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RICE UNIVERSITY

Some Non-Classical Problems
in Heat Conduction

by

John Rozier Cannon

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

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TO MY WIFE, MY MOTHER

AND MY FATHER
Acknowledgements:

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Thank you,

John Regier Cannon

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Determination of an Unknown Coefficient
in a Parabolic Differential Equation

by

J. R. Cannon

1. Introduction. In [3], B. F. Jones considered the problem of determining the conductivity of a medium if the conductivity was known a priori to be a function of time only. Specifically, Jones treated the problem

\[
\begin{aligned}
\begin{cases}
    u_t = a(t) \ u_{xx}, & 0 < x < 1, \quad 0 < t < T, \\
    u(0,t) = f_1(t), & 0 \leq t \leq T, \quad f_1(0) = 0, \\
    u(1,t) = f_2(t), & 0 \leq t \leq T, \quad f_2(0) = 0, \\
    u(x,0) = 0, & 0 \leq x \leq 1, \\
    -a(t) \lim_{x \downarrow 0} u_x(x,t) = g(t), & 0 < t < T,
\end{cases}
\end{aligned}
\]

(1.1)

where \( a(t) \) is the unknown conductivity. Jones gave conditions on the data \( f_1(t), f_2(t) \), and \( g(t) \) which enabled him to prove existence and uniqueness of a solution (a pair of functions \( u(x,t) \) and \( a(t) \) which satisfy (1.1)) of (1.1).

In this article a different approach to the problem of determining the conductivity \( a(t) \) is considered. This approach yields a simpler analysis of the existence and uniqueness problem. It also simplifies the numerical technique of approximating \( a(t) \) [1]. Consider the problem
\[ \begin{align*}
  u_t &= a(t) u_{xx}, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T, \\
  u(0,t) &= \psi_o, \quad 0 \leq t \leq T, \\
  u(1,t) &= \psi(t), \quad 0 \leq t \leq T, \\
  u(x,0) &= f(x), \quad 0 \leq x \leq 1, \quad f(0) = \psi_0, f(1) = \psi(0), \\
  a(t) \lim_{x \downarrow 0} u_x(x,t) &= h(t), \quad 0 \leq t \leq T,
\end{align*} \]

where \( \psi(t) \), \( f(t) \), and \( h(t) \) are known continuous functions of their arguments, and \( \psi_0 \) is a given constant. From physical experience, the conductivity is assumed to be positive for all time. A solution to (1.2) is defined as follows:

**Definition 1.1:** A pair of functions \( u(x,t) \) and \( a(t) \) is a solution of (1.2) if and only if the following conditions are satisfied:

(a.) \( a(t) \) is positive and continuous for \( 0 \leq t \leq T \);

(b.) \( u(x,t) \) is continuous in \( (x,t) \) for \( 0 \leq x \leq 1 \), \( 0 \leq t \leq T \);

(c.) \( u_x \) and \( u_{xx} \) are continuous in \( (x,t) \) for \( 0 \leq x \leq 1 \), \( 0 \leq t \leq T \);

(d.) \( \lim_{x \downarrow 0} u_x(x,t) \) exists for \( 0 \leq t \leq T \);

(e.) (1.2) is satisfied.

In order to simplify the analysis of problem (1.2), consider the problem

\[ \begin{align*}
  v_t &= a(t) v_{xx}, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T, \\
  v(0,t) &= 0, \quad 0 \leq t \leq T, \\
  v(1,t) &= \psi(t) - \psi(0), \quad 0 \leq t \leq T, \\
  v(x,0) &= f(x) - \left\{ \psi_0 [1 - x] + \psi(0) x \right\}, \quad 0 \leq x \leq 1, \\
  a(t) \lim_{x \downarrow 0} v_x(x,t) &= h(t) - a(t) \left\{ \psi(0) - \psi_0 \right\}, \quad 0 \leq t \leq T.
\end{align*} \]

**Definition 1.2:** A pair of functions \( v(x,t) \) and \( a(t) \) is a solution of (1.3) if and only if conditions (a.), (b.), (c.), and (d.) of Definition 1.1 hold when \( u(x,t) \) is replaced by \( v(x,t) \), and (1.3) is satisfied.
By considering the transformation

\[ v(x,t) = u(x,t) - [(1 - x) \psi_0 + x \psi(0)] , \]

it follows that in order to obtain a solution of problem (1.2), it suffices to study

\[
\begin{aligned}
    u_t &= a(t) u_{xx}, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T, \\
    u(0,t) &= 0, \quad 0 \leq t \leq T, \\
    u(1,t) &= \psi(t), \quad 0 \leq t \leq T, \quad \psi(0) = 0, \\
    u(x,0) &= F(x), \quad 0 \leq x \leq 1, \quad F(0) = F(1) = 0, \\
    a(t) \lim_{x \downarrow 0} u_x(x,t) &= h(t) + c a(t), \quad 0 \leq t \leq T,
\end{aligned}
\]

(1.4)

where \( \psi(t) \), \( F(x) \) and \( h(t) \) are known continuous functions of their arguments, and \( c \) is a given constant.

**Definition 1.3:** A pair of functions \( u(x,t) \) and \( a(t) \) is a solution of (1.4) if and only if conditions (a.), (b.), (c.), and (d.) of Definition 1.1 hold, and (1.4) is satisfied.

For the remainder of this section assume that \( F(x) \) is three times continuously differentiable. Assume also that problem (1.4) possesses a solution. Consider the transformation

\[
\begin{aligned}
    \theta(t) &= \int_0^t a(y) \, dy, \quad 0 \leq t \leq T. \\
    \theta'(t) &= a(t), \quad 0 \leq t \leq T,
\end{aligned}
\]

(1.5)

As

\[
\begin{aligned}
    \theta'(\theta(t)) &= a(t), \quad 0 \leq t \leq T, \\
    \theta(\theta(t)) &= t, \quad 0 \leq t \leq T; \\
    \theta'(\theta(t)) &= \frac{1}{\theta'(\theta(t))} = a(\theta(t)), \quad 0 \leq t \leq T.
\end{aligned}
\]

(1.6)

Let
(1.10) \[ U(x, \eta) = u(x, \theta(\eta)), \]
(1.11) \[ A(\eta) = a(\theta(\eta)), \]
(1.12) \[ \overline{\psi}(\eta) = \psi(\theta(\eta)), \]
and
(1.13) \[ H(\eta) = h(\theta(\eta)). \]

Note that

(1.14) \[ \frac{\partial U}{\partial \eta}(x, \eta) = \frac{1}{a(\theta(\eta))} \frac{\partial u}{\partial t}(x, \theta(\eta)). \]

Hence, \( U(x, \eta) \) satisfies

\[
\begin{cases}
U_\eta = U_{xx}, & 0 < x < 1, \quad 0 < \eta < \theta(T), \\
U(0, \eta) = 0, & 0 \leq \eta \leq \theta(T), \\
U(1, \eta) = \overline{\psi}(\eta), & 0 < \eta < \theta(T), \quad \overline{\psi}(0) = 0, \\
U(x, 0) = F(x), & 0 \leq x \leq 1, \quad F(0) = F(1) = 0, \\
A(\eta) \overline{\psi}(0, \eta) = H(\eta) + cA(\eta), & 0 < \eta < \theta(T).
\end{cases}
\]

From the fact that \( U(x, \eta) \) satisfies (1.15), a representation for \( u(x, t) \) in terms of \( a(t) \) and known functions will be derived.

It is well known \([2,4]\) that

(1.16) \[ U(x, \eta) = \sum_{k=1}^{\infty} A_k \exp \left\{ -k^2 \pi^2 \eta^2 \right\} \sin k \pi x 
\]
\[ + \int_0^{\eta} \frac{\partial M(x - \xi, \eta - \tau)}{\partial x} \overline{\psi}(\tau) \, d\tau, \]

where

(1.17) \[ A_k = 2 \int_0^1 F(x) \sin k \pi x \, dx \]
and

\[ M(x - \frac{3}{2}, \eta - \tau) = \frac{1}{\sqrt{\pi(\eta - \tau)}} \sum_{k=-\infty}^{+\infty} \exp \left\{ \frac{-(x - \frac{3}{2} + 2k)^2}{4(\eta - \tau)} \right\}, \eta > \tau, \]

the Green's function for the region \( 0 \leq x \leq 1, \ 0 < \eta < \Theta(T) \).

Hence, by (1.10),

\[ u(x, \Theta(\eta)) = \sum_{k=1}^{\infty} A_k \exp \left\{ -k^2 \eta^2 \right\} \sin k \pi x \]

\[ + \int_0^\eta \frac{\partial M(x - \frac{1}{2}, \eta - \tau)}{\partial x} \psi(\tau) \ d\tau \]

for \( 0 \leq \eta < \Theta(T) \). Setting \( \eta = \Theta(t) \), it follows that for \( 0 \leq t < T \),

\[ u(x, t) = \sum_{k=1}^{\infty} A_k \exp \left\{ -k^2 \Theta(t)^2 \right\} \sin k \pi x \]

\[ + \int_0^{\Theta(t)} \frac{\partial M(x - \frac{1}{2}, \Theta(t) - \tau)}{\partial x} \psi(\tau) \ d\tau. \]

Making the substitution \( \tau = \Theta(y) \), relabeling the dummy variable \( y \) as \( \tau \), and noting (1.5), it follows that

\[ u(x, t) = \sum_{k=1}^{\infty} A_k \exp \left\{ -k^2 \Theta(t)^2 \int_0^t a(y) \ dy \right\} \sin k \pi x \]

\[ + \int_0^t \frac{\partial M(x - \frac{1}{2}, \int_0^t a(y) \ dy)}{\partial x} \psi(\tau) a(\tau) \ d\tau. \]
Using this representation for \( u(x,t) \), an integral equation for \( a(t) \) will be derived.

From the continuity of \( a(t) \), the differentiability of \( F(x) \), and the properties of \( M(x - \xi, \gamma - \tau) \) \([2,4]\), it follows that for \( 0 < x < 1 \),

\[
(1.22) \quad u_x(x,t) = \sum_{k=1}^{\infty} k \xi A_k \exp \left\{ -k^2 \xi^2 \int_0^t a(y) \, dy \right\} \cos k \pi x \\
+ \int_0^t \frac{\partial^2 M(x - 1, \int_\tau^t a(y) \, dy)}{\partial x^2} \psi(\tau) a(\tau) \, d\tau.
\]

Assume that \( \psi(t) \) is continuously differentiable for \( 0 \leq t \leq T \). Then,

\[
(1.23) \quad \int_0^t \frac{\partial^2 M(x-1, \int_\tau^t a(y) \, dy)}{\partial x^2} \psi(\tau) a(\tau) \, d\tau =
\]

\[
= - \int_0^t M(x - 1, \int_\tau^t a(y) \, dy) \psi(\tau) \, d\tau
\]

\[
= \int_0^t M(x - 1, \int_\tau^t a(y) \, dy) \psi'(\tau) \, d\tau
\]

since

\[
(1.24) \quad \frac{\partial^2 M(x - 1, \int_\tau^t a(y) \, dy)}{\partial x^2} = - \frac{1}{a(\tau)} \frac{\partial M(x - 1, \int_\tau^t a(y) \, dy)}{\partial \tau},
\]
(1.25) \[ \lim_{x \to 1, \tau \to t} M(x - 1, \int_0^\tau a(y) \, dy) = 0, \quad x \neq 1, \]

and \( \psi(0) = 0 \). Thus, for \( 0 < x < 1, 0 \leq t \leq \tau \),

\[
(1.26) \quad u_x(x, t) = \sum_{k=1}^\infty k^2 A_k \exp \left\{ -k^2 x^2 \int_0^t a(y) \, dy \right\} \cos k x + \int_0^t M(x - 1, \int_\tau^t a(y) \, dy) \psi'(\tau) \, d\tau.
\]

By Lebesgue's dominated convergence theorem,

\[
(1.27) \quad \lim_{x \downarrow 0} u_x(x, t) = \sum_{k=1}^\infty k^2 A_k \exp \left\{ -k^2 x^2 \int_0^t a(y) \, dy \right\} + \int_0^t M(-1, \int_\tau^t a(y) \, dy) \psi'(\tau) \, d\tau.
\]

But, from (1.4),

\[
(1.28) \quad \lim_{x \downarrow 0} u_x(x, t) = \frac{h(t)}{a(t)} + c.
\]

Thus, \( a(t) \) satisfies the non-linear integral equation

\[
(1.29) \quad \frac{h(t)}{a(t)} + c = \sum_{k=1}^\infty k^2 A_k \exp \left\{ -k^2 x^2 \int_0^t a(y) \, dy \right\} + \int_0^t M(-1, \int_\tau^t a(y) \, dy) \psi'(\tau) \, d\tau, \quad 0 < t < T.
\]
Therefore, if (1.4) possesses a solution, $F(x)$ is three times continuously differentiable and $\psi(t)$ is continuously differentiable for $0 \leq t \leq T$, then the conductivity $a(t)$ must satisfy (1.29).

**Theorem 1.2:** If $F(x)$ is three times continuously differentiable and $\psi(t)$ is continuously differentiable for $0 \leq t \leq T$, then problem (1.4) possesses a unique solution if and only if for $0 \leq t \leq T$, (1.29) possesses a unique, positive, continuous solution $a(t)$ which is also positive and continuous for $t = 0$.

**Proof:** Assume that problem (1.4) possesses a unique solution.

Then, by previous analysis, (1.29) possesses a positive continuous solution $a(t)$ which is also positive and continuous for $t = 0$.

Assert that this solution is the only positive continuous solution of (1.29) which is continuous for $t = 0$. Suppose the contrary, i.e., that $a_i(t)$, $i = 1, 2$, are two positive continuous solutions of (1.29) which are positive and continuous for $t = 0$. Set

\[
(1.30) \quad u_i(x, t) = \sum_{k=1}^{\infty} A_k \exp \left\{ -k^2 x^2 \int_0^t a_i(y) \, dy \right\} \sin k \pi x
\]

\[
+ \int_0^t \frac{\partial M(x - l, \int_0^t a_i(y) \, dy)}{\partial x} \frac{\partial^2}{\partial x^2} \psi(\tau) a_i(\tau) \, d\tau,
\]

$i = 1, 2$.

From the properties of Green's function and the hypothesis on $F(x)$, it follows that

\[
(1.31) \quad \frac{\partial u_i}{\partial t} = a_i(t) \frac{\partial^2 u_i}{\partial x^2}, \quad 0 < x < l, \quad 0 < t < T, \quad i = 1, 2,
\]

\[
(1.32) \quad u_i(x, 0) = F(x), \quad 0 < x < l, \quad i = 1, 2,
\]

and

\[
(1.33) \quad u_i(0, t) = 0, \quad 0 < t < T, \quad i = 1, 2.
\]
At this point introduce the time transformations

\[ (1.34) \quad \Theta_i(t) = \int_0^t a_i(y) \, dy, \quad i = 1, 2, \quad 0 \leq t \leq T, \]

and their unique inverses \( \varphi_i(\Theta_i) \), \( i = 1, 2 \).

Using \( \Theta_i(T) \) as the variable of integration, it follows that

\[ (1.35) \quad u_i(x, t) = \sum_{k=1}^{\infty} A_k \exp \left\{ \int_0^t a_i(y) \, dy \right\} \sin k \pi x \]

\[ + \int_0^{\Theta_i(t)} \frac{\delta M(x - 1, \varphi_i(\Theta_i) - \Theta_i(T))}{\delta x} \psi(\varphi_i(\Theta_i(T))) \, d\Theta_i(T), \quad i = 1, 2. \]

Hence, by a well-known argument [4],

\[ (1.36) \quad \lim_{x \uparrow 1} u_i(x, t) = \psi(t), \quad i = 1, 2. \]

Finally, since \( a_i(t) \) are solutions of (1.29), it follows immediately that

\[ (1.37) \quad \lim_{x \downarrow 0} \frac{\partial u}{\partial x} i(x, t) = \frac{h(t)}{a_i(t)} + c, \quad i = 1, 2. \]

Hence, problem (1.4) has two solutions. This contradicts the hypothesis that it possesses only one solution. Therefore, if problem (1.4) possesses a unique solution, then (1.29) possesses exactly one positive continuous solution \( a(t) \) which is also positive and continuous for \( t = 0 \).

Assume now that (1.29) possesses exactly one positive continuous solution \( a(t) \) which is also positive and continuous for \( t = 0 \). By the analysis of the previous argument, it follows that problem (1.4) possesses a solution. Using an argument similar to the one above with the analysis which precedes the statement of this theorem, it follows that problem (1.4) possesses a unique solution.
By Theorem 1.2 in order to obtain a solution of (1.4), it suffices to consider the non-linear integral equation (1.29) for \(a(t)\). The remainder of this article is devoted to its study.

2. Some basic inequalities. In order to study the properties of a solution of (1.29), some bounds for

\[
(2.1) \quad M(-1, \sigma) = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{\sigma}} \exp \left\{ -\frac{(2k-1)^2}{4\sigma} \right\}, \sigma > 0,
\]

and

\[
(2.2) \quad N_\sigma(-1, \sigma) = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \left\{ -\frac{1}{\sigma^{3/2}} + \frac{(2k-1)^2}{2\sigma^{5/2}} \right\} \exp \left\{ -\frac{(2k-1)^2}{4\sigma} \right\}, \sigma > 0.
\]

are needed. Some useful estimates are given in the following theorem.

**Theorem 2.1:** For all \( \sigma > 0 \),

\[
(2.3) \quad 0 \leq M(-1, \sigma) \leq 2,
\]

and

\[
(2.4) \quad |M_\sigma(-1, \sigma)| \leq K,
\]

where

\[
(2.5) \quad K = 10^{5/2} \exp \left\{ -\frac{5}{2} \right\} + 2 \cdot 6^{3/2} \exp \left\{ -\frac{3}{2} \right\}.
\]

**Proof:** As

\[
(2.6) \quad M(-1, \sigma) = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{\sigma}} \exp \left\{ -\frac{(2k-1)^2}{4\sigma} \right\}
\]

\[
\leq \frac{2}{\sqrt{\pi} \sigma} \int_{0}^{\infty} \exp \left\{ -\frac{x^2}{4\sigma} \right\} dx,
\]

changing the variable of integration by the relation \( \eta = \frac{x}{2\sigma^{1/2}} \).
(2.7) \[ M(-1, \sigma') \leq \frac{4}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\eta^2} d\eta = 2. \]

Thus, (2.3) is valid.

To estimate the convergence of \( M_\sigma (-1, \sigma') \), it suffices to consider the two series

(2.8) \[ \sum_{1} = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{-1}{\sigma^{3/2}} \exp \left\{ \frac{-\left(2k - 1\right)^2}{4 \sigma} \right\} \]

and

(2.9) \[ \sum_{2} = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{\left(2k - 1\right)^2}{2 \sigma^{5/2}} \exp \left\{ \frac{-\left(2k - 1\right)^2}{4 \sigma} \right\}. \]

Since the function

(2.10) \[ F_1(\sigma') = \frac{1}{\sigma^{3/2}} \exp \left\{ \frac{-\left(2k - 1\right)^2}{4 \sigma} \right\} , \sigma' > 0, \]

has its maximum at

(2.11) \[ \sigma = \frac{(2k - 1)^2}{6}, \]

it follows that

(2.12) \[ F_1(\sigma') \leq \frac{6^{3/2} \exp\left\{ -\frac{3}{2} \right\}}{(2k - 1)^3}, \sigma' > 0. \]

Hence,

(2.13) \[ \left| \sum_{1} \right| \leq 6^{3/2} \exp\left\{ -\frac{3}{2} \right\} \sum_{k=1}^{\infty} \frac{1}{(2k - 1)^3} \]

\[ \leq 6^{3/2} \exp\left\{ -\frac{3}{2} \right\} \left[ 1 + \int_{1}^{\infty} \frac{1}{x^3} dx \right] \]

\[ = 2 \cdot 6^{3/2} \exp\left\{ -\frac{3}{2} \right\} . \]
As
\[(2.14) \quad F_2(\sigma') = \frac{(2k - 1)^2}{2\sigma^{5/2}} \exp \left\{\frac{-(2k - 1)^2}{4\sigma} \right\}, \quad \sigma > 0,\]
has its maximum at
\[(2.15) \quad \sigma = \frac{(2k - 1)^2}{10},\]
it follows that
\[(2.16) \quad F_2(\sigma') \leq \frac{10^{5/2} \exp \left\{\frac{5}{2} \right\}}{2(2k - 1)^{3/2}}, \quad \sigma > 0.\]
Therefore
\[(2.17) \quad |\sum_{k=1}^{\infty} \frac{1}{(2k - 1)^{3/2}}| \leq 10^{5/2} \exp \left\{\frac{5}{2} \right\} \sum_{k=1}^{\infty} \frac{1}{(2k - 1)^{3/2}} \leq 10^{5/2} \exp \left\{-\frac{5}{2} \right\}.\]
Thus, for all \(\sigma' > 0\),
\[(2.18) \quad |M_{\sigma'}(-1, \sigma')| \leq 10^{5/2} \exp \left\{-\frac{5}{2} \right\} + 2 \cdot 6^{1/2} \exp \left\{-\frac{3}{2} \right\}.\]
Therefore, (2.4) is valid.

3. Existence and uniqueness. The following conditions on \(F(x)\), \(\psi(t)\) and \(h(t)\) lead to an existence and uniqueness theorem for (1.29):
\[(3.1) \quad \begin{cases} 
(a.) \quad F(x) \neq 0, \text{ } F(x) \text{ is five times continuously differentiable} \\
\quad \text{with } F(0) = F(1) = F'(0) = F''(1) = 0, \text{ } F'(0) > c; \\
(b.) \quad \psi(t) \text{ is continuously differentiable for } 0 \leq t < \infty; \\
(c.) \quad h(t) \text{ is positive and continuously differentiable for } 0 \leq t < \infty.
\end{cases}\]
Unless it is otherwise specified, conditions (3.1) are assumed to hold throughout this section. Assume that (1.29) possesses a positive continuous solution $a(t)$ which is also positive and continuous for $t = 0$. Since (1.29) can be written in the form

\[(3.2) \quad a(t) = h(t) \left[ \sum_{k=1}^{\infty} k^2 A_k \exp \left\{ -k^2 t^2 \int_0^t a(y) \, dy \right\} - c \right. \]

\[\left. + \int_0^t M(-1, \int_\tau^t a(y) \, dy) \psi'(\tau) \, d\tau \right]^{-1}, \]

it follows that

\[(3.3) \quad a(0) = \frac{h(0)}{F'(0)} - c. \]

Let

\[(3.4) \quad R(T, B) = \left\{ (t, a) \mid 0 \leq t \leq T, \quad 0 \leq \frac{1}{B} \leq a \leq B \right\}. \]

Set

\[(3.5) \quad A = \frac{h(0)}{F'(0)} - c, \]

and take $B$ sufficiently large that $\frac{1}{B} < A < B$.

Since $h(t)$ is positive and continuously differentiable, both sides of (1.29) can be differentiated, and

\[(3.6) \quad a'(t) = a(t) \frac{h'(t)}{h(t)} + \left[ \frac{a(t)}{h(t)} \right]^3 \sum_{k=1}^{\infty} (k^2)^3 A_k \exp \left\{ -k^2 t^2 \int_0^t a(y) \, dy \right\}
\]

\[\left. - \frac{[a(t)]^2}{h(t)^2} \int_0^t M_t(-1, \int_\tau^t a(y) \, dy) \psi'(\tau) \, d\tau \right. \]

As

\[(3.7) \quad M_t(-1, \int_\tau^t a(y) \, dy) = a(t) M_\sigma(-1, \int_\tau^t a(y) \, dy), \]
it follows that as long as the graph of \( a(t) \) remains in \( R(T, B) \),

\[
(3.8) \quad |a'(t)| \leq B \frac{H'_*}{H_*} + \frac{B^3}{H_*} \left( \sum_{k=1}^{\infty} (k^3)^3 |A_k| + K \Psi^* T \right),
\]

\[
(3.9) \quad \Psi^* = \sup_{0 \leq t \leq T} |\Psi(t)|,
\]

\[
(3.10) \quad H'_* = \sup_{0 \leq t \leq T} |h'(t)|,
\]

and

\[
(3.11) \quad H_* = \inf_{0 \leq t \leq T} h(t).
\]

Let

\[
(3.12) \quad E = B \frac{H'_*}{H_*} + \frac{B^3}{H_*} \left( \sum_{k=1}^{\infty} (k^3)^3 |A_k| + K \Psi^* T \right).
\]

Let \( r(0, E, A) \) denote the subregion of \( R(T, B) \) which is bounded by the lines \( a = Et + A, \ a = -Et + A, \) and \( t = t_* \), where

\[
(3.13) \quad t_* = \min \left\{ \frac{B - A}{E}, \ \frac{A - \frac{1}{B}}{E}, \ T \right\}.
\]
By an argument similar to the one usually given for the initial value problem of first order ordinary differential equations, it follows that if \( a(t) \) does exist, then its graph must lie in \( r(0, B, A) \) for \( 0 \leq t \leq t^*_c \).

Let \( \mathcal{F} \) denote the family of all positive continuous functions \( g(t) \) whose graphs for \( 0 \leq t \leq t^*_c \) lie in \( r(0, B, A) \). Note that \( g(0) = A \) for all such functions. Define \( Lg(t) \) by the relation

\[
(3.14) \quad Lg(t) = h(t) \left[ \sum_{k=1}^{\infty} k^3 A_k \exp \left\{ -k^2 \int_0^t g(y) \, dy \right\} - c \right. \\
\left. + \int_0^t M(-1, \int_t^y g(y) \, dy) \psi'(\tau) \, d\tau \right]^{-1}
\]

It follows that for all \( g(t) \) which are members of \( \mathcal{F} \), \( Lg(t) \) is a positive continuously differentiable function for \( 0 \leq t < \epsilon \), where \( \epsilon \) is a sufficiently small number. Differentiating \( Lg(t) \) for \( 0 \leq t < \epsilon \),

\[
(3.15) \quad (Lg(t))' = Lg(t) \frac{h'(t)}{h(t)} + \frac{[Lg(t)]^2 g(t)}{h(t)}.
\]

\[
\sum_{k=1}^{\infty} k^3 A_k \exp \left\{ -k^2 \int_0^t g(y) \, dy \right\}
\]

\[
- \left\{ \frac{Lg(t)}{h(t)} \right\} \int_0^t M(-1, \int_t^y g(y) \, dy) \psi'(\tau) \, d\tau.
\]

Hence, as long as the graph of \( Lg(t) \) stays inside \( R(T, B) \) and \( 0 \leq t \leq t^*_c \),

\[
(3.16) \quad |(Lg(t))'| \leq B \frac{H^1_{t^*_c}}{H^1} + \frac{B^3}{H^*} \left( \sum_{k=1}^{\infty} k^3 |A_k| + K \tilde{\psi}^* T \right) = E.
\]
By an argument similar to the one for \( a(t) \), it follows that \( L^n g(t) \) is a member of \( \mathcal{F} \) if \( g(t) \) is a member of \( \mathcal{F} \). Thus, if \( g(t) \) is a member of \( \mathcal{F} \), then

\[
(3.17) \quad L^n g(t) = h(t) \left[ \sum_{k=1}^{\infty} k^x A_k \exp \left\{ -k^2 x^2 \int_0^t L^{n-1} g(y) \, dy \right\} \right.
- \left. c + \int_0^t M(-, \int_0^\tau L^{n-1} g(y) \, dy) \psi' (\tau) \, d\tau \right]^{-1}
\]

is a member of \( \mathcal{F} \).

Since members of \( \mathcal{F} \) are bounded away from zero, (3.17) can be written in the form

\[
(3.18) \quad \frac{h(t)}{L^n g(t)} + c = \sum_{k=1}^{\infty} k^x A_k \exp \left\{ -k^2 x^2 \int_0^t L^{n-1} g(y) \, dy \right\}
+ \int_0^t M(-, \int_0^\tau L^{n-1} g(y) \, dy) \psi' (\tau) \, d\tau.
\]

Subtracting the similar expression for \( L^{n+1} g(t) \) from (3.18),

\[
(3.19) \quad \frac{h(t)}{L^n g(t)} - \frac{h(t)}{L^{n+1} g(t)} = \sum_{k=1}^{\infty} k^x A_k \left\{ \exp \left\{ -k^2 x^2 \int_0^t L^{n-1} g(y) \, dy \right\} \right.
- \left. \exp \left\{ -k^2 x^2 \int_0^t L^n g(y) \, dy \right\} \right\}
+ \int_0^t \left[ M(-, \int_0^\tau L^{n-1} g(y) \, dy) - M(-, \int_0^\tau L^n g(y) \, dy) \right] \psi' (\tau) \, d\tau.
\]

Thus,
\begin{align}
(3.20) \quad L^{n+1}_g(t) - L^n_g(t) &= \frac{L^{n+1}_g(t)L^n_g(t)}{h(t)} \left\{ \sum_{k=1}^{\infty} k^3 \mu_k \exp\left\{-k^2 \mu_k \right\} \right\} \\
&\quad \cdot \int_0^t \left\{ L^{n-1}_g(y) - L^n_g(y) \right\} dy \\
&\quad + \frac{L^{n+1}_g(t)L^n_g(t)}{h(t)} \int_0^t M_n(-1, \xi(\tau))(L^{n-1}_g(y) - L^n_g(y)) dy \\
&\quad \psi'_{\xi}(\tau) d\tau,
\end{align}

where \( \mu_k, \xi(\tau) \) are the positive numbers called for by the mean value theorem. Therefore, it follows that

\begin{align}
(3.21) \quad ||L^{n+1}_g - L^n_g||_t &\leq \frac{B^2}{H_*} \left\{ \sum_{k=1}^{\infty} k^3 |A_k| + \\
&\quad + K \psi_{\xi}^{\ast T} \right\} \int_0^t ||L^n_g - L^{n-1}_g||_\tau d\tau,
\end{align}

where \( ||f||_t \) for any continuous function \( f(\tau) \) defined on \( 0 \leq \tau \leq t \) is given by the relation

\begin{align}
(3.22) \quad ||f||_t &= \sup_{0 \leq \tau \leq t} |f(\tau)|.
\end{align}

By induction, it follows that

\begin{align}
(3.23) \quad ||L^{n+1}_g - L^n_g||_t &\leq 2 \frac{B^{n+1}}{H_*^n} \left\{ \sum_{k=1}^{\infty} k^3 |A_k| + K \psi_{\xi}^{\ast T} \right\}^n \frac{t^n}{n!}
\end{align}

Hence, the sequence \( \{L^n_g(t)\} \) converges uniformly to a positive continuous function \( a(t) \) which is a member of \( \mathcal{F} \). Clearly, \( a(t) \) is a positive continuous solution of (1.29) for \( 0 < t < t_* \), which is also positive and continuous for \( t = 0 \). Also, \( a(t) \) is the only positive continuous solution of (1.29) for \( 0 < t < t_* \) which is positive and continuous for \( t = 0 \).
If \( b(t) \) were another positive continuous solution of (1.29) which were positive and continuous for \( t = 0 \), then \( b(t) \) would be a member of \( \mathcal{F} \) and

\[
(3.24) \quad \|a - b\|_t \leq \frac{B^2}{k_{\eta}} \left\{ \sum_{k=1}^{\infty} k^3 x^3 |A_k| + K \Psi^* T \right\} \int_0^t \|a - b\|_t \, dt.
\]

This implies that \( a(t) \equiv b(t) \) for \( 0 \leq t \leq t_* \).

By methods similar to those for first order ordinary differential equations, the solution \( a(t) \) can be continued uniquely across \( R(T,B) \). As \( T \) and \( B \) can be arbitrarily large, the following result is valid.

**Theorem 3.1:** If \( F(x), \Psi(t), \) and \( h(t) \) satisfy conditions (3.1), then there exists a positive time \( \mathcal{T} = \mathcal{F}(F, \Psi, h) \leq +\infty \) such that for \( 0 < t < \mathcal{T} \), (1.29) possesses exactly one positive continuous solution \( a(t) \) which is positive and continuous for \( t = 0 \).

**Remark 1:** An example of a finite \( \mathcal{T} \) is given by the solution

\[
(3.25) \quad a(t) = \frac{1}{(\frac{x}{h_0} + x^2 t)}
\]

of

\[
(3.26) \quad \begin{cases}
  u_t = a(t) u_{xx}, & 0 \leq x \leq l, \quad 0 \leq t < T,
  
  u(0,t) = u(l,t) = 0, & 0 \leq t < T,
  
  u(x,0) = \sin \pi x, & 0 \leq x \leq l,
  
  a(t) u_x(0,t) = h_0, & 0 \leq t < T,
\end{cases}
\]

where \( h_0 \) is a positive constant. Note that \( a(t) \) tends to infinity as \( t \) tends to infinity as \( h_0 \) is a positive constant. Note that \( a(t) \) tends to infinity as \( t \) tends to \( (x_h)^{-1} \).

**Remark 2:** The solution \( a(t) \) can never tend to zero as \( t \) tends to a finite positive number \( t_0 < \mathcal{T} \) since

\[
(3.27) \quad \lim_{t \to t_0} \frac{h(t)}{a(t)} + c = \infty
\]

and
\begin{equation}(3.28) \lim_{t \to t_0^-} \left| \sum_{k=1}^{\infty} k^x A_k \exp \left\{ -k^2x^2 \int_0^t a(y) \, dy \right\} \right. \\
\left. + \int_0^t M(-1, \int_0^{\tau} a(y) \, dy) \psi'(\tau) \, d\tau \right| \leq Q, \end{equation}

where \( Q \) is a positive constant.

**Remark 3:** Theorem 3.3 could be demonstrated for (3.1) - (b), (c) holding on \( 0 \leq t \leq T_1 \) with the corresponding change in the conclusion of \( \mathcal{J} \leq T_1 \).

Since the coefficient \( a(t) \) is the important part of the solution of (1.2), the next section deals with the stability of \( a(t) \) with respect to the data and a numerical procedure for approximating \( a(t) \).

4. **Stability.** For calculational purposes it is always desirable to know that the solution to a given problem depends continuously upon the data.

**Theorem 4.1:** If \( F_i(x), \psi_i(t), h_i(t), c_i, i = 1, 2 \), satisfy condition (3.1) and \( a_1(t) \) and \( a_2(t) \) are the corresponding solutions on \( 0 \leq t \leq T \) of (1.29), then

\begin{equation}(4.1) \quad ||a_2 - a_1||_T \leq \left\{ \frac{||a_1||_T}{H^*_x} ||h_2 - h_1||_T + \frac{||a_1||_T||a_2||_T}{H^*_x} ||c_2 - c_1||_T \\
+ \frac{||a_1||_T||a_2||_T}{H^*_x} \left( MN(N+1) \right) ||F_1 - F_2||_1 + 2T \frac{||a_1||_T||a_2||_T}{\mu^*_x} \right. \\
\left. \cdot ||\psi_1' - \psi_2'||_T \\
+ \frac{||a_1||_T||a_2||_T}{H^*_x} \left( \sum_{k=N+1}^{\infty} \left( k^x \left[ |A_k^{(1)}| + |A_k^{(2)}| \right] \right) \right) \right\} \end{equation}
where \( H_\alpha = \inf_{[0,T]} h_1(t) \), \( N \) is a positive integer, and \( A_k(i) \) are the Fourier sine coefficients of \( F_1(x) \), \( i = 1,2 \).

Proof: Since \( a_1(t), i = 1,2 \) satisfy

\[
(4.2) \quad \frac{h_1(t)}{a_1(t)} + c_i = \sum_{k=1}^{\infty} k^2 \, A_k(i) \exp \left\{ -k^2 \pi^2 \int_{0}^{t} a_1(y) \, dy \right\} \\
+ \int_{0}^{t} M(-1, \int_{0}^{t} a_1(y) \, dy) \, \psi_1(\tau) \, d\tau , \quad i = 1,2,
\]

where

\[
(4.3) \quad A_k(i) = 2 \int_{0}^{1} F_1(x) \sin k\pi x \, dx , \quad i = 1,2,
\]

for \( 0 \leq t \leq T \), it follows that

\[
(4.4) \quad a_2(t) - a_1(t) = \frac{a_1(t)}{h_1(t)} \left\{ h_2(t) - h_1(t) \right\} + \frac{a_1(t)a_2(t)}{h_1(t)} \left\{ c_2 - c_1 \right\} \\
+ \frac{a_1(t)a_2(t)}{h_1(t)} \sum_{k=1}^{N} k^2 \left[ A_k(1) - A_k(2) \right] \exp \left\{ -k^2 \pi^2 \int_{0}^{t} a_1(y) \, dy \right\} \\
+ \frac{a_1(t)a_2(t)}{h_1(t)} \sum_{k=1}^{N} k^2 A_k(2) \left[ \exp \left\{ -k^2 \pi^2 \int_{0}^{t} a_1(y) \, dy \right\} \\
- \exp \left\{ -k^2 \pi^2 \int_{0}^{t} a_2(y) \, dy \right\} \right] \\
+ \frac{a_1(t)a_2(t)}{h_1(t)} \sum_{k=N+1}^{\infty} k^2 A_k(1) \exp \left\{ -k^2 \pi^2 \int_{0}^{t} a_1(y) \, dy \right\}
\]
\[- \frac{a_1(t)a_2(t)}{h_1(t)} \sum_{k=N+1}^{\infty} k^2 A_k^{(2)} \exp \left\{ -k^2 \int_0^t a_2(y) \, dy \right\} \]
\[+ \frac{a_1(t)a_2(t)}{h_1(t)} \int_0^t M(-1, \int_0^\tau a_1(y) \, dy) \psi'_1(\tau) \, d\tau \]
\[- \frac{a_1(t)a_2(t)}{h_1(t)} \int_0^t M(-1, \int_0^\tau a_2(y) \, dy) \psi'_2(\tau) \, d\tau \]

Thus,

\[(4.5) \quad |a_2(t) - a_1(t)| \leq \frac{||a_1||_T}{H_*} ||h_2 - h_1||_T + \frac{||a_1||_T ||a_2||_T}{H_*} |c_2 - c_1| \]
\[+ \frac{||a_1||_T ||a_2||_T}{H_*} \left( N(N+1) \right) ||P_1 - P_2||_1 \]
\[+ \frac{||a_1||_T ||a_2||_T}{H_*} \left( \sum_{k=N+1}^{\infty} k^2 |A_k^{(1)}| + \sum_{k=N+1}^{\infty} k^2 |A_k^{(2)}| \right) \]
\[+ \frac{||a_1||_T ||a_2||_T}{H_*} \left( 2T ||\psi'_1 - \psi'_2||_T \right) \]
\[+ \frac{||a_1||_T ||a_2||_T}{H_*} \left( \sum_{k=1}^{\infty} (k^3)^3 |A_k^{(2)}| \right) \]
\[+ K ||\psi'_1||_T T \int_0^T ||a_2 - a_1||_T \, d\tau \]

By the monotonicity of the right side of (4.5),
\[(4.6) \quad \|a_2 - a_1\|_T \leq \frac{\|a_1\|_T}{\bar{a}_*} \|h_2 - h_1\|_T + \frac{\|a_1\|_T \|a_2\|_T}{\bar{h}_*} \|c_2 - c_1\|_T \]
\[\quad + \kappa \frac{\|a_1\|_T \|a_2\|_T}{\bar{H}_*} N(N + 1) \|P_1 - P_2\|_1 \]
\[\quad + \frac{\|a_1\|_T \|a_2\|_T}{\bar{H}_*} \left( \sum_{k=N+1}^{\infty} k^3 |A_k^{(1)}| + \sum_{k=N+1}^{\infty} k \|\psi_1\|_T \right) \]
\[\quad + 2T \left( \frac{\|a_1\|_T \|a_2\|_T}{\bar{H}_*} \right) \|\psi_1' - \psi_2'\|_T \]
\[\quad + \frac{\|a_1\|_T \|a_2\|_T}{\bar{H}_*} \left( \sum_{k=1}^{\infty} k^3 |A_k^{(2)}| + \kappa \|\psi_1\|_T \right) \int_0^t \|a_2 - a_1\|_T \, dt. \]

Hence, it follows from Gronwall's lemma that (4.1) holds.

**Remark 4:** If $F(x), \psi(t), h(t), c, F_n(x), \psi_n(t), h_n(t)$ and $c_n, \ n = 1, 2, \ldots$, satisfy condition (3.1)

\[
\begin{align*}
(a.) & \quad F_n(x) \rightarrow F(x) \text{ uniformly as } n \rightarrow \infty \\
(b.) & \quad \psi_n(t) \rightarrow \psi(t), \ \psi_n'(t) \rightarrow \psi'(t) \text{ uniformly on compact intervals as } n \rightarrow \infty, \\
(c.) & \quad h_n(t) \rightarrow h(t), \ h_n'(t) \rightarrow h'(t) \text{ uniformly on compact intervals as } n \rightarrow \infty, \\
(d.) & \quad c_n \rightarrow c \text{ as } n \rightarrow \infty, \\
(e.) & \quad \left\{ \frac{d^5 F_n(x)}{dx^5} \right\} \text{ is uniformly bounded},
\end{align*}
\]

and $a(t)$ and $a_n(t), \ n = 1, 2, \ldots$, are, respectively, the corresponding solutions of (1.29), then, $a_n(t) \rightarrow a(t) \text{ uniformly on compact intervals that are contained in the interval of definition of } a(t) \text{ as } n \rightarrow \infty.$
Remark 5: If \( f(x) \), \( \psi(t) \), \( h(t) \), \( \varphi_0 \), \( f_n(x) \), \( \psi_n(t) \), \( h_n(t) \), and \( \varphi_0(n) \), \( n = 1, 2, \ldots \), satisfy condition (3.1) [see (1.3)],

\[
\begin{align*}
(a.) & \quad f_n(x) \to f(x) \text{ uniformly as } n \to \infty, \\
(b.) & \quad \psi_n(t) \to \psi(t), \quad \psi'_n(t) \to \psi'(t) \text{ uniformly on compact intervals as } n \to \infty, \\
(c.) & \quad h_n(t) \to h(t), \quad h'_n(t) \to h'(t) \text{ uniformly on compact intervals as } n \to \infty, \\
(d.) & \quad \varphi_0(n) \to \varphi_0 \text{ as } n \to \infty \\
(e.) & \quad \left\{ \frac{d^5 f_n(x)}{dx^5} \right\} \text{ is uniformly bounded.}
\end{align*}
\]

and if \( a(t) \) and \( a_n(t) \), \( n = 1, 2, \ldots \), are, respectively, the corresponding solutions of (1.2), then \( a_n(t) \to a(t) \) uniformly on compact intervals that are contained in the interval of definition of \( a(t) \) as \( n \to \infty \).

Note that in the case that \( f(x) \) is seven times continuously differentiable with \( f(0) = \varphi_0 \), \( f(1) = \psi(0) \), \( f''(0) = f''(1) = f^{iv}(0) = f^{iv}(1) = 0 \), the sequence of functions \( \{ F_N(x) \} \), which are truncations of the Fourier sine expansion for \( F(x) = f(x) - \left\{ \varphi_0[1 - x] + \psi(0)x \right\} \), can be used as an approximating sequence for \( F(x) \), since the fifth derivatives of the functions \( F_N(x) \) are bounded uniformly. In this case, the task of approximating \( a(t) \) reduces to integrating numerically the ordinary first order integro-differential equation.
(4.9) \[ a_N'(t) = a_N(t) \frac{h'(t)}{h(t)} + \frac{[a_N(t)]^3}{h(t)} \sum_{k=1}^{N} (k^3) A_k \exp \left\{ -k^2 \int_0^t a_N(y) \, dy \right\} \]

\[ - \frac{[a_N(t)]^2}{n(t)} \int_0^t M_t(-1, \int_0^t a_N(y) \, dy) \psi'(\tau) \, d\tau , \ t > 0 , \]

with the initial condition

(4.10) \[ a_N(0) = \frac{h(0)}{f''_0(0) - \int \phi - \psi(0)} \]

where \( N \) has been chosen sufficiently large so that \( a_N(0) \) is positive. Moreover, if \( N \) is sufficiently large so that \( a(t) \) and \( a_N(t) \) are defined over \( 0 \leq t \leq T \), then it follows from (4.1) that

(4.11) \[ \|a - a_N\|_T = O(1/N^4) . \]

Let

(4.12) \[ \Delta t = TP^{-1} , \]

where \( P \) is a large positive integer. For arbitrary \( f(t) \) defined on \( 0 \leq t \leq T \), let

(4.13) \[ f_n = f(n \Delta t) , \ n = 0,1, \ldots , P . \]

Then, \( a_N(t) \) can be approximated at the points \( n\Delta t \), \( n = 1, \ldots , P \), by the sequence \( \{ y_n \} \), where
\[(4.14)\]

\[
y_{n+1} = y_n + y_n \frac{h_{n+1} - h_n}{h_n} + \Delta t \frac{y_n^3}{h_n} \sum_{k=1}^{N} (k\pi)^3
\]

\[
A_k \exp \left\{ -k^2 \pi^2 \sum_{j=0}^{n} y_j \Delta t \right\}
\]

\[
- \frac{y_n^3}{h_n} \sum_{j=0}^{n} M_\psi (-1, \sum_{\psi = j}^{n} y_{\psi} \Delta t) [\psi_{j+1} - \psi_j] \Delta t,
\]

\[
n = 0, 1, \ldots, P - 1,
\]

\[
y_0 = a_N(0).
\]
5. References.


Determination of Certain Parameters
in Heat Conduction Problems; Part I:
Diffusivities of Homogeneous Conductors

by

J. R. Cannon

1. **Introduction.** A simple method for the determination of the thermal diffusivity of a finite medium, a semi-finite medium, a part of a composite infinite medium, and a part of a composite semi-infinite medium will be presented. By over-specifying the usual boundary data, a non-linear algebraic equation for the diffusivity is derived. It will be shown that the numerical determination of the diffusivity follows from the construction of the graph of a certain function depending in an explicit fashion on the data.

2. **A finite medium.**

2.1. **Preliminaries.** Consider

\[
\begin{align*}
(a.) \quad & u_t = \kappa_{xx}, \quad 0 < x < l, \quad 0 < t < T, \\
(b.) \quad & u(x,0) = 0, \quad 0 < x < l, \\
(c.) \quad & u(0,t) = f(t), \quad 0 < t < T, \quad f(0) = 0, \\
(d.) \quad & u(l,t) = g(t), \quad 0 < t < T, \quad g(0) = 0, \\
(e.) \quad & -\rho c_\kappa \lim_{x \downarrow 0} u_x(x,t_o) = h, \quad 0 < t_o < T,
\end{align*}
\]

where \( \kappa \) is the unknown diffusivity, \( \rho \) is the known density, \( c \) is the known specific heat, \( h \) is a known constant, and \( f(t) \) and \( g(t) \) are known functions of \( t \). Note that it is necessary to measure the heat flow rate for just a single time.

**Remark:** As \( \kappa \) is regarded here as a constant, (2.1.1) is really a special case of the problem.
\[ u_t = a(t)u_{xx}, \quad 0 < x < l, \quad 0 < t < T, \]
\[ u(x,0) = 0, \quad 0 < x < l, \]
\[ u(0,t) = f(t), \quad 0 < t < T, \quad f(0) = 0, \]
\[ u(l,t) = g(t), \quad 0 < t < T, \quad g(0) = 0, \]
\[ -a(t) \lim_{x \downarrow 0} u_x(x,t) = h(t), \quad 0 < t < T, \]

where \( a(t) \) and \( u(x,t) \) are the unknowns. Problem (2.1.2) has been treated by B.P. Jones and J. Douglas, Jr. [3,7]. The method given for (2.1.2) seems unduly complicated in the opinion of the author to be applied to this simple case. Hence, the author is proposing an alternate approach for (2.1.1).

**Definition 2.1.1:** A function \( u(x,t) \) and a constant \( \kappa \) are a solution of (2.1.1) if and only if the following conditions are satisfied:

(a.) \( \kappa \) is a positive constant;

(b.) \( u(x,t) \) is continuous in \((x,t)\) for \( 0 < x < l, \quad 0 < t < T \);

(c.) \( u_x \) and \( u_{xx} \) are continuous in \((x,t)\) for \( 0 < x < l, \quad 0 < t < T \);

(d.) \( \lim_{x \downarrow 0} u_x(x,t_0) \) exists;

(e.) (2.1.1) is satisfied.

Assume that a solution of (2.1.1) exists. Then, [9] for continuous \( f(t) \) and \( g(t) \),

\[
(2.1.3) \quad u(x,t) = - \int_0^t \frac{\partial M(x,\kappa(t-\tau))}{\partial x} f(\tau) \kappa \, d\tau
\]

\[ + \int_0^t \frac{\partial M(x-l,\kappa(t-\tau))}{\partial x} g(\tau) \kappa \, d\tau, \]

where
\[ \begin{align*}
(2.1.4) \quad M(x - \xi, \kappa(t - \tau)) &= \frac{1}{\sqrt{\kappa \kappa(t - \tau)}} \sum_{n = -\infty}^{+\infty} 
\exp \left\{ \frac{-(x - \xi + 2n)^2}{4\kappa(t - \tau)} \right\}, \quad \tau < t,
\end{align*} \]

is the Green's function for the rectangle. If \( f(t) \) and \( g(t) \) are continuously differentiable, then Leibnitz's rule can be applied for \( 0 < x < 1, \ 0 < t < T \) to obtain

\[ \begin{align*}
(2.1.5) \quad u_x(x, t) &= -\int_0^t \frac{\partial^2 M(x, \kappa(t - \tau))}{\partial x^2} f(\tau) d\tau 
\quad + \int_0^t \frac{\partial^2 M(x - 1, \kappa(t - \tau))}{\partial x^2} g(\tau) d\tau.
\end{align*} \]

Since

\[ \begin{align*}
(2.1.6) \quad \frac{\partial M(x - \xi, \kappa(t - \tau))}{\partial \tau} &= -\kappa \frac{\partial^2 M(x - \xi, \kappa(t - \tau))}{\partial x^2}, \\
& \quad x \neq \xi, \ t \neq \tau,
\end{align*} \]

it follows that

\[ \begin{align*}
(2.1.7) \quad u_x(x, t) &= \int_0^t \frac{\partial M(x, \kappa(t - \tau))}{\partial \tau} f(\tau) d\tau 
\quad - \int_0^t \frac{\partial M(x - 1, \kappa(t - \tau))}{\partial \tau} g(\tau) d\tau.
\end{align*} \]

As \( f(0) = g(0) = 0 \) and
(2.1.8) \[ \lim_{t \to t} M(x - \xi, \kappa(t - \tau)) = 0, \quad x \neq \xi \] 
then

\[ u_x(x,t) = - \int_0^t M(x, \kappa(t - \tau)) f'(\tau) \, d\tau \]

\[ + \int_0^t M(x - 1, \kappa(t - \tau)) g'(\tau) \, d\tau. \]

By Lebesgue’s dominated convergence theorem

\[ \lim_{x \to 0} u_x(x,t) = - \int_0^t M(0, \kappa(t - \tau)) f'(\tau) \, d\tau \]

\[ + \int_0^t M(-1, \kappa(t - \tau)) g'(\tau) \, d\tau. \]

Hence, it follows from (e.) of Definition 2.1.1 that

\[ h = \rho c \kappa \int_{t_0}^t M(0, \kappa(t_0 - \tau)) f'(\tau) \, d\tau \]

\[ - \rho c \kappa \int_{t_0}^t M(-1, \kappa(t_0 - \tau)) g'(\tau) \, d\tau. \]

Thus, if (2.1.1) possesses a solution, then the diffusivity \( \kappa \) must satisfy (2.1.11).

**Theorem 2.1.1:** If \( f(t) \) and \( g(t) \) are continuously differentiable for \( 0 \leq t \leq T \), then problem (2.1.1) possesses a unique solution (in the sense of Definition 2.1.1) if and only if (2.1.11) possesses exactly one positive solution \( \kappa \).
Proof: Assume that (2.1.1) possesses a unique solution. By previous analysis, it follows that (2.1.11) possesses a positive solution \( \kappa \). If \( \kappa \) is not the only positive solution of (2.1.11), then there exists \( \kappa_1 > 0 \) and \( \kappa_2 > 0 \), \( \kappa_1 \neq \kappa_2 \), such that each is a solution of (2.1.11). Set

\[
(2.1.12) \quad u_i(x, t) = - \int_0^t \frac{\partial M(x, \kappa_i(t - \tau))}{\partial x} f(\tau) \kappa_i \, d\tau \\
+ \int_0^t \frac{\partial M(x - l, \kappa_i(t - \tau))}{\partial x} g(\tau) \kappa_i \, d\tau ,
\]

\( i = 1, 2. \)

Clearly, the pairs \( \kappa_i, u_i(x, t) \), \( i = 1, 2 \), form two distinct solutions of (2.1.1). This contradicts the hypothesis that (2.1.1) possesses a unique solution. Thus, \( \kappa \) is the only positive solution of (2.1.11).

Assume now that (2.1.11) possesses exactly one positive solution \( \kappa \). Then, \( \kappa \) and \( u(x, t) \), defined by (2.1.3), form a solution of (2.1.1). The uniqueness follows as above.

2.2. Solution of the transcendental equation for \( \kappa \). Consider the equation (2.1.11). Assume that

\[
\begin{align*}
(a.) \quad f(t) & \neq 0, \quad f'(t) \geq 0 \text{ for } 0 \leq t \leq t_o; \\
(b.) \quad g'(t) & \leq 0 \text{ for } 0 \leq t \leq t_o; \\
(c.) \quad h & \geq 0
\end{align*}
\]

Define the function \( F(\kappa) \) by the relation

\[
(2.2.2) \quad F(\kappa) = \rho c \kappa \int_0^{t_o} M(0, \kappa, t_o - \tau) \, f'(\tau) \, d\tau \\
- \rho c \kappa \int_0^{t_o} M(-l, \kappa, t_o - \tau) \, g'(\tau) \, d\tau , \quad \kappa > 0 .
\]
Since

\[(2.2.3) \quad M(0, \kappa(t_0 - t)) = \frac{1}{\sqrt{\pi \kappa (t_0 - t)}} + \]

\[+ 2 \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi \kappa (t_0 - t)}} \exp \left\{ \frac{-n^2}{\kappa(t_0 - t)} \right\} \]

and

\[(2.2.4) \quad M(-1, \kappa(t_0 - t)) = 2 \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi \kappa (t_0 - t)}} \exp \left\{ \frac{-(2n - 1)^2}{4\kappa(t_0 - t)} \right\} \]

for \( \kappa > 0 \) and \( t < t_0 \), it follows from [4] that \( F(\kappa) \) is continuously differentiable for \( \kappa > 0 \) and that

\[(2.2.5) \quad F'(\kappa) = \rho c \int_{0}^{t_0} \left\{ \frac{1}{\sqrt{2\pi \kappa (t_0 - \tau)}} + \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi \kappa (t_0 - \tau)}} \exp \left\{ \frac{-n^2}{\kappa(t_0 - \tau)} \right\} \right\} F'(\tau) \, d\tau \]

\[+ 2 \rho c \int_{0}^{t_0} \left\{ \sum_{n=1}^{\infty} \frac{n^2}{\pi \kappa^{3/2} (t_0 - \tau)^{3/2}} \exp \left\{ \frac{-n^2}{\kappa(t_0 - \tau)} \right\} \right\} F'(\tau) \, d\tau \]

\[- \rho c \int_{0}^{t_0} \left\{ \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi \kappa (t_0 - \tau)}} \exp \left\{ \frac{-(2n - 1)^2}{4\kappa(t_0 - \tau)} \right\} \right\} g'(\tau) \, d\tau \]

\[- \rho c \int_{0}^{t_0} \left\{ \sum_{n=1}^{\infty} \frac{(2n - 1)^2}{2\sqrt{\pi} \kappa^{3/2} (t_0 - \tau)^{3/2}} \exp \left\{ \frac{-(2n-1)^2}{4\kappa(t_0 - \tau)} \right\} \right\} g'(\tau) \, d\tau \]

\[> 0, \quad \kappa > 0. \]
Thus, \( F(\mathcal{K}) \) is a strictly increasing function. Hence, in order to show that (2.1.11) has exactly one positive solution \( \mathcal{K} \), it suffices to show that

\[
(2.2.6) \quad \lim_{\mathcal{K} \uparrow 0} F(\mathcal{K}) = 0
\]

and

\[
(2.2.7) \quad \lim_{\mathcal{K} \uparrow \infty} F(\mathcal{K}) = \infty.
\]

As

\[
(2.2.8) \quad \frac{1}{\sqrt{n}} \frac{1}{K(t_0 - \tau)} \exp \left\{ \frac{-n^2}{K(t_0 - \tau)} \right\} \leq x^{-1/2} K^{1/2}(t_0 - \tau)^{1/2} n^{-2}
\]

and

\[
(2.2.9) \quad \frac{1}{\sqrt{n}} \frac{1}{K(t_0 - \tau)} \cdot \exp \left\{ \frac{-(2n - 1)^2}{4K(t_0 - \tau)} \right\} \leq 4x^{-1/2} K^{1/2}(t_0 - \tau)^{1/2} (2n - 1)^{-2}
\]

it follows from (2.2.2), (2.2.3), and (2.2.4) that

\[
(2.2.10) \quad F(\mathcal{K}) \leq C K^{1/2},
\]

where \( C \) is a sufficiently large positive constant. Thus, (2.2.6) is valid. As

\[
(2.2.11) \quad F(\mathcal{K}) \geq \left\{ \rho C \int_0^{t_0} \frac{f'(\tau)}{\sqrt{n(t_0 - \tau)}} \, d\tau \right\} K^{1/2},
\]

relation to (2.2.7) holds.

**Theorem 2.2.1:** If \( f(t) \) and \( g(t) \) are continuously differentiable for \( 0 \leq t \leq T \) with \( f(t) \neq 0 \), \( f'(t) > 0 \), and \( g'(t) \leq 0 \) for \( 0 \leq t \leq t_0 \) and if \( h > 0 \), then (2.1.1) possesses a unique solution in the sense of Definition 2.1.1.
Note that constructing a graph of \( F(\kappa) \) suffices to provide a numerical solution for \( \kappa \).

2.3. *An a priori bound and stability.* Before the dependence of \( \kappa \) upon the data can be considered, an a priori bound for \( \kappa \) must be derived. Since

\[
(2.3.1) \quad \kappa = h \left\{ \rho c \int_0^\tau M(0,\kappa(t_o - \tau)) f'(\tau) \, d\tau 
- \rho c \int_0^\tau M(-1,\kappa(t_o - \tau)) g'(\tau) \, d\tau \right\}^{-1}
\]

it follows from (2.2.1) that

\[
(2.3.2) \quad \kappa \leq h \left\{ \rho c \int_0^\tau \frac{f'(\tau)}{\sqrt{\kappa(t_o - \tau)}} \, d\tau \right\}^{-1}.
\]

As \( \kappa \geq 0 \),

\[
(2.3.3) \quad \kappa \leq h^2 \left\{ \rho c \int_0^\tau \frac{f'(\tau)}{\sqrt{\kappa(t_o - \tau)}} \, d\tau \right\}^{-2} = \gamma.
\]

Let

\[
(2.3.4) \quad ||f||_{t_o} = \sup_{0 \leq t \leq t_o} |f(t)|
\]

for any function \( f(t) \) defined on \( 0 \leq t \leq t_o \). Let \( f_i(t) \), \( g_i(t) \), and \( h_i \), \( i = 1,2 \) satisfy (2.2.1). Consider

\[
(2.3.5) \quad h_i = F_i(\kappa), \quad i = 1, 2.
\]

Now,

\[
(2.3.6) \quad h_1 - h_2 = [F_1(\kappa_1) - F_1(\kappa_2)] + [F_1(\kappa_2) - F_2(\kappa_2)].
\]
As

\[(2.3.7) \quad |F_1(K_2) - F_2(K_2)| \leq 2x^{-1/2} \rho c Y_2^{1/2} t_0^{1/2} ||f'_1 - f'_2||t_0 \]

\[+ (\frac{8}{3}) \rho c x^{-1/2} Y_2^{3/2} t_0^{3/2} ||f'_1 - f'_2||t_0 \]

\[+ (\frac{32}{3}) \rho c x^{-1/2} Y_2^{3/2} t_0^{3/2} ||\epsilon'_1 - \epsilon'_2||t_0 \]

\[= c_1 ||f'_1 - f'_2||t_0 + c_2 ||\epsilon'_1 - \epsilon'_2||t_0 , \]

it follows from the differentiability of \( F_1(K) \) that

\[(2.3.8) \quad |K_1 - K_2| \leq c_3 \left\{ |h_1 - h_2| + c_1 ||f'_1 - f'_2||t_0 + c_2 ||\epsilon'_1 - \epsilon'_2||t_0 \right\}, \]

where

\[(2.3.9) \quad c_3 = \left\{ \frac{\rho c}{2} \int_0^{t_0} \min \left\{ \frac{f'_1(\tau), f'_2(\tau)}{\sqrt{x (t_0 - \tau) \max (Y_1, Y_2)}} \right\} d\tau \right\}^{-1} \]

This implies that the positive solution

\[(2.3.10) \quad K = K(h,f,g) \]

is locally Lipschitz continuous in its arguments. Therefore, the positive solution of (2.1.11) depends continuously upon the data.

3. A semi-infinite medium. Consider

\[
\begin{cases}
(a.) & u_t = K u_{xx}, \quad 0 < x < \infty, \quad 0 < t < T, \\
(b.) & u(x,0) = 0, \quad 0 < x < \infty, \\
(c.) & u(0,t) = f(t), \quad 0 \leq t < T, \quad f(0) = 0, \\
(d.) & -\rho c K \lim_{x \downarrow 0} u_x(x,t_0) = h, \quad 0 < t_0 < T.
\end{cases}
\]
Since

\[ u(x,t) = - \int_0^t \frac{\partial M(x, \kappa(t-\tau))}{\partial x} f(\tau) \kappa \, d\tau, \]

where

\[ M(x, \kappa(t-\tau)) = \frac{1}{\kappa(t-\tau)} \exp \left\{ \frac{-x^2}{4\kappa(t-\tau)} \right\}, \quad t > \tau, \]

is the well known [9] solution of (3.1.a-c), the methods of section 2 can be applied to derive the following result.

**Theorem 3.1:** If

\[ \begin{align*}
(a.) & \quad f(t) \text{ is continuously differentiable for } 0 \leq t \leq t_0 \\
& \quad \text{and such that} \\
(b.) & \quad h \neq 0, \quad h \kappa > 0,
\end{align*} \]

then the diffusivity \( \kappa \) is given explicitly by the formula

\[ \kappa = h^2 \kappa^{-2}. \]

4. **A composite infinite medium.**

4.1. **Preliminaries.** Consider

\[ \begin{align*}
(a.) & \quad u_t = \kappa_1 \, u_{xx}, \quad -\infty < x < 0, \quad 0 < t < T, \\
(b.) & \quad u_t = \kappa_2 \, u_{xx}, \quad 0 < x < \infty, \quad 0 < t < T, \\
(c.) & \quad u(x,0) = f(x), \quad -\infty < x < \infty, \quad x \neq 0, \\
(d.) & \quad \lim_{t \to 0^+} u(x,t) = \lim_{t \to 0^-} u(x,t), \quad 0 < t < T, \\
(e.) & \quad \lim_{x \to 0^+} u(x,t) = \rho_1 \kappa_1 \lim_{x \to 0^-} \psi(x,t), \quad 0 < t < T, \\
(f.) & \quad u(x,0^+) = h, \quad 0 < x < 0, \quad 0 < t_0 < T,
\end{align*} \]

where \( \kappa_2 \) is the unknown and the remaining data known. In [8], W.A. Mersman derived a representation for the solution of (4.1.1.a-e).

**Theorem 4.1.1:** (Mersman) Let \( f(x) \) be bounded for all real \( x \) and continuous, except possibly at \( x = 0 \), and let
\[
\begin{align*}
(4.1.2) \quad \left\{ 
\begin{array}{l}
f_1 &= \lim_{x \to -\infty} f(x), \\
f_2 &= \lim_{x \to \infty} f(x).
\end{array}
\right.
\end{align*}
\]

Then, the solution of (4.1.1.a-e) is

\[
\begin{align*}
(4.1.3) \quad u(x,t) &= v_1(x,t) + \frac{\rho_1 c_1}{A(\sqrt{xt})^{1/2}} \int_{-\infty}^{0} f(\xi) \exp \left\{ \frac{-(x + \xi)^2}{4 K_1 t} \right\} d\xi + \frac{\rho_2 c_2}{A(\sqrt{xt})^{1/2}} \int_{0}^{t} f(\xi) \exp \left\{ \frac{-(x + \xi)^2}{4 K_2 t} \right\} d\xi ; \\
& \quad x < 0, \ t > 0 ; \\
u(x,t) &= v_2(x,t) + \frac{\rho_2 c_2}{A(\sqrt{xt})^{1/2}} \int_{0}^{t} f(\xi) \exp \left\{ \frac{-(x + \xi)^2}{4 K_2 t} \right\} d\xi + \frac{\rho_1 c_1}{A(\sqrt{xt})^{1/2}} \int_{-\infty}^{0} f(\xi) \exp \left\{ \frac{-(x + \xi)^2}{4 K_1 t} \right\} d\xi ; \\
& \quad x > 0, \ t > 0 ,
\end{align*}
\]

where

\[
(4.1.4) \quad A = (\rho_1 c_1 K_1^{1/2} + \rho_2 c_2 K_2^{1/2}) ,
\]

\[
(4.1.5) \quad v_1(x,t) = \frac{1}{2(\sqrt{\xi} K_1 t)^{1/2}} \int_{-\infty}^{0} f(\xi) \left\{ \exp \left\{ \frac{-(x - \xi)^2}{4 K_1 t} \right\} - \exp \left\{ \frac{-(x + \xi)^2}{4 K_1 t} \right\} \right\} d\xi ,
\]

and
\[ (4.1.6) \quad v_2(x, t) = \frac{1}{2^{(x \cdot K_2 t)^{1/2}}} \int_0^\infty f(\xi) \left\{ \exp \left\{ \frac{-(x - \xi)^2}{4K_2 t} \right\} - \exp \left\{ \frac{-(x + \xi)^2}{4K_2 t} \right\} \right\} d\xi. \]

**Definition 4.1.1:** A function \( u(x, t) \) and a constant \( K_2 \) are a solution of \((4.1.1)\) if and only if the following conditions are satisfied:

(a.) \( K_2 \) is a positive constant;

(b.) \( u(x, t) \) satisfies the conditions set forth in [8];

(c.) \( u(x, t) \) and \( K_2 \) satisfy \((4.1.1)\).

Assume that \( f(x) \) satisfies the hypothesis of Theorem 4.1.1 and that problem \((4.1.1)\) has a solution. Then \( K_2 \) satisfies the non-linear equation

\[ (4.1.7) \quad h = v_1(x_0, t_0) + \frac{c_1 c_2}{A(xt_0)^{1/2}} \int_{-\infty}^0 f(\xi) \exp \left\{ \frac{-(x_0 + \xi)^2}{4K_1 t_0} \right\} d\xi \]

\[ + \frac{c_2}{A(xt_0)^{1/2}} \int_0^\infty f(\xi) \exp \left\{ \frac{-(x_0 K_2^{1/2} - \xi K_1^{1/2})^2}{4K_1 K_2 t_0} \right\} d\xi. \]

Clearly, problem \((4.1.1)\) and equation \((4.1.7)\) are equivalent.

4.2. **Solution of the non-linear equation for \( K_2 \).**

Assume that

\[ (4.2.1) \begin{align*}
\begin{cases}
\text{(a.) } & f(x) \text{ satisfies the hypothesis of Theorem 4.1.1 such that } f(x) \neq 0, \ f(x) \geq 0, \ -\infty < x < 0 \text{ and } f(x) \equiv 0, \ 0 \leq x \leq \infty; \\
\text{(b.) } & v_1(x_0, t_0) \leq h \leq v_1(x_0, t_0) + \frac{1}{(x K_1 t_0)^{1/2}} \int_{-\infty}^0 f(\xi) \\
& \quad \quad \exp \left\{ \frac{-(x_0 + \xi)^2}{4K_1 t_0} \right\} d\xi.
\end{cases}
\end{align*} \]
Now, (4.1.7) becomes

\begin{equation}
(4.2.2) \quad h = v_1(x_0, t_0) + \frac{\rho_{1c1}}{(\rho_{1c1} \kappa_1^{1/2} + \rho_{2c2} \kappa_2^{1/2})(\ast t_0)^{1/2}} \int_{-\infty}^{0} f(\xi) \exp \left\{ \frac{-(x_0 + \xi)^2}{4 \kappa_1 t_0} \right\} d\xi.
\end{equation}

As \( h \) is in the open interval of (4.2.1)-(b.),

\begin{equation}
(4.2.3) \quad \kappa_2 = \left\{ \frac{\rho_{1c1}}{\rho_{2c2}(h - v_1(x_0, t_0))(\ast t_0)^{1/2}} \int_{-\infty}^{0} f(\xi) \exp \left\{ \frac{-(x_0 + \xi)^2}{4 \kappa_1 t_0} \right\} d\xi - \frac{\rho_{1c1} \kappa_1^{1/2}}{\rho_{2c2}} \right\}^2.
\end{equation}

Thus, problem (4.1.1) possesses a unique solution with \( \kappa_2 \) given by (4.2.3).

5. **A composite semi-infinite medium.** Consider

\begin{equation}
(5.1) \quad \begin{cases}
(a.) \quad u_t = \kappa_1 u_{xx}, & 0 < x < \gamma, \quad 0 < t < T, \\
(b.) \quad u_t = \kappa_2 u_{xx}, & \gamma < x < \infty, \quad 0 < t < T, \\
(c.) \quad \lim_{x \uparrow \gamma} u(x, t) = \lim_{x \downarrow \gamma} u(x, t), & 0 < t < T, \\
(d.) \quad \lim_{x \uparrow \gamma} \rho_{1c1} \kappa_1 u(x, t) = \lim_{x \downarrow \gamma} \rho_{2c2} \kappa_2 u(x, t), & 0 < t < T, \\
(e.) \quad u(x, 0) = 0, & 0 < x < \infty, \\
(f.) \quad u(0, t) = f(t), & 0 < t < T, \\
(g.) \quad \lim_{x \uparrow 0} \rho_{1c1} \kappa_1 u(x, t_0) = h, & 0 < t_0 < T,
\end{cases}
\end{equation}
where $K_2$ is unknown. In the appendix, the author derives a representation for the solution of (5.1.a-f). If $f(t)$ is continuous and Lebesgue integrable and if $f(0) = 0$,

$$
u(x, t) = \int_0^t f(\tau) \left\{ \sum_{n=0}^{\infty} \alpha^n \left[ \frac{\xi(2n + 1) \nu + x - \nu_3^2}{2 \sqrt{\pi} \sqrt{K_1}(t - \tau)^{3/2}} \right] \exp \left\{ -\frac{((2n + 1) \nu + x - \nu)^2}{4 K_1 (t - \tau)} \right\} \right\} d\tau,$$

(5.2)

$$-\alpha \frac{\xi(2n + 1) \nu - x + \nu_3^2}{2 \sqrt{\pi} \sqrt{K_1}(t - \tau)^{3/2}} \exp \left\{ -\frac{((2n + 1) \nu - x + \nu)^2}{4 K_1 (t - \tau)} \right\} \right\} d\tau,$$

$0 < x < \nu, \quad 0 < t < T,$

$$\nu < x < \infty, \quad 0 < t < T,$$

where

(5.3) \[ k = \frac{K_1^{1/2}}{K_2^{1/2}}, \]

(5.4) \[ \sigma = \frac{\rho_2 c_2 K_2^{1/2}}{\rho_1 c_1 K_1^{1/2}}, \]

and

(5.5) \[ \lambda = \frac{\sigma - \frac{1}{2}}{\sigma + \frac{1}{2}}. \]
If \( f(t) \) is assumed to be continuously differentiable for \( 0 \leq t \leq T \), then \( K_2 \) satisfies the non-linear equation

\[
(5.6) \quad h = -\rho_1 c_1 K_1 \int_0^t \frac{f'(\tau)}{\sqrt{\frac{1}{x K_1(t_0 - \tau)}}} \left[ 1 + 2 \sum_{n=1}^\infty \alpha^n \exp \left\{ -\frac{n^2 \gamma^2}{K_1(t_0 - \tau)} \right\} \right] d\tau.
\]

In order to demonstrate the existence of a solution of (5.6), assume that

\[
(5.7) \quad \begin{cases} 
\text{(a.)} & 0 \leq t_0 < \frac{2 \gamma^2}{K_1}; \\
\text{(b.)} & f(t) \text{satisfies the hypothesis of Theorem 5.1 and is differentiable for } 0 \leq t \leq T \text{ such that } f(t) \neq 0, f'(t) \geq 0 \text{ for } 0 \leq t \leq t_0; \\
\text{(c.)} & a < h < b,
\end{cases}
\]

where

\[
(5.8) \quad a = -\rho_1 c_1 K_1 \int_0^t \frac{f'(\tau)}{\sqrt{\frac{1}{x K_1(t_0 - \tau)}}} \left[ 1 + 2 \sum_{n=1}^\infty \exp \left\{ -\frac{n^2 \gamma^2}{K_1(t_0 - \tau)} \right\} \right] d\tau
\]

and

\[
(5.9) \quad b = -\rho_1 c_1 K_1 \int_0^t \frac{f'(\tau)}{\sqrt{\frac{1}{x K_1(t_0 - \tau)}}} \left[ 1 + 2 \sum_{n=1}^\infty (-1)^n \exp \left\{ -\frac{n^2 \gamma^2}{K_1(t_0 - \tau)} \right\} \right] d\tau.
\]
Define $F(\alpha)$ by the relation

$$
(5.10) \quad F(\alpha) = -\rho_1 c_1 K_1 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{K_1(t_0 - \tau)}} \left\{ 1 + 2 \sum_{n=1}^{\infty} \alpha^n \exp \left\{ -\frac{n^2 \nu^2}{K_1(t_0 - \tau)} \right\} \right\} d\tau, \quad -1 < \alpha < 1.
$$

It is obvious that $F(\alpha)$ is continuously differentiable and that

$$
(5.11) \quad \lim_{\alpha \to 1} F(\alpha) = a,
$$

and

$$
(5.12) \quad \lim_{\alpha \to -1} F(\alpha) = b.
$$

Thus, in order to show the existence of a unique solution of problem (5.6), it suffices to show that

$$
(5.13) \quad F'(\alpha) = -\rho_1 c_1 K_1 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{K_1(t_0 - \tau)}} \left\{ 2 \sum_{n=1}^{\infty} \alpha^{n-1} \exp \left\{ -\frac{n^2 \nu^2}{K_1(t_0 - \tau)} \right\} \right\} d\tau < 0, \quad -1 < \alpha < 1.
$$

Now, the function $x \exp \left\{ -\beta x^2 \right\}$, $x > 0$, $\beta > 0$, assumes its maximum at

$$
(5.14) \quad x = [2\beta]^{-1/2}.
$$

Moreover, for $x > [2\beta]^{-1/2}$, the function $x \exp \left\{ -\beta x^2 \right\}$ is decreasing.
Set

$$\beta = \frac{\nu^2}{\kappa_1(t_0 - t)}.$$  (5.15)

By condition (5.7.a), the parameter of integration varies only over the interval for which $[2\beta]^{-1/2} < 1$. This implies that the terms of the series in the integrand of $P'(\alpha)$ are in absolute value monotonically decreasing with respect to $n$. Hence, $P'(\alpha) \leq 0$ for $-1 < \alpha < 1$.

Remark: As in the preceding sections, the solution $\kappa_2$ of (5.6) depends continuously upon the data.
Determination of Certain Parameters in
Heat Conduction Problems; Part II: Length
of a Finite Conductor

by

J. R. Cannon

1. Introduction: As in section 2 of Part I of this paper, by over specifying the usual boundary data a simple method for the determination of the length of a finite conductor can be given. Consider

\[
\begin{align*}
(a.) \quad & u_t = u_{xx}, \quad 0 < x < \nu, \quad 0 < t < T, \\
(b.) \quad & u(x,0) = 0, \quad 0 < x < \nu, \\
(c.) \quad & u(0,t) = f(t), \quad 0 \leq t \leq T, \quad f(0) = 0, \\
(d.) \quad & u(\nu,t) = g(t), \quad 0 \leq t \leq T, \quad g(0) = 0, \\
(e.) \quad & \lim_{x \downarrow 0} u_x(x,t_0) = h, \quad 0 < t_0 < T, 
\end{align*}
\]

(1.1)

where \( \nu \) is the unknown length of the conductor. It is assumed that the boundary at \( x = 0 \) is accessible for temperature and heat flow measurements, while the behavior of these quantities at \( x = \nu \) must be assumed to be known a priori or to be measurable in some fashion which does not require the location of this boundary to be known.

2. A non-linear algebraic equation for \( \nu \). Let \( f(t) \) and \( g(t) \) be continuous for \( 0 \leq t \leq T \). Then, the solution of (1.1.a-d) is

\[
\begin{align*}
(2.1) \quad & u(x,t) = - \int_0^t \frac{\partial M(x,t - \tau, \nu)}{\partial x} f(\tau) \, d\tau \\
& \quad + \int_0^t \frac{\partial M(x - \nu, t - \tau, \nu)}{\partial x} g(\tau) \, d\tau,
\end{align*}
\]

(2.1)

where
(2.2) \[ M(x - \bar{x}, t - \bar{t}, \nu) = \frac{1}{\sqrt{x(t - \bar{t})}} \sum_{n=-\infty}^{+\infty} \exp \left\{ -\frac{(x-\bar{x}+2n\nu)^2}{4(t - \bar{t})} \right\}, \]

\[ t > \bar{t}, \]

is Green's function for the rectangle. Assume that \( f(t) \) and \( g(t) \) are continuously differentiable for \( 0 \leq t \leq T \). Using the argument of section 2.1 of Part I; i.e., differentiating expression (2.1) with respect to \( x \), using the fact that \( M(x - \bar{x}, t - \bar{t}, \nu) \) satisfies

(2.3) \[ M_{\bar{t}} = -M_{xx}, \]

integrating by parts and taking the limit as \( x \downarrow 0 \), it follows that \( \nu \) must satisfy

(2.4) \[ h = \int_0^{t_0} \left\{ M(0,t_0 - \bar{t}, \nu) f'(\bar{t}) - M(-\nu,t_0 - \bar{t}, \nu) g'(\bar{t}) \right\} d\bar{t}. \]

Also, it can be shown via the arguments of section 2.1 of Part I that problem (1.1) and equation (2.4) are equivalent in the sense that problem (1.1) possesses a unique solution \( u(x,t), \nu > 0 \), which satisfy (1.1)) if and only if equation (2.4) possesses exactly one positive solution \( \nu \).

3. **Solution of the non-linear algebraic equation for \( \nu \).**

Assume that

(3.1) \[
\begin{cases}
\text{(a.)} & f(t) \text{ is continuously differentiable for } 0 \leq t \leq T; \ f(t) \neq 0, f'(t) \geq 0 \text{ for } 0 \leq t \leq t_0; \\
\text{(b.)} & g(t) \text{ is continuously differentiable for } 0 \leq t \leq T, g'(t) \leq 0 \text{ for } 0 \leq t \leq t_0; \\
\text{(c.)} & a \leq h \leq \infty,
\end{cases}
\]

where
(3.2) \[ a = \int_0^{t_0} \frac{f'(\tau)}{\sqrt{\pi(t_0 - \tau)}} \, d\tau \]

Define the function \( F(\nu) \) by the relation

\[
(3.3) \quad F(\nu) = \int_0^{t_0} \left\{ M(0, t_0 - \tau, \nu) f'(\tau) - M(-\nu, t_0 - \tau, \nu) g'(\tau) \right\} d\tau,
\]

\( \nu > 0 \).

Since

\[
(3.4) \quad M(0, t_0 - \tau, \nu) = \frac{1}{\sqrt{\pi(t_0 - \tau)}} + 2 \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi(t_0 - \tau)}} \exp \left\{ -\frac{n^2 \nu^2}{t_0 - \tau} \right\}
\]

and

\[
(3.5) \quad M(-\nu, t_0 - \tau, \nu) = 2 \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi(t_0 - \tau)}} \exp \left\{ \frac{-(2n - 1)^2 \nu^2}{4(t_0 - \tau)} \right\}
\]

for \( \nu > 0 \) and \( \tau < t_0 \), it follows [4] that \( F(\nu) \) is continuously differentiable and that

\[
(3.6) \quad F'(\nu) = \int_0^{t_0} \left[ \left\{ \sum_{n=1}^{\infty} \frac{-4n^2 \nu}{\sqrt{\pi(t_0 - \tau)}} \exp \left\{ -\frac{n^2 \nu^2}{t_0 - \tau} \right\} \right\} f'(\tau) + \left\{ \sum_{n=1}^{\infty} \frac{(2n - 1)^2 \nu}{\sqrt{\pi(t_0 - \tau)}} \exp \left\{ \frac{-(2n - 1)^2 \nu^2}{4(t_0 - \tau)} \right\} \right\} g'(\tau) \right] d\tau
\]

\(< 0, \quad \nu > 0 \).
Thus, \( F(\nu) \) is a strictly decreasing function. Hence, in order to show that (2.4) possesses exactly one positive solution \( \nu \), it suffices to show that

\[
(3.7) \quad \lim_{\nu \to 0} F(\nu) = \infty
\]

and

\[
(3.8) \quad \lim_{\nu \to \infty} F(\nu) = a.
\]

Since

\[
(3.9) \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{s(t_0 - \tau)}} \exp \left\{ \frac{-n^2 \nu^2}{t_0 - \tau} \right\} < C(t_0 - \tau)^{1/2} \nu^{-2}
\]

and

\[
(3.10) \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{s(t_0 - \tau)}} \exp \left\{ \frac{(2n - 1)^2 \nu^2}{4(t_0 - \tau)} \right\} < C(t_0 - \tau)^{1/2} \nu^{-2},
\]

where \( C \) is a sufficiently large positive constant, expression (3.8) follows immediately. From (3.1), there exists an interval \( I, I \subset [0, t_0] \), such that

\[
(3.11) \quad \begin{cases} 
(a.) & f'(t) \geq \epsilon > 0 \text{ on } I, \\
(b.) & t_0 - \tau \geq \eta > 0 \text{ on } I.
\end{cases}
\]

On \( I, \ M(0, t_0 - \tau, \nu) \to \infty \text{ as } \nu \to 0. \)

Therefore, \( F(\nu) \to \infty \text{ as } \nu \to 0. \) Thus, (3.7) holds. Hence, equation (2.4) possesses exactly one positive solution \( \nu \). Thus, problem (1.1) possesses a unique solution \( f(t), g(t), \) and \( h \) satisfy condition (3.1).

Note that constructing the graph of \( F(\nu) \) suffices to provide a numerical solution for \( \nu \).
4. A priori bounds and stability. As in Part I, a priori bounds for \( \nu \) must be derived before the stability result can be given.

Since

\[
(4.1) \quad h - a = \int_0^t \left\{ \sum_{n=1}^{\infty} \frac{2}{\sqrt{\kappa(t_o - \tau)}} \exp \left\{ \frac{-n^2 \nu^2}{(t_o - \tau)} \right\} f'(\tau) \right. \\
\left. - \sum_{n=1}^{\infty} \frac{2}{\sqrt{\kappa(t_o - \tau)}} \exp \left\{ -\frac{(2n - 1)^2 \nu^2}{4(t_o - \tau)} \right\} g'(\tau) \right\} d\tau
\]

and

\[
(4.2) \quad \exp \left\{ -x^2 \right\} < x^{-1}, \quad x > 0,
\]

it follows that

\[
(4.3) \quad h - a \leq \frac{16 t_o^{3/2}}{3 \kappa \nu^2} \left\{ \| f' \|_{t_o} + 2 \| g' \|_{t_o} \right\},
\]

where the norm \( \| \cdot \|_{t_o} \) is defined by (2.3.4) in Part I.

Therefore,

\[
(4.4) \quad \nu \leq \left[ \frac{16 t_o^{3/2}}{3 \kappa(h - a)} \left\{ \| f' \|_{t_o} + 2 \| g' \|_{t_o} \right\} \right]^{1/2} = \Delta.
\]

From (3.1), it follows that

\[
(4.5) \quad h \geq 2 \int_0^t \frac{1}{\sqrt{\kappa(t_o - \tau)}} \sum_{n=1}^{\infty} \exp \left\{ \frac{-n^2 \nu^2}{(t_o - \tau)} \right\} f'(\tau) d\tau.
\]

Consider the series

\[
(4.6) \quad \sum_{n=1}^{\infty} \exp \left\{ -\beta n^2 \right\}, \quad \beta > 0.
\]
Since
\begin{equation}
\sum_{n=1}^{\infty} \exp \left\{ -\beta n^2 \right\} \geq \int_{1}^{\infty} \exp \left\{ -\beta x^2 \right\} \, dx
\end{equation}
and
\begin{equation}
\int_{1}^{\infty} \exp \left\{ -\beta x^2 \right\} \, dx > \left[ \int_{\pi/8}^{3\pi/8} \int_{2^{1/2}}^{\infty} \exp \left\{ -\beta r^2 \right\} r \, dr \, d\theta \right]^{1/2}
\end{equation}

\begin{align*}
= & \sqrt{\pi} \left[ 2^{1/2} \right] \frac{1}{\sqrt{2 \pi}} \exp \left\{ -4\beta \right\},
\end{align*}

it follows that
\begin{equation}
\sum_{n=1}^{\infty} \exp \left\{ -\frac{n^2 \nu^2}{(t_0 - \tau)} \right\} > \frac{\sqrt{\pi} (t_0 - \tau)^{1/2}}{2^{1/2} \sqrt{2 \pi} \nu} \exp \left\{ -\frac{4\Delta^2}{(t_0 - \tau)} \right\}.
\end{equation}

Thus,
\begin{equation}
\nu \geq \frac{1}{\sqrt{2} \, h} \int_{0}^{t_0} \exp \left\{ -\frac{4\Delta^2}{(t_0 - \tau)} \right\} f' (\tau) \, d\tau = \delta > 0.
\end{equation}

Let \( f_1 (t), g_1 (t), \) and \( h_i, i = 1, 2, \) satisfy (3.1). Using precisely the same type of argument that was used in section 2.3 of Part I, it follows that
\begin{equation}
|\nu_1 - \nu_2| \leq C_1 \left\{ |h_1 - h_2| + C_2 (||f_1' - f_2'||_{t_0} + 4||g_1' - g_2'||_{t_0}) \right\},
\end{equation}

where
\begin{equation}
C_1 = \left\{ \frac{2 \min (\delta_1, \delta_2)}{\sqrt{\pi}} \int_{0}^{t_0} \frac{1}{(t_0 - \tau)^{3/2}} \exp \left\{ -\frac{(\max (\Delta_1, \Delta_2))^2}{(t_0 - \tau)} \right\} \right. \\
\cdot \min (f_1' (\tau), f_2' (\tau)) \, d\tau \right\}^{-1}
\end{equation}
and
\[ c_2 = \left\{ \frac{2}{\sqrt{\pi t}} t^{1/2} + \frac{32}{3} \frac{t^{3/2}}{\sqrt{\pi} \min(\delta_1, \delta_2)^2} \right\}. \]

Thus, the positive solution
\[ \gamma = \gamma(h, f, g) \]

of (2.4) is locally Lipschitz continuous. Hence, the positive solution of (2.4) depends continuously upon the data.

5. **An alternate approach to the determination of the length of a finite conductor.** Consider
\[
\begin{align*}
\text{(a.)} & \quad u_t = u_{xx}, \quad 0 < x < \gamma, \quad 0 < t < T, \\
\text{(b.)} & \quad u(x, 0) = 0, \quad 0 < x < \gamma, \\
\text{(c.)} & \quad \lim_{x \to 0} u_x(x, t) = h(t), \quad 0 < t < T, \\
\text{(d.)} & \quad \lim_{x \to \gamma} u_x(x, t) = i(t), \quad 0 < t < T, \\
\text{(e.)} & \quad u(0, t_0) = f, \quad 0 < t_0 < T,
\end{align*}
\]

where \( \gamma \) is the unknown length, \( u(x, t) \) is unknown, and all of the rest is known data. For continuous \( h(t) \) and \( i(t) \), the solution of (5.1.a-d) is given by
\[
(5.2) \quad u(x, t) = \int_0^t \left\{ M(x, t-\tau, \gamma) h(\tau) + M(x-\gamma, t-\tau, \gamma) i(\tau) \right\} d\tau.
\]

Hence, it follows immediately that \( \gamma \) must satisfy
\[
(5.3) \quad f = \int_0^{t_0} \left\{ M(0, t_0-\tau, \gamma) h(\tau) + M(-\gamma, t_0-\tau, \gamma) i(\tau) \right\} d\tau.
\]
Moreover, if

\[
\begin{align*}
\text{(a.) } h(t) &\geq 0, \ h(t) \neq 0 \text{ for } 0 \leq t \leq t_0; \\
\text{(b.) } i(t) &\geq 0, \text{ for } 0 \leq t \leq t_0; \\
\text{(c.) } b &< \infty,
\end{align*}
\]

where

\[
(5.5) \quad b = \int_0^{t_0} \frac{h(\tau)}{\sqrt{\pi(t_0-\tau)}} \, d\tau,
\]

then

\[
(5.6) \quad F_*(\nu) = \int_0^{t_0} \left\{ M(0, t_0-\tau, \nu) \ h(\tau) + M(-\nu, t_0-\tau, \nu) \ i(\tau) \right\} d\tau
\]

is a strictly decreasing function of $\nu$. Hence, (5.3) possesses exactly one positive solution $\nu$. 

by

J. R. Cannon

1. Introduction. By over specifying the usual boundary data, the method of Parts I and II will be employed to determine the location of an interface between two semi-infinite homogeneous conductors. Consider

\[
\begin{align*}
(a.) \quad u_t &= K_i u_{xx}, \quad t > 0; \quad x < \gamma, \quad i = 1; \quad x > \gamma, \quad i = 2; \quad \gamma > 0; \\
(b.) \quad u(x,0) &= f(x); \quad x \neq \gamma; \\
(c.) \quad \lim_{x \uparrow \gamma} u(x,t) &= \lim_{x \downarrow \gamma} u(x,t); \quad t > 0; \\
(d.) \quad \lim_{x \uparrow \gamma} \rho_1 c_1 K_1 u_x(x,t) &= \lim_{x \downarrow \gamma} \rho_2 c_2 K_2 u_x(x,t); \quad t > 0; \\
(e.) \quad u(0,t_0) &= g; \quad t_0 > 0.
\end{align*}
\]

where \( \gamma \) is the unknown distance of the interface from the origin and all of the remaining data known. Note that \( \gamma \) is assumed positive.

2. Solution of (1.1) for the case \( \rho_1 c_1 K_1^{1/2} \neq \rho_2 c_2 K_2^{1/2} \). Assume

\[
\begin{align*}
(a.) \quad f(x) &= \begin{cases} \\
\Phi(x), & x \leq 0, \quad \Phi(0) = 0, \\
0, & x > 0,
\end{cases} \\
\end{align*}
\]

where \( \Phi(x) \) is a continuous bounded function for all \( x \) in \(-\infty < x \leq 0\), \( \Phi(x) \geq 0, \quad \Phi(x) \neq 0, \)

\[
\begin{align*}
(2.1) \quad \lim_{x \to -\infty} \Phi(x) &= c, \quad c > 0; \\
(b.) \quad \text{if} \quad \rho_1 c_1 K_1^{1/2} > \rho_2 c_2 K_2^{1/2}, \quad a < g < b; \\
(c.) \quad \text{if} \quad \rho_1 c_1 K_1^{1/2} < \rho_2 c_2 K_2^{1/2}, \quad b < g < a,
\end{align*}
\]

where

\[
(2.2) \quad a = \frac{1}{2(\pi K_1 t_0)^{1/2}} \int_{-\infty}^{0} \Phi(\xi) \exp \left\{ -\frac{\xi^2}{4K_1 t_0} \right\} d\xi
\]
and

\[ b = \frac{\rho_1 c_1}{A(x t_0)^{1/2}} \int_{-\infty}^{\infty} \varphi(\xi) \exp \left\{ -\frac{\xi^2}{4 K_1 t_0} \right\} d\xi, \]

\[ A = (\rho_1 c_1 K_1^{1/2} + \rho_2 c_2 K_2^{1/2}). \]

From Mersman's representation for the solution of (4.1.1.a-e) of Part I [8], namely (4.1.3), the translation \( \bar{x} = x - \gamma \), and (2.1), it follows that \( \gamma \) must satisfy

\[ g = \frac{1}{2(x K_1 t_0)^{1/2}} \int_{-\infty}^{\infty} \varphi(\xi) \left\{ \exp \left\{ -\frac{\xi^2}{4 K_1 t_0} \right\} - \exp \left\{ -(\xi - 2\gamma)^2 \right\} \right\} d\xi \]

\[ + \frac{\rho_1 c_1}{A(x t_0)^{1/2}} \int_{-\infty}^{0} \varphi(\xi) \exp \left\{ -(\xi - 2\gamma)^2 \right\} d\xi. \]

By analysis similar to that of section 2.1 of Part I, it can be shown that problem (1.1) possesses a unique solution (a pair: \( u(x,t), \gamma > 0 \), which satisfies (1.1)) if and only if (2.4) possesses exactly one positive solution \( \gamma \).

Define the function \( F(\gamma) \) by the relation

\[ F(\gamma) = \frac{1}{2(x K_1 t_0)^{1/2}} \int_{-\infty}^{\infty} \varphi(\xi) \left\{ \exp \left\{ -\frac{\xi^2}{4 K_1 t_0} \right\} - \exp \left\{ -(\xi - 2\gamma)^2 \right\} \right\} d\xi \]

\[ + \frac{\rho_1 c_1}{A(x t_0)^{1/2}} \int_{-\infty}^{0} \varphi(\xi) \exp \left\{ -(\xi - 2\gamma)^2 \right\} d\xi, \gamma > 0. \]
Now,

\[ F'(\gamma) = \left( \frac{\rho_{c1}^{1/2} K_1 - \rho_{c2}^{1/2} K_2}{2A(z K_1 t_o)^{1/2}} \right) \int _{-\infty} ^{0} \phi(\xi) \left( \frac{\xi - 2\gamma}{K_1 t_o} \right) \exp \left\{ -\frac{(\xi - 2\gamma)^2}{4 K_1 t_o} \right\} d\xi. \]

Hence, if \( \rho_{c1}^{1/2} > \rho_{c2}^{1/2} K_2 \), then \( F(\gamma) \) is a strictly decreasing function, or if \( \rho_{c1}^{1/2} < \rho_{c2}^{1/2} K_2 \), then \( F(\gamma) \) is a strictly increasing function. Also, it is obvious that

\[ \lim _{\gamma \to 0} F(\gamma) = b. \]

Moreover, since

\[ \lim _{\gamma \to \infty} \int _{-\infty} ^{0} \exp \left\{ -\frac{(\xi - 2\gamma)^2}{4 K_1 t_o} \right\} d\xi = 0, \]

it follows that

\[ \lim _{\gamma \to \infty} F(\gamma) = a. \]

Hence, (2.4) possesses exactly one positive solution for the case \( \rho_{c1}^{1/2} \neq \rho_{c2}^{1/2} K_2^{1/2} \) under the conditions (2.1). Thus, for \( f(x) \) and \( g \) satisfying (2.1), problem (1.1) possesses a unique solution.

3. **An a priori bound and stability.** By analysis similar to that of section 2.3 of Part II, it can be shown that

\[ \gamma \leq \left\{ \frac{B11 \| \Phi \| \sqrt{\Pi} (K_1 t_o)^{3/2}}{1 - g - a} \right\} ^{1/2} = \hat{F}, \]

where

\[ E = \left\{ \frac{1}{2 (z K_1 t_o)^{1/2}} + \frac{\rho_{c1}^{1/2}}{A(z t_o)^{1/2}} \right\} \]
and

\[ \| f \| = \sup_{-\infty < x < \infty} |f(x)| \]

for any function \( f(x) \) defined on \(-\infty < x < \infty\).

Let \( f_i(x) \) and \( g_i \), \( i = 1, 2 \), satisfy condition (2.1). By the same type of argument given in section 2.3 of Part I, it follows that

\[ |\gamma_1 - \gamma_2| \leq c_1 \left\{ |g_1 - g_2| + c_2 ||\varphi_1 - \varphi_2|| \right\} , \]

where

\[ c_1 = \left\{ \frac{1}{2A(z_1 K_1 t_0)^{1/2}} \int_{-\infty}^{0} \min(\varphi_1(\xi), \varphi_2(\xi)) \frac{-\xi}{K_1 t_0} \right\} \exp \left\{ \frac{-(\xi - 2 \max(P_1, P_2))^2}{4 K_1 t_0} \right\} d\xi \]

and

\[ c_2 = (1 + \frac{\rho_1 c_1 K_1^{1/2}}{A}) . \]

Thus, the positive solution

\[ \gamma = \gamma(g, \varphi) \]

of (2.4) is locally Lipschitz continuous. Therefore, the positive solution of (2.4) depends continuously upon the data.

4. Solution of (1.1) for the case \( \rho_1 c_1 K_1^{1/2} = \rho_2 c_2 K_2^{1/2} \).

From (2.6), it follows that under the previous hypothesis on \( f(x) \), \( F(\gamma) \equiv \text{constant} \). Thus, no information can be obtained from such conditions. Therefore, assume in addition to (1.1) that \( 0 < \gamma < D \), where \( D \) is known a priori. Moreover, assume that
\( (4.1) \) 
\[
\begin{align*}
\text{(a.) } f(x) &= \begin{cases} 0, & -\infty < x < D, \\ \phi(x), & D \leq x < \infty, \quad \phi(D) = 0, \end{cases} \quad \text{where } \phi(x) \text{ is a continuous bounded function for all } x \text{ in } D \leq x < \infty, \quad \phi(x) \geq 0, \quad \phi(x) \neq 0, \\ \lim_{x \to \infty} \phi(x) &= c, \quad c \geq 0; \\
\text{(b.) } &\text{if } \kappa_1 > \kappa_2, \quad \mathcal{J} < \varepsilon < \omega; \\
\text{(c.) } &\text{if } \kappa_1 < \kappa_2, \quad \omega < \varepsilon < \mathcal{J}, 
\end{align*}
\]

where

\[
(4.2) \quad \mathcal{J} = \frac{\rho^2 c^2}{A(x \ t_0)^{1/2}} \int_{D}^{\infty} \phi(\xi) \exp \left\{ \frac{-\xi^2}{4 \kappa_2 t_0} \right\} d\xi
\]

and

\[
(4.3) \quad \omega = \frac{\rho^2 c^2}{A(x \ t_0)^{1/2}} \int_{D}^{\infty} \phi(\xi) \\
\exp \left\{ -\frac{((\kappa_1^{1/2} - \kappa_2^{1/2}) D - \xi \kappa_1^{1/2})^2}{4 \kappa_1 \kappa_2 t_0} \right\} d\xi.
\]

By the same argument used to derive (2.4), it follows that \( \gamma \) must satisfy

\[
(4.4) \quad \gamma = \frac{\rho^2 c^2}{A(x \ t_0)^{1/2}} \int_{D}^{\infty} \phi(\xi) \\
\exp \left\{ -\frac{((\kappa_1^{1/2} - \kappa_2^{1/2}) \gamma - \xi \kappa_1^{1/2})^2}{4 \kappa_1 \kappa_2 t_0} \right\} d\xi
\]

which is equivalent to (1.1) with the added condition \( 0 < \gamma < D \).

Defining \( B(\gamma) \) by the relation
(4.5) \[ E(\gamma) = \frac{\rho_2 c_2}{A(x t_0)^{1/2}} \int_0^\varphi D \varphi(z) \]

\[ \exp \left\{ -\left( \frac{K_1^{1/2} - K_2^{1/2}}{4 K_1 K_2 t_0} \right) \gamma - \frac{3 K_1^{1/2}}{2} \right\} d\gamma \]

0 < \gamma < D, it can be shown that if \( K_1 > K_2 \), then \( E(\gamma) \) is a strictly increasing function while if \( K_1 < K_2 \), then \( E(\gamma) \) is strictly decreasing. As the range of \( E(\gamma) \) is (\( f \), \( \omega \)) if \( K_1 > K_2 \) or \( (\omega, f) \) if \( K_1 < K_2 \), (1.1) possesses a solution for this case also. As before, standard analysis shows that the positive solution of (4.4) depends continuously upon the data.
Determination of Certain Parameters in Heat Conduction Problems; Part IV: Interface between a Finite Conductor and a Semi-Infinite Conductor

by

J. R. Cannon

1. Introduction. By overspecifying the usual boundary data, a simple method for determining the location of an interface between a finite conductor and a semi-infinite conductor will be given. Consider

\[
\begin{align*}
(a.) & \quad u_t = \kappa_1 u_{xx}, \quad 0 < x < \nu, \quad 0 < t < T, \\
(b.) & \quad u_t = \kappa_2 u_{xx}, \quad \nu < x < \infty, \quad 0 < t < T, \\
(c.) & \lim_{x \uparrow \nu} u(x,t) = \lim_{x \downarrow \nu} u(x,t), \quad 0 < t < T, \\
(d.) & \lim_{x \uparrow \nu} \rho_1 c_1 \kappa_1 u_x(x,t) = \lim_{x \downarrow \nu} \rho_2 c_2 \kappa_2 u_x(x,t), \quad 0 < t < T, \\
(e.) & \quad u(x,0) = 0, \quad 0 < x < \infty, \\
(f.) & \quad u(0,t) = f(t), \quad 0 < t < T, \\
(g.) & \lim_{x \downarrow 0} \rho_1 c_1 \kappa_1 u_x(x,t_0) = h, \quad 0 < t_0 < T,
\end{align*}
\]

where \( u(x,t) \) and \( \nu \) are the unknowns. Assume that \( f(t) \) is continuously differentiable and Lebesgue integrable for \( 0 \leq t \leq T \) with \( f(0) = 0 \). If problem (1.1) possessed a solution (a pair: \( u(x,t), \ \nu > 0 \) which satisfies (1.1)), then using the representation of the solution of (1.1.a-f) which is given by (5.2) of Part I and an argument which is step by step similar to that of section 2.1 of Part I, it follows that \( \nu \) must satisfy the non-linear algebraic equation
(1.2) \[ h = -\rho c_1 K_1 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{x} K_1(t_0-\tau)} \left\{ 1 + 2 \sum_{n=1}^{\infty} \alpha^n \exp \left\{ -\frac{n^2 y^2}{K_1(t_0-\tau)^3} \right\} \right\} d\tau. \]

By an argument similar to that of Theorem 2.1.1, (1.1) and (1.2) are equivalent in the sense that (1.1) possesses a unique solution if and only if (1.2) possesses exactly one positive solution $\mathcal{V}$. The existence of exactly one positive solution of (1.2) will be broken into the cases: $\alpha > 0$, $\alpha < 0$, $\alpha = 0$ (defined by (5.5) Part I).

2. Solution of the non-linear algebraic equation for the case $\alpha > 0$.
In addition to hypothesis on $f(t)$, assume that

\[
2.1 \begin{cases} 
(a.) & f(t) \neq 0, f'(t) > 0 \text{ for } 0 \leq t \leq t_0, \\
(b.) & a < h < b,
\end{cases}
\]

where

\[
2.2 \quad a = -\rho c_1 K_1 \left( \frac{1 + \alpha}{1 - \alpha} \right) \int_0^{t_0} \frac{f'(\tau)}{\sqrt{x} K_1(t_0-\tau)} d\tau
\]

and

\[
2.3 \quad b = -\rho c_1 K_1 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{x} K_1(t_0-\tau)} d\tau.
\]

Define the function $F(\mathcal{V})$ by the relation

\[
2.4 \quad F(\mathcal{V}) = -\rho c_1 K_1 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{x} K_1(t_0-\tau)} \left\{ 1 + 2 \sum_{n=1}^{\infty} \alpha^n \exp \left\{ -\frac{n^2 y^2}{K_1(t_0-\tau)^3} \right\} \right\} d\tau, \quad \mathcal{V} > 0.
\]

Since $0 < \alpha < 1$, it follows that $F(\mathcal{V})$ is continuously differentiable for $0 < \mathcal{V} < \infty$ with derivative
(2.5) \[
P'(\nu) = 4 \rho_1 c_1 K_1 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{\pi} \left[ K_1 (t_0 - \tau) \right]^{3/2}} \left\{ \sum_{n=1}^{\infty} \nu^n \alpha^n \exp \left\{ -\frac{n^2 \nu^2}{K_1 (t_0 - \tau)} \right\} \right\} d\tau > 0.
\]

Thus, \(P(\nu)\) is strictly increasing. As

(2.6) \[
\lim_{\nu \downarrow 0} P(\nu) = a
\]

and

(2.7) \[
\lim_{\nu \uparrow \infty} P(\nu) = b,
\]

it follows that (1.2) possesses exactly one positive solution.

**Remark:** Using the techniques of the previous parts, it can be shown that the solution \(\nu\) depends continuously upon the data. In showing this, the following a priori bounds for the solution can be derived:

(2.8) \[
0 < \beta \leq \nu \leq \gamma,
\]

where

(2.9) \[
\gamma = \left\{ \frac{2 \rho_1 c_1 K_1^{3/2} \left( \sum_{n=1}^{\infty} \frac{\alpha^n}{n^2} \right) \int_0^{t_0} f'(\tau)(t_0 - \tau)^{1/2} d\tau}{\sqrt{1 - h - b}} \right\},
\]

(2.10) \[
\beta = \left\{ \frac{(h - a) \sqrt{V(1 - \alpha)} K_1^{3/2} \delta^{3/2}}{4 \rho_1 c_1 K_1 \left[ 2 ! f' \left| \begin{array}{l} 1 - \alpha \end{array} \right| t_0 \right]^{2/3} \alpha \left| 2 ! f' \left| \begin{array}{l} 1 - \alpha \end{array} \right| t_0 \right|^{1/3} \alpha^2} \right\}^{1/2},
\]

(2.11) \[
\delta = \left\{ \frac{(h - a)^2}{64} \cdot \frac{n K_1 (1 - \alpha)^2}{64 \rho_1 c_1 K_1 \left[ 2 ! f' \left| \begin{array}{l} 1 - \alpha \end{array} \right| t_0 \right]^{2/3} \alpha^2} \right\},
\]

and \(N\) is the least positive integer which satisfies
(2.12) \[ \alpha^{N+1} < \frac{\sqrt{x K_1 (1-\alpha)} (h-a)}{64 \rho_1 c_1 K_1 \ell_{1/2} f''_{1/2} t_0^3 \sqrt{t_0}}. \]

The norm \( \| \|_{t_0} \) is defined by (2.3.4) in Part I.

3. Solution of the non-linear algebraic equation for the case \( \alpha < 0 \).

In order to treat this case, assume the existence of an a priori lower bound for \( \nu \); i.e., to (1.1) add the condition that \( 0 < \beta \leq \nu \). This means that is order for (1.1) to have a unique solution (a pair: \( u(x,t) \), \( \nu \geq \beta \)) which satisfies (1.1) with the added condition \( 0 < \beta \leq \nu \) exactly one solution \( \nu \geq \beta \) of (1.2) must be found. Assume in addition to the hypothesis on \( f(t) \) in sections 1 and 2 that

\[
\begin{cases}
(a.) & 0 < t_0 < \frac{\beta^2}{K_1}, \\
(b.) & b < h \leq d
\end{cases}
\]

where

\[
d = -\rho_1 c_1 K_1 \int_0^{t_0} \frac{f'(t)}{\sqrt{x K_1 (t_0-\tau)}} \left\{ 1 + 2 \sum_{n=1}^{\infty} \alpha^n \exp \left\{ \frac{-n^2 \beta^2}{K_1 (t_0-\tau)} \right\} \right\} d\tau.
\]

As in section 2, \( F(\nu) \) defined by (2.4) for \( \nu \geq \beta > 0 \) is continuously differentiable with its derivative given by (2.5) for \( \nu \geq \beta > 0 \). Note here that the derivative certainly exists since \(-1 \leq \alpha < 0 \). Consider the series

\[
\sum_{n=1}^{\infty} n^2 \alpha^n \exp \left\{ \frac{-n^2 \nu^2}{K_1 (t_0-\tau)} \right\}
\]

which is an alternating series since \( \alpha < 0 \).
Now, the function

\[ S(x) = x^2 \exp \left\{ -\frac{\mu x^2}{k_1(t_o - \tau)} \right\}, \quad \mu > 0, \quad x > 0, \]

has its maximum at

\[ x = -\frac{1}{\mu}. \]

Setting

\[ \mu = \frac{\nu^2}{k_1(t_o - \tau)}, \]

the function

\[ x^2 \exp \left\{ -\frac{x^2 \nu^2}{k_1(t_o - \tau)} \right\}, \quad x > 0, \]

has its maximum at

\[ x = \frac{\sqrt{k_1(t_o - \tau)}}{\nu} \leq \frac{\sqrt{k_1 t_o}}{\nu} < \frac{\sqrt{k_1 \beta^2}}{\nu} = \frac{\beta}{\nu} \leq 1. \]

Hence, for \( x \geq 1 \), the function \( S(x) \) is strictly decreasing. Thus, the series in expression (4.6) converges and is negative. Therefore, \( F'(\nu) \leq 0 \). Hence, for this case \( F(\nu) \) is a strictly decreasing function.

As

\[ \lim_{\nu \to \beta} F(\nu) = d \]

and

\[ \lim_{\nu \to \infty} F(\nu) = 0, \]

it follows that (1.2) possesses exactly one positive solution \( \nu \geq \beta > 0 \).

Remark: The solution \( \nu \geq \beta > 0 \) depends continuously upon the data.

4. Discussion of the case \( \alpha = 0 \). In this case (note \( \sigma' = 1 \iff \alpha = 0 \)) it can be shown that the solution of (1.1.a-f) is
\[
\begin{align*}
E-6 \\
(4.1) \\
\begin{cases}
    u(x,t) = \frac{1}{2\sqrt{k} \kappa_1} \int_0^t \frac{x}{(t-\tau)^{3/2}} \exp \left\{ \frac{x^2}{4\kappa_1(t-\tau)} \right\} f(\tau) \, d\tau, \\
    &0 < x < \nu, \quad 0 < t < T, \\
    u(x,t) = \frac{1}{2\sqrt{k} \kappa_1} \int_0^t \frac{(\nu + kx - k\nu)}{(t-\tau)^{3/2}} \exp \left\{ \frac{-(\nu + kx - k\nu)^2}{4\kappa_1(t-\tau)} \right\} f(\tau) \, d\tau, \\
    &\nu < x < \infty, \quad 0 < t < T,
\end{cases}
\end{align*}
\]

where \( k \) is defined by (5.3) of Part I. It can be seen immediately that the position of the interface \( \nu \) has no effect upon the behavior of the solution of (1.1.a-f) in the neighborhood of the boundary \( x = 0 \). Hence, no heat flow test can be devised which will locate the interface from data taken at the boundary for the case \( \alpha = 0 \) \((\sigma' = 1)\).

It is possible that an a priori upper bound \( \nu \) for \( \nu \) might be known. Also, it is possible that at \( x = x_* \geq \nu \) heat flow measurements could be taken. Since for \( \alpha = 0 \) the solution of (1.1.a-f) in the neighborhood of \( x = x_* \) depends upon the location of the interface \( \nu \), it is reasonable to consider the following problem:
\[
\begin{align*}
\{ & \text{(a.) } u_t = \kappa_1 u_{xx}, \quad 0 < x < \nu, \quad 0 < t < T, \quad \nu \leq \nu, \\
& \text{(b.) } u_t = \kappa_2 u_{xx}, \quad \nu < x < \infty, \quad 0 < t < T, \\
& \text{(c.) } \lim_{x \uparrow \nu} u(x, t) = \lim_{x \downarrow \nu} u(x, t), \quad 0 < t < T, \\
& \text{(d.) } \lim_{x \uparrow \nu} \rho_1 c_1 \kappa_1 u_x(x, t) = \lim_{x \downarrow \nu} \rho_2 c_2 \kappa_2 u_x(x, t), \quad 0 < t < T, \\
& \text{(e.) } u(x, 0) = 0, \quad 0 < x < \infty, \\
& \text{(f.) } u(0, t) = f(t), \quad 0 < t < T, \\
& \text{(g.) } \rho_2 c_2 \kappa_2 u_x(x_*, t_0) = h, \quad 0 < t_0 < T, \quad x_* > \nu,
\end{align*}
\]

where \(u(x, t)\) and \(\nu\) are unknowns.

The equivalent non-linear algebraic equation for \(\nu\) is

\[
(4.3) \quad h = -\rho_2 c_2 \kappa_2 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{\pi \kappa_2(t_0-\tau)}} \exp\left\{\frac{-(x_*-\nu+\frac{\xi}{\kappa_2})^2}{4 \kappa_2(t_0-\tau)}\right\} \, d\tau.
\]

This equation possesses exactly one positive solution \(\nu \leq \nu\) if in addition to the hypothesis on \(f(t)\) in sections 1 and 2, it is assumed that

\[
(4.4) \quad \begin{cases} 
\text{(a.) if } \kappa_1 < \kappa_2, \quad p < h < q, \quad \text{or} \\
\text{(b.) if } \kappa_2 < \kappa_1, \quad q < h < p,
\end{cases}
\]

where

\[
(4.5) \quad p = -\rho_2 c_2 \kappa_2 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{\pi \kappa_2(t_0-\tau)}} \exp\left\{\frac{-x_*^2}{4 \kappa_2(t_0-\tau)}\right\} \, d\tau
\]

and

\[
(4.6) \quad q = -\rho_2 c_2 \kappa_2 \int_0^{t_0} \frac{f'(\tau)}{\sqrt{\pi \kappa_2(t_0-\tau)}} \exp\left\{\frac{-(x_*-\nu+\frac{\xi}{\kappa_2})^2}{4 \kappa_2(t_0-\tau)}\right\} \, d\tau.
\]

Finally, it can be shown that the solution \(\nu\) of (4.3) depends continuously upon the data.
APPENDIX

1. Introduction. The object of this appendix is to find an explicit representation of a solution of the problem

\[
\begin{align*}
(a.) & \quad u_t = \kappa_1 u_{xx}, \quad 0 < x < \gamma, \quad 0 < t < T, \hfill \\
(b.) & \quad u_t = \kappa_2 u_{xx}, \quad \gamma < x < \infty, \quad 0 < t < T, \hfill \\
(c.) & \quad \lim_{x \to \gamma} u(x,t) = \lim_{x \to \gamma} u(x,t), \quad 0 < t < T, \hfill \\
(d.) & \quad \lim_{x \to \gamma} \rho_i c_i \kappa_1 u_x(x,t) = \lim_{x \to \gamma} \rho_i c_i \kappa_2 u_x(x,t), \quad 0 < t < T, \hfill \\
(e.) & \quad u(x,0) = 0, \quad 0 < x < \infty, \hfill \\
(f.) & \quad u(0,t) = f(t), \quad 0 < t < T, \hfill 
\end{align*}
\]

where \( \rho_i, c_i, \) and \( \kappa_i, \ i = 1,2, \) are the densities, the specific heats and the thermal diffusivities respectively, and to specify conditions on solutions of (1.1) which will imply the uniqueness of the explicitly represented solution.

In the remainder of the introduction an outline of the procedure used to obtain the representation will be given. This procedure is only a formalism for obtaining the representation. Thus, when the representation has been formally obtained, the remainder of the paper will be concerned with establishing the fact that it is a solution and that under a given definition of a solution of (1.1) it is the only solution of (1.1).

If \( f(t) \equiv V, \) a constant, then Carslaw and Jaeger [2] showed that

\[
\begin{align*}
U(x,t) &= V - V \sum_{n=0}^{\infty} \alpha^n \left\{ \text{erf} \left( \frac{(2n+1)\gamma + x - \gamma}{2\sqrt{\kappa_1 t}} \right) \right\}, \\
& \quad \alpha \text{ erf} \left( \frac{(2n+1)\gamma - x + \gamma}{2\sqrt{\kappa_1 t}} \right), \quad 0 < x < \gamma, \quad 0 < t < T, \\
U(x,t) &= V - \frac{2V}{1 + \sigma} \sum_{n=0}^{\infty} \alpha^n \text{ erf} \left( \frac{(2n+1)\gamma + kx - k\gamma}{2\sqrt{\kappa_1 t}} \right), \quad \gamma < x < \infty, \quad 0 < t < T, 
\end{align*}
\]
where

\[(1.3) \quad k = \frac{K_1}{K_2^{1/2}}, \]

\[(1.4) \quad \sigma' = \frac{\rho_2 c_2 K_2^{1/2}}{\rho_1 c_1 K_1^{1/2}}, \]

\[(1.5) \quad \alpha = \frac{\sigma - 1}{\sigma + 1}, \]

and

\[(1.6) \quad \text{erf} \ [x] = \frac{2}{\sqrt{\pi}} \int_0^x \exp \left[-\frac{s^2}{2}\right] \, ds, \]

satisfied \((1.1)\). By defining \(U(x, t)\) in \((1.2)\) to be identically zero for \(t \leq 0\), it follows that \(U(x, t)\) satisfies

\[
(1.7) \quad \begin{cases}
U_t = K_1 U_{xx}, & 0 < x < \nu, \quad -\infty < t < T, \\
U_t = K_2 U_{xx}, & \nu < x < \infty, \quad -\infty < t < T, \\
\lim_{x \uparrow \nu} U(x, t) = \lim_{x \downarrow \nu} U(x, t), & -\infty < t < T, \\
\lim_{x \uparrow \nu} \rho_1 c_1 K_1 U_x(x, t) = \lim_{x \downarrow \nu} \rho_2 c_2 K_2 U_x(x, t), & -\infty < t < T, \\
U(x, 0) = 0, & 0 < x < \infty, \\
U(0, t) = V, & 0 < t < T, \\
U(0, t) = 0, & -\infty < t < 0.
\end{cases}
\]

Also, the function

\[(1.8) \quad F(x, t) = U(x, t-\tau_1) - U(x, t-\tau_2), \quad T > \tau_2 > \tau_1, \]

satisfies
\[
\begin{cases}
F_t = \kappa_1 F_{xx}, & 0 < x < \nu, \quad -\infty < t < T, \\
F_t = \kappa_2 F_{xx}, & \nu < x < -\infty, \quad -\infty < t < T, \\
\lim_{x \uparrow \nu} F(x,t) = \lim_{x \downarrow \nu} F(x,t), & -\infty < t < T, \\
\lim_{x \uparrow \nu} \rho_1 c_1 \kappa_1 F_x(x,t) = \lim_{x \downarrow \nu} \rho_2 c_2 \kappa_2 F_x(x,t), & -\infty < t < T, \\
F(x,t) \equiv 0, & 0 < x < \infty, \quad -\infty < t \leq \tau_1, \\
F(0,t) \equiv 0, & \tau_2 \leq t < T, \\
F(0,t) \equiv V, & \tau_1 \leq t \leq \tau_2.
\end{cases}
\]

Hence, it is evident that for \( V = 1, \)

\[
G(x,t) = \sum_{i=1}^{n} \phi_i \left\{ U(x,t-\tau_{i-1}) - U(x,t-\tau_i) \right\},
\]

where \( \phi_i, \ i = 1, \ldots, n \) are constants and

\[
\tau_0 < \tau_1 < \ldots < \tau_{n-1} < \tau_n < T,
\]

satisfies

\[
\begin{cases}
G_t = \kappa_1 G_{xx}, & 0 < x < \nu, \quad -\infty < t < T, \\
G_t = \kappa_2 G_{xx}, & \nu < x < -\infty, \quad -\infty < t < T, \\
\lim_{x \uparrow \nu} G(x,t) = \lim_{x \downarrow \nu} G(x,t), & -\infty < t < T, \\
\lim_{x \uparrow \nu} \rho_1 c_1 \kappa_1 G_x(x,t) = \lim_{x \downarrow \nu} \rho_2 c_2 \kappa_2 G_x(x,t), & -\infty < t < T, \\
G(x,t) \equiv 0, & 0 < x < \infty, \quad -\infty < t < 0, \\
G(0,t) \equiv \phi_i, & \tau_{i-1} < t < \tau_i, \ i = 1, \ldots, n, \ \tau_0 = 0, \tau_n = T.
\end{cases}
\]

By the mean value theorem,

\[
G(x,t) = -\sum_{i=1}^{n} \phi_i \frac{\partial U}{\partial t}(x,t-\tau_i) [\tau_i - \tau_{i-1}],
\]

where \( \tau_{i-1} \leq \mu_i \leq \tau_i \). This suggests immediately that formally a solution of (1.1) is given by the representation
\[
\begin{aligned}
&\left\{\begin{array}{l}
  u(x,t) = \int_0^t f(\tau) \left\{ \sum_{n=0}^{\infty} \alpha^n \left[ \frac{(2n+1)\nu + x - \nu \beta}{2 \sqrt{\xi} \kappa_1(t-\tau)^{3/2}} \right. \\
  \quad \left. - \frac{\xi(2n+1)\nu - x + \nu \beta}{2 \sqrt{\xi} \kappa_1(t-\tau)^{3/2}} \right] \exp \left\{ \frac{-((2n+1)\nu + x - \nu \beta)^2}{4 \kappa_1(t-\tau)} \right\} \right\} \, d\tau,

  0 < x < \nu, \quad 0 < t < T,
\end{array}\right.
\end{aligned}
\]

\[
\left\{\begin{array}{l}
  u(x,t) = \int_0^t f(\tau) \left\{ \sum_{n=0}^{\infty} \alpha^n \left[ \frac{(2n+1)\nu + kx - k\nu}{2 \sqrt{\xi} \kappa_1(t-\tau)^{3/2}} \right. \\
  \quad \left. - \frac{\xi(2n+1)\nu + kx - k\nu}{2 \sqrt{\xi} \kappa_1(t-\tau)^{3/2}} \right] \exp \left\{ \frac{-((2n+1)\nu + kx - k\nu)^2}{4 \kappa_1(t-\tau)} \right\} \right\} \, d\tau,

  \nu < x < \infty, \quad 0 < t < T,
\end{array}\right.
\]

where \( k, \sigma' \) and \( \alpha \) are defined by (1.3), (1.4), and (1.5), respectively.

**Remark:** This type of argument was used by Weber [9] to derive a representation of the solution of the heat equation for a rectangle.

2. Convergence of the integrals and their formal derivatives.

Formally, it can be seen by differentiating the functions in the series with respect to \( x \) and \( t \) that in order to show the convergence of the integrals and their formal derivatives, it suffices to consider the series of the form

\[
(2.1) \quad \sum_{n=0}^{\infty} \frac{c_n(x, \nu)^r}{\sigma' s/2} \exp \left\{ \frac{-c_n(x, \nu)^2}{4 \kappa_1 \sigma'} \right\}, \quad \sigma' > 0,
\]

where \( c_n(x, \nu) \) is a linear function of \( x \) and \( n \) with \( \nu \) multiplying the portion of the function containing \( n \) and \( (r,s) \) can be one of the following points:

\[
(2.2) \quad (0,3), (1,3), (1,5), (2,5), (3,7).
\]
The method of proof of the absolute and uniform convergence of the above series is the same for all of them. Hence, consider \((r,s) = (3,7)\).

Now, the function

\[
(2.3) \quad \Phi(\sigma') = \frac{c^3}{\sigma^{7/2}} \exp \left\{ \frac{-c^2}{4K_1\sigma} \right\}, \quad \sigma > 0, \ c > 0,
\]

has its maximum value at

\[
(2.4) \quad \sigma' = \frac{c^2}{14K_1}.
\]

Thus,

\[
(2.5) \quad \Phi(\sigma') \leq \frac{(14K_1)^{7/2} \exp \left\{ \frac{-7/2}{c^4} \right\}}{c^4}.
\]

Hence, the series in (2.1) is majorized by

\[
(2.6) \quad \sum_{n=0}^{\infty} \frac{(14K_1)^{7/2} \exp \left\{ \frac{-7/2}{c^4} \right\}}{c^4 n}.
\]

Using this argument, it follows that the series in (1.14) and their formal derivatives converge uniformly and absolutely for \(0 < \delta \leq x\).

Hence, it follows that if \(f(t)\) is integrable in the sense of Lebesgue, then the integrals in (1.14) converge. Moreover, it follows by Lebesgue's dominated convergence theorem that the derivatives of \(u(x,t)\) for \(0 < x < \gamma, \gamma < x < \infty\), and \(0 < t < T\) exist and are given by Leibniz's rule.

3. **Demonstration that \(u(x,t)\) is a solution of (1.1).**

**Definition 3.1:** A function \(v(x,t)\) is a solution of (1.1) if and only if the following conditions are satisfied:

(a.) \(v(x,t)\) is continuous in \((x,t)\) for \(0 \leq x < \infty\),
\(0 \leq t < T\), \(v(x,t)\) is bounded for \(0 \leq x < \infty\),
\(0 \leq t \leq t_1 < T\);

(b.) \(v_x, v_t, \text{ and } v_{xx}\) exist and are continuous in \((x,t)\) for
\(0 < x < \gamma, \gamma < x < \infty\), and \(0 < t < T\);

(c.) (1.1) is satisfied by \(v(x,t)\).
Now, it follows immediately from the absolute and uniform convergence of the series of section 2 that \( u(x,t) \) satisfies requirement of (b.) of the definition.

Consider part (c.) of the definition. Again from the convergence properties of the series of section 2, it can be verified directly that \( u(x,t) \) satisfies (1.1.a.b.e). Using the mean value theorem and Lebesgue's dominated convergence theorem, it can be shown that the limits (1.1.c.d) are taken on uniformly by \( u(x,t) \) and \( u_x(x,t) \) for \( 0 \leq s \leq t \leq t_1 \leq T \). Note that this uniformity is actually \( 0 \leq t \leq t_1 \leq T \) for \( u(x,t) \). Since \( u(x,t) \) can be written for \( 0 \leq x \leq \infty \) as

\[
(3.1) \quad u(x,t) = u_1(x,t) + u_2(x,t),
\]

where \( u_1(x,t) \) is the solution to the usual quarter plane problem with \( f(t) \) as data and \( u_2(x,t) \) is a perturbation term which tends to zero uniformly for \( 0 \leq t \leq t_1 \leq T \) as \( x \to 0 \), it follows that \( u(x,t) \) satisfies condition (c.) of the definition. Moreover, it follows from the uniformity of the limits and the convergence properties of the series that \( u(x,t) \) satisfies (a.) of the definition. Hence, \( u(x,t) \) is a solution of (1.1) in the sense of Definition 3.1.

4. **Uniqueness.** Consider the problem

\[
\begin{align*}
\frac{w_t}{w_t} &= K_1 w_{xx}, & 0 \leq x \leq \infty, & 0 \leq t \leq T, \\
\frac{w_t}{w_t} &= K_2 w_{xx}, & \infty \leq x \leq \infty, & 0 \leq t \leq T, \\
\lim_{x \to 0} w(x,t) &= \lim_{x \to 0} w(x,t), & 0 \leq t \leq T, \\
\lim_{x \to \infty} w(x,t) &= \lim_{x \to \infty} w(x,t), & 0 \leq t \leq T, \\
\lim_{x \to 0} \rho_{c_1} K_1 w_x(x,t) &= \lim_{x \to \infty} \rho_{c_2} K_2 w_x(x,t), & 0 \leq t \leq T, \\
w(x,0) &= 0, & 0 \leq x \leq \infty, \\
w(0,t) &= 0, & 0 \leq t \leq T.
\end{align*}
\]

(4.1)

It will be shown that any solution of (4.1) in the sense of Definition 3.1 must be identically zero.
Assume that \(w(x,t)\) is a solution of (4.1) in the sense of Definition 3.1. Let \(\Phi(t)\) denote \(w(y,t)\). Then,

\[
\begin{align*}
\frac{\partial M_1(x - y, K_1(t - \tau), \nu)}{\partial x} & \quad \Phi(\tau) K_1 \, d\tau,
0 < x < y, \quad 0 < t < T, \\
\frac{\partial M_2(x - y, K_2(t - \tau), \nu)}{\partial x} & \quad \Phi(\tau) K_2 \, d\tau,
\nu < x < \infty, \quad 0 < t < T,
\end{align*}
\]

where

\[
M_1(x - y, K_1(t - \tau), \nu) = \frac{1}{\sqrt{\nu x K_1(t - \tau)}} \sum_{n=-\infty}^{\infty} \exp\left\{ -\frac{(x - y + 2n \nu)^2}{4 K_1(t - \tau)} \right\}, \quad t > \tau, \\
M_2(x - y, K_2(t - \tau), \nu) = \frac{1}{\sqrt{\nu x K_2(t - \tau)}} \exp\left\{ -\frac{(x - y)^2}{4 K_2(t - \tau)} \right\}, \quad t > \tau.
\]

Now, it must be shown that the conditions of Definition 3.1 imply that \(\Phi \equiv 0\). Suppose the contrary, i.e., \(\Phi(t) \neq 0\). Let \(t_1, \, 0 < t_1 < T\), be sufficiently large so that \(\Phi(t) \neq 0\) in \(0 \leq t \leq t_1\). Let \(t_0, \, 0 < t_0 < t_1\), be a point at which \(\Phi(t)\) assumes either its maximum or minimum value for \(0 \leq t \leq t_1\). If \(t_0 < t_1\), then a direct application of Theorem 2 in Friedman's paper on the maximum principle for parabolic equations [5] implies that either
(4.5) \[ \lim_{x \uparrow \nu} w_x(x, t_0) > 0 \]
for a maximum of \( \varphi(t) \), or

(4.6) \[ \lim_{x \downarrow \nu} w_x(x, t_0) < 0 \]
for a minimum of \( \varphi(t) \). If \( t_0 = t_1 \), then, by using the function

\[
(4.7) \quad \Phi(t) = \begin{cases} 
\varphi(t), & 0 \leq t \leq t_0 \\
\varphi(t_0), & t_0 \leq t \leq t_0 + \epsilon, \quad \epsilon > 0,
\end{cases}
\]

as boundary data for the rectangle \( 0 < x < \nu, \ 0 < t < t_0 + \epsilon \) instead of \( \varphi(t) \), it follows again from Theorem 2 [5] that either \( (4.5) \) or \( (4.6) \) holds. From a direct application of the maximum principle for bounded solutions of the quarter plane problem, it follows that either

(4.8) \[ \lim_{x \downarrow \nu} w_x(x, t_0) \leq 0 \]
for a maximum of \( \varphi(t) \), or

(4.9) \[ \lim_{x \downarrow \nu} w_x(x, t_0) \geq 0 \]
for a minimum of \( \varphi(t) \). Thus, \( w(x, t) \) cannot satisfy \( (4.1) \) if \( \varphi(t) \neq 0 \). Hence, if \( w(x, t) \) is a solution of \( (4.1) \), then

\[ \varphi(t) \equiv 0 \]
which implies via \( (4.2) \) that \( w(x, t) \equiv 0 \). Therefore, the following theorem is valid.

**Theorem 4.1:** There exists one and only one solution of \( (1.1) \) in the sense of Definition 3.1. Moreover, the solution is given by \( (1.14) \).

**Remark:** The uniqueness argument can be generalized to cover the case of a composite region composed of a finite number of finite parts and a semi-infinite part.
References


