms. For this purpose, let \( H_1 \) be a Hilbert space with inner product \( (f, g)_1 \) and norm \( \|f\|_1 \). Let \( H_2 \) be a submanifold of \( H_1 \), and let the inner product \( (f, g)_2 \) and norm \( \|f\|_2 \) be defined on \( H_2 \). The space \( H_2 \) thus has two topologies defined on it, corresponding to the two norms. We will assume that these satisfy the two requirements:

\[
(2.1) \quad \|f\|_1 < \|f\|_2 \quad \text{for all } f \neq 0 \text{ in } H_2,
\]

and

\[
(2.2) \quad \text{if } \{f_n\} \text{ is a bounded sequence in } H_2, \text{ then there exists an } f_0 \text{ in } H_2 \text{ and a subsequence } \{f_{n_k}\} \text{ such that } \|f_{n_k} - f_0\|_1 \to 0 \text{ and } \|f_0\|_2 \leq \liminf_{n \to \infty} \|f_n\|_2.
\]

We shall say that a sequence \( \{\varphi_n\} \) of elements of \( H_2 \) is a **doubly orthogonal set** in \( H_1 \) and \( H_2 \) when

\[
(2.3) \quad \begin{cases}
(f, \varphi_n)_1 &= \delta_{nm} \\
(f, \varphi_n)_2 &= \lambda_n \delta_{nm} \\
(f, \varphi_n)_2 &= \lambda_n(f, \varphi_n)_1 \quad \text{all } f \text{ in } H_2, \text{ all } n.
\end{cases}
\]

Note that the second property follows immediately from the third, and conversely that the first two imply the third if \( \{\varphi_n\} \) is complete in \( H_2 \).

In order to prove the existence of such doubly orthogonal sets, we introduce:

**THE MINIMUM PROBLEM.** If \( \varphi_1, \varphi_2, \ldots, \varphi_n \) have
been chosen satisfying (2.3), from the set \( A_n \) of all \( f \) in \( \mathcal{H}_2 \) satisfying
\[
\|f\|_1 = 1
\]
(2.4)
\[
(f, \varphi_m)_1 = 0 \quad m = 1, 2, \ldots, n-1,
\]
choose \( \varphi_n \) to be an element minimizing \( \|f\|_2 \).

THEOREM 2. If \( \mathcal{H}_2 \) has infinite Hamel dimension, the minimum problem can be solved to produce a doubly orthogonal set. The \( \lambda_n \) form a non-decreasing sequence, and if \( \lambda_n \to \infty \), then \( \{\varphi_n\} \) is complete in \( \mathcal{H}_2 \).

PROOF. Let \( \lambda_n = \inf_{f \in A_n} \|f\|_2^2 \). This number exists, since the assumption on the dimensionality insures that \( A_n \) is non-empty. Choose a sequence \( \{f_n\} \) from \( A_n \) with
\[
\|f_n\|_2^2 \to \lambda_n.
\]
This sequence is bounded in \( \mathcal{H}_2 \), and hence from (2.2), there exists a subsequence \( \{f_{n_k}\} \) and a \( \varphi_n \) in \( \mathcal{H}_2 \) such that \( \|f_{n_k} - \varphi_n\|_1 \to 0 \) and
\[
\|\varphi_n\|_2^2 \leq \lambda_n. \]
But the set \( A_n \) is closed with respect to the \( \mathcal{H}_1 \) norm, and hence \( \varphi_n \) is in \( A_n \) and thus \( \|\varphi_n\|_2^2 = \lambda_n \); this element therefore solves the minimum problem. The sequence of \( \lambda_n \) of necessity will form a non-decreasing sequence. We note in particular that from (2.1) we have
\[
(2.5) \quad \lambda_n > 1.
\]
To show that the set \( \{\varphi_n\} \) is doubly orthogonal, we note first that the set is by construction orthonormal in \( \mathcal{H}_1 \). We prove next that if \( (f, \varphi_m)_1 = 0 \) for
m = 1, 2, ..., n-1, then \((f, \varphi_n)_2 = \lambda_n(f, \varphi_n)_1\) and then use this to prove the third property of (2.3).

We proceed by induction.

Let \(f\) in \(H_2\) be given. Since \(\varphi_n\) is a solution of the minimum problem,

\[(2.6) \quad \|f\|^2_2 \geq \lambda_1 \|f\|^2_1,\]

and for any (complex) \(a\)

\[(2.7) \quad \|f + a \varphi_i\|^2_2 \geq \lambda_1 \|f + a \varphi_i\|^2_1.\]

This gives

\[\|f\|^2_2 + |a|^2 \lambda_1 + 2 \Re \overline{a} (f, \varphi_i)_2 \geq \lambda_1 \|f\|^2_1 + |a|^2 \lambda_1 + 2 \Re \overline{a} \lambda_1 (f, \varphi_i)_1,\]

or

\[\|f\|^2_2 - \lambda_1 \|f\|^2_1 \geq -2 \Re \overline{a} [(f, \varphi_i)_2 - \lambda_1 (f, \varphi_i)_1].\]

The left hand member of this inequality is non-negative by (2.6), and independent of \(a\). Since \(a\) may be given arbitrary complex values, this gives a contradiction unless

\[(f, \varphi_i)_2 = \lambda_1 (f, \varphi_i)_1.\]

If the assertion holds for \(n = 1, 2, ..., k-1,\) then \((f, \varphi_m)_1 = 0, m = 1, 2, ..., k-1,\) implies \((f, \varphi_m)_2 = 0\) for \(m = 1, ..., k-1,\) and the same proof as above with \(k\) replacing \(1\) in the subscripts proves that \((f, \varphi_k)_2 = \lambda_k (f, \varphi_k)_1.\)

Now let \(f\) in \(H_2\) be arbitrarily given. The above proof shows that \((f, \varphi_i)_2 = \lambda_1 (f, \varphi_i)_1.\) Set

\[a_n = (f, \varphi_n)_1\]

and
\[ f_n = f - \sum_{k=1}^{n-1} a_k \varphi_k. \]

Then \((f_n, \varphi_m)_1 = 0 \) for \( m = 1, 2, \ldots, n-1 \) and hence
\[(f, \varphi_n)_2 = (f, \varphi_n)_2 - \sum_{k=1}^{n-1} a_k (\varphi_k, \varphi_n)_2 \]
\[= (f_n, \varphi_n)_2 \]
\[= \lambda_n (f_n, \varphi_n)_1 \]
\[= \lambda_n (f, \varphi_n)_1 - \lambda_n \sum_{k=1}^{n-1} a_k (\varphi_k, \varphi_n)_1 \]
\[= \lambda_n (f, \varphi_n)_1 \]

which proves the third property of (2.3). In this calculation, we have \((\varphi_k, \varphi_n)_2 = 0\) since the \(\varphi_k\) form an orthonormal set in \(H_1\).

Finally, let us assume that \(\lambda_n \to \infty\) and show that the set \(\{\varphi_n\}\) is complete in \(H_2\). Let \(f\) be any element of \(H_2\). Set \(a_n = (f, \varphi_n)_1\) and
\[f_n = \sum_{k=1}^{n} a_k \varphi_k.\]

Then for any \(n\), \((f - f_n, \varphi_m)_1 = 0, \ m = 1, 2, \ldots, n-1,\) and hence
\[(2.8) \quad \lambda_n \|f - f_n\|_1^2 \leq \|f - f_n\|_2^2.\]

However, \((f, \varphi_n)_2 = \lambda_n a_n\), and \(\{\varphi_n/\sqrt{\lambda_n}\}\) is an orthonormal set in \(H_2\). Hence
\[\|f - f_n\|_2^2 \leq \|f\|_2^2,\]

and from (2.8) we see
\[\|f - f_n\|_1 \to 0.\]

But from theorem 1, \(g = f - \sum_{k=1}^{n} a_k \varphi_k\) is in \(H_2\) and
\[\lim_{n \to \infty} \|f - f_n\|_2 = \|g\|_2.\]

From the above calculation \(\|g\|_1 = 0\), and hence \(g = 0\), which gives
\[\lim_{n \to \infty} \|f - f_n\|_2 = 0,\]
which proves the completeness.

From the remarks made in paragraph 1, we note that $\lambda_n \to \infty$ thus implies that the space $H_2$ is separable. It would be interesting to know whether there exists an example of a doubly orthogonal set with the $\lambda_n$ bounded, or for that matter, if doubly orthogonal sets exist having any preassigned set of eigenvalues $\{\lambda_n\}$.

3. In this paragraph, the quasi-unicity of the doubly orthogonal sets will be discussed. The exact results we will obtain depend on the distribution of the $\lambda_n$. If we assume that $\lambda_n \to \infty$, then only a finite number of the $\lambda_n$ can have the same value, and therefore there exist integers $n_0 = 0 < n_1 < n_2 < \ldots$ such that

$$
\begin{align*}
\lambda_k &= \lambda_{n_j} \quad n_{j-1} < k \leq n_j \\
\lambda_{n_j} &< \lambda_{n_{j+1}}
\end{align*}
$$

(3.1)

That is, the integers $n_j$ divide the $\lambda_n$ into blocks of equal values.

**THEOREM 3.** Let $\{\varphi_n\}$ be a doubly orthogonal set in $H_1$ and $H_2$, constructed as in theorem 2. Let this set be such that the $\lambda_n \to \infty$, and satisfy (3.1). If $\{\psi_n\}$ is any set which is doubly orthogonal in $H_1$ and $H_2$ with
\((\psi_n, \psi_m)_1 = \delta_{nm}\)
\((\psi_n, \psi_m)_2 = \mu_n \delta_{nm}\)
\[1 < \mu_1 \leq \mu_2 \leq \ldots\]

and if \(\{\psi_n\}\) is complete in \(H_2\), then
\[\mu_k = \lambda_k, \quad k = 1, 2, \ldots,\]

and
\[\psi_k = \sum_{i=n_{j-1}}^{n_j} a_{ki} \varphi_i, \quad n_{j-1} < k \leq n_j,\]

where the constants \(a_{ki}\) satisfy
\[\sum_i a_{ki} \overline{a_{ji}} = \delta_{kj}, \quad j, k = 1, 2, \ldots.\]

That is, the \(\varphi_n\) are unique up to finite linear combinations within blocks of equal \(\lambda_n\).

PROOF. Since both \(\{\varphi_n\}\) and \(\{\psi_n\}\) are complete in \(H_2\), we may write
\[\psi_k = \sum_{i=n_{j-1}}^{n_j} a_{ki} \varphi_i\]
\[\varphi_k = \sum_{i=1}^{\infty} b_{ki} \psi_i.\]

Noting that \((\varphi_k, \psi_i)_1 = b_{ki}\) and \((\psi_i, \varphi_k)_1 = a_{ik}\) we have
\[b_{ki} = \overline{a_{ik}}.\]

Next, define a set of integers \(m_j\) similar to the \(n_j\). Let \(m_0 = 0 < m_1 < m_2 < \ldots\) be such that
\[\mu_k = \mu_{m_j}, \quad m_{j-1} < k \leq m_j\]
\[\mu_{m_j} < \mu_{m_{j+1}}\]

It is of course conceivable that one of the \(m_j\) (even \(m_i\)) might be infinite, i.e. that the set of \(\mu_k\) might all be identical from some point on. That such is not the case will appear in the course of the proof.
The proof of the theorem is accomplished by induction on the $n_j$. For convenience, we will list here the relations which make up a complete stage in the induction.

\begin{align}
3.7 & \quad \mu_k = \lambda_k \quad \text{all } k \leq n_r, \\
3.8 & \quad \psi_k = \sum_{i=n_r}^{n_j} a_{ki} \varphi_i \quad \text{all } j \leq r, \\
3.9 & \quad \varphi_k = \sum_{i=n_r}^{n_j} b_{ki} \psi_i \quad \text{all } k > n_r, \\
3.10 & \quad \psi_k = \sum_{i=n_r}^{n_j} a_{ki} \varphi_i \quad \text{all } k > n_r.
\end{align}

We will first prove these for $r = 1$, and then show that if they hold for $r = j$, they hold for $r = j+1$. This will complete the proof of the theorem, since (3.7) and (3.8) are merely equations (3.2) and (3.3) of the theorem. The condition on the coefficients $a_{ki}$ follows from (3.3) and the fact that both the $\varphi_k$ and the $\psi_k$ are orthonormal in $\mathbb{H}_1$.

We have for each $k \leq n$

$$\|\varphi_k\|_1^2 = \sum_{i} |b_{ki}|^2 = 1,$$

and

$$\|\varphi_k\|_2^2 = \lambda_i = \sum_{i=1}^{m_1} \mu_i |b_{ki}|^2 \geq \mu_1,$$

since the $\mu_i$ are increasing. Furthermore, equality holds if and only if $b_{ki} = 0$ for all $i > m_1$. However, from the minimum property of $\varphi_i$ we have

$$\|\psi_i\|_2^2 = \mu_i \geq \lambda_i.$$

Thus we conclude $\mu_i = \lambda_i$, and for all $k \leq n$,

\begin{align}
3.11 & \quad \varphi_k = \sum_{i=1}^{m_1} b_{ki} \psi_i.
\end{align}
Since the $\varphi_k$ and the $\psi_i$ both form linearly independent sets, we conclude from (3.11) that $m_i \geq n_i$, which proves (3.7) for $r = 1$.

Next, for each $k \leq n_i$
\[
\|\psi_k\|_2^2 = \sum_{i=1}^{\infty} |a_{ki}|^2 = 1,
\]
\[
\|\psi_k\|_2^2 = \mu_k = \sum_{i=1}^{\infty} \lambda_i |a_{ki}|^2 \geq \lambda_i \sum_{i=1}^{n_i} |a_{ki}|^2 + \lambda_{n_i+1} \sum_{i=n_i+1}^{\infty} |a_{ki}|^2 \geq \lambda_i,
\]
and since $\mu_k = \lambda_i$ and $\lambda_{n_i+1} > \lambda_i$, we conclude that
(3.12) \quad $a_{ki} = 0$, \quad $k \leq n_i$, $i > n_i$.

This proves (3.8) for $r = 1$. From (3.12) and (3.5), we also have
\[
b_{ki} = 0, \quad i \leq n_i, \quad k > n_i,
\]
which implies (3.9) for $r = 1$.

Now we have
\[
\psi_k = \sum_{i=1}^{n_i} a_{ki} \varphi_i, \quad k = 1, 2, \ldots, n_i.
\]
But the linear independence of the $\psi_k$ implies that
\[
\Delta_1 = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n_i} \\ a_{21} & a_{22} & \cdots & a_{2n_i} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n_1} & a_{n_2} & \cdots & a_{n_{n_i}} \end{vmatrix} \neq 0
\]
and thus the above system of equations can be solved to obtain
\[
\varphi_i = \sum_{k=1}^{n_i} b_{ik} \psi_k, \quad i = 1, 2, \ldots, n_i.
\]
From this, $(\psi_k, \varphi_i)_1 = 0$ for $k > n_i$, $i \leq n_i$, and
hence (3.10) follows for \( r = 1 \).

Now assume that (3.7) through (3.10) hold for \( r = j \). Let \( n_j < k \leq n_{j+1} \). Then using (3.9),
\[
\| \phi_k \|_1^2 = \sum_{i=n_j+1}^{m_{j+1}} |b_{ki}|^2 = 1,
\]
and
\[
\| \phi_k \|_2^2 = \lambda_{n_{j+1}} \sum_{i=n_{j+1}}^{m_{j+1}} |b_{ki}|^2 = \lambda_{n_{j+1}}.
\]  
(3.13)

Again, equality holds if and only if \( b_{ki} = 0 \) for all \( i > m_{j+1} \). However, if \( m \leq n_j \), then from (3.10), \((\psi_k, \phi_m) \|_1 = 0\) and from the minimum property
\[
\| \psi_k \|_2^2 = \mu_k \geq \lambda_{n_{j+1}}, \quad n_j < k \leq n_{j+1}.
\]
This with (3.13) gives
\[
\mu_{n_{j+1}} = \lambda_{n_{j+1}}
\]  
(3.14)

and
\[
\phi_k = \sum_{i=n_j+1}^{m_{j+1}} b_{ki} \psi_i.
\]  
(3.15)

Since (3.15) must hold for all \( k, n_j < k \leq n_{j+1} \), we conclude as before that \( m_{j+1} \geq n_{j+1} \), which with (3.14) proves (3.7) for \( r = j+1 \).

Next we compute for \( n_j < k \leq n_{j+1} \)
\[
\| \psi_k \|_1^2 = \sum_{i=n_j+1}^{m_{j+1}} |a_{ki}|^2 = 1,
\]
\[
\| \psi_k \|_2^2 = \mu_k = \sum_{i=n_j+1}^{m_{j+1}} \lambda_i |a_{ki}|^2 \geq \lambda_{n_{j+1}},
\]
with equality holding if and only if \( a_{ki} = 0 \) for \( i > n_{j+1} \). But equality does hold by (3.14), hence (3.8) is proved for \( r = j+1 \). From this and (3.5) and
(3.9) for \( r = j \) we have (3.9) for \( r = j+1 \).

Finally, (3.10) follows just as before by solving the set of equations
\[
\psi_k = \sum_{i=n_j}^{n_{j+1}} a_{ki} \varphi_i, \quad n_j < k \leq n_{j+1}
\]
for the \( \varphi_i \).

**COROLLARY 3.1.** Under the conditions of theorem 3, \( \{ \psi_n \} \) is also a set of solutions of the minimum problem and \( \{ \psi_n \} \) is complete in \( H_1 \) if and only if \( \{ \varphi_n \} \) is.

**PROOF.** The fact that the \( \psi_n \) satisfy the minimum problem is clear. For the completeness, it suffices to note that from (3.3), if \( n_j < k \leq n_{j+1} \), and if \( a_1, a_2, \ldots, a_k \) are given, there exist \( b_i \), such that
\[
\sum_{i=1}^{k} a_i \varphi_i = \sum_{i=1}^{n_{j+1}} b_i \psi_i.
\]

On the basis of theorem 3, we may demonstrate the existence of a canonical decomposition of any element of \( H_2 \). First we make some definitions.

If \( \{ \varphi_n \} \) is a doubly orthogonal set in \( H_1 \) and \( H_2 \), and if we assume \( \lambda_n \to \infty \), then by the \( j \)-th \( \lambda \)-set of the \( \varphi_n \) we mean the set of \( \varphi_k \) with \( n_{j-1} < k \leq n_j \), i.e. the set of \( \varphi_k \) which have the \( j \)-th distinct value of \( \lambda_k \). By the \( j \)-th \( \lambda \)-space we mean the finite dimensional subspace spanned by the \( j \)-th \( \lambda \)-set.

Theorem 3 can thus be interpreted as stating that the collection of \( \lambda \)-spaces depends only on \( H_1 \) and \( H_2 \) and
not on the particular set \( \{ \varphi_n \} \).

If \( f \) is in \( H_2 \) (or in \( H_1 \) and \( \{ \varphi_n \} \) is complete in \( H_1 \)), then by the \( \lambda \)-decomposition of \( f \) we mean a decomposition of the form

\[
(3.16) \quad f = \sum_{j=1}^{\infty} f_j
\]

where each \( f_j \) is an element of the \( j \)-th \( \lambda \)-space, i.e. if \( f = \sum_{n=1}^{\infty} a_n \varphi_n \) then

\[
f_j = \sum_{k=n_j+1}^{n_j} a_k \varphi_k.
\]

From theorem 3 we have immediately:

**COROLLARY 3.2.** If \( \{ \varphi_n \} \) is complete in \( H_2 \) (in \( H_1 \)) and if \( f \) is in \( H_2 \) (\( H_1 \)) then the \( \lambda \)-decomposition (3.16) is unique, converges in the norm in \( H_2 \) (\( H_1 \)) and furthermore there exists some set \( \{ \psi_n \} \) doubly orthogonal in \( H_1 \) and \( H_2 \) such that

\[
(3.17) \quad f = \sum_{j=1}^{\infty} a_{n_j} \psi_{n_j},
\]

i.e. such that the orthogonal expansion is itself the \( \lambda \)-decomposition. The \( \psi_n \) may be so chosen that any or all of the \( a_{n_j} \) are real.

**PROOF.** The proof is clear when we note that 

\[ e^{i\theta} f_j/\| f_j \|_1 \]

is a solution of the minimum problem for the \( n_{j-1} + 1 \)-st \( \varphi \).

4. Let the space \( H \) be a space of complex valued functions on a set \( D \). We assume that there exists a real, non-negative, finite valued function \( m(Z) \) on \( D \)
such that for any \( f \) in \( H \) and any \( Z \) in \( D \)

\[
(4.1) \quad |f(Z)| \leq m(Z) \| f \|. 
\]

Let \( \{\varphi_n\} \) be a complete orthonormal sequence in \( H \). Define

the kernel function

\[
(4.2) \quad K(Z, W) = \sum_{n=1}^{\infty} \varphi_n(Z) \overline{\varphi_n(W)}. 
\]

As a series of complex numbers, (4.2) is absolutely
convergent for each \( Z \) and \( W \) in \( D \), for by lemma 1.1
and Schwarz's inequality

\[
\sum_{k=1}^{n} |\varphi_k(Z) \varphi_k(W)| \leq \left( \sum_{k=1}^{n} |\varphi_k(Z)|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} |\varphi_k(W)|^2 \right)^{\frac{1}{2}} 
\]

\[
\leq m(Z) \cdot m(W). 
\]

As defined by (4.2), the kernel function satisfies the
law of Hermitian symmetry

\[
(4.3) \quad K(Z, W) = \overline{K(W, Z)}. 
\]

Using this and theorem 1.1 along with lemma 1.1, we see
that for fixed \( W_0 \) in \( D \)

\[
(4.4) \quad \begin{cases} 
K(Z, W_0) \text{ is in } H, \\
K(W_0, Z) \text{ is in } H. 
\end{cases} 
\]

The kernel function has the reproducing property

\[
(4.5) \quad (f(W), K(W, Z)) = f(Z) \quad \text{for } f \text{ in } H. 
\]

Here, as elsewhere, the inner product is taken with
respect to the variable which appears in both positions.

The proof of (4.5) is as follows. If \( f(Z) = \sum_{n=1}^{\infty} a_n \varphi_n(Z) \),
then

\[
(f(W), K(W, Z)) = (\sum_{n=1}^{\infty} a_n \varphi_n(W), \sum_{k=1}^{\infty} \varphi_k(W) \overline{\varphi_k(Z)}) 
\]

\[
= \sum_{k=1}^{\infty} \varphi_k(Z) (\sum_{n=1}^{\infty} a_n \varphi_n(W), \overline{\varphi_k(W)}) 
\]

\[
= \sum_{k=1}^{\infty} a_k \varphi_k(Z) 
\]

\[
= f(Z). 
\]
The kernel function $K(Z,W)$ depends only on the space $H$ and not on the particular orthonormal set used in the definition (4.2). That is, $K$ is uniquely characterized by (4.3), (4.4), and (4.5). For suppose there exist two functions $K(Z,W)$ and $K'(Z,W)$ with properties (4.3), (4.4), and (4.5). Then for any $Z$ and $W$ in $D$

$$K'(Z,W) = (K'(T,W), K(T,Z))$$

$$= (K(T,Z), K'(T,W))$$

$$= K(W,Z)$$

$$= K(Z,W).$$

Finally, it may be noted that the kernel function is also uniquely determined by the following minimum problem. If $W$ is in $D$ and $f$ in $H$ is such that $f(W) = 1$ and $\|f\|$ is a minimum, then it can be shown that

$$f(Z) = K(Z,W)/K(W,W).$$

See for example, Bergman [3] p. 21. The proof which he gives there for spaces of analytic functions applies equally well in the more general case.

5. The existence proof, theorem 2, is unsuited for actual computation of doubly orthogonal sets, or for that matter for demonstrating that a given element is one of such a set. In the cases which are of interest to us, the kernel function defined above may be considered as being accessible. We wish to show in
this paragraph how a doubly orthogonal set of functions can be constructed by means of a suitable operation using the kernel function.

We assume that the function spaces $H_1$ and $H_2$ are given satisfying the requirements of paragraph 2. We further assume that the set of doubly orthogonal functions of theorem 2 satisfy the requirement that $\lambda_n \rightarrow \infty$; thus $\{\phi_n\}$ is complete in $H_2$. We also assume that the function space $H_2$ satisfies the requirement (4.1) which insures the existence of the Kernel function $K_2(Z,W)$ of $H_2$. Let $G$ be the domain of definition of the functions of $H_1$ and $B$ the domain of definition of $H_2$. We shall assume $G \subseteq B$.

By our assumptions, the set $\{\phi_n/\sqrt{\lambda_n}\}$ is a complete orthonormal set in $H_2$. Hence

\begin{equation}
K_2(Z,W) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \phi_n(Z) \overline{\phi_n(W)}, \quad Z, W \text{ in } B.
\end{equation}

For any $f$ in $H_1$, introduce the transformation

\begin{equation}
T_1f = (f(W), K_2(W,Z))_1.
\end{equation}

For any $f$ in $H_1$, $f = \sum_{n=1}^{\infty} a_n \phi_n + f_0$ where $(f_0, \phi_n)_1 = 0$ for all $n$, and $\sum_{n=1}^{\infty} |a_n|^2 < \infty$. Thus

\begin{equation}
T_1f = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n} \phi_n.
\end{equation}

Simple calculation gives

\begin{equation}
\| T_1f \|_1^2 = \sum_{n=1}^{\infty} \frac{|a_n|^2}{\lambda_n^2}
\end{equation}

\begin{equation}
\| T_1f \|_2^2 = \sum_{n=1}^{\infty} \frac{|a_n|^2}{\lambda_n}.
\end{equation}

Since all $\lambda_n > 1$, the last series converges, and hence
if $f$ is in $H_1$ then $T_1 f$ is in $H_2$.

This transformation gives us a start on the construction of the doubly orthogonal sets. The minimum problem of theorem 2 is essentially that of finding an $f$ in $H_2$ which minimizes $\| f \|_2 / \| f \|_1$. However, in this direction we may show:

**Lemma 5.1.** If $f$ is in $H_2$ then

$$\frac{\| T_1 f \|_2}{\| T_1 f \|_1} \leq \frac{\| f \|_2}{\| f \|_1}$$

with equality holding if and only if the $\lambda$-decomposition of $f$ contains only a single term.

**Proof:** Using (5.4) this lemma is equivalent to the statement that if $f = \sum a_n \phi_n$ then

$$\sum a_n^2 \sum \frac{|a_n|^2}{\lambda_n} \leq \sum \lambda_n |a_n|^2 \sum \frac{|a_n|^2}{\lambda_n^2}$$

with equality holding if and only if all of the $\lambda_n$ associated with non-zero $a_n$ are equal. But this follows immediately from Tchebyshev's inequality (see [6] p. 43):

If $p_n > 0$ and if $A_n$ and $B_n$ are oppositely ordered, then

$$\frac{\sum p_n A_n B_n}{\sum p_n} \leq \frac{\sum p_n A_n}{\sum p_n} \cdot \frac{\sum p_n B_n}{\sum p_n}$$

with equality holding if and only if for all $n$ and $m$,

$p_n p_m (A_n - A_m)(B_n - B_m) = 0$. Here, we set $p_n = |a_n|^2$, $A_n = \lambda_n$, $B_n = 1/\lambda_n^2$ and the desired result follows.
Iteration of the transformation $T_1$ would lead to a sequence of functions tending toward zero, as can be seen from (5.4) and the fact that $\lambda_i > 1$. We correct this, and unfortunately at the same time spoil the linearity of $T_1$ by defining the transformation

$$\text{(5.5)} \quad \mathbf{Tf} = T^{(n)}f = \frac{\|f\|_1}{\|T_1^nf\|_1} T_1f.$$  

This transformation is defined for all $f$ in $H_1$ when we introduce the convention $T^0 = 0$. The function $\mathbf{Tf}$ is merely a (real) constant times $T_1f$, the constant being so chosen that

$$\text{(5.6)} \quad \|\mathbf{Tf}\|_1 =\|f\|_1.$$ 

We also introduce the iterates of $T$ by means of the recursion relation

$$T^{(n)}f = Tn^{-1}f$$

and note that $Tf$ is merely a constant times the $n$-th iterate of $T_1$.

Lemma 5.1 does not allow us to prove convergence of the iterates of $T$, but we have

**Lemma 5.2.** If $f$ is in $H_2$ then

$$\lim_{n \to \infty} T^{(n)}f = g$$

exists, $g$ in $H_2$, the convergence being in the norm of $H_2$. If

$$f = \sum_{j=1}^{\infty} f_j$$

is the $\lambda$-decomposition of $f$, then

$$g = \frac{\|f\|_1}{\|f_r\|_1} f_r.$$
PROOF. By using corollary 3.2 and making a simplification in the subscripts, it obviously suffices to prove:

If \( \{\varphi_n\} \) is a doubly orthogonal set such that
\[
5.7 \quad f = \sum_{j=0}^{\infty} a_j \varphi_j, \quad a_j > 0 \text{ for all } j, \quad \sum_{j=0}^{\infty} a_j^2 = 1,
\]
and
\[
5.8 \quad \lambda_r < \lambda_{r+1},
\]
then \( T^n f \rightarrow \varphi_r \) in the norm of \( H_2 \).

Each \( T^n f \) is in \( H_2 \), hence we may set
\[
5.9 \quad T^n f = \sum_{j=0}^{\infty} a_{nj} \varphi_j.
\]

This representation is allowable, since from (5.3) no \( \varphi_j \) which does not appear in \( f \) will appear in \( T f \).

From (5.6) we have
\[
5.10 \quad \sum_{j=0}^{\infty} |a_{nj}|^2 = 1.
\]

Also from (5.3) and (5.7) we have
\[
5.11 \quad a_{nj} > 0, \quad \text{all } n,j.
\]

Each transformation \( T \) is within a multiplicative constant the transformation \( T_1 \). Hence from (5.3) and (5.10) we have
\[
a_{nj} = \frac{a_j}{\lambda_j} \frac{\lambda_j^n}{\left(\sum_{k=0}^{\infty} a_k^2 / \lambda_k^{2n}\right)^{1/2}}.
\]

Hence, from (5.7) and the fact that the \( \lambda_n \) are increasing
\[
\sum_{j \in r^*} a_{nj}^2 = \frac{\sum_{j \in r^*} a_j^2 / \lambda_j^{2n}}{\sum_{k \in r} a_k^2 / \lambda_k^{2n}}
\]
\[
\leq \frac{\lambda_r^{-2n} \sum_{j \in r^*} a_j^2}{\sum_{k \in r} a_k^2 / \lambda_k^{2n}}
\]
\[
= (\lambda_r / \lambda_{r+1})^{2n} (a_r^{-2} - 1).
\]
Noting (5.8), this gives
(5.12) \[\sum_{j \in r^*} a_{nj}^2 \to 0 \text{ as } n \to \infty.\]
With (5.10), this shows
(5.13) \[a_{nr} \to 1 \text{ as } n \to \infty.\]

From (5.3) and (5.5) we also have
\[
a_{n+1,j} = (a_{nj} / \lambda_j) \left( \sum_{n \in r} a_{nk} / \lambda_k^{2n} \right)^{-\frac{1}{2}}
\]
and hence
\[
\sum_{j \in r^*} \lambda_j a_{n+1,j}^2 = \frac{\sum_{j \in r^*} a_{nj}^2 / \lambda_j}{\sum_{kn \in r} a_{nk}^2 / \lambda_k^{2n}}
\]
\[
\leq \frac{\lambda_r^{-1} \sum_{j \in r^*} a_{nj}^2}{a_{nr}^2 / \lambda_r^{2n}}
\]
\[
= \lambda_r^2 \lambda_{r+1} a_{nr}^{-2} \sum_{j \in r^*} a_{nj}^2.
\]

Thus from (5.12) and (5.13)
(5.14) \[\sum_{j \in r^*} \lambda_j a_{nj}^2 \to 0 \text{ as } n \to \infty.\]

Finally, using (5.14) and (5.13) we have
\[
\| T_{nr} - \varphi_r \|_2^2 = \lambda_r (1 - a_{nr})^2 + \sum_{j \in r^*} \lambda_j a_{nj}^2
\]
\[
\to 0 \text{ as } n \to \infty.
\]
which proves the lemma.

We now describe a scheme (essentially a diagonal process) for the construction of doubly orthogonal sets. Suppose a sequence \( \{ \psi_n \} \) of functions in \( H_1 \) is given. Set

\[
\varphi_n = \mu_n / \| \mu_n \|_1
\]

where

\[
\begin{align*}
\mu_1 &= \lim_{n \to \infty} T^{(n)} \psi_1 \\
\mu_2 &= \lim_{n \to \infty} T^{(n)} \left[ \psi_1 - \varphi_1 ( \psi_1, \varphi_1 )_1 \right] \\
\mu_3 &= \lim_{n \to \infty} T^{(n)} \left[ \psi_2 - \sum_{j=1}^{n} \varphi_j ( \psi_j, \varphi_j )_1 \right] \\
\mu_4 &= \lim_{n \to \infty} T^{(n)} \left[ \psi_2 - \sum_{j=1}^{n} \varphi_j ( \psi_j, \varphi_j )_1 \right] \\
\mu_p &= \lim_{n \to \infty} T^{(n)} \left[ \psi_m - \sum_{j=1}^{n} \varphi_j ( \psi_j, \varphi_j )_1 \right].
\end{align*}
\]

(5.15)

and in general, if \( \varphi_1, \varphi_2, \ldots, \varphi_{p-1} \) have been constructed, with \( \frac{1}{2}k(k-1) < p \leq \frac{1}{2}k(k+1) \)

set \( p = \frac{1}{2}k(k-1) + m \) and

It will be observed that the diagonal nature of this scheme consists of extracting linearly independent functions from the sequence \( \psi_j \) with \( j = 1; 1, 2; 1, 2, 3; 1, 2, 3, 4; \) etc. We have

THEOREM 4. If the sequence \( \{ \psi_n \} \) is complete in \( H_2 \), then the set of all non-vanishing \( \varphi_n \) generated by the scheme (5.15) will be a doubly orthogonal set, complete in \( H_2 \).
PROOF. By lemma 5.2, the resulting set \( \{ \varphi_n \} \) will be functions of some doubly orthogonal set. It is necessary to show only that the set is complete.

To show this, we must show that there will be \( n_r - n_{r-1} \) linearly independent \( \varphi_j \) in the \( r \)-th \( \lambda \)-space. Suppose that after \( p \) stages, \( m \) of the \( \varphi_j \) have been produced in the \( r \)-th \( \lambda \)-space, with \( m < n_r - n_{r-1} \). For convenience, let us suppose these are \( \varphi_1, \varphi_2, \ldots, \varphi_m \).

Now the \( r \)-th \( \lambda \)-space has dimension \( n_r - n_{r-1} \), and the set \( \{ \psi_n \} \) is complete, hence the \( r \)-th term of the \( \lambda \)-decomposition of
\[
\psi_n = \sum_{j=1}^{m} \varphi_j (\psi_n, \varphi_j)_\lambda
\]
cannot be zero for all \( n \). But this means that after a sufficient number of stages (but at least by the time one such \( \psi_n \) has been considered \( r \) times) another \( \varphi_j \) in the \( r \)-th \( \lambda \)-set will be produced. Hence we conclude that after at most a finite number of stages, the set of \( \varphi_n \)'s will contain a base for the \( r \)-th \( \lambda \)-space, and the theorem is proved.

One final remark may be made about the transformation \( T_1 \). If the set \( \{ \varphi_n \} \) is complete in \( H_2 \), then it spans a closed subspace of \( H_1 \), say \( H_0 \). If \( H_0 \) is properly contained in \( H_1 \), then \( H_1 = H_0 \oplus K_0 \) and \( K_0 \) has the characteristic property that it is the null
space of the transformation $T_1$, i.e. $f$ is in $K_0$ if and only if $T_1 f = 0$.

6. In the above development a number of assumptions were made. In order to apply these results to actual function spaces these assumptions must be verified. The space must be shown to be a Hilbert space, complete and satisfying (1.1). When we introduce the two spaces $H_1$ and $H_2$, properties (2.1) and (2.2) must be verified before we can assert the existence of a doubly orthogonal set. The completeness of this set in $H_2$ follows when we prove $\lambda_n \to \infty$. For the development of paragraph 5 we must have the existence of the kernel function, and this requires the proof of property (4.1).

We consider now a function space of analytic functions on a Riemann surface. Let $B$ be a domain contained properly in a closed Riemann surface $\mathcal{R}$ and let $B$ be such that it has as its boundary on $\mathcal{R}$ a finite number of simple closed Jordan curves. Let $\gamma$ be a uniform analytic covariant in $B$, having at most a finite number of zeros and poles there. If we fix a parameter disc about each pole of $\gamma$, and for a given $\epsilon > 0$ let $B_\epsilon$ be the domain $B$ with the set $|z| \leq \epsilon$ removed at each pole, then we demand that $\gamma$ be such that for each $\epsilon > 0$

\begin{equation}
(6.1) \quad \iint_{B_\epsilon} |\gamma|^2 dA < \infty.
\end{equation}
Such a covariant will always exist, since we may for example use any Abelian covariant on $\mathcal{R}$ to satisfy these requirements.

If $G$ is any domain contained in $B$, or $B$ itself, by $G^*$ we will denote the domain $G$ with all of the zeros of $\gamma$ punched out.

The Hilbert space we shall consider is the space of all uniform meromorphic functions in $B$ such that

$$\iint_B |f|^2 |\gamma|^2 \, dA < \infty.$$  

This space will be denoted by $L^2(B; \gamma)$ or more simply by $L^2(B)$ where no confusion can arise. A function $f$ in $L^2(B)$ must have a zero of at least order $n$ at a point where $\gamma$ has a pole of order $n$, and at a point where $\gamma$ has a zero of order $m$ can have a pole of order at most $m$. In particular, $f$ in $L^2(B)$ is regular and uniform in $B^*$.

The space $L^2(B)$ is always non-empty under these restrictions. Indeed, it is infinite dimensional, since there are an infinite number of linearly independent rational functions on $\mathcal{R}$ satisfying the requirement (6.2). Constant functions are contained in $L^2(B)$ if and only if $\gamma$ has no poles.

The inner product is defined as

$$(f, g) = \iint_B f \overline{g} |\gamma|^2 \, dA.$$  

Schwarz's inequality shows that this inner product is defined for any pair of functions $f$ and $g$ in $L^2(B)$. 
The properties (1.1) follow immediately, the triangle inequality being merely Minkowski's inequality. We defer proof of the completeness of the space until after property (4.1) has been proved.

**Lemma 6.1.** There exists a continuous, non-negative, real valued function \( m(Z) \) in \( B^* \) such that if \( f \) is in \( L^2(B; \gamma) \) then for each \( Z \) in \( B^* \)

\[
(6.4) \quad |f(Z)| \leq m(Z) \|f\|.
\]

Furthermore, if \( z \) is a local parameter about a pole or zero of \( \gamma \), then \( m(Z(z)) \gamma \) is bounded in a neighborhood of \( z = 0 \).

**Proof.** We first observe that if \( g(z) \) is regular in the circle \( |z| < R \) and \( g(z) = \sum_{n=0}^{\infty} a_n z^n \) there, then

\[
\iint_{|z| < R} |g(z)|^2 \, dx \, dy = \iint_{r < R} \left| \sum_{n} a_n r^n e^{in \theta} \right|^2 r \, dr \, d\theta = 2 \pi \sum_{n} (2n+2)^{-1} |a_n|^2 \geq \pi R^2 |a_0|^2.
\]

That is

\[
(6.5) \quad \iint_{|z| < R} |g(z)|^2 \, dx \, dy \geq \pi R^2 |g(0)|^2.
\]

Now fix some point \( P \) in \( B \) which is neither a pole nor a zero of \( \gamma \) and consider

\[
(6.6) \quad w(Z) = \int_{\gamma} Z \, dZ.
\]

The function \( w(Z) \) will not in general be single valued, but if \( Z_0 \) is any point of \( B^* \) at which \( \gamma \) is regular,
then
\[(6.7) \quad z = w(Z) - w(Z_0)\]
will be regular, single valued, and schlicht in some neighborhood of \(Z_0\). Thus, \(z\) may be used as a local parameter at \(Z_0\). There will exist some \(R = R(Z_0) > 0\) such that \(|z| < r\) is a parameter neighborhood of \(Z_0\) for every \(r < R\), but will not be for any \(r > R\).
(The possibility that \(R = \infty\) has been eliminated by our assumptions on the domain \(B\).)

If \(f\) is in \(L^2(B)\) and \(Z_0\) is as above, we have
\[\|f\|^2 \leq \iint_{|z| < R} |f|^2 \gamma^2 \, dA = \iint_{|z| < R} |f|^2 \, dx \, dy\]
where we put \(z = x + iy\). Then from (6.5) we have
\[\|f\|^2 \leq \pi R^2 \|f(Z_0)\|^2\]
which is
\[(6.8) \quad |f(Z_0)| \leq m(Z_0) \|f\|\]
where \([m(Z_0)]^2 = 1/\pi R^2\). The function \(m(Z)\) is thus defined and positive for every point \(Z\) in \(B^*\) where \(\gamma\) does not have a pole.

First we show the continuity of \(m(Z)\). Let \(Z_0\) be in \(B^*\), \(\gamma(Z_0) \neq \infty\), let \(z\) be the local parameter (6.7) about \(Z_0\), and let \(Z_1\) be any point such that \(|z(Z_1)| < \epsilon\), \(0 < \epsilon < R(Z_0)\). Then clearly
\[R(Z_0) - \epsilon \leq R(Z_1) \leq R(Z_0) + \epsilon\]
and we conclude that \(m(Z)\) is continuous at \(Z_0\).

Now let \(Z_0\) be a pole or a zero of \(\gamma\). Choose a local parameter \(z\) about \(Z_0\) such that \(\gamma = z^2\) (i.e. \(\gamma\)
has a zero of order $n$ or a pole of order $-n$) in the parameter disc $|z| \leq 1$. Then for any $f$ in $L^2(B)$, $f \gamma$ is regular; hence in this parameter disc
\[
|f(Z(z))| \leq \max_{|z| = 1} |f(Z(z))z^n|
\leq M\|f\|
\]
where $M = \max m(Z(z))$ on $|z| = 1$. But then
\[
|f(Z(z))| \leq M|z^{-n}|\cdot\|f\|.
\]
Since this holds for all $f$ in $L^2(B)$, we can redefine $m(Z)$ to be $M|z^{-n}|$ in the disc $|z| \leq \frac{1}{2}$, the function constructed earlier on and outside $|z| = 1$, and continuous in the annulus $\frac{1}{2} \leq |z| \leq 1$ but always greater than or equal to the minimum of the two functions. With such a definition, the lemma is proved.

At this point, it is convenient to introduce the following terminology. A collection of functions $\{f_\alpha(Z)\}$ is called subuniformly bounded in a domain $B$ if the set is uniformly bounded in every compact subset of $B$. A sequence of functions $\{f_n(Z)\}$ is said to converge subuniformly to a function $g(Z)$ in a domain $B$ if the sequence converges uniformly in every compact subset of $B$.

**Lemma 6.2.** A family $\mathcal{F}$ of functions in $L^2(B)$ which is bounded in the norm is a normal family in $B^*$. 
PROOF. Let $K$ be any compact subset in $B^*$. Then the function $m(Z)$ of lemma 6.1, being continuous, is bounded in $K$, say by $M$. Then for any $f$ in and any $Z$ in $K$, from lemma 6.1

$$|f(Z)| \leq m(Z) \|f\| \leq M \|f\|.$$ 

Thus, since the family is bounded in the norm, it is subuniformly bounded in $B^*$, and by Montel's theorem it is therefore a normal family.

**Lemma 6.3.** If a sequence of functions $f_n$ in $L^2(B)$ converges in the norm to $g$ in $L^2(B)$, then the sequence converges to $g$ subuniformly in $B^*$.

PROOF. Again, let $K$ be a compact subset of $B^*$ and let $m(Z) \leq M$ for all $Z$ in $K$. Then for any $Z$ in $K$

$$|f_n(Z) - g(Z)| \leq M \|f_n - g\|$$

from which the desired result follows.

**Lemma 6.4.** The space $L^2(B)$ is complete.

PROOF. Let the sequence $\{f_n\}$ be Cauchy in $L^2(B)$. Then the sequence is bounded in the norm and by lemma 6.2 is a normal family in $B^*$. Thus there exists a subsequence which converges subuniformly to a function $g(Z)$, regular and uniform in $B^*$. But as in the proof of lemma 6.3, since $\{f_n\}$ is Cauchy in the norm, it
is subuniformly convergent in \( B^* \), and hence the entire sequence converges subuniformly to \( g \).

We must show next that \( g \) is in \( L^2(B) \), and to do this we must first establish that \( g(z) \) is meromorphic with poles of the proper order. Let \( Z_o \) be a point of \( B \) at which \( \gamma \) has a zero of order \( n \), and let \( z \) be a local parameter at \( Z_o \). Then the functions \( f_k z^n \) are all regular in a neighborhood of \( Z_o \) and converge in the punctured neighborhood to \( g z^n \). Thus \( g z^n \) is regular at \( Z_o \) and hence \( g \) has a pole of order at most \( n \) there.

Let \( K \) be a compact subset of \( B \). Then

\[
\iint_K |g|^2 |\gamma|^2 \, dA = \lim_{n \to \infty} \iint_K |f_n|^2 |\gamma|^2 \, dA
\]

\[
\leq \lim_{n \to \infty} \|f_n\|^2
\]

and letting \( K \to B \) we see that \( g \) is in \( L^2(B) \).

Finally, to show that \( f_n \to g \) in the norm, let \( \varepsilon > 0 \) be given. Choose \( N \) such that \( n, m > N \) implies

\[
\|f_n - f_m\| < \varepsilon/3,
\]

and suppose that \( n > N \). Choose \( K = K(n) \) a compact subset of \( B^* \) such that

\[
(6.9) \quad \|f_n - g\| \leq \left[ \iint_K |f_n - g|^2 |\gamma|^2 \, dA \right]^\frac{1}{2} \leq \varepsilon/3.
\]

Now \( f_p \to g \) uniformly on \( K \), hence there exists an \( m > n \) such that

\[
\left[ \iint_K |f_m - g|^2 |\gamma|^2 \, dA \right]^\frac{1}{2} < \varepsilon/3.
\]

But then from Minkowski's inequality
\[
\left(\sqrt{\sum_{k} |f_n - g|^2 \gamma^2 dA}\right)^{1/2} \leq \left[ \sqrt{\sum_{k} |f_n - f_m|^2 \gamma^2 dA} \right]^{1/2} + \left[ \sqrt{\sum_{k} |f_m - g|^2 \gamma^2 dA} \right]^{1/2} \\
\leq \|f_n - f_m\| + \epsilon/3 \\
\leq 2\epsilon/3 ,
\]

and this together with (6.9) proves the lemma.

Now let \( B \) be a domain of \( \mathcal{R} \) satisfying the above conditions and let \( G \) be a subset of \( B \) with the properties:

(6.10 a) \( G \) is either a domain or a finite number of domains with disjoint closure, each having as its boundary a finite number of disjoint simple closed Jordan curves.

(6.10 b) The closure of \( G \) is contained in \( B \).

In paragraph 7, we will also require that \( G \) satisfy:

(6.10 c) Any point of the boundary of \( G \) can be connected to some point of the boundary of \( B \) by a Jordan curve which except for its end points lies entirely in \( B - \mathcal{G} \).

With the conditions (6.10 a) and (6.10 b) satisfied, and a covariant \( \gamma \) fixed, we will let \( H_1 \) of paragraph 2 be the space \( L^2(G; \gamma) \) and \( H_2 \) the space \( L^2(B; \gamma) \). We observe that \( L^2(B) \) is indeed a submanifold of \( L^2(G) \) and that the property (2.1) holds.

To show that (2.2) holds, let \( \{f_n\} \) be a bounded
sequence in $L^2(B)$. From this sequence we may extract a subsequence such that
\[
\lim_{k \to \infty} \|f_{n_k}\|_B = \lim_{n \to \infty} \inf \|f_n\|_B = b.
\]
But then, by lemma 6.2, we can extract a further subsequence which converges subuniformly in $B^*$ to some function $f_0(Z)$. Nothing is lost by assuming that this is the subsequence $\{f_{n_k}\}$ itself. However, $f_0$ is in $L^2(B)$, for it can be shown to be meromorphic just as in the proof of lemma 6.4, and if $K$ is any compact subset of $B$
\[
\|f_0\|_K = \lim_{k \to \infty} \|f_{n_k}\|_K \leq \lim_{k \to \infty} \|f_{n_k}\|_B = b,
\]
and letting $K \to B$, we have the required result
\[
\|f_0\|_B \leq b.
\]
Finally, since $F$ is a compact subset of $B$, we have
\[
\|f_n - f_0\|_G \to 0
\]
and the proof of (2.2) is complete.

Theorem 2 now holds and we have the existence of a doubly orthogonal set $\{\varphi_n\}$ in $L^2(G)$ and $L^2(B)$. This set is complete in $L^2(B)$ as can be seen from theorem 2 and

**LEMMA 6.5.** If $G$ and $B$ satisfy (6.10 a,b) and is a doubly orthogonal set in $L^2(G)$ and $L^2(B)$ constructed as in theorem 2, then
\[
(6.11) \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty
\]
and thus in particular, $\lambda_n \to \infty$. 

PROOF. The set \( \{ \varphi_n / \sqrt{\lambda_n} \} \) is an orthonormal set in \( L^2(B) \), hence from lemma 1.1
\[
\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \sum_{n=1}^{\infty} \int_{G} \frac{1}{\lambda_n} |\gamma|^2 dA \leq \int_{G} \left[ m(Z) \right]^2 |\gamma|^2 dA
\]
and since \( G \subset B \), lemma 6.1 shows that the last integral is finite.

The above proof shows that a complete orthonormal set always exists in \( L^2(B) \) for any \( B \) satisfying the requirements listed on page 25, for we can construct the doubly orthogonal set in \( L^2(G) \) and \( L^2(B) \) where \( G \) is a parameter disc contained in \( B \).

The results of this paragraph may be summarized in the following theorem:

THEOREM 5. If \( G \) and \( B \) satisfy (6.10 a, b) and a covariant \( \gamma \) is fixed with the properties listed on page 25, then there exists a doubly orthogonal set in \( L^2(G; \gamma) \) and \( L^2(B; \gamma) \) with \( \sum_{n=1}^{\infty} 1/\lambda_n < \infty \) and hence complete in \( L^2(B; \gamma) \).

7. In this paragraph we wish to discuss the completeness of the set of doubly orthogonal functions in \( L^2(G) \). Lemma 6.5 proves the completeness in \( L^2(B) \) but leaves the rest of the question open. It would seem that the requirements (6.10) would be sufficient
to prove the completeness in $L^2(G)$, but unfortunately, at least at present, further restrictions must be added.

Before going into the proofs proper, we give some results which are needed. The first is a theorem of Behnke and Stein [2, theorem 5] which can be stated as follows.

**THEOREM 6.** If the domain $B_1$ contained in a Riemann surface $\mathcal{R}$ satisfies the requirements listed on page 25, $G_1 \subset B_1$ satisfies the requirements (6.10 a, b, c), and if $K$ is a compact subset of $G_1$, then every function $f(z)$ regular and uniform in $G_1$ can be approximated uniformly in $K$ by a function regular and uniform in $B_1$.

This theorem, which is essentially an extension of Runge's theorem, is proved in the paper cited in almost the form given here. The statement of the theorem has merely been simplified somewhat to fit the case which we are considering.

We also need

**LEMMA 7.1.** If the domain $B$ is contained in a closed Riemann surface $\mathcal{R}$ so that $\mathcal{R} - B$ is non-empty and if the finite number of points $z_k$, $k = 1, \ldots, n$ are given in $B$, then there exists a constant $K$ such
that if \( a_k, k = 1, \ldots, n \), is any given set of \( n \) complex numbers there is a function \( g(Z) \) regular and uniform on \( \bar{B} \) such that

\[
g(Z_k) = a_k, \quad k = 1, 2, \ldots, n,
\]

and for all \( Z \) in \( \bar{B} \)

\[
|g(Z)| \leq K \max |a_k|.
\]

PROOF. When we say a function is regular on a closed region \( R \) we mean that it is regular in some region containing \( R \) in its interior.

Under the hypotheses of this lemma there exists a sequence of linearly independent functions regular and uniform on \( \bar{B} \). Indeed, rational functions with their poles in \( \mathcal{R} - \bar{B} \) will suffice for this purpose. We can therefore obtain a set of \( n \) linearly independent functions from these satisfying

\[
g_k(Z_i) = \delta_{ki}, \quad k, i = 1, 2, \ldots, n,
\]

by taking finite linear combinations. If we set

\[
K_i = \max_{Z \in \bar{B}} |g_i(Z)|
\]

then \( K = n(K_1 + \ldots + K_n) \) clearly satisfies the requirements of the lemma.

This lemma can be generalized to specify not only the value of the function at the points in question, but also the values of derivatives (in terms of fixed parameters). The resulting generalization is clear
however, and need not be gone into here.

**Lemma 7.2.** There exists a constant $M$, depending only on the region $G$ and the covariant $\gamma$ such that if $g$ is in $L^2(G)$ and $g\gamma$ is regular on $G$, then

$$\| g \|_G \leq M \sup_{Z \in G} |g(Z)| .$$

**Proof.** The result is trivially true if $g$ has poles on $G$. Hence we need consider only the case of $g$ regular on $G$. Let $Z_k, \ k = 1, \ldots, m$, be the poles of $\gamma$ on $G$. At each pole choose a parameter $t_k$ and fix the parameter disc $D_k : |t_k| < 1$. These may be so chosen that the $D_k$ have disjoint closures. Let

$$A = \max \max_{|t_k| \leq 1} |\gamma(Z(t_k))| ,$$

$$B = \sup_{Z \in G} |g(Z)| .$$

Then if $G_1 = G - \overline{D}_1 - \overline{D}_2 - \ldots - \overline{D}_m$ we have by property (6.1) the existence of a constant $M_1$ such that

$$\iint_G |g|^2 |\gamma|^2 dA = \iint_{G_1} |g|^2 |\gamma|^2 dA + \iint_{G-G_1} |g|^2 |\gamma|^2 dA$$

$$\leq B^2 \iint_{G_1} |\gamma|^2 dA + \sum_{k=1}^m \iint_{D_k} |g| |\gamma|^2 dA$$

$$\leq B^2 M_1^2 + \sum_{k=1}^m A^2 B^2 \pi$$

$$= B^2 (M_1^2 + m \pi A^2)$$

and from this the lemma is proved.
LEMMA 7.3. Let $G$ and $B$ satisfy the requirements of (6.10) and let $f(Z)$ be in $L^2(G)$ with $f \gamma$ a regular covariant on $G$. If $\epsilon > 0$ is given, then there exists an $h(Z)$ in $L^2(B)$ such that

$$\| f - h \|_G < \epsilon.$$  

REMARK. From (1.6) and the minimum property for expansions in terms of orthonormal sets, this lemma is equivalent to: The set $\{ \varphi_n \}$, doubly orthogonal in $L^2(G)$ and $L^2(B)$ is complete in that subset of $L^2(G)$ whose elements $f$ are such that $f \gamma$ is a regular covariant on $G$.

PROOF. First we must eliminate the poles of $f(Z)$. For this purpose, let $p(Z)$ be a meromorphic function in $F$, regular and uniform in $\overline{B}$ except for poles at the same points as $f$ in $G$ and such that

$$(7.1) \quad f_1(Z) = f(Z) - p(Z)$$

is regular and uniform on $G$.

Let $Z_1, Z_2, \ldots, Z_n$ be the poles of $\gamma$ in $B$, so ordered that

- $Z_1, Z_2, \ldots, Z_m$ are in $G$
- $Z_{m+1}, \ldots, Z_n$ are in $B - G$.

About each pole $Z_k$ in $B - G$ put a parameter disc $D_k$ so that each $D_k$ is disjoint from $G$ and from all other $D_k$ and also so that $D_k \subset B$. Let

$$G' = G \cup D_{m+1} \cup \ldots \cup D_n$$
and in $G'$ define the function

$$f_2(Z) = f_1(Z) \quad Z \text{ in } G$$

$$= -p(Z) \quad Z \text{ in } D_k, k = m+1, \ldots, n.$$  

This function $f_2$ is regular and uniform on $G'$ and $G'$ satisfies the hypotheses of theorem 6 with respect to the domain $B$. Hence there is a $G_1 \supset G'$ and a $B_1 \supset B$ so that $f_2$ is regular and uniform in $G_1$, and $G_1$ and $B_1$ also satisfy the hypotheses of theorem 6.

From theorem 6 we therefore have the existence of a function $f_3(Z)$, regular and uniform on $B$ and such that for any $Z$ in $G'$

$$|f_2(Z) - f_3(Z)| < \epsilon.$$  

(7.3)

The function $f_3 + p$ is not in $L^2(B)$, but only because it may not have zeros at the poles of $\gamma$. As was mentioned at the end of lemma 7.1, we may assume without loss of generality that all of the poles of $\gamma$ are of first order.

At every point $Z_k$, $f_2(Z_k) = -p(Z_k)$, hence

$$|f_3(Z_k) + p(Z_k)| < \epsilon$$

for every $k$. But then, from lemma 7.1, there exists a function $g(Z)$ regular and uniform on $\overline{B}$ such that

$$|g(Z)| < K \epsilon$$

(7.4)

for all $Z$ in $\overline{B}$ and such that

$$h(Z) = f_3(Z) + p(Z) - g(Z)$$

(7.5)

has a zero at each pole of $\gamma$.

The function $h(Z)$ defined by (7.5) will therefore
be in $L^2(B)$. But for any $Z$ in $G^*$

$$|f(Z) - h(Z)| = |f_2(Z) + p(Z) - h(Z)|$$

$$
\leq |f_2(Z) - f_3(Z)| + |g(Z)|
$$

$$< (K + 1) \epsilon .$$

Thus $f - h$ is bounded in each neighborhood of the zeros of $\gamma$, and hence is bounded on $\overline{G}$. Therefore by lemma 7.2,

$$\|f - h\|_G \leq M(K + 1) \epsilon ,$$

but $M$ and $K$ are independent of $f$ and $\epsilon$, hence the lemma is proved.

For our next proof, it is necessary that we identify the kernel function. As was seen in paragraph 4, the kernel function is characterized by the reproducing property, hence if a function is found with this property and satisfying (4.3) and (4.4), it must be the kernel function.

If we introduce the Wirtinger operators

$$\frac{\partial}{\partial Z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$

$$\frac{\partial}{\partial \overline{Z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

then it has been shown (see for example, Schiffer and Spencer [7], p. 121) that the bilinear differential

$$L_B(Z, W) = -\frac{2}{\pi} \frac{\partial^2 G(Z, W)}{\partial Z \partial \overline{W}} ,$$

where $G(Z, W)$ is the Green's function of the region $B$,.
has the reproducing property: if \( \psi \) dZ is a differential, regular and uniform in B, such that
\[
\iint_B |\psi|^2 dA < \infty
\]
then
\[
(7.7) \quad \psi(Z) = \iint_B L_B(Z, W) \psi(W) dA.
\]

Noting that if \( f \) is in \( L^2(B) \), then \( f \gamma dZ \) is a differential regular and uniform in B, we see that
\[
(7.8) \quad K_B(Z, W) = \frac{L_B(Z, W)}{\gamma(Z) \gamma(W)}
\]
possesses the desired reproducing property (4.5); however, from (7.6) and (7.8) this function satisfies the required properties (4.4) and (4.5). It is therefore the kernel function of \( L^2(B; \gamma) \).

**THEOREM 7.** If G and B satisfy the requirements (6.10), and if G possesses an analytic boundary, then the doubly orthogonal set \( \{ \varphi_n \} \) is complete in \( L^2(G) \).

**PROOF.** From (1.6) we see that in view of the fact that the set \( \{ \varphi_n \} \) is complete in \( L^2(B) \), we need merely show that given any \( f \) in \( L^2(G) \) and any \( \varepsilon > 0 \) there exists a function \( g \) in \( L^2(B) \) with \( ||f - g||_G < \varepsilon \). However, from lemma 7.3, it suffices to show that a function \( f_1 \) in \( L^2(G) \) exists with \( f_1 \gamma \) regular on G and \( ||f - f_1||_G < \varepsilon \).
Let $G_1$ be any region in $G$ such that $\overline{G_1} \subset G$. Then if $\Gamma_k$ is any boundary component of $G$, since is analytic there will exist an $r_k$, $0 < r_k < 1$, and a domain $D_k$ on the surface such that the annulus
$$\left\{ z \mid r_k < |z| < 1/r_k \right\}$$
on the $z$-plane can be mapped conformally onto $D_k$ with $|z| = 1$ mapping onto $\Gamma_k$, and such that $\overline{D_k} \cap \overline{G_1} = 0$. We may assume all of the $\overline{D_k}$ disjoint. Then if we set $D = G \cup D_1 \cup D_2 \cup \ldots \cup D_n$, where there are $n$ boundary components of $G$, we see that $D$ is open and $\overline{G} \subset D$. Furthermore, if $W$ is in $\overline{G_1}$, then by Schwarz's reflection principle the Green's function $G(Z,W)$ of the region $G$ will be regular (except at $W$) for all $Z$ in $D$. But then the bilinear differential $L_G(Z,W)$ given by (7.6) will be regular for all $Z$ in $D$ and all $W$ in $\overline{G_1}$.

Therefore, by (7.8), the function

$$f_1(Z) = \iint_{G_1} K_G(Z,W) f(W) |\gamma(W)|^2 dA_W$$

will be regular and uniform in $D$ except for poles of the same order and at the same points as $\gamma$ has zeros. Also $f_1(Z)$ will have zeros at the poles of $\gamma$, i.e. $f_1$ will be in $L^2(G)$. All that remains to be shown is that $G_1$ can be so chosen that $\|f - f_1\|_G < \epsilon$ for any given $\epsilon$.

For any $Z$ in $G$

$$f(Z) - f_1(Z) = \iiint_{G-G_1} K_G(Z,W) f(W) |\gamma(W)|^2 dA_W$$
and hence
\[ |f(z) - f_1(z)|^2 \]
\[ = \sum_{w} \sum_{s} K_G(z,w)K_G(z,s)f(w) \overline{f(s)} |\gamma(w)|^2 |\gamma(s)|^2 dA_w dA_s \]
and therefore
\[ \|f - f_1\|_G^2 \]
\[ = \sum_{w} \sum_{s} \sum_{z} K_G(z,w)K_G(s,z)f(w) \overline{f(s)} |\gamma(w)|^2 |\gamma(s)|^2 |\gamma(z)|^2 dA_w dA_s dA_z \]
However
\[ \sum_{G} K_G(s,z)K_G(z,w) |\gamma(z)|^2 dA_z = K_G(s,w) \]
and since all of the above integrals converge absolutely, as follows from (4.4), we may reverse the order of integration to obtain
\[ \|f - f_1\|_G^2 \]
\[ = \sum_{w} \sum_{s} K_G(s,w)f(w) \overline{f(s)} |\gamma(w)|^2 |\gamma(s)|^2 dA_w dA_s \]
But we see that
\[ \sum_{G} \sum_{w} K_G(s,w)f(w) \overline{f(s)} |\gamma(w)|^2 |\gamma(s)|^2 dA_w dA_s \]
\[ = \sum_{G} |f(w)|^2 |\gamma(w)|^2 dA_w \]
\[ = \|f\|_G^2 < \infty , \]
and hence, by the very definition of the integral, if we let \( G_1 \rightarrow G \) we must have \( \|f - f_1\|_G \rightarrow 0 \), and the required result follows.
8. The results of the previous two paragraphs hold equally well for the spaces \( \mathbb{H}^2(B; \gamma) \) consisting of all harmonic functions \( h(Z) \) regular and uniform in \( B \) except for isolated poles of at most the same order as the zeros of the covariant \( \gamma \) and such that

\[
\| h \|_B^2 = \iint_B |h|^2 |\gamma|^2 dA < \infty.
\]

Detailed study of these spaces will not be made here, but we will make mention of a few of the changes necessary for the development.

The proof of lemma 6.1 is unchanged when we note that if \( h(re^{i\theta}) = a_0 + \sum_{n=1}^{\infty} a_n r^n \cos n\theta + b_n r^n \sin n\theta \)

then

\[
\iint_{r < R} |h(re^{i\theta})|^2 r dr d\theta = \pi R^2 |a_0|^2 + \pi \sum_{n=1}^{\infty} \frac{R^{2n+2}}{2n+2} (|a_n|^2 + |b_n|^2) \\
\geq \pi R^2 |a_0|^2.
\]

The result of lemma 7.3 can easily be extended to this case by the following simple device. If \( h(Z) \) is a regular uniform harmonic function on \( \mathcal{C} \), then we can write

\[ h(Z) = \mathcal{R}_\alpha f(Z) \]

where \( f(Z) \) is a regular analytic function on \( \mathcal{C} \), but may have imaginary periods. Then there exists a function \( f_1(Z) \) regular analytic on \( \mathbb{B} \), and having the same periods as \( f(Z) \); indeed, there exist Abelian integrals of the second kind on \( \mathcal{R} \) having arbitrary periods. But then \( f(Z) - f_1(Z) \) can be approximated by a function \( g(Z) \) regular and uniform in \( \mathbb{B} \) by lemma 7.3. Then
\[ \mathcal{R}_a \left[ g(z) + f_1(z) \right] \]

will approximate \( h(z) \) as desired. By use of this device, the remaining changes in the proof of lemma 7.3 can easily be made.

Theorem 7 can be extended to this case by use of exactly the same device.

9. Two interesting consequences of the above results may be quoted. As was noted, there was no restriction that the region \( G \) be connected, but only that it consist of at most a finite number of domains having disjoint boundaries. If \( G = G_1 \cup G_2 \cup \ldots \cup G_n \) therefore, we see that a function \( f \) is in \( L^2(G) \) if and only if the function \( f \) restricted to the domain \( G_k \) is in \( L^2(G_k) \) for each \( k \). Thus \( f \) may be \( n \) "pieces" of entirely distinct functions.

With \( \gamma \) given, we can find a set \( \{ \phi_n \} \) doubly orthogonal in \( L^2(G) \) and \( L^2(B) \) such that for any set of \( n \) distinct functions \( f_k, k = 1, 2, \ldots, n \), with each \( f_k \) in \( L^2(G_k) \), the series

\[
\sum_{i=1}^{\infty} a_i \phi_i
\]

(9.1)

where

\[
a_i = \sum_{k=1}^{n} \int_{G_k} f_k \bar{\phi_i} |\gamma|^{2} dA
\]

(9.2)

will converge subuniformly to \( f_k \) in \( G_k^* \) for each \( k \), provided that for each \( k \), either \( f_k \gamma \) is regular
and uniform in \( \mathcal{G}_k \) (by the remark after lemma 7.3), or the boundary of \( \mathcal{G}_k \) is analytic (because of theorem 7).

This result is interesting not so much because the functions \( f_k \) may be simultaneously approximated by a function in \( L^2(B) \), but because the sequence \( \{ \varphi_n \} \) is independent of the \( f_k \), and the coefficients in the expansion (9.1) are given by the simple formula (9.2).

If we look at theorem 7 in a slightly different way, we may obtain the following result.

**COROLLARY 7.1.** Let the finite number of domains \( \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n \) be given (not necessarily disjoint), each being a proper subset of a closed Riemann surface \( \mathcal{R}_k \) and each having a finite number of Jordan curves as its boundary. In these domains, let a finite number of points divided into two sets, \( P_1, P_2, \ldots, P_r \) and \( Q_1, Q_2, \ldots, Q_s \), and associated positive integers, \( n_1, n_2, \ldots, n_r \) and \( m_1, m_2, \ldots, m_s \) be given. Then there exists a closed Riemann surface \( \mathcal{R} \), a domain \( B \) in \( \mathcal{R} \) and a covariant \( \gamma \), regular and uniform in \( B \) except for poles, such that domains \( \mathcal{G}_k \), respectively conformally equivalent to \( \mathcal{G}_k \), are imbedded in \( \mathcal{R} \) with disjoint closures, and are contained in \( B \), the covariant \( \gamma \) having poles of order \( n_i \) at the \( P_i \) and zeros of order \( m_i \) at the \( Q_i \), and no other poles or zeros in \( \mathcal{G} \), where \( \mathcal{G} = \bigcup_{i=1}^{n} \mathcal{G}_i \). There also exists a set
of functions \( \{\varphi_k\} \) in \( L^2(B;\gamma) \), orthogonal in \( L^2(B;\gamma) \) and orthonormal in \( L^2(G;\gamma) \), with the following properties: If \( f_k \) is regular and uniform in \( G_k \), 
\( 1 \leq k \leq n \), except for poles of at most order \( m_i \) at the points \( Q_i \) and has zeros of at least order \( n_i \) at the \( P_i \), and is such that for each \( k \) one of the two conditions

\begin{itemize}
  \item[a)] \( f_k \) is regular on the boundary of \( G_k \),
  \item[b)] \( f_k \) is in \( L^2(G_k;\gamma_k) \) where \( \gamma_k \) is a covariant having the same poles and zeros as \( \gamma \) in \( G_k \) and is otherwise regular and uniform in \( \overline{G_k} \), and \( G_k \) has analytic boundaries,
\end{itemize}

holds, then there exists a sequence of constants given by (9.2) such that the series (9.1) is subuniformly convergent in \( G_k^* \) to \( f_k \) for each \( k \).

**PROOF.** The surface \( \mathcal{R} \) can be constructed simply by letting each \( \mathcal{R}_k \) be given as a covering of the \( z \)-sphere. Introduce two branch points and a cut into each component of \( \mathcal{R}_k - \overline{G_k} \) and cross connect with the \( z \)-sphere. This may always be done so that the resulting cuts are disjoint on the \( z \)-sphere. Choose a schlicht disc \( D \) in \( \mathcal{R} - \overline{G} \) and set \( B = \mathcal{R} - D \). Then \( \overline{G} \subset B \subset \mathcal{R} \), the components of \( G \) have disjoint closures, and the boundaries of \( G \) are "accessible" to the boundary of \( B \) as required in (6.10).
The covariant $\gamma$ is easily chosen to have the required properties. Indeed, $\gamma$ can be an Abelian covariant, since any number of poles or zeros can be specified in the domain $D$.

Either condition (a) or (b) shows that $f_k$ is in $L^2(G_k; \gamma)$ and the corollary follows immediately from (a) lemma 7.3 or (b) theorem 7.

10. One possible application of doubly orthogonal functions is to the problem of analytic continuation. For this application we have the following theorem.
(See Bergman [3] p. 18)

**THEOREM 8. Let $G$ and $B$ be a pair of regions in the closed Riemann surface , satisfying (6.10).**
Let the function $f(Z)$ in $L^2(G)$ be given in one of the two alternative forms

(10.1 a) $f(Z) = \int_a^b \Phi(Z,t)d\alpha(t)\quad -\infty \leq a < b \leq +\infty$

or

(10.1 b) $f(Z) = \sum_{k=1}^{\infty} a_k \psi_k(Z)$

where, whichever form is given, the representation converges uniformly absolutely in any compact subset of $G$ not containing a zero of $\gamma$, and further, at each zero of $\gamma$, in terms of a fixed parameter $\int_a^b \Phi \gamma d\alpha(t)$ or $\sum_{k=1}^{\infty} a_k \psi_k \gamma$ converge uniformly absolutely in a neighborhood of this zero. Then a necessary and sufficient
The condition that \( f(Z) \) be in \( L^2(B) \) is that

\[(10.2 \text{ a}) \quad \sum_{n=1}^{\infty} \lambda_n \left| \int_a^b c_n(t) d\alpha(t) \right|^2 < \infty \]

or

\[(10.2 \text{ b}) \quad \sum_{n=1}^{\infty} \lambda_n \left| \sum_{k=1}^{\infty} a_k c_{kn} \right|^2 < \infty \]

respectively, where the \( \lambda_n = \| \varphi_n \|_B \) for the doubly orthogonal set \( \{ \varphi_n \} \) complete in \( L^2(B) \), and

\[(10.3 \text{ a}) \quad c_n(t) = \langle \Phi(Z,t), \varphi_n(Z) \rangle_G \]

or

\[(10.3 \text{ b}) \quad c_{kn} = \langle \psi_k, \varphi_n \rangle_G. \]

REMARK: No conditions on \( \Phi, \alpha, \) or \( \psi_k \) are assumed in this theorem other than that the representations (10.1) are uniformly absolutely convergent, and that the function so represented is in \( L^2(G) \).

PROOF. The series case follows from the integral case if we let \( \alpha(t) \) be a step function and define \( \Phi(Z,t) \) appropriately.

The set \( \{ \varphi_n / \sqrt{\lambda_n} \} \) is a complete orthonormal set in \( L^2(B) \), hence by theorem 1, a necessary and sufficient condition that \( f \) be in \( L^2(B) \) is that if \( f = \sum_{n=1}^{\infty} b_n \varphi_n \) then \( \sum_{n=1}^{\infty} \lambda_n |b_n|^2 < \infty \). But if \( f \) is in \( L^2(G) \)

\[ b_n = \langle f, \varphi_n \rangle_G = \int_G \left[ \int_a^b \Phi(Z,t) d\alpha(t) \varphi_n(Z) \right] \gamma^2 d\alpha \]

\[ = \int_a^b d\alpha(t) \int_G \Phi(Z,t) \varphi_n(Z) \gamma^2 d\alpha \]
\[ \int_{a}^{b} c_n(t) d\alpha(t) \]

from which the desired result follows. The uniform absolute convergence makes the change of order of integration allowable.

Two consequences of this theorem will be given here. Unfortunately, the only explicitly known doubly orthogonal sets are those for concentric circles in the plane (\( \gamma = z' \), the prime meaning differentiation with respect to the local parameter) and concentric annuli. But even these can be used to produce some interesting results. The following theorems are stated only for functions given as Laplace-Stieltjes integrals, but of course include Dirichlet series.

**THEOREM 9.** Let \( \alpha(x) \) be of bounded variation in \([0, \infty)\) and let

\[ f(s) = \int_{0}^{\infty} e^{-xs} d\alpha(x) \]

possess a non-void half plane of absolute convergence. Let \( A \) and \( R \) be arbitrary positive numbers and set

\[ (10.4) \quad b_n(x) = e^{-x(A+it_0)}(Rx)(Rx + 1) \cdots (Rx + n - 1). \]

If there exists \( A_0 < \infty \) such that
\[(10.5) \sum_{n=0}^{\infty} \frac{\delta^{2n+2}}{(n+1)(n+1)} \int_{0}^{\infty} |b_n(x)d\alpha(x)|^2 < \infty \]

for all $\delta < 1$ and $A > A_0$, then $f(s)$ is holomorphic in the strip

\[(10.6) \quad |t - t_0| < \pi R/2 , \quad (s = \sigma + it). \]

The condition (10.5) is equivalent to

\[(10.7) \quad \lim_{n \to \infty} \left[ \frac{1}{n} \log \left| \int_{0}^{\infty} b_n(x)d\alpha(x) \right| - \log n \right] \leq -1 \]

for all $A > A_0$.

PROOF. Set

\[s = R \log(1 - z) + A + it_0.\]

This maps the circle $|z| < \frac{1}{2}$ onto a domain $G$ interior to the strip (10.6) and contained in the half plane $\Re(s) \geq A - R \log 2$. Now if we put $\gamma = z'$, then for $A$ sufficiently large, $f$ is in $L^2(G)$. The circle $|z| < 1$ maps onto a domain $B'$ which tends to the strip (10.6) as $\sigma \to -\infty$ or as $A \to +\infty$. The circle $|z| < \delta < 1$ maps into a domain $B$ which tends to $B'$ as $\delta \to 1$.

Here

\[\varphi_n(Z) = \sqrt{\frac{n+1}{\pi}} 2^{n+1} z^n, \quad n \geq 0\]

\[= 2^{2n+2} \delta^{2n+2}.\]

In applying theorem 8, set $\Phi(Z,x) = e^{-xs}$ and use

\[\int_{|z| < \frac{1}{2}} f(z)z^n dx dy = \frac{\pi}{2^{2n+2}(n+1)} \frac{1}{2\pi i} \oint f(z) \frac{dz}{z^{n+1}}.\]

Then
\[ c_n(u) = \sqrt{\frac{n+1}{n}} 2^{n+1} \sum_{|\xi| < \frac{1}{2}} e^{-u\xi} z^n d\xi d\gamma \]

\[ = \sqrt{\frac{\pi}{n+1}} \frac{1}{2^{n+1}} \frac{1}{2\pi i} \int_{|z|<\frac{1}{2}} e^{-u\xi} z^n d\xi \]

\[ = \sqrt{\frac{\pi}{n+1}} \frac{1}{2^{n+1}} \frac{1}{2\pi i} e^{u(A+i\tau)} \int_{|z|<\frac{1}{2}} \frac{d\xi}{z^{n+1}(1-z)^{Ru}} \]

\[ = \sqrt{\frac{\pi}{n+1}} \frac{1}{2^{n+1}} e^{u(A+i\tau)} \frac{1}{n!} (Ru)(Ru+1)\cdots(Ru+n-1). \]

Putting these values into condition (10.2) gives the condition (10.5).

Condition (10.7) is merely the requirement that (10.5), as a power series in \( S \) have a radius of convergence at least one.

**THEOREM 10.** If \( \alpha(x) \) is of bounded variation in \([0, \infty)\) and

\[ f(s) = \int_0^\infty e^{-xs} d\alpha(x) \]

converges for \( R_\alpha s > 0 \) and converges absolutely for \( R_\alpha s \geq \sigma_0 \) for some \( \sigma_0 < 1 \), then a necessary and sufficient condition that \( f(s) \) has a branch point of order \( p - 1 \) at \( s = 0 \) (i.e. \( f(s) \) is holomorphic in \( z \) at \( z = 1 \), where \( s = (1 - z)^p \) is that

\[ \sum_{n=1}^{\infty} \frac{\delta_{n+1}^{2n+2}}{(n+1)(n+2)!} \left| \int_0^\infty b_n(x) d\alpha(x) \right|^2 < \infty \]

for some \( \delta > 1 \) where

\[ b_n(x) = \sum_{m \geq n \frac{p}{p}} (-x)^m \frac{(mp)!}{m! (mp-n)!} \]

or equivalently that

\[ \lim_{n \to \infty} \left[ \frac{1}{n} \log \left| \int_0^\infty b_n(x) d\alpha(x) \right| - \log n \right] < -1. \]
Proof. Here we put
\[ s = (1 - z)^p \]
and set
\[ G = \{ s \mid |z| < \rho \} \]
\[ B = \{ s \mid |z| < \delta \} \]
where \( \rho \) is chosen small enough that \( G \) is in the half plane \( \Re s > \sigma_0 \) and \( \delta > 1 \) (for the theorem in question). The domain \( B \) will then be a \( p \)-sheeted covering of \( s = 0 \). Here,
\[ \varphi_n(z) = \sqrt{\frac{n+1}{n}} \left( \frac{1}{\rho} \right)^{n+1} z^n \]
\[ \lambda_n = \left( \frac{\delta}{\rho} \right)^{2n+2} \]
\[ c_n(u) = \frac{\pi}{n+1} \rho^{n+1} \frac{1}{z^{n+1}} \int \frac{e^{-u(z-\bar{z})^n}}{z^{n+1}} \, dz \]
\[ = \frac{\pi}{n+1} \rho^{n+1} \sum_{m \geq \frac{n}{p}} \frac{(-u)^m}{m^!} \frac{(mp)!}{n^! (mp-n)!} \]
\[ = \frac{\pi}{n+1} \rho^{n+1} \frac{1}{n^!} b_n(u) \]
and (10.8) follows immediately from condition (10.2).
Condition (10.10) follows from (10.8) just as before.

11. In conclusion, we will make a few remarks about the construction method of paragraph 5 for producing doubly orthogonal sets of functions. This construction would hardly be expected to be a practical method of producing a set of doubly orthogonal functions. It would probably be more useful in finding
some properties of these sets and in showing that a particular function is actually a member of some doubly orthogonal set.

For example, let us consider the plane case, \( \gamma = z' \), with the regions

\[
B = \left\{ z \mid 1/r < |z| < r \right\}
\]
\[
G = \left\{ z \mid |z - 1| < d \right\}
\]

(11.1)
\[
d < \begin{cases} r - 1 \\ 1 - 1/r \end{cases}
\]

It is easily shown that here the kernel function is

(11.2) \[ K_B(z,w) = \sum_{n=-\infty}^{\infty} A_n z^n w^{-n} \]

where

(11.3) \[ A_{-1} = (4\pi \log r)^{-1} \]
\[ A_n = \frac{(n + 1) r^{2n+2}}{\pi (r^{2n+4} - 1)} \quad n \neq -1, \]

(See Bergman [3] p. 9, for example).

A simple calculation shows that if \( f = \sum_{n=-\infty}^{\infty} a_n z^n \) then the transformation \( T_1 f \) (5.2) is given by

\[
T_1 f = \sum_{n=-\infty}^{\infty} A_n C_n z^n
\]

(11.4) \[ C_n = \sum_{k=-\infty}^{\infty} a_k B_{kn} \]
\[ B_{kn} = 2\pi \sum_{j=0}^{\infty} \binom{k}{j} \binom{n}{j} \frac{d^{2j+2}}{2j + 2} \]

The last series of course reducing to a finite series if either \( k \) or \( n \) is a non-negative integer. In particular, if we set \( f = 1 \), then
\[(11.5) \quad T_1 f = \pi d^2 \sum_{n=-\infty}^{\infty} A_n z^n,\]

hence the constant cannot be a member of the doubly orthogonal set for G and B. This result however is suggestive of the Weierstrass \(\wp\)-functions. The expansion given by Bergman, [3] p. 10, for \(\wp(\log z)\) is closely related to (11.5). It might therefore be conjectured that we should look among the \(\wp\)-functions for the desired doubly orthogonal set.
BIBLIOGRAPHY


