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PROPERTIES OF FUNCTIONS WHICH UNIFORMIZE CERTAIN
CLASSES OF SIMPLY CONNECTED RIEMANN SURFACES

by

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A THESIS
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INTRODUCTION

Three classes of simply connected Riemann surfaces are considered in the following study. They are designated as Class III, Class IV, and Class V in keeping with the notation of Classes I, II, and III already considered by Taylor in [9,10]. Class III is a subclass of Class IV; it is shown in [9] only that the surfaces of Class III are parabolic, while here the forms of the uniformizing function and its derivative are found for Class III. For Classes IV and V the forms of the uniformizing function and its derivative are obtained, and the type of the surface is determined. Finally, a proof is given to show that every mapping function of the form obtained for Class V defines a surface of this class.

The principal method used in the proofs is that of approximating the given surface by a sequence of compact surfaces. This procedure has been used previously by Elfving and MacLane in [3,5,6].

The term "subuniform convergence on a set D" as used throughout the following pages means uniform convergence on any compact subset of D. By convergence of a sequence of domains or a sequence of Riemann surfaces is meant convergence to the kernel in the
sense of Carathéodory as defined in \([1,6]\).

The following well-known theorems are stated for future reference.

**Theorem A.** Let \(w = f(z) = z + a_2 z^2 + \ldots\) be holomorphic and schlicht in \(|z| < R\) and map \(|z| < R\) onto a domain \(D_w\). Then \(D_w\) contains the disc \(|w| < \frac{1}{4} R\).

**Theorem B.** Let \(w = f(z) = z + a_2 z^2 + \ldots\) be holomorphic and schlicht in \(D_z\), the \(z\)-plane slit along the real axis from \(z = R > 0\) to \(+\infty\). If \(D_w\) is the domain which is the image of \(D_z\) by this map, then the distance from \(w = 0\) to the boundary of \(D_w\) is \(\geq R\).

**Theorem C.** 1) Let \(D\) be the non-degenerate kernel with respect to \(z = 0\) of a sequence of domains \(\{D_n\}\) on the \(z\)-sphere each of which contains the origin.
Let \(D_n \to D\). Let \(E\) be the kernel with respect to \(w = 0\) of a sequence of domains \(\{E_n\}\) on the \(w\)-sphere each of which contains the origin. Let \(w = f_n(z)\) map \(D_n\) schlichtly and conformally onto \(E_n\) with \(f_n(0) \to 0\) and \(f_n'(0) \to c \neq 0, \infty\).

2) If \(f_n(z) \to f(z)\) subuniformly in \(D\), then \(E_n \to E\) and \(w = f(z)\) maps \(D\) schlichtly and conformally onto \(E\) with \(f(0) = 0\) and \(f'(0) = c\).
3) On the other hand, if $E_n \rightarrow E$ and $E$ is not the complete sphere, then $f_n(z) \rightarrow f(z)$ subuniformly in $D$ and $w = f(z)$ maps $D$ schlichtly and conformally onto $E$ with $f(0) = 0$ and $f'(0) = c$.

**Theorem D.** With the hypothesis of 1) in Theorem C, if $D = \{ |z| < \infty \}$ and if $\infty \notin E_n$, then $E = \{ |w| < \infty \}$ and $f_n(z) \rightarrow cz$ subuniformly in $D$.

**Theorem E.** Every family of functions meromorphic in a domain $D$ and satisfying the condition that no member of the family assumes the distinct values $A$, $B$, or $C$, is a normal family.

**Theorem F.** Let $f(z)$ be holomorphic in the domain $D$ between two Jordan arcs $C_1$ and $C_2$ ending at $P$. Let

$$\lim_{z \to P} f(z) = \alpha_1$$

Then if $U$ is an arbitrary neighborhood of $P$, $f(z)$ assumes every value with the possible exception of two in $U \cap D$.

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CHAPTER I
SURFACES OF CLASS III

Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences such that
\[ 0 < a_1 < b_1 < a_2 < a_3 < b_3 < b_4 < \ldots. \]
A surface \( F \) of Class III consists of sheets \( S_1, S_2, \ldots, \)
\( S_n, \ldots \) over the \( w \)-sphere where

(a) \( S_1 \) is a copy of the \( w \)-sphere slit along the real
axis from \( a_1 \) to \( b_1 \), \( 0 < a_1 < b_1 \); for \( n > 1 \) and
odd, \( S_n \) is a copy of the \( w \)-sphere slit along
the real axis from \( b_{n-1} \) to \( a_{n-1} \) and from \( a_n \) to \( b_n \),
\( b_{n-2} < b_{n-1} < a_{n-1} < a_n < b_n \); for \( n \geq 2 \) and \( n \)
even, \( S_n \) is a copy of the \( w \)-sphere slit along the
real axis from \( a_{n-1} \) to \( b_{n-1} \) and from \( b_n \) to \( a_n \),
\( a_{n-2} < a_{n-1} < b_{n-1} < b_n < a_n \).

(b) \( S_n \) is joined to \( S_{n+1} \) by connecting the slits
between \( a_n \) and \( b_n \) to form first order branch
points over \( a_n \) and \( b_n \) for \( n \geq 1 \). (See Figure 1.)

\( F \) is simply connected and open. Hence by the
general uniformisation theorem there exists a function
\[ s = \phi(w) \]
Figure 1
which maps $F$ 1-1 and conformally onto $|z| < R \leq \infty$,

where for $w = f(z) = \phi^{-1}(z)$, $f(0) = 0 \in S_1$ and $f'(0) = 1$.

It is shown in [3] that:

(a) $f(s)$ is real for $s$ real.

(b) if $s = \alpha_k$ and $s = -\beta_k$ denote the images, respectively, of the branch points over $a_k$ and $b_k$, if $s = (-1)^k \gamma_k$ denotes the image of $w = \infty$ in $S_k$, and if for $k \geq 2$ the symbol $(-1)^{k-1} \gamma_k$ denotes the image of $w = 0$ in $S_k$, then

- $-\beta_{k+1} < -\beta_k < 0 < \alpha_k < \alpha_{k+1}$; for $k$ even,
- $\alpha_{k-1} < \gamma_k < \gamma_k < \alpha_k$; for $k$ odd,
- $-\beta_k < -\gamma_k < -\beta_{k-1}$.

(c) the image of $S_k$ is a simply connected region which contains the origin, is bounded by a simple closed curve $C_k$, and is symmetric about the real axis. For $k \geq 2$, the image of $S_k$ is an annular region which is symmetric about the real axis and is bounded by two simple, non-intersecting closed curves $C_{k-1}$ and $C_k$. (See Figure 2.)

(d) $F$ is parabolic, i.e., $R = \infty$. 
Lemma 1.1. There exists a sequence of rational functions \( \{R_n(z)\} \) such that \( R_n(z) \rightarrow f(z) \) subuniformly in \( |z| < \infty \) as \( n \rightarrow \infty \), where \( R_n(z) \) has the form (1) and \( R'_n(z) \) has the form (2) given below.

Proof: Let \( F_n \) be the first \( 2n + 1 \) sheets of \( F \) with the slit in \( S_{2n+1} \) between \( a_{2n+1} \) and \( b_{2n+1} \) deleted. \( F_n \) is a simply connected closed surface with branch points over \( a_1, a_2, \ldots, a_{2n}, b_1, b_2, \ldots, b_{2n} \), and with \( 2n + 1 \) sheets. Then \( F_n \) is a Riemann surface of the inverse of a unique rational function \( w = R_n(z) \) such that \( R_n(0) = 0 \in S_1, R'_n(0) = 1, \) and \( R_n(\infty) = \infty \in S_{2n+1} \).

Let \( \gamma_{1,n}, \gamma_{2,n}, \gamma_{3,n}, \ldots, \gamma_{2n,n} \) denote the values of \( z \) at which \( R_n(z) \) has its \( 2n + 1 \) poles, and let \( 0, \delta_{2,n}, -\delta_{3,n}, \ldots, \delta_{2n,n}, -\delta_{2n+1,n} \) denote the values of \( z \) at which \( R_n(z) \) has its \( 2n + 1 \) zeros. Also, let \( \alpha_{1,n}, \alpha_{2,n}, \ldots, \alpha_{2n,n}, -\beta_{1,n}, -\beta_{2,n}, \ldots, -\beta_{2n,n} \) denote the values of \( z \) for which \( R'_n(z) \) has simple zeros to produce first order branch points at \( a_1, a_2, \ldots, a_{2n}, b_1, b_2, \ldots, b_{2n} \), respectively. It is clear from the form of the surface \( F_n \) and the normalization of \( R_n(z) \) that all the numbers \( \alpha_{i,n}, \beta_{i,n}, \gamma_{i,n}, \) and \( \delta_{i,n} \)
are positive and, for fixed $n$, ordered in a manner similar to that of the $\alpha^1_1$, $\beta^1_1$, $\gamma^1_1$, and $\delta^1_1$. Hence

$$R_n(s) = \frac{2^n}{1 + \frac{s}{\gamma^1_{1,n}}} \prod_{k=2}^{2n} \left( 1 - (-1)^k \frac{s}{\gamma^k_{k,n}} \right) \left( 1 + \frac{s}{\delta^k_{2n+1,n}} \right)$$

(1)

and

$$R'_n(s) = \prod_{k=1}^{2n} \frac{(1 - \frac{s}{\alpha^k_{k,n}})(1 + \frac{s}{\beta^k_{k,n}})}{(1 - (-1)^k \frac{s}{\gamma^k_{k,n}})^2}.$$  

(2)

Now let $D_n$ denote the $s$-plane cut along the positive real axis from $\alpha_{2n,n}$ to $\infty$. $R_n(z)$ maps this region onto $F_n$ with the sheet $S_{2n+1}$ slit from $\alpha_{2n}$ to $+\infty$ along the real axis. Moreover, $\zeta = \phi(w)$ maps $F_n$ schlichtly onto a domain $\Delta_n$ of the $\zeta$-plane whose boundary is $C_{2n+1}$ and the segments of the real axis $(-\beta_{2n+1}, -\gamma_{2n+1})$ and $(\alpha_{2n}, \alpha_{2n+1})$. Since $F$ is parabolic, $\{\Delta_n\}$ converges to $\{ |\zeta| < \infty \}$ and from Theorem D it follows that $z = R^{-1}_n[f(\zeta)] \rightarrow z$ subuniformly in $\{ |\zeta| < \infty \}$ and $\{D_n\}$ converges to $\{ |z| < \infty \}$.

Then because $\{D_n\}$ converges to $\{ |z| < \infty \}$, another application of Theorem D implies that $\zeta = \phi[R_n(z)] \rightarrow z$
subuniformly and thus \( R_n(s) \rightarrow f(s) \) subuniformly in 
\( \{ |s| < \infty \} \). Also, \( R'_n(s) \rightarrow f'(s) \). \|

**Lemma 1.2**. \( \alpha_{k,n} \rightarrow \alpha_k', \beta_{k,n} \rightarrow \beta_k', \gamma_{k,n} \rightarrow \gamma_k' \)
and \( \delta_{k,n} \rightarrow \delta_k \) as \( n \rightarrow \infty \).

Proof: This is an immediate consequence of Hurwitz's Theorem. \|

**Lemma 1.3**. The infinite products

\[
M(z) = \prod_{k=1}^{\infty} \frac{1 - \frac{s}{\delta_{2k}}}{1 - \frac{s}{\gamma_{2k}}},
\]

\[
N(z) = \prod_{k=1}^{\infty} \frac{1 + \frac{s}{\delta_{2k+1}}}{1 + \frac{s}{\gamma_{2k+1}}},
\]

and

\[
\Pi(z) = \frac{s}{1 + \frac{s}{\delta_1}} \prod_{k=2}^{\infty} \frac{1 - (-1)^k \frac{s}{\delta_k}}{1 - (-1)^k \frac{s}{\gamma_k}} = \frac{s}{1 + \frac{s}{\delta_1}} M(z) N(z)
\]

all converge subuniformly for \( |s| < \infty \).

Proof: Since \( F \) is parabolic and \( \{ \Delta_n \} \rightarrow \{ |z| < \infty \} \),
\( \delta_k \rightarrow \infty \) and \( \gamma_k \rightarrow \infty \) as \( k \rightarrow \infty \); therefore, for any \( R > 0 \)
there exists \( n_0 = n_0(R) \) such that \( \delta_k > R \) and \( \gamma_k > R \)
for \( k \geq n_0 \). Then consider
\[ M_p(z) = \prod_{k=n_0}^{n_0+p} \frac{1 - \frac{z}{\gamma_{2k}}}{1 - \frac{z}{\delta_{2k}}} \]

and

\[ N_p(z) = \prod_{k=n_0}^{n_0+p} \frac{1 + \frac{z}{\delta_{2k+1}}}{1 + \frac{z}{\gamma_{2k+1}}} \]

Both \( M_p(z) \) and \( N_p(z) \) are holomorphic, while neither is zero, for \( |z| \leq R \).

A necessary and sufficient condition for the uniform convergence of \( M_p(z) \) as \( p \to \infty \) is the uniform convergence of

\[ \sum_{k=n_0}^{n_0+p} \log \left[ \frac{1 - \frac{z}{\delta_{2k}}}{1 - \frac{z}{\gamma_{2k}}} \right] \quad \text{as} \quad p \to \infty, \]

where the logarithm is the principal value. By the Cauchy criterion, this last limit converges uniformly in \( |z| \leq R \) provided for all \( z \) such that \( |z| \leq R \)

\[ \left| \sum_{k=n_0+n}^{n_0+n+p} \log \frac{1 - \frac{z}{\delta_{2k}}}{1 - \frac{z}{\gamma_{2k}}} \right| \]

is arbitrarily and uniformly small for all \( n \) sufficiently large and for all \( p > 0 \).

Because \( \ldots \delta_{2k} < \gamma_{2k} < \delta_{2k+2} < \gamma_{2k+2} \ldots \), then
for \( m \geq 1 \) and for all \( p > 0 \),

\[
0 < \sum_{k = n_0 + 1}^{n_0 + n + p} \left[ \left( \frac{1}{\delta_2 k} \right)^m - \left( \frac{1}{\gamma_2 k} \right)^m \right] < \left( \frac{1}{\delta_2 n_0 + 2n} \right)^m.
\]

Hence for all \( p > 0 \) and \( |z| \leq R \),

\[
\left| \sum_{k = n_0 + 1}^{n_0 + n + p} \log \frac{1 - \frac{z}{\delta_2 k}}{1 - \frac{z}{\gamma_2 k}} \right| = \left| \sum_{n=1}^{\infty} z^n \sum_{k = n_0 + 1}^{n_0 + n + p} \left( \frac{\frac{1}{\delta_2 k}}{\frac{1}{\gamma_2 k}} \right)^m \right|
\]

\[
\leq \sum_{n=1}^{\infty} z^n \left( \frac{R}{\delta_2 n_0 + 2n} \right)^m \leq \sum_{n=1}^{\infty} \left( \frac{R}{\delta_2 n_0 + 2n} \right)^m = \frac{R}{\delta_2 n_0 + 2n} - R.
\]

However, this last fraction is arbitrarily small for all \( n \) sufficiently large because \( \delta_k \to \infty \) as \( k \to \infty \). Hence \( M_p(z) \) converges subuniformly for \( |z| < \infty \) as \( p \to \infty \).

Similarly, \( M_p(z) \) converges subuniformly for \( |z| < \infty \) as \( p \to \infty \). So the product \( M_p(z) N_p(z) \) converges

subuniformly for \( |z| < \infty \) as \( p \to \infty \), and since

\( \lim_{p \to \infty} M_p(z) \) and \( \lim_{p \to \infty} N_p(z) \) are \( M(z) \),

\( N(z) \), and \( \Pi(z) \), respectively, with a finite number of factors omitted, \( M(z) \), \( N(z) \), and \( \Pi(z) \) converge

subuniformly for \( |z| < \infty \). ||
Lemma 1.4. \( \Pi(z) = f(z) \).

Proof: Lemma 1.2 shows that \( \delta_{2,n} \to \delta_2 > 0 \) and
\[ -\gamma_{1,n} \to -\gamma_1 < 0 \] where \( \delta_{2,n} > 0 \) and \( \gamma_{1,n} > 0 \) for all \( n > 0 \), and from this follows the existence of \( r > 0 \)
such that \( \frac{R_n(z)}{z} \neq 0 \) and \( \frac{\Pi(z)}{z} \neq 0 \) for \( |z| < r \).

Then using the principal determination of the logarithm, for \( |z| < r \)

\[
\log \frac{R_n(z)}{z} = -\log \left(1 + \frac{z}{\gamma_{1,n}}\right)
\]

\[-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{k=1}^{\infty} \left(\frac{1}{\delta_{2k,n}} - \frac{1}{\gamma_{2k,n}}\right)\]

\[-\sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n} \left[ \sum_{k=1}^{n} \frac{1}{\delta_{2k+1,n}} - \sum_{k=1}^{n-1} \frac{1}{\gamma_{2k+1,n}} \right] \]

and

\[
\log \frac{\Pi(z)}{z} = -\log \left(1 + \frac{z}{\gamma_1}\right)
\]

\[-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{k=1}^{\infty} \left(\frac{1}{\delta_{2k}} - \frac{1}{\gamma_{2k}}\right)\]

\[-\sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n} \sum_{k=1}^{\infty} \left(\frac{1}{\delta_{2k+1}} - \frac{1}{\gamma_{2k+1}}\right) \]
so that

$$\log \frac{R_n(z)}{\Pi(z)} = \log \frac{1 + \frac{z}{\gamma_1}}{1 + \frac{z}{\gamma_{1,n}}}$$

$$- \sum_{n=1}^{\infty} \frac{z^n}{n} \left[ \sum_{k=1}^{n} \left( \frac{1}{\delta_{2k,n}} - \frac{1}{\gamma_{2k,n}} \right) - \sum_{k=1}^{\infty} \left( \frac{1}{\delta_{2k}} - \frac{1}{\gamma_{2k}} \right) \right]$$

$$- \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n} \left[ \sum_{k=1}^{n} \frac{1}{\delta_{2k+1,n}} - \sum_{k=1}^{\infty} \frac{1}{\gamma_{2k+1,n}} - \sum_{k=1}^{\infty} \left( \frac{1}{\delta_{2k+1}} - \frac{1}{\gamma_{2k+1}} \right) \right].$$

From Lemma 1.2, the first term approaches 0 as $n \to \infty$.

Now for $n \geq 1$,

$$0 < \sum_{k=1}^{n} \left( \frac{1}{\delta_{2k}} - \frac{1}{\gamma_{2k}} \right) \leq \frac{1}{\delta_{21}}$$

and

$$0 < \sum_{k=1}^{n} \left( \frac{1}{\delta_{2k,n}} - \frac{1}{\gamma_{2k,n}} \right) < \frac{1}{\delta_{21,n}}.$$

and Lemma 1.2 with these results implies for every $n_0 > 1$, 
\[ 0 \leq \lim_{n \to \infty} \left| \sum_{k=1}^{n} \left( \frac{1}{\delta_{2k,n}} - \frac{1}{\gamma_{2k,n}} \right) - \sum_{k=1}^{\infty} \left( \frac{1}{\delta_{2k}} - \frac{1}{\gamma_{2k}} \right) \right| \]

\leq \lim_{n \to \infty} \left| \sum_{k=n_{0}}^{\infty} \left( \frac{1}{\delta_{2k,n}} - \frac{1}{\gamma_{2k,n}} \right) - \sum_{k=n_{0}}^{\infty} \left( \frac{1}{\delta_{2k}} - \frac{1}{\gamma_{2k}} \right) \right| \]

\leq \lim_{n \to \infty} \left| \frac{1}{\delta_{2n_{0}},n} + \frac{1}{\delta_{2n_{0}}} \right| = \frac{2}{\delta_{2n_{0}}}.

However, for \( n_{0} \) sufficiently large, \( \delta_{2n_{0}} \) is arbitrarily large, and thus

\[ \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \left( \frac{1}{\delta_{2k,n}} - \frac{1}{\gamma_{2k,n}} \right) - \sum_{k=1}^{\infty} \left( \frac{1}{\delta_{2k}} - \frac{1}{\gamma_{2k}} \right) \right] = 0. \]

Similarly, for \( m \geq 1 \), since

\[ 0 < \sum_{k=1}^{n} \frac{1}{\delta_{2k+1,n}} - \sum_{k=1}^{n-1} \frac{1}{\gamma_{2k+1,n}} < \frac{1}{\delta_{2i+1}} \]

then

\[ \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \frac{1}{\delta_{2k+1,n}} - \sum_{k=1}^{n-1} \frac{1}{\gamma_{2k+1,n}} - \sum_{k=1}^{\infty} \frac{1}{\delta_{2k+1}} \right] = 0. \]
Since \( \lim_{n \to \infty} \log \frac{R_n(z)}{\Pi(z)} \) exists and all the coefficients of the Taylor expansion of \( \log \frac{R_n(z)}{\Pi(z)} \) tend to zero as \( n \to \infty \), it follows that

\[
\lim_{n \to \infty} \log \frac{R_n(z)}{\Pi(z)} = 0
\]

and

\[
\lim_{n \to \infty} R_n(z) = f(z) = \Pi(z).
\]

**Lemma 1.5.** The infinite products

\[
G(s) = \prod_{k=1}^{\infty} \frac{(1 - \frac{s}{\alpha_{2k}})(1 - \frac{s}{\alpha_{2k+1}})}{(1 - \frac{s}{\gamma_{2k+2}})^2},
\]

\[
H(s) = \prod_{k=1}^{\infty} \frac{(1 + \frac{s}{\beta_{2k-1}})(1 + \frac{s}{\beta_{2k}})}{(1 + \frac{s}{\gamma_{2k+1}})^2},
\]

\[
P(s) = \prod_{k=1}^{\infty} \frac{(1 - \frac{s}{\alpha_k})(1 + \frac{s}{\beta_k})}{[1 - (-1)^k \frac{s}{\gamma_k}]^2}
\]

converge subuniformly for \(|s| < \infty\).
Proof: Since \( \alpha_k \to \infty, \beta_k \to \infty, \) and \( \gamma_k \to \infty \) as \( k \to \infty, \) for any \( R > 0 \) there exists \( n_0 = n_0(R) \) such that for \( k \geq n_0, \ \alpha_k > R, \ \beta_k > R, \) and \( \gamma_k > R. \) Then consider

\[
G_p(z) = \prod_{k=n_0}^{n_0+p} \frac{(1 - \frac{z}{\alpha_{2k}})(1 - \frac{z}{\alpha_{2k+1}})}{(1 - \frac{z}{\gamma_{2k+2}})^2}
\]

and

\[
H_p(z) = \prod_{k=n_0}^{n_0+p} \frac{(1 + \frac{z}{\beta_{2k-1}})(1 + \frac{z}{\beta_{2k}})}{(1 + \frac{z}{\gamma_{2k+1}})^2}
\]

Both \( G_p(z) \) and \( H_p(z) \) are holomorphic, while neither is zero, for \( |z| \leq R. \)

As in the proof of Lemma 1.3, the uniform convergence of \( G_p(z) \) is assured provided for \( |z| \leq R \)

\[
\sum_{k=n_0+n}^{n_0+n+p} \log \frac{(1 - \frac{z}{\alpha_{2k}})(1 - \frac{z}{\alpha_{2k+1}})}{(1 - \frac{z}{\gamma_{2k+2}})^2}
\]

is arbitrarily and uniformly small for all \( n \) sufficiently large and for all \( p > 0. \)
Because $\ldots \alpha_{2k} < \alpha_{2k+1} < \gamma_{2k+2} < \alpha_{2k+2} < \ldots$, then for $m \geq 1$ and all $p > 0$,

$$0 < \sum_{k=n_0+n}^{n_0+n+p} \left( \frac{1}{\alpha_{2k}} + \frac{1}{\alpha_{2k+1}} - \frac{1}{\gamma_{2k+2}} \right) < \frac{2}{2n_0+2n}.$$ 

Hence for all $p > 0$ and $|z| < R$,

$$\left| \sum_{k=n_0+n}^{n_0+n+p} \log \frac{(1 - \frac{z}{\alpha_{2k}})(1 - \frac{z}{\alpha_{2k+1}})}{(1 - \frac{z}{\gamma_{2k+2}})^2} \right|$$

$$= \left| - \sum_{m=1}^{\infty} \frac{R^m}{m^{n_0+n+p}} \sum_{k=n_0+n}^{n_0+n+p} \left( \frac{1}{\alpha_{2k}} + \frac{1}{\alpha_{2k+1}} - \frac{2}{\gamma_{2k+2}} \right) \right|$$

$$\leq \sum_{m=1}^{\infty} \frac{R^m}{m^{\alpha_{2n_0+2n}}} \leq 2 \sum_{m=1}^{\infty} \left( \frac{R}{2n_0+2n} \right)^m = \frac{2R}{2n_0+2n} - R^m.$$ 

However, this last fraction is arbitrarily small for all $m$ sufficiently large because $\alpha_k \to \infty$ as $k \to \infty$.

Thus $G_p(z)$ converges subuniformly for $|z| < \infty$ as $p \to \infty$.

Similarly, $H_p(z)$ converges subuniformly for $|z| < \infty$ as $p \to \infty$. So the product $G_p(z) H_p(z)$ converges subuniformly for $|z| < \infty$ as $p \to \infty$, and since $\lim_{p \to \infty} G_p(z)$,
\[ \lim_{p \to \infty} H_p(z), \text{ and } \lim_{p \to \infty} G_p(z) H_p(z) \text{ are } G(z), H(z), \text{ and } P(z), \text{ respectively, with a finite number of factors omitted, } G(z), H(z), \text{ and } P(z) \text{ converge subuniformly for } |z| < \infty. \]

**Lemma 1.6.** \( P(z) = f'(z). \)

**Proof:** Lemma 1.2 shows that \( \alpha'_{1,n} \to \alpha'_1 > 0 \) and \( -\gamma'_{1,n} \to -\gamma'_1 \) as \( n \to \infty \), where \( \alpha'_{1,n} > 0 \) and \( \gamma'_{1,n} > 0 \) for all \( n > 0 \), and from this follows the existence of \( r > 0 \) such that \( R_n'(z) \not\equiv 0 \) and \( P(z) \not\equiv 0 \) for \( |z| < r \). Then using the principal determination of the logarithm, \( R_n'(z) \)

\[
\begin{align*}
&= \frac{\left(1 - \frac{z}{\alpha_{1,n}}\right)\left(1 - \frac{z}{\alpha_{2,n}}\right)}{\alpha_{1,n}} \left[ \frac{n - 1}{\alpha_{2k,n}} \left(1 - \frac{z}{\alpha_{2k,n}}\right) \right] \frac{n}{\alpha_{2k-1,n}} \left(1 + \frac{z}{\alpha_{2k-1,n}}\right) \left(1 + \frac{z}{\alpha_{2k,n}}\right) \\
&= \frac{\left(1 + \frac{z}{\gamma_{1,n}}\right)\left(1 + \frac{z}{\gamma_{2,n}}\right)}{\gamma_{1,n}} \left[ \frac{2k-1}{(1 - \gamma_{2k,n})^2} \right] \frac{1}{\gamma_{2k+2,n}} \left(1 + \frac{z}{\gamma_{2k+2,n}}\right) \left(1 + \frac{z}{\gamma_{2k,n}}\right)
\end{align*}
\]

implies for \( |z| < r \):

\[ \log R_n'(z) = \log \frac{1 - \frac{z}{\alpha_{1,n}}}{\left(1 + \frac{z}{\gamma_{1,n}}\right)^2} \left(1 + \frac{z}{\gamma_{2,n}}\right)^2 \]

\[
- \sum_{m=1}^{\infty} \frac{z^n}{n} \left( \sum_{k=1}^{n} \frac{1}{\alpha_{2k,n}} - \sum_{k=1}^{n-1} \frac{1}{\alpha_{2k+1,n}} \right) - \sum_{m=1}^{\infty} (-1)^m \frac{z^n}{n} \left( \frac{1}{\gamma_{2k-1,n}} + \frac{1}{\gamma_{2k,n}} - \frac{2}{\gamma_{2k+1,n}} \right) .
\]
Also, $P(z)$

\[
= \frac{1 - \frac{z}{\alpha_i}}{(1 + \frac{z}{\beta_i})^2} \prod_{k=1}^{\infty} \frac{1 - \frac{z}{\alpha_{2k}} (1 - \frac{z}{\alpha_{2k+1}})}{(1 - \frac{z}{\gamma_{2k+1}}) (1 + \frac{z}{\gamma_{2k}})} \prod_{k=1}^{\infty} \frac{1 + \frac{z}{\beta_{2k-1}} (1 + \frac{z}{\beta_{2k}})}{(1 + \frac{z}{\gamma_{2k+1}})^2}
\]

implies for $|z| < r$:

\[
\log P(z) = \log \frac{1 - \frac{z}{\alpha_i}}{(1 + \frac{z}{\beta_i})^2} \prod_{k=1}^{\infty} \left( \frac{1}{\alpha_{2k}} + \frac{1}{\alpha_{2k+1}} - \frac{2}{\gamma_{2k+1}} \right)
\]

\[
= \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{k=1}^{\infty} \left( \frac{1}{\alpha_{2k}} + \frac{1}{\alpha_{2k+1}} - \frac{2}{\gamma_{2k+1}} \right).
\]

Hence for $|z| < r$:

\[
\log \frac{R^i(z)}{P(z)} = \log \frac{\prod_{k=1}^{\infty} \frac{1 - \frac{z}{\alpha_{2k}} (1 - \frac{z}{\alpha_{2k+1}})}{(1 - \frac{z}{\gamma_{2k+1}}) (1 + \frac{z}{\gamma_{2k}})} \prod_{k=1}^{\infty} \frac{1 + \frac{z}{\beta_{2k-1}} (1 + \frac{z}{\beta_{2k}})}{(1 + \frac{z}{\gamma_{2k+1}})^2}}{(1 - \frac{z}{\alpha_i}) \prod_{k=1}^{\infty} \left( \frac{1}{\alpha_{2k}} + \frac{1}{\alpha_{2k+1}} - \frac{2}{\gamma_{2k+1}} \right)}
\]

\[
= \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{k=1}^{\infty} \left( \frac{1}{\alpha_{2k}} + \frac{1}{\alpha_{2k+1}} - \frac{2}{\gamma_{2k+1}} \right) \sum_{k=1}^{\infty} \left( \frac{1}{\alpha_{2k}} + \frac{1}{\alpha_{2k+1}} - \frac{2}{\gamma_{2k+1}} \right).
\]

From Lemma 1.2, the first term approaches 0 as $n \to \infty$. 
Now for all \( i > 0 \) and \( n \geq 1 \),

\[
0 < \sum_{k=1}^{n} \frac{1}{\alpha_{2k,n}} + \sum_{k=1}^{n-1} \frac{1}{\alpha_{2k+1,n}} - \sum_{k=1}^{n-1} \frac{2}{\gamma_{2k+2,n}} < \frac{2}{\alpha_{2i,n}}
\]

and

\[
0 < \sum_{k=1}^{\infty} \left( \frac{1}{\alpha_{2k,n}} + \frac{1}{\alpha_{2k+1,n}} - \frac{2}{\gamma_{2k+2,n}} \right) \leq \frac{2}{\alpha_{2i,n}}.
\]

Then Lemma 1.2 with these results implies for every \( n_0 > 1 \),

\[
0 \leq \lim_{n \to \infty} \left| \sum_{k=1}^{n} \frac{1}{\alpha_{2k,n}} + \sum_{k=1}^{n-1} \frac{1}{\alpha_{2k+1,n}} - \sum_{k=1}^{n-1} \frac{2}{\gamma_{2k+2,n}} - \sum_{k=1}^{\infty} \left( \frac{1}{\alpha_{2k,n}} + \frac{1}{\alpha_{2k+1,n}} - \frac{2}{\gamma_{2k+2,n}} \right) \right|
\]

\[
\leq \lim_{n \to \infty} \left| \sum_{k=n_0}^{n} \frac{1}{\alpha_{2k,n}} + \sum_{k=n_0}^{n-1} \frac{1}{\alpha_{2k+1,n}} - \sum_{k=n_0}^{n-1} \frac{2}{\gamma_{2k+2,n}} - \sum_{k=n_0}^{\infty} \left( \frac{1}{\alpha_{2k,n}} + \frac{1}{\alpha_{2k+1,n}} - \frac{2}{\gamma_{2k+2,n}} \right) \right|
\]

\[
\leq \lim_{n \to \infty} \left( \frac{2}{\alpha_{2n_0,n}} + \frac{2}{\alpha_{2n_0,n}} \right) = \frac{4}{\alpha_{2n_0,n}}.
\]

However, for \( n_0 \) sufficiently large \( \alpha_{2n_0} \) is arbitrarily large and thus

\[
\lim_{n \to \infty} \left[ \sum_{k=1}^{n} \frac{1}{\alpha_{2k,n}} + \sum_{k=1}^{n-1} \frac{1}{\alpha_{2k+1,n}} - \sum_{k=1}^{n-1} \frac{2}{\gamma_{2k+2,n}} - \sum_{k=n}^{\infty} \left( \frac{1}{\alpha_{2k,n}} + \frac{1}{\alpha_{2k+1,n}} - \frac{2}{\gamma_{2k+2,n}} \right) \right]
\]

\[= 0.\]
Similarly, for \( m \geq 1 \),

\[
\lim_{n \to \infty} \left[ \sum_{k=1}^{n} \left( \frac{1}{\beta_{2k-1,n}} + \frac{1}{\beta_{2k,n}} - \frac{2}{\gamma_{2k+1,n}} \right) - \sum_{k=1}^{m} \left( \frac{1}{\beta_{2k-1}} + \frac{1}{\beta_{2k}} - \frac{2}{\gamma_{2k+1}} \right) \right] = 0.
\]

Since \( \lim_{n \to \infty} \log \frac{R_n(z)}{P(z)} \) exists and all the coefficients of the Taylor expansion of \( \log \frac{R_n(z)}{P(z)} \) tend to zero as \( n \to \infty \), it follows that

\[
\lim_{n \to \infty} \log \frac{R_n(z)}{P(z)} = 0 \quad \text{and} \quad \lim_{n \to \infty} R_n(z) = f'(z) = P(z). \]

**Theorem I.** A Riemann surface of class III is parabolic and its mapping function is given by

\[
f(z) = \frac{z}{1 + \frac{z}{\gamma_1}} \prod_{k=2}^{\infty} \left[ \frac{1 - (-1)^k \frac{z}{\beta_k}}{1 - (-1)^k \frac{z}{\gamma_k}} \right]
\]

where

\[
f'(z) = \prod_{k=1}^{\infty} \left[ \frac{(1 - \frac{z}{\alpha_k})(1 + \frac{z}{\beta_k})}{(1 - (-1)^k \frac{z}{\gamma_k})^2} \right].
\]

For \( k > 0 \),

\[
0 < \alpha_{2k-1} < \delta_{2k} < \gamma_{2k} < \alpha_{2k} < \alpha_{2k+1}.
\]

and

\[
0 < \gamma_{2k-1} < \beta_{2k-1} < \beta_{2k} < \delta_{2k+1} < \gamma_{2k+1}.
\]
Furthermore, \( \alpha_k \to \infty, \beta_k \to \infty, \gamma_k \to \infty, \) and \( \delta_k \to \infty \) as \( k \to \infty. \)

**Remark.** It should be noted that the factors in the infinite products of Lemmas 1.3 and 1.5 are quotients. Hence the representations in Theorem I are products of quotients, not quotients of products.
CHAPTER II
SURFACES OF CLASS IV

Let \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) be two infinite sequences and for each \( n \geq 1 \) let \( \{a_1(n)\}_{i=1}^{q_n} \) be a finite sequence such that:

\[
0 < a_1 < b_1 < b_2 < a_2 < a_3 < b_3 < b_4 < \ldots ;
\]

\[
0 < a_1(1) < b_1(1) < a_2(1) < b_2(1) < \ldots < a_{q_1}(1) < b_{q_1}(1) < a_1;
\]

for \( n > 1 \),

\[
a_{2n-1} < b_{2n-1} < b_1(2n) < a_1(2n) < b_2(2n) < a_2(2n) < \ldots < b_{q_{2n}}(2n) < a_{q_{2n}}(2n) < b_{2n} < a_{2n}
\]
and

\[
b_{2n} < a_{2n} < a_1(2n+1) < b_1(2n+1) < a_2(2n+1) < b_2(2n+1) < \ldots < a_{q_{2n+1}}(2n+1) < b_{q_{2n+1}}(2n+1) < a_{2n+1} < b_{2n+1}.
\]

A surface \( F \) of Class IV consists of sheets \( S_n \) and \( S_i(n) \), \( n = 1, 2, 3, \ldots \), \( i = 1, 2, 3, \ldots, q_n \), over the \( w \)-sphere where, for \( S_n \) and \( S_i(n) \) copies of the \( w \)-sphere,

(a) \( S_1 \) is slit along the real axis from \( a_1(1) \) to \( b_1(1), i = 1, 2, \ldots, q_1 \), and from \( a_1 \) to \( b_1 \), where
0 < a_1(1) < b_1(1) < a_2(1) < b_2(1) < \ldots < a_q(1) < b_q(1) < a_1 < b_1.

(b) for n > 1, \( S_{2n} \) is slit along the real axis from 
\( a_{2n-1} \) to \( b_{2n-1} \), from \( b_{2n} \) to \( a_{2n} \), and from \( b_1(2n) \) 
\( a_i(2n) \), \( i = 1, 2, \ldots, q_{2n} \), where \( 0 < a_{2n-1} < b_{2n-1} \) 
\( < b_1(2n) < a_1(2n) < b_2(2n) < a_2(2n) < \ldots \) 
\( < b_{q_{2n}}(2n) < a_{q_{2n}}(2n) < b_{2n} < a_{2n} \).

(c) for n > 1, \( S_{2n+1} \) is slit along the real axis 
from \( b_{2n} \) to \( a_{2n} \), from \( a_{2n+1} \) to \( b_{2n+1} \), and from 
\( a_i(2n+1) \) to \( b_i(2n+1) \), \( i = 1, 2, \ldots, q_{2n+1} \), where 
\( 0 < b_{2n} < a_{2n} < a_1(2n+1) < b_1(2n+1) < a_2(2n+1) \) 
\( < b_2(2n+1) < \ldots < a_{q_{2n+1}}(2n+1) < b_{q_{2n+1}}(2n+1) \) 
\( < a_{2n+1} < b_{2n+1} \).

(d) for n > 1 and i = 1, 2, \ldots, q_n, \( S_i(n) \) is slit 
along the real axis between \( a_i(n) \) and \( b_i(n) \) as 
defined for \( S_n \).

Join \( S_n \) and \( S_{n+1} \) along the slits between \( a_n \) and \( b_n \) 
to form first order branch points at the end points. 
Join \( S_1(n) \) to \( S_n \) along the slits between \( a_1(n) \) and 
\( b_1(n) \) to form first order branch points at the end.
\[ S_1 \]
\[ \begin{array}{c}
    a_i^{(1)} \quad b_i^{(1)} \\
    \hspace{1cm} a_i \quad b_i
    \end{array} \]

\[ S_1^{(1)} \]
\[ \begin{array}{c}
    a_i^{(1)} \quad b_i^{(1)}
    \end{array} \]

\[ S_2 \]
\[ \begin{array}{c}
    a_i \quad b_i \quad a_i^{(2)} \quad b_i^{(2)} \quad a_i \quad b_i \\
    \hspace{1cm} a_i \quad b_i \quad b_i \quad a_i
    \end{array} \]

\[ S_1^{(2)} \]
\[ \begin{array}{c}
    b_i^{(2)} \quad a_i^{(2)}
    \end{array} \]

\[ S_3 \]
\[ \begin{array}{c}
    \hspace{1cm} b_i \quad a_i \\
    \end{array} \]

\[ \vdots \]

\[ S_{2n} \]
\[ \begin{array}{c}
    a_{2n-1} \quad b_{2n-1} \quad b_{2n} \quad a_{2n}
    \end{array} \]

\[ S_1^{(2n)} \]
\[ \begin{array}{c}
    b_{2n}^{(2n)} \quad a_{2n}^{(2n)}
    \end{array} \]

\[ S_{2n+1} \]
\[ \begin{array}{c}
    b_{2n} \quad a_{2n} \quad a_{2n+1} \quad b_{2n+1}
    \end{array} \]

\[ S_1^{(2n+1)} \]
\[ \begin{array}{c}
    a_{2n+1}^{(2n+1)} \quad b_{2n+1}^{(2n+1)}
    \end{array} \]

Figure 3
points. A surface formed in this way will be a member of Class IV. This formation is shown schematically for $q_n = 1$ in Figure 3. If $q_n = 0$ for all $n$, then the surface of Class IV is a member of Class III. Hence Class IV $\supset$ Class III.

$F$ is simply connected and by the Fundamental Mapping Theorem there exists a unique $z = \phi(w)$, where $w = f(z)$, such that $f(0) = 0 \in S_1$ and $f'(0) = 1$, so that $F$ is mapped schlichtly onto $|z| < R \leq \infty$. With an argument similar to that given on pages 9 and 10 of [9] it follows that $f(z)$ is real for $z$ real. The image of $F$ in the $z$-plane satisfies the following properties. The image of $S_n$ or $S_i(n)$ is a region symmetric about the real axis. $S_1$ and $S_i(1)$, $i = 1, 2, \ldots q_1$, are mapped onto a domain containing $z = 0$ and bounded by a simple closed curve $C_1$ which intersects the real axis at $\alpha_1$ and $-\beta_1$, $-\beta_1 < 0 < \alpha_1$. For $n > 1$, $S_n$ and $S_i(n)$, $i = 1, 2, \ldots, q_n$, are mapped onto an annular region about $z = 0$ bounded by two simple closed curves $C_{n-1}$ and $C_n$ which intersect the real axis at $-\beta_{n-1}$ and $\alpha_{n-1}$, and $-\beta_n$ and $\alpha_n$, respectively, where $-\beta_n < -\beta_{n-1} < 0 < \alpha_{n-1} < \alpha_n$. Each $S_i(n)$ is
is mapped onto a domain bounded by a simple closed curve \( C_i(n) \) which intersects the real axis at \((-1)^{n+1} \beta_i(n)\) and \((-1)^{n+1} \alpha_i(n)\). If \((-1)^n \gamma_n\) and \((-1)^{n+1} \gamma_i(n)\) denote the image in the \(z\)-plane of \( w = \infty \) for \( S_n \) and \( S_i(n) \), while \((-1)^n \delta_n\) and \((-1)^{n+1} \delta_i(n)\) denote the image of \( w = 0 \) for \( S_n \) and \( S_i(n) \), respectively, then the following inequalities are satisfied:

\[
0 < \alpha_1 < \delta_2 < \gamma_2 < \alpha_2 < \delta_3 < \gamma_3 < \alpha_3 < \delta_4 < \gamma_4 < \alpha_4 < \ldots
\]

and

\[
0 > -\gamma_1 > -\beta_1 > -\delta_2 > -\gamma_2 > -\beta_2 > -\delta_3 > -\gamma_3 > -\beta_3 > -\delta_4 > -\gamma_4 > \ldots
\]

Furthermore,

\[
0 < \alpha_1(1) < \beta_1(1) < \alpha_2(1) < \beta_2(1) < \ldots < \alpha_q(1) < \beta_q(1) < \alpha_1.
\]

For \( n \) even,

\[
-\beta_{n-1} > -\beta_1(n) > -\alpha_1(n) > -\beta_2(n) > -\alpha_2(n) > \ldots
\]

\[
> -\beta_{q_n}(n) > -\alpha_{q_n}(n) > -\beta_n,
\]

and for \( n \) odd,

\[
\alpha_{n-1} < \alpha_1(n) < \beta_1(n) < \alpha_2(n) < \beta_2(n) < \ldots
\]

\[
< \alpha_{q_n}(n) < \beta_{q_n}(n) < \alpha_n.
\]
Let $F_n$ be the part of $F$ formed from $S_k$, $i = 1, 2, \ldots, 2n+1$, and $S_i(k)$, $k = 1, 2, \ldots, 2n$ and $i = 1, 2, \ldots, q_{2n+1}$, with the slits in $S_{2n+1}$ from $a_{2n+1}$ to $b_{2n+1}$ and $a_{1}(2n+1)$ to $b_{1}(2n+1)$, $i = 1, 2, \ldots, q_{2n+1}$, deleted.

**Lemma 2.1.** It is possible to construct a function $R_n(z)$ which maps the $z$-plane onto $F_n$.

Proof: $F_n$ is a simply connected closed surface and hence is the Riemann surface of the inverse of a unique rational function $w = R_n(z)$ such that $R_n(0) = 0 \in S_1$, $R_n'(0) = 1$, and $R_n(\infty) = \infty \in S_{2n+1}$.

Let $(-1)^k \delta_{k,n}$ and $(-1)^{k+1} \delta_{1,n}(k)$ denote the zeros of $R_n(z)$, while $(-1)^k \gamma_{k,n}$ and $(-1)^{k+1} \gamma_{1,n}(k)$ denote the poles of $R_n(z)$, corresponding to points in $S_k$ and $S_i(k)$ over 0 and $\infty$, respectively. Similarly, let $\alpha_{k,n}$ and $-\beta_{k,n}$ be the zeros of $R'_n(z)$ for branch points in $S_k$, and let $(-1)^{k+1} \alpha_{1,n}(k)$ and $(-1)^{k+1} \beta_{1,n}(k)$ be the zeros of $R'_n(z)$ for branch points in $S_i(k)$. Then

$$R_n(z) = \frac{z}{1 + \frac{z}{\gamma_{1,n}(k)}} \prod_{k=2}^{q_{2n+1}} \frac{1 - (-1)^k \frac{z}{\gamma_{k,n}(k)}}{1 - (-1)^k \frac{z}{\gamma_{k,n}(k)}} \prod_{i=1}^{q_{2n+1}} \frac{z}{1 + (-1)^k \frac{z}{\gamma_{1,n}(k)}} \left(1 + \frac{z}{\delta_{2n+1,n}}\right)$$
and $R'(s)$

$$R'(s) = \prod_{k=1}^{2n} \left[ \frac{(1 - \frac{2}{\alpha_{k,n}})(1 + \frac{2}{\beta_{k,n}})}{(1 + (-1)^{k+1})^{2}} \prod_{i=1}^{q_k} \frac{(1 + (-1)^{k} \frac{2}{\alpha_{i,n}(k)})}{(1 + (-1)^{k} \frac{2}{\beta_{i,n}(k)})} \right]^{1/2} \tag{32}$$

**Lemma 2.2.** $F$ is parabolic.

**Proof:** Suppose that $F$ is not parabolic and thus $R < \infty$. Let $D_n$ be the $z$-plane slit along the real axis from $\alpha_{2n,2n}$ to $+\infty$. $\xi = \psi(z) = \phi[R_n(z)]$ is a schlicht map of $D_n$ onto a simply connected region $\Delta_n$ of the $\xi$-plane bounded by $C_{2n+1}$, $C_i(2n+1)$, $i = 1, 2, \ldots, q_{2n+1}$, and the segments $(\alpha_{2n}, \alpha_{2n+1})$ and $(-\beta_{2n+1}, -\gamma_{2n})$.

By Theorem B,

$$\alpha_{2n,2n} \leq D(0, C_{2n+1}) \leq R < \infty,$$

where $D(0, C_{2n+1})$ is the distance from $s = 0$ to the curve $C_{2n+1}$, and thus there exists a subsequence $
 \{\alpha_{2n_i,2n_i}\}$ such that $\alpha_{2n_i,2n_i} \to A \leq R$. Then $\psi_{n_i}(z)$

is a schlicht map of $D_{n_i}$ into $\Delta_{n_i}$, where for $n_i$
sufficiently large $\Delta_{n_i} \subset \{s \mid |s| \leq R\}$. If $D$ is the

$s$-plane slit along the real axis from $s = A$ to $+\infty$,

then $\{\psi_{n_i}(z)\}$ forms a normal family by Montel's Theorem

and $\psi_{n_j}(z) \to \psi(z)$ subuniformly in $D$. $\psi(z)$ maps $D$
lichtly onto \( \{ \zeta \mid |\zeta| \leq R \} \). Then \( R_n^j(z) \rightarrow f(z) \), a function meromorphic in \( D \). \( F(z) \neq \infty \) because \( R_n(0) = 0 \).

Let \( D^* \) be the \( z \)-plane slit along the real axis from \( z = A \) to \( -\infty \). Now \( R_n^j(z) \) is defined in \( D^* \) and for \( j \) sufficiently large assumes no negative real values in any compact subset of \( D^* \). Then \( \{ R_n^j(z) \} \) is a normal family in \( D^* \) by Theorem C, so that a subsequence \( \{ R_n^k(z) \} \) of \( \{ R_n^j(z) \} \) converges subuniformly in \( D^* \) to a function \( G(z) \). Because of a common domain of convergence, \( G(z) \) is the analytic continuation of \( F(z) \). Then \( G(z) \) maps the \( z \)-plane punched at \( z = A \) and \( \infty \) one to one and conformally onto an open doubly connected Riemann surface \( F^* \) of which \( F \) is a subsurface obtained by inserting some slits in \( F^* \) over the real axis. Clearly this is impossible, and thus \( R = \infty \). ||

**Lemma 2.3.** \( R_n(z) \rightarrow f(z) \) subuniformly in \( |z| < \infty \) as \( n \to \infty \).

**Proof:** Because \( F \) is parabolic and \( \{ \Delta_n \} \) converges to \( |\zeta| < \infty \), by Theorem D it follows that \( z = R_n^{-1}(f(\zeta)) \rightarrow \zeta = \phi[R_n(z)] \) subuniformly in \( \{ |\zeta| < \infty \} \). Then
again by Theorem 4, \( \{D_n\} \) converges to \( \{ |z| < \infty \} \) and
thus \( R_n(z) \to f(z) \) subuniformly in \( \{ |z| < \infty \} \).

**Lemma 2.4.** \( \alpha_k, n \to \alpha_k' \), \( \beta_k, n \to \beta_k' \), \( \gamma_k, n \to \gamma_k' \),
\( \delta_k, n \to \delta_k' \), \( \alpha_{1, n}(k) \to \alpha_{1}(k) \), \( \beta_{1, n}(k) \to \beta_{1}(k) \),
\( \gamma_{1, n}(k) \to \gamma_{1}(k) \), and \( \delta_{1, n}(k) \to \delta_{1}(k) \) as \( n \to \infty \).

**Proof:** This is a consequence of Hurwitz's Theorem. \( \| \)

**Lemma 2.5.** The infinite products

\[
M_1(s) = \prod_{k=1}^{\infty} \frac{1 - \frac{\zeta}{\delta_{2k}}}{1 - \frac{\zeta}{\gamma_{2k}}} ,
\]

\[
M_2(s) = \prod_{k=1}^{\infty} \frac{1 + \frac{\zeta}{\delta_{2k+1}}}{1 + \frac{\zeta}{\gamma_{2k+1}}} ,
\]

\[
M_3(s) = \prod_{k=1}^{\infty} \left[ \prod_{i=1}^{q_{2k-1}} \frac{1 - \frac{2^i}{\delta_{i}(2k-1)}}{1 - \frac{2^i}{\gamma_{i}(2k-1)}} \right] ,
\]

\[
M_4(s) = \prod_{k=1}^{\infty} \left[ \prod_{i=1}^{q_{2k}} \frac{1 + \frac{2^i}{\delta_{i}(2k)}}{1 + \frac{2^i}{\gamma_{i}(2k)}} \right] ,
\]

and \( \Pi(s) \)

\[
= \frac{\zeta}{1 + \frac{\zeta}{\gamma_{i}}} \left[ \prod_{k=2}^{\infty} \frac{l(-1)^k}{1 - (-1)^k} \right] \prod_{k=1}^{\infty} \left[ \prod_{i=1}^{q_k} \frac{1 + (-1)^k \frac{2^i}{\delta_{i}(k)}}{1 + (-1)^k \frac{2^i}{\gamma_{i}(k)}} \right]
\]

\[
= \frac{\zeta}{1 + \frac{\zeta}{\gamma_{i}}} M_1(s) M_2(s) M_3(s) M_4(s) \text{ converge subuniformly}
\]
for \(|s| < \infty\).

Proof: The first two products converge subuniformly for \(|s| < \infty\) by Lemma 1.3. The values of \(\delta_1(k)\) and \(\gamma_1(k)\) are arranged in such a way that the proof of Lemma 1.3 can also be used to show that \(M_3(z)\) and \(M_4(z)\) converge subuniformly for \(|s| < \infty\).

\[\text{Lemma 2.6.} \quad \Pi(z) = f(z).\]

Proof: Lemma 2.4 shows that \(\delta_{1,n}(1) \rightarrow \delta_1 > 0\) and \(\gamma_{1,n} \rightarrow \gamma_1 > 0\) as \(n \rightarrow \infty\), where \(\delta_{1,n}(1) > 0\) and \(\gamma_{1,n} > 0\) for all \(n > 0\), and from this follows the existence of \(r > 0\) such that

\[
\frac{R_n(z)}{z} \not\rightarrow 0 \quad \text{and} \quad \frac{\Pi(z)}{z} \not\rightarrow 0
\]

for \(|s| < r\). Then using the principal determination of the logarithm it can be shown by the same method used in the proof of Lemma 1.4 that

\[
\log \frac{R_n(z)}{\Pi(z)} \not\rightarrow 0 \quad \text{as} \quad n \not\rightarrow \infty.
\]

Hence

\[
R_n(z) \rightarrow f(z) = \Pi(z)
\]

subuniformly for \(|s| < \infty\) as \(n \not\rightarrow \infty\).
Lemma 2.7. The infinite products

\[ G_1(s) = \prod_{k=1}^{\infty} \frac{(1 - \frac{x}{\alpha_{2k}})(1 - \frac{x}{\alpha_{2k+1}})}{(1 - \frac{x}{\beta_{2k+1}})^2}, \]

\[ G_2(s) = \prod_{k=1}^{\infty} \frac{(1 + \frac{x}{\beta_{2k-1}})(1 + \frac{x}{\beta_{2k}})}{(1 + \frac{x}{\beta_{2k+1}})^2}, \]

\[ G_3(s) = \prod_{k=1}^{\infty} \prod_{i=1}^{2k-1} \frac{(1 - \frac{x}{\alpha_{i}(2k-1)})(1 - \frac{x}{\beta_{i}(2k-1)})}{(1 - \frac{x}{\delta_{i}(2k-1)})^2}, \]

\[ G_4(s) = \prod_{k=1}^{\infty} \prod_{i=1}^{2k} \frac{(1 + \frac{x}{\alpha_{i}(2k)})(1 + \frac{x}{\beta_{i}(2k)})}{(1 + \frac{x}{\delta_{i}(2k)})^2}, \]

and \( P(s) \)

\[ = \prod_{k=1}^{\infty} \left[ \frac{(1 - \frac{x}{\alpha_{k}})(1 + \frac{x}{\beta_{k}})}{(1 - ( -1)^k \frac{x}{\delta_{k}})^2} \right]^{2k} \prod_{i=1}^{\infty} \frac{(1 + (-1)^k \frac{x}{\alpha_{i}(k)})(1 + (-1)^k \frac{x}{\beta_{i}(k)})}{(1 + (-1)^k \frac{x}{\delta_{i}(k)})^2} \]

\[ = G_1(s) G_2(s) G_3(s) G_4(s) \]

converge subuniformly for \(|s| < \infty\).

Proof: Lemma 1.5 assures the subuniform convergence of \( G_1(s) \) and \( G_2(s) \) in \(|s| < \infty\). An argument similar to that used in the proof of Lemma 1.5 will show the
subuniform convergence of \( G_3(z) \) and \( G_4(z) \) in \(|z| < \infty\).
Thus \( P(z) \) converges subuniformly in \(|z| < \infty\). ||

**Lemma 2.8.** \( P(z) = f'(z) \).

Proof: Using the principal determination of the logarithm, it is possible to show by methods similar to those of the proof of Lemma 1.6 that

\[
\log \frac{R_n'(z)}{P(z)} \to 0 \text{ as } n \to \infty,
\]

and thus

\[
R_n'(z) \to f'(z) = P(z)
\]

subuniformly for \(|z| < \infty\) as \( n \to \infty \). ||

**Theorem II.** A Riemann surface of Class IV is

parabolic and its mapping function is given by

\[
f(z) = \frac{z}{1 + \frac{z}{\delta_i}} \prod_{k=2}^{\infty} \left[ \frac{l - (-1)^k \frac{\delta_k}{\delta_k}}{l - (-1)^k \frac{\delta_k}{\delta_k}} \right] \prod_{i=1}^{\infty} \left[ \frac{l + (-1)^k \frac{\delta_k}{\delta_k}}{l + (-1)^k \frac{\delta_k}{\delta_k}} \right]
\]

where

\[
f'(z) = \prod_{k=1}^{\infty} \left[ \frac{(1 - \frac{\delta_k}{\delta_k}) (1 + \frac{\delta_k}{\delta_k})}{(1 - (-1)^k \frac{\delta_k}{\delta_k})^2} \right] \prod_{i=1}^{\infty} \left[ \frac{(1 + (-1)^k \frac{\delta_k}{\delta_k}) (1 + (-1)^k \frac{\delta_k}{\delta_k})}{(1 + (-1)^k \frac{\delta_k}{\delta_k})^2} \right].
\]

The \( \alpha, \beta, \gamma, \) and \( \delta \) are ordered as indicated on page 30.
CHAPTER III
SURFACES OF CLASS V

Let \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_{2n-1}\}_{n=1}^{\infty} \) be two sequences such that for \( n > 0 \), \( a_n > 0 \) and \( a_{2n-1} < b_{2n-1} \). For each sheet a copy of the \( w \)-sphere, let a surface \( F \) of Class V consist of sheets \( S_1, S_2, \ldots, S_n, \ldots \) over the \( w \)-sphere where

(a) \( S_1 \) is slit along the real axis from \( a_1 \) to \( b_1 \), \( 0 < a_1 < b_1 \); for \( n \geq 1 \), \( S_{2n} \) is slit along the real axis from \( -a_{2n} \) to \( -\infty \) and from \( a_{2n-1} \) to \( b_{2n-1} \), \( -a_{2n} < 0 < a_{2n-1} < b_{2n-1} \); for \( n \geq 1 \), \( S_{2n+1} \) is slit along the real axis from \( -a_{2n} \) to \( -\infty \) and from \( a_{2n+1} \) to \( b_{2n+1} \), \( -a_{2n} < 0 < a_{2n+1} < b_{2n+1} \).

(b) \( S_n \) is joined to \( S_{n+1} \) be connecting the slits which have one endpoint at \( \pm a_n \) to form first order branch points at the endpoints of the slits. (See Figure 5.)

\( F \) is simply connected and open. Hence by the General Uniformization Theorem there exists a unique function \( z = \phi(w) \) which maps \( F \) one to one and conformally onto \( \{ |z| < R < \infty \} \).
where for \( w = f(z) = \phi^{-1}(z) \),

\[
f(0) = 0 \in S_1 \text{ and } f'(0) = 1.
\]

Let \( \alpha_{2i} \) and \( -\beta_{2i-1} \) be the zeros of \( f'(z) \) corresponding to first order branch points over \((-1)^{i+1}a_i \) and \( b_{2i-1} \), respectively. Let \( f(\gamma_{2i}) = +\infty \in S_1 \), and let \( f(\gamma_{2i}) = -\infty \), a first order branch point over \( -\infty \) in the sheets \( S_{2i} \) and \( S_{2i+1} \). Let \( f(\delta_i) = 0 \in S_1 \), \( i = 1, 2, \ldots \).

**Lemma 3.1.** \( f(z) \) is real for \( z \) real, and for \( k \geq 1 \),

\[
\delta_{2k+1} < -\gamma_{2k} < -\beta_1 < -\gamma_1 < 0 < \alpha_1 < \delta_{2k} < \alpha_{2k}.
\]

**Proof:** This can be proved by methods similar to those given on pages 9 and 10 of [9].

Thus the image of \( S_1 \) is a simply connected region which contains the origin, is bounded by a simple closed curve \( C_1 \), and is symmetric about the real axis.

For \( k \geq 2 \), the image of \( S_k \) is an annular region which is symmetric about the real axis and is bounded by two simple, non-intersecting closed curves \( C_{k-1} \) and \( C_k \).

For \( k \) odd, \( C_k \) intersects the real axis at \( -\beta_k \) and \( \alpha_k \).

For \( k \) even, \( C_k \) intersects the real axis at \( -\gamma_k \) and \( \alpha_k \). (See Figure 6.)
Figure 5
**Lemma 3.2.** Let $F_n$ be the first $2n + 2$ sheets of $F$ with the slit from $-a_{2n+2}$ to $-\infty$ deleted. Then there exists a rational function which maps the $z$-plane onto $F_n$.

Proof: $F_n$ is a simply connected closed surface with branch points over $a_1, a_2, \ldots, a_{2n+1}, b_1, b_2, \ldots, b_{2n+1}$, and with $n$ branch points over $\infty$. $F_n$ has $2n + 2$ points over the origin and two points over $\infty$ which are not branch points, one in $S_1$ and the other in $S_{2n+2}$. Then $F_n$ is the Riemann surface of the inverse of a unique rational function $w = R_n(z)$ such that $R_n(0) = 0 \in S_1$, $R_n'(0) = 1$, and $R_n(\infty) = \infty \in S_{2n+2}$.

Let $-\gamma_{1,n}$ and $\infty$ denote the values of $z$ where $R_n(z)$ has its simple poles and let $-\gamma_{2,n}, -\gamma_{4,n}, \ldots, -\gamma_{2n,n}$ be the values of $z$ for double poles. If $0, \delta_2, n', \delta_3, n', \ldots, \delta_{2n+1}, n', \delta_{2n+2}, n$ are the zeros of $R_n(z)$ while $\alpha_1, n', \alpha_2, n', \ldots, \alpha_{2n+1}, n', -\beta_1, n', -\beta_3, n', \ldots, -\beta_{2n+1, n}$ are the zeros of $R_n'(z)$, then

$$R_n(z) = \frac{z}{1 + \frac{z}{\gamma_{1,n}}} \prod_{k=1}^{n} \left[ \frac{(1 - \frac{z}{\delta_{2k+1}, n})(1 - \frac{z}{\delta_{2k+2}, n})}{(1 + \frac{z}{\alpha_{2k+1}, n})} \right] \left(1 - \frac{z}{\delta_{2n+1}, n}\right)$$
and

\[ R_1^n(s) = \frac{1}{(1 + \frac{z}{\delta_{1,n}})^2} \prod_{k=1}^{2n+1} \left( 1 - \frac{z}{\alpha_{k,n}} \right) \prod_{k=1}^{n} \left( 1 + \frac{z}{\gamma_{k,n}} \right)^3 \left( 1 + \frac{z}{\beta_{2n+1,n}} \right) \]

**Lemma 3.3**. For \( k = 1, 2, \ldots, 2n + 1, \quad \alpha_{k,n} > k a_1 \).

**Proof:** For \( 0 < s < \alpha_{1,n}^* \),

\[ \frac{1 + \frac{z}{\beta_{1,n}}}{(1 + \frac{z}{\gamma_{1,n}})^2} < 1 \quad \text{and} \quad \frac{1 + \frac{z}{\beta_{2n+1,n}}}{1 + \frac{z}{\gamma_{2n+1,n}}} < 1 \]

while

\[ 1 + \frac{z}{\gamma_{2n+1,n}} > 1 \quad \text{and} \quad 1 - \frac{z}{\alpha_{k,n}} > 0. \]

Then for the same values of \( s \),

\[ 0 < R_1^n(s) < \prod_{k=1}^{2n+1} \left( 1 - \frac{z}{\alpha_{k,n}} \right). \]

If \( \frac{1}{\alpha_{n^*}} = \frac{1}{2n+1} \left( \frac{1}{\alpha_{1,n}} + \frac{1}{\alpha_{2,n}} + \ldots + \frac{1}{\alpha_{2n+1,n}} \right) \), then

\[ \alpha_{1,n} < \alpha_{n^*} < \alpha_{2n+1,n} \]

and
\[ \left[ R_n(s) \right]^{\frac{1}{2n+2}} < \left[ \prod_{k=1}^{2n+1} \left(1 - \frac{z}{\alpha_{k,n}} \right) \right]^{\frac{1}{2n+2}} < \frac{1}{2n+1} \sum_{k=1}^{2n+1} \left(1 - \frac{z}{\alpha_{k,n}} \right) \]

\[ = 1 - \frac{z}{\alpha_{n}^{*}}. \]

Then

\[ a_1 = \int_0^{\alpha_{n}^{*}} R_n(s) \, ds < \int_0^{\alpha_{n}^{*}} \left(1 - \frac{z}{\alpha_{n}^{*}} \right)(2n+1) \, dz \]

\[ < \int_0^{\alpha_{n}^{*}} \left(1 - \frac{z}{\alpha_{n}^{*}} \right)(2n+1) \, dz = \frac{\alpha_{n}^{*}}{2n+2}, \]

which implies

\[ \frac{1}{\alpha_{n}^{*}} < \frac{1}{a_1(2n+2)} \]

and thus

\[ \frac{1}{\alpha_{1,n}} + \frac{1}{\alpha_{2,n}} + \ldots + \frac{1}{\alpha_{2n+1,n}} < \frac{2n+1}{2n+2} \frac{1}{a_1} < \frac{1}{a_1} . \]

Thus for \( k = 1, 2, \ldots, 2n+1, \)

\[ \frac{k}{\alpha_{k,n}} < \frac{1}{\alpha_{1,n}} + \frac{1}{\alpha_{2,n}} + \ldots + \frac{1}{\alpha_{k,n}} \]

and

\[ \alpha_{k,n} > ka_1. \]

\[ \| \]
Lemma 3.4. \( F \) is parabolic.

Proof: Let \( D_n \) be the \( z \)-plane slit along the positive real axis from \( -\alpha_{2n+1,n} \) to \( \infty \). Then \( R_n(z) \) maps \( D_n \) one to one and conformally onto \( F_n \) with \( S_{2n+2} \) slit from \( -\alpha_{2n+1} \) to \( -\infty \) through the origin along the real axis. Now \( \zeta = \phi(w) \) maps \( F_n \) (slit thus) one to one and conformally onto a domain \( \Delta_n \) of the \( \zeta \)-plane where \( \Delta_n \) is bounded by \( C_{2n+2} \) and the segment \((\alpha_{2n+1}, \alpha_{2n+2})\). Then \( \zeta = \phi[R_n(z)] = \psi_n(z) \) is a schlicht map of \( D_n \) onto \( \Delta_n \) where \( \psi_n(0) = 0 \) and \( \psi_n'(0) = 1 \). Then by Theorem 3 the distance from \( \zeta = 0 \) to \( C_{2n+2} \) is greater than \( \alpha_{2n+1,n} \), but from Lemma 3.3, \( \alpha_{2n+1,n} > (2n+1)a_1 \). Thus the distance from \( \zeta = 0 \) to \( C_{2n+2} \) \( \to \infty \) as \( n \to \infty \) and \( F \) is parabolic. It follows readily that \( \omega_k \geq c_k \) where \( c > 0 \) is a constant and \( \omega \) is \( \alpha, \beta, \gamma \), or \( \delta \).

Lemma 3.5. \( R_n(z) \to f(z) \) subuniformly for \( |z| < \infty \) as \( n \to \infty \).

Proof: Since \( \Delta_n \) converges to \( \{ |\zeta| < \infty \} \), then by Theorem D, \( \psi_n(z) = \phi[R_n(z)] \to z \) subuniformly for \( |z| < \infty \) which implies
\[ R_n(z) \rightarrow \phi^{-1}(z) = f(z); \]

also,

\[ R_n'(z) \rightarrow f'(z) \]

subuniformly for \(|z| < \infty\). \]

**Lemma 3.6.** For all \( k \geq 1 \), \( \alpha_{k,n} \rightarrow k' \beta_{2k-1,n} \rightarrow \beta_{2k-1} \), \( \gamma_{2k,n} \rightarrow \gamma_{2k} \), and \( \delta_{k,n} \rightarrow \delta_k \).

**Proof:** This is an immediate consequence of Lemma 3.5 and Hurwitz's Theorem. \]

**Lemma 3.7.** \( \lim_{n \rightarrow \infty} \sum_{k=1}^{2n+1} \frac{1}{\alpha_{k,n}} < \infty, \]

\[ \lim_{n \rightarrow \infty} \sum_{k=0}^{n} \frac{1}{\beta_{2k+1,n}} < \infty, \lim_{n \rightarrow \infty} \sum_{k=1}^{n} \frac{1}{\gamma_{2k,n}} < \infty \]

\[ \lim_{n \rightarrow \infty} \sum_{k=2}^{2n+2} \frac{1}{\delta_{k,n}} < \infty, \sum_{k=1}^{\infty} \frac{1}{\alpha_k} < \infty, \sum_{k=0}^{\infty} \frac{1}{\beta_{2k+1}} < \infty, \]

\[ \sum_{k=1}^{\infty} \frac{1}{\gamma_{2k}} < \infty, \text{ and } \sum_{k=2}^{\infty} \frac{1}{\delta_k} < \infty. \]

**Proof:** Because of Lemma 3.6, there exists \( r > 0 \) such that \( R_n(z) \) is holomorphic and not zero for \(|z| < r\).

Then for \(|z| < r\),
\[ \log R_n'(s) \]

\[ = \frac{\Theta}{\sum_{n=1}^{\infty}} \sum_{n=1}^{\infty} \left[ \frac{2n+1}{\sum_{k=1}^{\infty}} \alpha_k, n - \frac{1}{\beta_{1k, n}} + \sum_{k=0}^{n} \left( -\frac{1}{\beta_{2k+1, n}} \right)^{m+1} + \frac{2(\beta_{1, n})^m}{\beta_{2k+1, n}} + \frac{3}{\sum_{k=1}^{\infty}} \frac{(-1)^m}{\beta_{1, n}} \right] . \]

Since \( \log R_n'(s) \to \log f(z) \) subuniformly in \( |z| < r \),

\[ \lim_{\left| z \right| \to \infty} \left| \frac{2n+1}{\sum_{k=1}^{\infty}} \frac{1}{\alpha_k, n} + \sum_{k=0}^{n} \frac{1}{\beta_{2k+1, n}} - \frac{2}{\beta_{2, n}} - 3 \sum_{k=1}^{\infty} \frac{1}{\beta_{2k+1, n}} \right| < \infty. \]

Because \( 0 < \beta_{1, n} < \beta_{2, n} \) and \( 0 < \beta_{2k+1, n} < \beta_{2k+1, n} \),

it follows that

\[ \frac{2n+1}{\sum_{k=1}^{\infty}} \frac{1}{\alpha_k, n} + \sum_{k=0}^{n} \frac{1}{\beta_{2k+1, n}} - \frac{2}{\beta_{2, n}} - 3 \sum_{k=1}^{\infty} \frac{1}{\beta_{2k+1, n}} \]

\[ < \frac{2n+1}{\sum_{k=1}^{\infty}} \frac{1}{\alpha_k, n} + \sum_{k=0}^{n} \frac{1}{\beta_{2k+1, n}} - \frac{2}{\beta_{2, n}} - 3 \sum_{k=1}^{\infty} \frac{1}{\beta_{2k+1, n}} \]

\[ = \frac{2n+1}{\sum_{k=1}^{\infty}} \frac{1}{\alpha_k, n} - \frac{1}{\beta_{2, n}} - 2 \sum_{k=1}^{\infty} \frac{1}{\beta_{2k+1, n}} < 0 . \]

Thus

\[ -\infty < -\lim_{n \to \infty} \left[ \frac{2n+1}{\sum_{k=1}^{\infty}} \frac{1}{\alpha_k, n} + \frac{1}{\beta_{2, n}} + \sum_{k=1}^{n} \frac{2}{\beta_{2k+1, n}} \right] \leq 0 , \]

which implies
\[
\lim_{n \to \infty} \sum_{k=1}^{2n+1} \frac{1}{\alpha_{k,n}} < \infty \quad \text{and} \quad \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{\beta_{2k+1,n}} < \infty.
\]

Since \(0 < \beta_{2k-1,n} < \gamma_{2k,n}\), then

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{\alpha_{k,n}} \leq \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{\beta_{2k-1,n}} \leq \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{\beta_{2k+1,n}}< \infty.
\]

Also, since \(0 < \alpha_{k,n} < \delta_{k+1,n}, \ k = 1, 2, \ldots, 2n+1\), then

\[
\lim_{n \to \infty} \sum_{k=2}^{2n+2} \frac{1}{\delta_{k,n}} \leq \lim_{n \to \infty} \sum_{k=1}^{2n+1} \frac{1}{\alpha_{k,n}} < \infty.
\]

Now for all \(N > 0\),

\[
\sum_{k=1}^{N} \frac{1}{\alpha_{k,n}} = \sum_{k=1}^{N} \lim_{n \to \infty} \frac{1}{\alpha_{k,n}} \leq \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{\alpha_{k,n}} < \infty,
\]

and thus

\[
\sum_{k=1}^{\infty} \frac{1}{\alpha_{k,n}} < \infty.
\]

Similarly,

\[
\sum_{k=0}^{\infty} \frac{1}{\beta_{2k+1,n}} < \infty, \quad \sum_{k=1}^{\infty} \frac{1}{\gamma_{k,n}} < \infty, \quad \text{and} \quad \sum_{k=2}^{\infty} \frac{1}{\delta_{k}} < \infty.
\]
Lemma 3.8. If

\[ \Pi(z) = \frac{1}{(1 + \frac{z}{x})^n} \prod_{k=1}^{n} \left( 1 - \frac{x}{\beta_{2k+1}} \right) \prod_{k=0}^{n+1} \left( 1 + \frac{x}{\gamma_{2k}} \right)^3 \]

then \( f'(z) = e^{\delta' x} \Pi(z) \) where \( \delta' \) is real and \( \delta' = \lim_{n \to \infty} s_n \),

with \( s_n \)

\[ = -s \left( \sum_{k=1}^{n} \frac{1}{\gamma_{2k}} - \sum_{k=1}^{n} \frac{1}{\gamma_{2k}} \right) = \left( \sum_{k=1}^{n} \frac{1}{\gamma_{2k}} - \frac{1}{\gamma_{2k+1}} \right) \]

\[ + \left( \sum_{k=0}^{n} \beta_{2k+1} - \sum_{k=0}^{n} \beta_{2k+1} \right) \]

Proof: The subuniform convergence of each factor of \( \Pi(z) \) is assured by the last relations of Lemma 3.7. Furthermore, Lemma 3.6 implies the existence of \( r > 0 \) such that \( R_n(z) \neq 0 \) and \( \Pi(z) \neq 0 \) for \( |z| < r \).

Now for \( |z| < r \),

log \( \Pi(z) \)

\[ = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left( \sum_{k=1}^{n} \frac{1}{\gamma_{2k}} - \sum_{k=0}^{n} \left( \frac{1}{\beta_{2k+1}} + \frac{2(-1)^{n+1}}{\gamma_{2k}} + \frac{3(-1)^n}{\gamma_{2k}} \right) \right) \]

Then for \( |z| < r \),
\[ \log \frac{R_\mu(s)}{\Pi(s)} = \sum_{n=1}^{\infty} \frac{s^n}{n!} \left[ -\left( \frac{2\mu+1}{\sigma \mu} \frac{1}{x_1} - \frac{1}{\alpha_\mu} \right) \right. \\
\quad + (-1)^{n+1} \left( \frac{n}{k=0} \frac{1}{\beta_{2k+1}} - \frac{n}{k=0} \frac{1}{\beta_{2k}} \right) + \left( 2\frac{(-1)^{n}}{y_{1,1}^m} - 2\frac{(-1)^{n}}{y_{1,1}^m} \right) \\
\left. + 3\frac{(-1)^{n}}{\gamma_{2k+1}} \left( \frac{n}{k=1} \frac{1}{\gamma_{2k,n}^m} - \frac{n}{k=1} \frac{1}{\gamma_{1,k}^m} \right) \right] \] \\

Because \( \gamma_{2k,n} \to \delta_{2k} \), then for every \( N > 1 \), \\

\[ 0 \leq \lim_{n \to \infty} \left| \frac{n}{k=1} \frac{1}{\gamma_{2k,n}^m} - \frac{n}{k=1} \frac{1}{\gamma_{1,k}^m} \right| = \lim_{n \to \infty} \left| \frac{n}{k=N} \frac{1}{\gamma_{2k,n}^m} - \frac{n}{k=N} \frac{1}{\gamma_{1,k}^m} \right| \] \\

Moreover, since \( \lim_{n \to \infty} \frac{n}{k=1} \frac{1}{\gamma_{2k,n}^m} < \infty \) and \( \frac{n}{k=1} \frac{1}{\gamma_{1,k}^m} < \infty \), \\

there exists \( M > 0 \) such that for all \( n \geq 1 \), \\

\[ \sum_{k=1}^{n} \frac{1}{\gamma_{2k,n}^m} < M \] \\

and \\

\[ \sum_{k=1}^{\infty} \frac{1}{\gamma_{1,k}^m} < M. \] \\

For every \( k < n \) and for \( k' = 1, 2, 3, \ldots, k-1 \),
\[
\frac{1}{\gamma_{2k,n}} < \frac{1}{\gamma_{2k',n}} \quad \text{and} \quad \frac{1}{\gamma_{2k}} < \frac{1}{\gamma_{2k'}},
\]
so
\[
\frac{k}{\gamma_{2k,n}} < M \quad \text{and} \quad \frac{k}{\gamma_{2k}} < M.
\]
Thus
\[
\frac{1}{\gamma_{2k,n}} < \frac{k^n}{k^n} \quad \text{and} \quad \frac{1}{\gamma_{2k}} < \frac{k^n}{k^n}.
\]
Now for every \( N > 1 \),
\[
\lim_{n \to \infty} \left| \sum_{k=N}^{n} \frac{1}{\gamma_{2k,n}} - \sum_{k=N}^{\infty} \frac{1}{\gamma_{2k}} \right| \leq 2^{n-N} \sum_{k=N}^{\infty} \frac{1}{k^n},
\]
which implies for \( m \geq 2 \),
\[
\lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{\gamma_{2k,n}} - \sum_{k=1}^{\infty} \frac{1}{\gamma_{2k}} \right) = 0.
\]
Similarly for \( m \geq 2 \),
\[
\lim_{n \to \infty} \left( \sum_{k=0}^{n} \frac{1}{\beta_{2k+1,n}} - \sum_{k=0}^{\infty} \frac{1}{\beta_{2k+1}} \right) = 0 \quad \text{and} \quad \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{\alpha_{2k,n}} - \sum_{k=1}^{\infty} \frac{1}{\alpha_{2k}} \right) = 0.
\]
Then
\[
\lim_{n \to \infty} \log \frac{R_n(z)}{\Pi(z)} = \log \frac{f'(z)}{\Pi(z)} = \delta' z,
\]
and thus
\[
f'(z) = e^{\delta' z} \Pi(z).
\]

**Lemma 3.9.** \(\delta' \leq 0\).

**Proof:** Because the factors of \(\Pi(z)\) are canonical products of genus zero, for every \(\varepsilon > 0\) and \(0 < \rho \leq |\arg z| \leq \Pi - \rho\),

\[
\Pi(z) = O(e^{\varepsilon |z|}) \quad \text{and} \quad \frac{1}{\Pi(z)} = O(e^{\varepsilon |z|}).
\]

Then for \(\arg z\) satisfying these conditions and \(|z| > R\) sufficiently large,

\[
\delta' R(z) - \varepsilon |z| \leq |f'(z)| \leq \delta' R(z) + \varepsilon |z|.
\]

If \(\delta' > 0\) and \(|z| > R\), there exists \(\phi_1 > 0\) such that

\[
|f'(z)| \geq e^{\phi_1 |z|} \quad \text{for} \quad \frac{\pi}{6} \leq \arg z \leq \frac{\pi}{3}
\]

and

\[
|f'(z)| \leq e^{-\phi_1 |z|} \quad \text{for} \quad \frac{2\pi}{3} \leq \arg z \leq \frac{5\pi}{6}.
\]

If \(\delta' < 0\) and \(|z| > R\), there exists \(\phi_2 > 0\) such that

\[
|f'(z)| \leq e^{-\phi_2 |z|} \quad \text{for} \quad \frac{\pi}{6} \leq \arg z \leq \frac{\pi}{3}
\]

and

\[
|f'(z)| > e^{\phi_2 |z|} \quad \text{for} \quad \frac{2\pi}{3} \leq \arg z \leq \frac{5\pi}{6}.
\]
Since the distance from the origin to \( C_{n+1} \to \infty \) as \( n \to \infty \), then there exists a sequence of positive numbers \( \{r_n\} \) such that \( r_n \to \infty \) as \( n \to \infty \) and such that for \( z \) on \( C_{n+1} \), \( |z| \geq r_n \).

Let \( z_{1,2j+1} \) and \( z_{2,2j+1} \) be points of \( C_{2j+1} \) such that \( \arg z_{1,2j+1} = \frac{\pi}{6} \) and \( \arg z_{2,2j+1} = \frac{\pi}{3} \). As \( z \) traverses \( C_{2j+1} \) from \( z_{1,2j+1} \) to \( z_{2,2j+1} \), \( f(z) \) is real and increasing, and thus \( f'(z) \, dz \geq 0 \) for these values of \( z \).

Let \( z_{1,2j} \) and \( z_{2,2j} \) be points of \( C_{2j} \) such that \( \arg z_{1,2j} = \frac{5\pi}{6} \) and \( \arg z_{2,2j} = \frac{2\pi}{3} \). As \( z \) traverses \( C_{2j} \) from \( z_{1,2j} \) to \( z_{2,2j} \), \( f(z) \) is real and increasing, and thus \( f'(z) \, dz \geq 0 \) for these values.

If \( \delta' > 0 \), then for \( z_1 \) and \( z_2 \) in \( \{ \frac{2\pi}{3} \leq \arg z \leq \frac{5\pi}{6} , \ |z| > R \} \),

\[
|f(z_2) - f(z_1)| = \left| \int_{z_1}^{z_2} f'(t) \, dt \right| \leq \int_{z_1}^{z_2} |f'(t)| \, dt \leq e^{-\delta' R} |z_2 - z_1|.
\]

Thus \( f(z) \to k \), a constant, uniformly in \( \{ \frac{2\pi}{3} \leq \arg z \leq \frac{5\pi}{6} \} \) as \( z \to \infty \). \( k = 0 \), because as \( z \to \infty \), say along
the ray arg $z = \frac{3\pi}{4}$, $f(z) < 0$ where the ray crosses $C_{2n}$ and $f(z) > 0$ where the ray crosses $C_{2n+1}$. Thus for $j$ sufficiently large,

$$0 < f(z_{2,2j+1}) < 1.$$ 

For $|z| \geq r_{2j+1}$ sufficiently large,

$$b_{2j+1} - a_{2j+1} \geq f(z_{2,2j+1}) - f(z_{1,2j+1}) = \int_{z_{1,2j+1}}^{z_{2,2j+1}} f'(t) \, dt \geq \phi_1 r_{2j+1} \frac{\pi}{6} r_{2j+1},$$

and hence

$$b_{2j+1} - a_{2j+1} \geq f(z_{2,2j+1}) - f(z_{1,2j+1}) \to \infty \quad \text{as} \quad j \to \infty.$$ 

Because $0 < a_{2j+1} < b_{2j+1}$, then $f(z_{1,2j+1}) > 0$, and thus $f(z_{2,2j+1}) \to \infty$ as $j \to \infty$, which contradicts a previously established inequality. Thus $\delta' \not\in (0, \frac{\pi}{2})$ or $\delta' \not\in (0, \frac{\pi}{3})$.

However, if $\delta' < 0$, then as $z \to \infty$, say along $\arg z = \frac{3\pi}{4}$, then $|f(z)| \to 0$ as before, while

$$f(z_{2,2j}) - f(z_{1,2j}) \to \infty \quad \text{as} \quad j \to \infty$$

where $f(z_{2,2j})$ $< 0$. Then $f(z_{1,2j}) \to -\infty$ as $j \to \infty$ which is certainly possible. \|
Lemma 3.10. If $\delta' < 0$, then $a_n = o(1)$ as $n \to \infty$ and there exist positive constants $p$ and $A$ such that for $n$ sufficiently large, $b_{2n+1} \geq Ae^{pn}$.

Proof: If $\delta' < 0$, then for $z_n$ a point on $C_n$ and $\frac{\pi}{6} \leq \arg z_n \leq \frac{\pi}{3}$,

$$0 < a_n < |f(z_n)|,$$

and from the proof of Lemma 3.9, $|f(z_n)| \to 0$ as $n \to \infty$. Hence $a_n = o(1)$ as $n \to \infty$.

Let $z_{1,2j+1}$ and $z_{2,2j+1}$ be points of $C_{2j+1}$ such that $\arg z_{1,2j+1} = \frac{5\pi}{6}$ and $\arg z_{2,2j+1} = \frac{2\pi}{3}$. Using the inequalities in the proof of Lemma 3.9, for $j$ sufficiently large

$$b_{2n+1} - a_{2n+1} \geq f(z_{2,2j+1}) - f(z_{1,2j+1}) \geq \int_{z_1,z_{j+1}} |f'(t)| \, dt \geq e^{\delta' z_{2j+1}} \frac{\pi}{6} r_{2j+1} \geq Ae^{pn}.$$

Lemma 3.11. If

$$P(z) = \frac{\prod_{k=2}^{\infty} \left( 1 - \frac{z}{\delta_k x} \right)}{1 + \frac{z}{\delta_k x}} \prod_{k=1}^{\infty} \left( 1 + \frac{z}{\delta_k x} \right)^2,$$
then \( f(z) = e^{\delta z} P(z) \), where \( \delta \) is real and \( \delta = \lim_{n \to \infty} s_n \),

with

\[
s_n = -2 \left( \sum_{k=1}^{n} \frac{1}{\gamma_{2k,n}} - \sum_{k=1}^{\infty} \frac{1}{\gamma_k} \right) - \left( \sum_{k=2}^{2n+2} \frac{1}{\delta_{k,n}} - \sum_{k=2}^{\infty} \frac{1}{\delta_k} \right).
\]

Proof: The subuniform convergence of \( P(z) \) in \(|z| < \infty\) is given as a consequence of Lemma 3.7, and the existence of \( r > 0 \) such that

\[
\frac{P(z)}{z} \neq 0 \quad \text{and} \quad \frac{R_n(z)}{z} \neq 0
\]

for \(|z| < r\) is given as a consequence of Lemma 3.6.

Then for \(|z| < r\),

\[
\log \frac{P(z)}{z} = - \log \left( 1 + \frac{z}{\gamma} \right)
\]

\[
+ \sum_{m=1}^{\infty} \frac{z^m}{m} \left( \sum_{k=1}^{\infty} \frac{2(-1)^m}{\gamma_k} - \sum_{k=2}^{2n+2} \frac{1}{\delta_k} \right)
\]

and

\[
\log \frac{R_n(z)}{z} = - \log \left( 1 + \frac{z}{\gamma_{i,n}} \right)
\]

\[
+ \sum_{m=1}^{\infty} \frac{z^m}{m} \left( \sum_{k=1}^{\infty} \frac{2(-1)^m}{\gamma_{2k,n}} - \sum_{k=2}^{2n+2} \frac{1}{\delta_{k,n}} \right).
\]

Hence
\[ \log \frac{R_n(z)}{P(z)} = \log \frac{1 + \frac{z}{\gamma_z}}{1 + \frac{z}{\gamma_{1,n}}} \]

\[ + \sum_{m=1}^{\infty} \frac{z^m}{m} \left[ \left( \sum_{k=1}^{\infty} \frac{2(-1)^m}{\gamma_{2,k,n}} - \sum_{k=1}^{\infty} \frac{2(-1)^{2m}}{\gamma_{2,k,n}} \right) \right. \]

\[ - \left( \sum_{k=2}^{\infty} \frac{1}{\delta_{k,n}} - \sum_{k=2}^{\infty} \frac{1}{\delta_{2,n}} \right) \left. \right] . \]

Now by proof similar to that of Lemma 3.8, for \( m \geq 2 \),

\[ \lim_{n \to \infty} \left( \sum_{k=1}^{\infty} \frac{1}{\gamma_{2,k,n}} - \sum_{k=1}^{\infty} \frac{1}{\gamma_{2,k,n}} \right) = 0 \]

and

\[ \lim_{n \to \infty} \left( \sum_{k=2}^{\infty} \frac{1}{\delta_{k,n}} - \sum_{k=2}^{\infty} \frac{1}{\delta_{k,n}} \right) = 0. \]

Hence \( \lim_{n \to \infty} \log \frac{R_n(z)}{P(z)} = \delta z \), where \( \delta \) is real, and thus

\[ f(z) = e^{\delta z}P(z). \]

Lemma 3.12. \( \delta = \delta'. \)

Proof: From Lemmas 3.8 and 3.11,
\[ \delta - \delta' = \lim_{n \to \infty} (s_n - s_n') \]

where

\[
\begin{align*}
s_n - s_n' &= \left( \sum_{k=1}^{n} \frac{1}{\gamma_{k,n}} - \frac{\infty}{\sum_{k=1}^{\infty} \frac{1}{\gamma_k}} \right) - \left( \sum_{k=0}^{\infty} \frac{1}{\beta_{k+n}} - \frac{\infty}{\sum_{k=0}^{\infty} \frac{1}{\beta_{k+1}}} \right) \\
&\quad + \left( \sum_{k=1}^{2n+1} \frac{1}{\alpha_{k,n}} - \frac{\infty}{\sum_{k=1}^{\infty} \frac{1}{\alpha_k}} \right) - \left( \sum_{k=2}^{\infty} \frac{1}{\delta_{k,n}} - \frac{\infty}{\sum_{k=2}^{\infty} \frac{1}{\delta_k}} \right).
\end{align*}
\]

Using Lemma 3.6, for \( n_0 > 2 \),

\[
0 \leq \lim_{n \to \infty} |s_n - s_n'| \leq \left| \frac{n}{\sum_{k=n_0}^{\infty} \frac{1}{\gamma_{k,n}}} - \frac{\infty}{\sum_{k=n_0}^{\infty} \frac{1}{\gamma_k}} \right|
\]

\[
- \left( \sum_{k=n_0}^{2n+1} \frac{1}{\beta_{k+n}} \right) + \left( \sum_{k=n_0}^{2n+1} \frac{1}{\alpha_{k,n}} \right) - \left( \sum_{k=n_0}^{\infty} \frac{1}{\delta_{k,n}} \right)
\]

\[
- \left( \sum_{k=n_0}^{\infty} \frac{1}{\delta_k} \right)
\]

\[
\leq \left| \sum_{k=n_0}^{\infty} \left( \frac{1}{\gamma_{k,n}} - \frac{1}{\beta_{k+n+1,n}} \right) \right| + \left| \sum_{k=n_0}^{\infty} \left( \frac{1}{\gamma_k} - \frac{1}{\beta_{k+1}} \right) \right|
\]

\[
+ \left| \sum_{k=n_0}^{2n+1} \left( \frac{1}{\delta_{k,n}} - \frac{1}{\alpha_{k,n}} \right) \right| + \left| \sum_{k=n_0}^{\infty} \left( \frac{1}{\delta_k} - \frac{1}{\alpha_k} \right) \right|
\]
\[
\leq \lim_{n \to \infty} \frac{1}{\gamma_{\nu_n, \gamma_n}} + \frac{1}{\delta_{\nu_n, \gamma_n}} + \lim_{n \to \infty} \frac{1}{\delta_{\nu_n}} = \frac{2}{\gamma_{\nu_n}} + \frac{2}{\delta_{\nu_n}}.
\]

Since \( \gamma_{\nu_n} \) and \( \delta_{\nu_n} \to \infty \) as \( n \to \infty \), then \( \lim_{n \to \infty} (s_n - s_n') = 0 \), which implies

\[
\lim_{n \to \infty} s_n = \lim_{n \to \infty} s_n' = 0 = 0'.
\]

**Lemma 3.13.** If \( \delta < 0 \), then there exists a positive constant \( p \) such that \( a_n = O(e^{-pn}) \) as \( n \to \infty \).

**Note:** This improves the results of Lemma 3.10.

**Proof:** Now \( |f(s)| \) satisfies the same inequalities that \( |f'(s)| \) satisfies in the first part of the proof of Lemma 3.9. Thus if \( \delta < 0 \), there exists a positive \( p \) such that for \( z_n \) on \( C_n \) and \( \arg z_n = \frac{\pi}{4} \),

\[
|f(z_n)| = O(e^{-pn}) \text{ as } n \to \infty.
\]

But for \( n > 0 \),

\[
0 < a_n < |f(z_n)|.
\]

Thus \( a_n = O(e^{-pn}) \) as \( n \to \infty \).

**Theorem III.** A surface of Class V is parabolic and the mapping function is given by

\[
f(s) = e^{\delta s} \frac{\sum_{k=1}^{\infty} \left(1 - \frac{s}{\delta_k}\right)}{1 + \frac{s}{\delta_k}} \frac{\sum_{k=2}^{\infty} \left(1 - \frac{s}{\delta_k}\right)}{1 + \frac{s}{\delta_k}}.
\]
where
\[ f'(s) = e^{\delta s} \frac{1}{(1 + \frac{\delta}{\gamma_s})^2} \prod_{k=1}^{\infty} \left(1 - \frac{\gamma_s}{\delta_s k}\right) \prod_{k=0}^{\infty} \left(1 + \frac{\gamma_s}{\delta_s k}\right) \]

with \( \delta \leq 0 \). If \( \delta < 0 \), then there exist positive constants \( A \) and \( p \) such that
\[ a_n = O(e^{-pn}) \quad \text{as} \quad n \to \infty \]
and
\[ b_{2n+1} \geq Ae^{pn} \quad \text{for} \quad n \quad \text{sufficiently large}. \]

Also,
\[ \sum_{k=1}^{\infty} \frac{1}{\varpi_k} < \infty, \quad \sum_{k=0}^{\infty} \frac{1}{\beta_{2k+1}} < \infty, \quad \frac{1}{\sum_{k=1}^{\infty} \gamma_{1k}} < \infty, \quad \text{and} \quad \sum_{k=2}^{\infty} \frac{1}{\delta_{k}} < \infty, \]
and for \( k > 0 \),
\[ 0 < \alpha_k < \delta_{k+1} < \alpha_{k+1} \]
and
\[ 0 < \gamma_1 < \beta_{2k-1} < \gamma_{2k} < \beta_{2k+1}. \]

The remainder of this chapter is devoted to the proof of the following theorem.
Theorem IV. Let

\[
f(z) = e^{\delta z} \frac{\prod_{k=1}^{\infty} \left(1 - \frac{\delta}{\delta_{3k}} \right) \left(1 - \frac{\delta}{\delta_{3k+1}} \right)}{(1 + \frac{\delta}{\delta_{3k}}) \left(1 + \frac{\delta}{\delta_{3k}} \right)}
\]

where

\[
\sum_{k=1}^{\infty} \frac{1}{\delta_k} < \infty, \quad \sum_{k=1}^{\infty} \frac{1}{\gamma_{3k}} < \infty, \quad \delta \leq 0,
\]

\[0 < \delta_2 < \delta_3 < \ldots < \delta_{2n} < \delta_{2n+1} < \ldots\]

and

\[0 < \gamma_1 < \gamma_2 < \gamma_4 < \ldots < \gamma_{2n} < \gamma_{2n+2} < \ldots\]

The Riemann surface of the inverse of \( f(z) \) is a surface of Class V.

Case 1: \( \delta = 0 \).

Lemma 3.14. It is possible to construct a sequence of rational functions \( R_n(z), R_n(z) \) of degree \( n \), such that \( R_n(z) \rightarrow f(z) \) subuniformly in \( |z| < \infty \) and such that the paths other than the real axis on which \( R_n(z) \) is real are \( n-1 \) simple, closed, non-intersecting curves \( C_1, C_2, \ldots, C_{n-1}, C_n \). \( C_1 \) bounds a domain containing the origin, and \( C_{j+1}, C_j \), \( 1 \leq j \leq n-2 \), bound an annular region about the origin.

Proof: Let

\[
R_n(z) = \frac{\prod_{k=1}^{n-1}}{1 + \frac{\delta}{\delta_{3k}}} \frac{\prod_{k=1}^{n-1} \left(1 - \frac{\delta}{\delta_{3k}} \right) \left(1 - \frac{\delta}{\delta_{3k+1}} \right)}{(1 + \frac{\delta}{\delta_{3k}}) \left(1 + \frac{\delta}{\delta_{3k}} \right)}
\]
where \( n > 2 \) and \( n \) is an odd integer.

Hence \( R_n'(z) \) certainly has at least the following:

\[
\frac{n-1}{2} \text{ zeros at } -\beta_{2k+1,n}, \quad k = 0, 1, \ldots, \frac{n-3}{2},
\]
such that

\[
-\gamma_{n-1} < -\beta_{n-2,n} < -\gamma_{n-3} < \ldots < -\beta_{3,n} < -\gamma_2 < -\beta_{1,n} < -\gamma_1 < 0,
\]

and

\[
(n-1) \text{ zeros at } \alpha_{k,n}, \quad k = 1, 2, \ldots, n-1, \text{ such that }
\]

\[
0 < \alpha_{1,n} < \delta_2 < \alpha_{2,n} < \delta_3 < \alpha_{3,n} < \ldots < \alpha_{n,n} < \delta_n.
\]

\( R_n(z) \) also has \( \frac{n-1}{2} \) first order branch points over the poles at \(-\gamma_{2k}\). The indicated critical points of \( R_n(z) \) account for a branch order of

\[
\frac{n-1}{2} + (n-1) + \frac{n-1}{2} = 2n - 2.
\]
which is the exact total branch order for $R_n(z)$; thus $R'_n(z)$ has no further zeros. Then through each of the $\alpha'$s, $\beta'$s, and $\gamma'$s passes a curve, in addition to the real axis, on which $R_n(z)$ is real. Because $R_n(\bar{z}) = \overline{R_n(z)}$, these curves are symmetric about the real axis, and the absence of branching over $\infty$ implies that the curves are simple, closed, non-intersecting ones each of which intersects the real axis at two points.

One such curve cannot intersect the real axis at $\alpha'_{j,n}$ and $\alpha'_{k,n}$; $j \neq k$. For if this were the case, either an odd or even number of $\alpha'$s would be enclosed inside the curve. An odd number of $\alpha'$s inside the curve would imply the intersection of another curve $C$ with the given curve:

![Diagram 1](image1)

An even number of $\alpha'$s (including no $\alpha'$s) inside the curve would imply the existence of two adjacent $\alpha'$s,

![Diagram 2](image2)
and \( \alpha_{i,n} \) and \( \alpha_{i+1,n} \), such that another of these curves \( C \) passes through them. As \( s \) traverses the segment from \( \alpha_{i,n} \) to \( \alpha_{i+1,n} \) and then back along the upper half of this second curve \( C \), \( R_n(z) \) either increases or decreases through finite values from \( R_n(\alpha_{i,n}) \) back to \( R_n(\alpha_{i,n}) \).

This is impossible.

Since the number of critical points to the left of the origin is the same as the number to the right of the origin, the \( n-1 \) curves \( C_{k,n} \) on which \( R_n(z) \) is real consist of simple, closed, non-intersecting ones which intersect the real axis at \( \alpha_1 \) and \( -\beta_1 \), or \( \alpha_{2k} \) and \( -\gamma_{2k} \), or \( \alpha_{2k+1} \) and \( -\beta_{2k+1} \). \( \|

\underline{Lemma 3.15.} Each curve \( C_{k,n} \) is star-shaped with respect to the origin.

\underline{Proof:} \( R_n(z) = \frac{P_n(z)}{Q_n(z)} \), where \( P_n(z) \) and \( Q_n(z) \) are each polynomials of degree \( n \) with real coefficients. The paths of reality, i.e., the curves \( C_{k,n} \) and the real axis, have equation

\[
21 \text{ } \frac{\partial}{\partial [R_n(z)]} \frac{P_n(z)}{Q_n(z)} \frac{P_n(z)}{Q_n(z)} = \frac{P_n(z)}{Q_n(z)} - \frac{P_n(z)}{Q_n(z)} = 0,
\]
or
\[ F(x, y) = P_n(z) Q_n(z) - P_n(z) Q_n(z) = 0. \]

\( F(x, y) \) is of degree at most \((2n - 1)\) in \( x \) and \( y \) simultaneously. Now any line \( y = mx \) or \( x = ny \) intersects each \( C_{k,n} \) at least twice, and these \( 2(n-1) \) intersections together with the one intersection at the origin make a total of \( 2n - 1 \) intersections, the maximum number of solutions of

\[ F(x, mx) = 0 \quad \text{and} \quad F(ny, y) = 0. \]

Hence each \( C_{k,n} \) is star shaped with respect to the origin. ||

**Lemma 3.16.** The points of \( C_{k,n} \) tend to the points of a curve \( C_k \) as \( n \to \infty \), where \( C_k \) intersects the real axis at \( \alpha_k \) and \(-\beta_k \) or \(-\gamma_k \). \( C_k \) is star shaped with respect to \( z = 0 \), \( C_k \) is symmetric about the real axis, and \( C_k \) does not pass through \( \infty \).

**Proof:** Since \( R_n(z) \to f(z) \), then \( R_n'(z) \to f'(z) \), and by Hurwitz's Theorem, \( \alpha_k, n \to \alpha_k \) and \( \beta_k, n \to \beta_k \) as \( n \to \infty \). Then each point of \( C_{k,n} \), which is symmetric about the real axis and star shaped with respect to \( z = 0 \), tends along the line \( y = mx \) to a point of a limiting curve \( C_k \) which passes through \( \alpha_k \) and \(-\beta_k \) or \(-\gamma_k \), is star shaped with respect to the origin,
and is symmetric about the real axis.

To show that \( C_k \) has no points at \( \infty \), suppose there exists a \( C_{k^*} \) which has \( z = \infty \) as one of its points. Then let \( C_{k^*} \) be the curve with minimum subscript such that \( C_{k^*} \) has \( z = \infty \) as one of its points. As \( z \to \infty \) along the upper half of \( C_{k^*} \) from \( \alpha_{k^*} \), \( f(z) \to \infty \) because otherwise the upper parts of \( C_{k^*} \) and \( C_{k^*+1} \) and the segment \( (\alpha_{k^*}, \alpha_{k^*+1}) \) bound a region on whose boundary \( f(z) \) steadily decreases or increases through finite real values from \( f(\alpha_{k^*}) \) back to \( f(\alpha_{k^*}) \), which is impossible. Now as \( z \to \infty \) along the upper part of \( C_{k^*} \) from its intersection with the negative real axis, \( f(z) \to \infty \), for if \( f(z) \to \infty < \infty \), the absence of any paths of reality in the interior of the region bounded by \( C_{k^*} \), the real axis, and \( C_{k^*+1} \) in the upper half plane offers a contradiction to Theorem \( F \). Now let \( C_k \) be the curve with minimum even subscript and such that \( C_{k-1} \) also has \( z = \infty \) as one of its points.
As $z$ traverses the path from $-\gamma_k$ to $-\beta_{k-1}$ along the real axis and from $-\beta_{k-1}$ to $\infty$ along $C_{k-1}$, $f(z)$ decreases steadily through real values from $+\infty$ to $-\infty$. As $z$ traverses the path from $-\gamma_k$ to $\infty$ along $C_k$, $f(z)$ increases steadily from $-\infty$ to $+\infty$. These two paths then bound a region where $f(z)$ takes every real value twice on the boundary and there is no branch point interior to the region. This is impossible. ||

**Lemme 3.17.** $f(z)$ is a schlicht and conformal map of the upper half of the annular region between $C_j$ and $C_{j+1}$ onto $\mathcal{D}(-1)^j(d(w) > 0$.

**Proof:** If $j$ is even,

or if $j$ is odd,

as $z$ traverses the boundary of this region in the positive sense, starting from the $\gamma$ on the boundary,
then \( f(z) \) varies steadily from \((-1)^{j+1} \infty\) to \((-1)^j \infty\).

In this region \( f(z) \neq (-1)^{j+1} i \), and the hypothesis of Darboux's Theorem is satisfied, which establishes the truth of the lemma. 

The preceding lemmas establish Theorem IV for \( \delta = 0 \).

Case 2: \( \delta < 0 \).

**Lemma 3.18.** It is possible to construct a sequence of rational functions

\[
R_n(z) = \left(1 + \frac{\hat{z}}{\lambda_n} \right) \frac{z^{-1}}{1 + \frac{z}{\delta}} \prod_{k=1}^{n-1} \frac{(1 - \frac{z}{\delta_k}) (1 - \frac{z}{\delta_{k+1}})}{(1 + \frac{z}{\delta_k})^2}
\]

where \( \{\lambda_n\} \) is an increasing sequence of integers such that \( R_n(z) \to f(z) \) subuniformly in \( |z| < \infty \). The paths on which \( R_n(z) \) is real, other than the real axis, are \( n + \lambda_n \) simple closed curves \( C_{1,n}, C_{2,n}, \ldots, C_{n-1,n} \)

\( D_{1,n}, D_{2,n}, \ldots, D_{\lambda_n,n} \). The curves \( C_{k,n}, k = 1, 2, \ldots, n-1 \), are non-intersecting. \( C_{1,n} \) bounds a domain containing the origin. \( C_j \) and \( C_{j+1}, 1 \leq j \leq n-2 \), bound an annular region about the origin. \( D_{k,n}, k = 1, 2, \ldots, \lambda_n \),
are curves through \( \infty \) and lie outside \( C_{k,n} \).

Proof: For \( n \) odd and \( n > 2 \), let \( \lambda_n \) be an integer such that 
\[
\frac{\lambda_n}{n} > \delta_n.
\]
Since \( \delta_n \rightarrow \infty \), then \( \lambda_n \rightarrow \infty \) as \( n \rightarrow \infty \). Also,
\[
R_n(z) \rightarrow f(z)
\]
subuniformly in \( |z| < \infty \) as \( n \rightarrow \infty \).

\[
R_n'(z) \text{ has at least } \frac{n+1}{2} \text{ zeros at } -\beta_{2k+1,n}, k = 0, 1, \ldots, \frac{n-1}{2}, \text{ such that } \\
-\beta_{n,n} < -\gamma_{n-1} < -\beta_{n-2,n} < \ldots < -\beta_{3,n} < -\gamma_2 < -\beta_{1,n} < \gamma_1 < 0; \\
\text{at least } n-1 \text{ zeros at } \alpha_{k,n}, k = 1, 2, \ldots, n-1, \text{ such that }
\]
\begin{align*}
0 < \alpha_{1,n} < \delta_2 < \alpha_{2,n} < \ldots < \alpha_{n-1,n} < \delta_n;
\end{align*}

at least one zero of order one at \( \varepsilon_n \) and a zero of order \( \lambda_n - 1 \) at \(-\frac{\lambda_n}{\delta}\). \( R_n(z) \) has \( \frac{n-1}{2} \) branch points of order one over \( \infty \) and a branch point of order \( \lambda_n - 1 \) over \( \infty \). The indicated critical points of \( R_n(z) \) account for a branch order of

\begin{equation}
\frac{n+1}{2} + (n-1) + 1 + (\lambda_n - 1) + \frac{n-1}{2} + (\lambda_n - 1) = 2(n + \lambda_n) - 2,
\end{equation}

which is the exact total branch order for \( R_n(z) \). Thus \( R_n'(z) \) has no further zeros.

In addition to the real axis, one other curve on which \( R_n(z) \) is real passes through each of the \( \alpha \)'s, \( \beta \)'s, \( \gamma \)'s, and \( \varepsilon_n \), while \( \lambda_n - 1 \) such curves pass through \(-\frac{\lambda_n}{\delta}\) and \( z = \infty \). These curves are simple, closed, and symmetric about the real axis. Because of the argument given in Lemma 3.14, no curve can pass through two \( \alpha \)'s or an \( \alpha \) and \( \varepsilon_n \). The curve through \( \varepsilon_n \) cannot pass through \( z = \infty \), for then the \( \lambda_n - 1 \) curves through \(-\frac{\lambda_n}{\delta}\) must also pass through \( z = \infty \), making a total of \( \lambda_n \) curves through \( z = \infty \), which is impossible. Then \( \varepsilon_n, \alpha_{n-1,n}, \ldots, \alpha_{1,n}, \alpha_{1,n} \) must be connected to...
by \( n-1 \) curves \( C_{k,n}, k = 1,2,\ldots,n-1 \). \( \epsilon_n \) must be connected to \( -\beta_{n,n} \) by a curve \( D_{1,n} \) and \( -\frac{\lambda_n}{\delta} \) must be joined to \( z = \infty \) by \( \lambda_n - 1 \) curves \( D_{k,n}, k = 2,3,\ldots \lambda_n \).

\[ \text{Lemma 3.19.} \] The points of \( C_{k,n} \) tend to the points of a curve \( C_k \) as \( n \to \infty \), where \( C_k \) intersects the real axis at \( \alpha_k \) and \( -\beta_k \) or \( -\gamma_k \). \( C_k \) is symmetric about the real axis and \( C_k \) does not pass through \( \infty \).

Proof: The proof is essentially the same as that of Lemma 3.16.

\[ \text{Lemma 3.20.} \] For every \( k \), the distance from \( z = 0 \) to \( D_{k,n} \) \( \to \infty \) as \( n \to \infty \).

Proof: Suppose not. Then for some \( k_0 \), there exists \( M > 0 \) such that \( D_{k_0,n} \) tends to \( D_{k_0} \) where the distance from \( z = 0 \) to \( D_{k_0} \) is less than \( M \). Moreover, since for all \( j \leq n \) the distance from \( z = 0 \) to \( C_{j,n} \) is less than the distance from \( z = 0 \) to \( D_{k_0,n} \), the distance from \( z = 0 \) to \( C_j \) is at most \( M \). Then for \( |z| < M \), there are infinitely many paths, whose
points must have a finite limit point \( z_0 \), such that in a neighborhood of \( z_0 \) the function \( f(z) \) has infinitely many non-intersecting paths on which \( f(z) \) is real. This is impossible. \( \|
\)

**Lemma 3.21.** \( f(z) \) is a schlicht and conformal map of the upper half of the annular region between \( C_j \) and \( C_{j+1} \) onto \( \mathbb{D} \) \((-1)^j(w) > 0\).

Proof: The proof is the same as that of Lemma 3.17. \( \|
\)

These lemmas prove the preceding theorem.
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