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The Mixed Problem for Harmonic Functions
with Discontinuous Boundary
Conditions

by

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The Mixed Problem for Harmonic Functions with Discontinuous Boundary Conditions.

1. Introduction.

The mixed problem is a combination of the Dirichlet and Neumann problems. A function harmonic in a closed region is to be determined by its values on part of the boundary of the region and by the boundary values of its normal derivative on the remainder of the boundary.

Hadamard suggested the problem and it has been considered by Zaremba, Lichtenstein, and Villat. Following are synopses of their results. We shall use the notation and terminology of the authors.

Zaremba considers the mixed problem for a simply connected space region \( \mathcal{D} \), which admits at each point a unique tangent plane, such that the angle formed by two normals is less than the product of the distance of the feet of the normals by a constant. A parallel to the normal of the boundary of \( \mathcal{D} \) meets the portion of this boundary at most in one point interior to a sphere of radius not larger than a fixed length having for center the foot of the normal considered; and it surely meets it if the distance to the normal is small enough. Denote by \( S \) the parts of the boundary where the function is given by \( S' \) the rest. The points exterior to \( \mathcal{D} \), denote by \( (\mathcal{D}') \). Denote by \( F \) the boundary between \( S \) and \( S' \). \( F \) must have a unique normal plane at each point such that the acute angle formed by two of these planes is inferior to the product of a constant by the distance of the points of \( F \) where the planes are normal. Suppose that a plane is normal to \( F \) at \( A \), then all planes in a sphere with center \( A \) and radius given parallel to it cut \( F \) in one point at most and surely if the distance to the plane considered is small enough. First the problem \( \mathcal{A} \) is solved. To determine a function \( V \) harmonic at all interior points of \( \mathcal{D} \) or \( (\mathcal{D}') \), continuous on \( S + S' \) and equal to zero at infinity, such that

\[ V = 0 \quad \text{on} \quad S \]

and such that in all points of \( S' \) distinct from \( F \)

\[ \left( \frac{dV}{dn} \right)_c - \left( \frac{dV}{dn} \right)_i = \varphi' \]

where \( \left( \frac{dV}{dn} \right)_c \) is the limit of the derivative of \( V \) along the exterior normal with a like definition for \( \left( \frac{dV}{dn} \right)_i \) and \( \varphi' \) is bounded and continuous on \( S' \). In order to solve this problem let

\[ V(A) = \frac{1}{4\pi} \int_{S'} \frac{\varphi(A')}{A' B} \]
and 
\[ V = V'_l - W. \]

Now the problem reduces to finding \[ W = V, \text{ on } S \] and 
\[ \left( \frac{dW}{dn} \right)_c + \left( \frac{dW}{dn} \right)_i = 0 \text{ on } S'. \]

By the alternate method of Schwarz \( W \) can be determined and

finally it is shown that 
\[ W(A) = \frac{1}{4\pi} \int_S \left\{ \left( \frac{dW}{dn} \right)_c - \left( \frac{dW}{dn} \right)_i \right\} \frac{d\omega_{AB}}{A+B} \]

A green's function 
\[ g(A,B) = \frac{1}{4\pi A B} - G(A,B) \]

is shown to be determined uniquely where \( G(A,B) \) is a green's function for problem \( (A) \) and 
\[ V = V'_l + N = \int_{S'} \sigma'(B) G(A,B) \, d\omega_B \]

The problem \( U = \mathcal{K} \) on \( S \) and \( \sigma'(B) \mathcal{K}' \) on the interior of \( S' \) is then discussed. \( \mathcal{K} \) and \( \mathcal{K}' \) are continuous on \( S \) and \( S' \) respectively. By means of a transformation \( U = A + G \)

it is equivalent to 
\[ (U)_c = 0 \text{ on } S \]

Therefore the problem is reduced to that of determining \( V \) given that 
\[ (V)_c = (V)_i = 0 \text{ on } S \]
\[ \left( \frac{dV}{dn} \right)_c - \left( \frac{dV}{dn} \right)_i + \lambda \left\{ \left( \frac{dV}{dn} \right)_c + \left( \frac{dV}{dn} \right)_i \right\} + 2 \mathcal{K}' = 0 \]

on the interior of \( S' \). Let \( \sigma' = \left( \frac{dV}{dn} \right)_c - \left( \frac{dV}{dn} \right)_i \).

Then 
\[ V(A) = \int_{S'} \sigma'(B) G(A,B) \, d\omega_B. \]

Let 
\[ \mathcal{K}(A,B) = \left( \frac{dG}{dn} \right)_c + \left( \frac{dG}{dn} \right)_i \]

On \( S' \), \( \mathcal{K}(A,B) = 2 \left( \frac{dG(A,B)}{dn} \right)_c = 2 \left( \frac{dG(A,B)}{dn} \right)_c \)

Therefore 
\[ \sigma'(A) + \lambda \int_{S'} \sigma'(B) \mathcal{K}(A,B) \, d\omega_B + 2 \mathcal{K}' = 0. \]

and the problem is reduced to that of solving an integral equation of the second kind of Fredholm. Now \( |\mathcal{K}(A,B)| \leq \frac{C}{A+B} \) and the
general theory for the solution of this equation applies. The author proves that \( A = -1 \) is not a singular value and that the problem therefore has a unique solution.

In the case where \( S \) and \( S' \) have no common points the problem is easily solved by means of the potential of a double layer on \( S \) and of a single layer on \( S' \). If \( S \) and \( S' \) do have common points the integral equation resulting from this method of solution has a kernel of the order \( \frac{1}{r^2} \), so that a unique solution may not exist. Lichtenstein considers the problem relative to a linear partial differential equation of elliptic type.

Let \( \Sigma \) be a closed continuous curve in the \( \mathbb{C} \) plane, without double points and \( \Theta \) the finite region bounded by it. Let \( \Sigma \) be the arc length of the curve \( x = x(\Theta) \) and \( y = y(\Theta) \) its equation. Assume for simplicity that \( x(\Theta) \) and \( y(\Theta) \) have continuous derivatives of the first three orders.

Now let \( \Sigma = S + S' \). The problem is that of determining a bounded solution regular in \( \Theta \) of

\[
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} + \Delta u = f
\]

which on \( S \) takes on given continuous values \( \nu(\Theta) \) and such that its normal derivative takes on given bounded continuous values \( \alpha(\Theta) \) on \( S' \). \( \Delta \) and \( \nabla \) have continuous derivatives of the first and second orders in \( \Theta \) and on \( \Sigma \), \( \alpha \) and \( f \) those of the first order.

Map the region \( \Theta \) on a unit circle \( \mathbb{T} \) by means of a function \( \chi : \Theta \rightarrow \mathbb{T} \) and take \( (r, \theta) \) as new variables and the above problem becomes the following one.

Denote by \( S \) the boundary of \( \mathbb{T} \). There is given on \( S \) a finite number of connected pieces \( S_1, S_2, \ldots, S_m \) which we denote totally by \( S \) and the remainder of \( S \) by \( S' \). To determine the bounded solution \( u(\Theta) \) of the differential equation \( \Delta u = f \), regular in \( \mathbb{T} \) which takes on, on \( S \) a continuous, given sequence of values \( \nu(\Theta) \) while on \( S' \) the condition

\[
\frac{\partial u}{\partial n} = \frac{\partial u(\Theta)}{\partial r} = -\alpha(\Theta)
\]

is satisfied. No special hypothesis is made on \( \frac{\partial u}{\partial r} \) at the common end points.

Let \( \mathcal{G}_\nu(\Theta) \) be any continuous function defined on \( S \) and satisfying the relation \( \int_S \mathcal{G}_\nu(\Theta) \, d\Theta = 0 \) which on \( S' \) is identical with \( \mathcal{G}(\Theta) \), and let \( W(\Theta) = W(\Theta, \Theta) \) be the potential function regular in \( \mathbb{T} \) such that \( \partial W / \partial S_1 = \mathcal{G}_\nu(\Theta) \) on \( S \) and \( W(\Theta, \Theta) = 1 \). Let \( W(\Theta) = W(\Theta, \Theta) \) on \( S \) and put

\[
u(\Theta) = \mathcal{U}_0(\Theta) + W(\Theta, \Theta)
\]

where \( \mathcal{U}_0(\Theta) \) is the bounded solution regular in \( \mathbb{T} \) of the differential equation

\[
\frac{\partial^2 \mathcal{U}_0}{\partial x^2} + \frac{\partial^2 \mathcal{U}_0}{\partial y^2} - \alpha \frac{\partial \mathcal{U}_0}{\partial x} + \beta \frac{\partial \mathcal{U}_0}{\partial y} + C \mathcal{U}_0 = f - \alpha \frac{\partial W}{\partial x} - \beta \frac{\partial W}{\partial y} - CW = f.
\]
which on \( \Sigma \) takes on the value
\[
\varphi(\xi) - \mathcal{W}(\xi) = \Psi(\xi)
\]
and whose normal derivative vanishes on \( \Sigma \) except perhaps at the end points of \( \Sigma \). Continue \( U_0(\xi, \eta) \) across the entire plane assigning each time the same value at conjugate points \( U_0(\xi, \eta) = U_0(\xi', \eta') \) and \( \xi \to \infty \to U_0(\xi, \eta) \).

In \( \tau \) and \( \bar{\tau} \), \( U_0(\eta) \) satisfies
\[
\Delta_0 U_0(\eta) = \frac{\partial^2 U_0}{\partial \xi^2} + \frac{\partial^2 U_0}{\partial \eta^2} + 4 \xi \partial_\xi U_0 + 4 \eta \partial_\eta U_0 + C_0 = f_0
\]
where \( C_0 \), \( C_0 \), \( C_0 \), and \( f_0 \) are discontinuous on \( \Sigma \). \( U_0 \) takes on the value \( \varphi(\xi) \) on \( \Sigma \).

Now let \( \Sigma = \{ \xi, \eta \} = \mathcal{R}(\xi) = \mathcal{G}(\eta) \) be an analytic function by means of which the \( \{ \xi, \eta \} \) plane with the exception of \( m \) pieces on the unit circle is mapped on a finite region \( \Omega \), bounded by \( \infty \) closed regular analytic curves \( \Sigma '_m \). In this mapping the arcs \( \Sigma \) go into \( \infty \) regular analytic pieces of curves, from boundary to boundary and the totality of new pieces denote by \( \Sigma '_m \). Introduce \( \{ \xi', \eta \} \) as new variables and put \( U_0(\xi', \eta) = U_0(\xi', \eta) \).

This is a solution regular inside \( \Omega \) with the exception perhaps of \( \Sigma '_m \) of the differential equation
\[
\Delta_0 U_0 = \frac{\partial_\xi^2 U_0}{\partial \xi^2} + \frac{\partial_\eta^2 U_0}{\partial \eta^2} + a \partial_\xi U_0 + b \partial_\eta U_0 + c = f_0
\]
The coefficients are finitely discontinuous on \( \Sigma \) and have in the interior and on the boundary of the subregions into which \( \Omega \) is split by \( \Sigma '_m \) continuous derivatives of first order. The function \( U_0(\xi, \eta) \) takes on on \( \Sigma '_m \) a given continuous sequence of values \( Q_0(\xi, \eta) \). If \( f_0(\xi, \eta) = 0 \), \( f_0(\xi, \eta) = 0 \), then a particular \( U_0(\xi, \eta) \) is the solution of \( \Delta_0 U_0 = 0 \) regular in \( \Omega \) which vanishes on \( \Sigma '_m \). The problem is thus reduced to the first boundary value problem which has been solved by Lichtenstein, Dini, Gervay and others.

Lichtenstein shows that this solution is unique, if the first boundary value problem for \( \Delta_0 U_0 = 0 \) is unique.

A green's function may now be found for the problem.

In what follows suppose that the first boundary problem for \( \Delta_0 U_0 = 0 \) and the mixed problem for \( \Delta_0 U_0 = f \) has a unique solution.

Let \( \{ \xi', \eta \} \) be a point in \( \tau \) or \( \bar{\tau} \) and \( \{ \xi', \eta \} \) its conjugate with respect to \( \Sigma \). The corresponding points of \( \Omega \) are denoted by \( \{ \xi', \eta \} \) and \( \{ \xi', \eta \} \). Let \( G(\xi', \eta, \xi', \eta) \) be the green's function for \( \Delta_0 U_0 = 0 \) vanishing on \( \Sigma '_m \).
and let \[ \Gamma(n', \eta) = G(n', \eta) + G(n, \eta) \]
where \((\eta, \eta')\) is conjugate to \((\eta, \eta)\). Then
\[ \Gamma(n', \eta, \eta) = \Gamma(n', \eta, \eta) = \Gamma(n, \eta) \]
and \(\Gamma\) has a logarithmic singularity for \((\eta', \eta) \equiv (\eta, \eta)\). Therefore for all \((\eta, \eta)\) in \(S\), \(\partial \Gamma / \partial n = 0\). For all \((\eta, \eta)\) in \(T\) let
\[ F(\eta, \eta') = \Gamma(n, \eta, \eta) \]
then \(\Gamma\) is the solution of \(\Delta u = \eta\) which except in \((\eta, \eta')\) is continuous in \(T\) and on \(S\), regular in \(T\), has a logarithmic singularity for \((\eta', \eta) \equiv (\eta, \eta)\), vanishes on \(S\) and satisfies on \(S\) the relation \(\partial \Gamma / \partial n = 0\). It is the Green's function for the mixed problem.

Assume now that \(E(\eta, \eta')\) is the Green's function for the mixed problem for \(M(\eta) = \eta\) where \(M(\eta)\) is the adjoint of \(L(\eta) = \eta\). Now substitute in the Green's formula
\[ \int_T \left[ w'M'(w') - wM(w) \right] dx dy = -\int_S \left[ w''(\partial w / \partial n) - \partial w / \partial n \right] dx \]
and obtain
\[ u(\eta, \eta') = \int_T \left[ H(\eta, \eta', \eta) \right] dx dy + \int_S H(\eta, \eta', \eta) \eta dy \]
\[ \int_S \left[ H(\eta, \eta', \eta) \eta dy \right] \]
\[ = \left( \eta, \eta' \right) + \int_T H(\eta, \eta', \eta) \eta dy \]
Lichtenstein uses this formula to extend his results to boundary conditions that have a finite number of finite discontinuities, but he does not consider the uniqueness of the function determined by these more general conditions. Villat studies the determination of an analytic function given that its real part takes on given values on one part of the boundary of a region and that its imaginary part takes on given values on the remainder of the boundary.

In a unit circle having for center the origin of the plane \(z = \eta + \eta\) a function which has for frontier values \(f(\eta)\) to within an additive pure imaginary constant is
\[ A(z) = \frac{i}{2 \pi} \int_0^{2 \pi} \overline{f(\eta)} \frac{d\eta}{\eta - z} \]
\[ \phi(\eta) \text{ may be merely summable in which case } A(z) \rightarrow f(\eta) \text{ as } z \rightarrow 1. \]
nearly everywhere. If \( f(\varphi) \) satisfies a Lipschitz condition the imaginary part \( g(\varphi) \) of \( H(\varphi) \) is given by
\[
g(\varphi) = \frac{i}{2\pi} \int_0^{2\pi} \frac{f(\varphi) - f(\varphi - \epsilon)}{\tan \frac{\varphi - \epsilon}{2}} d\epsilon
\]

Now to give the integral of the normal derivative of a harmonic function along an arc is the same as giving the values of the conjugate harmonic along the arc. Let \( \mathcal{C} \) be part of the circumference of \( |z| = 1 \) and \( \mathcal{C}_1 \), be the rest. Let \( \mathcal{C}_2 \), \( \mathcal{C}_3 \) be the extremities of \( \mathcal{C} \) and denote the points of \( \mathcal{C} \) by \( z, (z, \varphi) \). Suppose \( f(z) \) the given value of the real part on \( \mathcal{C}_1 \) and suppose \( f(\varphi) \) the (unknown) value on \( \mathcal{C}_3 \) of the function sought; and finally let \( g(\varphi) \) be the imaginary part of the same function on \( \mathcal{C} \). Then
\[
g(\varphi) = \frac{i}{2\pi} \int_0^{2\pi} \frac{f(z) - f(\varphi)}{\tan \frac{\varphi - \epsilon}{2}} d\epsilon + \text{const}
\]

The integral of \( f(z) \) is equal to its principal value and therefore
\[
\frac{i}{2\pi} \int_0^{2\pi} \frac{f(z) - f(\varphi)}{\tan \frac{\varphi - \epsilon}{2}} d\epsilon + \text{const} = f(\varphi) - \frac{i}{2\pi} \int_0^{2\pi} \frac{f(z) - f(\varphi)}{\tan \frac{\varphi - \epsilon}{2}} d\epsilon + \text{const}
\]

In order to find the unknown \( f(\varphi) \) a transformation is made from the unit circle to the unit semi-circle by means of the transformation
\[
\sqrt{\frac{\varphi - \epsilon}{\varphi + \epsilon}} = \left(\frac{z^2 + 1}{z - 1}\right) \quad \text{where} \quad \sqrt{\cdot}
\]
is a constant. The arc \( \mathcal{C}_1 \) goes into the semi-circumference and \( \mathcal{C}_2 \) goes into the diameter. If \( \sqrt{\cdot} = -z \) the point \( \mathcal{F} \) corresponds to \( z = \left(\frac{\varphi + \epsilon}{\varphi - \epsilon}\right) \). Then to \( f(z) \) correspond \( f(\varphi) \) and to \( g(\varphi) \) corresponds, \( G(\varphi) \)
\[
F(\varphi) = \frac{1}{2\pi i} \int \frac{\tan \frac{\varphi - \epsilon}{2}}{\tan \frac{\varphi + \epsilon}{2}} \frac{\cos \phi - \tan \frac{\varepsilon - \phi}{2}}{\frac{\varepsilon + \phi}{\varepsilon - \phi}} d\epsilon
\]
\[
G(\varphi) = \frac{1}{2\pi i} \int \frac{\tan \frac{\varphi - \epsilon}{2}}{\tan \frac{\varphi + \epsilon}{2}} \frac{1 + \tan \frac{\varepsilon + \phi}{2} - 2e^{i\phi} \tan \frac{\varepsilon - \phi}{2}}{1 + \varepsilon^2 + 2e^{i\phi} \tan \frac{\varepsilon + \phi}{2} \tan \frac{\varepsilon - \phi}{2}} d\epsilon
\]
The problem is reduced to finding in a semi-circle an analytic function given the real part \( F(\xi) \) on the circumference and \( G(\eta) \) on the diameter. The solution is
\[
K(Z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{G(\mu)}{\mu - Z} \, d\mu + \frac{i}{2\pi} \int_{0}^{2\pi} \left[ \frac{f(e^i\xi)}{1 - Z e^{i\xi}} - f'(e^i\xi) \right] \frac{d\xi}{1 - Z e^{i\xi}}.
\]

\( G(\mu) \) is summable from \(-\infty \) to \( +\infty \). It must be assumed finite in the neighborhood of the points \( \xi = \pm i \), continuous and satisfying a Lipschitz condition. Then \( K(Z) \) which is the real part of the first term on \( C(\xi) \) is bounded and therefore summable. We shall study the problem with the boundary conditions as functions of limited variations which are the limits of integrals, and find necessary and sufficient conditions for the determination of a harmonic function by means of those conditions. For this purpose we find an explicit formula for the problem by methods which are a combination of Lichtenstein's and Villat's.

A green's function for the problem can be determined for a semi-circular region. Consider a unit circle with center at the origin of a rectangular coordinate system. A point \( M \) may be given coordinates \( (x, y) \) or \( (r, \theta) \) where \( x = r \cos \theta \), \( y = r \sin \theta \).

Denote by \( (r, \theta) \) the polar coordinates of a point \( P \). Now consider the points \( M(1, \theta) \), \( M(1, -\theta) \), \( P(\xi, \eta) \) and \( \overline{P}(\xi, \eta) \). \( M \) is the reflection of \( M \) in the x-axis and \( \overline{P} \) is the conjugate of \( P \) with respect to the circle. \( M \) and \( \overline{P} \) are taken interior to the semi-circle. Let \( \phi(M, P) \) denote the classical green's function for the complete circle and let \( \phi(M, \overline{P}) \) be the green's function for the mixed problem, for the semi-circle. Then
\[
\phi(M, \overline{P}) = \phi(M, P) + \phi(M, \overline{P}).
\]

This function is zero for \( M \) on the circumference \( \partial \Omega \) and its normal derivative is zero for \( M \) on \( (-\infty, 1) \) if \( P \) is interior to the region. It has a logarithmic singularity for \( M = \overline{P} \) and \( \phi(M, P) = \phi(M, \overline{P}). \)

This function leads to a solution of the problem which is virtually the potential of a double layer on the complete arc \( (0, \pi) \) and a single layer on \( (-\infty, \infty) \). Zarema shows that an attempt at a solution directly by means of potentials of single and double layers leads to an integral equation with unbounded kernel. Our formula gives a solution to this equation.

By means of a conformal transformation our results may be extended to more general boundaries.

If a variable is a coordinate of \( M \) e.g. \( \theta \) we put that fact in evidence by the notation \( \partial M \). The points of the semi-circle arc have coordinates \( (r, \theta) \) and \( (r, \pi + \theta) \) we shall denote by \( f(\theta) \).

A function \( \phi \) of the points on the diameter we shall denote by \( \phi(\xi) \) and \( \phi(\xi = 0) = \phi(\xi = \pi) \) or more briefly \( \phi(\xi) = \phi(0) \); \( \phi(-i) = \phi(\pi) \).

( in Villat's word the \( F(\theta) \) and \( G(\mu) \) are summable our he does not consider the unique determination of an analytic function by means of such general boundary conditions.)
Suppose our boundary conditions are given in the form
\[ \lambda = 1 \int_{\Omega} u(\lambda, \theta) \, d\theta = F(\theta) - F(\theta) \quad (0 < \theta, \theta_2 \leq \pi) \]
and
\[ \lambda \int_{\Omega} \frac{\partial u(\lambda)}{\partial \theta} \, d\theta = H(\lambda) - H(\lambda) \quad (-1 \leq \lambda, \lambda_2 \leq 1) \]
The last condition is independent of the curve joining \( A \) and \( B \) and the limit is the same for all curves joining \( x, x_2 \), \( F \) and \( H \) are functions of limited variation on their respective ranges with regular discontinuities.

By means of the above conditions and an application of Green's theorem to \( U(\lambda) \) and \( G(\lambda, \eta) \), a solution is obtained in the form
\[ U(\lambda) = \frac{1}{2\pi} \int_{\Omega} \left[ \log \left( \frac{1 - \lambda^2}{1 - \lambda_1^2} \right) + \frac{1 - \lambda^2}{1 - \lambda^2} \right] dF(\eta) \]
\[ - \frac{1}{2\pi} \int_{\Omega} \left[ \log \left( \frac{1 - \lambda^2}{1 - \lambda_1^2} \right) \frac{x_2^2}{x_2^2} \right] dH(\eta) \]
where the indicated integrals are Stieltjes integrals. The coordinates of \( \eta \) are \( (\lambda, \eta) \) and \( \lambda \) is the variable point of integration. In the first integral the coordinates of \( \eta \) are \( (\lambda, \eta) \) and in the second they are \( (\lambda_2, \eta) \).
2. The classical green's function and the green's function for the mixed problem for the semi-circle

Consider the upper unit semi-circle and the points $M, M'$, $p$ and $p'$ with polar coordinates as indicated in the figure. $M'$ is the reflection of $M$ in the diameter and $p'$ is the inverse of $p$ with respect to the circle. In what is to follow it is often convenient to give these points rectangular coordinates as indicated.

Denote by $G(p)$ the classical green's function for the complete circle by $G(p)$ the classical green's function for the semi-circle and by $J(p)$ the green's function for the mixed problem, for the semi-circle. We define $J(p)$ by the relation

$$J(p) = 1 - (M, p') + J(M, p')$$

The classical green's function is given by the relation

$$G(p) = J(p) - G(M, p)$$

Let $K(p)$ be the conjugate to $G(p)$. The curves $G(p) = c$ for the pole $p$ an interior point are a one parameter family of convex curves in the semi-circle and $G = 0$ is its boundary. $K(p) = c$ for $p$ an interior point of $K$ as is a one parameter family of curves orthogonal to the curves $G = c$ for $c = 0$ and orthogonal to the boundary except at the points $q = -1$ and there they bisect the right angle between the diameter and circumference. The curves $K = K$ may be put in one to one correspondence with the values of $\theta$ between 0 and $\pi$ and with $\lambda$ between $-1$ and $1$. $J(q) = K(\theta) = K(\theta + K(\pi))$

$J(p)$ is clearly zero for $p$ on the circumference ($z = 1$) and $p$ interior to the semi-circle. It is easily verified that

$$\frac{\partial J(p)}{\partial y} = 0$$

on the diameter ($z = 1$) for $p$

interior to the semi-circle. This fact is obvious because of the relation $J(p) = J(M, p)$. $J(p)$ has a logarithmic singularity for $M = p$ and is symmetric on $M$ and $p$. It therefore satisfies the conditions of a Green's function for the mixed problem. We shall employ it to obtain a tentative solution of our problem.
3. A form of solution.

Now given $U(m)$ harmonic in the semi-circle satisfying the boundary conditions

$$
L \int_{\theta_1}^{\theta_2} U(\varphi, \theta) \, d\theta = -F(\varphi) - F(\theta_1), \quad (\varphi \leq \varphi_2, \varphi_2 = \pi).
$$

$$
L \int_{\theta_1}^{\theta_2} \frac{\partial U(m)}{\partial m} \, d\theta = H(\varphi) - H(\theta_1), \quad \varphi \leq \varphi_1, \varphi_1 = 1.
$$

Where $F$ and $H$ are functions of limited variation on their respective ranges which for simplicity we shall assume have regular discontinuities. The last condition is independent of the path and might have been taken along any rectifiable path joining the points $\chi_1$ and $\chi_2$.

Apply Green's theorem to $U(m)$ and $S(m)$ on a circular sector parallel to the boundary of the semi-circle at a distance $s$ from the boundary. Call this sector $C_s$. Then

$$
U(m) = \frac{1}{2\pi} \int_{C_s} \left( \frac{\partial U(m)}{\partial n_1} - \frac{\partial U(m)}{\partial n_2} \right) \, ds.
$$

This suggests that for the semi-circle for $\varphi = \varphi_2$

$$
U(m) = \frac{1}{2\pi} \int_{C_s} \frac{1}{2} \left( -\frac{\partial^2 U(m)}{\partial \varphi \partial \theta} \right) \, dF(\varphi) - \frac{1}{2\pi} \int_{C_s} \frac{\partial U(m)}{\partial \varphi} \, dF(\theta).
$$

Now

$$
S(m) = \frac{1}{2} \partial U(m) = \frac{1}{2} \frac{\partial U(m)}{\partial m} = \log \frac{m_1}{m_2} - \log \frac{m_2}{m_1}.
$$

$$
\Rightarrow \quad \frac{1 + \frac{x}{m_2} \pm 2m_1 \cos(4\theta)}{1 + \frac{x}{m_2} \mp 2m_1 \cos(4\theta)} = \frac{1 + \frac{x}{m_1} \pm 2m_2 \cos(4\theta)}{1 + \frac{x}{m_1} \mp 2m_2 \cos(4\theta)}.
$$

From this it is easily verified that

$$
L \int_{\theta_1}^{\theta_2} \frac{\partial S(m)}{\partial \theta} = \frac{1 - x^2}{1 + x \mp 2m \cos(4\theta)} + \frac{1 - x^2}{1 + x \mp 2m \cos(4\theta)}
$$

and

$$
S(m, \chi_1) = S(m, \chi_2) = \log \left( \frac{m_1}{m_2} \right) - \log \left( \frac{m_2}{m_1} \right).
$$
The first of these relations is evident from the classical fact that

\[
\sum_{\rho=1}^{\mathcal{L}} \left( -\frac{\partial g(\mathbf{p})}{\partial \mathbf{r}} \right) = \frac{i-r^2}{i+r^2} \frac{i\mp 2}{2\pi \mp 2\pi \cos(\phi - \theta)}.
\]

We shall therefore study

\[
\mathcal{L}(M) = \frac{i}{2\pi} \int_0^{\tau} \left[ \frac{1-r^2}{1+r^2} \frac{1}{2\pi \cos(\phi - \theta)} + \frac{i-r^2}{i+r^2} \frac{1}{2\pi \cos(\phi + \theta)} \right] dF(\theta_0)
\]

\[
= \mathcal{L}_1(M) + \mathcal{L}_2(M)
\]

where we denote by \( \mathcal{L}_1(M) \) the first integral and by \( \mathcal{L}_2(M) \) the second.
4. Properties of $U$ and $\frac{\partial U}{\partial y}$ as given by (1).

$U(M)$ as given by (1) is the sum of two functions harmonic in the circle and it therefore is harmonic in the same region. It is of interest to know the values of $U$ and $\frac{\partial U}{\partial y}$ as they approach the arc $(\pi, \phi)$ and the diameter $(-1, 1)$ respectively.

Let $Q$ be an interior point of $(\phi, 1)$ i.e., an interior point of the point set composing the circumference of the semi-circle. Consider

$$\mathcal{L} U(M) = \mathcal{L} \int_{1}^{\pi} \left[ \frac{1 - 1}{1 + 1 - 2 \cos(\phi - \phi)} + \frac{1 - 2}{1 + 1 - 2 \cos(\phi + \phi)} \right] d(F, \phi).$$

The second term approaches zero since the denominator remains bounded but greater than zero. The first term is a Poisson's integral and approaches $F'(Q)$, if $F'(Q)$ exists. It is easily seen that

$$\mathcal{L} U(M) = \mathcal{L} U(M)$$

and if these limits exist for any approach of $M$ to $Q$ they exist in the wide sense. Therefore $\mathcal{L} U(M) = F'(Q)$ wherever $F'(Q)$ exists in $(\phi, 1)$ for approach in the wide sense.

Now consider

$$\frac{\partial U(M)}{\partial y} = \frac{1}{2\pi i} \int_{\gamma} \frac{\phi - y}{(\phi - y)^2 - 1} d(F, \phi).$$

and let $M \to Q$ an interior point of $(-1, 1)$. As $M \to Q$; $\frac{\partial U}{\partial y}$ and

$$\mathcal{L} \frac{\partial U(M)}{\partial y} = 0.$$

Now

$$\frac{\partial U(M)}{\partial y} = \frac{1}{\pi} \int_{\gamma} \frac{\phi - y}{(1 - x)^2 + y^2} d(F, \phi).$$

As $M \to Q$ interior to $(-1, 1)$ the first term in the integral approaches zero. The second term is the Poisson's integral for a straight line and approaches $F'(Q)$ if $F'(Q)$ exists. If $F'(Q)$ exists all of these limits exist for approach in the wide sense, and therefore

$$\mathcal{L} \frac{\partial U(M)}{\partial y} = \mathcal{L} \frac{\partial U(M)}{\partial y} = H'(Q).$$
wherever $\mathcal{H}(\mathcal{O})$ exists for approach in the wide sense of $M$ to a point $\mathcal{O}$ strictly interior to $(-\mathcal{O})$. 
5. Properties of \[ f_u, d\theta \] and \[ f_u, d\theta \]

In order to obtain a unique representation of a harmonic function in the form \( \phi \) it is necessary to utilize some properties of the integrals

\[ f_u, d\theta, \int \frac{\partial u}{\partial \phi} d\phi, \int u_{,\phi} d\phi \text{ and } \int \frac{\partial^2 u}{\partial \phi^2} d\phi. \]

Moreover if formula \( \phi \) is a solution of the mixed problem it must satisfy the given boundary conditions.

We shall first study \( f_u, d\theta \). Let \( F(-\phi) = -F(\phi) \), \( F(\phi) \) thus defined is of bounded variation on the complete circumference \( (-\pi, \pi) \) with a regular discontinuity at \( \phi \). Make a change of variable from \( \varphi \) to \( -\varphi \) in the second term of the integral of \( f_u, d\theta \).

We have

\[
\phi_1 (M) = \frac{1}{2\pi} \int_0^{\pi} \int_0^{\pi} \frac{1 - \cos(\varphi - \theta)}{1 + \cos(\varphi - \theta)} \, d\varphi \, d\theta.
\]

\[
= \frac{1}{2\pi} \int_0^{\pi} \int_0^{\pi} \frac{1 - \cos(\varphi - \theta)}{1 + \cos(\varphi - \theta)} \, d\varphi \, d\theta.
\]

\[
= \frac{1}{2} \int_0^{\pi} \int_0^{\pi} \frac{1 - \cos(\varphi - \theta)}{1 + \cos(\varphi - \theta)} \, d\varphi \, d\theta.
\]

\[
= \frac{1}{2} \int_0^{\pi} \int_0^{\pi} \frac{1 - \cos(\varphi - \theta)}{1 + \cos(\varphi - \theta)} \, d\varphi \, d\theta.
\]

\[
\phi_1 (M) \text{ is thus the difference of two non-negative harmonic functions in the complete circle and}
\]

\[
L \int_0^{\pi} \phi_1 (M) d\theta = \int_0^{\pi} \phi_1 (M) d\theta \leq \int_0^{\pi} \phi_1 (M) d\theta.
\]

This equation is of course true for \( \theta \), \( \phi \) anywhere in the complete arc \( (-\pi, \pi) \) but we are interested ultimately in \( L \int f_u, d\theta \).

Clearly \( L \int_0^{\pi} \phi_1 (M) d\theta = 0 \) for \( \theta \in (\pi, -\pi) \).

Now consider

\[
\int_0^{\pi} \phi_1 (M) d\theta = \int_0^{\pi} \phi_2 (M) d\theta = 0.
\]

Evidently

\[
L \int_0^{\pi} \phi_1 (M) = 0.
\]
By a change of order of integration

\[ I_2(x) = \frac{1}{2\pi} \int_0^{2\pi} dH(x,\theta) \int_0^1 r^2 dr \theta \left\{ -\frac{c}{2} \left[ \log \left( \frac{r - \sqrt{1 - r^2}}{2} \right) \right] \right\} \]

Consider the part of this integral in a circle of radius \( \eta \) with center at \((l, s)\).

\[ I_2(x) = \frac{1}{2\pi} \int_0^{2\pi} dH(x,\theta) \int_0^1 r^2 dr \theta \left\{ -\frac{c}{2} \left[ \log \left( \frac{r - \sqrt{1 - r^2}}{2} \right)^2 \right] \right\} \]

\[ = I_2(x) - I_2''(x). \]

Now

\[ \int_0^{2\pi} I_2''(x) \, d\theta = 0 \quad \text{and} \]

\[ \int_0^{2\pi} I_2(x) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} dH(x,\theta) \left\{ -\log \left( \frac{r - \sqrt{1 - r^2}}{2} \right)^2 \right\} \]

\[ \geq \frac{1}{2\pi} \int_0^{2\pi} dH(x,\theta) \left\{ -\log \left( \frac{r - \sqrt{1 - r^2}}{2} \right)^2 \right\} \]

\[ \geq \frac{1}{2\pi} \int_0^{2\pi} dH(x,\theta) \left\{ -\log \left( \frac{r - \sqrt{1 - r^2}}{2} \right)^2 \right\} \]

Now for the point \( P \) at \( x = 1 \); \( I_2''(x) \) can be made as small as we please by taking \( \eta \) small enough. Therefore

\[ \int_0^{2\pi} I_2(x) \, d\theta = 0 \]

and

\[ \int_0^{2\pi} I_2(x) \, d\theta = \int_1^{2\pi} I_2(x) \, d\theta = \int_1^{2\pi} I_2(x) \, d\theta = \int_1^{2\pi} \theta \, d\theta = \frac{1}{2}\pi(2\pi - \pi) = \pi. \]
6. Properties of \( \int \frac{\partial \xi}{\partial \eta} \, ds \) and \( \int \frac{\partial \eta}{\partial \eta} \, ds \).

The integrals of this section are independent of the path of integration, being functions merely of the end points of the path. We shall however consider these integrals along paths for their connection in a later section.

Take the pole of \( p(M) = c \) at \( (\xi = 1, \eta = \xi_2) \) and consider

\[
\int_{\Gamma_{1} : x_1 = c} \frac{\partial \xi}{\partial \eta} \, ds = \int \frac{\partial \xi}{\partial \eta} \, ds.
\]

Denote by \( y(M) \) the conjugate of \( y(M) \).

\[
\int_{\Gamma_{1} : x_1 = c} \frac{\partial \xi}{\partial \eta} \, ds = \int \frac{\partial \xi}{\partial \eta} \, ds = \int \frac{\partial \xi}{\partial \eta} \, ds.
\]

Then

\[
\int_{\Gamma_{1} : x_1 = c} \frac{\partial \xi}{\partial \eta} \, ds = \int \frac{\partial \xi}{\partial \eta} \, ds = \int \frac{\partial \xi}{\partial \eta} \, ds.
\]

where \( A \) and \( B \) are the points of intersection of \( y = c \) and

\[
\int_{\Gamma_{1} : x_1 = c} \frac{\partial \xi}{\partial \eta} \, ds = \int \frac{\partial \xi}{\partial \eta} \, ds.
\]

As \( c \to 0 \), \( \Theta_A \) and \( \Theta_B \) approach \( 0 \) and therefore

\[
\int_{\Gamma_{1} : x_1 = c} \frac{\partial \xi}{\partial \eta} \, ds = 0 \quad \text{for} \quad (-1 < \xi, \xi_2 < 1)
\]

Let \( \phi_0 \) and \( \phi_1(\xi) \) denote the positive and negative variation functions respectively of \( \phi(\xi) \). By substitution in the formula for \( \int_{\Gamma_{1} : x_1 = c} \frac{\partial \xi}{\partial \eta} \, ds \) it is easily seen that \( \int_{\Gamma_{1} : x_1 = c} \frac{\partial \xi}{\partial \eta} \, ds \) is the difference of two non-negative functions \( l_1 \) and \( l_1' \) in the semi-circle which have the properties

\[
\int_{\Gamma_{1} : x_1 = c} \frac{\partial \xi}{\partial \eta} \, ds = 0 \quad \text{and} \quad \int_{\Gamma_{1} : x_1 = c} \frac{\partial \xi}{\partial \eta} \, ds = 0.
\]

for \( (-1 < \xi, \xi_2 < 1) \).
Now consider
\[ \int_{k \neq k_1} \frac{2}{n} \frac{dU}{dx} \, dx \]

Let \( V_2^1(M) \) be the conjugate of \( U_2^1(M) \) i.e.
\[ V_2^1(M) = -i \int_{k}^{1} \left[ \tan^{-1} \frac{i \lambda x}{1 - i \lambda x} - \tan^{-1} \frac{i \lambda x}{x - k} \right] \, dH(x) \]

Let \( H(x) = \int_{-\infty}^{x} \frac{dF}{dx} \) be the function of limited variation on the whole \( x \)-axis \( (-\infty, \infty) \). Take a change of variable from \( \lambda x \) to \( \lambda x - k \) in the first term of the integral of \( V_2^1(M) \). We have
\[ V_2^1(M) = -i \int_{-\infty}^{1} \tan^{-1} \frac{\lambda x}{1 - \lambda x} \, dH(x) \]
\[ + i \int_{-1}^{1} \tan^{-1} \frac{\lambda x}{x - k} \, dH(x) \]
\[ = -i \int_{-\infty}^{\infty} \tan^{-1} \frac{\lambda x}{1 - \lambda x} \, dH(x) \]
\[ - \frac{i}{\lambda} \int_{-\infty}^{\infty} \tan^{-1} \frac{\lambda x}{x - k} \, dH(x) \]

Now
\[ \int_{k \neq k_1} \frac{2}{n} \frac{dU}{dx} \, dx = \int_{k \neq k_1} \frac{dV}{dx} \, dx = -V_2^1(M) - V_2^1(\delta) \]
\[ \delta = c \]
\[ \delta = c \]
where \( \delta \) and \( \beta \) have same meaning as before.

Therefore
\[ \int_{k \neq k_1} \frac{2}{n} \frac{dU}{dx} \, dx = -i \int_{-\infty}^{\infty} \left[ \tan^{-1} \frac{\lambda x}{1 - \lambda x} - \tan^{-1} \frac{\lambda x}{x - k} \right] \, dH(x) \]
\[ \delta = c \]
\[ = -i \int_{-\infty}^{\infty} \left( V_2^1(M) - V_2^1(\delta) \right) \, dH(x) \].
and 
\[ \int_{c=0}^{h} \frac{\partial \tilde{h}}{\partial \eta} \, ds = \int_{-\infty}^{\infty} \psi(p) \, dH(x_p) \]

where
\[ \psi(p) = \begin{cases} 0 & \text{if } p \text{ is outside } (x_1, x_2) \\ \infty & \text{if } p \text{ is either } x_1 \text{ or } x_2 \\ 1 & \text{if } p \text{ is interior to } (x_1, x_2) \end{cases} \]

Therefore
\[ \int_{c=0}^{h} \frac{\partial \tilde{h}}{\partial \eta} \, ds = H(x_2) - H(x_1) \quad (x_1 = x_2 = 1). \]

and
\[ \int_{c=0}^{h} \frac{\partial \tilde{h}}{\partial \eta} \, ds = \int_{c=0}^{h} \frac{\partial \tilde{h}}{\partial \eta} \, ds = H(x_2) - H(x_1) \]

\[ \int_{c=0}^{h} \frac{\partial \tilde{h}}{\partial \eta} \, ds = \int_{c=0}^{h} \frac{\partial \tilde{h}}{\partial \eta} \, ds = H(x_2) - H(x_1) \]

\[ \int_{c=0}^{h} \frac{\partial \tilde{h}}{\partial \eta} \, ds = \int_{c=0}^{h} \frac{\partial \tilde{h}}{\partial \eta} \, ds = H(x_2) - H(x_1) \]

We can deduce a property of \[ \int \frac{\partial \tilde{h}}{\partial \eta} \, ds \] that is useful also as a sufficient condition for the determination of \[ x_2(x_1) \].

To prove this it is only necessary to note that \[ \tilde{h}(x_1, x_2) \] is a convex curve and that for a fixed \( x_1 \) and fixed \( x_2 \), the \( \tilde{h}(x_1, x_2) \) above increases to a point \( \tilde{c} \), and then decreases to \( 0 \) as \( x_2 \) increases to \( 1 \). The increase is at most \( \tilde{c} \). Therefore
\[ \int \frac{\partial \tilde{h}}{\partial \eta} \, ds = \int \frac{\partial \tilde{h}}{\partial \eta} \, ds \leq 2 \int_{-\infty}^{\infty} H(x_1) \, ds \]

\[ = 2 \int_{-\infty}^{\infty} H(x_1) \, ds \leq 2 \int_{x_1}^{x_2} H(x_1) \, ds \]

Therefore
\[ \int \frac{\partial \tilde{h}}{\partial \eta} \, ds \leq \kappa. \]

The result is true of course for any convex path in the semi-circle.
7. Necessary and sufficient conditions for formula 1.

The properties of \( \int \int \) 5 and 6 are sufficient as well as necessary for formula 1. We have the theorem.

Theorem 1. The necessary and sufficient condition that \( U'(m) \) harmonic in the semi-circle, may be represented by formula 1, are that

(a) \( U'(m) = U_1'(m) + i U_2'(m) \) where \( U_1, U_2, \) and \( i U_2 \) are each harmonic in the semi-circle.

(b) \( U_1'(m) = U_1'(m) - i U_2'(m) \) where \( U_1, \) and \( i U_2 \) are each non-negative harmonic functions in the semi-circle, such that

\[
\int_{-\pi}^{\pi} \int_{\gamma}^{\infty} \frac{1}{r^2} \frac{d\gamma}{\pi} \, d\alpha = 0
\]

for all closed intervals in \( \gamma, \alpha \).

(c) \( \int_{\gamma}^{\infty} U_2'(m) \, d\gamma = 0 \) for any point of the open interval \( (\gamma, \infty) \)

and

\[
\int_{\gamma}^{\infty} \frac{dU_2'(m)}{\pi} \, d\gamma < \infty
\]

Let \( V_1' \) and \( V_2' \) denote the conjugates of \( U_1' \) and \( U_2' \) respectively.

Since

\[
\int_{\gamma}^{\infty} \frac{dU_1'(m)}{\pi} \, d\gamma = \int_{\gamma}^{\infty} \frac{dV_2'(m)}{\pi} \, d\gamma = - \int_{\gamma}^{\infty} \frac{dV_1'(m)}{\pi} \, d\gamma
\]

This constant is not \( \infty \) for taking the integral along any path joining \( \gamma, \infty \) and \( \gamma, \infty \) in the semi-circle we could have \( V_1'(m) \) infinite at points interior to the semi-circle, and for a similar reason \( V_2'(m) \) infinite. Now define \( U_1' \) and \( U_2' \) in the lower semi-circle as the conjugate of \( V_1' \) and \( V_2' \) extended across \( (-\pi) \). \( U_1'(m) \) is then the difference of two non-negative functions in the complete circle and \( U_1'(\gamma, \infty) = U_1'(\gamma, -\infty) \).

Therefore

\[
U_1'(m) = \frac{\pi}{2} \int_{\gamma}^{\infty} \frac{1 - \lambda^2}{\lambda^2 + 16 \cos^2 (\gamma - \theta)} \, d\theta
\]

where

\[
F(\gamma) + F(\gamma) = \int_{\gamma}^{\infty} U_1'(m) \, d\theta.
\]

Since

\[
U_1'(\gamma, \infty) = U_1'(\gamma, -\infty)
\]

we have

\[
\int_{\gamma}^{\infty} U_1'(\gamma, \theta) \, d\theta = \int_{\gamma}^{\infty} \frac{1 - \lambda^2}{\lambda^2 + 16 \cos^2 (\gamma - \theta)} \, d\theta
\]

or

\[
F(\theta_1) - F(\theta_2) = - \int [F(\theta_2) - F(\theta_1)]
\]

Let \( \theta = 0 \) then we have

\[
F(0) = - F(-\theta)
\]

Now

\[
U_1'(m) = \frac{\pi}{2} \int_{\gamma}^{\infty} \frac{1 - \lambda^2}{\lambda^2 + 16 \cos^2 (\gamma - \theta)} \, d\theta
\]
Make a change of variable of integration from \( \varphi_p \) to \( -\varphi_p \) in the first integral using the fact that \( F(\varphi) = -F(-\varphi) \), and we have

\[
U_{0}(\mu) = \frac{1}{2\pi i} \int_{0}^{\infty} \left[ \frac{1+\mu^2}{1+\mu^2 \cos(\varphi-\varphi)} + \frac{1-\mu^2}{1+\mu^2 \cos(\varphi+\varphi)} \right] dF(-\varphi).
\]

To determine \( U_{2}(\mu) \), assume \( \int_{\phi}^{\infty} \frac{d\mu}{\sqrt{\mu}} = \gamma(\mu) \). \( \gamma(\mu) \) is a one parameter family of level curves of the Green's function for the upper half plane with pole at \( x = \sigma \), \( \sigma > 0 \). The points \( A \) and \( B \) are the points of intersection of \( y = c \) with the arc \( \phi \xi \). \( \xi \in \mathbb{C} \) is orthogonal to the arc \( (\xi, x) \) and \( y = 0 \) is the x-axis. We also assume \( U_{0}(\mu) = 0 \) on the open interval \( (\xi, x) \). Then \( U_{0}(\mu) \) can be extended harmonically across the arc \( (\xi, x) \) and \( U_{0}(\xi, x) = U_{0}(\xi, x) \). An inversion such as this followed by a reflection in the x-axis constitutes a conformal transformation \( \gamma \) except for sign \( \int_{\phi}^{\infty} \frac{d\mu}{\sqrt{\mu}} \) is equal to \( \int_{A}^{B} \frac{d\mu}{\sqrt{\mu}} \) and \( \int_{\phi}^{\infty} \frac{d\mu}{\sqrt{\mu}} \). Now transform the upper half plane into the unit circle by means of a conformal transformation. The upper semi-circle goes into the lower semi-circle and the curves \( \gamma(\mu) \) go into circles concentric with the unit circle. Then we have \( \int_{\phi}^{\infty} \frac{d\mu}{\sqrt{\mu}} \) independently of \( \mu \) on these circles. Therefore

\[
\int_{\phi}^{\infty} \frac{d\mu}{\sqrt{\mu}} = \gamma(\mu), \quad (-\pi \leq \varphi, x_{0} = \pi)
\]

and there is one and only one such function satisfying this boundary condition. Transforming back we have one and only one function of the class

\[
\int_{A}^{B} \frac{d\mu}{\sqrt{\mu}} < \kappa
\]

such that

\[
\int_{A}^{B} \frac{d\mu}{\sqrt{\mu}} = \gamma(\mu) - \gamma(\mu), \quad (-\pi \leq \varphi, x_{0} = \pi)
\]

Therefore

\[
U_{2}(\mu) = -\kappa \int_{\phi}^{\infty} \left[ \log(1-\kappa \gamma) - \log(1+\kappa \gamma) \right] d\gamma.
\]

We therefore have theorem 2.

**Theorem 2.** Among the class of functions satisfying the conditions of theorem 1; there is one and only one satisfying the boundary conditions

\[
\int_{A}^{B} \frac{d\mu}{\sqrt{\mu}} (\varphi, x) = 0, \quad (d = 0, \varphi_{0} = \pi), \quad (\xi \geq 0, x_{0} \geq 1)
\]

\[
\int_{A}^{B} \frac{d\mu}{\sqrt{\mu}} (\varphi, x) = H(x) - H(x), \quad (\xi \geq 0, x_{0} \geq 1)
\]

It is given by formula 1 which suggests the possibility of a solution of the problem directly by means of potentials of single and double distributions. Accordingly, for the function so determined

\[
\int_{A}^{B} \frac{d\mu}{\sqrt{\mu}} (\varphi, x) = F(\varphi) - F(\varphi), \quad (0 \leq \varphi, x_{0} \leq 1)
\]

\[
\int_{A}^{B} \frac{d\mu}{\sqrt{\mu}} (\varphi, x) = H(\varphi) - H(\varphi), \quad (\xi \geq 0, x_{0} \geq 1).
\]
8. Solution by means of potentials of single and double layers.

Imagine a general distribution of a single layer \( \mu(x) \) on the \( x \)-axis and a general distribution of a double layer \( \mu(x) \) on the total circumference of the unit circle. These distributions are functions of limited variation and determine a function harmonic in the whole plane except on the circumference and \( x \)-axis. In particular this function is harmonic in the upper semi-circle and we shall assume it is regular at \( \infty \). Then we have

\[
\mu(L) = \int_{\infty}^{\infty} \frac{\cos \left( \frac{m_1 \pi r}{2} \right)}{m_1 \pi} d\mu(x) + \int_{-\infty}^{\infty} \log \frac{1}{m_1 \pi} d\mu(x) + \chi.
\]

Impose the conditions

\[
\int_{-\infty}^{\infty} \mu(x) \, dx = \varphi_1 - \varphi_2
\]

where \( \varphi_1 \) and \( \varphi_2 \) are any two points in the open interval \((-\infty, \infty)\).

\[
\int_{-\infty}^{\infty} \frac{\mu(x)}{x - y} \, dx = \varphi_1 - \varphi_2,
\]

where \( \varphi_1 \) and \( \varphi_2 \) are any two points in \((-\infty, \infty)\).

\[
\frac{\mu(x)}{x - y} \text{ and } \frac{\mu(x)}{x - y} \text{ are functions of limited variation with regular discontinuities on their respective ranges, and we assume that } \frac{\mu(x)}{x - y} \text{ exists for } x = y.
\]

From the first of conditions (4)

\[
\int_{-\infty}^{\infty} \frac{\mu(x)}{x - y} \, dx = \int_{-\infty}^{\infty} \frac{\cos \left( \frac{m_1 \pi r}{2} \right)}{m_1 \pi} \, dx\int_{-\infty}^{\infty} \log \frac{1}{m_1 \pi} \, dx\mu(x) + \chi = \varphi_1 - \varphi_2.
\]

Now

\[
\frac{\cos \left( \frac{m_1 \pi r}{2} \right)}{m_1 \pi} = \frac{1}{2} \frac{1 - \frac{1}{1 + \frac{1}{2}}} = 2 \pi \cos \left( \frac{\pi r}{2} \right)
\]

Changing the order of integration in the above condition we have
\[ \sum_{n=1}^{n} \int_{x}^{\infty} \frac{dU(x)}{dx} \int_{\Theta}^{\Theta + \frac{2\pi}{n}} \frac{d\Theta}{2 (1 + \frac{n}{2} \cos(\Theta))} \Theta d\Theta + \sum_{n=1}^{n} \int_{x}^{\infty} \frac{dU(x)}{dx} \int_{\Theta}^{\Theta + \frac{2\pi}{n}} \frac{d\Theta}{2 (1 + \frac{n}{2} \cos(\Theta))} \Theta d\Theta + \int_{\Theta - \Theta}^{\Theta - \Theta} \frac{d\Theta}{2 (1 + \frac{n}{2} \cos(\Theta))} \Theta d\Theta + \int_{\Theta - \Theta}^{\Theta - \Theta} \frac{d\Theta}{2 (1 + \frac{n}{2} \cos(\Theta))} \Theta d\Theta. \]

Passing to the limit,

\[ \lim_{\Theta \to \infty} \left[ \int_{\Theta}^{\infty} \frac{dU(x)}{dx} \int_{\Theta}^{\Theta + \frac{2\pi}{n}} \frac{d\Theta}{2 (1 + \frac{n}{2} \cos(\Theta))} \Theta d\Theta \right] = \int_{0}^{\infty} \frac{dU(x)}{dx} \int_{0}^{\frac{2\pi}{n}} \frac{d\Theta}{2 (1 + \frac{n}{2} \cos(\Theta))} \Theta d\Theta. \]

From the second of conditions,

\[ \int_{0}^{\frac{2\pi}{n}} \frac{dU(x)}{dx} \int_{0}^{\frac{2\pi}{n}} \frac{d\Theta}{2 (1 + \frac{n}{2} \cos(\Theta))} \Theta d\Theta = \int_{0}^{\frac{2\pi}{n}} \frac{dU(x)}{dx} \int_{0}^{\frac{2\pi}{n}} \frac{d\Theta}{2 (1 + \frac{n}{2} \cos(\Theta))} \Theta d\Theta. \]

Interchanging the order of integration and passing to the limit we have

\[ \int_{0}^{\frac{2\pi}{n}} \frac{dU(x)}{dx} \int_{0}^{\frac{2\pi}{n}} \frac{d\Theta}{2 (1 + \frac{n}{2} \cos(\Theta))} \Theta d\Theta = \int_{0}^{\frac{2\pi}{n}} \frac{dU(x)}{dx} \int_{0}^{\frac{2\pi}{n}} \frac{d\Theta}{2 (1 + \frac{n}{2} \cos(\Theta))} \Theta d\Theta. \]
We therefore have the two integral equations

\[ (a) \quad \pi \left[ \mu(\theta) - \nu(\theta) \right] - \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\mu(x_p)}{dx_p} \int_0^\theta \frac{\log (1 + x_p^2 - 2 x_p \cos \theta)}{x_p^2} d\theta \]

\[ + \frac{\pi}{2} \left[ \mu(\theta) - \nu(\theta) \right] = \pi(\theta) - \pi(\theta_0). \]

\[ (b) \quad -\pi \left[ \mu(x_m) - \nu(x_m) \right] + \int_{-\infty}^{\infty} \frac{d\mu(x_p)}{dx_p} \int x_p \frac{1}{x_p^2} \frac{d\mu(x_m)}{dx_m} \frac{1}{1 + x_m^2 - 2 x_m x_p^2} d\theta = \pi(x_m) - \pi(x_m_0). \]

which \( \mu(x) \) and \( \nu(x) \) must satisfy simultaneously in order that \( \pi(x) \)
given by (3) may satisfy (4). In order to find a solution of (5)
restrict \( \pi(\theta) \) and \( \pi(x) \) by the following conditions:

\[ \pi(\theta) = -\pi(-\theta), \]

\[ \pi(x) = -\pi(-x). \]

and let \( \chi_p = \frac{1}{2} \int \frac{d\mu(x_p)}{dx_p} \int \frac{d\mu(x_m)}{dx_m} \pi(x_m) \pi(x_m_0). \)

This integral converges because \( \mu(x) \) is regular at \( \infty \).

In equation (5) replace \( \pi \) by \( \frac{1}{\pi} \) and \( \chi \) by \( \frac{1}{\chi} \in \) the kernel and right hand member. After a change of variable \( \chi_p = \frac{1}{\chi} \)
the equation that \( \mu(x) \) satisfies is the same except for sign as the
one that \( \mu(x) \) satisfies. We may therefore take \( \mu(x) = -\mu(x) \).

Since the kernel \( \to \infty \) like \( \frac{1}{\pi} \) \( \pi \) we cannot say that this solution is
unique. Now in \( \pi(x) \), write the kernel as the sum of three
integrals

\[ -\frac{1}{2} \int_{-\infty}^{\infty} \frac{d\mu(x_p)}{dx_p} \int_0^\theta \frac{d\mu(x_m)}{dx_m} \frac{\log (1 + x_p^2 - 2 x_p \cos \theta)}{x_p^2} d\theta \]

and make the change of variable \( \chi_p \) to \( \frac{1}{\chi_p} \) in the first and last integrals.

We have

\[ -\frac{1}{2} \int_{-\infty}^{\infty} \frac{d\mu(x_p)}{dx_p} \int_0^\theta \frac{d\mu(x_m)}{dx_m} \frac{\log (1 + x_p^2 - 2 x_p \cos \theta)}{x_p^2} d\theta \]

\[ -\frac{1}{2} \int \log x_p d\mu(x_p) \int_0^\theta d\theta. \]
Substituting in (2) we see that the last two terms of the left-hand member of (2) cancel and
\[ 2 \left[ \nu (\Theta) - \nu (\Theta, \varphi) \right] = F (\Theta) - F (\Theta, \varphi) \]

Now since
\[ F (\Theta) = F (\Theta, \varphi) \]
\[ \nu (\Theta) = \nu (\Theta, \varphi) \]
Substituting in (2), the integral term cancels by symmetry and
\[ -\pi \int \left[ \nu (x, y) - \nu (x, y_0) \right] = \mu (x) - \mu (x_0). \]

Substitute these values of \( \nu (x, y) \) and \( \nu (\Theta) \) in (3) and add and subtract
\[ \frac{i}{2} \int_{-\pi}^{\pi} \nu (x, y) \, d\varphi = 0 \]

We obtain
\[ L (\Theta) = \frac{i}{2} \int_{-\pi}^{\pi} \frac{e^{-i \varphi}}{\sin \varphi} \, dF (\varphi) \]
\[ = -\frac{i}{2} \int_{-\pi}^{\pi} \frac{e^{-i \varphi}}{\sin \varphi} \, dF (\varphi) \]
\[ = \frac{i}{2} \int_{-\pi}^{\pi} \log \left( 1 + e^{-i \varphi} \right) - \log \left( 1 + e^{i \varphi} \right) \, dF (\varphi) \]

In account of the relations \( F (\Theta) = F (-\Theta) \) and \( \mu (\Theta) = -\mu (\Theta) \),
\[ L (\Theta) = \frac{i}{2} \int_{-\pi}^{\pi} \left[ \frac{e^{-i \varphi}}{\sin \varphi} + \frac{e^{i \varphi}}{\sin \varphi} \right] \, dF (\varphi) \]
\[ = \frac{i}{2} \int_{-\pi}^{\pi} \log \left( 1 + e^{-i \varphi} \right) - \log \left( 1 + e^{i \varphi} \right) \, dF (\varphi) \]

This solution is slightly less general than the one we have obtained by means of the Green's function since \( g (\varphi) \) must exist. In order to get a solution of our integral equations we have assumed that
\[ F (\Theta) = -F (-\Theta) \] and \( \mu (\Theta) = -\mu (\Theta) \) which is equivalent to
assuming that the limit of the integral of the term
\[ \int_{-\infty}^{\infty} \frac{\cos (n \cdot \varphi)}{n^2} \, d\varphi \]
is zero along the x-axis and that
\[ \lim_{M \to \infty} \int_{0}^{\pi} \left[ \int_{-\infty}^{\infty} \frac{\log \frac{1}{n} \, d\lambda \varphi + \pi} \right] \]
is zero for \( \varphi = \pi \) . With these added conditions our theorem of \( \mathcal{F} \) shows that we have the only solution of our integral equations.
9. The solution for a simply connected plane region.

Let \( T \) be a simply connected plane region with boundary \( S \). Let \( w = f(z) \) be a holomorphic transformation which transforms the semicircle into \( T \); the transformation is one-one and conformal at interior points, and transforms the points of the boundary of the semicircle in a one-one manner into the prime ends constituting the boundary of \( T \). Let \( S_1 \) be the closed set in \( S \) which corresponds to the semicircumference \( (0 \leq \theta \leq \pi) \) and \( S_2 \), the closed set in \( S \) which corresponds to the diameter \((r = x = 0)\). The order of the accessible points of such boundaries has been discussed by G. C. Evans with reference to the conjugate function to the Green's function for the region, with arbitrarily given pole \( Q \).

In the semicircle we can express the relations \( y = c \) and \( r = c \), and the relations which describe the orthogonal trajectories of these families of curves, in terms of the Green's function \( G(\omega) \) and its conjugate for the semicircle, taken as curvilinear coordinates, the coordinates of points of these curves interior to the semicircle being analytic functions of the curvilinear coordinates. In \( T \), the images of the curves \( y = c \) and \( r = c \), which we call \( \zeta \) and \( \zeta \), provide one parameter families of analytic cross-cuts of \( T \); their orthogonal trajectories are also analytic cross-cuts; we may denote their parameters by \( \zeta \) and \( \zeta \).

Two curves of any one family cut each other at points of \( T \), and if a point is taken on each curve of the family the limit points of the set thus constituted lie always in \( S \). The families constitute therefore nests of cross-cuts.

If \( h \) corresponds to a value of \( \theta \), \( 0 < \theta < \pi \), we say that \( h \) corresponds to an interior prime end of \( S_1 \); if \( h \) corresponds to a value of \( x \), \( -1 < x < 1 \), we say that \( h \) corresponds to an interior prime end of \( S_2 \).

The theorems just given now apply to this general region, since all their statements and hypotheses may obviously be stated in terms which are invariant of the conformal transformation from the semicircle to \( T \).

Theorem 3. Among the functions of the class that are harmonic in \( T \) which are the sum of two functions \( u \) and \( v \) harmonic in \( T \), with the following properties:
(a) \( u = u'_1 + u'_2 \), where \( u'_1 \) and \( u'_2 \) are harmonic and \( \geq 0 \) in \( T \)
(b) \( \int_{c_0}^{c_1} \frac{\partial u'}{\partial n} \, ds = 0 \), \( \int_{c_0}^{c_1} \frac{\partial u''}{\partial n} \, ds = 0 \)

On \( \gamma \)

where \( h'_1 \) and \( h'_2 \) correspond to interior points of \( S_2 \).
(c) \( u_2 \) takes on continuously boundary values zero for interior prime ends of \( S_1 \).
(d) \( \int |\nabla u_2|^2 \, ds < K \)

where the integration is extended along the complete cross-cut \( \gamma \); there is one and only one which satisfies the boundary conditions:
(\( \gamma \))

where \( h_1 \) and \( h_2 \) correspond to interior points of \( S_1 \).
\[
\mathcal{L} \int_{c=0}^{h_{e'}} \frac{\partial u}{\partial n} \, ds = H(h_{e'}) - H(h_{i'}). 
\]

where \( h' \) and \( h_{e'} \) correspond to interior points of \( S \) and \( F(h) \) and \( H(h) \) are of limited variation as functions of \( h \) on their respective open intervals.

\[
\begin{align*}
\mathcal{L} \int_{c=0}^{h_{e}} U_{1}(u) \, dh &= C_{1}, \quad \mathcal{L} \int_{c=0}^{h_{i}} U_{2}(u) \, dh = C_{2} \\
\mathcal{L} \int_{c=0}^{h_{e}} \frac{\partial u}{\partial n} \, ds &= \Gamma_{1}, \quad \mathcal{L} \int_{c=0}^{h_{i}} \frac{\partial u}{\partial n} \, ds = \Gamma_{2} \\
\end{align*}
\]

where \( C_{1}, C_{2}, \Gamma_{1}, \Gamma_{2} \) are arbitrary, and the subscript 1 or 2 in \( G_{c} \) or \( \gamma_{c} \) indicates that the integration is extended over that part of the cross-cut \( G_{c} \) or \( \gamma_{c} \) which corresponds to the right hand or left hand halves of the images of these curves in the semicircle.

In fact,

\[
\begin{align*}
\mathcal{L} \int_{h_{i}}^{h_{e}} U_{1}(u) \, dh &= \int_{h_{i}}^{h_{e}} v_{1}(u) \, dh \\
\mathcal{L} \int_{h_{i}}^{h_{e}} \frac{\partial u}{\partial n} \, ds &= \int_{h_{i}}^{h_{e}} \frac{\partial v}{\partial n} \, ds \\
\end{align*}
\]

when \( h_{i}, h_{e}, h_{i}', h_{e}' \) correspond to interior points of their respective intervals.
References

(1) Hadamard, Sur la propagation des ondes et les équations hydrodynamique, pp 56,57.


(7) Gevrey, Détermination et emploi des fonctions de Green dans les problemes aux limites relatifs aux équations linéaires du type elliptique, pp. 1-80, Journal de Mathématiques, 1930.

(8) Picard, Sur un théorem général relatif aux équations intégrales de premièr espèce etc. Rendiconti del Circolo matematico di Palermo, T XXIX, 1910.