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UMI
NUMERICAL STUDY OF THE DIRECT
INTERACTION THEORY OF DEUTERON STRIPPING

by

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I. INTRODUCTION

A. Statement of the Problem

If a target is bombarded with a beam of deuterons it is observed, in certain cases, that protons are present among the reaction products. There may be, in fact, several groups of protons corresponding to the different possible levels of excitation of the residual nucleus. At the present time, two competing mechanisms by which this reaction takes place have been proposed.

The first is the compound nucleus model. According to this model the deuteron is captured to form a compound nucleus consisting of the original A nucleons plus the deuteron. This process is much more probable if there exists a metastable state in the A+2 nucleus at an energy equal to the total energy of the system. After this state has existed for some time the proton is emitted leaving the A+1 nucleons in either the ground state or some excited state. By the use of this model, it is possible to understand the variation in the total number of protons as a function of energy for some cases. One difficulty, however, is that if the excited state of the compound nucleus and the final state of the residual nucleus are assumed to have definite parities, or if it is assumed that there are contributions from a large number of metastable states of the compound nucleus, then the angular distribution should be symmetric about 90°. This is very often not the case.
The second mechanism is known as the direct interaction model. In this picture the deuteron is pictured as spending very little time in the vicinity of the nucleus before the proton is emitted. The neutron is, so to speak, "transferred in flight" from the deuteron to the nucleus. The neutron is assumed to be captured into a shell model state\(^{(3)}\). Since, as we shall see later, the angular distribution depends very strongly on the particular shell into which the neutron is captured, a very powerful tool for the study of nuclear spectroscopy is provided. A great deal of the evidence for the order of shell filling in the shell model has been derived in this way from deuteron stripping experiments. Except in rare instances this model predicts a smooth variation of the total cross section with energy.

This paper is concerned entirely with the direct interaction model. A study is made to determine the dependence of the differential cross section and polarization for the \((d,p)\) reaction on the various parameters characterizing the interaction of the deuteron with the target nucleus and the proton with the residual nucleus. For this purpose we assume that these interactions can be represented by the optical model\(^{(4)}\). We will be particularly concerned with the effect of including a spin-orbit interaction in the optical potential.
B. The History of Deuteron Stripping

The first experiments which involved deuteron stripping were those by Lawrence, McMillan and Thornton\(^5\) in which several elements were bombarded with deuterons at varying energies. The radiation produced did not increase with energy as rapidly as predicted by the Gamow barrier penetration function. To explain this Oppenheimer and Phillips\(^6\) in 1935 proposed the stripping mechanism. They suggested that the cross section for stripping should be proportional to the probability of finding a neutron at the surface of the nucleus. If the nucleus is treated as having zero size, this becomes the probability of finding the neutron at the origin. When this was calculated on the basis of an adiabatic approximation, it was found that the experimental data could be fitted.

In 1938 H. A. Bethe\(^7\) made a refinement of the theory to include the finite size of the nucleus. Paislee\(^8\), in 1948, found an improved treatment resulted from using the incident wave function in place of one calculated from the adiabatic approximation.

In 1952 Bhatia et al\(^9\) published a calculation of the \((d,p)\) angular distribution based on the Born approximation. They took the \((d,p)\) matrix element to be the integrand of the Born integral evaluated at the nuclear surface. This gave a spherical Bessel function dependence
which is very similar to the angular distributions observed. However, in order to fit the experimental data, a very large nuclear radius must be assumed.

At the same time S. T. Butler(10) formulated a theory which gave a second term to the amplitude*. At the time, the Butler formulation did not appear to be related to the Born approximation but it was later shown(11,12,13,14,15) that it may be obtained from the Born approximation if one carries out the integration from the nuclear surface to infinity instead of assuming that the only contribution comes from the nuclear surface.

In 1957 Owen and Madansky(16,17) gave an explanation of the strong backward peaking that is sometimes observed by assuming that the target nucleus can be stripped as well as the deuteron.

The average effect of the nucleus on the wave functions was treated by Tobacman and Kalos(18) in 1955 and the effect of the interior of the nucleus was treated by Tobacman(19) in 1959. Both of these treatments were based on the distorted wave Born approximation(20,21).

A full review of deuteron stripping theories at low and intermediate energies has been given by Tobacman(22).

*See Section I-C.
C. Fundamentals of the Theory

Consider the deuteron stripping process in progressive steps of approximation. Take first the plane wave approximation. Classically, this means the deuteron approaches the nucleus along a straight line with constant velocity and the proton leaves the nucleus along a straight line with constant velocity. As the binding between the neutron and proton is relatively weak, the angular distribution will be peaked strongly forward; that is, the path of the proton will not be much disturbed. However, it can be seen that there is a restriction to the amount of forward peaking which comes from the requirement for the conservation of angular momentum\(^{(23)}\).

Let the momentum of the incident deuteron be \(\vec{P}_D = \vec{k}_D\hbar\) and the momentum of the outgoing proton be \(\vec{P}_p = \vec{k}_p\hbar\) then the momentum transfer to the nucleus will be \(\vec{K}_n = (\vec{k}_p - \vec{k}_D)\hbar\). Now assume that the neutron is captured with a definite value of angular momentum, \(\lambda\hbar\), at a radius less than or equal to the radius of the nucleus, R. Then:

\[
\lambda(\lambda + 1) \leq R^2 k^2,
\]
or

\[
\lambda(\lambda + 1) \leq R^2 (k_D^2 + k_p^2 - 2k_p k_D \cos \theta)
\]

where \(\theta\) is the angle of scattering with respect to the direction of the incident deuteron.
This implies that

\[ \cos \theta \leq \frac{K_D^2 + K_P^2}{2K_D K_P} - \frac{\lambda(\lambda+1)}{2K_D K_D R^2} \].

Considering the fact that the peaking should be strongly forward a maximum in the angular distribution would be expected to occur when the condition

\[ K_D^2 R^2 = \lambda(\lambda+1) \]

is met. We see that the peak must move to larger angles as \( \lambda \) is increased.

Indeed if one treats this problem by the Born approximation using plane waves for initial and final states and assumes the stripping occurs only outside the radius \( R \), one obtains the Butler theory prediction of the stripping cross section.

This is

\[ \frac{d\sigma}{d\Omega} = \frac{\gamma M_{fp} M_{ID} (2 J + 1)}{(2 \pi \hbar^2)^2 2 (2 \lambda + 1)(2 I + 1)} |K_P| |B^\lambda|^2 \]

where

\[ B^\lambda = \frac{4\pi \hbar}{M_{np}} \frac{\sqrt{(2 \lambda + 1)\lambda M_{in} R^2}}{(K^2 + K_N^2)} \left[ \frac{4}{3} \frac{\partial}{\partial Y} |n|^{(\bar{u})}_{\lambda}(iK_NY) - \frac{2}{3} \frac{\partial}{\partial Y} j^\lambda(K_NY) \right]_{Y = \infty}. \]

\( \gamma \) is the neutron reduced width. \( M_{pp}, M_{ID}, M_{IN}, M_{NP} \) are reduced masses of the final nucleus and the proton, the
initial nucleus and the deuteron, the initial nucleus and
the neutron, and the neutron and the proton, respectively.

\[ I \] is the spin of the initial nucleus,
\[ J \] is the spin of the final nucleus,
\[ K_N = \sqrt{\frac{2M_{IN}(Q+\epsilon_D)}{\hbar^2}} } \]
\[ Q \] is the "Q-value" of the reaction,
\[ \epsilon_D \] is the binding energy of the deuteron,
\[ a = \sqrt{\frac{2M_{IN}\epsilon_D}{\hbar^2}} } , \]
\[ j_\lambda \] is the spherical Bessel function of order \( \lambda \), and
\[ h_\lambda(1) \] is the spherical Hankel function of the first
kind of order \( \lambda \). This expression has been
derived on the assumption that the neutron-proton interaction
has zero range.

The second term in the brackets is usually small in
comparison with the first and the coefficient is a slowly
varying function of \( \Theta \) so

\[ B_\lambda \sim j_\lambda(KR) \]

It may be noted that this function has a large relative
maximum at \( KR = [\lambda(\lambda+1)]^{1/2} \).

There is no polarization of the outgoing protons
predicted by the Butler theory.
Now consider the relaxation of two approximations. The first is the use of plane waves and the second is the neglect of the interior of the nucleus. To do this we must use numerical techniques.

The plane waves used in the Born approximation may be improved by taking those which seem to fit elastic scattering data well, i.e. Optical Model Wave functions. To obtain these wave functions the Schrödinger wave equation is solved with a potential of the form:

$$\Psi_n = \psi_n + i \mathcal{W}_n + \psi_C + g \frac{1}{r} \frac{\partial \Psi_n}{\partial r} \mathbf{e} \cdot \mathbf{z}$$

where $\psi_C$ is the electrostatic potential due to a uniform sphere of charge.

The stripping reaction, without the spin-orbit term, may be regarded as a three-step process. First, the deuteron moves into the field of the nucleus and is deflected and partially absorbed (scattered into non-elastic channels) by this field. Second, the neutron is captured with the proton liberated in accordance with the angular momentum rules set out previously. Third, the proton leaves the nucleus and is deflected and partially absorbed. A simple physical argument due to Newns(24) shows that the first and third steps of this process, in conjunction with the second,
produce a polarization of the outgoing protons. It can be proved\(^{(25)}\) that the maximum polarization that can be obtained by this form of the theory is 33 percent. For capture in which the angular momentum transfer is zero, the polarization is predicted to be zero as in the Butler case.

If the spin-orbit potential is included, the deuteron will be polarized upon entering the nuclear field and the proton will be polarized upon leaving. These polarization effects will be in addition to the effects already discussed. There is no limit to the polarization that may be produced by spin-orbit interactions and even in the case of no angular momentum transfer, a polarization is predicted.

Except in the special case of \(\lambda = 0\) the resultant polarization will be the result of the four competing mechanisms described above—incident deuteron deflection, outgoing proton deflection, incident deuteron polarization, and outgoing proton polarization.

We include a contribution due to the stripping processes occurring in the interior of the nucleus by postulating a wave function for the captured neutron inside the nucleus and assuming that the optical model wave functions are still valid there. The bound state function taken is an harmonic oscillator wave function chosen to have the correct slope and value at the nuclear surface.
II. STRIPPING WITH DISTORTED WAVES

A. The D.W.B.A. Treatment of Stripping

The distorted wave Born approximation treatment \(^{21}\) is based on the following exact expression for the differential cross section for stripping.

\[
\frac{d\sigma}{d\Omega} = \frac{M_d M_p K_D}{(2\pi h^2)^2} |A(\theta)|^2,
\]

where

\[
A(\theta) = \left< \Phi_p^{(-)}(\vec{k_p}) | V_{NP} | \Psi_D^{(+)}(\vec{k_D}) \right> ,
\]

and

\( \theta \) is the angle between \( \vec{k_D} \) and \( \vec{k_p} \),

\( V_{NP} \) is the neutron-proton interaction potential

\( \Psi_D^{(+)}(\vec{k_D}) \) is the solution of the Schrödinger equation for the total system with the boundary condition: as \( \gamma \rightarrow \infty \) the wave function reduces to that of an incident wave of deuterons impinging upon the target nucleus in the direction \( \vec{k_D} \), and

\( \Phi_p^{(-)}(\vec{k_p}) \) is the solution of the Schrödinger equation with \( V_{NP} \) removed from the Hamiltonian with the boundary condition: as \( \gamma \rightarrow \infty \) the wave function reduces to that of a wave of protons leaving the target nucleus in the direction \( \vec{k_p} \).
The Born approximation consists in replacing
\[ \psi'_0(\vec{k}_b) \] by \[ \phi'_0(\vec{k}_b) \] which is the elastic scattering wave function for deuterons and thus neglects all other outgoing channels.

These wave functions still cannot be calculated from first principles but the success of the optical model suggests that the wave functions obtained from this model are a good approximation to the actual wave functions. We may then write:

\[ \Phi_p^{(r)} = \chi_p(\vec{\xi}_p, \vec{\xi}_N) \phi_p^{(r)} | \frac{1}{2} \vec{\xi}_p \rangle \]
\[ \Phi_o^{(r)} = \chi_o(\vec{\xi}_o) \phi_o^{(r)} \bar{g} | 1 \vec{\xi}_o \rangle \]

where \( \chi_p \) and \( \chi_o \) are the wave functions of the final and initial nucleus, \( \phi_p^{(r)} \) and \( \phi_o^{(r)} \) are the optical elastic scattering wave functions for the proton and deuteron. \( | \frac{1}{2} \vec{\xi}_p \rangle \) and \( | 1 \vec{\xi}_o \rangle \) are the spin states of the proton and deuteron, \( \bar{g} \) is the deuteron bound wave function and \( \vec{\xi} \) denotes the nuclear coordinates.

If \( \chi_p \) is formed by adding the last neutron into a shell model state, we may write

\[
\int \chi_p^*(\vec{\xi}, r_N) \chi_o(\vec{\xi}) d\vec{\xi} = C(I, I_z, j, J_z; J, J_z) Z_\lambda(r_N) \\
\times \sum_{\mathcal{G}_N} C(\lambda, m, \frac{1}{2}, \mathcal{G}_N; J, J_z) Y_{\lambda m}(\vec{r}_N) \left| \frac{1}{2} \mathcal{G}_N \right|
\]
Where $C$ denotes the Clebsch-Gordan coefficient\(^{(26)}\), $Y^m_\lambda$ is the spherical harmonic and $|\uparrow \uparrow \uparrow \uparrow \rangle$ the spin state of the neutron.

$Z_\lambda (r_N)$ is the radial wave function which is a decaying Hankel function outside the nucleus. Using these results, the amplitude may now be written

$$A(\theta) = \sum_{\ell m} \int \Phi_\ell \phi_j \phi_{m^*} \left( C_\ell (I, I_z, j, j_z, \bar{J}, J_z) Z_\lambda (r_N) \right) \times C(\lambda, m, 1/2, \sigma, j, j_z) Y^m_\lambda (r_N) \langle 1/2 \sigma | \uparrow \uparrow \uparrow \uparrow \rangle d\bar{r}_N d\bar{r}_p$$

The product function $g(\bar{r}_N - \bar{r}_p) V_{np}(\bar{r}_N - \bar{r}_p)$ has a strong relative maximum at $\bar{r}_N = \bar{r}_p$. By considering the limit of the Hulthen potential and wave function, the proper normalization for the replacement of this product by a Dirac delta function is found to be

$$g V_{np} \rightarrow \frac{\hbar^2}{2M_{np}} \sqrt{8 \pi \alpha} \delta (\bar{r}_N - \bar{r}_p)$$

In order to calculate the $\phi_i$'s, each is expanded in partial waves.

$$\phi (\vec{K}, \vec{r}) = \sum_{\ell m} C_{\xi} \frac{\ell + \ell^*}{\ell} \frac{X^l_\lambda}{\gamma} Y^m_\lambda (i \vec{K}) \gamma^m (\vec{r})$$

where $C_{\xi}$ is the proper normalization such that $C_{\xi} \frac{X^l_\lambda}{\gamma}$ reduces to a sum of Coulomb wave functions at infinity.
Using the properties of the Clebsch-Gordan coefficients\(^{(27)}\) the cross section can be written as follows.

\[
\frac{d\sigma}{d\omega} = \frac{(2J+1)M_{\text{eff}} M_{z\text{D}} E_p}{(2J+1)\frac{\lambda^2}{\lambda} K_{\text{D}} K_p (M_{\text{eff}}/M_p)} \sum_{\lambda'=-\lambda}^{\lambda} \left| \beta_{\lambda}^m (\theta) \right|^2
\]

\[
\beta_{\lambda}^m (\theta) = \sum_{i=0}^{\infty} \frac{B_{\lambda, i}^m P_{\lambda}^m (\cos \theta)}{i!}
\]

\[
B_{\lambda, i}^m = \int_{0}^{\infty} \sum_{i=0}^{\lambda} \mathcal{f}_{i, \lambda}^\lambda (-1)^{\frac{\lambda-1}{2}}
\]

\[
\mathcal{f}_{i, \lambda}^\lambda = C_{i, \lambda} \int_{0}^{\infty} \chi_{\lambda} \Xi_{\lambda} \chi_{\lambda} \, d\gamma
\]

\[
\sum_{i=0}^{\lambda} \frac{(\frac{\lambda-1}{2})! (\frac{\lambda+1}{2})!}{(\frac{i+1}{2})! (\frac{i-1}{2})!} \frac{(2i+1)(2i+3)}{(2i+1)} C_{i, \lambda, \lambda, \lambda, \lambda}
\]

\[
x C(\bar{\lambda}, 0, 0, 0; \lambda, 0).
\]

The polarization is given by

\[
P(\theta) = \frac{4}{3} \frac{(2J+1)}{(2J+1)} \sum_{\lambda'=-\lambda}^{\lambda} \sqrt{\lambda(\lambda+1)-m(m+1)} \frac{2}{2} \frac{\lambda^2}{\lambda} \left( \frac{\lambda^2}{\lambda^2} \beta_{\lambda}^m (\theta) \beta_{\lambda}^m (\theta)^* \right)
\]

where \( P_{\lambda}^m (\cos \theta) \) is the associated Legendre function and all other symbols are defined in Appendix A.
B. Results of the Calculations

Calculations were carried out using the distorted wave Born approximation as described in the previous section. Appendix A gives the details of this calculation.

The optical wave functions are taken to be the solution of the Schrödinger equation with a potential of the form

\[ \sqrt{V} \approx (V + iW)f(r) \]

where

\[ f(r) = \frac{1}{e^{\frac{\beta}{\alpha^2}} + 1} \].

Thus the (d,p) cross section and the elastic cross sections will depend on the parameters V, W, R and a. There is one set of parameters for the deuteron and one set for the proton. The ideal procedure would be to fit the deuteron and proton elastic scattering data first by variation of the two sets of parameters separately and then calculate the predicted (d,p) cross section for the wave functions implied by the fits. In this paper, however, we are interested in a study of the variation of these cross sections with the parameters.

In addition, provision is made for excluding contributions to stripping from the inside of the nucleus by the use of the parameter \( R' \). \( R' \) is the lower limit to the
radial integrals so that if it is set equal to the radius of the nucleus, there is no contribution from this region.

To find the dependence on the various factors discussed above, a systematic variation was carried out and compared with a particular set of data. In the experiment to be compared the target was Ca$^{40}$ and the incident deuteron energy was 4.13 MEV. The Q-value for this reaction is 4.19 MEV and the $\lambda$ of the captured neutron is 1. The data were taken by Class and Rusk(28).

Figure 1 shows a comparison of Butler theory, cut-off D.W.B.A. and D.W.B.A. The circles are the experimental points. Note that the second peak is more pronounced in the $R' = 0$ case indicating that the interior of the nucleus plays an important role in this respect. Note also that the second peak of the Butler curve is in the wrong place and much too low.

The next two curves (Figures 2 and 3) show the variation with the depth of the real potential. Note that the peaks move in relation to one another and, in fact, go in opposite directions at times. It is suspected that this effect is the result of complicated reflections and refractions inside and around the nucleus and cannot be directly understood in simple terms at the present time. Note that the relative height of the peaks is not much affected.
The next two curves (Figures 4 and 5) show the variation with the nuclear absorption, \( W \). This parameter has the main effect of altering the relative heights of the main and second peak. This would tend to indicate that the second peak comes from stripping taking place very close to the surface or inside the nucleus since the imaginary part of the potential has its strongest effect in this region.

Figure 6 and 7 show the effect of changing the range of the well. It may be seen that the effect is a combination of the previous two as might be expected since a change in \( R \) changes the effective strength of both wells.

Figures 8 and 9 show the effect of the diffuseness. It is seen to cause little change but what effect is seen is of the same character as that of \( W \). This might be expected since this is the same result that is observed in optical model calculations of elastic scattering.

The most striking thing which is predicted is the second peak for the \( R' = 0 \) distorted wave case. This second peak has been observed many times experimentally and the Butler theory does not fit it. It is seen to arise naturally with the addition of the interior of the nucleus and nuclear and Coulomb distorted wave functions.
C. Figures for Section II

Except when otherwise indicated, the optical model parameters used in the calculation of Figures 1 through 9 were as given below.

\[ V_D = -60 \quad V_P = -62 \]
\[ W_D = -18 \quad W_P = -10 \]
\[ R_D = 5.17 \quad R_P = 4.10 \]
\[ a_D = 0.65 \quad a_P = 0.40 \]

All quantities are given in MEV and Fermis. The polarization is plotted taking the positive direction of the Z-axis to be \( k_D \times k_P \).
\[ \text{FIGURE 1} \]
FIGURE 2
III. STRIPPING WITH SPIN-ORBIT DISTORTED WAVES

A. The Spin-orbit Interaction in Deuteron Stripping

The spin-orbit potential has been used for many years in the calculation of atomic spectra\(^{(29)}\) and in more recent times in the fitting of elastic scattering of spin \(1/2\) particles from nuclei\(^{(30)}\). The fits to nuclear elastic scattering by means of the optical model are improved as a result of the introduction of a spin-orbit potential and, more important, the polarization is correctly fitted. The introduction of a spin-orbit potential is also necessary to split the levels of the shell model in order that the order and spacing of the nuclear energy levels be correct.

We are interested here primarily in the spin-orbit interaction used for optical model calculations of elastic scattering. For this work the spin-orbit potential is taken to be of the form

\[
V_{SO} = (V_{LS} + iW_{LS}) \frac{1}{r} \frac{df(r)}{dr} \bar{L} \cdot \bar{S}
\]

where the central optical potential is given by

\[
V_c = (V + iW)f(r)
\]

\[
f(r) = \frac{1}{1 + \exp(r - R/a)}
\]

and \(\bar{L} = \vec{F} \times \vec{p}\) is the orbital angular momentum operator of the scattered particle and \(\bar{S}\) is its spin operator. The constants \(V\) and \(W\) are negative for an attractive potential in this notation.

The maximum value of \(V_{SO}\) is given by

\[
V_{SO\ (peak)} = \frac{1}{4} \frac{V_{LS}}{Ra} \bar{L} \cdot \bar{S} = \nu \cdot \bar{L} \cdot \bar{S}
\]
The central and spin-orbit potentials are shown in Figure 10.

![Diagram](image)

**Figure 10**

However, optical model polarization is not very sensitive to the exact shape of the well.

A better measure of the strength of an optical well is its integral overall space.

\[
\frac{\int_{\text{all space}} V_{so} \, d\tau}{\int_{\text{all space}} V_c \, d\tau} = \frac{3V_{LS} \bar{L} \bar{S}}{V(R^2 + \pi a^2)} \quad \text{if } \varepsilon \frac{R}{a} \ll 1.
\]

Rodberg\(^{(31)}\) has given a simple rule by which one can derive the general behavior of the polarization due to the spin-orbit potential by observing the cross section alone.

Consider the drawing in Figure 11. Let us choose the quantization axis to be in the direction \( \vec{K}_f \times \vec{K}_i \). Then this axis will be pointing out of the paper. Now consider a particle with its spin up following the path drawn. Note that this causes the orbital angular momentum to be up also so that the spin angular momentum and orbital angular momentum are aligned. If, as in the case of the shell model,
it is assumed that the spin orbit portion of the potential is attractive for the aligned configuration, the angular distribution for spin up particles will be shifted toward the forward direction and the angular distribution for spin down particles will be shifted toward backward angles. Thus if at a given angle, $\theta$, the angular distribution with no spin orbit term has a positive slope, the particles with spin up will predominate at this angle when a spin-orbit term is added. Thus

$$P \sim \frac{d}{d\theta} \left( \frac{d\sigma}{dN} \right), \bar{\mathbf{k}}_f \times \bar{\mathbf{k}}_i$$

where $\frac{d\sigma}{dN}$ is the differential cross section and $P$ is the polarization of the scattered particles. Since the usual convention for the quantization axis is just the opposite of the one we have taken here let us write:

$$P \sim -\frac{d}{d\theta} \left( \frac{d\sigma}{dN} \right), \bar{\mathbf{k}}_i \times \bar{\mathbf{k}}_f$$

![Figure 11](image)

This relation can not hold in the forward direction because the polarization goes to zero here and the slope of the Rutherford scattering curve is infinite. Likewise it
might not be expected to hold in the extreme backward direction since this is the region in which the particle enters the optical well and suffers reflections. However, in the middle region this relation is found to hold fairly well.

Biedenharn has pointed out$^{(32)}$ that it is to be expected that a similar relation holds for $\lambda = 0$ stripping. For $\lambda \neq 0$ there is a polarization without the introduction of a spin orbit potential. The polarization which results would not necessarily be expected to be a simple function of the polarizations of the incident deuterons and outgoing protons.

In addition to the spin-orbit potential acting on the outgoing proton, there is no reason to exclude a spin-orbit potential acting on the incoming deuteron. The spin-orbit potential will be assumed to be of the same form for the deuteron as for the proton.

B. The D.W.B.A. Treatment of Stripping When Spin-orbit Interactions are Present

In order to treat this problem we must first find the optical wave functions for spin $1/2$ and spin $1$ particles. We will expand the wave function in a complete set of spin-angular momentum eigenfunctions defined by the following equation.
\[ |(\lambda S)J_J \rangle = \sum_{\ell^\prime = m} C(\lambda, m, S, \ell^\prime, J, J_\lambda) \, Y^m_{\ell^\prime}(\varpi) |S \rangle \]

The vector \( \bar{\mathbf{r}} \) appearing in the argument of the spherical harmonic refers only to the spherical angles \((\theta, \phi)\) of the vector. This set of states was chosen because it is diagonal in the operator \( \bar{L} \cdot \vec{S} \). Expanding the solution to the Schrödinger equation which is asymptotically an incoming wave along the direction \( \bar{\mathbf{r}} \) with spin projection \( \sigma \) we may write

\[ \Psi^\sigma(\bar{\mathbf{r}}, \mathbf{r}) = \sum_{\lambda J \frac{\ell}{2}} \frac{1}{\sqrt{\gamma}} \phi^{\frac{\ell}{2} \sigma}_{\lambda J}(\bar{\mathbf{r}}, \gamma) |(\lambda S)J_J \rangle \]

where this may be regarded as the definition of \( \phi^{\frac{\ell}{2} \sigma}_{\lambda J}(\bar{\mathbf{r}}, \gamma) \)

Substitution of \( \Psi^\sigma(\bar{\mathbf{r}}, \mathbf{r}) \) into the Schrödinger equation gives

\[ \left\{ \frac{d^2}{dr^2} + \mathbf{K}^2 - \frac{2m}{\hbar^2} \left[ V_c + i W_c + (V_{L,S} + i W_{L,S})(\frac{\lambda(\lambda + 1) - 5(\lambda + 1) - 8(8\pi)}{2}} \right] \right\} \]

\[ \times \phi^{\frac{\ell}{2} \sigma}_{\lambda J}(\bar{\mathbf{r}}, \gamma) = 0 \]

From this we may see that the functional dependence of \( \phi^{\frac{\ell}{2} \sigma}_{\lambda J}(\bar{\mathbf{r}}, \gamma) \) depends only on \( \lambda, J \) and the magnitude of \( \bar{\mathbf{r}} \). We may then write

\[ \phi^{\frac{\ell}{2} \sigma}_{\lambda J}(\bar{\mathbf{r}}, \gamma) = 4\pi i^{\lambda} C^{\frac{\ell}{2} \sigma}_{\lambda J}(\bar{\mathbf{r}}) \chi_{\lambda J}(\gamma) \]

where \( \chi_{\lambda J}(\gamma) \) satisfies the same differential equation as \( \phi^{\frac{\ell}{2} \sigma}_{\lambda J}(\bar{\mathbf{r}}, \gamma) \).
Asymptotically*

\[ \Psi^{(\sigma)}_{(K, r)} \rightarrow e^{iKz} \left| S_{\sigma} \right> + \frac{e^{iKz}}{r} \sum_{\sigma'} F^{(\sigma', \sigma)}(\theta, \phi) \left| S_{\sigma} \right> \]

\[ = \sum_{l,m} 4\pi c^{l} \mathcal{J}_{l}(Kr) Y_{l}^{m}(\hat{r}) Y_{l}^{m*}(\hat{K}) \left| S_{\sigma} \right> \]

\[ + \sum_{l,m}\mathcal{J}^{(2)}_{l}(Kr) Y_{l}^{m}(\hat{r}) B_{l}^{\sigma}_{m}(\hat{K}) \left| S_{\sigma} \right> \]

where

\[ F^{(\sigma', \sigma)}(\theta, \phi) = \frac{-i\pi}{\kappa \nu} \sum_{l,m} B_{l}^{\sigma}_{m}(\hat{K}) Y_{l}^{m}(\theta, \phi) \]

Comparing the forms for large values of \( r \) we see that

\[ B_{l}^{\sigma}_{m}(\hat{K}) = Y_{l}^{m+\psi}(\hat{K}) \sum_{j} C_{\lambda_{j}}^{(j)}(\lambda, \sigma) \sum_{\lambda} C_{\lambda}(\lambda_{j}, \sigma_{j}^{\nu}, \lambda_{j}^{m+\psi}) B_{\lambda_{j}}^{\lambda_{j}} \]

and

\[ C_{\lambda_{j}}^{(j)}(\hat{K}) = Y_{l}^{j}(\hat{K}) \sum_{\lambda} C_{\lambda}(\lambda_{j}, \sigma_{j}^{\nu}, \lambda_{j}^{m+\psi}) \]

where

\[ C_{\lambda_{j}}^{(j)}(\lambda, \sigma) \rightarrow \gamma_{\lambda_{j}}^{(2)}(\lambda, \sigma) + \gamma_{\lambda_{j}}^{(2)}(\lambda, \sigma) B_{\lambda_{j}}^{\lambda_{j}} \]

for large \( r \).

*In the presence of a Coulomb field such an equation cannot be written but the entire analysis goes through by replacing the spherical Bessel and Hankel functions by their Coulomb analogues.
By the means outlined above we may obtain both of the wave functions necessary for the calculation of the stripping matrix element and the elastic cross section and polarization. Using these wave functions in the expression for the stripping amplitude (D.W.B.A.) we may obtain it as a function of the radial integrals

$$ f_{\lambda j \lambda' j} \equiv C_{\lambda j} C_{\lambda' j} \int_0^\infty \chi_{\lambda j}(r) \chi_{\lambda' j}(r) Z_\lambda(r) \, dr. $$

$Z_\lambda(r)$ is the bound state wave function of the captured neutron. The final expression is given at the end of this section.

There is one difficulty in the comparison with experimental results. As may be noticed in section III the sign of the proton spin-orbit potential required to fit the d-p results is just opposite to that required to fit elastic proton scattering. This is not understood at the present time. It is felt that too much importance should not be placed on this since the only strong evidence is based on a single data point (see Figure 12).

$$ A^{oo}(\theta) = \frac{\hbar^2}{M_{np}} \sqrt{\frac{\alpha}{2}} 4\pi i^{L} \sum_{m} C(s, e-\tilde{e}, -m, I, I; J, J_2) P_{\lambda}^m(\theta) $$
\[ \beta_{\lambda}(\theta) = \sum \frac{[i-|m|]!}{\sqrt{(i+|m|)!}} C \left( \frac{i}{2}, \frac{i}{2}, \lambda, \frac{i}{2}, \frac{i}{2}, m, \frac{j}{2}, \frac{j}{2}, m + \frac{\lambda}{2} \right) \times C \left( s, 3 - \delta - m, \frac{j}{2}, \frac{j}{2}, \lambda + m, \frac{j}{2}, \frac{j}{2}, \lambda, 0 \right) X \left( \frac{i}{2}, \frac{j}{2}, \frac{j}{2}, \frac{s}{2} \right) \]
\[ \times \frac{1}{(2l+1) \sqrt{3(2s+1)(2\lambda+1)(2\lambda+1)(2\lambda+1)}} (-1)^{l+\lambda-i} \hat{c}(m) \int_{\lambda \frac{j}{2} \frac{j}{2} \frac{s}{2}}^\lambda \beta_{\lambda}^{nl} \cos \theta \]

where

\[ \hat{c}(m) = \begin{cases} (-1)^{m} & m > 0 \\ 1 & m < 0 \end{cases} \]

I is the total angular momentum of the target nucleus

\( I_Z \) is the Z-projection of I

J is the total angular momentum of the final nucleus

\( J_Z \) is the Z-projection of J

S is the total angular momentum of the capture neutron

\( \lambda \) is the orbital angular momentum of the captured neutron

\( \sigma \) is the spin projection of the incoming deuteron

\( \bar{\sigma} \) is the spin projection of the outgoing proton

\[ X \left( \frac{i}{2}, \frac{j}{2}, \frac{j}{2}, \frac{s}{2} \right) \] is the x - coefficient

C. Results of the Spin-orbit Calculations

Figure 12 shows the results of calculations on the reaction \( ^{12}\text{C}(d,p)^{13}\text{C}^* \). \( V_{LS} \) for the proton is -20. Since
this is \( \lambda = 0 \) stripping, there is no polarization predicted unless spin-orbit terms are included. The derivative rule is seen to hold very well. The experimental points are from Hensel and Parkinson\(^{34} \).

The next curve (Figure 13) is the same case as shown in Figure 12 with \( V_{LS} = 20 \). Note that the only effect is to change the sign of the polarization.

Figure 14 shows the proton elastic scattering from \( ^{13}\text{C} \) for the two potentials used in Figure 12 and 13. Again the main effect of the change in sign of \( V_{LS} \) is a change in the sign of the polarization. The derivative rule holds well\(^* \).

This calculation (Figure 15) is for \( V_{LS} = 30 \) and when compared with Figure 13 shows little difference. That is, the polarization is not very sensitive to the magnitude of \( V_{LS} \).

In Figure 16 is plotted the elastic scattering of protons corresponding to Figure 15. When compared with Figure 14 it shows that the elastic scattering cross section is more sensitive to changes in magnitude than changes in sign of the spin-orbit potential and the polarization is more sensitive than in the stripping case. The derivative rule does not hold.

The cut-off case of \( ^{12}\text{C}(d,p)\, ^{13}\text{C}^* \) (Figure 17)

\*Note that the derivative rule compares the derivative of the cross section to the polarization. The ratio of the elastic cross section to the Rutherford cross section is plotted in Figures 14, 16, 20 and 27.
shows only qualitatively the same shape and does not fit as well as the zero range case. Note that in every case the behavior beyond $120^\circ$ is almost invariant to non-spin-orbit parameters and gives a large polarization. The derivative rule does not hold.

Figure 18 shows the results of a deuteron spin-orbit potential. Very little change would be expected in the region of the two experimental points if this potential were included in the calculation shown in Figure 12. The derivative rule holds fairly well.

Figure 19 shows the cut-off calculation with deuteron spin-orbit included. Note that the entire character of the polarization is changed including a reversal of sign for large angles. At forward angles the curves have the same sign. The derivative rule is very poor here.

Here (Figure 20) the deuteron elastic scattering corresponding to Figures 18 and 19 is plotted. As on all of these curves the reflection symmetry is striking. The derivative rule is fair.

$^3\text{He}^3(d,p)^4\text{He}$ is also a $\chi = 0$ capture and we see in Figure 21 that large polarizations can be obtained with $V_{LS} = -5$ which is small. The derivative rule is very good.

Figure 22 shows the cut-off case corresponding to Figure 21. The polarization is again reduced at most angles but left the same near $180^\circ$. The derivative rule is very poor.
Figure 23 shows the elastic proton scattering from He$^4$ corresponding to Figures 21 and 22. The derivative rule seems to work quite well here.

Figure 24 shows the plot for Ca$^{40}(d,p)ca^{41}$, $\lambda = 1$ capture, with no spin orbit effects.

The next curve (Figure 25) shows the same reaction with spin orbit effects for the proton included. There is no change in cross section and small change in polarization.

Figure 26 shows the cut-off calculation corresponding to Figure 25.

Figure 27 shows the elastic scattering of protons from Ca$^{41}$ at 8.32 MEV. This curve compares favorably with polarization of elastic protons from Ti$^{48}$ at 8 MEV. The derivative rule is poor here.

In Figure 28 is shown calculation for the case C$^{12}(d,p)c^{13}$, $\lambda = 1$ capture. The data points are by Juveland and Jentschke. The fit is seemingly poor but it is good by comparison with a calculation with the opposite sign to the spin-orbit potential.

This curve (Figure 29) shows the same calculation as Figure 28 with the opposite sign of the spin-orbit potential. Note that very forward angles are not affected while backward angles are completely dominated by the spin-orbit potential.
Figure 30 shows the same calculation as Figure 29 with a larger spin-orbit potential. There is little difference between the two.

Figure 31 shows the effect of making the deuteron optical well deeper. This has a larger effect than the changes in the spin-orbit potential. The backward angles, however, are left unchanged.

Figure 32 shows the cut-off case corresponding to Figure 28. The curve is greatly changed except in the backward direction.

Figure 33 shows the elastic scattering of protons with $V_{LS} = 20$ and $V_{LS} = 30$ on $^{13}C$. The derivative rule is only fair.

D. Figures for Section III

**FIGURE 12**

$^1_{12}(d,p)^{13}C$\ *

<table>
<thead>
<tr>
<th>$E_D$</th>
<th>$Q$</th>
<th>$R'$</th>
<th>$\lambda$</th>
<th>$j$</th>
<th>$n$</th>
</tr>
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<tbody>
<tr>
<td>7.8</td>
<td>-0.36</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>3</td>
</tr>
<tr>
<td>$V_D$</td>
<td>$W_D$</td>
<td>$R_D$</td>
<td>$a_D$</td>
<td>$V_{DLS}$</td>
<td>$W_{DLS}$</td>
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<td>-13</td>
<td>4.0</td>
<td>0.75</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$V_p = -50$, $W_p = -6$, $R_p = 3.03$, $a_p = 0.52$, $V_{PLS} = -20$, $W_{PLS} = 0$
FIGURE 13

$^1\text{Cl}^{12}(d,p)^{13}\text{Cl}^*$

\begin{align*}
E_D &= 7.8 \\
\lambda &= 0 \\
V_D &= -40 \\
W_D &= -13 \\
R_D &= 4.0 \\
a_D &= 0.75 \\
V_{DL} &= 0 \\
W_{DL} &= 0 \\
Q &= -0.36 \\
J &= 1/2 \\
R' &= 0 \\
N &= 3 \\
V_p &= -50 \\
W_p &= -6 \\
R_p &= 3.03 \\
a_p &= 0.52 \\
V_{PL} &= 20 \\
W_{PL} &= 0
\end{align*}

FIGURE 14

$^{13}\text{Cl}(p,p)$

\begin{align*}
E_p &= 7.44 \\
V &= -50 \\
W &= -6 \\
R &= 3.03 \\
a &= 0.52 \\
a) \quad V_{LS} &= 20, \quad W_{LS} = 0 \\
& \quad b) \quad V_{LS} &= -20, \quad W_{LS} = 0
\end{align*}
FIGURE 15
\[ c^{12}(d,p)c^{13*} \]

\[ E_D = 7.8 \quad Q = -0.36 \quad R' = 0 \]
\[ \lambda = 0 \quad J = 1/2 \quad n = 3 \]
\[ V_D = -40 \quad V_p = -50 \]
\[ W_D = -13 \quad W_p = -6 \]
\[ R_D = 4.0 \quad R_p = 3.03 \]
\[ a_D = 0.75 \quad a_p = 0.52 \]
\[ V_{DLS} = 0 \quad V_{PLS} = 30 \]
\[ W_{DLS} = 0 \quad W_{PLS} = 0 \]

FIGURE 16
\[ c^{12}(p,p) \]

\[ E_p = 7.44 \]
\[ V = -50 \]
\[ W = -6 \]
\[ R = 3.03 \]
\[ a = 0.52 \]
\[ V_{LS} = 30 \]
\[ W_{LS} = 0 \]

FIGURE 17
\[ c^{12}(d,p)c^{13*} \]

\[ E_D = 7.8 \quad Q = -0.36 \quad R' = 6.612 \]
\[ \lambda = 0 \quad J = 1/2 \quad n = 3 \]
\[ V_D = -40 \quad V_p = -50 \]
\[ W_D = -13 \quad W_p = -6 \]
\[ R_D = 4.0 \quad R_p = 3.03 \]
\[ a_D = 0.75 \quad a_p = 0.52 \]
\[ V_{DLS} = 0 \quad V_{PLS} = -20 \]
\[ W_{DLS} = 0 \quad W_{PLS} = 0 \]
**FIGURE 18**

$c^{12}(d,p)c^{13*}$

<table>
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<td>$R_D$</td>
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<td>$a_D$</td>
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<td>$R_p$</td>
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<tr>
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</tr>
<tr>
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<tr>
<td>$W_{DLS}$</td>
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</table>

**FIGURE 19**

$c^{12}(d,p)c^{13*}$

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<td>$R_D$</td>
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<tr>
<td>$a_D$</td>
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<tr>
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<td>$R'$</td>
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<td>$n$</td>
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</tr>
<tr>
<td>$V_p$</td>
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</tr>
<tr>
<td>$W_p$</td>
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</tr>
<tr>
<td>$R_p$</td>
<td>3.03</td>
</tr>
<tr>
<td>$a_p$</td>
<td>0.52</td>
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<tr>
<td>$V_{DLS}$</td>
<td>-20</td>
</tr>
<tr>
<td>$W_{DLS}$</td>
<td>0</td>
</tr>
</tbody>
</table>
**FIGURE 20**

$^{12}\text{C}(d,d)$

$E_D = 7.8$

a) $v_{LS} = -20$, $w_{LS} = 0$

$V = -40$

$b) v_{LS} = 20$, $w_{LS} = 0$

$W = -13$

$R = 4.0$

$a = 0.75$

**FIGURE 21**

$\text{He}^3(d,p)\text{He}^4$

$E_D = 12.3$

$Q = 18.4$

$R^i = 0$

$\lambda = 0$

$j = 1/2$

$n = 1$

$V_{DP} = -85$

$W_{DP} = -7$

$\alpha_{DP} = 0.02$

$V_{PLS} = 0$

$W_{PLS} = 0$

$V_P = -90$

$W_P = -3$

$R_P = 1.92$

$\alpha_P = 0.3$

$V_{PLS} = -5$

$W_{PLS} = 0$
### FIGURE 22
\( \text{He}^3(d,p)\text{He}^4 \)

<table>
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<tr>
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<tr>
<td>( W_{DLS} )</td>
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</table>

\( R' = 3.4 \)

### FIGURE 23
\( \text{He}^4(p,p) \)

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<td>( V )</td>
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</tr>
<tr>
<td>( W )</td>
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</tr>
<tr>
<td>( R )</td>
<td>1.92</td>
</tr>
<tr>
<td>( a )</td>
<td>0.3</td>
</tr>
<tr>
<td>( V_{LS} )</td>
<td>-5</td>
</tr>
<tr>
<td>( W_{LS} )</td>
<td>0</td>
</tr>
</tbody>
</table>

### FIGURE 24
\( \text{Ca}^{40}(d,p)\text{Ca}^{41} \)

<table>
<thead>
<tr>
<th>Parameter</th>
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</tr>
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</tr>
<tr>
<td>( W_D )</td>
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<td>( R_D )</td>
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<td>( a_D )</td>
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</tr>
<tr>
<td>( V_{DLS} )</td>
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</tr>
<tr>
<td>( W_{DLS} )</td>
<td>0</td>
</tr>
</tbody>
</table>

\( Q = 4.19 \)

\( R' = 0 \)

\( n = 4 \)

\( V_p = -62 \)

\( W_p = -4 \)

\( R_p = 4.1 \)

\( a_p = 0.4 \)

\( V_{PLS} = 0 \)

\( W_{PLS} = 0 \)
FIGURE 25

\( Ca^{40}(d,p)Ca^{41} \)

\[
\begin{align*}
E_D &= 4.13 & Q &= 4.19 & R' &= 0 \\
\lambda &= 1 & j &= 3/2 & n &= 4 \\
V_D &= -60 \\
W_D &= -10 & V_p &= -62 & W_p &= -4 \\
R_D &= 5.55 & R_p &= 4.1 & a_p &= 0.4 \\
a_D &= 0.72 & V_{pLS} &= -10 & W_{pLS} &= -2 \\
V_{DLS} &= 0 & W_{DLS} &= 0
\end{align*}
\]

FIGURE 26

\( Ca^{40}(d,p)Ca^{41} \)

\[
\begin{align*}
E_D &= 4.13 & Q &= 4.19 & R' &= 6.728 \\
\lambda &= 1 & j &= 3/2 & n &= 4 \\
V_D &= -60 \\
W_D &= -10 & V_p &= -62 & W_p &= -4 \\
R_D &= 5.55 & R_p &= 4.1 & a_p &= 0.4 \\
a_D &= 0.72 & V_{pLS} &= -10 & W_{pLS} &= -2 \\
V_{DLS} &= 0 & W_{DLS} &= 0
\end{align*}
\]

FIGURE 27

\( Ca^{41}(p,p) \)

\[
\begin{align*}
E_p &= 8.32 & V &= -62 \\
W &= -4 & R &= 4.1 \\
a &= 0.4 & V_{LS} &= -10 & W_{LS} &= -2
\end{align*}
\]
FIGURE 28
\[ ^{12}\text{(d,p)}^{13} \]

\[ E_D = 11.9 \quad Q = 2.72 \]
\[ \lambda = 1 \quad J = 1/2 \]
\[ V_D = -40 \]
\[ W_D = -13 \]
\[ R_D = 4.0 \]
\[ a_D = 0.75 \]
\[ V_{DLS} = 0 \]
\[ W_{DLS} = 0 \]
\[ R' = 0 \]
\[ n = 2 \]
\[ V_p = -50 \]
\[ W_p = -6 \]
\[ R_p = 3.03 \]
\[ a_p = 0.52 \]
\[ V_{PLS} = -20 \]
\[ W_{PLS} = 0 \]

FIGURE 29
\[ ^{12}\text{(d,p)}^{13} \]

\[ E_D = 11.9 \quad Q = 2.72 \]
\[ \lambda = 1 \quad J = 1/2 \]
\[ V_D = -40 \]
\[ W_D = -13 \]
\[ R_D = 4.0 \]
\[ a_D = 0.75 \]
\[ V_{DLS} = 0 \]
\[ W_{DLS} = 0 \]
\[ R' = 0 \]
\[ n = 2 \]
\[ V_p = -50 \]
\[ W_p = -6 \]
\[ R_p = 3.03 \]
\[ a_p = 0.52 \]
\[ V_{PLS} = 20 \]
\[ W_{PLS} = 0 \]
FIGURE 30

$c^{12}(d,p)c^{13}$

\[ E_D = 11.9 \quad Q = 2.72 \]
\[ \lambda = 1 \quad J = 1/2 \]
\[ V_D = -40 \]
\[ W_D = -13 \]
\[ R_D = 4.0 \]
\[ a_D = 0.75 \]
\[ V_{DLS} = 0 \]
\[ W_{DLS} = 0 \]
\[ R' = 0 \]
\[ n = 2 \]
\[ V_p = -50 \]
\[ W_p = -6 \]
\[ R_p = 3.03 \]
\[ a_p = 0.52 \]
\[ V_{PLS} = 30 \]
\[ W_{PLS} = 0 \]

FIGURE 31

$c^{12}(d,p)c^{13}$

\[ E_D = 11.9 \quad Q = 2.72 \]
\[ \lambda = 1 \quad J = 1/2 \]
\[ V_D = -60 \]
\[ W_D = -13 \]
\[ R_D = 4.0 \]
\[ a_D = 0.75 \]
\[ V_{DLS} = 0 \]
\[ W_{DLS} = 0 \]
\[ R' = 0 \]
\[ n = 2 \]
\[ V_p = -50 \]
\[ W_p = -6 \]
\[ R_p = 3.03 \]
\[ a_p = 0.52 \]
\[ V_{PLS} = 20 \]
\[ W_{PLS} = 0 \]
FIGURE 32

c^{12}(d,p)c^{13}

$E_D = 11.9$  
$\lambda = 1$  
$V_D = -40$  
$W_D = -13$  
$R_D = 4.0$  
$A_D = 0.75$  
$V_{DLS} = 0$  
$W_{DLS} = 0$  

$Q = 2.72$  
$j = 1/2$  
$R' = 7.134$  
$n = 2$  

$V_p = -50$  
$W_p = -6$  
$R_p = 3.03$  
$a_p = 0.52$  

$V_{PLS} = 20$  
$W_{PLS} = 0$

FIGURE 33

c^{12}(p,p)

$E_p = 14.6$  
$V = -50$  
$W = -6$  
$R = 3.03$  
$a = 0.52$

a) $V_{LS} = 30, W_{LS} = 0$

b) $V_{LS} = 20, W_{LS} = 0$
FIGURE 14
FIGURE 21
IV. Appendix A

DESCRIPTION OF THE STANDARD PROGRAM
IV. DESCRIPTION OF THE PROGRAM

The program described here calculates the differential cross section and polarization of the outgoing protons from the d-p reaction. It assumes a central complex potential with no spin-orbit coupling to get the proton and deuteron wave functions. In addition the program calculates the elastic cross sections for both the deuteron and the proton.

The program consists of a control program plus 13 computing sections. These sections load one after the other so that flow of information and program is linear through them. Here the sections are called: Control, A, B, C₁, C₂, D₁, D₂, E, F, G, H, I, J, and K. The names used when writing the program were different so anyone wishing to consult the Fortran listings should refer to the correspondence table at the end of this description (Table 4).

CONTROL

This is the first program appearing on tape and serves to add flexibility to the program as well as allow certain intermediate results to be obtained. This program has a tape loader just before it so it may be loaded into core by pressing "load tape". If sense switch 1 is down, the program operates in manual mode, if it is up the program takes command information from the card reader.
In manual mode the information is entered into the program by setting the binary representation of a control number \((N)\) into the last 4 sense switches. The first section of this program reads \(N\) cards and prints them on line as headers or extra identification. This section can "read and print" a maximum of 15 cards when operating in the manual mode. If sense switch 2 is down the program stops (HPR 2) after "reading and printing" to allow another control number to be entered for part 2. If it is up, the program returns to card control after "reading and printing".

In card control mode the program reads one card for control. The first two columns of this card contain \(N\) and the third column contains a number \((M2)\) which controls which section is to be executed. If \(M2 = 1\) the program will read and print \(N\) cards. If \(M2 = 2\) the program will use \(N\) as a control number in part 2. If \(M2\) is zero the program will immediately execute the load card sequence. Thus a blank card will cause the program to transfer complete control to the card reader.
The control numbers for part 2 are given in Table 1.

<table>
<thead>
<tr>
<th>N</th>
<th>Operation</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>Print bound state wave function</td>
</tr>
<tr>
<td>01</td>
<td>Print Coulomb phase shifts</td>
</tr>
<tr>
<td>02</td>
<td>Print distorted phase shifts</td>
</tr>
<tr>
<td>03</td>
<td>Print last points of wave function</td>
</tr>
<tr>
<td>04</td>
<td>Print optical potentials</td>
</tr>
<tr>
<td>05</td>
<td>Print proton wave function*</td>
</tr>
<tr>
<td>06</td>
<td>Print deuteron wave function*</td>
</tr>
<tr>
<td>07</td>
<td>Run D-P program from tape</td>
</tr>
</tbody>
</table>

**TABLE 1**

*For N = 5 or N = 6 a second card must be supplied which contains N and M2 in the same format as before. If M2 is zero the program prints only the wave function with L = N. If M2 > 0 the program prints all of the wave functions from 0 to N. When operating in the manual mode, if 5 or 6 is entered into the console as a control number for section 2, the program will stop (HPR 3) for a second number to be entered. This second number (N) is entered into all 6 sense switches. When the 5 or 6 is entered into the console, if sense switch 3 is down, the program will print only the wave function for L = N; if it is up the program will print all of the wave functions from 0 to N.*
The program tape rewinds itself and starts the control program when it finishes each problem so that the control program is the first and the last program to be loaded. Thus it may head up a case with comments then run the D-P program or transfer control to a one-card program in the card reader which will find another program on tape and run it, and then may be used to obtain the intermediate results listed in Table 1.

Section A - Data Reduction

This program performs data reduction on the input variables and computes the following quantities:

1) Wave numbers for proton, deuteron, and neutron
2) Reduced mass of the proton, deuteron and neutron
3) Coulomb constants $\gamma$ for the proton, deuteron and neutron
4) Certain constants involved in the computation of the optical potential
5) The radius ($r_p^*r_D$) for which the Baxon potential is negligible
6) The interval size $\varepsilon$ and $\delta = 2\varepsilon$
7) The Coulomb normalization constants $C_o (\gamma)$
8) The radius ($r_o$) for which further contribution to the radial integrals would be negligible
9) The proper integral values of $\varepsilon$ for $r_o$, $R$ (the cut-off radius), and $R_N$ (the nuclear radius)
10) The normalization constants for the cross-section and polarization.
The formulas used to calculate these quantities are:

1) \[ K_p = \left[ \frac{2(M_i/M_F)^2 M_p \varepsilon \left( \frac{E_{M_D}}{M_{D_D}} \right)}{h^2} \right]^{\frac{1}{2}} \]

\[ K_D = \left[ \frac{2M_{D_D}^2 E_D}{(h^2 M_D)} \right]^{\frac{1}{2}} \]

\[ K_N = \left[ \frac{2M_{IN} \varepsilon_D \varepsilon_{Q}}{h^2} \right]^{\frac{1}{2}} \]

2) \[ M_{FP} = M_F M_p/(M_i + M_p) \]

\[ M_{ID} = M_I M_D/(M_i + M_D) \]

\[ M_{IN} = M_I M_N/(M_i + M_N) \]

3) \[ \gamma_p = \frac{Z_p(Z_i+Z_D)M_{FP} a_Z}{(K_{ FP})} \]

\[ \gamma_D = \frac{Z_I(Z_i+Z_D)M_{DI} a_Z}{K_D} \]

\[ \gamma_N = \frac{Z_N Z_{IN} a_Z}{K_N} \]

4) \[ \gamma_I = \frac{M_{FP} a_p}{a_D} \]

\[ \gamma_D = 1/a_D \]

\[ \lambda_p = \frac{V_p (1+e^{-m}) h^2}{(2M_{pF})(M_i/M_F)^2} \]

\[ \lambda_D = \frac{V_D (1+e^{-m}) h^2 M_D}{(2M_{DI})} \]

\[ \lambda_{FP} = \frac{W_p (1+e^{-m}) h^2}{(2M_{pF})(M_i/M_F)^2} \]

\[ \lambda_{FD} = \frac{W_D (1+e^{-m}) h^2 M_D}{(2M_{DI})} \]

\[ \phi_p = \frac{-a_I \ln(1+2e^{-m})}{a_p} \]

\[ \phi_D = \frac{-a_D \ln(1+2e^{-m})}{a_D} \]

\[ \bar{\phi}_p = \frac{M_D \bar{\phi}_p}{M_I} \]

\[ \bar{\phi}_D = \frac{-a_D \bar{\phi}_D}{a_D} \]
5) \[ r_p = \bar{R}_p + a_p (0.69315 + n_9 \times 2.302585) \]
\[ r_D = \bar{R}_D + a_D (0.69315 + n_9 \times 2.302585) \]

6) \[ \varepsilon = \delta_o/(K_p + K_D) \quad \delta = 2\varepsilon \]

7) \[ C_0(\eta_p) = \left[ \frac{2\pi \eta_p}{(e^{2\pi \eta_p} - 1)} \right]^{1/2} \]
\[ C_0(\eta_D) = \left[ \frac{2\pi \eta_D}{(e^{2\pi \eta_D} - 1)} \right]^{1/2} \]

8) \[ r_o = R + n_9/K_N \]

9) \[ \bar{R}_o = 6\varepsilon \left\lfloor \frac{r_o}{6\varepsilon} \right\rfloor \text{ Integer} \]
\[ \bar{R} = \left\lfloor \frac{R}{6\varepsilon} \right\rfloor \text{ Integer} \times 6\varepsilon - 2\varepsilon \]
\[ \bar{R}_N = \left\lfloor \frac{R_N}{6\varepsilon} \right\rfloor \times 6\varepsilon - 2\varepsilon \]

10) \[ \bar{\eta} = (2J+1)M_pM_ID_F^d M_F^e M_F^f / ((2I+1)M_N^d M_D^2 K_P^2 M_1^2) \]
\[ C_p = 4(J - \lambda)/(3(2J - 1)) \]

The program reads 11 cards and prints several lines of unlabeled results. The identification of these quantities is contained in Tables 2 and 3. The input data are read with a 4E17.8 format for the first 10 cards and an I5 format for the 11th card.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_0$</td>
<td>power series point*</td>
</tr>
<tr>
<td>$R$</td>
<td>cut-off radius</td>
</tr>
<tr>
<td>$Z_p$</td>
<td>proton charge</td>
</tr>
<tr>
<td>$R_N$</td>
<td>nuclear radius</td>
</tr>
<tr>
<td>$a_D, a_p$</td>
<td>Saxon diffuseness</td>
</tr>
<tr>
<td>$R_D, R_p$</td>
<td>Saxon radius</td>
</tr>
<tr>
<td>$V_D, V_p$</td>
<td>Saxon depths, real</td>
</tr>
<tr>
<td>$W_D, W_p$</td>
<td>Saxon depths, imaginary</td>
</tr>
<tr>
<td>$\delta_0$</td>
<td>interval size*</td>
</tr>
<tr>
<td>$M_N$</td>
<td>neutron mass</td>
</tr>
</tbody>
</table>

(Continued)
Q        Q-value
M_I       target mass
M_P       proton mass
M_D       deuteron mass
E_D       incident deuteron energy
\varepsilon_D  deuteron binding energy
\lambda   \lambda of captured neutron
n         energy state of neutron
n_2       convergence for y_L^*
z         unit conversion constant^*
n_5       convergence for bound state^*
n_6       convergence for \chi_L^*
n_8       controls size of r_0^*
n_9       neglect of Saxon potential^*
z_n       neutron charge
\alpha    fine structure constant^*
z_I       charge of target nucleus
I          J of initial nucleus
J          J of final nucleus
J          J of captured neutron
M_F       mass of final nucleus
m          matching of Saxon potential^*
n_1       LDD = K_D R_D + n_1^*
n_3       LCD = r_0 K_D + n_3^*
n_4       not used^*, n_7 LDP = K_p R_p + n_7^*

MOLD number of log. ders.^*

^This parameter is normally held fixed
| \( \lambda \) | \( 10^{-n_6} \) | \( 3\varepsilon/4 \) | \( 10^{-n_2} \) | \( r_p, r_D \) | Saxon potential is taken to be zero after this point. |
| \( \eta_p, \eta_D \) | \( K_p, K_D \) | \( \rho_p, \rho_D \) | Saxon potential is taken to be constant before this point. |
| \( r_0, R_N, \varepsilon, R \) | \( C_o(\eta), C_o(\eta_D) \) | Normalization for Coulomb wave functions |
| \( \sqrt{p_p}, \sqrt{D} \) | \( \sqrt{\eta_p}, \sqrt{\eta_D} \) | \( N_1 \) | Number of mesh points between zero and \( R_N \) |
| \( R_D, R_p, r_p, r_D \) | \( N_2 \) | Total number of mesh points |
| \( \ell_p, \ell_D \) | \( C_o(\eta_p), C_o(\eta_D) \) | LDD, LDP | Number of distorted waves taken |
| \( K_N, \eta_N, -R_p/\alpha_p, -R_D/\alpha_D \) | \( LCD, LCP \) | Total number of partial waves taken |
| \( R_p, R_D \) | \( x \) | \( y = A \int_{R_N}^{\infty} R \frac{d^2}{d^2} \) |
| \( N_1 \) | \( N_2 \) | \( y \) | \( y = A \int_{0}^{R_N} R \frac{d^2}{d^2} \) |
| \( LDD \) | \( LDP \) | \( LCD \) | \( LCP \) | \( z = \frac{\hbar^2}{2M_{IN}} \) |
| \( x, y, z, w \) | \( w \) | contains nonsense. |
Section B - Bound State Wave Function

This section calculates the bound state wave function which is assumed to be the asymptotically vanishing solution of

\[ \left( \frac{d^2}{dr^2} - K_N - \frac{2\eta N K_N}{r} - \frac{\lambda(\lambda+1)}{r^2} \right) \gamma_\lambda = 0. \]

for \( r > R_N \) and a harmonic oscillator wave function which fits smoothly onto \( \gamma_\lambda \) for \( r < R_N \). Thus for \( r < R_N \)

\[ \gamma_\lambda = A r^{\lambda+1} e^{-\frac{r^2}{\beta^2}} F(1/2(\lambda+1-n), \lambda+3/2, \beta r^2) \]

where \( F \) is the confluent hypergeometric function and \( A \) and \( \beta \) are chosen to match the value and logarithmic derivative of \( \gamma_\lambda \) at \( r = R_N \).

The calculation of \( \gamma_\lambda \) starts at \( r = r_0 \). \( r_0 \) is defined to be about \( R_N + \delta / K_N \) so that \( \gamma_\lambda(r)/r \) is about \( e^{\eta \delta} \) times as large as \( \gamma_\lambda(r_0)/r_0 \) making the contribution to the radial integral from \( r > r_0 \) negligible. The first step in calculating \( \gamma_\lambda \) is to use the asymptotic expansion to evaluate \( \gamma_\lambda \) at \( r_0 \) and \( r_0 - 2 \varepsilon \).

\[ \gamma_\lambda = \frac{e^{-\chi \lambda}}{(2\pi)^{\lambda+1/2}} \left( 1 + \frac{(\lambda-\eta)(\lambda+\eta+1)}{1! (2\pi)^2} \right) \]

\[ + \frac{(\lambda-\eta)(\lambda-\eta-1)(\lambda+\eta+1)(\lambda+\eta+2)}{2! (2\pi)^3} \]

\[ + \frac{(\lambda-\eta)(\lambda-\eta-1)(\lambda-\eta-2)(\lambda+\eta+1)(\lambda+\eta+2)(\lambda+\eta+3)}{3! (2\pi)^3} + \ldots \]
where \( x = K_N r \). The value of \( \eta_\lambda \) for smaller values of \( r \) is then calculated step by step by the finite difference method*.

\[
\eta_\lambda(r=\delta) = \frac{(12-10q(\gamma)) \eta_\lambda(\gamma) - q(\gamma+\delta) \eta_\lambda(\gamma+\delta)}{q(\gamma+\delta)}
\]

\[
q(r) = 1 - \frac{\delta^2}{12} \left( K_N^2 + \frac{2K_N \eta_N}{r} + \frac{\lambda(\lambda+1)}{r^2} \right)
\]

*See Section D1
When the calculation reaches $R_N$ the program evaluates the logarithmic derivative of $\frac{d}{dx} \varphi_{\lambda \nu}$. The tape is then searched to find the closest value of the logarithmic derivative there and the corresponding value of $\beta r^2$. The program compares values to find $A$ and then continues calculating $\frac{d}{dx} \varphi_{\lambda \nu}$ using the harmonic oscillator wave function with $F$ given by the terminating series

$$F(a | c | x) = 1 + ax/c + a(a+1)x^2/(c(c+1)2! + \ldots$$

For $r > R_N$, the interval size is $2\varepsilon$ and for $r < R_N$, the interval size is $\varepsilon$. The program also computes and prints

$$\int_0^{R_N} \varphi_{\lambda \nu}^2 d\nu \quad , \quad \int_{R_N}^{\infty} \varphi_{\lambda \nu}^2 d\nu \quad , \quad \text{and} \quad \int_{0}^{\infty} \varphi_{\lambda \nu}^2 d\nu$$

It then replaces $\varphi_{\lambda \nu}$ by $\varphi_{\lambda \nu} \sqrt{\int_0^{\infty} \varphi_{\lambda \nu}^2 d\nu}$. If sense switch 2 is down the program prints the bound state wave function.

If sense switch 1 is down the program prints $\lambda, n$, the number of the entry in the table where the logarithmic derivative was found, $\beta r^2$, and the value of the logarithmic derivative.

If sense switch 3 is down the program will not search the tape but just print out $\lambda, n$, the logarithmic derivative, and $R_N$ and then stop (HPR O). If the value of logarithmic derivative is known it may be placed in a card replacing $R_N$ and with the same format as the printed line. If the program cannot find the logarithmic derivative on the tape it will behave as if sense switch 3 were down.
The program writes the bound state wave function on drum 1 in locations 100 through 599.

Sections C₁ and C₂ - Optical Potentials

These sections compute the optical potentials for the proton and deuteron distorted waves. There are three quantities calculated at each point, \( V \) the real part of the optical potential (including \( K^2 \)), \( W \) the imaginary part of the optical potential, and \( V_C \) the potential due to a point charge. The quantities calculated in the various regions are:

\[
\begin{align*}
\gamma > Y_p(Y_b) \quad V &= \frac{e'}{12} K^2 - \frac{2e'}{12} \frac{\eta K}{Y} \\
W &= 0 \\
V_C &= \frac{e'}{12} K^2 - \frac{2e'}{12} \frac{\eta K}{Y} \\
\gamma > \vec{R}_p(\vec{R}_b) \quad V &= \frac{e'}{12} K^2 - \frac{2e'}{12} \frac{\eta K}{Y} - \frac{e'}{12} \frac{\sqrt{Y}}{(1+e^{Y/\alpha})} \\
W &= -\frac{e'}{12} \frac{\sqrt{Y}}{(1+e^{Y/\alpha})} \\
V_C &= \frac{e'}{12} K^2 - \frac{2e'}{12} \frac{\eta K}{Y} \\
\gamma > \vec{R}_p(\vec{R}_b) \quad V &= \frac{e'}{12} K^2 - \frac{2e'}{12} \eta K \left( \frac{3}{\bar{R}^2} - \frac{\gamma^2}{\bar{R}^3} \right) - \frac{e'}{12} \frac{\sqrt{Y}}{(1+e^{Y/\alpha})} \\
W &= \frac{\omega}{12} \frac{\sqrt{Y}}{(1+e^{Y/\alpha})}
\end{align*}
\]
\[ V_C = \frac{e'^2}{12} K^2 - \frac{2e'^2}{12} \frac{\gamma K}{Y} \]

\[ p_x(p_0) > \gamma \]

\[ v = \frac{e'^2}{12} K^2 - \frac{2e'}{12} \frac{\gamma K}{12} \left( \frac{3}{\alpha^3} - \frac{\gamma^2}{\alpha^2} \right) - \frac{e'^2}{12} \frac{\gamma V}{(1 + e^{-m})} \]

\[ w = - \frac{e'^2}{12} \frac{\gamma V}{(1 + e^{-m})} \]

\[ V_C = \frac{e'^2}{12} K^2 - \frac{2e'^2}{12} \frac{\gamma K}{Y} \]

where

\[ e' = \delta \text{ if } r < R_N \]

\[ e = \delta \text{ if } r > R_N. \]

These programs write the proton potentials on drum 3 in locations 1 through 1499 and the deuteron potentials on drum 2 in locations 1 through 1499. Both programs use the same subroutine for calculating the optical potentials. This subroutine is called LFD.

Sections D1 and D2 - Free Wave Functions

These sections compute the proton and deuteron elastic scattering wave functions. Both sections use the same subroutine to calculate the wave functions and merely insert the necessary arguments into the calling sequence for the subroutine. After obtaining the wave functions they are written one at a time on tape. Section D1 writes the proton wave functions on tape 3 and Section D2 writes the deuteron wave functions on tape 2. These sections also place the
last two points of the wave function on drum 3 starting at location 1501.

The subroutine used by both sections is called 2AB and will now be described.

**Subroutine 2AB**

There are two operating modes for this subroutine depending on the variable M10. If M10 is zero or negative, wave functions will be distorted by the complex optical potential while if M10 is greater than zero, pure Coulomb wave functions will be calculated. Calculation starts at \( r = 0 \) and proceeds outward with different methods being used in different regions. For the optical potential calculation the methods are as follows.

1. \( Y \leq n_o L / \kappa \)

The power series is used here being carried to the first five terms only. This is all the accuracy needed in this region since the wave function is small here. The full expression used is:

\[
\chi_L = C_L Y^{L+1} \sum_{n=0}^{10} \phi_n y^n = \chi_L^+ + i \chi_L^-
\]

\[
\phi_0 = 1 \quad \phi_1 = 0 \quad \phi_n = \phi^+_n + i \phi^-_n
\]
\[ \Phi_n^+ = (\lambda_0^- \Phi_{n-2}^- - \lambda_0^+ \Phi_{n-2}^+ - \lambda_2^+ \Phi_{n+4}^+ ) \div n(n+2L+1) \]

\[ \Phi_n^- = (\lambda_0^- \Phi_{n-2}^+ - \lambda_0^+ \Phi_{n-2}^- - \lambda_2^+ \Phi_{n+4}^- ) \div n(n+2L+1) \]

\[ \lambda_0^+ = k^2 - \frac{3nK}{k} - \sqrt{1+e^{-m}} \]

\[ \lambda_0^- = -\sqrt{1+e^{-m}} \]

\[ \lambda_2^+ = \frac{nK}{3} \]

\[ c_L = \frac{\sqrt{L^2 + \gamma^2}}{L(n+1)} c_{L-1} \]

\[ c_0 = \sqrt{2\pi n \div (e^{2\pi n} - 1)} \]

2. \[ \gamma = n_0 L + \varepsilon, \gamma = \frac{n_0 L}{K} + 2\varepsilon \]

At these two points the power series is also used but it is carried to convergence so that two accurate starting values may be obtained for the difference method. The sum is continued until the condition

\[ |\Phi_n \gamma^n| = \frac{\sum_{n=0}^{N} \Phi_n \gamma^n}{\sum_{n=0}^{N} |\Phi_n \gamma^n|} \leq 10^{-n_0} \]

is met.

3. \[ \gamma > n_0 L + 2\varepsilon \]

In this region the general form of the difference method\(^{(36)}\) is used. These equations follow.
\[ \chi_L^+(r+\delta) = \frac{q(r+\delta) \chi^+(r) + p(r+\delta) \chi^-(r)}{q^2(r+\delta) + p^2(r+\delta)} \]

\[ \chi_L^-(r+\delta) = \frac{q(r+\delta) \chi^-(r) - p(r+\delta) \chi^+(r)}{q^2(r+\delta) + p^2(r+\delta)} \]

\[ \chi^+(r) = (12-10q(r)) \chi_L^+(r) - q(r-\delta) \chi_L^+(r-\delta) + 10p(r) \chi_L^-(r) + p(r-\delta) \chi_L^-(r-\delta) \]

\[ \chi^-(r) = (12-10q(r)) \chi_L^-(r) - q(r-\delta) \chi_L^-(r-\delta) - 10p(r) \chi_L^+(r) - p(r-\delta) \chi_L^+(r-\delta) \]

\[ q(r) = 1 + V(r) - \frac{S^2}{\Lambda^2} \frac{L(L+1)}{\Lambda^2} \]

\[ p(r) = W(r) \]

4. \( \gamma > \tilde{\gamma} \); \( W=0 \) and \( V=V_C \) so the equations may be simplified.

\[ \chi_L^+(r+\delta) = \frac{\chi^+(r)}{q(r+\delta)} \]

\[ \chi_L^-(r+\delta) = \frac{\chi^-(r)}{q(r+\delta)} \]

\[ \chi^+(r) = (12-10q(r)) \chi_L^+(r) - q(r-\delta) \chi_L^+(r-\delta) \]

\[ \chi^-(r) = (12-10q(r)) \chi_L^-(r) - q(r-\delta) \chi_L^-(r-\delta) \]

\[ q(r) = 1 + V(r) - \frac{S^2}{\Lambda^2} \frac{L(L+1)}{\Lambda^2} \]
For the case in which \( M > 0 \) and Coulomb wave functions are to be calculated the following methods are used. \( \chi_L^- \) is set to zero in all cases.

1. \( Y \leq \frac{\eta_L}{K} \)

   The power series for Coulomb wave functions \( (F_L^-) \) expanded about \( r = 0 \) is
   \[
   \chi_L^{+} = C_L(\mathbf{k}\mathbf{r})^{L+1} \phi_L
   \]
   \[\phi_L = \sum_{n=0}^\infty A_n(\mathbf{k}\mathbf{r})^n\]
   \[A_0 = 1, \quad A_1 = \frac{\eta}{(l+1)}\]
   \[A_n = \frac{2\eta A_{n-1} - A_{n-2}}{n(n+2L+1)}\]

2. \( Y = \frac{\eta_L}{K} + \epsilon, \quad Y = \frac{\eta_L}{K} + 2\epsilon \)

   The power series is used. This time we take as many terms as are necessary to attain the following criterion.
   \[
   \left( |A_{N-1}(\mathbf{k}\mathbf{r})^{N-1}| + |A_N(\mathbf{k}\mathbf{r})^N| \right)^{\frac{1}{N}} \left| \sum_{n=0}^{N} A_n(\mathbf{k}\mathbf{r})^n \right| < 10^{-n_6} .
   \]

3. \( Y > \frac{\eta_L}{K} + 2\epsilon \)

   Method 4 of the first case is now used. In order for these wave functions to be calculated correctly \( V \) must be replaced by \( V_C \) throughout the range before calculation is started.
**Section E - Coulomb Phase Shifts**

This section calculates $\xi_{L}$ and $\xi_{2L}$ from $L = 0$ to $L = 50$ for both the proton and deuteron. This is accomplished by using Stirling's approximation for $L = 50$ to get the argument of the $\Gamma$ function in the expression.

$$\xi_{L} = \text{arg} \left( \Gamma \left( L + 1 + i \eta \right) \right)$$

This procedure yields

$$\xi_{L} = \left( L + 1 \right) \beta + \eta \ln \sqrt{\frac{1}{(L+1)^2 + \eta^2}} - \eta - \frac{\xi_1 \eta^2}{12 \alpha}$$

$$+ \frac{\sin \beta}{360 \alpha^2} - \frac{\sin 5 \beta}{1260 \alpha^5} + \frac{\sin 7 \beta}{1680 \alpha^7}$$

$$- 0.00084175084 \frac{\sin 9 \beta}{\alpha^9} + \cdots$$

where

$$\beta = \tan^{-1} \frac{\eta}{L+1}$$

$$\alpha = \sqrt{(L+1)^2 + \eta^2}$$

The rest of the phase shifts are calculated by the recursion relation

$$\xi_{L-1} = \xi_{L} - \tan^{-1} \frac{\eta}{L}$$

The actual quantities desired are

$$\beta_{L} = \xi_{L}$$

$$\gamma_{L} = e^{\xi_{L}}$$

These are calculated and written on drum 2 from location 1001 to location 1409.
Section F - Distorted Phase Shifts

This section compares the last points of the wave functions calculated using the optical potential with a Coulomb wave function to determine the normalization constants for the wave functions and the phase shifts for elastic scattering. The Coulomb wave function is calculated by an asymptotic series and gives virtually the only mathematical difficulty encountered in running the program since it may not converge rapidly enough for high L if $r_o$ is not large enough.

The quantities $\hat{C}_L$ and $\hat{S}_L$ are defined by

$$\hat{C}_L \chi_L = \frac{k_r}{z} \left\{ y_L^* \hat{S}_L y_L + \hat{S}_L y_L \right\}$$

where

$$y_L \rightarrow \frac{\mathcal{C} \cdot \mathcal{D}}{z \cdot k_r} \left\{ 1 + \frac{(i \eta - L)(i \eta + L + 1)}{11, (21k_r)^2} + \ldots \right\}$$

and

$$\Theta = k_r - \eta \ln 2k_r - \frac{\pi L}{2}$$

If we write

$$\frac{k_r}{2} y_L = a - i \beta \quad \hat{C}_L = c^+ + i c^-$$

$$\chi_L = x + iy \quad \hat{S}_L = \delta^+ + i \delta^-$$

$$f(r_0) = f_1 \quad f(r_0 - 5) = f_2$$
\[
\begin{align*}
\mathbb{C}^+ &= c_- \Delta_- \div (\Delta_+^2 + \Delta_-^2) \\
\mathbb{C}^- &= c_- \Delta_+ \div (\Delta_+^2 + \Delta_-^2) \\
\mathbb{B}^+ &= (b_- \Delta_+ + b_+ \Delta_-) \div (\Delta_+^2 + \Delta_-^2) \\
\mathbb{B}^- &= (b_- \Delta_+ - b_+ \Delta_-) \div (\Delta_+^2 + \Delta_-^2)
\end{align*}
\]

where

\[
\begin{align*}
\Delta_+ &= x_2 a_1 + y_2 \beta_1 - x_1 a_2 - y_1 \beta_2 \\
\Delta_- &= y_2 a_1 - x_2 \beta_1 - y_1 a_2 + x_1 \beta_2 \\
c_- &= 2(\beta_2 a - \beta_1 a_2) \\
b_+ &= x_1 a_2 - x_2 a_1 - y_1 \beta_1 + y_2 \beta_2 \\
b_- &= y_1 a_2 + x_1 \beta_2 - y_2 a_1 - x_2 \beta_1
\end{align*}
\]

This section writes $\hat{\mathbb{S}}_L - \mathbb{S}_L$ and $\hat{\mathbb{C}}_L$ for the deuteron on drum 2 starting at 1409 and $\hat{\mathbb{S}}_L - \mathbb{S}_L$ and $\hat{\mathbb{C}}_L$ for the proton on drum 2 starting at 1613. It also prints on line $\hat{\mathbb{S}}_L - \mathbb{S}_L$ for the largest value of $L$ which was used in calculating optical wave functions under the heading of "Distortion Check" to serve as a guide as to the number of waves which should be distorted by the optical potential.
Section G - Integration

This program computes the radial integrals. The unnormalized radial integral is given by

\[ \tilde{\gamma}_{\lambda} = \int_{R'}^{\infty} \chi_{\lambda} \gamma \chi_{\lambda} \, d\gamma \]

where the integrals are computed for \( R' = 0 \) and \( R' = \bar{R} \).

The integration is carried out by the 3/8's rule which is

\[ \int_{x_0}^{x_N} y(x) \, dx = \frac{9}{8} \delta \left\{ \frac{y_0}{3} + y_1 + y_2 + \frac{2}{3} y_3 + y_4 + y_5 + \frac{2}{3} y_6 + \cdots \right\} \]

\[ + y_{N-2} + y_{N-1} + \frac{y_N}{3} \]

where \( N \) is a multiple of 3.

The integrals are written on drum 4 starting at location 1.

Section H - Normalization and Summation

This section computes the normalized radial integrals by combining the correct information from drums 2 and 4. If

\[ C'_L = \hat{C}_L \quad \text{if } \chi_L \text{ is distorted by the optical potential} \]

\[ C'_L = C_L \quad \text{if } \chi_L \text{ is a Coulomb wave function} \]

then the normalized radial integral is given by

\[ \tilde{\gamma}_{\lambda} = C'_L \, \hat{C}_L \, \tilde{f}_{\lambda} \]
After being normalized the integrals are summed to form the quantities \( \beta_{x}^{\lambda m} \),

\[
\beta_{x}^{\lambda m} = \sum_{l=|x-\lambda|}^{-1} \lambda m \int_{1}^{x} (-1)^{\frac{x+l-\lambda}{2}} \int_{1}^{x} \text{l} \%
\]

\( \int_{1}^{x} \lambda m \) is read from the table tape.

\[
\int_{1}^{x} \lambda m = \frac{(2x+1)(2l+1)}{(2l+1)} \sqrt{(\lambda-1m)!} \cdot C(\lambda,0,\bar{x},x;\lambda,m) \newline \times C(\lambda,0,\bar{x},x;\lambda,0)
\]

where \( C \) is the Clebsch-Gordan coefficient. \( \int_{1}^{x} \lambda m \) was calculated from the following equation.

\[
\int_{1}^{x} \lambda m = \frac{1}{(2+\lambda+m+1)!} \frac{(\lambda+1)!}{(\lambda+m)!} \frac{(x+2l-\lambda)!}{(x+\lambda-m)!} \frac{1}{(x-m)!} \newline \times \sum_{t} \frac{(-1)^{t}}{t!} \frac{(x+t)!}{(x-t)!} \frac{(\lambda+\bar{x}-t)!}{(\lambda-m-t)!} \frac{1}{(x-\lambda+t)!} \newline \times \sum_{s} \frac{c^{\bar{s}}}{s!} \frac{(x+s)!}{(x-s)!} \frac{(x+s)!}{(x-s)!} \frac{(\lambda+s)!}{(x+s)!} \frac{(x+s)!}{(x-s)!} \frac{(\lambda+s)!}{(x+s)!}
\]

where the sums are limited by zeros of the factorials and

\( c = 0 \) unless

\( x \leq \bar{x} + \lambda, \)

\( \bar{x} \leq x + l, \) and

\( \lambda \leq \bar{x} + \lambda. \)
Section I - Differential Cross Section

This section computes the differential cross section and the polarization as a function of angle. First the quantities $\beta_{\lambda m}^{(e)}$ are computed.

$$\beta_{\lambda m}^{(e)} = \sum_{l} P_{l}^{\lambda m} P_{l}^{m l} \left( \cos \Theta \right)$$

where $P_{l}^{m} \left( \cos \Theta \right)$ is the associated Legendre function. These quantities are punched in cards on line and contain information useful in other calculations such as the correlation between the outgoing proton and a subsequent $\gamma$-ray.

The cross section and polarization are then computed by the following formulae.

$$\frac{dS(\Theta)}{d\Omega} = \sum_{l=\lambda}^{\lambda} \left| \beta_{\lambda m}^{(e)} \right|^2$$

$$P(\Theta) = \frac{\sum_{m=-\lambda}^{\lambda} \left( \lambda(\lambda+1)-m(m+1) \right) \sum_{m=-\lambda}^{\lambda} \left| \beta_{\lambda m}^{(e)} \right|^2}{\sum_{m=-\lambda}^{\lambda} \left| \beta_{\lambda m}^{(e)} \right|^2}$$

These quantities are printed on line for both the case in which $R^b = 0$ and $R^b = R$. The polarization is computed according to the convention $\vec{k}_p \times \vec{k}_D$.

This section reads one data card specifying the angle for which the cross section is to be computed. This card contains $\Theta_0$, $\Delta \Theta$, and NOT in a $(2E10.3, I10)$ format.
is the smallest angle desired, \( \Delta \theta \) is the spacing between angles and \( \text{NOT} \) is the number of angles \( m \) for which the cross section is to be computed. For example, if \( \theta_0 = 0 \), \( \Delta \theta = 5 \), \( \text{NOT} = 37 \), the program will compute the cross section at 5\(^\circ\) intervals from 0 through 180\(^\circ\).

**Section J - Total Cross Section**

This section computes the total cross section according to

\[
\sigma = \frac{4\pi}{2\lambda+1} \sum_{\lambda \text{ to } m} \left| \beta_{\lambda}^{*} \right|^2 (2m)!
\]

and a check of the accuracy to which this summation has been carried,

\[
\frac{\sum_{\lambda} \left| \beta_{\lambda}^{*} \right|^2 (2m)!}{\sum_{\lambda \text{ to } m} \left| \beta_{\lambda}^{*} \right|^2 (2m)!}
\]

These are printed on line under the heading of "Total Cross Section" and "Convergence Test."

**Section K - Elastic Cross Sections**

This program computes the elastic cross sections for both the proton and deuteron. The Rutherford cross section is also calculated along with the ratio of the elastic cross section to the Coulomb cross section.
\[ g_{\text{elastic}} = \left| f_c + f_o \right|^2 / K^2 \]
\[ g_{\text{Coulomb}} = \left| f_c \right|^2 / K^2 \]

where

\[ f_c = -\frac{\eta}{2 \sin^2 \frac{\Theta}{2}} \left( 1(2 \xi - \eta \ln 2 \sin^2 \frac{\Theta}{2} \right) \]

\[ f_o = \frac{1}{i} \sum (L+1/2) P_L(\cos \Theta) (\hat{B}_L - \hat{B}_L) \].

The results of these calculations are printed on line. The reduced width for the neutron is also calculated and printed in this section. At the beginning of this section the program tape is rewound and the load tape sequence is executed at the end.

This section reads one data card giving the values of NOT, \( \Theta_o \), and \( \Delta \Theta \) in a (I5,2E17.8) format. The definition of these symbols is given in Section I.
**TABLE 4**

Correspondence Table

<table>
<thead>
<tr>
<th>Name used in this write up</th>
<th>Name used originally</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>I(1)</td>
</tr>
<tr>
<td>B</td>
<td>M1Z</td>
</tr>
<tr>
<td>C₁</td>
<td>CPP</td>
</tr>
<tr>
<td>C₂</td>
<td>CDP</td>
</tr>
<tr>
<td>D₁</td>
<td>C2ABP</td>
</tr>
<tr>
<td>D₂</td>
<td>C2ABD</td>
</tr>
<tr>
<td>E</td>
<td>I(3)</td>
</tr>
<tr>
<td>F</td>
<td>C3A</td>
</tr>
<tr>
<td>G</td>
<td>II</td>
</tr>
<tr>
<td>H</td>
<td>III(1)</td>
</tr>
<tr>
<td>I</td>
<td>III(2)</td>
</tr>
<tr>
<td>J</td>
<td>CT</td>
</tr>
<tr>
<td>K</td>
<td>ECS</td>
</tr>
</tbody>
</table>
V. Appendix B

DESCRIPTION OF THE SPIN-ORBIT PROGRAM
DESCRIPTION OF THE SPIN-ORBIT PROGRAM

There are two fundamental differences between the ordinary program and the spin-orbit program purely from the viewpoint of the machine calculation. The first is that the wave functions must now be calculated as a function of \( j \) and \( \bar{j} \), the deuteron and proton total angular momentum, as well as \( \lambda \) and \( \bar{\lambda} \). Thus the number of radial integrals is multiplied by a factor of 6. These are classified and ordered in the following manner:

1. \( j = \lambda - 1 \)
   \( \bar{j} = \lambda - 1/2 \)

2. \( j = \lambda - 1 \)
   \( \bar{j} = \lambda + 1/2 \)

3. \( j = \lambda \)
   \( \bar{j} = \lambda - 1/2 \)

4. \( j = \lambda \)
   \( \bar{j} = \lambda + 1/2 \)

5. \( j = \lambda + 1 \)
   \( \bar{j} = \lambda - 1/2 \)

6. \( j = \lambda + 1 \)
   \( \bar{j} = \lambda + 1/2 \)

In practice the order of calculation is as follows. The proton wave functions for \( j = \lambda - 1/2 \), and \( j = \lambda + 1/2 \) are calculated and written on separate tapes. Their normalization constants and distorted phase shifts are calculated and the elastic cross section for protons is calculated. The program then enters a loop which is programmed so that two sets of radial integrals are calculated each time through the loop. The two sets of integrals correspond to \( j = \lambda - 1/2 \) and \( j = \lambda + 1/2 \). The program passes through the loop three times corresponding to \( j = \lambda - 1 \), \( j = \lambda \), and
\( \lambda + 1 \). The integrals are normalized immediately after being calculated and written on tape so that the drum space is free for the next calculation. At the end of the loop the last time through the radial integrals have been written on tape in the order indicated above.

The second difference is that the sum to form the scattering amplitude \( S(\theta) \) is more complex. This sum is now done in two parts with an intermediate result being written on tape.
The order of the program follows.

Control
A'
B
E
D_1'
F'
K'

Program
D_2'
F''

Loop
G'
H'
G''
H''
L
M
K''
N
P
I'
Dump J
Dump B

Sections with primes are the same as programs previously described with minor modifications.
should be made to the write up of the unprimed program
before reading the explanation following. Sections with
names identical to those described previously are identical
and explanation of them will not be given again. Sections
with entirely new names are different programs and
explanations of their complete workings will be given.

Section \( A' \)

This is the same as \( A \) except for modifications as
described below.

One more card is read following the one containing
\( (n_1, n_3, n_4, n_7) \) which contains \( (V_{PLS}, V_{DLS}, W_{PLS}, W_{DLS}) \). These
numbers are processed in the same manner as \( V_p \). More
constants used in \( D_1 \) and \( D_2 \) are calculated here to allow
more room in these programs.

Section \( D_1' \)

This program calculates proton wave functions for
\( j = \frac{\hbar}{2} - \frac{1}{2} \) and \( j = \frac{\hbar}{2} + \frac{1}{2} \) and writes them on tapes 3 and 4.
The optical potential subroutine (modified to include the
\( L \cdot \bar{S} \) potential) is now contained within \( D_1 \) since the
potential changes with \( \bar{\alpha} \) and \( \bar{j} \). Both sets of final points
of the wave function are written on drum 3 starting at
location 1501. The \( L \cdot \bar{S} \) form factor is taken to be the
derivative of the same approximation which is used for the
Woods-Saxon potential.
Section F'

This section computes \( \hat{c}_{\frac{1}{2}} \) and \( \hat{c}_{\frac{3}{2}} \) in exactly the same manner as before and writes them on drum 2 starting at location 1409. The only real difference between the programs F and F' is that F' computes two sets for the proton while F computes one set for the deuteron and one for the proton.

Section K'

This program computes the elastic cross section and polarization for the proton only. With the addition of the spin-orbit potential the equations are altered somewhat. Let \( f_c \) be defined as before.

\[
\begin{align*}
  f'_{\frac{1}{2}} &= \frac{1}{K} \left\{ \frac{1}{2} \sum_L \rho_L (\cos \theta) \left[ L \left( \hat{b}_{L+\frac{1}{2}} - \hat{b}_L \right) - (L-1) \left( \hat{b}_{L-\frac{1}{2}} - \hat{b}_L \right) \right] + f_c \right\} \\
  f'_{\frac{3}{2}} &= \frac{1}{K1} \sum_L \rho_L (\cos \theta) \left[ \left( \hat{b}_{L+\frac{3}{2}} - \hat{b}_L \right) - \left( \hat{b}_{L-\frac{3}{2}} - \hat{b}_L \right) \right] \\
  \frac{dG(\mathbf{e})}{d\Omega} &= |f'_{\frac{1}{2}}|^2 + |f'_{-\frac{1}{2}}|^2 \\
  \text{P}(\theta) &= 2 \text{Im} \left( f_{-\frac{1}{2}}^{\frac{1}{2}} f_{-\frac{1}{2}}^{\frac{3}{2}} \right) \left/ \frac{dG(\mathbf{e})}{d\Omega} \right|
\end{align*}
\]

These results are printed as soon as they are obtained so this program could be used to calculate optical scattering of spin 1/2 particles with a spin-orbit potential with fair efficiency.
This section reads one data card. The format and definition of quantities are the same as in Section K.

**Program Loop**

The following six programs (H' is counted twice) are repeated three times for the conditions j = \( l - 1 \), j = \( l \), and j = \( l + 1 \).

**Section D'_2**

This program computes the deuteron wave functions and writes them on tape 2. The alterations are similar to those made in D'_1 except that only one set of wave functions is computed.

**Section F''**

This program computes the \( \hat{C}_{\lambda \lambda} \) and \( \hat{C}'_{\lambda \lambda} \) necessary and writes them on drum 2 starting in location 1001.

**Section G'**

This section integrates the product of the wave functions to obtain the unnormalized radial integrals and writes them on drum 4 as before. G' takes its proton wave function from tape 3.
Section H'

Section H' computes the normalized radial integrals and writes them on tape 5.

Section G''

This program is identical with G' except that the proton wave functions are taken from tape 4.

Section I

This program does the first part of the sum to compute the stripping amplitude. The result of the computation is called B.

\[
B_{\lambda, \lambda_1, \lambda_2, \lambda_3}^{\sigma_1 \sigma_2 \sigma_3} (-1)^{\lambda_2 + \lambda_3 + 1} \frac{1}{\sqrt{2}} \sum_{\lambda_4} \frac{(-1)^{\lambda - \lambda_4}}{\lambda_4 \lambda_1 \lambda_2 \lambda_3} \binom{\lambda_4 + \lambda_3 + \lambda - 1}{\lambda_4 - \lambda_3 - \lambda - 1} \binom{\lambda - \lambda_4 - \lambda_3 - \lambda - 1}{\lambda_4 - \lambda - 1} \binom{\lambda - \lambda_4 - \lambda_3 - \lambda - 1}{\lambda_4 - \lambda - 1} \\
\times C_{\lambda_1, \lambda_2, \lambda_3}^{\sigma_1, \sigma_2, \sigma_3} \left( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \right) X_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}^{\sigma_1 \sigma_2 \sigma_3 \sigma_4}
\]

Here C is the Clebsch-Gordan coefficient and X is the \(X\)-coefficient.

\[C_{\lambda_1, m_1, \lambda_2, m_2, \lambda_3, m_3}^{\sigma_1, \sigma_2, \sigma_3} \] is calculated by a general subroutine. In this subroutine C is obtained from the recurrence relation
$$N(l_3, m_3) C(l_1, m_1, l_2, m_2; l_3, m_3 - 1) =$$

$$N(l_1, m_1 + 1) C(l_1, m_1 + 1, l_2, m_2; l_3, m_3) + N(l_2, m_2 + 1) C(l_1, m_1, l_2, m_2 + 1; l_3, m_3)$$

where

$$N(L, M) = \sqrt{(L+M)(L-M+1)}$$

are the starting values for the relation are obtained from the formula which applies when $$m_3 = l_3$$.

$$C(l_1, m_1, l_2, m_2; l_3, l_3) = (-1)^{l_1-m_1} x$$

$$\sqrt{\frac{(l_1 + l_2 - l_3)! (l_3 + m_1)! (l_3 + l_2 - m_1)! (2l_3 + 1)!}{(l_1 + l_2 + l_3 + 1)! (l_3 - m_1)! (l_1 - m_1)! (l_3 + l_1 - l_2)! (l_3 - l_1 + l_2)!}}$$

If $$m_1 = m_2 = m_3 = 0$$, $$C$$ is calculated directly from the following relation

$$C(l_1, 0, l_2, 0; l_3, 0) = (-1)^{G+l_3}$$

$$x = \frac{G!}{(G-l_1)!} \frac{1}{(G-l_2)!} \frac{1}{(G-l_3)!}$$

$$G = \frac{(l_1 + l_2 + l_3)}{2}$$ if $$l_1 + l_2 + l_3$$ is even.

$$C = 0$$ if $$l_1 + l_2 + l_3$$ is odd. The X-coefficient was also calculated by subroutine.
\[
\begin{align*}
\left( \begin{array}{cc}
A & 1/2 \\
C & 1/2 \\
E & 1 \\
\end{array} \right)
\left( \begin{array}{cc}
B \\
D \\
F \\
\end{array} \right)
&= X(A, B, C, D, E, F) \\
X(A, B, C, D, E, E)
&= (-1)^{3E+B+C+1.5} x \\
&\cdot \frac{(B(B+1)-A(A+1)-D(D+1)+C(C+1))}{\sqrt{2(2E+1)^3E(E+1)}} \\
&\cdot W(A, B, C, D, 1/2, E) \\
X(A, B, C, D, E-1, E)
&= \\
&\left\{ \begin{array}{c}
\frac{(2E-2)!}{(2E+1)!} \cdot \frac{(A+B-1/2)!}{(A+B+3/2)!} \cdot \frac{(C+D-1/2)!}{(C+D+3/2)!} \cdot \frac{(A+C-E+1)!}{(B+D-E)!} \cdot \frac{(E+D+E+1)!}{(A+C+E)!} \\
\frac{(B-D+E)!}{(A-C+E-1)!} \cdot \frac{(D-B+E)!}{(C-A+E-1)!} \cdot \frac{1}{3} \\
\end{array} \right\}^{1/2}
\end{align*}
\]
The results of this calculation are written on tape 2.
Section M

Here the second part of the sum is carried out.

The result of this computation is called \( J \).

\[
J^{\lambda, \sigma, m}_{\lambda_1, \sigma_1, I_1} = \sum_{j} B^{\sigma, \sigma_1, m}_{\lambda, \lambda_1, j, I_1, \sigma_1} C (\lambda, -m, j, \sigma + m; 1/2, \xi) \frac{(\lambda - m)!}{(\lambda + m)!}
\]

The notation which differs from the previous program or is new is:

- \( S \) is the total angular momentum of the capture neutron.
- \( \sigma \) is the spin projection of the incoming deuteron (-1,0,1).
- \( \overline{\sigma} \) is the spin projection of the outgoing proton (-1/2,1/2).
- \( j \) is the total angular momentum of the incoming deuteron.
- \( \overline{j} \) is the total angular momentum of the outgoing proton.

The results of this calculation are written on tape 5.

Section K''

This program computes the elastic scattering cross section for deuterons from an optical well with a spin-orbit term. If \( f_C \) is defined as before and if we use + and - to denote real and imaginary parts and write

\[
B'_{LJ} = \hat{\mathcal{B}}_{LJ} - \mathcal{B}_L
\]

(Except \( B_{00}' = B_{0-1}' = 0 \)) then with
\[ f_{11} = \frac{1}{2\epsilon} \sum_k \mathcal{P}_k (\cos \Theta) \left[ \mathcal{B}_{k,j+k}(\lambda+2) + \mathcal{B}_{k,j+k}(2\lambda+1) + \mathcal{B}_{k,j-1}(\lambda-1) \right] + f_c \]

\[ f_{22} = \frac{1}{\epsilon} \sum_k \mathcal{P}_k (\cos \Theta) \left[ \mathcal{B}_{k,j+1}(\lambda+1) + \mathcal{B}_{k,j-1}(\lambda-1) \right] + f_c \]

\[ f_{12} = \frac{z}{\epsilon} \sum_k \mathcal{P}_k (\cos \Theta) \left[ \mathcal{B}_{k,j+1}(\lambda+1) - \mathcal{B}_{k,j-1}(\lambda-1) \right] \]

\[ f_{21} = \frac{z}{\epsilon} \sum_k \mathcal{P}_k (\cos \Theta) / \lambda(\lambda+1) \left[ \mathcal{B}_{k,j+1}(\lambda+2) - \mathcal{B}_{k,j}(2\lambda+1) - \mathcal{B}_{k,j-1}(\lambda-1) \right] \]

\[ f_{13} = \frac{1}{\epsilon} \sum_k \mathcal{P}_k (\cos \Theta) / \lambda(\lambda+1) \left[ \mathcal{B}_{k,j+1}(\lambda+1) - \mathcal{B}_{k,j}(2\lambda+1) + \mathcal{B}_{k,j-1}(\lambda+1) \right] \]

The cross section and polarization are given by

\[ \frac{d\sigma}{d\Omega} (\Theta) = \frac{1}{3} \left( |f_{22}|^2 + 2 |f_{21}|^2 + 2 |f_{13}|^2 + 2 |f_{12}|^2 + 2 |f_{21}|^2 \right) \]

\[ P(\Theta) = \frac{2\sqrt{2}}{3} \left( f_{12}^* f_{11} - f_{11}^* f_{12} + f_{13}^* f_{12} - f_{12}^* f_{13} + f_{21}^* f_{22} - f_{22}^* f_{21} \right) / \frac{d\sigma}{d\Omega} \]

This program places the amplitudes \( f_{\text{geo}} \) on drum 3 starting at location 0. The cross section and polarization are printed on line after they are calculated. This program reads one data card which is of the same form as the one read in \( K \).

**Section N**

This program simply reads the deuteron elastic amplitudes from drum 3 and prints them on line.
Section P

This program computes \( \beta_{\lambda}^{m,m',s'} (\Theta) \) where

\[
\beta_{\lambda,s}^{m,m',s'} (\Theta) = \sum_{\gamma} P_{\gamma}^{m} (\cos \Theta) J_{\lambda,s}^{m,\gamma} \gamma_{s',m',\gamma}.
\]

The calculations are carried out at 5° intervals from 0° to 180°. The results are written on tape 4.

Section I

This section computes the differential cross section and polarization from the quantities \( \beta_{\lambda,s}^{m,m',s'} (\Theta) \) computed in Section P.

\[
\frac{d\sigma(\Theta)}{d\Omega} = \frac{10^2}{2S+1} \sum_{m,\gamma} \left| \beta_{\lambda,s}^{m,m',s'} (\Theta) \right|^2
\]

\[
P(\Theta) = 4 \left( \sum_{m,\gamma} m_{\gamma} \beta_{\lambda,s}^{m,m',s'} (\Theta) \right)^2 + \beta_{\lambda,s}^{m,m',s'} (\Theta) \left( \sum_{m,\gamma} \left| \beta_{\lambda,s}^{m,m',s'} (\Theta) \right|^2 \right)
\]

These results are printed on line as they are computed. The polarization is computed according to the convention: \( \bar{R} D \times \bar{R} D \).

Dump J and Dump B

These programs dump on line the quantities named. Only the \( R' = 0 \) sections are dumped and not all of the \( B \)'s are dumped for that case. These can be used for checking and to see how many partial waves are really important. The values of \( J \) may be used to compute other quantities of interest.
VI. ACKNOWLEDGMENTS

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