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UMI
SOME GENERALIZATIONS FOR DIRICHLET SERIES OF
HADAMARD'S THEOREM, WITH APPLICATIONS

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A thesis presented to the faculty of the Rice Institute in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Houston, Texas
February, 1944
INTRODUCTION
The problem of obtaining information concerning the singular points of a function given by a Dirichlet series is similar in nature to the corresponding problem for functions given by Taylor series. There are, however, many important differences; in fact, the Dirichlet series may be regarded as a generalization of the Taylor series. It is true that every analytic function has a Taylor series, while only almost periodic functions have Dirichlet series. However, if a function \( F(s) \) is analytic in a complete neighborhood of the origin, the function \( f(s) = F(e^{i\theta}) \) has a Dirichlet series expansion: for a Dirichlet series is one of the form \[ \sum a_n e^{-\lambda_n s}, \] where the \( a_n \) are complex constants, the \( \lambda_n \) are non-negative numbers, with \( \lambda_n < \lambda_{n+1} \) for each \( n \), \( \lim_{n \to \infty} \lambda_n = \infty \) (for elementary results in the theory of Dirichlet series, see [1], chapter 1, or [5]). Thus if \( F(s) = \sum a_n s^n \), then \( f(s) = F(e^{i\theta}) = \sum a_n e^{-\lambda_n s} \).

Such a Dirichlet series, whose sequence of exponents is composed entirely of integers, has been called by S. Mandelbrojt a Taylor-Dirichlet series.
An equation of the type \( f(s) = \sum a_n e^{-\lambda s} \) implies, throughout this paper, that the function is given by the series where the series converges, and is defined by analytic continuation elsewhere.

Many results pertaining to the theory of singular points of functions given by Taylor series have immediate generalisations for the theory of Dirichlet series; others do not. One well-known theorem for Taylor series whose obvious generalisation for Dirichlet series fails to hold, is the "multiplication theorem" of Hadamard. In terms of Taylor-Dirichlet series, this theorem states that if

\[
f(s) = \sum a_n e^{-\lambda_n s}, \quad \psi(s) = \sum b_n e^{-\psi_n s},
\]

then the singular points of \( H(s) \) are of the form \( \lambda + \psi \), where \( \lambda \) is singular for \( f(s) \), \( \psi \) is singular for \( \psi(s) \); or, stated more accurately, if \( A_f \) and \( A_\psi \) are the abscissae of convergence of \( f(s) \) and \( \psi(s) \) respectively, and if

\[ S_{f, \psi} \]

denotes the set of all points of the form \( \lambda + \psi \), where \( \lambda \) is singular for \( f(s) \), \( \psi \) is singular for \( \psi(s) \), then \( H(s) \) is holomorphic in that part of the complement of \( S_{f, \psi} \) which is connected with the half-plane \( R(s) > Q_f + Q_\psi \).

The obvious generalisation of this theorem for Dirichlet
series is not true. A counter-example oftentimes is furnished by the Riemann Zeta-function, \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\sigma} \log n \). Here \( H(s) = H(1, \sigma) \equiv \zeta(s) \). But the only singular point of \( \zeta(s) \) is the point \( s = 1 \), so that the only point of \( S_{\zeta} \) is the point \( 2 \); the singular point \( s = 1 \) of \( H(s) \) does certainly not belong to \( S_{\zeta} \).

3. Mandelbrojt has given theorems ([1] and [2]) which generalise Hadamard's "multiplication theorem". Another such theorem is given by Bohr (Annals of Math., v. 41, 1940, p. 711), and applies to almost periodic functions in general, as well as to functions given by Dirichlet series. These theorems define regions of holomorphy of \( H(f, \zeta) \), the "composite" function, in terms of regions of holomorphy of \( f(s) \) and \( \zeta(s) \), the "generating" functions. One theorem of Mandelbrojt ([3], p. 5), in particular, defines regions of holomorphy of \( H(f, \zeta) \) in terms of regions in certain half-planes in which \( f(s) \) and \( \zeta(s) \) satisfy certain boundedness conditions. The conditions imposed by Bohr on the "generating" functions are also boundedness conditions, but of a somewhat different
nature from those imposed by Mandelbrojt, in that they relate to the "stars" of the functions. Mandelbrojt's results are perhaps more comprehensive, in that they give wider regions of holomorphy for the composite function.

The first theorem of the present paper is a generalization of the theorem of Mandelbrojt, and is proved by his method; this generalization is then used in defining regions of holomorphy of \( H(f, \varphi) \) in terms of regions where \( f(s) \) and \( \varphi(s) \) fail to take certain values, with the aid of a theorem of Schottky which has been used by Mandelbrojt in a similar manner ([4], p. 86). The first theorem of the present paper is also used in proving a theorem (1.5) very similar to that of Mandelbrojt mentioned just above, with somewhat less restrictive conditions on \( f(s) \) and \( \varphi(s) \). The theorems of the second and third chapters are largely applications of theorem 1.5.

As has been noted above, Hadamard's "multiplication theorem" does not state that \( H(f, \varphi) \) is holomorphic in the complete complement of the set \( S_{f, \varphi} \), but only in that part of the complement of \( S_{f, \varphi} \) which is
connected with the half-plane \( \mathbb{H}(e) \supset \Omega_f + \Omega \). Similar statements are made in many of the theorems of this paper. By the statement that a certain set is connected with a certain half-plane is to be understood that the point-set sum of the set and the half-plane is a connected set. The term "curve", when used, refers to a simple, rectifiable, Jordan curve. The only use of the term "curve" in this paper is in connection with the fact that a necessary and sufficient condition that a set be connected is that every pair of points of the set can be connected by a simple, rectifiable, Jordan curve. The term "curve" could therefore be replaced everywhere in the paper by the term "polygonal line".

All of the functions to be considered will be assumed to be uniform, unless otherwise stated. A point will be considered singular for a function, if the function can not be continued analytically throughout a complete neighborhood of that point.

No distinction will be made between complex numbers and the points of the complex s-plane (s = \( \sigma + it \)) which they represent. No distinction
in language is made between points of a set and the equation
or inequality satisfied by those points: for example, the
set of points with real part greater than \( c \) is referred
to as the half-plane \( Re(s) > c \). Often the symbol
\( ( Re(s) > c ) \) is used to refer to the set of points of
this half-plane.

The symbols \( +, \cdot, \supset, \subset, \in, \notin \) are
borrowed from set theory: \( A + B \) is the set of all points
each of which is in either \( A \) or \( B \); \( A \cdot B \) is the set
common to both; \( A \supset B \) and \( A \subset B \) mean respectively,
\( A \) contains \( B \) and \( A \) is contained in \( B \); \( x \in A \) means
\( x \) belongs to \( A \); \( x \notin A \) means \( x \) fails to belong to \( A \).
For example, \( A \cdot ( I(s) > t ) \) is the set of points in
\( A \) with imaginary part greater than \( t \).

The symbols \( A_f \) and \( H_f \) are borrowed from
V. Bernsteins' exposition [1] of a theorem of
Mandelbrojt. \( A_f \) is the abscissa of absolute convergence
of the Dirichlet series which represents the function
\( f(s) \). \( H_f \) is the abscissa of holomorphism of \( f(s) \).
It will be assumed throughout this paper that every
function \( f(s) \) which is represented by a Dirichlet
series has \( A_f < \infty \).
Numbers in brackets ([ ] ) refer to the bibliography, which appears at the end of the paper.

I wish to acknowledge my indebtedness to Professor S. Mandelbrojt for his invaluable assistance during the period of research which led to the writing of this thesis. Many of the ideas developed in the thesis came as suggestions from Professor Mandelbrojt; others were inspired by some of his published papers, while others grew naturally out of the research as it progressed. Professor Mandelbrojt also read the manuscript carefully, made valuable suggestions in several instances regarding the method of presentation, and helped in correcting errors in the proofs of some of the theorems.
Certain properties of the two "generating" functions 
\[ f(s) = \sum c_n e^{-\lambda_n s} , \quad \varphi(s) = \sum h_n e^{-\lambda_n s} \] in certain regions permit the use of those regions in defining a region of holomorphy of the "composite" function 
\[ H(f,\varphi) = H(s) = \sum c_n h_n e^{-\lambda_n s} . \] It will be seen later, for example, that if \( f(s) \), \( \varphi(s) \) fail to take certain values in certain regions, those regions may be so used. Also, the theorem of Mandelbrojt \(^3\) uses half-planes where \( f(s) \) and \( \varphi(s) \) satisfy certain conditions of boundedness in defining a region in which \( H(s) \) is holomorphic. Professor Mandelbrojt suggested that it might also be interesting to obtain a general theorem which defines a region of holomorphy for \( H(s) \) using regions where the functions \( f(s) \) and \( \varphi(s) \) are bounded; it was found that such a theorem can easily be given, using the method of Mandelbrojt in his paper \(^3\) referred to above.

The introduction of an operator on sets, \([A, B]\) will simplify the notation; this symbol represents a set of points determined by the sets \( A \) and \( B \). The set \([A, B]\) is the closure (the set plus its limit points) of the set of all points \( \alpha + \beta \), where \( \alpha \) is any point of \( A \), \( \beta \) any point of \( B \). It is, by
definition, a closed set.

Let $M(r)$ be any positive function of the real positive variable $r$, strictly increasing to infinity as $r$ becomes infinite (it will later be convenient to write $M(r) = M_1(1/r) \sim M_1(s)$, so that $M_1(s)$ is a monotone function of real positive $s$, increasing to infinity as $s \to \infty$).

Let $B_\varphi(r)$ be a connected set defined for $r > r_0$ (for some positive number $r_0$) containing the half-plane $R(s) > c$ for some real number $c$ greater than $A_\varphi$, such that in $B_\varphi(r)$ $f(s)$ is holomorphic and $|f(s)| < M(r)$; further let the sets be so chosen that $B_\varphi(r') \supset B_\varphi(r)$ for $r' > r$. Such sets certainly exist, if $f(s) = \sum a_n e^{-\lambda n s}$ has a finite abscissa of absolute convergence $A_\varphi$; for let $c$ be any real number greater than $A_\varphi$; then since $M(r)$ increases to infinity as $r \to \infty$, there exists a positive $r_0$ such that $|f(s)| \leq \sum |a_n| e^{-\lambda n s} < M(r_0)$ in $R(s) > c$. Hence

$B_\varphi(r_0)$ may even be defined so as to include the half-plane $R(s) > c$. It is clearly possible, since $M(r)$ increases to infinity as $r \to \infty$, to define $B_\varphi(r)$ for $r > r_0$ so that $B_\varphi(r') \supset B_\varphi(r)$ for $r' > r$.  


By $B_\rho(r)$, then, will be understood a member of any one-parameter (\(r\)) family of connected sets, defined for \(r > r_1\), with the following properties:

1. \(f(s)\) is holomorphic, and \(|f(s)| \leq M(r)\) in $B_\rho(r)$;

2. $B_\rho(r)$ contains the half-plane $R(s) \geq \epsilon$ for some real number $\epsilon$ greater than $A_\rho$;

3. $B_\rho(r') \supset B_\rho(r)$ for $r' > r$.

Define similarly $B_\alpha(r)$ for $r > r_2$ so that

1'. $\alpha(s)$ is holomorphic, and \(|\alpha(s)| \leq M(r)\) in $B_\alpha(r)$;

2'. $B_\alpha(r)$ contains the half-plane $R(s) \geq \epsilon'$ for some real number $\epsilon'$ greater than $A_\alpha$;

3'. $B_\alpha(r') \supset B_\alpha(r)$ for $r' > r$.

Put \(r_0 = \max(r_1, r_2)\) so that (1), (2), (3), (1'), (2'), and (3') hold for $r > r_0$.

Let $U_\rho(r)$ denote the complement (in the complex plane) of $B_\rho(r)$, $U_\alpha(r)$ the complement of $B_\alpha(r)$. $U_\rho(r)$ is contained in the half-plane $R(s) < \epsilon$, $U_\alpha(r)$ in the half-plane $R(s) < \epsilon'$. Define

$$U_{\rho\alpha}(r) = [U_\rho(r), U_\alpha(r)],$$

the closure of the set of all points $\alpha + \rho$, where
\( \forall \in U_\alpha(r), \beta \in U_\alpha(r); U_{\beta \alpha}(r) \) is a closed set in the half-plane \( R(s) \leq s + \epsilon' \). Since for \( \epsilon' > r \)

\( B_r(\epsilon') \supset B_{\epsilon}(r) \) and \( B_{\epsilon}(\epsilon') \supset B_{\alpha}(r) \), also

\( U_r(\epsilon') \subset U_r(r) \) and \( U_{\epsilon}(\epsilon') \subset U_{\alpha}(r) \). It follows that \( U_{\beta \alpha}(r') \subset U_{\epsilon \alpha}(r) \) for \( r' > r \).

Each set \( U_{\beta \alpha}(r) \) is by definition closed (not necessarily bounded). The limit set as \( r \) becomes infinite exists, is the intersection of all sets \( U_{\beta \alpha}(r) \) (\( r \) takes on all values greater than \( r_0 \)), and is therefore necessarily closed. Define

\[ U_{\beta \alpha} = \lim_{\epsilon \to \alpha} U_{\beta \alpha}(r). \]

Let \( B_{\beta \alpha}(r) \) denote that part of the open complement of \( U_{\beta \alpha}(r) \) which is connected with the half-plane

\[ R(s) > s + \epsilon'. \] Since \( B_{\beta \alpha}(r) \) is a component (a maximum connected subset, by the definition of Hausdorff in "Mengenlehre", Gruyter and Co., Berlin, 1927) of the closed set \( U_{\beta \alpha}(r) \), \( B_{\beta \alpha}(r) \) is open. Also, for \( r' > r \), \( B_{\beta \alpha}(r') \supset B_{\beta \alpha}(r) \), since \( U_{\beta \alpha}(r') \subset U_{\beta \alpha}(r) \).

The limit set as \( r \) becomes infinite exists, is the sum of all sets \( B_{\beta \alpha}(r) \) as \( r \) takes on all values greater than \( r_0 \), and is therefore necessarily open. This is the set \( B_{\beta \alpha} = \lim_{\epsilon \to \alpha} B_{\beta \alpha}(r) \), the set of all points, each
of which belongs to at least one set \( B_{f}(r) \) (hence to \( B_{f}(r') \) for all \( r'>r \)).

The set \( B_{f} \) is that part of the complement of \( U_{f} \) which is connected with the half-plane \( R(s)>c+s' \). For if \( x \in B_{f} \), then there exists a positive number \( r_{x}>r_{o} \) such that \( x \in B_{f}(r_{x}) \). Then \( x \) can be joined to the half-plane \( R(s)>c+s' \) by a curve which contains no point of \( U_{f}(r_{x}) \), hence no point of \( U_{f} \), the intersection of all sets \( U_{f}(r) \). Also, if \( x \notin U_{f} \), and if there is a curve \( L \) joining \( x \) to \( R(s)>c+s' \) which contains no point of \( U_{f} \), then \( x \in B_{f} \). For to every point \( y \) of \( L \) corresponds a (circular) neighborhood of \( y, N_{y} \), and a positive number \( r_{y} \) such that \( N_{y} \) contains no point of \( U_{f}(r_{y}) \) \((y \) belongs to the complement of \( U_{f} \) and hence to the complement of \( U_{f}(r_{y}) \) for \( r_{y} \) sufficiently large--this complement is an open set). By the Borel-Lebesgue theorem, there are then a finite number \( K \) of such neighborhoods \( N_{y_{n}} \) \((n \geq 1, 2, ..., K) \) which cover the curve \( L \). If \( \gamma = \max_{n \leq K} r_{y_{n}} \), then for each \( n \leq K \), \( N_{y_{n}} \) contains no point of \( U_{f}(r_{y_{n}}) \), hence no point of \( U_{f}(\gamma) \). Therefore \( L \) contains no point of \( U_{f}(\gamma) \), \( x \in B_{f}(\gamma), x \in B_{f} \).
Thus $E_{\Re \alpha}$ is that part of the complement of $U_{R(s)}$ which is connected with the half-plane $R(s) > \alpha + \alpha'$.

The set of points $E_{\Re \alpha}$ can be shown to constitute a region of holomorphism for $H(s)$. In fact, a point set $\tilde{E}_{\Re \alpha}$ which contains $E_{\Re \alpha}$ ($E_{\Re \alpha}$ and $\tilde{E}_{\Re \alpha}$ may be identical) can be defined, which is also a region of holomorphism for $H(s)$. In order to define $\tilde{E}_{\Re \alpha}$, let $U_{\mathbb{C}}(r; t)$ be the intersection of $U_{\mathbb{C}}(r)$ and the half-plane $I(s) > t$: $U_{\mathbb{C}}(r; t) = U_{\mathbb{C}}(r) \cap (I(s) > t)$; $U_{\mathbb{C}}(r; t)$ is the set of those points of $U_{\mathbb{C}}(r)$ having imaginary part greater than $t$. Define $U_{\mathbb{C}}(r; t) = \tilde{E}_{\Re \alpha}(r; t)$, $U_{\mathbb{C}}(\infty)$. As before, $U_{\mathbb{C}}(r''; t) \subseteq U_{\mathbb{C}}(r; t)$ for $r'' > r$. Each set $U_{\mathbb{C}}(r; t)$ is closed, so that again the limit set $U_{\mathbb{C}}(\infty; t)$ exists, is closed, and is the set of all points, each of which belongs to $U_{\mathbb{C}}(r; t)$ for all $r > R(s)$.

Again, it is clear that for $t'' > t$, $U_{\mathbb{C}}(\infty; t'')$ is contained in $U_{\mathbb{C}}(\infty; t)$. Hence the limit set as $t$ becomes infinite exists, is closed, and is the set of all points, each of which belongs to $U_{\mathbb{C}}(\infty; t)$ for every real $t$. This set will be denoted by $\tilde{E}_{\Re \alpha}$.

Let $E_{\Re \alpha}(r; t)$ be that part of the complement of $U_{\mathbb{C}}(r; t)$ which is connected with $R(s) > \alpha + \alpha'$. 
Then \( B_{f_a}(r'; t) \supset B_{f_a}(r; t) \) for \( r' > r \), and the limit set as \( r \) becomes infinite exists, is open, and is the sum of all sets \( B_{f_a}(r; t) \) for all \( r > r_o \); it will be denoted by \( B_{f_a}(\infty; t) \). Here too, as can be shown by an argument similar to that employed above, \( B_{f_a}(\infty; t) \) is that part of the complement of \( U_{f_a}(\infty; t) \) which is connected with the half-plane \( R(s) > c + c' \). Also, 
\( B_{f_a}(\infty; t') \supset B_{f_a}(\infty; t) \) for \( t' > t \). The limit set as \( t \) becomes infinite exists, is open, and is the set of all points, each of which belongs to some \( B_{f_a}(\infty; t_o) \), hence to \( B_{f_a}(\infty; t) \) for all \( t > t_o \). It will be denoted by \( \hat{B}_{f_a} \), and is that part of the complement of \( U_{f,a} \) which is connected with the half-plane \( R(s) > c + c' \).

It is clear that the set \( \hat{B}_{f,a} \) contains the set \( B_{f,a} \) defined previously. Indeed \( B_{f,a} \supset B_{f,a}(\infty; -\infty) \).
Theorem 1.1: If \( f(s) \equiv \sum \alpha_n e^{-\lambda_n s} \), \( \Phi(s) \equiv \sum \beta_n e^{-\lambda_n s} \),
if both series have finite abscissas of absolute convergence, then
\( H(f, \Phi) = H(s) \sum \alpha_n \beta_n e^{-\lambda_n s} \)
is holomorphic in \( \mathcal{D}_f \mathcal{D}_\Phi \), hence also in \( \mathcal{D}_f \mathcal{D}_\Phi \).

Proof: Let \( \xi \) be the real number, greater than \( \mu_f \), of the definition of \( \mathcal{D}_f \mathcal{D}_\Phi \). Let to be an arbitrary real positive number. Consider

\[
(1) \lim_{T \to \infty} \frac{1}{T} \int_{c+iT}^{c+iT_0} f(s) \Phi(s-s) \, ds,
\]
the path of integration being the straight line segment, parallel to the imaginary axis, which joins \( c + i \xi \) and \( c + iT \). It will first be demonstrated that the limit (1) exists and is equal to \( H(s) \sum \alpha_n \beta_n e^{-\lambda_n s} \) for \( R(s) > (\mu_f + \xi) \). It will appear incidentally that the series \( \sum \alpha_n \beta_n e^{-\lambda_n s} \) converges absolutely for \( R(s) > \mu_f + \xi \). Since \( \xi \) is any real number greater than \( \mu_f \), it follows that \( \mu_f < \mu_f + \xi \).

For fixed \( s \) in the range \( R(s) > (\mu_f + \xi) \), the series
\[
\sum b_n e^{\lambda_n(s-s)}
\]
converges uniformly to \( \Phi(s-s) \) for \( s \) on the path of integration. Hence term-wise integration is permissible:
\[
H(T,s) = \frac{1}{iT} \int_{c+iT}^{c+iT} f(s) e^{\lambda_n s} ds
\]

the series converging for \( R(s) > \Re \alpha + \varepsilon \). It is desired to show that \( \lim_{T \to \infty} H(T,s) \) exists and is equal to \( H(s) \) for \( R(s) > \Re \alpha + \varepsilon \).

It is well known (cf. for example [5], p. 15) that

\[
\lim_{T \to \infty} \frac{1}{iT} \int_{c+iT}^{c+iT} f(s) e^{\lambda_n s} ds = a_n \quad \text{thus}
\]

\[
\frac{1}{iT} \int_{c+iT_0}^{c+iT} f(s) e^{\lambda_n s} ds = a_n + e_n \quad \text{where}
\]

\[
\lim_{T \to \infty} e_n(T) = 0
\]

Also

\[
\left| \frac{1}{iT} \int_{c+iT_0}^{c+iT} f(s) e^{\lambda_n s} ds \right| \leq M e^{-\lambda_n T}
\]

for all \( T > T_0 \)

where \( M = \sum |a_n| e^{-\lambda_n T} \). Also \( \sum a_n b_n e^{-\lambda_n T} \) converges absolutely for \( R(s) > \Re \alpha + \varepsilon \). For

\[
\sum_{n \neq \ell} \left| a \right|^2 \sum_{n \neq \ell} \left| b_\ell \right|^2 \sum_{n \neq \ell} \left| a_n b_\ell \right| e^{-\lambda_n T} \leq M \sum_{n \neq \ell} \left| b_\ell \right|^2 e^{-\lambda_n T}
\]

\[
\sum_{n \neq \ell} \left| a_n \right|^2 e^{-\lambda_n T}, \quad \text{this last}
\]

series approaching zero as \( p \) becomes infinite, since \( R(s-\varepsilon) > \Re \alpha \).
Let \( f \) be arbitrary, positive. Choose a positive integer \( p_0 \) so that \( \sum_{n=1}^{\infty} \left| b_n \right| e^{-\lambda_n R(z-c)} \leq \frac{\delta}{3} \). Then also
\[
\sum_{n=1}^{\infty} \left| a_n b_n e^{-\lambda_n^2} \right| \leq \frac{\delta}{3}.
\]
But
\[
\left| H(T,s) - H(s) \right| = \left| \sum_{n=0}^{\infty} b_n \cdot e^{-\lambda_n^2} \left[ \epsilon_n \cdot \epsilon_{\eta}(T) \right] \right|
\]
\[
= \sum_{n=0}^{\infty} a_n b_n e^{-\lambda_n^2} \cdot \epsilon_n \cdot \epsilon_{\eta}(T)
\]
\[
\leq \frac{\delta}{3}.
\]
\[
\sum_{n=0}^{\infty} b_n \cdot e^{-\lambda_n^2} \int_{c+iT}^{c+iT+1} f(s) \cdot e^{\lambda_n s} \, ds
\]
\[
\leq M \sum_{n=0}^{\infty} \left| b_n \right| e^{-\lambda_n R(z-c)} \leq \frac{\delta}{3}.
\]
Let \( \eta_0 = \sum_{n=0}^{\infty} \left| b_n \right| e^{-\lambda_n R(z-c)} \). Then for \( n \leq p_0 \), \( T > T_0 \),
\[
\left| e_{\eta}(T) \right| < \frac{\delta}{3 M_0}.
\]
Hence for \( T > T_0 \)
\[
\sum_{n=0}^{\infty} b_n \cdot e^{-\lambda_n^2} \left| \epsilon_{\eta}(T) \right| \leq \frac{\delta}{3 M_0} \sum_{n=0}^{\infty} \left| b_n \right| e^{-\lambda_n R(z-c)} = \frac{\delta}{3}.
\]
Therefore, given an arbitrary positive \( \delta \), there exists a positive number \( T_0 \) such that \( \left| H(T_2 s) - H(s) \right| < \delta \) for \( T > T_0 \). It has now been established that

\[
(2) \quad H(s) = \lim_{T \to \infty} \frac{1}{iT} \int_{C+iT} f(s) \varphi(s-s) \ ds, \quad \text{for } R(s) > \alpha + \epsilon. 
\]

The first part of the proof is completed. The second part consists of showing that the function \( H(s) \) given by (2) can be continued analytically in \( \hat{B}_{\infty} \).

Let \( s \) be any point of \( \hat{B}_{\infty} \). There then exists a positive \( t_0 \) such that \( s \in B_{\infty \alpha}(\alpha^2 t_0) \) (since \( \hat{B}_{\infty \alpha} \) is the sum of all sets \( B_{\infty \alpha}(\alpha^2 t) \)). There exists also a positive \( r_0 \) such that \( s \in B_{\infty \alpha}(r_0, t_0) \) (since \( B_{\infty \alpha}(\alpha^2; t_0) \) is the sum of all sets \( B_{\infty \alpha}(r; t_0) \)).

Then there exists a curve \( L \) joining \( s \) to \( H(s) > \epsilon + \epsilon' \) which contains no point of \( U_{\infty \alpha}(r_0, t_0) \). Since \( L \) and \( U_{\infty \alpha}(r_0, t_0) \) are disjoint, closed sets, the distance between them is positive. It is therefore possible to choose a positive number \( \epsilon \) so that \( L \) is completely interior to an open region \( \Delta \), whose boundary is at a distance greater than \( \delta \) from every point of \( U_{\infty \alpha}(r_0, t_0) \); since \( U_{\infty \alpha}(r_0, t_0) \) lies in the half-plane \( H(s) < \alpha + \epsilon' \).
the domain $\Delta$ may be chosen so as to include points with arbitrarily large real part. It is now desired to show that $R(s)$ can be continued analytically throughout $\Delta$.

Let $x_0 \in \text{bd } R(s)$ which is finite; choose $\sigma'$ real so that $x_0 = \sigma' - U_{e'} > 6 e$; then for all $s$ in $\Delta$, $R(s) = \sigma' - U_{e'} < e$. The reason for this choice of $\sigma'$ will appear later.

Pave the strip $\sigma' < R(s) < \sigma$ with squares of side $e$ (choose so that $\sigma \equiv \sigma' \text{ (mod } e) \text{ )}. Extract those (closed) squares, with their neighbors, which contain points of $U_{r_\sigma}(s_0; t_0)$; extract also those bordering on the line $R(s) = \sigma'$. Denote by $D_e$ that part of the open region in the half-plane $R(s) > \sigma'$ which remains, and is connected with the half-plane $R(s) > \sigma$; and by $C_e$ its boundary. Let $\{e_m\}$ be a sequence of positive numbers greater than $t_0$, monotonically increasing to infinity as $m$ becomes infinite (for convenience, choose the sequence so that $e_m \equiv t_0 \text{ (mod } e) \text{ for each } m$). Denote by $D_{e_m}^\infty$ that part of $D_e$ contained in the strip $t_0 < I(s) < \infty$ which is connected with the half-plane $R(s) > \sigma$; by $C_{e_m}^\infty$ its boundary, excluding segments of lines $I(s) = t_0$, $I(s) = \sigma_m$, and by $L_{e_m}$ the length of $C_{e_m}^\infty$. Note
that the length $L_{\epsilon}^{\Lambda}$ is certainly less than the sum of the sides of the squares of the paving in $r^{x} R(s) \leq \epsilon$, $t_{0} < I(s) \leq \epsilon$; thus $L_{\epsilon}^{\Lambda} > \frac{\Lambda}{\epsilon} (\epsilon - \sigma') + \frac{\Lambda}{\epsilon} (\epsilon - \sigma') \Lambda_{\epsilon}$, $L_{\epsilon}^{\Lambda} < \Lambda_{\epsilon}^{\Lambda}$, where $\Lambda_{\epsilon}$ depends only on $\epsilon$ (and on $\epsilon, \sigma'$).

Define

$$U_{\epsilon}(s) = \frac{1}{\Lambda_{\epsilon}^{\Lambda}} \int_{\Lambda_{\epsilon}^{\Lambda}} f(s) \eta (s-a) ds .$$

If $s$ is on $\Lambda_{\epsilon}^{\Lambda}$, either

(a) $|s - \lambda| < \delta \in$ for some $\lambda \in U_{\epsilon}(r_{\delta} s t_{0})$; or

(b) $|s - \sigma'| - \delta < \delta \in$ for some $t > t_{0}$.

But for $s$ in $\Lambda$, $|s - (\lambda + \beta)| > \delta \in$ for all $\lambda \in U_{\epsilon}(r_{\delta} s t_{0}), \beta \in U_{\epsilon}(r_{\delta})$, since every point of $\Lambda$ is at distance greater than $\delta \in$ from $U_{\epsilon}(r_{\delta} s t_{0})$, which is the set $\left[ U_{\epsilon}(r_{\delta} s t_{0}), U_{\epsilon}(r_{\delta}) \right]$.

If (a) holds, then

$$|s - s - \beta| > |s - \lambda - \beta| = |s - \lambda| > \delta \in ,$$

for all points $\beta \in U_{\epsilon}(r_{\delta})$. If (b) holds, then

$R(s) = \sigma' < \delta \in$. But $R(s) = \sigma' \cap \delta \sigma > \delta \in$ ,

$R(s) > \sigma' + \delta \sigma > \delta \in$; $R(s - \epsilon - \beta) > \sigma' + \beta \sigma + \delta \in = \sigma'$

$- \delta \in = R(\beta)$. 

But \( R(\beta) \leq A_q \), hence

\[
\{ s - s - \beta \} \geq R(s - s - \beta) > 3 \in .
\]

Therefore if \( s \) is on \( \mathcal{C}_a \), \( s-s \) is exterior to \( \mathcal{U}_q(\mathcal{R}_s) \), hence interior to \( \mathcal{B}_\varphi(\mathcal{R}_s) \), the complement of \( \mathcal{U}_{cp}(\mathcal{R}_s) \) (\( \mathcal{R}_s \), it will be remembered, is fixed, and depends only on \( \Delta \), is independent of \( s \) in \( \Delta \)). Thus for fixed \( s \) in \( \Delta \), \( s \) on \( \mathcal{C}_a \), \( \varphi(s-s) \) is holomorphic and

\[
\| \varphi(s-s) \| < M(\mathcal{R}_s).
\]

Also, since \( \mathcal{C}_a \) is interior to \( \mathcal{B}_\varphi(\mathcal{R}_s) \), \( \varphi(s) \) is holomorphic and

\[
\| \varphi(s) \| < M(\mathcal{R}_s)
\]

for \( s \) on \( \mathcal{C}_a \). It follows that \( H_m(s) \) is holomorphic in \( \Delta \). Moreover, \( H_m(s) \) is bounded in \( \Delta \), uniformly for all \( m \). For \( L^a < K \alpha \), hence

\[
\| H_m(s) \| < \frac{1}{\alpha} [M(\mathcal{R}_s)]^2 K \alpha \times K [M(\mathcal{R}_s)]^2,
\]

which is independent of \( m \).

The sequence of functions \( \{ H_m(s) \} \) thus forms a bounded family of holomorphic functions in \( \Delta \), hence forms a normal family in \( \Delta \). It will now be shown that the sequence \( \{ E_m(s) \} \) converges to \( H(s) \) for \( R(s) \) sufficiently large. The Stieltjes-Vitali theorem then assures its uniform convergence in \( \Delta \) to a function holomorphic in \( \Delta \), thus proving that \( H(s) \) can be
continued analytically throughout $\Delta$. Denote by $J_m$ that part of the boundary of $D^{\alpha_m}_c$ which is made up of segments of the lines $I(s) \equiv t_0$, $I(s) \equiv \alpha_m$. Then

$$\lim_{m \to \infty} \frac{1}{i\alpha_m} \int_{J_m} f(s) \varphi(s-s) \, ds = 0 \quad \text{for } R(s)$$

sufficiently large; for on $J_m$, $|f(s)| < M(r_0)$; $\varphi(s-s)$ is bounded for $R(s)$ sufficiently large, and the length of $J_m$ is bounded uniformly for all $m$.

Since $D^{\alpha_m}_c$ is connected with the half-plane $R(s) > 0$, the curve $C$ formed of $C^{\alpha_m}_c$, $J_m$, and the line segment $s = a + it$, $t_0 \leq t \leq \alpha_m$, is composed of a finite number of closed curves, so that

$$\int_C f(s) \varphi(s-s) \, ds = 0 \quad \text{for } R(s)$$

sufficiently large (since for $s$ in $R(s) \leq 0$, $R(s-a) > \alpha m$ for $R(s)$ sufficiently large). It follows from (3) and (4) that

$$\lim_{m \to \infty} \frac{1}{i\alpha_m} \int_{C^{\alpha_m}_c} f(s) \varphi(s-s) \, ds = \lim_{m \to \infty} \frac{1}{i\alpha_m} \int_{C+it_0} f(s) \varphi(s-s) \, ds,$$
the integral around \( C^e \) being taken in the appropriate sense; thus, \( \lim_{m \to \infty} \mathcal{E}_m(s) = \mathcal{E}(s) \) for \( s \) sufficiently large.

Since the family of functions \( \mathcal{E}_m(s) \) is normal in \( \Delta \) (\( \Delta \) was chosen so as to contain points with arbitrarily large real part), and since it has now been demonstrated that \( \mathcal{E}_m(s) \to \mathcal{E}(s) \) on those points of \( \Delta \) with sufficiently large real part, it follows from the Stieltjes-Vitali theorem that \( \{ \mathcal{E}_m(s) \} \) converges uniformly in every closed region in \( \Delta \) to a function holomorphic in \( \Delta \), which is the analytic continuation of \( \mathcal{E}(s) \). This completes the proof of the theorem.

It may be noted that, interchanging the roles of \( f(s) \) and \( \phi(s) \), \( \mathcal{E}(s) \) is also holomorphic in \( \hat{B}_{\phi f} \), hence in \( \hat{B}_{\phi f} \cup \hat{B}_{\phi f} \) (since both \( \hat{B}_{\phi f} \) and \( \hat{B}_{\phi f} \) are connected with the half-planes \( \Re(s) > c \pm \epsilon \), so also is \( \hat{B}_{\phi f} \cup \hat{B}_{\phi f} \).
Before stating the theorem of Mandelbrojt referred to above (page 1), it will be necessary to make the following definitions:

A function \( f(s) \) is said to be bounded except for singularities (bounded e.f.s.) in a region if to every positive \( \varepsilon \) corresponds a positive \( M \in \) such that when circles of radius \( \varepsilon \) about the singular points of \( f(s) \) are extracted from the region, \( |f(s)| < M \varepsilon \) in the part of the region remaining.

If \( f(s) = \sum a_n s^{-n} \), \( g(s) = \sum b_n s^{-n} \) have finite abscissae of absolute convergence \( A_f \) and \( A_g \) respectively, denote by \( S_f \subset S_f^o \) the set of singular points of \( f(s) \) in \( \sigma < B(s) \leq A_f \), and by \( S_g \subset S_g^o \) the set of singular points of \( g(s) \) in \( \sigma < B(s) \leq A_g \). Denote by \( S_f(t) \) that part of \( S_f \) which lies in the half-plane \( R(s) > t \). Define \( S_{f, g} = [S_f, S_g] \), the closure of the sets of all points \( \alpha + \beta \), where \( \alpha \in S_f, \beta \in S_g \). Define also \( S_{f, g}(t) = [S_f(t), S_g(t)] \). Evidently \( S_{f, g}(t^+) \subset S_{f, g}(t) \) for \( t^+ > t \); hence \( \lim_{t \to \infty} S_{f, g}(t) \) exists, is closed, and is the intersection of all sets \( S_{f, g}(t) \) as \( t \) takes on all real values.
For a given \( t \), let \( \sigma_t \) be the real part of the
singular point of \( f(s) \) on the horizontal line \( I(s) = t \)
with greatest real part. If there is no singular point
with real part greater than \( \sigma_t \) on \( I(s) = t \), put \( \sigma_t = \sigma_t \).
Let \( \sigma_t^\circ = \max \{ \lim_{t \to -\infty} \sigma_t, \lim_{t \to \infty} \sigma_t \} \). (This is
not precisely the definition given by Mandelbrojt; in
[3] Mandelbrojt defines \( \sigma_t^\circ = \lim_{t \to \infty} \sigma_t \). Define
similarly \( \sigma_\varphi^\circ \) with respect to \( \varphi(s) \) and \( \sigma_\varphi \). With
these definitions, a theorem which is essentially that
of Mandelbrojt \(( [3], \text{page 5}) \) can be stated:

If \( f(s) = \sum a_n e^{-\lambda_n s} \) and \( \varphi(s) = \sum b_n e^{-\lambda_n s} \)
are bounded e.f.s. in \( H(s) > \sigma_t \), \( R(s) > \sigma_\varphi \) respectively,
then \( H(f, \varphi) = H(s) = \sum a_n b_n e^{-\lambda_n s} \) is holomorphic
in that part of the complement of \( \Sigma \lambda \), which lies in the
half-plane \( R(s) > \max (\sigma_t + \sigma_\varphi, \sigma_t + \sigma_\varphi) \) and which
is connected with the half-plane of absolute convergence
of \( H(s) \).

The theorem as given by Mandelbrojt imposes somewhat
less restrictive conditions on the function \( f(s) \), and
states the result somewhat differently; however, it is
in essence the same as that stated above. The method of
proof of theorem 1.1 of this paper is patterned on Mandelbrojt's proof of this theorem.

It will be convenient now, for use in establishing further results later, to make use of a condition slightly less restrictive than that implied by the term "bounded e.f.s.". It involves no radical change, and the theorem of Mandelbrojt still holds true under the new condition. Suppose \( f(s) \) is a function uniformly bounded in some half-plane \( R(s) > \sigma \). Let \( \sigma_t \) be a real number less than \( \sigma \). Let \( L(\delta) \) be a positive, decreasing function of the real positive variable \( \delta \), strictly increasing to infinity as \( \delta \to 0 \). As above, \( S_{t'} \) is the set of singular points of \( f(s) \) in \( R(s) > \sigma_t \). Define \( \Phi_{t'}(\delta) \) as the set of points in \( R(s) > \sigma_t + \delta \), each of which can be joined to the half-plane \( R(s) > \sigma_t \) by a curve of length less than \( L(\delta) \) in \( R(s) > \sigma_t + \delta \), at distance greater than \( \delta \) from the set \( S_{t'} \). This set, \( \Phi_{t'}(\delta) \), is the set of all curves of length less than \( L(\delta) \) in \( R(s) > \sigma_t \) having one end-point in \( R(s) > \sigma_t \), which are central lines of (open) channels in \( R(s) > \sigma_t \) of width greater than \( 2\delta \) which contain no points of \( S_{t'} \). The boundary of such a channel consists of a curve.
on each side of $L$, similar to $L$, these curves being obtained by displacing each point of $L$ a distance $\rho > \delta$ ($\rho$ is half the width of the channel) in a direction normal to $L$ at the point, also two semicircles, each with center at one end-point, radius $\rho$. In other words, such a channel is the open region consisting of all circles of radius $\rho$ having centers on $L$.

The complement in $R(s) > \sigma_1$ of $\Theta_\rho(\delta)$ is denoted by $S_\rho(\delta)$, and is the set of points in $\sigma_1 < R(s) < \delta$ consisting of points of closed circles of radius $\delta$ about points of $S_\rho$, and also those points of $\sigma_1 < R(s) < \delta$ which cannot be joined to the half-plane $R(s) > \sigma_1$ by curves of length less than $L(\delta)$ in $R(s) > \sigma_1 + \delta$ which contain no points of these circles.

Definition The function $f(s)$ is "$M$" in $R(s) > \sigma_1$, if there exists a positive decreasing function $L(\delta)$ of the positive variable $\delta$, strictly increasing to infinity as $\delta \to 0$, such that to each positive $\delta$ corresponds a positive number $M(\delta)$ such that $|f(s)| < M(\delta)$ for $s$ in $\Theta_\rho(\delta)$. 
Thus if \( f(s) \) is "M" in \( R(s) > \sigma_1 \), if \( x \) is a point of this half-plane, and if \( x \) can be joined to \( R(s) > c \) by a curve in \( R(s) > \sigma_1 \) of length less than \( L(s) \) at distance greater than \( \delta \) from every singular point of \( f(s) \), then \( |f(x)| < M(\delta) \).

If \( f(s) \) is "M" in \( R(s) > \sigma \), for functions \( L(s) \) and \( M(s) \), then \( f(s) \) is also "M" in \( R(s) > \sigma \) for functions \( L_1(s) \) and \( M_1(s) \), where \( M_1(s) > M(s) \) (for each positive \( \delta \)), and where \( L_1(s) \) increases to infinity as \( \delta \to 0 \), \( L_1(s) < L(s) \) (for each positive \( \delta \)). Evidently, then, if \( \phi(s) \) also is "M" in some half-plane \( R(s) > \sigma_2 \), there are functions \( L(s), M(s) \) such that both \( f(s) \) and \( \phi(s) \) are "M" for the same functions \( L(s) \) and \( M(s) \). For fixed \( \delta \), let the value of the common \( L(s) \) be the smaller of the values of the two separate \( L \)'s, and the value of the common \( M(s) \) be the larger of the values of the two separate \( M \)'s.

It is clear also that if a function is bounded e.f.f., in a certain half-plane, it is also "M" in that half-plane.

The converse is not necessarily true. It is possible to envisage a set of singular points such that the corresponding function is "M" in a certain half-plane, but not bounded e.f.f.
If $S_\sigma^\alpha \ominus \mathcal{O}$ denotes that part of $S_\sigma^\alpha (\mathcal{O})$ consisting of circles of radius $\rho$ about points of $S_\sigma^\alpha$, then $f(s)$ is bounded e.f.s. in $R(s) > \sigma_i$ if to each $\mathcal{O}$ corresponds an $M(\mathcal{O})$ such that $|f(s)| < M(\mathcal{O})$ in the complement (in $R(s) > \sigma_i$) of $S_\sigma^\alpha \ominus \mathcal{O}$. This is not implied by the hypothesis that $f(s)$ is $M$ in $R(s) > \sigma_i$. For it is possible to imagine a set of singular points of $f(s)$ with the property that, given any function $L(\mathcal{O})$ which increases to infinity as $\rho \to 0$, and given any positive $\delta$, there are points of the half-plane $R(s) > \sigma_i$ which are not included in $S_\sigma^\alpha \ominus \mathcal{O}$ and which can not be reached by curves in $R(s) > \sigma_i$ of length less than $L(\mathcal{O})$ which pass through no points of $S_\sigma^\alpha \ominus \mathcal{O}$. Such points belong to $S_\sigma^\alpha (\mathcal{O})$, the complement of $S_\sigma^\alpha (\mathcal{O})$. The function $f(s)$ need not then be bounded on this set of points, so that it can not be concluded that $f(s)$ is bounded e.f.s. in $R(s) > \sigma_i$.

The theorem of Mandelbrojt stated above was originally proved by the method on which the proof of theorem 1.1 is based. It is not surprising then, that this theorem can be derived as a consequence of theorem 1.1. The purpose of the discussion in the following pages is to
prove, using theorem 1.1, that the theorem of Mandelbrojt still holds true when the "bounded e.f.s." condition is replaced by the somewhat less restrictive "M" condition. The accomplishment of this aim is facilitated by the introduction of another operator $[A, B] L$, similar to the operator $[A, B]$ (page 2). A lemma relating the two will prove useful.

Let $A$ be a set of points in the strip $c < R(s) < c_1$, $B$ a set in $c < R(s) < c_2$. The real numbers $c_1, c_2$ are supposed finite, either $c_1$ or $c_2$ or both may be negatively infinite. Let $L(\delta)$ be a positive, decreasing function of the positive variable $\delta$, strictly increasing to infinity as $\delta \to 0$. Let $A(\delta)$ be the complement in $R(s) > c$ of the set of all curves of length less than $L(\delta)$, having one end-point in $R(s) > c_1$, which are central lines of (open) channels in $R(s) > c$ of width greater than $\delta$ containing no points of $A$. A point $x$ in $R(s) > c$ fails to belong to $A(\delta)$ if and only if there is a curve in $R(s) > c + \delta$ of length less than $L(\delta)$ which joins $x$ to $R(s) > c_2$, and which is at distance greater than $\delta$ from every point of $A$. The set $A(\delta)$ consists of closed circles of radius $\delta$ about points of
A, plus every point such that every curve in \( R(s) > \sigma_1 + \delta \)
of length less than \( L(\delta) \) which joins the point to
\( R(s) > \sigma \) contains points of these circles. Define
similarly \( B(\delta) \) with respect to \( B, \sigma_2 \)

Since \( L(\delta) \) increases to infinity as \( \delta \to 0 \), it
is clear that \( A(\delta') \subseteq A(\delta) \) for \( \delta' < \delta \), and
\( B(\delta') \subseteq B(\delta) \) for \( \delta' < \delta \). Hence also
\[
\left[ A(\delta'), B(\delta') \right] \subseteq \left[ A(\delta), B(\delta) \right] \text{ for } \delta' < \delta.
\]

Here again the limit set (as \( \delta \to 0 \)) exists, is the set
of points common to all \( \left[ A(\delta), B(\delta) \right] \) as \( \delta \) takes on
all positive values, and will be denoted by \( [A, B]^L \):
\[
[A, B]^L \equiv \lim_{\delta \to 0} \left[ A(\delta), B(\delta) \right].
\]

Let \( \sigma' \) be any real number greater than or equal to
\( \max (\sigma_1 + \sigma_1, \sigma_2 + \sigma_2) \); if \( \sigma_1 = -\infty \), \( \sigma_2 = -\infty \),
then \( \sigma' \) may be negatively infinite. Denote by \( T_{\sigma'}^L \)
that part of \( [A, B]^L \) which lies in the half-plane
\( R(s) > \sigma' \), and by \( T_{\sigma}^L \) that part of \( [A, B]^L \) which lies
in this half-plane. Let \( T_{\sigma} \) denote that part of the
complement in \( R(s) > \sigma' \) of \( T_{\sigma}^L \), which is connected with
the half-plane \( R(s) > \sigma_1 + \sigma_2 \) (there are no points of
\[ [A, B]^L \] or \((A, B)^L\) in \(R(s) > e_1 + e_2\). Let \(\mathcal{A}_{L}^{s} \) denote that part of the complement in \(R(s) > \sigma'\) of \(\mathcal{A}_{L}^{s} \), which is connected with the half-plane \(R(s) > e_1 + e_2\). There may then be points of the strip \(\sigma' < R(s) < e_1 + e_2\) which belong neither to \(T_{s}^{L} \), nor to \(T_{s}^{L} \). Denote the set of these points by \(J_{s}^{L} \). If \(x\) is any point of \(J_{s}^{L} \), then \(x \notin T_{s}^{L} \), but every curve in \(R(s) > \sigma'\) which joins \(x\) to \(R(s) > e_1 + e_2\) contains points of \(T_{s}^{L} \). Define similarly \(J_{s}^{L} \), as the set of points in \(R(s) > \sigma'\) which belong neither to \(T_{s}^{L} \), nor to \(\mathcal{A}_{L}^{s} \). If a point \(x\) fails to belong to \(T_{s}^{L} \) \((T_{s}^{L} \) it belongs to \(\mathcal{A}_{L}^{s} \) or \(J_{s}^{L} \) or \(T_{s}^{L} \), \(J_{s}^{L} \)), according as there is or is not a curve in \(R(s) > \sigma'\) which joins \(x\) to \(R(s) > e_1 + e_2\) and contains no point of \(T_{s}^{L} \).

It can then be shown that \(\mathcal{A}_{L}^{s} \subseteq \mathcal{A}_{L}^{s} \); or, what is equivalent, that \(T_{s}^{L} + J_{s}^{L} \subseteq \mathcal{A}_{L}^{s} \subseteq J_{s}^{L} \). This means that that part of the complement in \(R(s) > \sigma'\) of \([A, B]^L \) is \((R(s) > \sigma')\) which is connected with \(R(s) > e_1 + e_2\) is the same as that part of the complement in \(R(s) > \sigma'\) of \([A, B] \) \((R(s) > \sigma')\) which is
connected with \( R(s) > \sigma_1 + \sigma_2 \).

In fact, somewhat more can be shown, and will be used. Let \( t \) be any real number, let \( A_\delta(\delta) \) be the set \( A(\delta) = \{ I(s) > t + \delta \} \), the set of points in \( A(\delta) \) with imaginary part greater than \( t + \delta \); as before,

\[
[A, B]_c^L = \lim_{\delta \to 0} [A_\delta(\delta), B(\delta)]
\]
exists. Also

\[
[A, B]_c^L = \lim_{\delta \to 0} [A_\delta B]_c^L
\]
extists. Let \( A_t = A^\circ(I(s) > t) \).

For a given \( t \), let \( \alpha_t \) be the real part of the point of \( A \) on the line \( I(s) = t \) with greatest real part. If there is no such point in \( R(s) > \sigma_t \), put \( \sigma_t = \infty \). Define

\[
\sigma_t^* = \max \left( \lim_{\epsilon \to 0^+} \alpha_t, \lim_{\epsilon \to 0^-} \alpha_t \right)
\]
(cf. \( \sigma_t^* \), page 18). Define similarly \( \sigma_B^* \).

For \( \sigma = \max(\sigma_t + \alpha_t, \alpha_t + \alpha_B) \), define

\[
[A, B]_c^L = [A_B^L]_{\infty}^{\infty}(R(s) > \sigma).
\]

Let \( \tilde{T} \) be that part of the complement in \( R(s) > \sigma \) of \( \tilde{T} \), which is connected with \( R(s) > \sigma_1 + \sigma_2 \). \( \tilde{T} \) that part of the complement in \( R(s) > \sigma \) of \( \tilde{T} \), which is connected with \( R(s) > \sigma_1 + \sigma_2 \). Define as before \( J_{\tilde{T}^\circ} \), \( J_{\tilde{T}^\circ} \), so that \( \tilde{T}^\circ \) \( (\tilde{T}^\circ) \) and \( \tilde{T}^\circ + J_{\tilde{T}^\circ} \) \((\tilde{T}^\circ + J_{\tilde{T}^\circ}) \) are complementary in \( R(s) > \sigma \). It can then be shown that \( \tilde{T}^\circ + J_{\tilde{T}^\circ} = \tilde{T}^\circ + J_{\tilde{T}^\circ} \), or, what is equivalent, that \( \tilde{T}^\circ + J_{\tilde{T}^\circ} = \tilde{T}^\circ + J_{\tilde{T}^\circ} \).
Lemma 1.2: \( \hat{\mathcal{Z}}_{\sigma} \subseteq \hat{\mathcal{Z}}_{\sigma} \); also, \( \hat{\mathcal{Z}}_{\sigma} \subseteq \hat{\mathcal{Z}}_{\sigma} \).

The first statement is equivalent to the statement
\( \hat{\mathcal{Z}}_{\sigma} \subseteq \hat{\mathcal{Z}}_{\sigma} \) \( \hat{\mathcal{Z}}_{\sigma} \subseteq \hat{\mathcal{Z}}_{\sigma} \) and it is this statement which will be proved. The idea is comparatively simple. Effectively, the set \( \hat{\mathcal{Z}}_{\sigma} \) is obtained by extracting from the half-plane \( \mathbb{R}(s) \geq \sigma \) all points of \( \hat{\mathcal{Z}}_{\sigma} \) \( \hat{\mathcal{Z}}_{\sigma} \), which is the set of all points in \( \mathbb{R}(s) \geq \sigma \) which can not be connected to \( \mathbb{R}(s) > \sigma_{1} + \sigma_{2} \) by curves in \( \mathbb{R}(s) \geq \sigma \) containing no points of \( \mathbb{R}(s) \geq \sigma \). The question is, all these points having been extracted, do any of the remaining points belong to \( \hat{\mathcal{Z}}_{\sigma} \)? The answer is no, and it follows, as will be seen later, that all points of the remaining region can be joined to \( \mathbb{R}(s) > \sigma_{1} + \sigma_{2} \) by curves in \( \mathbb{R}(s) \geq \sigma \) which contain no points of \( \hat{\mathcal{Z}}_{\sigma} \), hence belong to \( \hat{\mathcal{Z}}_{\sigma} \) as well as \( \hat{\mathcal{Z}}_{\sigma} \).

It is evident from the construction of the sets \( \mathbb{A}_{\sigma}^{L} \) and \( \mathbb{A}^{L}_{\sigma} \) (page 23) that
\( \mathbb{A}_{\sigma}^{L} \supset \mathbb{A}^{L}_{\sigma} \), hence also
\( \hat{\mathcal{Z}}_{\sigma} \subseteq \hat{\mathcal{Z}}_{\sigma} \); also, \( \hat{\mathcal{Z}}_{\sigma} \subseteq \hat{\mathcal{Z}}_{\sigma} \).
It is now sufficient then, to show that
\[ \widetilde{T}_\sigma^L \cap \widetilde{T}_\sigma^L \subseteq \widetilde{T}_\sigma^L \cap \widetilde{T}_\sigma^L \]. It will be shown first that
\[ \widetilde{T}_\sigma^L \subseteq \widetilde{T}_\sigma^L \cup \widetilde{T}_\sigma^L \]; that also \[ \widetilde{T}_\sigma^L \subseteq \widetilde{T}_\sigma^L \cap \widetilde{T}_\sigma^L \] will later be seen to follow immediately.

Suppose the contrary, that there is a point \( x \) of \( \widetilde{T}_\sigma \) which belongs neither to \( \widetilde{T}_\sigma \) nor to \( \widetilde{T}_\sigma \).

There is then a curve \( L \), of length \( l \), in \( R(s) > \sigma \), which joins \( x \) to \( R(s) > c_1 + c_2 \) and which contains no point of \( \widetilde{T}_\sigma \). Since both \( L \) and \( \widetilde{T}_\sigma \) are closed sets, the distance between them is positive. Since
\[ \widetilde{T}_\sigma = \lim_{\varepsilon \to 0} (A_\varepsilon B \cap (R(s) > c)) \], there exists a \( t_1 \) such that the distance between \( L \)
and \( [A_t, B] \) is positive, for each \( t > t_1 \). For to
each point \( y \) on \( L \) corresponds a (circular) neighborhood \( H_y \) of \( y \) and a real number \( t_y \) such that \( H_y \) contains no point of \( [A_{t_y}, B] \) (each \( [A_t, B] \) is closed, its complement open). The curve \( L \) is covered by these
neighborhoods, hence (Borel-Lebesgue theorem) by a finite
number \( K \) of neighborhoods \( H_{y_j} \) of points \( y_j \)
\((j = 1, 2, \ldots, K)\). Put \( t_1 = \max_{1 \leq j \leq K} t_{y_j} \); then the
distance \( d \) between \( L \) and \( [A_{t_1}, B] \) is positive;
for each \( t > t_1 \) the distance between \( L \) and \( [A_t, B] \)
is greater than or equal to $\rho$.  

Let $A_\delta \otimes$ denote that part of $A_\delta(\delta')$ which is made up of points of (closed) circles of radius $\delta$ about points of $A_\delta$; define similarly $B \otimes$ as that part of $B(\delta)$ made up of points of circles of radius $\delta$ about points of $B$. Then
\[
[A_\delta, B] \geq \lim_{\delta \to 0} [A_\delta \otimes, B \otimes];
\]
for if $y \in [A_\delta \otimes, B \otimes]$ for every positive $\delta'$, then to every positive $\epsilon$, and to every positive $\delta$, correspond points $\alpha$ of $A_\delta$, $\alpha'$ of $A_\delta \otimes$, $\beta$ of $B \otimes$, such that $|y - (\alpha + \epsilon')| < \epsilon$, $|\alpha - \alpha'| < \delta$, $|\beta - \epsilon'| < \delta$; hence $|y - (\alpha + \beta)| < \epsilon + 2\delta$; hence also $y \in [A_\delta, B]$; conversely, if $y \in [A_\delta, B]$, also $y \in \lim_{\delta \to 0} [A_\delta \otimes, B \otimes]$.

The Borel-Lebesgue theorem may be used as above to show that there then exists a positive $\delta'$, such that the distance $p'$ between $L$ and $[A_{\delta'} \otimes, B \otimes]$ is positive. For each $\delta < \delta'$, the distance between $L$ and $[A_{\delta} \otimes, B \otimes]$ is greater than or equal to $\rho'$; since $[A_{\delta} \otimes, B \otimes] \subset [A_{\delta'} \otimes, B \otimes]$ for $t > t_1$, then for each $\delta < \delta'$, for each $t > t_1$, the distance between
L and \( \tilde{T}_\delta \) is greater than or equal to \( \tilde{\rho}_1 \).

Since \( L \) lies in \( R(s) > \sigma \), there is a positive number \( \rho_2 \) such that \( L \) lies in \( R(s) > \sigma + 4 \rho_2 \). Choose \( \rho = \min(\rho_1, \rho_2) \); then for \( \delta < \delta_1 \), \( t > t_1 \), the distance between \( L \) and \( \tilde{T}_\delta \) is greater than \( \rho \), and \( L \) lies in \( R(s) > \sigma + 4 \rho \).

Since \( x \in \widetilde{T}_{\delta} \supseteq A_{\tilde{T}} \), \( \rho \), and \( x \in \widetilde{A}_{\tilde{T}} \) for every real \( t \).

Hence \( x \in \tilde{T}_{\delta} \) for every real \( t \), every positive \( \delta \). Hence \( x \) has in every neighborhood, for every real \( t \), every positive \( \delta \), a point \( \alpha + \beta \), where \( \alpha \in \tilde{A}_{\tilde{T}}(\delta) \), \( \beta \in B(\delta) \); that is, if \( \delta \), \( \in \) are any two positive numbers, \( t \) any real number, there is a point \( \alpha \) of \( \tilde{A}_{\tilde{T}}(\delta) \) and a point \( \beta \) of \( B(\delta) \) such that \( |x - (\alpha + \beta)| < \varepsilon \).

It is now desired to show that it is possible to so choose \( \varepsilon, \delta, \gamma \) that the above statements involve a contradiction; this contradiction must then arise from the assumption of the existence of a point \( x \) of \( \tilde{T}_{\delta} \) which fails to belong to \( \tilde{T}_{\delta} + J\tilde{T}_{\delta} \). The choice of \( \varepsilon, \delta, \gamma \) will be made as follows:
Let \( \varepsilon \) be fixed, less than \( \varepsilon_2 \).

Let \( d \geq \max \left( \sigma_1 - \sigma, \sigma_2 - \sigma_2 \right) \). There exists a positive number \( \delta_2 \) such that \( l(\delta) > l + d + \rho \) for \( \delta < \delta_2 \). Let \( \delta \leq \min \left( \delta_1, \delta_2, \delta_2 \right) \). Let \( \delta \) be fixed, less than \( \delta_1 \).

It will be recalled (page 25) that

\[
\sigma_2 = \max \left[ \lim_{n \to \infty} \sigma_n, \lim_{n \to \infty} \sigma_n \right].
\]

There therefore exists a positive number \( t_2 \) such that

\( R(\sigma) = \sigma < \rho \) for \( \sigma \in A \), \( I(\sigma) > t_2 \); hence such that

\( R(\sigma') = \sigma < \rho + \delta \) for \( \sigma' \in A_n \), \( t > t_2 \).

Similarly there exists a positive number \( t_3 \) such that

\( R(\beta) = \beta < \rho \) for \( \beta \in B \), \( I(\beta) \leq I(x) \leq t + \rho \), \( t > t_3 \); that is, such that

\( R(\beta') = \beta < \rho + \delta \) for \( \beta' \in B_2 \), \( I(\beta') < I(x) \leq t + \rho \), \( t > t_3 \).

Let \( t' = \min \left( I(a) - \delta_1 \right) \). There then exists a positive number \( t_4 \) such that \( t' > I(x) \geq t + \rho > t_1 + \rho \) for \( t > t_4 \).

Let \( t \) be fixed, greater than \( t_1, t_2, t_3, \) and \( t_4 \).

The reasons for these choices of \( \varepsilon, \delta, t \) will appear in the following discussion.

From the discussion prior to that above which resulted in the choices of \( \varepsilon, \delta, t \), it is seen that there exist a point \( \sigma \in A_n(\delta) \) and a point \( \rho \in B(\delta) \) such that
Then one of the four possibilities below must occur:

**Case 1:** \( \alpha \in A_t(\delta), \beta \in B(\delta) \).

Then \( |x - (\alpha + \beta)| < \epsilon < \rho \), but \( \lambda \neq \beta \) belongs to \([A_t(\delta), B(\delta)]\), and since \( \delta < \delta_1, t > t_1 \), the distance between \( L \) and \([A_t(\delta), B(\delta)]\), is greater than \( \rho \), a contradiction.

**Case 2:** \( \alpha \in A_t(\delta), \alpha \in A_t(\delta), \beta \in B(\delta) \).

Since \( |x - (\alpha + \beta)| < \epsilon < \rho \), there is a curve \( L' \) (the curve \( L \) plus the line segment joining \( x \) and \( (\alpha + \beta) \)) which lies in \( R(s) \geq \sigma + 3\rho \) (\( L \) lies in \( R(s) \geq \sigma + 4\rho \), page 50), joins \( \alpha + \beta \) to \( R(s) > e_1 + e_2 \), and is at distance greater than \( \ell/2 \) from all points of \([A_t(\delta), B(\delta)]\). Subtract \( \ell \) from each point of \( L' \), obtaining the curve \( L' - (\beta) \). The curve \( L' - (\beta) \) is of length less than \( \ell + \rho \) (hence less than \( L(\delta) \), see page 51), joins \( \alpha \) to \( R(s) > e_1 + e_2 - R(\delta) \) (hence to \( R(s) > e_1 \), since \( R(\delta) < e_2 \)), and is the central line of a channel of width greater than \( \ell/2 \) (hence greater than \( \delta \), see page 51) which contains no point of \( A_{t_1}(\delta) \) (for if this channel contains a point
\( \alpha \in A_t(\delta) \), then \( L' \) is at distance less than or equal to \( \frac{\rho}{2} \) from a point \( \alpha_1 \in \bigcup_{2} \bigcap_{B(\delta)} \). For all points \( y \) on \( L' \),

\[
R(y) > \sigma \quad + \quad 3\rho \quad > \quad \sigma \quad + \quad \sigma_B \quad + \quad 3\rho .
\]

Also
\[
I(\alpha) > \gamma, \quad I(\beta) < I(x) = I(\alpha) + \epsilon < I(x) < \gamma + \rho .
\]

Hence \( R(\beta) = \sigma_B < \rho + \delta < 2\rho \) (page 31). But for \( y \) on \( L' \), \( y - \beta \) on \( L' - (\beta) \),

\[
R(y) - R(\beta) > \sigma_1 \quad + \quad \sigma_B \quad + \quad 3\rho = R(\beta)\
\]

\[
> \sigma_1 \quad + \quad \rho \quad > \quad \sigma_1 \quad + \quad \delta \quad (\text{page 31}).
\]

Hence \( L' - (\beta) \) lies in \( R(s) > \sigma_1 + \delta \).

Let \( t' = \min \) \( \begin{array}{cc} I(y) - \frac{\rho}{2} \end{array} \). Then \( L \) contains only points with imaginary part greater than \( t' + \frac{\rho}{2} \).

\( L' \) contains only points with imaginary part greater than \( t' \). But \( I(\beta) < I(x) = I(\alpha) + \epsilon < I(x) = I(\alpha) + \rho \), and \( I(\alpha) > t_1 \) hence \( L' - (\beta) \) contains only points with imaginary part greater than \( t' = I(x) + t - \rho \)

which is greater than \( t_1 + \delta \) (page 31).

But \( A_t(\delta) \subseteq A(\delta) \). Thus if \( \alpha \in A_t(\delta) \), also \( \alpha \in A(\delta) \). But the curve \( L' - (\beta) \) lies in \( R(s) > \sigma_1 + \delta \), has length less than \( L(\delta) \), joins \( \alpha \) to \( R(s) > \sigma_1 \), and is the central line of a channel of width greater than \( \delta \) which contains no point of \( A_t(\delta) \subseteq A(\delta) \).
every point of this curve having imaginary part greater
than \( t_1 + \delta \). Hence this curve contains no point of
\( A(\delta) \), so that \( \lambda \notin A(\delta) \), a contradiction.

**Case 3:** \( \lambda \in A_1(\delta) \), \( \beta \in B(\delta) \), \( \beta \notin B(\delta) \).

Consider here the curve \( L' = (\lambda) \) which joins 3
to \( R(s) > \sigma_2 \) and which contains no point of \( B(\delta) \) for
reasons similar to those discussed in case 2. Here,
for \( y \) on \( L' \), \( R(y) > \delta - 3 \rho > \sigma_2 + \sigma_1 + 3 \rho \); for
\( y = \lambda \) on \( L' = (\lambda) \), \( R(\gamma) - R(\lambda) > \sigma_2 + \sigma_1 - R(\lambda) + 3 \rho \);
but \( I(\lambda) > t \), \( R(\lambda) = \sigma_0 + 2 \rho \) (page 51), hence
\( R(\gamma) = R(\lambda) = \sigma_2 + 3 \rho > \sigma_2 + \delta \), so that \( L' = (\lambda) \)
lies in \( R(s) > \sigma_2 + \delta \). The curve \( L' = (\lambda) \) lies in
\( R(s) > \sigma_2 + \delta \), has length less than \( L(\delta) \), joins \( \lambda \) to
\( R(s) > \sigma_2 \) and contains no point of \( B(\delta) \); hence
\( \beta \notin B(\delta) \), a contradiction.

**Case 4:** \( \lambda \in A_1(\delta) \), \( \lambda \notin A_1(\delta) \), \( \beta \in B(\delta) \), \( \beta \notin B(\delta) \).

Let \( \lambda^+ \) be the point of \( A_1(\delta) \) nearest \( \lambda \) with
greater real part, the same imaginary part; there is
certainly such a point \( \lambda^+ \), for otherwise there would
be a curve (a horizontal straight line) of length less
than $L(\ell')$ (which is greater than $d$, page 31)

joining $\lambda$ to $R(s) + e_1$ which would contain no point
of $A(\delta)$, and $\lambda$ would not belong to $A(\delta)$, a
contradiction.

Let $\beta$ be the point (if any) of $B$ nearest
with smaller real part, the same imaginary part, if
there is no such point in $R(s) > \sigma_2$. Put $\beta = \sigma_2 + i I(\beta)$.

Since $I(\lambda) < I(\lambda') > t$, $R(\lambda') < \sigma_2 + 2 \rho$ (page 31).

But $R(x) > \sigma_2 + 4 \rho > \sigma_2 + \sigma_2 + 4 \rho$.

Thus $R(\beta) > R(x) - R(\lambda) + \epsilon > R(x) - R(\lambda) - \rho$

> $\sigma_2 + \sigma_2 = R(\lambda) + 3 \rho$

Thus if $\beta = \sigma_2 + i I(\beta)$, $|\beta - \beta| > |\lambda' - \lambda|$. Multiply.

Suppose, then, that $|\beta' - \beta| > |\lambda' - \lambda|$. Subtract
$\lambda'$ from each point of $L'$, obtaining $L' - (\lambda')$. This
curve joins $\nu = \beta + (\lambda' - \lambda)$ to $R(s) > e_1 + e_2 = R(\lambda')$
(hence to $R(s) > e_2$). By the argument of case 2, it
lies in $R(s) > \sigma_2 + \delta$, is at distance greater than $\delta$
from all points of $B$, and is of length less than $L' + \rho$. Since $I(\beta) < I(\nu) < I(\beta')$, and
$R(\beta) > R(\nu) > R(\beta')$, the straight line-segment joining
$\beta$ to $\nu$ contains no point of $B$; it is of length
less than \( d \). There is therefore a curve \( (L' = (\alpha^+) ) \) plus this line segment) which lies in \( R(s) > \sigma_2 + \delta \), joins \( \beta \) to \( R(s) > \sigma_2 \), has length less than \( \ell + d + \rho \) (hence less than \( L(\delta) \), see page 31) and which contains no point of \( B \delta \). Hence \( \delta \in B(\delta) \), a contradiction.

Suppose \( |(\beta - \beta^-)| < |\alpha^- - \alpha'| \). Translate the curve \( L' \) by subtracting \( \beta^- \) from each point, obtaining \( L' = (\beta^-) \).

This curve joins \( u = \alpha + (\beta - \beta^-) \) to \( R(s) > \sigma_1 \), has length less than \( \ell + \rho \), lies in \( R(s) > \sigma + \delta \), and contains no point of \( A_{\alpha_1} \delta \). Since \( I(\alpha) \geq I(u) = I(\alpha^+) \), and \( R(\alpha) < R(u) < R(\alpha^+) \) (\( u < \alpha^+ \) since then \( L' = (\beta^-) \) would contain a point of \( A_{\alpha} \delta \)), the line segment joining \( \alpha \) to \( u \) contains no point of \( A_{\alpha_1} \delta \), its length is less than \( d \), and it lies in \( R(s) > \sigma + \delta \) (by an argument similar to that found on page 33). There is therefore a curve \( L'' = (L' = (\beta^-)) \) plus this line segment) which lies in \( R(s) > \sigma + \delta \), joins \( \alpha \) to \( R(s) > \sigma_1 \), has length less than \( L(\delta) \), and which contains no point of \( A_{\alpha_1} \delta \). As in case 2, \( L'' \) contains only points with imaginary part greater than \( t_1 + \delta \) so that it contains no point of \( A \delta \), so that \( \alpha \in A(\delta) \), a contradiction.
Each of the four possibilities leads to a contradiction.

The original assertion, that \( \mathcal{T}_\sigma^L \subset \mathcal{T}_\sigma + J\mathcal{T}_\sigma \), is proved. It follows at once that also \( \mathcal{J}\mathcal{T}_\sigma^L \subset \mathcal{T}_\sigma + J\mathcal{T}_\sigma \). For suppose \( x \in J\mathcal{T}_\sigma^L \). Then every curve in \( R(s) > \sigma \) which joins \( x \) to \( R(s) > \sigma_1 + \sigma_2 \) contains a point \( y \) of \( \mathcal{T}_\sigma^L \). Either \( y \in \mathcal{T}_\sigma \) or every curve joining \( y \) to \( R(s) > \sigma_1 + \sigma_2 \) contains a point of \( \mathcal{T}_\sigma^L \); hence every curve joining \( x \) to \( R(s) > \sigma_1 + \sigma_2 \) contains a point of \( \mathcal{T}_\sigma^L \), \( x \in \mathcal{T}_\sigma + J\mathcal{T}_\sigma \).

The first part of the lemma, that \( \mathcal{T}_\sigma^L = \mathcal{T}_\sigma \) has been demonstrated; the proof of the second part, that \( \mathcal{T}_\sigma^L = \mathcal{T}_\sigma \), follows the same lines, without the complications introduced by \( t \). It is accomplished by showing first that \( \mathcal{T}_\sigma^L \subset \mathcal{T}_\sigma - J\mathcal{T}_\sigma \), then that \( \mathcal{T}_\sigma^L + J\mathcal{T}_\sigma \) is contained in \( \mathcal{T}_\sigma - J\mathcal{T}_\sigma \). Also, it is again evident from the definitions of the sets involved that \( [A_\sigma B]^L \supset [A_\sigma B] \), hence \( \mathcal{T}_\sigma^L \subset [A_\sigma B]^L \cap (R(s) > \sigma) \) contains \( \mathcal{T}_\sigma^L \subset [A_\sigma B]^L \cap (R(s) > \sigma) \), and \( \mathcal{T}_\sigma^L + J\mathcal{T}_\sigma \subset \mathcal{T}_\sigma + J\mathcal{T}_\sigma \); it follows that \( \mathcal{T}_\sigma^L + J\mathcal{T}_\sigma \subset \mathcal{T}_\sigma + J\mathcal{T}_\sigma \) and that \( \mathcal{T}_\sigma^L = \mathcal{T}_\sigma \).

Arguing by contradiction, assume the existence of a point \( x \in \mathcal{T}_\sigma^L \) which belongs neither to \( \mathcal{T}_\sigma \) nor to
\textbf{JT}_{\sigma'} \text{. Here if } \delta, \varepsilon \text{ are any two positive numbers, there is a point } \alpha \text{ of } A(\delta) \text{ and a point } \beta \text{ of } B(\delta) \text{ such that } |x - (\alpha + \beta)| < \varepsilon \text{. Case 1 proceeds as before; in case 2, for } y \text{ on } L', \ y = \xi \text{ on } L' = (\beta), R(y) = R(\beta) > \sigma + \varepsilon = R(\beta) + 3\varepsilon > \sigma + \delta \text{, hence } L' = (\beta) \text{ lies in } R(s) > \sigma + \delta \text{. Everything else is as before, except that } t \text{ does not enter in, so that it is not necessary to show that } L\delta = (\beta) \text{ contains only points with imaginary part greater than a certain number. The discussions of cases 3 and 4 proceed with the same simplifications, and the proof is completed in the same way.}
Theorem 1.5: If \( f(s) \equiv \sum a_n s^{-\lambda_n s} \) and \( \varphi(s) \equiv \sum b_n s^{-\lambda_n s} \)
are \(^\mathbb{N}\) in \( R(s) > \sigma_1 \), \( R(s) > \sigma_2 \) respectively, then
\( R(s) \equiv \sum a_n b_n s^{-\lambda_n s} \) is holomorphic in that part of
the half-plane \( R(s) > \overline{\sigma} \equiv \max(\sigma_1 + \sigma_2, \sigma_2 + \sigma_2) \)
which contains no point of \( S_{\sigma, \varphi} \), and which is connected
with the half-plane \( R(s) > \overline{\sigma} \).

Proof: It has already been seen that if \( f(s) \) and
\( \varphi(s) \) are \(^\mathbb{N}\), the functions \( L(s) \) and \( M(s) \) may
be so chosen that both \( f(s) \) and \( \varphi(s) \) are \(^\mathbb{N}\) for the
same functions \( L(s) \) and \( M(s) \). Let \( L(s) \) and \( M(s) \)
be these functions ( \( M(s) \) is used to avoid confusion
with \( M(r) \) of theorem 1.1). Then in \( \phi(s) \) (see
page 19) \( f(s) \) is holomorphic, and \( |f(s)| < M(s) \).

Put \( \sigma \equiv 1/\psi \). Then \( \phi(s) \) is a set \( B_\varphi(r) \equiv B_{\varphi}^\prime(r) \equiv B_{\varphi}(s) \) (the prime does not mean the derivative
here; for a definition of \( B_\varphi(r) \) see page 2), where
\( M(r) \equiv M_\varphi(\frac{1}{r}) \equiv M_\varphi(s) \). Similarly, \( \phi_\varphi(s) \) is a set
\( B_\varphi(r) \equiv B_\varphi^{\prime}(\frac{1}{r}) \equiv B_\varphi(s) \).

Denote by \( P_1 \) the half-plane \( R(s) > \sigma_1 \), by \( P_2 \)
the half-plane \( R(s) > \sigma_2 \). Then \( P_1 + S_{\sigma, \varphi}(s) \) is the
corresponding set \( U_{\varphi}(r) \equiv U_{\varphi}^{\prime}(\frac{1}{r}) \equiv U_{\varphi}(s) \), where \( S_{\sigma, \varphi}(s) \)
is the complement in \( R(s) > \sigma \) of \( \Theta_r^\sigma (\delta) \). Similarly
\[
P_2 + S_\phi^\sigma (\delta) \quad \text{is the corresponding set} \quad U_\phi (\mathcal{F}) = U_\phi (\mathcal{F}) \oplus U_\phi (\delta).
\]

Denote by \( P_1(t) \) the set of points in \( R(s) \leq \sigma \)
with imaginary part greater than \( t \), and by \( S_\phi^\sigma (\delta ; t) \)
the set of those points of \( S_\phi^\sigma (\delta) \) which have imaginary
part greater than \( t \).

Here \( S_\phi^\sigma \) plays the role of the set \( A \) of lemma
1.2, \( S_\phi^\sigma \) plays the role of \( B \). Then \( S_\phi^\sigma (\delta) \oplus A_\phi (\delta) \).
\( S_\phi^\sigma (\delta) \oplus B(\delta) \). Also \( S_\phi^\sigma (\delta ; t) \oplus A_\phi (\delta) \).

If \( \epsilon \) is any real number greater than \( \bar{A}_\phi \), then \( \epsilon \) may play
the role of \( c_1 \) (page 28) in lemma 1.2; if \( \epsilon > \bar{A}_\phi \),
then \( \epsilon \) may play the role of \( c_2 \).

Since \( U_\phi (\delta) = P_1 + S_\phi^\sigma (\delta) \),
\[
U_\phi (P_1, t) = U_\phi (\delta ; t) = P_1(t) + S_\phi^\sigma (\delta ; t).
\]

Then \( U_\phi (P_1, t) = U_\phi (\delta ; t) = [P_1(t) + S_\phi^\sigma (\delta ; t), P_2 + S_\phi^\sigma (\delta)] \).

It is clear from the definition of the operator
\( \left[ A, B \right] \) that \( [A + C, B] = [A, B] + [C, B] \). Thus
\[
U_\phi (P_1, t) = U_\phi (\delta ; t) \equiv [P_1(t) + S_\phi^\sigma (\delta ; t), P_2 + S_\phi^\sigma (\delta)]
+ [S_\phi^\sigma (\delta ; t), P_2] = [S_\phi^\sigma (\delta ; t), S_\phi^\sigma (\delta)] + [S_\phi^\sigma (\delta ; t), S_\phi^\sigma (\delta)] \].
\[ U_{g_0}(\infty; t) = U_{\infty}^0(0; t) \leq [P_1(t), P_2] + \lim_{\delta \to 0} [P_1(t), S_\phi^\delta(\delta)] \]
\[ + \lim_{\delta \to 0} [S_\phi^\delta(\delta; t), P_2] + \left[ S_\phi^\delta, S_\phi^\delta \right]_t^L \] (page 26).

(1) \[ \hat{U}_{g_0} \leq U_{g_0}(\infty; \infty) \leq U_{g_0}(0; \infty) \]
\[ \leq \lim_{t \to \infty} \left[ P_1(t), P_2 \right] + \lim_{t \to \infty} \lim_{\delta \to 0} [P_1(t), S_\phi^\delta(\delta)] \]
\[ + \lim_{t \to \infty} \lim_{\delta \to 0} [S_\phi^\delta(\delta; t), P_2] \]
\[ + \left[ S_\phi^\delta, S_\phi^\delta \right]_t^L \] (page 26).

Suppose now that \( x \in \hat{U}_{g_0} \circ (R(x) > \delta) \). Here \( \sigma^0 = \sigma_1, \delta^0 = \sigma_2 \) (pages 18 and 26);
\[ \delta \leq \max (\sigma_1 + \sigma_2, \sigma_1 + \sigma_2) = \max (\sigma_1 + \sigma_2, \sigma_2 + \sigma_2). \]

If \( x \in \lim_{t \to \infty} [P_1(t), P_2] \), then \( x \) has in every neighborhood, for every real \( t \), a point \( \alpha + \beta \), where \( \alpha \in P_1(t) \), \( \beta \in P_2 \). Then \( R(x) \leq \sigma_1 + \sigma_2 \). But \( R(x) > \delta > \sigma_1 + \sigma_2 \), a contradiction.

If \( x \in \lim_{t \to \infty} \lim_{\delta \to 0} [P_1(t), S_\phi^\delta(\delta)] \), then \( x \) has in every neighborhood, for every real \( t \) and every positive \( \delta \), a point \( \alpha + \beta \), where \( \alpha \in P_1(t) \), \( \beta \in S_\phi^\delta(\delta) \). In other words, given arbitrary positive \( \delta \), \( \epsilon \) and an arbitrary real \( t \), there is a point
\( \alpha \in P_1(t) \) and a point \( \beta \) of \( S_{\varphi}^2(\delta) \) such that \( |x - (\alpha \ast \beta)| < \varepsilon \). But \( I(\alpha) > t \), hence \( I(\beta) < I(x) - t + \varepsilon \). Choose \( \rho \) so that \( R(x) - \sigma > 4 \rho \).

Then (cf. page 24) fix \( \varepsilon, \delta \) less than \( \rho_2 \cdot t \) greater than \( t_0 \) (page 31). Then (cf. page 33)

\[
R(\beta) = \sigma \ast \rho + \delta < 2 \rho \text{ and, since } \alpha \in P_1.
\]

\[
R(x) = R(\alpha) + R(\beta) + \varepsilon < \sigma + \sigma \ast \rho + 2 \rho + \rho.
\]

But \( R(x) > \sigma + 4 \rho > \sigma + \sigma \ast \rho + 4 \rho \), a contradiction.

If \( x \in \lim_\delta \lim_{t \to \infty} [S_{\varphi}^2(\delta; t), P_2(\sigma)] \), then to every positive \( \varepsilon, \delta, t \), every real \( t \), correspond an \( \alpha \in S_{\varphi}^2(\delta; t) \) and a \( \beta \in P_2 \) such that \( |x - (\alpha \ast \beta)| < \varepsilon \).

With the same choices of \( \varepsilon, \delta, t \) as above,

\( I(\alpha) > t, R(\alpha) = \sigma \ast \rho + \delta < 2 \rho \) (page 34);

hence \( R(x) < R(\alpha) + R(\beta) + \varepsilon < \sigma \ast \rho + 2 \rho + \sigma \ast \rho \).

But \( R(x) > \sigma + 4 \rho > \sigma \ast \rho + 4 \rho \), a contradiction.

It follows that if \( x \in \overline{\omega}_{2\varphi} \circ (R(s) > \sigma) \), then

\[
x \in \left[ S_{\varphi}^2, S_{\varphi}^2 \right]_\infty^L;
\]

that is,

\[
\overline{\omega}_{2\varphi} \circ (R(s) > \sigma) \subseteq \left[ S_{\varphi}^2, S_{\varphi}^2 \right]_\infty^L \circ (R(s) > \sigma).
\]

But from (1) on page 41, also

\[
\overline{\omega}_{2\varphi} \circ (R(s) > \sigma) \subseteq \left[ S_{\varphi}^2, S_{\varphi}^2 \right]_\infty^L \circ (R(s) > \sigma).
\]
Hence \( \hat{U}_{f \Phi} \cdot (R(s) > \sigma) = [S_{f \Phi}^\sigma, S_{\Phi}^\infty] \cdot (R(s) > \sigma) \)

\( = [A_1 B]_\infty \cdot (R(s) > \sigma) = \hat{\gamma}_\sigma \) (page 26).

\( \tilde{B}_{f \Phi} \) is that part of the complement of \( \hat{U}_{f \Phi} \)
which is connected with \( R(s) > \sigma_1 + \sigma_2 = \sigma + \sigma' \).

Let \( \tilde{B}_{f \Phi}^{\sigma} \) be that part of the complement in \( R(s) > \sigma \)
of \( \hat{U}_{f \Phi} \cdot (R(s) > \sigma) = \hat{\gamma}_\sigma \) which is connected
with \( R(s) > \sigma + \sigma' \). Then \( \tilde{B}_{f \Phi}^{\sigma} \subset \tilde{B}_{f \Phi} \). But
\( \tilde{B}_{f \Phi}^{\sigma} = \hat{\gamma}_\sigma \) (page 26); and by lemma 1.2,
\( \tilde{B}_{f \Phi}^{\sigma} \subset \tilde{B}_{f \Phi} \), where

\[ \tilde{\gamma}_\sigma = \left( \lim_{t \to -\infty} [A_{f \Phi} B] \right) \cdot (R(s) > \sigma) \]

\[ = \left( \lim_{t \to -\infty} [S_{f \Phi}^\sigma(t), S_{\Phi}^\sigma(t)] \right) \cdot (R(s) > \sigma) \]

\[ = \tilde{S}_{f \Phi} \cdot (R(s) > \sigma) \) (page 17).

Therefore \( \tilde{B}_{f \Phi}^{\sigma} \) is also that part of the complement
in \( R(s) > \sigma \) of \( \tilde{S}_{f \Phi} \cdot (R(s) > \sigma) \) which is connected
with \( R(s) > \sigma + \sigma' \). But \( \tilde{B}_{f \Phi}^{\sigma} \subset \tilde{B}_{f \Phi} \), and \( H(s) \)
is holomorphic in \( \tilde{B}_{f \Phi} \).

Therefore \( H(s) \) is holomorphic in that part of the
half-plane \( R(s) > \max(\sigma_1, \sigma_2, \sigma_3) \) which is connected
with the half-plane \( R(s) > \sigma + \sigma' \) (hence with the
half-plane \( R(s) > A_f + A_{\phi} \) and which contains no point of \( \hat{S}_{f,\phi} \). This completes the proof of theorem 1.3.

Theorem 1.3 states that if \( f(s) \) and \( Q(s) \) are "N" in \( R(s) > \sigma ; \) \( R(s) > \sigma_\lambda \), respectively, then the only singular points of \( H(s) \) in \( R(s) > \overline{\sigma} = \max (\sigma_1 + \sigma_\phi^*, \sigma_\lambda + \sigma_\phi^*) \) are points of \( \hat{S}_{f,\phi} + JS_{f,\phi} \), where \( JS_{f,\phi} \) is that part of the complement in \( R(s) > \overline{\sigma} \) of \( \hat{S}_{f,\phi} \) which is not connected with \( R(s) > A_f + A_{\phi} \); it is the set of points \( x \) in \( R(s) > \overline{\sigma} \) with the property that every curve in \( R(s) > \overline{\sigma} \) which joins \( x \) to \( R(s) > A_f + A_{\phi} \) contains a point of \( \hat{S}_{f,\phi} \), the point \( x \) being not itself a point of \( \hat{S}_{f,\phi} \).

The set \( \hat{S}_{f,\phi} \) (page 17) is the closure of the set of points representable as sums \( \lambda + \phi \), where \( \lambda \in S_\phi \), \( \phi \in S_{\phi} \), \( \alpha \) has arbitrarily large imaginary part.

The evident symmetry between \( f(s) \) and \( Q(s) \) justifies the restatement of the conclusion in terms of \( \hat{S}_{Q,\phi} \) instead of \( \hat{S}_{f,\phi} \).

Among the immediate consequences of this theorem is the fact that if \( f(s) \) and \( Q(s) \) are "N" in \( R(s) > \sigma ; \) \( R(s) > \sigma_\lambda \) respectively, and if \( H(s) \) has
any singular point in $R(s) > \max (\sigma_i + \sigma_i^\circ, \sigma_2 + \sigma_2^\circ)$, then both $f(s)$ and $\phi(s)$ have infinitely many singular points in the half-planes $R(s) > \sigma_i$, $R(s) > \sigma_2$ respectively. For if $f(s)$ has only a finite number, $\hat{S}_{2\phi}$ is empty, and if $\phi(s)$ has only a finite number, $\hat{S}_{\phi\phi}$ is empty (in fact, if either has only a finite number, both $\hat{S}_{2\phi}$ and $\hat{S}_{\phi\phi}$ are empty).

For example, if the $\alpha_n$ are all real, $f(s) = \sum \alpha_n e^{-\lambda_n n}$, $H(s) = H(f, s) = \sum \alpha_n e^{-\lambda_n n}$, and if $f(s)$ is "$n$" in $R(s) > \sigma_i$ for some $\sigma_i < \frac{1}{2} A_H$, then $f(s)$ has infinitely many singular points in $R(s) > \sigma_i$.

For suppose the contrary, then $\sigma_i^* < \sigma_i$; here $f(s) = \phi(s)$, $\sigma_i < \sigma_2 < \sigma_i^* < \sigma_2^*$. Also $\hat{S}_{2\phi}$ is empty, hence $H(s)$ has no singular point in $R(s) > \max (\sigma_i + \sigma_i^\circ, \sigma_2 + \sigma_2^\circ) = 2 \sigma_i$. But $2\sigma_i < A_H$, hence $H(s)$ has no singular point on $R(s) = A_H$. This gives a contradiction, for a well-known theorem of Landau states that, if the coefficients of a Dirichlet series are all positive, the real point on the axis of absolute convergence is singular.
In the foregoing pages, regions in which \( H(s) \) is holomorphic have been defined in terms of regions of boundedness of \( f(s) \) and \( \Omega(s) \). These results can be used in defining regions of holomorphism for \( H(s) \), not directly in terms of regions in which \( f(s) \) and \( \Omega(s) \) are bounded, but in terms of regions where \( f(s) \) and \( \Omega(s) \) fail to take on certain values. In so doing, the following definitions and notations will be used:

\( S_f \) designates the set of singular points of \( f(s) \).

Let \( \varepsilon \) be any real number greater than \( A_f \), and let \( E^\varepsilon(a) \) denote the set of points in the half-plane \( R(s) < \varepsilon \) where \( f(s) \) takes the value \( a \). Let \( L \in L(\varepsilon) \) be a real, positive function of the real positive variable \( \varepsilon \), strictly increasing to infinity as \( \varepsilon \to 0 \). Define \( E^\varepsilon(a,b) \) as the set of points, each of which can be joined to the half-plane \( R(s) > \varepsilon \) by a curve of length less than \( L(\varepsilon) \) at a distance greater than \( \varepsilon \) from every point of \( E^\varepsilon(a) + E^\varepsilon(b) + S_f \). It includes the half-plane \( R(s) > \varepsilon \). It is the set of all curves of length less than \( L(\varepsilon) \) which have one end-point in \( R(s) > \varepsilon \) and are central lines of channels of width \( 2\varepsilon \) which contain
no points of $E^c(a) + E^c(b) + S^c$. Let $G^c_S(a, b)$ be
the complement of $E^c_S(a, b)$. If $A \subseteq E^c(a) + E^c(b) + S^c$,
then the set $A(S)$ (cf. page 203) is the set
$G^c_S(a, b)$; here $c = \infty$.

Lemma 1.4: The function $f(s)$ is bounded in $G^c_S(a, b)$
for every positive $S$.

That to every positive $S$ corresponds a positive
number $M = M(S)$ such that $|f(s)| < M$ for $s$ in
$G^c_S(a, b)$ can be seen with the use of a theorem of
Scheffy in a form given by Montel (cf. [4], p. 88,
where this theorem is used for a similar purpose). The
theorem states that if $F(s)$ is a function holomorphic
in a circle of radius $R$, is in absolute value less than
a fixed number $N$ at the center of the circle, and fails
to take two distinct values $a$ and $b$ in the circle,
then in every concentric circle of radius $\Theta R$ ($\Theta < 1$)
$F(s)$ is in absolute value less than a fixed number
$K(N, \Theta)$ depending only on $\Theta$ and $N$ (the values $a$ and
$b$ being regarded as fixed).
It follows from the definition of \( \varphi^e_S(a, b) \) that every point \( z \) of \( \varphi^e_S(a, b) \) either lies in \( R(s) \supset e \) or can be joined to a point of the line \( R(s) \supset e \) by a curve of length less than \( L(S) \); this curve being the central line of a channel of width \( 2S \) which contains no point of \( E^e(a) + E^e(b) + S' \). If \( R(s) \supset e \), evidently

\[
|f(z)| \leq \sum |a_n| e^{-\lambda_n} < N \quad \text{for some positive number } N.
\]

If not, cover the curve described above with circles of radius \( \frac{1}{2}S \), having centers on the curve a distance on the curve \( \frac{1}{2}S \) apart, the center of the first circle lying on \( R(s) \supset e \). There exists a positive integer \( n = n(S) \) such that the curve can be covered by at most \( n \) such overlapping circles, and

\[
|f(s)| < N \quad \text{on } R(s) \supset e. \quad \text{By applying at most } n \text{ times the theorem of Schottky referere to above, the desired conclusion is reached; for put } R = \frac{1}{2}S, \quad \Theta = \frac{1}{2}, \text{ then if } s_j \text{ is the center of the } j \text{th circle } (j = 1, 2, \ldots, n), \text{ then}
\]

\[
|f(s_j)| < K(N, \frac{1}{2}); \quad |f(s'_{j})| < K\left[K(N, \frac{1}{2})\frac{1}{2}\right], \quad \text{etc.}
\]

As has been seen, if \( A = E^e(a) + E^e(b) + S' \), then \( A(S) \supset \varphi^e_S(a, b) \). Similarly, let \( e' \) be a
real number greater than $\alpha_q$, and define $C^0_S(a', b')$
($a', b'$ are distinct) as the set $B(S)$ (cf. page 23)
where $B = E^0(a') + E^0(b') + S_q$. Here $c_1 = -\infty$,
$c_2 = -\infty$. Let $\Phi^0_S(a', b')$ be the complement of
$C^0_S(a', b')$.

By lemma 1.4, there exist functions $N_1(S)$, $N_2(S)$,
strictly increasing to infinity as $S$ approaches zero,
such that

$$|f(s)| < N_1(S) \quad \text{in } \Phi^0_S(a, b), \quad \text{and}$$

$$|g(s)| < N_2(S) \quad \text{in } \Phi^0_S(a', b').$$

Define $N'(S) = \max \left[ N_1(S), N_2(S) \right]$. Put

$\delta = 1/r$, and let $N(r) = N(1/\delta) = N'(S)$ be the
function $N(r)$ of theorem 1.1. Then $\Phi^0_S(a, b)$ is a
region $B_\delta B(r)$ = $B_\delta^S(S)$. Similarly $\Phi^0_S(a', b')$ is a
region $B_\delta B'(r)$ = $B_\delta^S(S)$; and $U_\delta^S(S)$ = $U_\delta^S(r)$ = $C^0_S(a, b),
U_\delta^S(r)$ = $U_\delta^S(r)$ = $C^0_S(a, b)$.

Define $C^f_S = C^f_S(a, b; a', b') = [A^f_S, C^f_S]
= [A(S), B(S)]$.

Then $U_{f_S}(r) \subseteq U_{f_S}(S)$ = $[U_{f_S}(S), U_{f_S}(S)] = C^f_S$.

Let $\Phi^f_S$ be that part of the complement of $C_S$
which is connected with the half-plane $R(s) > 0 + a'$;
it contains the half-plane $R(s) > 0 + a'$. 
Then \( \phi_{f_0} = \phi_{f_0} (s, b, b') \star B_{f_0} (S) \star B_{f_0} (r) \).

Define \( C_{f_0} = C_{f_0} (L, x, b, b') \star \lim_{s \to 0} C_{s}^{f_0} = [A, B] \).

Then \( C_{f_0} = U_{f_0} \star \lim_{s \to 0} U_{f_0} (S) \star \lim_{s \to 0} U_{f_0} (r) = U_{f_0} \).

Define \( A_{f_0} = A_{f_0} (L, x, b, a, b') \) as that part of the complement of \( C_{f_0} \) which is connected with \( R(s) > 0 \).

Then \( B_{f_0} = B_{f_0} \star \lim_{s \to 0} B_{f_0} (S) \star \lim_{s \to 0} B_{f_0} (r) = B_{f_0} \).

Hence \( B(s) \) is holomorphic in \( \phi_{f_0} \).

But \( C_{f_0} = L = [A, B] \)

\[ = \left[ B_{f_0} (a) + B_{f_0} (b) + S_{f_0}, B_{f_0} (a') + B_{f_0} (b') + S_{f_0} \right] \]

and \( \phi_{f_0} = L \), that part of the complement of \( L \)
which is connected with the half-plane \( R(s) > 0 + \).

By lemma 1.2, if \( T = [A, B] \), then \( \phi_{f_0} = L \); hence
\( \phi_{f_0} \) is also that part of the complement of \( T \) which is connected with the half-plane \( R(s) > 0 + \).

The following theorem has been demonstrated:

**Theorem 1.9.** \( \psi(s) \) is holomorphic in \( \phi_{f_0} \) in \( \Phi \), that part of the complement of

\[ = \left[ B_{f_0} (a) + B_{f_0} (b) + S_{f_0}, B_{f_0} (a') + B_{f_0} (b') + S_{f_0} \right] \]

which is connected with the half-plane \( R(s) > 0 + \).

The numbers \( c \) (greater than \( a_f \)), and \( c' \)
(greater than \( q \)), may be taken arbitrarily near \( a_f \) and \( q \).
respectively; the sets $E^f(a)$, etc., being defined accordingly.

It follows from the definition of $[A, B]$ that

\[
[A, B^C] = [A, B] + [A, C].
\]

Hence

\[
E^f(a) + E^g(b) + E^q, E^q(a') + E^q(b') + S_q
\]

\[
= [S_p, S_q] + [S_p, E^q(a')] + [S_p, E^q(b')]
\]

\[
+ [S_q, E^f(a)] + [S_q, E^f(b)] + [E^f(a), E^q(a')]
\]

\[
+ [E^f(b), E^q(b')]
\]

Hence if $a, b$ are any two distinct values, and if $a', b'$ are any two distinct values, then the only possible singular points of $E(s)$ are points $P_{f, q}(a, b; a', b')$, each of which has in every neighborhood a point of one of the following forms: $\alpha + \beta\alpha'$, $\alpha + \beta\beta'$, $\alpha + \beta\alpha'$, $\alpha + \beta\beta'$, or $\beta + \beta\alpha'$, where $\alpha \in S_p$, $\beta \in S_q$, $(x \in E^f(x))$, $\beta \in E^q(x')$ (where $x \in a, b$; $x' \in a', b'$); also those points which can not be connected with the half-plane $R(s) \succ 0 + s'$ by curves containing no points $P_{f, q}(a, b; a', b')$ may be singular.

In the statement of theorem 1.5, the values $a, b$, $a'$, $b'$ are all arbitrary, except that $a, b$ are distinct, and $a', b'$ are distinct. Since $E(s)$ is holomorphic in $\Phi$, ...
where \( T = T(a, b; a', b') = [A, B] \)
\[ \in \left[ E^2(a) \times E^2(b) \times S_\alpha, \quad E^q(a') \times E^q(b') \times S_\beta \right]. \]

\( R(s) \) is also holomorphic on the sum, over all values \( a, b \) (distinct) and \( a', b' \) (distinct), of all sets \( T(a, b; a', b') \). Denote this sum by \( \mathcal{G} \equiv \mathcal{G}_{f \mathcal{G}} \).

Since \( \mathcal{G} \) is the complement of \( T + JT \), the complement of \( \mathcal{G}_{f \mathcal{G}} \) is \( \bigcap \bigcup \mathcal{G}_{f \mathcal{G}} \), the intersection of all sets \( T + JT \) (over all values \( a, b \) (distinct) and \( a', b' \) (distinct)).

**Definition:** A point \( x \) will be said to be a point \( P_{f \mathcal{G}} \) if \( x \in T + JT \) for all \( a, b \) (distinct), \( a', b' \) (distinct); i.e., if \( x \in \bigcap \bigcup \mathcal{G}_{f \mathcal{G}} \).

A point \( P_{f \mathcal{G}} \) has one of the following two properties for any given \( a, b \) (distinct), \( a', b' \) (distinct):

1. the point \( P_{f \mathcal{G}} \) has in every neighborhood a point of one of the following forms: \( \alpha + b, \alpha + p' \), \( \alpha + b', \Theta + p, \Theta + p', \Theta + p'' \), \( \Theta + p'' \), or \( \Theta + p' \) for \( \alpha \in S_\beta, \Theta \in S_\alpha \), \( p, p' \in E^2(x), p'' \in E^2(x') \);

2. every curve which joins \( P_{f \mathcal{G}} \) to \( R(s) > e + s' \) contains a point which has property (1).
Theorem 1.6: The only possible singular points of \( H(s) \) are points \( P_{\phi} \).

If one of the two functions, \( f(s) \) or \( \phi(s) \), is bounded e.f.s. in a half-plane, theorem 1.6 yields a theorem simpler in form than the general theorem itself. Suppose, for example, that \( \phi(s) \) is bounded e.f.s. in \( R(s) > \sigma^2 \). There then exists a function \( M(\delta) \) such that if, for some \( \phi(a) \leq E(\mathbb{Q}(a')) \), \( R(p_{\phi}) > \sigma^2 \),

\[
|\phi(a)| \geq |a'| > M(\delta), \quad \text{then} \quad |\phi'(a') - \beta| < \delta
\]

for some \( \delta \in S_{\phi} \), for each positive \( \delta \). Hence if

\[
|a'| > M(\delta), \quad |b'| > M(\delta), \quad \text{then}
\]

\[
(E(\phi(a')) + E(\phi(b')) + S(\phi)) \circ (R(s) > \sigma^2) \subseteq S_{\phi}
\]

where \( S_{\phi} \) is the set of points contained in circles of radius \( \delta \) about points of \( S_{\phi} \circ (R(s) > \sigma^2) \). Hence, if \( |a'| > M(\delta), \quad |b'| > M(\delta), \quad R(p_{\phi}) > \sigma^2 \),

\[
R(p_{\phi}) > \sigma^2 \]

then every point in \( R(s) > \sigma^2 + \delta \) of

\[
T \equiv \{ a, b \} \equiv [E(\phi(a)) + E(\phi(b)) + S_{\phi} \}
\]

is contained in \( T'(a, b, \delta) \equiv [E(\phi(a)) + E(\phi(b)) + S_{\phi}] \)

(the set \( E(\phi(a)) + E(\phi(b)) + S_{\phi} \) lies in the half-plane \( R(s) < \phi \)).

If \( R(x) > \sigma^2 + \delta \), and if \( x \in T + JT \) for every \( (a, b; a', b') \)

then for every \( a, b, \delta \), either \( x \in T' \) (x has in every
neighborhood a point \( \alpha \ast \beta \) for \( \alpha \in A \in E^d(a) + E^d(b) + S_{S^3} \),
\( \beta \in S_{\delta_n} \), or \( x \in JT' \) (every curve in \( R(s) > \sigma_n \ast \epsilon \) joining \( x \) to \( R(s) > \epsilon + \epsilon' \) passes through a point of 
\( T' \)). There is then a monotone sequence \( \{ \delta_n \} \rightarrow 0 \) as 
\( n \rightarrow \infty \) such that

1. \( x \in T'(a,b,\delta_n) \) for each \( n \), or
2. \( x \in JT'(a,b,\delta_n) \) for each \( n \).

For if not, there is a \( \delta' > 0 \) such that \( x \not\in T'(a,b,\delta') \)
for \( \delta < \delta' \), and a \( \delta'' > 0 \) such that \( x \in JT'(a,b,\delta') \)
for \( \delta < \delta'' \), a contradiction.

Suppose (1) is true. Then \( x \) has in every
neighborhood a point \( \alpha \ast \beta \) for \( \alpha \in A \in E^d(a) + E^d(b) + S_{S^3} \),
\( \beta \in S_{\delta_n} \), for every \( n \); hence \( x \) has in every
neighborhood a point \( \alpha \ast \beta \) for \( \alpha \in A \), \( \beta \in S_{\delta_n} \), so
that \( x \in [A, S_{\delta_n}] \in [E^d(a) + E^d(b) + S_{S^3}, S_{\delta_n}] \not\in T'(a,b) \).

Suppose (2) is true. If \( L \) is any curve in
\( R(s) > \sigma_n \ast \epsilon \) joining \( x \) to \( R(s) > \epsilon + \epsilon' \), then to
each \( n \) corresponds a point \( y_n \) on \( L \) such that \( y_n \)
has in every neighborhood a point \( \alpha \ast \beta \) for \( \alpha \in A \),
\( \beta \in S_{\delta_n} \). Let \( y \) be any limit point of points \( y_n \).
Then \( y \) has in every neighborhood a point \( \alpha \ast \beta \) for
\( \alpha \in A \), \( \beta \in S_{\delta_n} \).
Thus every curve \( L \) which joins \( x \) to \( R(s) \cdot e^{s'} \) passes through a point \( y \) of \( T'(a,b) \equiv [A, S_{D}] \equiv [E^f(a) + E^f(b) + S_{g}, S_{g}] \); i.e., \( x \in JT'(a,b) \).

Thus if \( x \) is singular for \( H(s) \), \( R(x) \cdot \sigma_2 \cdot e \), then \( x \in T + JT \) for every \((a,b; a', b')\), hence \( x \in T'(a,b; b') \cdot JT'(a,b; b') \) for every \( a, b, \), hence \( x \in T'(a,b) + JT'(a,b) \) for every distinct pair \( a, b \). Let \( D \equiv (S_{D}, E^f) \) be the set of points common to all sets \( T'(a,b) + JT'(a,b) \) (for all distinct pairs \( a, b \)).

The following theorem has been demonstrated:

**Theorem 1.7:** If \( D(a) \) is bounded e.f.s. in \( R(s) \cdot \sigma_2 \cdot e \), the only possible singular points of \( H(s) \) in \( R(s) \cdot \sigma_2 \cdot e \) are points of \( D \equiv (S_{D}, E^f) \).

Here \( e \) may be taken to be any number greater than \( \sigma_2 \), the sets \( E^f(a), E^f(b) \) being defined accordingly.

If \( x \in D \equiv (S_{D}, E^f) \), then for every distinct pair of complex numbers \( a, b \), either

1. \( x \in T'(a,b) \equiv [E^f(a) + E^f(b) + S_{g}, S_{g}] \) or
2. \( x \in JT'(a,b) \); that is, every curve in \( R(s) \cdot \sigma_2 \cdot e \) joining \( x \) to \( R(s) \cdot e^{s'} \) contains a point of \( T'(a,b) \).

This theorem is similar in form to a theorem of Mandelbrojt (4, p. 69) in which he expresses the singular points of a function given by a Dirichlet series.
\[ \sum b_n e^{-\alpha_n s} \] in terms of the singular points of the
series \[ \sum e^{-\alpha_n s} \] and those of a certain Taylor-Dirichlet
series.

The results of the last few paragraphs relating to
sets \( A, B \) and their product set \( T = [A, B] \) can be
stated more simply if \( A \) and \( B \) are such that \( JT \) is
empty; that is, every point which does not belong to
\( T \) can be joined to \( R(s) \supset e \circ e' \) by a curve which
passes through no point of \( T \), so that \( T \) and \( T' \) are
complementary. Suppose, for example, that \( A^* \) denotes
the "star" of \( A \); the maximum set of points to each of
which corresponds a point of \( A \) with equal imaginary
part and greater or equal real part. If \( x \) is a point
of \( A \), all points \( x' \) with \( R(x') \supset R(x) \), \( I(x') = I(x) \)
(that is, all points of the half-line \( R(s) \supset R(x) \),
\( I(s) = I(x) \)), are points of \( A^* \).

If \( A^*, B^* \) are the "stars" of \( A, B \) respectively,
and if \( T^* = [A^*, B^*] \), then \( JT^* \) is empty, so that
\( T^* \) and \( T' \) are complementary. For suppose the contrary,
that there is a point \( x \) which belongs to \( JT^* \); that is,
\( x \) does not belong to \( T^* \), but every curve joining \( x 
\) to \( R(s) \supset e \circ e' \) passes through points of \( T^* \). Then
on the horizontal line joining \( x \) to \( R(s) \supset e \circ e' \) there
is a point \( y \) of \( T^* \). The point \( y \) has in every
neighborhood a point \( a' \circ b' \), for \( a' \in A^*, b' \in B^* \).
Let \( \alpha = \alpha' = (x' - x), \beta = \beta' = (x' - x); \) then
\[
I(\alpha) \subseteq I(\alpha'), I(\beta) \subseteq I(\beta'),
\]
so that also \( \alpha \in A^\circ, \beta \in B^\circ, \) and \( x \) has in every neighborhood a point
\( \alpha + \beta, \alpha \in A^\circ, \beta \in B^\circ; \) hence \( x \in T^\circ, \) a contradiction.

Evidently \( T^\circ \subseteq [A^\circ, B^\circ] \) contains \( T \subseteq [A, B] \)
as well as \( JT, \) so that \( T \) is contained in \( \bar{T}^\circ, \) the complement of \( T^\circ. \) If \( A \subseteq E^\circ(a) + E^\circ(b) + S^\circ, \)
\( A^\circ \subseteq E^\circ(a) + E^\circ(b) + S^\circ, R \subseteq E^\circ(a') + E^\circ(b') + S^\circ, \)
\( B^\circ \subseteq E^\circ(a') + E^\circ(b') + S^\circ, \) then the following is a corollary of theorem 1.5:

**Corollary 1.5':** \( H(s) \) is holomorphic in \( \bar{T}^\circ, \) the complement
of \( T^\circ = T^\circ(a, b; a', b') = [A^\circ, B^\circ], \)
\[
\subseteq [E^\circ(a) + E^\circ(b) + S^\circ, E^\circ(a') + E^\circ(b') + S^\circ].
\]
Or, the only singular points of \( H(s) \) are points of \( T^\circ. \)

Corollary 1.5' holds (as does 1.5) for arbitrary
\( a, b \) (distinct), \( a', b' \) (distinct). Hence if \( x \) is
singular for \( H(s), \) \( x \in T^\circ(a, b; a', b') \) for all such
\( a, b, a', b'. \) Let \( \bigcup T^\circ \) be the intersection of all sets
\( T^\circ(a, b; a', b') \) (\( a, b \) take all distinct values, \( a', b' \) take
all distinct pairs of values). All singular points of
\( H(s) \) are found among points of the set \( \bigcup T^\circ. \)

It is evidently not necessary to suppose here that
the functions are uniform.
Any point \( x \) of \( \mathcal{E}_{f_0}^* \) must satisfy at least one of
the following four conditions; conversely, if \( x \) satisfies
one of these four conditions, \( x \in \mathcal{E}_{f_0}^* \).

1. \( x \in S_{f_0}^* \subseteq S_{f_0}^0, S_{f_0}^* \); in this case \( x \) has
   in every neighborhood a point \( \alpha + \rho \) where \( \alpha \in S_{f_0}^0,
   \rho \in S_{f_0}^* \). Such a point will be called a point \( F^0(\alpha, \rho) \).

2. \( x \in [S_{f_0}^0, E_{f_0}^0(a')] \) for every \( a' \) except at
   most one (the exceptional value depends on \( x \)); here
   \( x \) has in every neighborhood, for every \( a' \) except at
   most one, a point \( \alpha + \rho a' \) where \( \alpha \in S_{f_0}^0, \rho a' \in E_{f_0}^0(a') \)
   \( \alpha \) is a singular point for \( f(a) \), \( \rho a' \) is a point where
   \( f(a) \) takes the value \( a' \); \( x \) is a point \( F^0(\alpha, E_{f_0}^0) \).

3. \( x \in [S_{f_0}^0, E_{f_0}^0(a)] \) for every \( a \) except at most
   one (depending on \( x \)); \( x \) has in every neighborhood,
   for every \( a \) except at most one, a point \( \rho + \rho a \)
   where \( \rho \in S_{f_0}^0, \rho a \in E_{f_0}^0(a) \); \( x \) is a point
   \( F^0(\rho, E_{f_0}^0) \).

4. There exist at least two values \( e, d \), and at
   least two values \( e', d' \), such that \( x \in S_{f_0}^0 \).
   \( x \in [E_{f_0}^0(e), E_{f_0}^0(e')] + [S_{f_0}^0, E_{f_0}^0(d)] + [S_{f_0}^0, E_{f_0}^0(d')] \)
   \( [S_{f_0}^0, E_{f_0}^0(d)] \). For all such \((e, d; e', d')\)
   \( x \in [E_{f_0}^0(e), E_{f_0}^0(e')] + [E_{f_0}^0(e), E_{f_0}^0(d')] + [E_{f_0}^0(d), E_{f_0}^0(e')] \)
   \( + [E_{f_0}^0(d), E_{f_0}^0(d')] \).
Or, there exist at least two values \( e, d \) and at least two values \( e', d' \) such that there is a neighborhood of \( x \) which contains no point \( \alpha \cdot p'_{e'}, \alpha \cdot p'_{d'}, \beta \cdot p'_{e}, \beta \cdot p'_{d} \) or \( \beta \cdot p'_{d}, \) for \( \alpha = S_{e}, \beta = S_{d}, p'_{e} \in E_{\phi}^{e}(e'), p'_{d} \in E_{\phi}^{e}(d'), p'_{e} \in E_{\phi}^{e}(e'), p'_{d} \in E_{\phi}^{e}(d'). \) For all such \((e, d, e', d')\) \( x \) has in every neighborhood a point \( p_{e} \cdot p'_{e'}, \) a point \( p_{e} \cdot p'_{d'}, \) a point \( p_{d} \cdot p'_{e}, \) or a point \( p_{d} \cdot p'_{d} \). \( x \) is a point \( F^{e}(E_{\phi}^{e}, E_{\phi}^{e}). \)

That if \( x \in \mathcal{Q} \) then \( x \) must be a point of one of the four categories can be seen by the following considerations: suppose \( x \) is not a point of any of the first three categories. Then (I) \( x \in S_{\phi}^{e} \); (II) there exist two distinct values \( e \) and \( d \) such that \( x \in [S_{\phi}^{e}, E_{\phi}^{e}(e')] + [S_{\phi}^{e}, E_{\phi}^{e}(d')] \); (III) there exist two distinct values \( e' \) and \( d' \) such that \( x \in [S_{\phi}^{e}, E_{\phi}^{e}(e')] + [S_{\phi}^{d}, E_{\phi}^{d}(d')] \). But since \( x \in [E_{\phi}^{e}(e) + E_{\phi}^{e}(d) + S_{\phi}^{e}, E_{\phi}^{e}(e')] + [E_{\phi}^{d}(e) + E_{\phi}^{d}(d') + S_{\phi}^{d}, E_{\phi}^{d}(d')] \), then \( x \in [E_{\phi}^{e}(e) + E_{\phi}^{e}(e')] + [E_{\phi}^{d}(e) + E_{\phi}^{d}(d')] \), and thus \( x \) is a point of category IV.

The following corollary of theorem 1.5 has been demonstrated:
Corollary 1.6's \( H(s) \) is holomorphic in \( \mathbb{C}_{\alpha} \), the complement of \( \mathbb{C}_{\alpha} \).

Or,

The only possible singular points of \( H(s) \) are points \( P^0(\alpha, E^0), P^0(\beta, E^0), P(\pi_0, E^0) \).

It has been seen (page 55) that if \( \mathcal{O}(s) \) is bounded \( e.f.s. \) in \( R(s) > \alpha \), and if \( x \) is singular for \( H(s) \), \( R(x) > \alpha + \epsilon \), then \( x \in T'(a, b) \cup JT'(a, b) \) for every distinct pair \( (a, b) \), where

\[
T'(a, b) = [E^s(a) + E^s(b) + S_{\pi_0}, S_{\alpha}] \quad \text{Evidently}
\]

\[
T'(a, b) \cup JT'(a, b) \quad \text{contains}
\]

\[
T'(a, b) \quad \text{for every distinct pair \( (a, b) \). Then either} \quad x \in S'_{\pi_0} = [S_{\pi_0}, S_{\alpha}] \quad \text{or} \quad x \in [E^s(a), S_{\alpha}] \quad \text{for every} \ a \ \text{except at most one. For suppose there are two distinct values} \ a, b \ \text{such that} \ x \notin [S_{\pi_0}, S_{\alpha}], x \in [E^s(a), S_{\alpha}], x \in [E^s(b), S_{\alpha}]. \ \text{Then} \ x \in [E^s(a), S_{\alpha}] + [E^s(b), S_{\alpha}] + [E^s(a), S_{\alpha}] \quad \text{contains} \quad T'(a, b), \ \text{a contradiction. Thus if} \ x \ \text{is singular for} \ H(s), \ R(x) > \alpha + \epsilon, \ \text{either} \ x \in S^0 \ (x \ is \ a) \ \text{point} \ P^0(\alpha, E^0) \ \text{or} \ x \in [E^s(a), S_{\alpha}] \ \text{for every}
value a except at most one \( x \) is a point \( F^s(\beta; E^s_\beta) \).

**Corollary 1.7'**

If \( \varphi(s) \) is bounded e.s.o. in \( R(s) > \alpha \), the only singular points of \( E(s) \) in \( R(s) > \alpha + \varepsilon \) are points \( F^s(\alpha; \beta) \) and \( F^s(\beta; E^s_\beta) \).

Before finishing the chapter, it will be well to note some conditions on \( f(s) \) and \( \varphi(s) \) which, in the light of Schottky's theorem (cf. lemma 1.4), imply that \( f(s) \) and \( \varphi(s) \) are "\( \mathbb{H} \)" in certain half-planes, and that hence the conclusion of theorem 1.3 holds.

It will be recalled that \( f(s) \) is "\( \mathbb{H} \)" in \( R(s) \) if there exists a real positive function \( L(\delta) \) strictly increasing to \( \infty \) as \( \delta \to 0 \), such that to each positive \( \delta \) corresponds a positive number \( M(\delta) \) such that
\[
|f(s)| < M(\delta) \quad \text{in} \quad E^s_\delta(\delta).
\]
The set \( E^s_\delta(\delta) \) is the set of all curves of length less than \( L(\delta) \) having one end-point in \( R(s) > \alpha \) which are central lines of channels in \( R(s) > \alpha \) of width greater than \( 2\delta \) containing no points of \( E^s_\delta(\delta) \). Suppose now that to each \( \delta \) correspond two distinct complex values \( a(\delta) \), \( b(\delta) \) such that \( f(s) \neq a(\delta) \), \( f(s) \neq b(\delta) \) in \( E^s_\delta(\delta) \).
It follows immediately from the application of Schetky's theorem discussed in lemma 1.4, that there is a positive number $M(\delta)$ such that $|f(s)| < M(\delta)$ in $\mathbb{C}_{\alpha}(\delta)$, hence $f(s)$ is "M" in $\mathbb{R}(s) > \alpha$.

If $f(s)$ is bounded e.f.s. in $\mathbb{R}(s) > \alpha$, $f(s)$ certainly satisfies the above condition, and is "M" in $\mathbb{R}(s) > \alpha$.

If $f(s)$ fails to take two distinct values $a$, $b$ in $\alpha < \mathbb{R}(s) < \alpha$, then $f(s)$ satisfies the above condition, with $a = a(\delta)$ (each positive $\delta$), $b = b(\delta)$ (each positive $\delta$), hence $f(s)$ is "M" in $\mathbb{R}(s) > \alpha$.

With the above remarks, theorem 1.3 can be used to give results concerning the values taken on by functions having Dirichlet series expansions. For example, the following theorem:

The function $\sum (s) = \sum \frac{1}{n^s} = \sum e^{-s \log n}$ takes every value except at most one in every strip $\frac{1}{2} < \mathbb{R}(s) < 1 + \delta$ ($\delta > 0$).

Let $f(s) = \theta(s) g(s)$. Then $H(f, \theta) = H(s) g_1(s)$. Here $A_f = A_{\theta} = A_s = 1$. Since $1$ is the only singular point of $g_1(s)$, $a_1^\alpha$ and $a_{\delta}^\gamma$ may be taken equal to $\alpha$ respectively, where $\alpha > \gamma > \delta > 0$. Let
\[ e = \bar{\mathcal{E}}_f + \bar{\mathcal{E}}_h = 1 + \bar{\mathcal{E}}_h. \] Then \( \overline{\mathcal{E}} = \max (\sigma^1 + \sigma^2, \sigma^3 + \sigma^4) = 1 - 2\bar{\mathcal{E}}_h. \)

Now suppose the theorem is not true, that there exist distinct values \( a, b \) not taken by \( \hat{f}(s) \) in the strip \( \sigma^1 \leq R(s) < \sigma^2 \). Then, by the above remarks,

\[ \hat{f}(s) \cdot \mathcal{E}(s) = \mathcal{E}(s) \text{ is } "M" \text{ in } R(s) > \sigma^1, \] and theorem 1.3 applies. But theorem 1.3 states that the only singular points of \( \mathcal{H}(s) \) in \( R(s) > \sigma^1 \) are points of \( \overline{\mathcal{S}}_{f,\mathcal{E}} \) (which is contained in \( \mathcal{S}_{f,\mathcal{E}} \)) and those points which can not be connected with \( R(s) > \sigma^1 + \sigma^2 \) by curves passing through no points of \( \overline{\mathcal{S}}_{f,\mathcal{E}} \). But the only point of \( \mathcal{S}_{f,\mathcal{E}} \) \( 1 + 1 \geq 2 \), hence the only singular point of \( \mathcal{H}(s) = \hat{f}(s) \) is 2, a contradiction.
CHAPTER 2
Applications of the theorem of Mandelbrojt, or the somewhat more general theorem 1.5, form the subject matter of this chapter. The results obtained yield information concerning the role played by the sequence of exponents of a Dirichlet series in determining the nature and position of the singularities of the function it represents.

One purpose for which theorem 1.5 proves useful is to investigate the similarities between certain functions given by Dirichlet series, and functions periodic with respect to the imaginary part of the variable. Functions given by Dirichlet series will in a certain sense be compared with those given by Taylor-Dirichlet series (period $2\pi i$; the sequence $\{\lambda_n\}$ is a sequence of positive integers). A comparison with functions having other periods ($k i$, for real $k$) follows the same lines.

It will be noticed that theorem 1.5 can be used only if $f(s) = \sum a_n e^{\lambda n s}$, $\varphi(s) = \sum b_n e^{\lambda n s}$, the two series having the same sequence of exponents. If, however, $f(s) = \sum a_n e^{\lambda n s}$, $\varphi(s) = \sum b_n e^{\lambda n s}$, there is a minimum sequence $\{\nu_n\}$ which contains both sequences $\{\lambda_n\}$ and $\{\mu_n\}$. The functions $f(s)$ and $\varphi(s)$
can each be expressed as the sum of a Dirichlet series
whose sequence of exponents is \( \lambda_n \):

\[
f(s) = \sum A_n e^{\lambda_n s}; \quad A_n = a_n \quad \text{for} \quad \nu_n \leq \lambda_n; \quad A_n = 0 \quad \text{if} \quad \nu_n > \lambda_n \text{for all } n;\]

\[
\varphi(s) = \sum B_n e^{\mu_n s}; \quad B_n = b_n \quad \text{for} \quad \nu_n \leq \mu_n; \quad B_n = 0 \quad \text{if} \quad \nu_n > \mu_n \text{for all } n.
\]

If the sequences \( \lambda_n \) and \( \mu_n \) have an infinite sequence
\( \lambda_n \) and \( \mu_n \) in common (\( \mu_n \) is the sequence
of all positive numbers each of which appears in both
\( \lambda_n \) and \( \mu_n \)), let \( \nu_i = \lambda_{n_i} = \mu_{m_i} (i = 1, 2, \ldots) \).

If \( f_1(s) = \sum a_{n_1} e^{\lambda_{n_1} s}, \quad \varphi(s) = \sum b_{m_1} e^{\mu_{m_1} s}, \)
then \( H(f, \varphi) = H(f_1, \varphi) = \sum a_{n_1} b_{m_1} e^{s(n_1 + m_1)} \).

If \( \lambda_n \) and \( \mu_n \) have no infinite sequence in
common, the result obtained by the use of theorem 1.3
is of course trivial, since then \( H(f, \varphi) \) is entire
or constant. Use will often be made in the following
pages of theorem 1.3 applied to functions whose
sequences of exponents are not identical, but have an
infinite subsequence in common. One immediate result is
the following:
Theorem 2.1: Given \( f(s) = \sum a_n s^{-\lambda_n s} \), let
\[
H(s) = H(f, s) = \sum a_n^2 s^{-\lambda_n s}.
\]
If \( f(s) \) is \( \mathcal{H} \) in \( \mathbb{R}(s) > \sigma \) for some \( \sigma < \frac{1}{2} \mathcal{H}_H \), then \( f(s) \) has infinitely many singular points in \( \mathbb{R}(s) > \sigma \).

(compare also with page 45).

Suppose the contrary is true, that \( f(s) \) has only a finite number of singular points in \( \mathbb{R}(s) > \sigma \). Then
\[
\sigma^* = \sigma \quad \text{(page 18)};
\]
hence \( f(s) = g(s) \) so that
\[
\sigma = \max (\sigma_0, \sigma_0^*, \sigma_0^*, \sigma_0^*) = 2 \sigma \quad \text{(page 39)}.
\]
Also \( I(\alpha) \) is bounded for \( \alpha \in S_f \), so that \( S_f(t) \) is empty for \( t \) sufficiently large, and \( \mathcal{H}_f \) is empty. Hence \( H(s) \) is holomorphic in \( \mathbb{R}(s) > \sigma \), a contradiction. For \( \mathcal{H}_H > 2 \sigma \geq \sigma \), this proves the theorem.

Corollary 2.2: Suppose that \( f(s) = \sum a_n s^{-\lambda_n s} \) is \( \mathcal{H} \) in \( \mathbb{R}(s) > \sigma \) for some real number \( \sigma \) less than \( \mathcal{H}_H \); suppose also that \( \lim_{\lambda_n} \log n = 0 \), and that the index of condensation \( \text{see [1], page 25} \) of the sequence \( \sum \lambda_n \) is zero. Then \( f(s) \) has infinitely many singular points in \( \mathbb{R}(s) > \sigma \).

That this follows from theorem 2.1 is readily seen by using well-known theorems \( \text{cf. [1], pages 4} \).
sufficient to assure that

\[ H_n = A_n = \lim_{\lambda_n} \frac{1}{\lambda_n} \log |a_n^2| = 2 \lim_{\lambda_n} \frac{1}{\lambda_n} \log |a_n| = 2 \sigma_f > 2 \sigma_f. \]

The hypotheses of \( \Omega \) are clearly satisfied.

In using theorem 1.3 to compare certain classes of functions given by Dirichlet series with functions given by Taylor-Dirichlet series, it will be convenient to speak of "classes" (modulo \( 2\pi \)) of singular points (cf. [3], page 50, where this idea of "classes" of singular points is used in a different connection).

Two points will be said to belong to the same class (mod \( 2\pi \)) if either may be obtained by the addition of an integral multiple of \( 2\pi \) from the other. Each class of points has a representative in any given horizontal, half-open strip of width \( 2\pi \); namely, that point of the strip from which each point of the class may be obtained by the addition of an integral multiple of \( 2\pi \). In this manner a many-one correspondence may be set up between points of any set in the plane and points of a "representative" set in a given horizontal, half-open strip of width \( 2\pi \).
As in chapter I, let $S_{\sigma} \equiv S_{\sigma}^{\infty}$ denote the set of singular points of $f(s)$ in $R(s) > \sigma^{-}$; let $R_{\sigma}(\sigma > \sigma^{-})$ denote the corresponding representative set (modulo $2\pi i$) in the strip $0 \leq I(s) < 2\pi$. Thus to each point of $R_{\sigma}(\sigma > \sigma^{-})$ corresponds one or more points of $S_{\sigma}$ with the same real part, whose imaginary parts differ from its imaginary part by multiples of $2\pi$.

Let $R_{\sigma}(\sigma \geq \sigma^{-})$ denote the set representative of all points of $S_{\sigma}$ with real part $\sigma^{-}$. It consists of points on the line segment $R(s) = \sigma^{-}, 0 \leq I(s) < 2\pi$.

The statement of the next theorem involves not simply the set $R_{\sigma}(\sigma > \sigma^{-})$, but its closure; and it will simplify the statement if also that part of the complement in the strip $R(s) > \sigma^{-}, 0 \leq I(s) < 2\pi$, of $R_{\sigma}(\sigma > \sigma^{-})$, which is not connected with the half-plane $R(s) > \gamma_{\sigma}^{-}$, is included. The symbol $R_{\sigma}(\sigma > \sigma^{-})$, then, will denote the closure of the set $R_{\sigma}(\sigma > \sigma^{-})$; plus every point $x$ such that every curve in the strip $R(s) > \sigma^{-}, 0 < I(s) < 2\pi$ which joins $x$ to $R(s) > \gamma_{\sigma}^{-}$ contains a point of the closure of $R_{\sigma}(\sigma > \sigma^{-})$. The symbol $R_{\sigma}(\sigma < \sigma^{-})$ (for $\sigma^{-} > \sigma^{-}$) denotes the set of those points of $R_{\sigma}(\sigma > \sigma^{-})$ which lie on the line $\sigma^{-} = \sigma^{-}$.
It will have been noticed that if \( \sum a_n e^{-\lambda_n s} \) is a Taylor-Dirichlet series, the set \( R_\sigma(\sigma > \sigma_0) \) and its closure are identical. If \( \sum a_n e^{-\lambda_n s} \) is not a Taylor-Dirichlet series, this is not necessarily true; for a point may be a limit point of points of the form \( \alpha + 2\pi ki \ (\alpha \in B, \ k \ \text{integral}) \), without being itself representable in this form.

This notation is useful in stating the following theorem:

**Theorem 2.5:** Suppose \( f(s) = \sum a_n e^{-\lambda_n s} \) satisfies the following hypotheses:

1. The sequence \( \lambda_n \) contains an infinite subsequence of integers \( \lambda_1^2, \lambda_2^2, \ldots \), with \( \lambda_n = \frac{1}{n_i} \) (\( i > 0 \));
2. \( \sum \frac{1}{n_i} \log |a_{n_i}| \rightarrow -\infty \);
3. \( f(s) \) is \( M \) in \( B(s) > \sigma, \) for some real number \( \sigma_0 < \sum \frac{1}{n_i} \log |a_{n_i}| \);

then \( f(s) \) has infinitely many singular points in \( B(s) > \sigma_1 \); moreover, if \( H(s) = \sum a_n e^{-\beta_n s} \), the set \( R_H(\sigma > \sigma_0) \) is contained in the set \( R_f(\sigma > \sigma_0) \).

It follows, of course, that if \( \sigma' > \sigma_0 \), then \( R_f(\sigma > \sigma') \supset R_f(\sigma > \sigma_0) \). In particular, if \( \sigma < A_\mu \).
\[ \mathcal{H}_f(\sigma > \sigma_0) \subseteq \mathcal{H}_R(\sigma = \sigma_0). \]

The theorem states that if \( \gamma \) is a point of \( \mathcal{H}_R(\sigma > \sigma_1) \), the representative set of the singular points of \( H(s) \) in \( R(s) > \sigma_1 \), then either \( \gamma \) belongs also to the closure of the representative set of singular points of \( f(s) \) in \( R(s) > \sigma_1 \), or every curve in the strip \( R(s) > \sigma_1 \), \( 0 < I(s) < 2\pi \), which joins \( \gamma \) to \( R(s) > \mathcal{H}_f \) contains a point of the closure of \( \mathcal{H}_f(\sigma > \sigma_1) \).

**Proof:** Put \( \varphi(s) = \sum e^{-2\pi s} \). Then the sequences \( \{n_j\} \) and \( \{A_n\} \) have the sequence \( \{M_j\} \) in common, and \( H(s) = H(f(s)) = \sum A_n e^{-2\pi i s} \). The function \( \varphi(s) \) is "in" in \( R(s) > \sigma_2 \) for \( -\sigma_2 \) arbitrarily large; also \( \sigma_2 > 0 \). Hence, if \( \sigma_2 \) is so chosen that \( \sigma_2 < \sigma_1 \cdot \sigma_2^2 \), then \( \max (\sigma_1 + \sigma_2, \sigma_2 + \sigma_2^2) < \sigma_2 \).

Hypothesis (ii) assures the existence of a singular point of \( H(s) \) in \( R(s) > \sigma_1 \), for

\[ \mathcal{H}_H = A_n \in \mathbb{H} \quad \lim_{\mathcal{N}} \frac{\log |A_n|}{H_2} > \sigma_1. \]

It follows from theorem 1.3 that the set \( \hat{\mathcal{H}} \) is not empty. The set \( \hat{\mathcal{H}} \) here is the set of all points \( \varphi \), each of which has in every neighborhood, for each real positive \( \tau \),
a point $\alpha + 2\pi k i$ for some $\alpha \in S_f, I(\alpha) > t$, $k$ an integer, since the set $S_f$ consists of points $2\pi k i$. That the set $\delta_{2f}$ is not empty implies that there are infinitely many points $\alpha$ of $S_f$.

If now $Y \in R(s) > \sigma_I$ is singular for $H(s)$, either (1) $Y \in S_f$, or (2) every curve in $R(s) > \sigma_I$ which joins $Y$ to $R(s) > C + C'$ (where $C = \hat{C}, C' = \hat{C}'$) contains a point of $\delta_{2f}$; $\delta_{2f}$ contains no point of $R(s) > \hat{C}'$. If (1), then $Y$ has in every neighborhood of a point $\alpha + 2\pi k i$ for $\alpha \in S_f, k$ an integer, hence the representative $\mod 2\pi i$, $Y''$, of $Y$ belongs to $\overline{R}_f(\sigma > \sigma_i)$. If (2), then every curve in the strip $R(s) > \sigma_I, 0 < I(s) < 2\pi$ which joins $Y''$ to $R(s)$ contains a point of $\overline{R}_f(\sigma > \sigma_i)$; if not, there is certainly a curve in $R(s) > \sigma_I$ which joins $Y$ to $R(s) > \hat{C}'$ and which contains no point of $\delta_{2f}$; hence $Y''$ belongs to $\overline{R}_f(\sigma > \sigma_i)$. This completes the proof of the theorem.

**Corollary 2.4:** If (i), (ii), and (iii) of theorem 2.3 hold, and if further $\lim_{t \to \infty} \frac{B(t)}{t} = \infty$, then $\overline{R}_f(\sigma > \sigma_I)$ contains every point of the line segment $R(s) \leq \overline{R}_f \frac{1}{\pi} \log \frac{1}{|w_n|}, 0 < I(s) < 2\pi$. 
Proof: Under the hypotheses, \( H(s) = \sum a_i e^{-\lambda_i s} \)
has the line \( \Re(s) = \alpha(s) = \frac{1}{\lambda_i} \log |a_i| \)
as a cut (a theorem of Fabry; cf. S. Mandelbrojt, Série
lacunaires, Actualités Scientifiques et Industrielles,
Exposés sur la théorie des fonctions, 1936).

Corollary 2.5: If (i), (ii), and (iii) hold, then
\[ \Im \sum \frac{\log |a_i|}{n_i} \]

For \( H(s) \), being a Taylor-Dirichlet series, has certainly
a singular point on its axis of convergence.

Corollary 2.6: If (i), (ii), and (iii) hold, if
further \( \lim_{\lambda_n} \frac{\log n}{\lambda_n} = 0 \), \( \Im \sum \frac{\log |a_i|}{n_i} \) is
equal to \( \Im \sum \frac{\log |a_i|}{\lambda_n} \), then
\[ \mathcal{H}_f = \mathcal{A}_f = \Im \sum \frac{\log |a_i|}{\lambda_n} \]

For in this case \( \mathcal{A}_f \leq \mathcal{A}_f \leq \Im \sum \frac{\log |a_i|}{\lambda_n} \)
(of [1], p. 4, and [5]). But \( \mathcal{H}_f \leq \mathcal{A}_f \), also by
corollary 2.8, \( H_f > A_f \) hence \( H_f \geq A_f \).

Theorem 1.5 may also be used to obtain sufficient conditions on \( f(s) = \sum a_n e^{-\lambda n^\beta} \) in order that \( R\big(-\frac{\pi}{\lambda}\big) \) may contain at least two points. In his thesis [2] Mandelbrojt uses the multiplication theorem of Hadamard in a similar manner to obtain results relative to lacunary Taylor series.

**Definition:** Consider a function \( g(s) = \sum e^{-\lambda n^\beta} \).

It will be said that \( g(s) \) can be completed, if there exists a function \( g_\lambda(s) = \sum e^{-\lambda n^\beta} \) with the following properties:

(i) \( \{\lambda_n\} \subseteq \lambda_i \), with \( \lambda_n \sim \lambda_i \), \( \lambda_n \leq 1 \), \( i = 1, 2, \ldots \);

(ii) \( A_g > 0 \), and \( g_\lambda(s) \) is "M" in \( R(s) > \gamma \) for some \( \gamma > 0 \);

(iii) There exists a positive number \( h \) such that \( g_\lambda(s) \) is holomorphic in a circle of radius \( h \) about every point \( 2\pi ki, \ k \) takes on all integral values.

It follows from property (i) that \( H(g_\lambda) \leq g(s) \); indeed, if \( f(s) = \sum a_n e^{-\lambda n^\beta} \), \( H(f, g_\lambda) \leq f(s) \).
Theorem 2.7.1 Suppose \( f(s) = \sum \phi(s) \) is holomorphic in \( R(s) > \sigma \), for \( \sigma < \mathcal{H}_f \), \( \mathcal{H}_f > -\infty \), and suppose that \( S_{\sigma}^f \) contains only points on \( R(s) > \mathcal{H}_f \). Then if \( g(s) = \sum \phi(s) \) can be completed, \( R(s) > \mathcal{H}_f \) contains at least two points.

Proof: From the definition of \( \mathcal{H}_f \), it follows that under the hypothesis, \( f(s) \) has at least one singular point \( s_0 \) on \( R(s) > \mathcal{H}_f \). Suppose the theorem is not true. Then \( S_{\sigma}^f = S_{\sigma}^{g(s)} \) has as its only points in \( R(s) > \sigma \) points of the form \( s_0 + 2\pi k i \), for integral \( k \).

Let \( g_1(s) \) be the function of the above definition. Then \( H(f \circ g_1) = f(s) \). Let \( f(s) \) play the role of \( f(s) \), \( g_1(s) \) the role of \( g(s) \) in theorem 1.3.

Here \( \sigma_2 > \gamma \), \( \sigma_0 < 0 \), \( \sigma_1 < \mathcal{H}_f \). By theorem 1.3, \( f(s) \) is holomorphic in that part of the complement in \( R(s) > \sigma \) of \( \Delta_{f, g} \) (cf. page 17) which is connected with \( R(s) > \sigma \). Here \( \sigma > \mathcal{H}_f \).

Clearly, the set \( \Delta_{f, g} \) contains no point of \( R(s) > \sigma_1 + \sigma_0^\circ \). Here, since \( \sigma_0 < 0 \), \( \sigma_1 < \mathcal{H}_f \), \( \Delta_{f, g} \) contains no point of \( R(s) > \mathcal{H}_f \). Since \( s_0 \) is singular.
with \( R(s_0) \neq \mathcal{H}_f \), \( s_0 \in S_{f_	heta} \). Hence \( s_0 \) has in every neighborhood a point of the form \( s_0 + 2\pi ki + \beta \), for integral \( k \), \( \beta \in S_{f_\theta} \neq S_{g_\theta} \). Since, by property (iii) of \( S_{g_\theta}(s) \), \( S_{g_\theta} \) is bounded away from all points \( 2\pi ki \), this is impossible. This contradiction proves the theorem.

If to the hypotheses of theorem 2.7 is added:
\[
\mathcal{H}_{[\mathcal{H}_f, \mathcal{H}_f]} \leq 2 \mathcal{H}_f \text{, it follows from theorem 2.1 that } f(s)
\]
has infinitely many singular points on \( R(s) = \mathcal{H}_f \). This does not imply, however, that \( R(s, s' = \mathcal{H}_f) \) contains even two points; so that theorem 2.7 is not contained in theorem 2.1. Besides, it is not in general true that
\[
\mathcal{H}_{[\mathcal{H}_f, \mathcal{H}_f]} \leq 2 \mathcal{H}_f \text{ if } f(s) = \sum \frac{(-1)^n}{n^s} = \sum (-1)^n e^{-s \log n},
\]
then \( H(f, s) \leq \sum e^{-s \log n} = \sum \frac{1}{n^s} \cdot \mathcal{H}_{[\mathcal{H}_f, \mathcal{H}_f]} \leq 1 \); but \( \mathcal{H}_f \geq \infty \).

In approaching the question of what types of series
\[
g(s) = \sum e^{-\lambda n^s}
\]
can be completed, a suggestion of Professor Mandelbrojt proves useful. He suggested that it might be interesting to consider the case in which any singularities of \( g(s) \) which occur at points \( 2\pi ki \) are simple poles. In this case, with suitable hypotheses on
g(s), the function $g_1(s) = g(s) (1 - e^{-s})$ then satisfies the conditions of the definition. In proving this, the following lemma will be used:

**Lemma 2.8** If $f(s) \sum a_n e^{\lambda_n s}$ has $A_f < \infty$ and is "$\mathbb{H}$" in $R(s) > \sigma_i$, if $\gamma(s) \sum b_n e^{\gamma_n s}$ has $A_\gamma < A_f$, and is holomorphic and "$\mathbb{H}$" in $R(s) > \sigma_i$, and if the minimum distance between two zeros of $\gamma(s)$ in $R(s) > \sigma_i$ is greater than a positive number $d$, then $F(s) \sum f(s) \gamma(s)$ has $A_f < A_f$, and is "$\mathbb{H}$" in $R(s) > \sigma_i$.

If $\gamma(s)$ has no zeros in $R(s) > \sigma_i$, the theorem is still true.

**Proof:** That the product $F(s) \sum f(s) \gamma(s)$ is representable by a Dirichlet series having $A_f < A_f$ under the above hypotheses is well-known (cf. [5], p. 29). It remains to show that $F(s)$ is "$\mathbb{H}$" in $R(s) > \sigma_i$.

As was seen in chapter 1, (page 21), if any two functions are "$\mathbb{H}$", they may be regarded as being "$\mathbb{H}$" for the same function $L(f)$. Let $L(f)$ here be a function for which both $f(s)$ and $\gamma(s)$ are "$\mathbb{H}$" in $R(s) > \sigma_i$. If $f$ is sufficiently small, say $f < f_0$, then $L(f) > \sigma_i$, so that since $\gamma(s)$ is holomorphic
in $R(s) > \sigma_i$, $\Phi^F_1(S)$ (page 19) is the half-plane $R(s) > \sigma_i + \delta$. Thus to say that $\gamma(s)$ is holomorphic and "$M$" in $R(s) > \sigma_i$ is to say that to each positive $\delta < \delta_0$ corresponds a positive number $M_\gamma(\delta)$ such that $|\gamma(s)| < M_\gamma(\delta)$ in $R(s) > \sigma_i + \delta$.

It is required to show that there exists a positive function $M_\gamma(S)$ such that $|F(s)| < M_\gamma(S)$ in $\Phi^F_\delta(S)$ for each positive $\delta$. Since $\Phi^F_\delta(S)$ in $\Phi^F_\delta(S)$ for $\delta < \delta_0$, it is sufficient to show that the above is true for $\delta$ sufficiently small. Let $\delta$, then, be any positive number less than $\min \left( M, \delta_0 \right)$.

Note first that if $x \in \Phi^F_\delta(S)$, and if also $x \in \Phi^F_{\delta_0}(S)$, then $|f(x)| < M_\mu(S_\delta)$, also $\gamma(x) < M_\gamma(S)$, hence $|F(x)| < M_\mu(S_\delta)M_\gamma(S)$.

Let $x$ now be any point of $\Phi^F_\delta(S)$. There is then a curve $L$ of length less than $L(S)$ (hence less than $L(S_{\delta_0})$) which joins $x$ to $R(s) > \sigma_i + \delta_0$. Let $L$ be the central line of a channel $C$ in $R(s) > \sigma_i$ of width $\delta$ which contains no point of $S^\sigma_\delta$. But since $\gamma(s)$ is holomorphic in $R(s) > \sigma_i$, a point $y$ in $R(s) > \sigma_i$ can be singular for $\gamma(s)$ and regular for $F(s)$ only if $\gamma(y) = 0$. Thus if $C$ contains any singular points of

...
f(s), \( \gamma(s) \) has a zero at each such point. Since the minimum distance between two zeros of \( \gamma(s) \) is greater than \( d \), there are at most a finite number \( n \) of zeros \( y_1, y_2, \ldots, y_n \) of \( \gamma(s) \) in \( C \); for any channel whose central line is of length less than \( L(\mathcal{S}) \) \( (\mathcal{S} < \frac{3}{4} d) \) can be covered by a finite number of circles of radius \( d \); each such circle contains at most one zero of \( \gamma(s) \).

If \( y_1 \) is any zero of \( \gamma(s) \) in \( C \), let \( K_1 \) be a circle of radius \( \frac{\sqrt{s}}{4} \) about \( y_1 \). Since \( s < \frac{3}{4} d \), no two such circles intersect. If each \( K_1 \) fails to intersect \( L \), or intersects \( L \) in at most one point, then \( L \) is the central line of a channel of width greater than \( \frac{s}{4} \) which contains no point of \( \mathcal{S}_x^{\mathcal{S}} \), so that \( x \in \mathcal{S}_x^{\mathcal{S}}(\mathcal{S}) \), \[ |F(x)| < M_x(\mathcal{S}) M_y(\mathcal{S}). \]

If a circle \( K_1 \) intersects \( L \), \( K_1 \) is at distance greater than \( \frac{s}{4} \) from the boundary of \( C \); for a circle with center at an intersection of \( L \) and \( K_1 \), and radius \( \frac{s}{2} \), contains \( K_1 \).

If no \( K_1 \) contains \( x \), \( L \) may be replaced by a curve \( L' \), which is obtained by replacing every arc of \( L \) (if any) which is in a circle \( K_1 \) by an arc of \( K_1 \) between the two points of intersection of \( L \) and \( K_1 \).
The curve $L'$ joins $x$ to $R(s) > \delta$, and is the central line of a channel of width greater than $\delta/4$ which contains no point of $S^C_I$.

If a circle $K_j$ contains $x$, there is only one such circle. By the discussion of the paragraph directly above, each point $x'$ on the boundary of $K_j$ lies in $S^C_I(\delta/4)$, hence $|F(x')| < M_F(\delta/4) \cdot M_\gamma(\delta')$ for each $x'$ on the circumference of $K_j$. But $F(s)$ is holomorphic in $K_j$, hence takes its maximum value on the boundary, so that also $|F(x)| < M_F(\delta/4) \cdot M_\gamma(\delta')$.

It has now been demonstrated that $|F(s)| < M_F(\delta/4)M_\gamma(\delta')$ in $S^C_I(\delta')$. Writing $M_F(\delta') = M_F(\delta/4)M_\gamma(\delta')$, the lemma is proved.

Suppose now that $g(s) = \sum e^{-\lambda_s s}$ has $A_q = 0$, and is "N" in $R(s) > -\gamma$ for some $\gamma > 0$. Suppose also that any singular points of $g(s)$ which occur at points $2\pi ki$ (for $k$ integral) are simple poles, and that $\lambda_n \neq \lambda_m + 1$ for every pair $m, n$. Under the additional hypothesis that the only singularities of $g(s)$ in $R(s) > \sigma_I$ are simple poles at certain points $2\pi ki$ ($k$ integral), the use of theorem 2.7 as indicated below shows that a
contradiction is involved; hence \( R_g(\sigma \neq 0) \) contains at least two points.

Let \( g(s) \) be the function of \( f(s) \) of lemma 2.8, 
- \( \gamma \neq \sigma \); and \((1-e^{-s})\) the function \( \gamma (s) \). Then \( g_1(s) = F(s) \equiv f(s) \gamma (s) = g(s) (1-e^{-s}) \) satisfies the conditions of the definition on page 73, so that \( g_1(s) \) "completes" \( g(s) \). For here

\[
g_1(s) = \sum e^{-s} - \sum e^{-(\lambda_n+1)s} = \sum e^{-\mu_n s};
\]

\( \phi_{\mu_n} \sum_{\lambda_n} \lambda_n \) since \( \lambda_n \neq \lambda_m + 1 \) for all \( n \neq m \) and \( \epsilon_n = 1 \) for \( \mu_n = \lambda_n \). Also \( A_g = 0 \), and \( g_1(s) \) is holomorphic in a circle of any radius less than \( 2 \pi \) about every point \( 2\pi ki \), since the zeros of \((1-e^{-s})\) " cancell" the simple poles of \( g(s) \). Moreover, by lemma 2.8, \( g_1(s) \) is "\( M \)" in \( R(s) > \sigma \). The application of theorem 2.7, in which \( g(s) \) plays the role of \( f(s) \), then yields a contradiction.

It is therefore impossible that such a function \( g(s) = \sum_{\lambda_n} e^{\lambda_n s} \) with \( \lambda_n \neq \lambda_m + 1 \), \( A_g = 0 \), which is "\( M \)" in \( R(s) > \gamma \), should have as its only singular points simple poles at points \( 2\pi ki \). In fact, it can be shown that if \( g(s) \) has \( A_g = 0 \), is "\( M \)" in \( R(s) > \gamma \).
and has only such singular points, it is the sum of a finite number of functions of the form \( e^{\alpha s} \sum e^{-\lambda_n s} \).

**Theorem 2.9:** Let \( g(s) = \sum e^{-\lambda_n s} \) satisfy the following conditions:

1. \( \eta_d \geq 0 \), \( g(s) \) is "R" in \( R(s) > -\eta \) for some \( \eta > 0 \).
2. All the singularities of \( g(s) \) in \( R(s) > -\eta \) lie on the line \( R(s) = 0 \), those at points \( 2\pi ik \) (k integral) being simple poles.
3. There exists an infinite subsequence \( \lambda_{n_k} \) of \( \lambda_n \) such that \( \lambda_{n_k} \neq \lambda_n + 1 \) for all \( n \).

Then \( R_\sigma (\sigma = 0) \) has at least two points.

**Proof:** Let \( \lambda_{n_k} \) be the maximum subsequence of \( \lambda_{n_k} \) such that \( \lambda_{n_k} \neq \lambda_n + 1 \) for all \( n \). Suppose the theorem not true; then the only singular points of \( g(s) \) in \( R(s) > \sigma \) are simple poles at certain points \( 2\pi ik \) (k integral).

There is certainly at least one, namely the point \( s \geq 0 \), by Landau's theorem (cf. [1], p. 80).

Let \( g_1(s) = g(s) (1-e^{-s}) \approx \sum e^{-\lambda_n s} - \sum e^{-\lambda_n (s+1)} \) 

\[ \approx \sum a_n e^{-\lambda_n s} \quad (a_n \geq 1), \] 

(this function \( g_1(s) \) does not complete \( g(s) \); it is not intended...
to play the role of the function $g_1(s)$ of the definition on page 77. The zeros of $\gamma(s)$ (1-\$e^{-s}$) "cancel" the simple poles of $g(s)$, so that $g_1(s)$ is holomorphic in $R(s) > \gamma$; moreover, $g_1(s)$ is "$\infty$" in $R(s) > \gamma$.

By lemma 2.3.

Since $\sum \frac{1}{\delta^+} \leq \sum \frac{1}{\delta^+}$, $H(g_{\sum \delta^+}) \leq H(g) \sum e^{-\lambda s}$.

Apply theorem 1.3, with $f(s) = g(s)$, $\phi(s) = g_1(s)$.

Here $\sigma_2 = \sigma_1 = \gamma$, $\sigma_\phi = 0$, $\sigma_\phi = \gamma$; also, $\delta^{\phi_{\sum \delta^+}}$ is empty, so that also $\delta^{\phi_{\sum \delta^+}}$ is empty. Hence in $R(s) \geq \gamma = \max (\sigma_1 + \sigma_\phi, \sigma_\phi + \sigma_\phi) = \gamma$, $H(s)$ is holomorphic. But this is manifestly impossible, for since $A_g = 0$, $\lim_{n \to \infty} \frac{1}{\lambda_n} \log n \geq 0$ (cf. [1], page 5); hence $0 \leq \lim_{c \to \infty} \frac{1}{\lambda_n} \log 1 \leq \lim_{c \to \infty} \frac{1}{\lambda_n} \log 1 \leq 0$.

But $H(s) = \sum e^{-\lambda s}$, hence $A_g = 0$, and by Landau's theorem, so $0$ is singular for $H(s)$. This contradiction proves the theorem.

Suppose now that $g(s) = \sum e^{-\lambda s}$ has $A_g > 0$, is "$\infty$" in $R(s) > \gamma$ ($\gamma > 0$) and has as its only singular points in $R(s) > \gamma$ simple poles at points $2 \pi \Gamma k$, where $k$ takes on certain integral values. It follows from the above theorem, that there exists no
infinite subsequence \( \lambda_{n_i} \) such that \( \lambda_{n_i} \not= \lambda_{n} + 1 \) for all \( i \), \( m \). There then exists a positive integer \( n_0 \), such that for \( n > n_0 \), to each \( \lambda_{n} \) corresponds a \( \lambda_{n_i} \) such that \( \lambda_{n_i} \not= \lambda_{n} + 1 \); if, for a given \( \lambda_{n} \), \( n_2 \) is also greater than \( n_0 \), there is a \( \lambda_{n_2} \) such that \( \lambda_{n_2} \not= \lambda_{n_2} + 1 \); if \( n_2 > n_0 \), there is a \( \lambda_{n_3} \) such that \( \lambda_{n_3} \not= \lambda_{n_3} + 1 \). It is clear that the following situation obtains: to each \( \lambda_{n} \) corresponds a positive integer \( p(n) \leq n_0 \), and a finite decreasing sequence \( \lambda_{n} > \lambda_{n_1} > \lambda_{n_2} > \ldots > \lambda_{n_k} \) \( n_k \equiv p(n) \), such that \( \lambda_{n_i} \not= \lambda_{n_i} + 1 \) \( (i = 1, 2, \ldots, k-1) \). Thus to each positive integer \( q \leq n_0 \) corresponds a sequence (which may have no terms, a finite number, or infinitely many terms) of \( \lambda_{n} \)'s, each term of which can be obtained by adding 1 to the preceding. Thus the sequence \( \{\lambda_{n}\} \) consists of a finite number \( n_0 \) of finite and infinite sub-sequences of the form \( \{\lambda_{q} + k(q)\} \) \( (q = 1, 2, \ldots, n_0) \); \( k(q) = 0, 1, 2, \ldots, f_q \); \( 0 \leq f_q < \infty \). The following corollary of theorem 2.9 has been demonstrated:

Theorem 2.10: If \( g(s) \equiv \sum e^{-\lambda_{n}s} \) has \( G \equiv 0 \), is \( \mu \) in \( \Re(s) > -\gamma \) for some positive \( \gamma \), and has as its only singular points in \( \Re(s) > -\gamma \) simple poles at points \( 2 = k \ell \) where \( k \) takes on certain integral values, then there exists a positive integer \( n_0 \) such that

\[
g(s) \equiv \sum_{\ell=0}^{n_0} e^{-\lambda_{\ell}s} \sum_{\ell=0}^{n_0} e^{-\mu\ell s}, \quad 0 \leq \mu \leq \infty.
\]
In other words, if \( g(s) \) has the above-mentioned properties, it can be expressed as the sum of a finite number of functions, each of which is the product of a function of the form \( e^{-\lambda s} \) and a finite or infinite Taylor-Dirichlet series; \( g(s) \) is the sum of a finite number of functions, each periodic of period \( 2\pi i \). If \( T_q(s) = \sum_{n=0}^{\infty} e^{\pi n} \), a Taylor-Dirichlet series which may or may not be finite and which has no gaps, the conclusion of theorem 2.10 may be rewritten

\[
g(s) = \sum_{q=1}^{\infty} \sum_{n=0}^{\infty} e^{-\lambda ns} T_q(s).
\]

A similar result is obtained if the hypotheses that the poles of \( g(s) \) are simple is replaced by the hypothesis that they are of finite maximum order (a precise definition of this term is given below). Similar theorems also hold when \( g(s) \) is replaced by more general series of the form \( \sum e^{\lambda_n s} \).

**Lemma 2.11:** If \( \phi(s) \) is holomorphic and \( \mathbb{H} \) in \( \mathbb{R}(s) > \sigma_1 \), if \( f(s) \) is \( \mathbb{H} \) in \( \mathbb{R}(s) > \sigma \), then \( \mathbb{R}(f(s) \phi) \in \mathbb{H} \) is holomorphic and \( \mathbb{H} \) in \( \mathbb{R}(s) > \sigma_1 + \sigma_1 \).

Note that a necessary and sufficient condition that \( \phi(s) \) be holomorphic in \( \mathbb{R}(s) > \sigma_1 \) is that the
set \( S^\pm_\phi (\delta) \) (page 19) be identical with the half-plane \( \mathbb{R}(s) > \sigma_2 + \delta \) for each positive \( \delta \) sufficiently small.

In applying theorem 1.3, here \( \sigma_1 = \sigma_\phi^0 \), so that \( \sigma^- = \max (\sigma_1 + \sigma_\phi^0, \sigma_2 + \sigma_\phi^0) \geq \sigma_2 + \sigma_\phi^0 \), since \( \sigma_2 + \sigma_\phi^0 > \sigma \). It is first easy to see that \( \mathbb{R}(s) \) is holomorphic in \( \mathbb{R}(s) > \sigma^- \). For evidently \( S^\pm_\phi \) is empty, so also then is \( \widehat{S}^\pm_\phi \); so that by theorem 1.3 \( \mathbb{R}(s) \) has no singular point in \( \mathbb{R}(s) > \sigma^- \).

It remains to show that to each positive \( \delta \) corresponds a positive number \( M_H(\delta) \) such that \(|H(s)| < M_H(\delta)\) on the set \( \overline{S}^\pm_H (\delta) = (\mathbb{R}(s) > \sigma^- + \delta) \).

As was seen (pages 39) in the proof of theorem 1.3, the sets \( B_\phi(r), B_\phi^*(r) \) of theorem 1.1 can be so chosen that if \( r = \frac{\delta}{\delta} \), \( B(r) \supseteq S^\pm_\phi (\delta) \), \( B_\phi^*(r) \supseteq S^\pm_\phi (\delta) \), and that then \( \widehat{B}_\phi^* \) contains that part of the complement in \( \mathbb{R}(s) > \sigma^- \) of \( \widehat{S}_\phi^* \cap (\mathbb{R}(s) > \sigma^- ) \) which is connected with \( \mathbb{R}(s) > 0_1 + 0_2 \). Hence here \( \widehat{B}_\phi^* \) contains the half-plane \( \mathbb{R}(s) > \sigma^- \), \( \widehat{B}_\phi^* \subseteq (\mathbb{R}(s) < \sigma^- ) \). Then for any given positive \( \delta \), there is a positive number \( r \) (depending only on \( \delta \)) and a positive number \( t \) (depending only on \( \delta \)) such that \( U_{\phi, \delta} (r; t) \subseteq (\mathbb{R}(s) < \sigma^- + \frac{\delta}{2}) \).

Hence if \( s \) is any point in \( \mathbb{R}(s) > \sigma^- + \delta \), and if \( \epsilon \in \mathbb{R} \setminus \frac{\delta}{2} \), there is a domain \( \Delta \) in \( \mathbb{R}(s) > \sigma^- + \frac{\delta}{2} \) (see...
page 13 containing \( z \), at a distance greater than 
\( 4 \in \circ \delta / 3 \) from every point of \( U_{\delta \varphi}(r; t) \), which contains points with arbitrarily large real part. Using the notation of theorem 1.1, it is seen (page 14) that 
\[ |E(z)| < \frac{M(r)}{2} \] for all \( z \) in \( \Delta \). But \( r, \in \) depend only on \( \delta \) (not on \( \Delta \)) so that, if 
\[ M_0(\delta) = \frac{M(r)}{2}, \] then \( |E(z)| < M_0(\delta) \) in \( R(s) > \delta + \delta \), which completes the proof of the lemma.

**Definition:** The poles of a function in a region are said to have a **finite maximum order** if there exists a positive integer \( K \) such that the order of every pole in the region is less than or equal to \( K \). The smallest such integer is the **maximum order** \( M \) of the poles of the function.
Theorem 2.32: Let \( g(s) = \sum e^{-\lambda_n s} \) satisfy the following conditions:

(i) \( \Re \alpha > 0 \), \( g(s) \) is "H" in \( \Re(s) > -\gamma \) for some \( \gamma > 0 \).

(ii) All the singularities of \( g(s) \) in \( \Re(s) > -\gamma \) lie on the line \( \Re(s) = 0 \), those at points \( 2 \pi ki \) \( (k \text{ integral}) \) being poles of finite maximum order \( M \).

(iii) There exists an infinite subsequence \( \{ \lambda_n^{(m)} \} \) of \( \lambda_n \) such that \( \lambda_n^{(m)} < \lambda_n + k \) for all \( m, n, \) and for all positive integers \( k < M \).

Then \( R_\sigma(\sigma \geq 0) \) has at least two points.

Proof: Suppose the contrary, that the only singular points of \( g(s) \) in \( \Re(s) > -\gamma \) are poles of finite maximum order \( M \) at certain points \( 2 \pi ki \). Set \( g_1(s) = g(s)(1-e^{-s})^M \).

As in theorem 2.3, \( g_1(s) \) is holomorphic and "H" in \( \Re(s) > -\gamma \), and \( g_2(s) = H(g, g_1) \) has no singular points in \( \Re(s) > -\gamma \).

The sequence \( \{ \lambda_n^{(m)} \} \) of hypothesis (iii) is assumed to be the largest such subsequence of \( \{ \lambda_n \} \).

The function \( g_2(s) \) may be expressed as follows:
\[ \sigma_1(s) = \sum_{j=0}^{M} (-1)^j c_j^d e^{-js} = \sum_{n=0}^{\infty} \lambda_n e^{-ns} \sum_{j=0}^{M} (-1)^j c_j^d e^{-js} \]

\[ = \sum_{n=0}^{\infty} \lambda_n e^{-ns} - \sum_{n=0}^{\infty} e^{(\lambda_n + 1)s} + \sum_{n=0}^{2} e^{(\lambda_n + 2)s} + \ldots + (-1)^M \sum_{n=0}^{\infty} e^{(\lambda_n + M)s} \]

(The \( c_j^d \) are binomial coefficients).

\[ \sigma_1(s) = \sum_{j=0}^{M} (-1)^j c_j^d \sum_{n=0}^{\infty} e^{(\lambda_n + j)s} \]

Denote by \( \{\lambda_n^{(k, \ldots, \ell)}\} \) the sequence of \( \lambda_n \)'s (if any) each of which belongs to the sequences \( \{\lambda_n + k\} \), \( \{\lambda_n + \ell\} \), and to no others;

\( \{\lambda_n + k\} \), \( \ldots \), \( \{\lambda_n + \ell\} \) and to no others;

\( \{\lambda_n \} = \{\lambda_n + 1\} \), \( \{\lambda_n + 2\} \), \ldots , \( \{\lambda_n + j\} \), \( \{\lambda_n + k\} \), \ldots , \( \{\lambda_n + \ell\} \)

(\( j, k, \ldots, \ell \) are all positive integers less than or equal to \( M \)). For example, \( \{\lambda_n^{(o)}\} \) is the sequence which belongs to \( \{\lambda_n\} \) but to no \( \{\lambda_n + k\} \) (\( k = 1, 2, \ldots, M \))

(cf. hypothesis iii); \( \{\lambda_n^{(o)}\} \) is the sequence of \( \lambda_n \)'s each of which belongs only to \( \{\lambda_n\} \) and to \( \{\lambda_n + 2\} \); \( \{\lambda_n^{(o, 1)}\} \) is the sequence of \( \lambda_n \)'s, each of which belongs only to \( \{\lambda_n\} \), \( \{\lambda_n + 1\} \), \( \{\lambda_n + 2\} \). If \( \sigma_1(s) \) is written as a single Dirichlet series \( \sigma_1(s) = \sum \lambda_n e^{-\lambda_n s} \), then
the only $\gamma_i$'s which are also $\lambda_i$'s are members of the sequences $\{n_i^{(1)}, n_i^{(2)}, \ldots, n_i^{(s)}\}$. If $\{\mu_{n_i}^{(j)}\}$ is the sequence of all terms of $\{\gamma_i\}$ which do not belong to $\{\lambda_i\}$, then $g_1(s)$ may be written:

$$
\sum_{n=1}^{\infty} d_n e^{-\gamma_n s} + \sum_{k=1}^{s} e^{-\lambda_i^{(k)}} \cdot (1 - C_n^2) e^{-\gamma_n^{(k)}} s
+ (1 - C_n^2) \sum_{k=1}^{s} e^{-\lambda_i^{(k)}} s + \ldots + \left[1 + (1 - H_n) C_n^2 \right] \sum_{k=1}^{s} e^{-\lambda_i^{(k)}} s
+ (1 - C_n^2 + C_n^4) \sum_{k=1}^{s} e^{-\lambda_i^{(k)}} s + \ldots + \left(1 - C_n^2 + (-1)^n C_n^4 \right) \sum_{k=1}^{s} e^{-\lambda_i^{(k)}} s + \ldots
$$

the sum of a finite number of series.

Then

$$
g_2(s) = H(g_2) = \sum_{n=1}^{\infty} e^{-\lambda_n^{(1)}} s + (1 - C_n^2) \sum_{k=1}^{s} e^{-\lambda_n^{(k)}} s
+ (1 + C_n^2) \sum_{k=1}^{s} e^{-\lambda_n^{(k)}} s + \ldots
$$

the sum of a finite number of series.

In lemma 2.11 put $g(s) = f(s)$, $g_1(s) = \emptyset(s)$.

Here $\sigma_\gamma = \sigma_\gamma f = \gamma$, $\sigma_\gamma f = 0$, the hypotheses of lemma 2.11 are satisfied, hence $g_2(s)$ is $H(g_2, g_1)$ is $\gamma$ in $\Re(s) > \gamma$. 

Let \( g(s) = \sum e^{-\lambda_n s} \). Since \( A_\eta \geq 0 \), \( \lim_{n \to \infty} \lambda_n \log n = 0 \) (cf. page 82); then also \( \lim_{n \to \infty} \frac{\lambda_n}{\log n} \) log \( n \geq 0 \) since, for each \( n, \lambda_n^{(a)} > \lambda_n \). Hence \( A_{g(s)} \geq 0 \), and \( g(s) \) is \( "M" \) in \( R(s) > k \eta \). Apply theorem 1.3, replacing \( f(s) \) by \( g(s) \), \( \eta(s) \) by \( g(s) \). Here \( \eta_\gamma ^{\delta} > \eta \). \( \zeta_\eta \) is empty, but both \( S_{\eta}^{\delta} \) and \( S_{\eta}^{\gamma} \) are empty; hence \( R(g(s) g(s)) \geq 0 \) is holomorphic in \( R(s) > \eta = \max (\eta_\gamma + \eta_\delta, \eta_\delta + \eta_\gamma) \geq \frac{h}{2} \eta < 0 \). But by Landau's theorem \( e = 0 \) is singular for \( g(s) \), a contradiction which proves the theorem.

Suppose now that \( g(s) = \sum e^{-\lambda_n s} \) has \( A_\eta \geq 0 \), is \( "M" \) in \( R(s) > k \eta \), has as its only singular points in \( R(s) > k \eta \) poles of finite maximum order \( M \) at points \( 2 \pi k \eta \), where \( k \) takes on certain integral values. It follows from the above theorem that there exists no infinite subsequence \( \{ \lambda_n \} \) of \( \{ \lambda_n \} \) such that \( \lambda_n^{(a)} = \lambda_m + k \) for all \( m, n \), and for all positive integers \( k \leq M \). There then exists a positive integer \( n_\eta \), such that \( n > n_\eta \) to each \( \lambda_n \eta \) corresponds a \( \lambda_{n_\eta} \) less than \( \lambda_\eta \) which differs from \( \lambda_{n_\eta} \) by a positive integer less than \( M \). To each \( \lambda_{n_\eta} \eta \) \( (n > n_\eta) \) corresponds a \( \lambda_{n_\eta} \) and a
positive integer \( n_1 \) \((0 \leq n_1 \leq M)\) such that
\[ \lambda \subset \lambda^{n_1} + R_{n_1}. \]
If \( n_1 > n_0 \), then \( \lambda \subset \lambda^{n_1} + R_{n_1} \)
\((0 \leq R_{n_1} \leq M, R_{n_1} \text{ a positive integer}). \)
If \( n_2 > n_0 \),
then \( \lambda \subset \lambda^{n_2} + R_{n_2}, \) etc. Evidently an \( n_k \)
will be reached after repeating this a finite number of
times; \( \lambda \subset \lambda^{n_k} + (R_{n_1} + R_{n_2} + \cdots + R_{n_k}). \)
Thus to
every \( n > n_0 \) corresponds a positive integer \( p(n) \geq R_{n_k}, \)
p(n) \leq n_0, and a positive integer \( R(n), \)
\( R(n) = R_{n_1} + R_{n_2} + \cdots + R_{n_k}, \) such that \( \lambda \subset \lambda^{n_k} + R(n). \)
Hence to each positive integer \( q \leq n_0 \) corresponds a
finite or infinite sequence of \( \lambda^{n_k}'s, \) each term of which
is obtained from the preceding by adding a positive
integer; the sequence \( \lambda^{n_k} \) is the sum of all these
sequences \( q \leq 0, 1, 2, \ldots, n_0 \). Thus the sequence \( \lambda^{n_k} \)
is composed of a finite number \( n_0 \) of finite or
infinite subsequences \( \lambda_{n_k}^{1} + n_1(q) \), where \( \lambda_{n_k}^{1} \)
is for each \( q \leq n_0 \) a finite or infinite sequence of
positive integers. Hence \( g(s) \) may be written:
\[ g(s) = \sum_{n_0} e^{-\lambda^{n_1} s} \sum_{n_1} e^{-n_1(q)s}, \quad 0 \leq r_q \leq \infty. \]
Or, writing \( T_q(s) \)
\[ T_q(s) = \sum_{n_0} e^{-n_1(q)s}, \quad 0 \leq r_q \leq \infty. \]
\[ g(s) = \sum_{q=0}^{\infty} e^{\lambda q} T_q(s). \quad T_q(s) \text{ is a finite or infinite (according as } r_q < \infty \text{ or } r_q = \infty) \text{ Taylor-Dirichlet series which may or may not have gaps. The following corollary has been demonstrated:} \]

**Theorem 2.13:** If \( g(s) = \sum e^{\lambda q} T_q(s) \) has \( Q = 0 \),

is \( "n" \) in \( R(s) > \gamma \) for some positive \( \gamma \), and has

as its only singular points in \( R(s) > \gamma \) poles of

finite maximum order \( M \) at points \( 2\pi k i \) where \( k \) takes

on certain integral values, then there exists a positive

integer \( n_0 \) such that

\[ g(s) = \sum_{q=0}^{\infty} e^{\lambda q} T_q(s), \quad 0 < r_q < \infty. \]

Theorems similar to the above hold in the general case, where a function \( f(s) = \sum a_n e^{\lambda q} \) is considered, with some added restrictions on the sequence \( \lambda, \lambda^2 \). The following theorem is an example of such a theorem:
Theorem 2.14: Let \( f(s) = \sum a_n e^{\lambda_n s} \) satisfy the following conditions:

(i) \( R_+ > 0 \), \( f(s) \) is "H" in \( R(s) > -\gamma \) for some \( \gamma > 0 \);

(ii) \( f(s) \) has a singular point at some point \( 2\pi ki \) for \( k \) an integer; further all singularities of \( f(s) \) in \( R(s) > -\gamma \) lie on \( R(s) = 0 \), those at points \( 2\pi ki \) being poles of finite maximum order \( N \);

(iii) There exists a subsequence \( \{\lambda_n^{(k)}\} \) of \( \{\lambda_n\} \) such that \( \lambda_n^{(k)} = \lambda_n + k \) for all \( m, n \), and for all positive integers \( k < K \) and such that \( \lim_{\lambda_n} \frac{1}{\lambda_n^{(k)}} \log |a_n^{(k)}| = 0 \), where \( a_n^{(k)} \) is the coefficient of the \( e^{-\lambda_n^{(k)} s} \) term in \( \sum a_n e^{\lambda_n s} \).

(iv) \( \lim_{\lambda_n} \frac{1}{\lambda_n^{(k)}} \log n = 0 \);

(v) The index of condensation of \( \{\lambda_n^{(k)}\} \) is zero.

Then \( R_+(s > 0) \) contains at least two points.

Proof: Suppose the contrary, that the only singular points of \( f(s) \) in \( R(s) > -\gamma \) are poles of finite maximum order \( N \) at certain points \( 2\pi ki \). Put \( f_1(s) = f(s) (1 - e^{-s})^N \). As in theorems 2.9 and 2.12,
$f_1(s)$ is holomorphic and $R^m$ (by lemma 2.8) in $R(s) > -\gamma$.
hence $f_2(s) = R(f_1)$ is holomorphic in $R(s) > -\gamma$.

But $f_1(s) = f(s) \sum_{j=0}^{\infty} (-1)^j c_j^\omega e^{-js}$

$$= \sum_{n=0}^{\infty} a_n e^{-\lambda_n s} \sum_{j=0}^{\infty} (-1)^j c_j^\omega e^{-js}$$

$$= \sum_{j=0}^{\infty} (-1)^j c_j^\omega \sum_{n=0}^{\infty} a_n e^{-(\lambda_n + j) s}$$

Using again the notation $\{\lambda_n, \gamma, \ldots, p\}$ to designate the
set of $\lambda_n$'s (if any) each of which belongs to the sets
$\{\lambda_0\}, \{\lambda_0 + j\}, \{\lambda_0 + k\}, \ldots, \{\lambda_0 + p\}$ and to no
others, and designating by $a_{\lambda_0, \ldots, p}$ the
coefficient of $e^{-\lambda_0 (t_1, \ldots, p)}$ if $f_1(s) \equiv \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}$,

then $f_1(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s} + \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}$

$$+ (1-C_{l1}^\omega) \sum_{n=0}^{\infty} a_n e^{-\lambda_n s} + \ldots$$

the sum of a finite number of series, where $\mathbf{\mu} + \mathbf{\gamma}$ is
the maximum subsequence of $\mathbf{\mu}$ having no member in
common with $\mathbf{\gamma}$. Hence
\[ f_2(s) = H(f,f_1) = \sum [a_n]^{(a)} 2 \cdot e^{-\lambda_n s} + (1-C_M^1) \sum [a_n]^{(a)} 2 \cdot e^{-\lambda_n s} + \ldots + (1-C_M^1 - C_M^2) \sum [a_n]^{(a)} 2 \cdot e^{-\lambda_n s} + \ldots \]

The sum of a finite number of series. In lemma 2.11, replace \( \varphi(s) \) by \( f_1(s) \). Here \( \sigma_2 \leq \sigma_1 = \gamma, \sigma_f \leq 0 \).

Hence \( f_2(s) = H(f,f_1) \) is holomorphic and \( "M" \) in \( R(s) > -\gamma \). Let \( f_3(s) = \sum [a_n]^{(a)} 2 \cdot e^{-\lambda_n s} \). Since

\[
\lim \log \frac{n}{\lambda_n^{(a)}} = c, \quad (\text{cf. } [1], \text{ page } 4),
\]

\[ = \lim \frac{\log |a_n|}{\lambda_n^{(a)}} = \lim \frac{\log |a_n|}{\lambda_n^{(a)}} = 0, \]

by hypothesis (iii). But since the index of condensation of \( \{\lambda_n^{(a)}\} \) is zero, \( G_f = H_f \), so that \( H_f \in C \).

On the other hand, in theorem 1.3 replace \( f(s) \) by \( f_2(s) \), \( \varphi(s) \) by \( f_4(s) = \sum e^{-\lambda_n s} \). Here

\( f_2(s) \) is \( "M" \) and holomorphic for \( R(s) > \sigma; \gamma = \gamma \).

\( f_4(s) \) is \( "M" \) and holomorphic in \( R(s) > \frac{3}{2} \gamma \). Here \( \sigma_f = \sigma_1 = \gamma, \sigma_f = \sigma_1 = \frac{3}{2} \gamma \). Both \( S_{\sigma_f} \) and \( S_{\sigma_1} \) are empty, therefore \( H(f_2,f_4) = f_3(s) \) is holomorphic.
in \( R(s) > \sigma \), \( s \max (\alpha_1 + \alpha_2, \alpha_2 + \alpha_3) + \frac{1}{\gamma} < 0 \);
a contradiction which proves the theorem.

The hypothesis \( \frac{1}{\lambda_n} \log |a_n| \leq 0 \) is certainly not one which can not be satisfied, in view of the fact that \( A_f = \frac{1}{\lambda_n} \log |a_n| \leq 0 \). It is essential, as the following example shows:

Let \( \varphi(s) = \sum e^{-s^n} \). Define \( \gamma(s) = \sum a_n e^{s^{\mu_n}} \) so that the sequence \( \{ \mu_n \} \) has an index of condensation zero, \( \lim \frac{1}{\mu_n} \log n \neq 0 \). \( \lambda_n < -\gamma \) \((\gamma > 0)\), no \( \mu_n \) is an integer, \( \mu_n \neq \mu_m + 1 \) for all \( n, m \) (for example, \( \mu_n = n + 1/n \), \( c_n = e^{-n} \)). Let \( f(s) = \varphi(s) + \gamma(s) = \sum a_n e^{s^{\mu_n}} \). Hypothesis (i) is satisfied; so is hypothesis (ii), the poles being simple \( (M = 1) \); so also is hypothesis (iii), the subsequence \( \{ \lambda_n \} \) being the sequence \( \{ \mu_n \} \), except that

\[
\frac{\log |a_n|}{\lambda_n} \leq \frac{\log |c_n|}{\mu_n} \leq -\gamma \neq 0.
\]

Hypotheses (iv) and (v) are also satisfied. Yet the conclusion is not valid; in fact, \( R_2(\sigma, \gamma) \neq 0 \) contains only one point, \( \gamma \neq 0 \). The theorem therefore fails if the hypothesis \( \frac{1}{\lambda_n} \log |a_n| \leq 0 \) is not included.
It is not essential, of course, that $A_f \neq 0$. The theorem may be stated as follows:

Theorem 2.14': Let $f(s) = \sum a_n e^{-\lambda_n s}$ satisfy the following conditions:

(i) $f(s)$ is $\nu^+$ in $R(s) > A_f - \gamma$ for some $\gamma > 0$;

(ii) there is a point $s_0$ with $R(s_0) = A_f$ which is singular for $f(s)$; all singularities of $f(s)$ in $R(s) > A_f - \gamma$ lie on $R(s) = A_f$ at points $s_0 + 2\pi k i$ ($k$ integral) being poles of finite maximum order $N$;

(iii) there exists a subsequence $\lambda_n^{(0)}$ of $\lambda_n$ such that $\lambda_n^{(0)} \neq \lambda_m + k$ for all $m, n$, and for all positive integers $k \leq N$; and such that $\text{Im} \frac{1}{\lambda_n^{(0)}} \log a_n^{(0)} A_f$ is the coefficient of the $e^{-\lambda_n^{(0)} s}$ term in $\sum a_n e^{-\lambda_n s}$;

(iv) $\text{Im} \frac{1}{\lambda_n^{(0)}} \log n \neq 0$;

(v) the index of condensation of $\lambda_n^{(0)}$ is zero;

then $R_f(s \geq A_f)$ contains at least two points.

For $F(s) = f(s + s_0) = \sum a_n e^{-\lambda_n (s+s_0)} = \sum a_n e^{-\lambda_n s}$ satisfies the hypotheses of theorem 2.14, hence $R_F(s \geq 0)$ contains at least two points. But $R(s_0) = A_f$, hence $R_f(s \geq A_f)$ contains at least
two points. In other words, there are singular points on $R(s) = A_f$ other than those at points $s_0 + 2\pi k i$.

By using precisely the same argument with which corollary 2.15 was shown to be a consequence of theorem 2.12, the following theorem may be shown to follow from theorem 2.14':

**Theorem 2.15:** If $f(s) = \sum a_n e^{-\lambda_n s}$ is $M$ in $R(s) > A_f = \gamma$ for some positive $\gamma$, and has as its only singular points in $R(s) > A_f = \gamma$ poles of finite maximum order at points $s_0 + 2\pi k i$ for some $s_0$ with $R(s_0) = A_f$ ($k$ taking on certain integral values), and if further the index of condensation of $\sum_n [f]$ in zero, $\sum_n \frac{1}{\lambda_n} \log n = 0$, and $\lim \frac{1}{\lambda_n} \log |a_n| = A_f$, then there exists a positive integer $n_0$ such that $f(s) = \sum a_n e^{-\lambda_n s} T_q(s)$, where $T_q(s)$ is a finite or infinite Taylor-Dirichlet series.

For all the hypotheses of theorem 2.14' are satisfied except (iii), and the conclusion fails to hold. Hence either there is no such subsequence $\sum_n [f]$, or if there is one, $\sum_n \frac{1}{\lambda_n} \log |a_n| \neq A_f$; but this is clearly impossible, since $\lim \frac{1}{\lambda_n} \log |a_n| < A_f$. 
Hence there exists no such subsequence \( \{ \lambda_{n}^{(m)} \} \), and the rest of the argument is exactly the same as that preceding corollary 2.13.

Theorem 2.14 gives sufficient conditions in order that \( R_{\sigma}(\sigma = 0) \) may contain two points; by replacing hypotheses (ii) and (iii) by (ii') and (iii') below are obtained sufficient conditions that \( R_{\sigma}(\sigma = 0) \) contains infinitely many points.

(iii'): All singular points (of which there is at least one) of \( f(s) \) in \( R(s) > -\gamma \) are poles of finite maximum order \( M \) on \( R(s) = \sigma = 0 \).

(iii'): There exists a subsequence \( \{ \lambda_{n}^{(m)} \} \) of such that \( \lambda_{n}^{(m)} \rightarrow \lambda_{n} + k \) for all positive integers \( m, n, k \), and such that \( \prod_{\lambda_{n}^{(m)}} \frac{1}{\lambda_{n}^{(m)}} \log |a_{n}^{(m)}| < 0 \).

Suppose the contrary, that \( R_{\sigma}(\sigma = 0) \) contains only a finite number \( p \) of points, \( s_{1}, s_{2}, \ldots, s_{p} \).

Then \( f_{1}(s) \equiv f(s) \prod_{j=1}^{p} (1 - e^{-(s-s_{j})})^{M} \) is holomorphic and \( \lambda^{(m)} \) in \( R(s) > -\gamma \) (lemma 2.8). But

\[
f_{1}(s) \equiv f(s) \prod_{i=1}^{p} \left( \sum_{k=0}^{M} (-1)^{k} C_{M}^{k} e^{-k(s-s_{j})} \right)
\]
\( f_1(s) \cdot f(s) = \sum_{n=0}^{M} \frac{c_n}{s^n} \) (where the \( c_n \) are complex constants)

\[ = \sum_{n=0}^{\infty} \alpha_n \cdot e^{\lambda_n s} \sum_{n=0}^{M} \frac{c_n}{s^n} \]

\[ = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \alpha_n \cdot e^{(\lambda_n + p)s} \]

\[ = c_0 \sum_{n=0}^{\infty} \alpha_n \cdot e^{\lambda_n s} + c_1 \sum_{n=0}^{\infty} \alpha_n \cdot e^{(\lambda_n + 1)s} \]

\[ + \cdots + c_{PM} \sum_{n=0}^{\infty} \alpha_n \cdot e^{(\lambda_n + PM)s} \]

Using again the symbol \( \lambda_n^{(l_1, l_2, \ldots, l_r)} \) to denote the set of \( \lambda_n \)'s (if any) each of which belongs to the sets \( \lambda_n \cdot \lambda_n + k \), \( \lambda_n + k \), \( \lambda_n + kp \) and to no others, and denoting by \( a_n(j, k, \ldots, p) \) the coefficient of \( e^{\lambda_n^{(j, k, \ldots, p)} s} \) (cf. theorem 2.16),

\[ f_2(s) = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} [a_n]^{(p)} \cdot a^{(n)} \]

\[ + (c_0 + c_1) \sum [a_n]^{(1)} \cdot e^{\lambda_n s} \]

\[ + \cdots \]

the sum of a finite number of series. The remainder of
the argument is exactly that of theorem 2.14.

Again, it is evidently not essential that $R_f \neq 0$ if hypotheses (ii) and (iii) of theorem 2.14 are replaced by

(ii"') All singularities (of which there is at least one) of $f(s)$ in $R(s) \uparrow R_f = \gamma, \gamma > 0$, are poles of finite maximum order $N$ on $R(s) \uparrow R_f$;

(iii") There exists a subsequence $\{\lambda_n^m\}$ of $\{\lambda_n^m\}$ such that $\lambda_n^m \neq \lambda_m^k$ for all positive integers $m$, $n$, $k$; and such that

$$\text{Im} \left( \frac{1}{\lambda_n^m} \log |z_n^m| \right) \in A_f;$$

it then follows that $R_f(\cdot + R_f)$ contains infinitely many points.
Theorem 2.13: If the following hypotheses are made:

(i) \( f(s) = \sum a_n s^{-\lambda_n - s} \), \( \varphi(s) = \sum b_n s^{-\mu_n - s} \), \( \varphi(s) \neq 0 \), \( f(s) \) is "\( \mathbb{M} \)" in \( \Re(s) > -\gamma \) for some \( \gamma > 0 \);

(ii) \( f(s) - \varphi(s) \) is holomorphic and "\( \mathbb{M} \)" in \( \Re(s) > -\gamma \);

(iii) \( \lim \frac{1}{\lambda_n} \log |a_n| = 0 \);

(iv) \( \lim \frac{1}{\lambda_n} \log n = 0 \);

(v) The index of condensation of the sequence \( \xi \lambda_n \) is zero;

Then all the members of the sequence \( \xi \lambda_n \) except at most a finite number belong also to the sequence \( \xi \mu_n \).

Proof: Suppose the contrary, that there exists an infinite subsequence \( \xi \lambda_n \) of \( \xi \lambda_n \) such that \( \lambda_n \neq \mu_m \) for all \( i, m \). Let \( \xi \lambda_n \) be the maximum such subsequence of \( \xi \lambda_n \). Let \( \xi \lambda_n \) be the complement in \( \xi \lambda_n \) of \( \xi \lambda_n \); each \( \lambda_n \) belongs both to \( \xi \lambda_n \) and to \( \xi \mu_n \). Let \( \xi \mu_n \) be the subsequence of \( \xi \mu_n \) which is identical with \( \xi \lambda_n \); and let \( \xi \mu_n \) denote the complement in \( \xi \mu_n \) of \( \xi \mu_n \); here \( \lambda_n = \mu_n \) (\( j > 0 \)).

Define \( \gamma(s) = f(s) - \varphi(s) \). Then \( \gamma(s) \) is holomorphic and "\( \mathbb{M} \)" in \( \Re(s) > -\gamma \), hence \( f_\gamma(s) \in \mathbb{M}(f, \gamma) \).
has no singular points in \( R(s) > \gamma \). But

\[ \gamma(s) = \sum c_n e^{\lambda_n s} - \sum b_n e^{\gamma_n s} \]

\[ = \sum c_n e^{\lambda_n s} + \sum (c_m - b_m) e^{\lambda_m s} \]

\[ - \sum b_n e^{\gamma_n s} \]

Then \( f_2(s) = H(f, \gamma) = \sum c_n^2 e^{\lambda_n s} \]

\[ + \sum c_m (c_m - b_m) e^{\lambda_m s} \]

Let \( f_2(s) = \sum c_n^2 e^{\lambda_n s} \). Then under hypotheses

(iii), (iv), (v),

\[ A_{f_3} = \frac{\text{Im} \log |c_n|^2}{\lambda_n} = 2 \frac{\text{Im} \log |c_n|}{\lambda_n} \]

\[ = 0 \]

On the other hand, in lemma 2.11 replace \( \phi(s) \) by \( \gamma(s) \), here \( \sigma_1 = \sigma; g = \gamma, \sigma_0 = 0 \); hence

\( f_2(s) \in H(f, \gamma) \) is holomorphic and "M" in \( R(s) > \gamma \).

In theorem 1.3 replace \( f(s) \) by \( f_2(s), \phi(s) \) by \( f_2(s) \).
where \( f_q(s) = \sum e^{\lambda_n f_q} \), with \( \sigma_2 = \frac{1}{2} \) (cf. page 18), \( \sigma_2 \geq \sigma_1 = -\gamma \). Then (cf. theorem 2.14) \( H(f_2 f_4) \otimes f_3(s) \) is holomorphic in \( R(s) \supset \sigma \), where \( \sigma = \max (\sigma_1 + \sigma_2, \sigma_1 + \sigma_2') = -\frac{1}{2} \gamma \); a contradiction which proves the theorem.

If to the hypotheses of theorem 2.16 are added

(iii') \( \lim \frac{\mu_n}{\nu_n} \log |b_n| = 0 \),

(iv') \( \lim \frac{\mu_n}{\nu_n} \log n = 0 \), and

(v') the index of condensation of the sequence \( \tilde{\lambda}_n \tilde{f} \) is zero, then the two sequences \( \tilde{\lambda}_n \tilde{f} \) and \( \tilde{\mu}_n \tilde{f} \) are identical, with the exception of at most a finite number of terms of each sequence. For (iii') and (iv') together imply that also \( A_2 \geq 0 \). Hence with hypotheses (iii'), (iv'), and (v'), interchanging \( f(s) \) and \( \Phi(s) \) in theorem 2.16, it follows that also all the elements of \( \tilde{\mu}_n \tilde{f} \) except at most a finite number belong also to \( \tilde{\lambda}_n \tilde{f} \), which yield the desired conclusion.

It is, of course, again not essential that \( A_\Phi \geq A_\Phi = 0 \). Theorem 2.16 may be rewritten:
Theorem 2.16':  Suppose

\[ f(s) = \sum a_n e^{-\lambda_n s}, \quad \varphi(s) = \sum b_n e^{-\mu_n s}, \]

\( f(s) \) is \( t \) in \( R(s) > \delta - \gamma \) for some positive number \( \delta \); \n
\( f(s) - \varphi(s) \) is holomorphic and \( t \) in \( R(s) > \delta - \gamma \); \n
\( \lim \frac{1}{\lambda_n} \log |a_n| \leq A \), \n
\( \lim \frac{1}{\lambda_n} \log n \leq 0 \), \n
\( \nu \) the index of condensation of the sequence \( \{\lambda_n\} \)

is zero, then all of the terms of the sequence \( \{\lambda_n\} \) except at most a finite number belong also to \( \{\mu_n\} \).

For then \( F(s) = f(s + \delta) = \sum a_n e^{-\lambda_n(s + \delta)} = \sum a_n e^{-\lambda_n s} \), and \( \varphi(s + \delta) = \sum b_n e^{-\mu_n(s + \delta)} = \sum b_n e^{-\mu_n s} \) satisfy all the hypotheses of theorem 2.16.

Again, if to the hypotheses of theorem 2.16' are added (iii'): \( \lim \frac{1}{\lambda_n} \log |b_n| = A_f \), and (iv') and (v') of page 104, the two sequences \( \{\lambda_n\} \) and \( \{\mu_n\} \) are identical, with the exception of at most a finite number of terms of each sequence.

By making conditions (i) and (ii) more restrictive, the conclusion of theorem 2.19' may be obtained with
less restrictive hypotheses (iii) and (iv).

**Theorem 2.17**: Suppose

1. \[ f(s) = \sum a_n e^{-\lambda_n s}, \quad \varphi(s) = \sum b_n e^{-\lambda_n s}, \quad A_e < \infty; \]
2. \( f(s) \) is "N" in the entire plane;
3. \( \gamma(s) = f(s) - \varphi(s) \) is entire, and "N" in the entire plane;
4. \[ \lim_{\lambda_n} \frac{\varphi(n)}{\lambda_n} = 0; \]
5. the index of condensation of \( \mathcal{F}_{\lambda_n} \) is finite;

then all the terms of the sequence \( \mathcal{F}_{\lambda_n} \) except at most a finite number belong also to \( \mathcal{F}_{\mu_n} \).

**Proof**: Suppose the contrary; define \( \mathcal{E}_{\lambda_n}, \mathcal{E}_{\lambda_n}, \mathcal{E}_{\mu_n}, \mathcal{E}_{\mu_n} \), \( \mathcal{F}_{\lambda_n}, \mathcal{F}_{\lambda_n}, \mathcal{F}_{\mu_n}, \mathcal{F}_{\mu_n} \) as in the proof of theorem 2.16. Then \( \gamma(s) \) is entire, and "N" in the entire plane, so that \( H(s) = H(f, \gamma) \) is entire. But, as in theorem 2.16,

\[ H(f, \gamma) = \sum a_{21} e^{-\lambda_n s} + \sum a_{21} e^{-\lambda_n s} - b_{21} e^{-\lambda_n s} + \sum c_n e^{-\lambda_n s}. \]

Then under hypothesis (iv),

\[ \Omega_n = \lim_{\lambda_n} \frac{\log |c_n|}{\lambda_n} > \lim_{\lambda_n} \frac{\log |a_{21}|^2}{\lambda_n}. \]
\[ a_n > \frac{2 \log |a_n|}{\lambda_n} \rightarrow 2 \lim_{n \to \infty} \log |a_n| > -\infty \]

(\{a_{n_k}\} is a subsequence of \{a_n\}). Hence \( a_n \) is finite. But it is known that the difference between \( a_n \) and \( H_n \) is not greater than the index of condensation of the sequence \( \lambda_n \) (see [1], page 54). Hence \( H_n \) is also finite, \( H(s) \) is not entire; this contradiction completes the proof of the theorem.

If to the hypotheses of theorem 2.17 are added

(iii') \( \lim \log \|s_k - b_n\| > -\infty \)

(iv') \( \lim \mu_n \log n = 0 \)

(v') the index of condensation of the sequence \( \mu_n \) is finite;

then the two sequences \( \lambda_n \), \( \mu_n \) are identical, with

the exception of at most a finite number of terms of each

sequence. It is sufficient to interchange \( f(s) \) and \( \phi(s) \)

in theorem 2.17.

Theorem 2.16' may be used to show that under certain

conditions, if \( |\lambda_n - \mu_n| \) is sufficiently small for each

n, and if \( \lambda_n \) contains an infinity of terms not in \( \mu_n \) ,

then the functions \( f(s) = \sum a_n e^{\lambda_n s} \) and \( \phi(s) = 0 \),

where \( \phi(s) = \sum a_n e^{\mu_n s} \), can not both be "N" in
in \( A(s) > \tilde{\alpha}_f - \gamma \) for any positive number \( \gamma \). In \([1]\)
(page 30), V. Bernstein shows the existence of a sequence of positive numbers \( \xi \in \mathbb{N}_+ \) such that if \( |\lambda_n - \mu_n| \in \xi \), then \( f(s) = \phi(s) \) is entire. This gives the following corollary of theorem 2.16:

**Corollary 2.16:** Suppose

\[
(1) \quad f(s) = \sum a_n e^{\lambda_n s}, \quad \phi(s) = \sum a_n e^{\mu_n s},
\]

\( f(s) \) is "\( \gamma \)" in \( R(s) > \tilde{\alpha}_f - \gamma \) for some positive number \( \gamma \);

\( (ii) \quad |\lambda_n - \mu_n| \in \xi \), where \( \xi > 0 \), \( \lim_{\lambda_n} \frac{1}{\xi} \log \xi = -\infty \) \( (n \in \{1, 2, \ldots\}) \);

\( (iii) \quad \lim_{\lambda_n} \frac{1}{\xi} \log |a_n| = \bar{\alpha}_f \);

\( (iv) \quad \lim_{\lambda_n} \frac{1}{\xi} \log n = 0 \);

\( (v) \quad \) the index of condensation of the sequence \( \xi \lambda_n \xi \)

is zero;

then all of the terms of the sequence \( \xi \lambda_n \xi \) except at most a finite number belong also to \( \xi \mu_n \xi \).

For if \( \gamma (s) = \sum \xi \in \gamma e^{\lambda_n s} \), then under hypothesis \( (iv) \), \( \tilde{\alpha}_f \geq \lim_{\lambda_n} \frac{1}{\xi} \log \xi |s| = -\infty \); it follows

(see \([1]\), page 30) that \( f(s) = \phi(s) \) is entire. Then \( (ii) \) of 2.16 implies \( (ii) \) of 2.16'. Hence all conditions of 2.16' are satisfied, and the conclusion follows.
In view of the fact that in theorem 1.3 and in all the theorems of chapter 2 the functions concerned are supposed "H" in some half-plane, it is pertinent to ask whether such functions exist. Obviously all periodic (with respect to the imaginary part of the variable) functions have this property, in particular those given by Taylor-Dirichlet series. However, if these were the only "H" functions, these theorems would indeed be trivial. Fortunately, the class of functions which are "H" in a strip outside their half-planes of absolute convergence is a far wider class than that of periodic functions. The following few paragraphs are devoted to a discussion of an extended class of functions which are "H" in some half-plane which includes the half-plane of absolute convergence.

Let \( \psi(s) = \sum b_n e^{i s b_n} \) be a periodic (period \( 2\pi i \)) function with \( -\infty < a \psi(s) < \infty \). Let \( f(s) = \sum a_n e^{-i s b_n} \) have \( a_f < a \psi \). Then \( \psi(s) = f(s) + \phi(s) = \sum c_n e^{i s b_n} \) is bounded e.s. (page 17) in \( R(s) > \sigma \), where \( \sigma_f < \sigma < a \psi \), and has \( a \psi \geq a \phi \) for evidently \( \sum c_n e^{i s b_n} \) converges absolutely in \( R(s) > \sigma \), and has singular points on \( R(s) = a \phi \), hence \( a \psi \geq a \phi \).
But $f(s)$ is bounded in $\mathbb{R}(s) > \sigma_\tau$, and $\phi(s)$ is bounded e.f.s. in the whole plane, so that

$$\psi(s) = f(s) + \phi(s)$$

is bounded e.f.s. in $\mathbb{R}(s) > \sigma_\tau$.

The example given by Mandelbrojt ([5], page 4) of a class of functions which are bounded e.f.s., is far more general. Suppose that the function $\phi(s)$ above has only isolated singularities in $\mathbb{R}(s) > \sigma_\tau$ for some $\sigma_\tau$ such that $\mathcal{A}_f < \sigma_\tau < \mathcal{A}_\phi$. Let $T(x) = \sum d_n x^n$ be any entire function. If circles of radius $\epsilon > 0$ is arbitrary, positive) about the singular points of $\phi(s)$ are extracted from the half-plane $\mathbb{R}(s) > \sigma_\tau$, $f(s)$ and $\phi(s)$ are bounded in the remaining part of this half-plane. Hence the function $T_1(s) = T(f(s) + \phi(s))$ is here holomorphic and bounded. Also, for $\mathbb{R}(s) > \mathcal{A}_\phi$,

$$T_1(s) = \sum d_n \left( \sum a_n e^{-\lambda_n s} + \sum b_j e^{-j\omega s} \right)^n.$$ 

All the series converge uniformly and absolutely, hence $T_1(s) = \sum k_n e^{-\lambda_n s}$, the series having $\mathcal{A}_T < \mathcal{A}_\phi$. Moreover, the singularities of $\phi(s)$ in $\mathbb{R}(s) > \sigma_\tau$ are also singular for $T_1(s)$, hence $\mathcal{A}_T < \mathcal{A}_\phi$. For if $s_0$ is a pole of $\phi(s)$, $\phi(s)$ is unbounded in every neighborhood of $s_0$, hence $s_0$ is singular for $T_1(s)$.
the same is true if $a_0$ is an isolated essential singularity, by Heierstrass' theorem on isolated essential singularities.

Hence $\Omega_1(s)$ is bounded e.f.s. in $R(s) > c^r$.

It is worth noting that if $\psi(s) = f(s) + \varphi(s) = \sum a_n e^{-\lambda_n s} + \sum b_n e^{-\mu_n s} = \sum c_n e^{-\gamma_n s}$ is defined as on page 110, then $\lim \frac{\lambda}{\lambda_n} \log |a_n|$ does not exist. For the sequence $\frac{\lambda}{\lambda_n} \log |a_n|$ has at least two distance limit points, namely $\text{Im} \frac{\lambda}{\lambda_n} \log |a_n|$, and $\text{Im} \frac{\lambda}{\lambda_n} \log |b_n|$. These are certainly distinct if $\lim \frac{\lambda}{\lambda_n} \log |a_n|$ exists, for then $A = \text{Im} \frac{\lambda}{\lambda_n} \log |a_n|$, $A > \text{Im} \frac{\lambda}{\lambda_n} \log |b_n|$.

To assure the non-triviality of some of the theorems, it will be well to have examples of non-periodic functions $\sum a_n e^{-\lambda_n s}$ which are "M" in half-planes including their half-planes of absolute convergence, and for which $\lim \frac{\lambda}{\lambda_n} \log |a_n|$ exists. Such an example would be an "M" function $g(s) = \sum e^{-\lambda_n s}$. That such functions do exist is readily shown by the following example:

Let $\mu_1, \mu_2, \ldots, \mu_m$ be a finite set of linearly independent positive numbers; so that $x_1 \mu_1 + x_2 \mu_2 + \ldots + x_m \mu_m = 0$ for $x_1, \ldots, x_m$
non-negative integers only if \( x_1 = x_2 = \ldots = x_M = 0 \).

Then \( x_1^{\mu_1} + x_2^{\mu_2} + \ldots + x_M^{\mu_M} = y_1^{\mu_1} + \ldots + y_M^{\mu_M} \)
for \( x_i, y_i \ (i = 1, 2, \ldots, M) \) non-negative integers only
if \( x_i \geq y_i \ (i = 1, 2, \ldots, M) \).

Put \( f_i(s) = \sum_{\sigma_i} e^{-\sigma_i s} = \frac{1}{1 - e^{-\mu_i s}} \quad (i = 1, 2, \ldots, M) \).

Each \( f_i(s) \) is "M" in the entire plane; in fact, each
\( f_i(s) \) is pure periodic with period \( \frac{2\pi i}{\mu_i} \); also, no \( f_i(s) \)
has a finite zero. Hence \( F(s) = \sum_{i=1}^{M} \frac{f_i(s)}{\mu_i} \) \( f_i(s) \) is singular
at each point which is singular for some \( f_i(s) \), and at
no others. It follows that for every \( \sigma > -\infty \), \( F(s) \) is
bounded e.f.s. in \( R(s) > \sigma \); that is, \( F(s) \) is bounded
e.f.s. in the entire plane. But clearly \( F(s) = \sum e^{-\nu_i s} \)
where the set of positive numbers \( \nu_i \) is the set of all
numbers of the form \( x_1^{\mu_1} + x_2^{\mu_2} + \ldots + x_M^{\mu_M} \) where
each \( x_i \ (i = 1, 2, \ldots, M) \) takes on all non-negative
integral values. The function \( F(s) \) is evidently not
periodic, but is quasi-periodic in the sense of Bohl and
Esclangon (cf. E. Bohr, Zur Theorie der fast periodischen

It seems probable that the class of functions of
the form \( g(s) = \sum e^{-a_n s} \) with \( A_q = 0 \) which are "M"
in some half-plane $R(s) > -\gamma$ for some positive $\gamma$ is wider than the class of functions indicated above; in particular it is possible that every quasi-periodic function which can be continued along some curve having points with arbitrarily small (large negatively) real part is $\Omega$ in the entire plane. However, I have not been able to prove that this is true.
An application of a type somewhat different from those of the second chapter gives the singularities of the function \( F(s) = \sum \psi(a_n) e^{-\lambda_n s} \) in terms of those of the function \( f(s) = \sum a_n e^{-\lambda_n s} \), where \( \psi(s) = \sum a_n e^{s^m} \) is an entire function satisfying certain conditions. Purely formal operations yield:

\[
F(s) = \sum \psi(a_n) e^{-\lambda_n s} = \sum e^{-\lambda_n s} \sum a_n e^{s^m}
\]

\[
= \sum a_m \sum e^{s^m} e^{-\lambda_n s}. \quad \text{But}
\]

\( f_m(s) = \sum e^{s^m} e^{-\lambda_n s} \) is the composite function \( H(f, f_{m-1}) \). This recurrence relation, with theorem 1.3, makes it possible to express the singular points of \( f_m(s) \) and hence of \( F(s) \) in terms of those of \( f(s) \), if certain convergence conditions are satisfied.

As before, \( S_f \) will denote the set of singular points of \( f(s) \) in \( \mathbb{R}(s) \). The set \( S_{f^k} = \left[ S_f, S_f \right] \) will be denoted by \( (2S)_f = (2S)_f \); and \( (kS)_f = (kS)_f \) \((k \text{ a positive integer})\) will be defined by the recurrence relationship \( (k+1)S)_f = \left[ (kS)_f, S_f \right] \). Define

\[
\overline{\mathcal{S}_f} = S_f + (2S)_f + (3S)_f + \ldots, \quad \text{the set of points, each of which belongs to at least one set } (kS)_f. \quad \text{The symbol } \overline{\mathcal{S}_f} \text{ denotes the closure of the set } \mathcal{S}_f.\]
It will be convenient to use a slight modification of the idea of an \( \Phi \) function. Suppose \( R(s) \) is a function, uniformly bounded in some half-plane \( R(s) > c \). Let \( \sigma \) be a real number less than \( c \), and \( T \) a closed set of points in \( R(s) \leq c \). Let \( L(\delta) \) be a positive function of the real variable \( \delta \), strictly increasing to infinity as \( \delta \to 0 \). Define \( R(\delta) \) as the set of points in \( R(s) > \sigma \) each of which can be joined to the half-plane \( R(s) > c \) by a curve in \( R(s) > \sigma \) of length less than \( L(\delta) \) at a distance greater than \( \delta \) from every point of the set \( T \). It may be described as the set of curves of length less than \( L(\delta) \) having one end-point in \( R(s) > c \), which are central lines of channels in \( R(s) > \sigma \) of width \( 2\delta \) containing no points of \( T \).

Its complement in \( R(s) > \sigma \), \( T(\delta) \), is the set of points in \( \sigma \leq R(s) \leq c \) consisting of closed circles of radius \( \delta \) about points of \( T \), and also all points in \( R(s) > \sigma \) which can not be joined to \( R(s) > c \) by curves in \( R(s) > \sigma \) of length less than \( L(\delta) \) containing no points of these circles.
**Definition:** The function $F(s)$ is "$\mu_L^m$" in $\mathbb{R}(s) > \sigma_1$ with respect to the closed set $T$, if there exists a real positive function $L(\delta)$ strictly increasing to infinity as $\delta \to 0$, such that to each positive $\delta$ corresponds a positive number $M(\delta)$ such that $F(s)$ is holomorphic and $|F(s)| < M(\delta)$ in $\Phi(\delta)$.

To say that a function $f(s)$ is "$\mu_m$" in $\mathbb{R}(s) > \sigma_1$ is to say that it is "$\mu_L^m$" in $\mathbb{R}(s) > \sigma_1$ with respect to $S_0^\infty$ for some function $L(\delta)$.

Evidently if a function $F(s)$ is "$\mu_L^m$" in $\mathbb{R}(s) > \sigma_1$ with respect to a set $T$, and if $T$ is contained in $T'$ (so that $\Phi(\delta)$ contains $\Phi'(\delta)$), then $F(s)$ is "$\mu_L^m$" in $\mathbb{R}(s) > \sigma_1$ with respect to $T'$ also.

If it is possible to take $L(\delta) \leq \infty$, then $F(s)$ is "$\mu_m$" in $\mathbb{R}(s) > \sigma_1$ with respect to $T$. Here $\Phi(\delta)$ is the set of all curves in $\mathbb{R}(s) > \sigma_1 + \delta$ (no restriction on the length), having one end-point in $\mathbb{R}(s) > \sigma_1$, which are central lines of channels in $\mathbb{R}(s) > \sigma_1$, of width $2\delta$ containing no points of $T$. 
The idea of a function being \( \mathcal{K}_\infty \) in \( R(s) > \sigma_\tau \) with respect to \( S_\tau^{\sigma_\tau} \) is very close to the idea of a function being bounded except for singularities in \( R(s) > \sigma_\tau \) (cf. page 17). For the function is \( \mathcal{M}_\infty \) in \( R(s) > \sigma_\tau \) with respect to \( S_\tau^{\sigma_\tau} \) if and only if, after circles of radius about points of \( S_\tau^{\sigma_\tau} \) have been extracted from the half-plane \( R(s) > \sigma_\tau \), the function is bounded in that part of the remaining region which is connected with the half-plane \( R(s) > \sigma_\tau \).

The only difference between this idea and the idea of a function being bounded except for singularities in \( R(s) > \sigma_\tau \) is that the latter idea implies that the function is bounded in the complete complement in \( R(s) > \sigma_\tau \) of the region composed of circles of radius \( \delta \) about points of \( S_\tau^{\sigma_\tau} \).
It is clear that if a function $f(s)$ is $M_\infty$ in $R(s) > \sigma_1$, with respect to a closed set $T$, then $S_\sigma^f \subset T + JT$, where $JT$ is that part of the complement in $R(s) > \sigma_2$ of $T$ which is not connected with the half-plane $R(s) > c$. For if there is a point $x$ of $S_\sigma^f$ not in $T + JT$, it is easily verified that $x \in \mathcal{U}(\delta)$ for $\delta$ sufficiently small, hence $f(s)$ is holomorphic at $x$, a contradiction.

Suppose $f(s)$ is $M_\infty$ in $R(s) > \sigma_1$, with respect to $T_1$, a closed set of points in $R(s) < c_1$, where $c_1 < c_1$, $c$ is any real number greater than $A_1$. Then there is by definition a function $M_2(\cdot)$ such that $f(s)$ is holomorphic and $|f(s)| < M_2(\cdot)$ in $S_1(\cdot)$. Similarly, if $g(s)$ is $M_\infty$ in $R(s) > \sigma_2$, with respect to $T_2$, a closed set in $R(s) < c_2$, where $c_2 < c_2$, $(c_2 > A_2)$, then $g(s)$ is holomorphic and $|g(s)| < M_2(\cdot)$ in $S_2(\cdot)$. If $H(s)$ is $M_\infty$ in $R(s) > \sigma_2 > \max (\sigma_1 + c_2, \sigma_1 + c_1)$ with respect to $T$, a closed set in $R(s) < c_1$ $(c_1 > A_1)$, then $H(s)$ is holomorphic and $|H(s)| < M_2(\cdot)$ in $S(\cdot)$. 
Theorem 5.1: If \( f(s) \) is \( \mathcal{N}_c \) in \( \mathbb{R}(s) > c \), with respect to closed \( T_1 \), \( g(s) \) is \( \mathcal{N}_c \) in \( \mathbb{R}(s) > c_3 \) with respect to closed \( T_2 \), then \( \mathcal{R}(f, g) \subseteq \mathcal{R}(s) \) is \( \mathcal{N}_c \) in \( \mathbb{R}(s) > c' \) if \( \max \{ c_1 + c_2, c_2 + c_3 \} \) with respect to \( T = [T_1, T_2] \). Moreover, there is a function \( K(\delta) = 2(c_1+c_3)/\delta \) such that

\[
\mathcal{M}_\mathbb{E}(\delta' + c_1, c_2) \leq K(\delta) \mathcal{M}_\mathbb{F}(\gamma) \mathcal{M}_\varphi(j, \lambda) .
\]

Here \( \delta' \), \( \gamma_1 \), \( \gamma_2 \) are arbitrary, positive; \( T_1 \) is a closed set of points in \( \mathbb{R}(s) \leq c_1 \), where \( c_1 < c \) \( (c > G_F) \), \( T_2 \) is a closed set of points in \( \mathbb{R}(s) \leq c_2 \), where \( c_2 < c' \) \( (c' > G_F) \). \( T_1(\gamma) \) is the set of curves having one end-point in \( \mathbb{R}(s) > c \), which are central lines of channels in \( \mathbb{R}(s) > c \) of width greater than \( 2 \gamma \), passing through no points of \( T_1; T_1(\gamma) \) is its complement in \( \mathbb{R}(s) > c \). \( T_2(\gamma) \) and \( T_2(\gamma_2) \) are defined similarly, and are complementary sets in \( \mathbb{R}(s) > c \).

Proof: Note that \( T = [T_1, T_2] \) is a closed set of points which lies in \( \mathbb{R}(s) < c + c' \) since \( T_1 \) lies in \( \mathbb{R}(s) < c \), \( T_2 \) in \( \mathbb{R}(s) < c' \). The set \( S(\gamma) \) is the set of curves having one end-point in \( \mathbb{R}(s) > c + c' \).
(no restriction as to length) which are central lines of channels in \( R(s) > \sigma' \) of width greater than \( 2 \gamma \) containing no points of \( T \). Its complement in \( R(s) > \sigma' \), \( T(\gamma) \), is the set consisting of all closed circles of radius \( \gamma \) about points of \( T \), and also all points which can not be connected with \( R(s) > e + e' \) by curves in \( R(s) > \sigma' + \gamma \) containing no points of such circles.

In order to show that \( H(s) \) is \( M_\infty \) in \( R(s) > \sigma' \) (note that \( G_{\infty} < G_{c} + G_{d} < e + e' \)), it is sufficient to show that there is a function \( M_H(\gamma) \) such that \( H(s) \) is holomorphic and \( |H(s)| < M_H(\gamma) \) in \( \mathcal{F}(\gamma) \). It will appear from the proof that \( M_H(\gamma) \) can be so chosen that

\[
M_H(\delta \gamma + \gamma + \gamma') = E(\delta) M_R(\gamma) M_\phi(\gamma')
\]

for arbitrary, positive \( \delta, \gamma, \gamma' \). This is, of course, a very rough estimate of \( M_H \) in terms of \( M_R \) and \( M_\phi \).

Suppose then, that \( s_\circ \in \mathcal{E}(\gamma) \). Then there is a curve \( L \) joining \( s_\circ \) to \( R(s) > e + e' \) which is the central line of a channel \( C \) in \( R(s) > \sigma' \) of width greater than \( 2 \gamma \) which contains no point of \( T \in [T_1, T_2] \).
It will first be shown that if \( \gamma_1 + \gamma_2 < \gamma \), then L is the central line of a channel \( C' \) of width \( \gamma = (\gamma_1, \gamma_2) \) containing no points of \( [T_1(\gamma), T_2(\gamma)] \). Denote by \( T_1(\gamma) \), \( T_2(\gamma) \) those parts of \( T_1(\gamma), T_2(\gamma) \) respectively consisting of closed circles of radii \( \gamma_1 \), \( \gamma_2 \) respectively.

Suppose now that the above statement is false, that the channel \( C' \) does contain a point of \( [T_1(\gamma), T_2(\gamma)] \). Since the channel is open, \( C' \) contains a point \( \alpha + \beta \) for some \( \alpha \in T_1(\gamma), \beta \in T_2(\gamma) \). The following four possibilities arise, each of which will be seen to lead to a contradiction, proving the original assertion.

Case 1: \( \alpha \in T_1(\gamma), \beta \in T_2(\gamma) \).

There is then an \( \alpha' \in T_1 \) and a \( \beta' \in T_2 \) such that \( |\alpha' - \alpha| < \gamma_1 \), \( |\beta' - \beta| < \gamma_2 \). But \( \alpha + \beta \) is in \( C' \), \( (\alpha' + \beta') = (\alpha + \beta) - (\gamma_1, \gamma_2) \), hence \( \alpha' + \beta' \) is in \( C \). But \( \alpha' + \beta' \in T_1 \), a contradiction.

Case 2: \( \alpha \in T_1(\gamma), \alpha \in T_1(\gamma), \beta \in T_2(\gamma) \).

There is then a \( \beta' \in T_2 \) such that \( |\beta - \beta'| < \gamma_2 \).

Since \( \alpha + \beta \) is in \( C' \), there is a curve \( L' \) (the curve \( L \) plus the line-segment joining \( \alpha \) to \( \alpha + \beta \)).
which joins \( \alpha + \beta \) to \( R(s) \supset c + e' \) and is at
distance greater than \( \gamma, + \gamma_2 \) from all points of

\[ T = \left[ T_1, T_2 \right]. \]

Subtract \( \beta' \) from each point of \( L' \), obtaining
the curve \( L' = (\beta') \). This curve joins \( y = \alpha + (\beta - \beta') \)
to the half-plane \( R(s) \supset c + c' = R(\beta') \) (hence to the
half-plane \( R(s) \supset c, \) since \( R(\beta') \supset e' \) and is at
distance greater than \( \gamma, + \gamma_2 \) from every point of
\( T_1 \); for if not, the point \( \alpha' + \beta' \) is at distance less
than or equal to \( \gamma, + \gamma_2 \) from \( L' \); but \( \alpha' + \beta' \in T_2 \)
so that this is impossible.

But \( |y - \alpha| = \left| \beta - \beta' \right| \) hence there is a
curve \( L'' \) (the curve \( L' = (\beta') \) plus the line segment
joining \( y \) to \( \alpha \) ) which joins \( \alpha \) to \( R(s) \supset c \) and is at
distance greater than \( \gamma, \) from all points of \( T_1 \).

Moreover, the curve \( L'' \) lies in the half-plane \( R(s) \supset \sigma_1 + \gamma \).

For the channel \( C \) lies in \( R(s) \supset \sigma' \geq \max \{ \sigma_1 + c_2, \sigma_2 + e' \} \),
hence the channel \( C' \) and the curve \( L' \) lie in
\( R(s) \supset \sigma' + \gamma, + \gamma_2 \supset \sigma' + c_2 + \gamma, + \gamma_2 \) for \( y' \) on
\( L' = (\beta') \), \( R(y') \supset \sigma_1 + c_2 + \gamma, + \gamma_2 = R(\beta') \).

But \( R(\beta') \supset c_2 \), hence \( R(y') \supset \sigma_1, + \gamma, + \gamma_2 \) thus for
\( y'' \) on \( L'' \), \( R(y'') > \sigma, + \eta, + \gamma_2 + 0 = \sigma + \eta, \).

Therefore \( \lambda \in S_2(\gamma) \), the complement of \( T_1(\gamma) \), a contradiction.

Case 3: \( \lambda \in T_1(\gamma) \), \( \beta \in T_2(\gamma_2) \), \( 3 \in T_2(\gamma_2) \).

It is sufficient to interchange \( \lambda, \beta \) etc., in the discussion of case 2 to arrive at a contradiction.

Case 4: \( \lambda \in T_1(\gamma) \), \( \lambda \in T_1(\gamma) \), \( 3 \in T_2(\gamma_2) \).

Let \( \lambda^+ \) be the point of \( T_1(\gamma) \) nearest \( \lambda \) with the same imaginary part and greater real part. There is certainly such a point, for otherwise the horizontal line joining \( \lambda \) to \( R(s) > c \) contains no point of \( T_1(\gamma) \); hence \( \lambda \in T_1(\gamma) \), a contradiction.

Let \( \beta^- \) be the point (if any) of \( T_2(\gamma_2) \) nearest \( 3 \) with the same imaginary part, smaller real part,

\( 3(\beta^-) > \sigma_2 \). If there is no such point, let

\( \beta = \sigma_2 + i I(\beta) \); in this case

\( |\beta - \beta^-| = R(\beta - \beta^-) = R(\beta) - \sigma_2 \).

But since \( \lambda + \beta \) lies in the channel \( C' \),

\( R(\alpha + \beta) > \sigma, + \gamma, + \gamma_2 > \sigma_2 + \sigma_1 + \gamma, + \gamma_2 \).

\( R(\beta) > \sigma_2 + \sigma_1 + \gamma, + \gamma_2 > R(\alpha) \).
\[ R(\beta - \gamma) > \sigma_2 + e_1 - R(\alpha) = \sigma_2 + e_1 = R(\alpha). \]

But \( e_1 > R(\alpha^+), \) hence
\[ |\beta - \gamma| < R(\beta - \gamma) > R(\alpha^+) \alpha | \alpha^+ - \alpha |, \text{ if } \beta = \sigma_2 + e_1 - R(\beta). \]

Having defined \( \alpha^+, \beta, \) suppose first that
\[ |\beta - \gamma| < |\alpha^+ - \alpha| \].

Then subtract \( \beta^+ \) from each point of \( \gamma \) obtaining the translated curve \( \gamma' = (\gamma - \beta) \). This curve joins \( y = \alpha + (\beta - \gamma) \) to the half-plane \( R(s) > \gamma \)
and contains no point \( \alpha' \in T_1(\gamma) \) (for if so, the point
\( \alpha' + \beta \) is on \( \gamma' \); there is a point \( \alpha'' \in T_1 \) such that
\[ |\alpha'' - \alpha'| < \eta, \text{ and a point } \beta_2 \in T_2 \text{ such that } |\beta_2 - \beta| < \eta, \]
\[ |(\alpha'' + \beta_2) - (\alpha' + \beta)| < \eta, + \eta_2 \text{, hence } \gamma' \text{ is at distance less than } \eta, + \eta_2 \text{ from a point } \alpha'' + \beta_2 \text{ of } T_3 = [T_1, T_2], \text{ a contradiction.} \]

But \( I(\alpha) \subset I(y) \subset I(\alpha^+) \),
\[ R(\alpha) < R(y) < R(\alpha^+) \] since then the curve
\( \gamma' = (\gamma - \beta) \) contains a point \( \alpha^+ \in T_1(\gamma) \). Hence there is no point of \( T_1(\gamma) \) on the line segment joining \( y \) to \( \alpha \).

Hence \( \alpha \) can be joined to the half-plane \( R(s) > \gamma \) by a curve \( (\gamma' = (\gamma - \beta) \) plus the line segment) which contains no point of \( T_1(\gamma) \). Moreover, as in case (2), this curve lies in \( R(s) > \sigma \). Thus \( \alpha \in T_1(\gamma), \text{ a contradiction.} \)

Suppose, then, that
\[ |\beta - \gamma| > |\alpha^+ - \alpha| \].

Subtract
from each point of \( L' \), obtaining the curve \( L' = (\alpha^+) \).

This curve joins \( w \) to \( \gamma = (\alpha^+ - \alpha^-) \) to the half-plane \( R(s) > c' \), and contains no point of \( T_2(\gamma_2) \) (by an argument similar to that for the curve \( L' = (\beta^-) \)).

But \( I(\beta) \not\subseteq I(w) \supseteq I(\beta^-) \), \( R(\beta) > R(w) > R(\beta^-) \).

Hence there is no point of \( T_2(\gamma_2) \) on the line segment joining \( w \) and \( \beta^- \). Therefore \( \beta \) can be joined to the half-plane \( R(s) > c' \) by a curve (the curve \( L' = (\alpha^+) \) plus the line segment) which contains no point of \( T_2(\gamma_2) \). This curve lies in \( R(s) > \sigma_1 \) by an argument similar to that showing that \( L'' \) lies in \( R(s) > \sigma_1 \), in case 2. Hence \( \beta \not\subseteq T_2(\gamma_2) \), a contradiction.

Each of the four possibilities leads to a contradiction; the original assertion, that if \( s_0 \in \mathcal{P}(\gamma) \) then there is a curve \( L \) in \( R(s) > c' \) joining \( s_0 \) to \( R(s) > c + c' \) which is the central line of a channel of width \( \gamma = (\gamma_1, \gamma_2) \) containing no point of \( [T_1(\gamma), T_2(\gamma_2)] \), is proved.

It remains to show that to each positive \( \gamma \) corresponds a positive number \( m_\gamma(\gamma) \) such that in \( \mathcal{P}(\gamma) \), \( \mathcal{H}(s) \) is holomorphic, and \( |H(s)| < m_\gamma(\gamma) \).

First, \( S^{\sigma_1}_s \subseteq T_1 + JT_1 \), \( S^{\sigma_2}_s \subseteq T_2 + JT_2 \) (page 118), hence \( S^{\sigma_1}_s(f) \subseteq T_1(f) \), \( S^{\sigma_2}_s(f) \subseteq T_2(f) \), for each
positive \( \delta \). Therefore (here \( L(\delta) \equiv \infty \))

\[
\left[ S_T, S_S \right] \subseteq \left[ T_1, T_2 \right] \subseteq \left[ S_T, S_S \right].
\]

Write \( T_S = S_T \delta = S_T \delta \).

\[ T_S^L \subseteq \left[ S_T, S_S \right] \subseteq \left[ T_1, T_2 \right]. \]

Then \( T_S^L \subseteq \left[ T_1, T_2 \right] \), \( S_T^L = S_S^L \). But by lemma 1.2, with \( L(\delta) = \infty \), \( S_T^L \subseteq S_{\delta} \), \( S_T^L = S_{\delta} \).

hence \( S_{\delta} \subseteq S_{\delta} \). But by theorem 1.3, \( H(s) \) is holomorphic in \( S_{\delta} = (R(s) > \sigma') \), that part of the complement in \( R(s) > \sigma' \) of \( S_{\delta} \circ (R(s) > \sigma') \), which is connected with \( R(s) > \sigma' \).

But \( \mathcal{F}(\gamma) \subseteq \mathcal{C} \subseteq (R(s) > \sigma') \) for each \( \gamma \), hence \( \mathcal{S} \subseteq (R(s) > \sigma') \subseteq \mathcal{F}(\gamma) \). \( H(s) \) is holomorphic in \( \mathcal{F}(\gamma) \) for each \( \gamma \).

It will now be shown that \( M_H(\gamma) \) can be so chosen that for all positive \( \delta \), \( \gamma \), \( \gamma_2 \),

\[
M_H(5 \delta + \gamma, + \gamma_2) < K(\delta) M(\gamma) M(\gamma_2).
\]

Let \( \delta_0 \) again be any point of \( \mathcal{F}(\gamma) \), and let \( L, C, C', \) etc., have the meanings ascribed to them in the paragraphs above. Put \( \gamma = 5 \delta + \gamma, + \gamma_2 \). Then \( C' \) has width \( 5 \delta \) and contains no point of \( \left[ T_1(\gamma), T_2(\gamma_2) \right] \).

There is then an open region \( \Delta \) containing \( L \) at distance greater than \( 4 \delta \) from \( \left[ T_1(\gamma), T_2(\gamma_1) \right] \), and which contains points with arbitrary large real part. Let this be the region \( \Delta \) of theorem 1.1.

Favor the strip \( \sigma \subseteq R(s) \leq \delta \) with squares of
side \( s \), extracting those containing points of \( T_1(\gamma, \lambda) \) as well as their neighbors. For \( s \) in \( \Delta \), there is a positive number \( h \) such that \( R(s) > \sigma_1 + \epsilon_2 + 2h \).

Extract from the strip all points with real part less than \( \sigma_1 + h \). Define as in theorem 1.1 \( D_\sigma^s \), \( D_\epsilon^s \), \( C_s^{\alpha_m} \), and \( L_s^{\alpha_m} \), noting that

\[
L_s^{\alpha_m} < 2^{\alpha_m - 1} (\epsilon - \epsilon + 1) / \eta < K(\epsilon, \lambda) \alpha_m.
\]

Then for \( s \) in \( \Delta \) let

\[
M(s) = \frac{1}{2^{\alpha_m}} \int f(s) \Theta(s-s) \, ds.
\]

If \( s \) is on \( C_s^{\alpha_m} \), either

(a) \( |s - \alpha| < 3 \eta \) for some \( \alpha \in T_1(\gamma, \lambda) \), or

(b) \( s = \sigma_1 + h + it \) for some \( t \).

In any event \( R(s) > \sigma_1 \), and \( s \in T_1(\gamma, \lambda) \), hence \( s \in \mathcal{Q}(\gamma, \lambda) \), and \( |f(s)| < M_\epsilon(\gamma, \lambda) \).

If (a), then, since for \( s \) in \( \Delta \),

\[
|s - (\alpha + \beta)| > \eta \delta \quad \text{for all } \alpha \in T_1(\gamma, \lambda), \quad \beta \in T_2(\gamma, \lambda), \quad |s - s - \beta| > \eta \delta \quad \text{for all } \beta \in T_2(\gamma, \lambda)
\]

Moreover, \( R(s) > \sigma_1 + \epsilon_1 + \delta \), \( R(s) > \epsilon_1 + \delta \), hence

\( R(s-s) > \sigma_2 \); therefore \( (s-s) \in \mathcal{Q}(\gamma, \lambda) \),

\[
|f(s-s)| < M_\Theta(\gamma, \lambda).
\]

If (b), then, since for \( s \) in \( \Delta \),

\[
R(s) > \sigma_1 + \epsilon_2 + 2h,
\]

\( R(s-s) > \sigma_1 + \epsilon_2 + 2h - \sigma_1 - h > \epsilon_2 + h \).
\[ s \sim s - \beta \succ \mathcal{H}(s - s - \beta) \succ c_2 + h - e_2 - h_1 \]

hence \((s-s) \in \mathcal{S}_2(\gamma_2)\). \(\mathcal{Q}(s-s) \prec \mathcal{M}(\gamma_2)\).

Therefore

\[
\left| H_m(s) \right| \sim \frac{1}{x_m} M_\mathcal{P}(\gamma_1) M_\mathcal{Q}(\gamma_2) K(\delta) \alpha_m .
\]

\[
\left| H(s) \right| \sim \lim_{m \to \infty} H_m(s) \prec K(\delta) M_\mathcal{P}(\gamma_1) M_\mathcal{Q}(\gamma_2)
\]

in \(\Delta\), and in particular on the line \(L\). It has now been demonstrated that in \(\mathcal{H}(s + \gamma, + \gamma_2)\),

\[
\left| H(s) \right| \prec K(\delta) M_\mathcal{P}(\gamma_1) M_\mathcal{Q}(\gamma_2). 
\]

Put

\(x \sim \gamma, x \sim \gamma_2 \sim \gamma_1\); then in \(\mathcal{H}(\gamma)\),

\[
\left| H(s) \right| \prec K(\gamma_1) M_\mathcal{P}(\gamma_2) M_\mathcal{Q}(\gamma_1). 
\]

There is therefore a function \(M_H(\gamma)\) such that in \(\mathcal{H}(\gamma)\), \(H(s)\) is holomorphic, and \(\left| H(s) \right| \prec M_H(\gamma)\). It has been shown that this function can be so chosen that for arbitrary

\(x \sim \gamma, x \sim \gamma_2\),

\(M_H(s + \gamma, + \gamma_2) \prec K(\delta) M_\mathcal{P}(\gamma_1) M_\mathcal{Q}(\gamma_2)\).

This completes the proof of the theorem.
Lemma 3.2: Let \( f_k(s) = \sum_{n=0}^{\infty} a_n s^{-\lambda_n s} \) (\( k = 1, 2, \ldots, \)).

and \( f(s) \neq f_1(s) \), \( G_f < \infty \). If \( f(s) \) is "\( \mathcal{N}_\infty \)

in \( R(s) > \sigma \), and if \( \sigma_k < \sigma + (k-l) \frac{\sigma}{2} \),

then \( S^\sigma_{\lambda f} \) is contained in \( T^k + J^k \), where \( T^k = (kS)^\sigma \) (\( k = 1, 2, \ldots \)).

Proof: Let \( f(s) \) here play the role of both \( f(s) \) and \( \varphi(s) \). \( S^\sigma_{\lambda f} \) is the set of singular points of \( f_k(s) \) in \( R(s) > \sigma \), \( (kS)^\sigma \) is defined by the recurrence relation \( (kS)^\sigma = \left[ S^\sigma f (k-1 S)^\sigma \right] \).

\( J^k \) is that part of the complement in \( R(s) > \sigma \) of \( T^k \) which is not connected with the half-plane \( R(s) > \sigma \) \( \varphi \).

Let \( T_1 \) of theorem 3.1 be the set \( S^\sigma_{\lambda f} \).

\( T_1 \equiv T_2 \equiv \varphi^\sigma \equiv \varphi \). Then \( R(s) \neq f_2(s) \),

\( T = \left[ T_1 \cap T_2 \right] = \left[ \varphi^\sigma \cap \varphi^\sigma \right] = (2S)^\sigma \equiv T^2 \).

Since \( f(s) \) is "\( \mathcal{N}_\infty \)" with respect to \( T_1 \equiv S^\sigma_{\lambda f} \),
in \( R(s) > \sigma \), it follows from theorem 3.1 that \( R(s) \neq f_2(s) \) is "\( \mathcal{N}_\infty \)" with respect to \( T^2 \equiv (2S)^\sigma \).
in \( R(s) > \max (\sigma, \sigma_2, \sigma_3) \). Here \( T_1, T_2 \) lie in the strip \( \sigma < R(s) \leq \varphi \), thus \( \sigma_1, \sigma_2 \)
may be taken equal to \( \varphi \). Theorem 1.3 implies that \( R(s) \neq f_2(s) \) is holomorphic in \( S^2 \), that is, that the
singular points of \( H(s) \) in \( R(s) > \sigma_e \cap \sigma_f \). are points of \( T^2 + JT^2 \). Theorem 3.1 implies that \( H(s) \) is "\( M_\infty \" with respect to \( T^2 \) in \( R(s) > \sigma_i + \frac{I}{\sigma_f} \). Now apply theorems 1.3 and 3.1 again, with \( f(s) \) playing the role of \( f_3(s) \), \( f_2(s) \) playing the role of \( \psi(s) \).

Then \( H(s) = f_3(s) \). Let \( T_1 = S_{T^2}^{\sigma_r} \), \( T_2 = T^2 \cap (28)^{\sigma_r} \).

Then \( T = T^3 = [T_1, T_2] = (38)^{\sigma_r} \). By theorem 1.3, since \( T^2 \) contains \( S_{T^2}^{\sigma_r} \), the only possible singularities of \( H(s) = f_3(s) \) in \( R(s) > \sigma^* \cap \max (\sigma_i, \frac{I}{\sigma_f}, \sigma_e + \frac{I}{\sigma_f}) \) are among points of \( T^3 + JT^3 \). But \( \sigma_f \leq 2 \frac{I}{\sigma_f} \).

\( \sigma_r = \sigma_i = \sigma_{3} \leq \sigma_1 \). Hence \( \sigma^* \geq \sigma_i + 2 \frac{I}{\sigma_f} \geq \sigma_0 \).

Theorem 3.1 implies that \( H(s) = f_3(s) \) is "\( M_\infty \" in \( R(s) > \sigma_3 \) with respect to \( T = T^3 = (38)^{\sigma_r} \). An evident induction completes the proof.

Define \( M(\delta) = N(\delta) = N_{\sigma_r}^{\sigma_r}(\delta) \) as the least upper bound of \( |f(s)| \) in \( T_3(\delta) \) (cf. page 116), where \( T_3 = S_{T^3}^{\sigma_r} \). Similarly let \( M(\delta) = M_{\sigma_r}(\delta) \) be the least upper bound of \( |f_k(s)| \) in \( T_k(\delta) \), where \( T_k = (kS)^{\sigma_r} \) \( (k \in \{1, 2, \ldots \}) \). By theorem 3.1, since \( f_k(s) \subset H(f, f_{k-1}) \).
\[ M_k(5 \delta_1^* + \gamma_1 + \gamma_2) \leq K(\delta_1^*) M(\gamma_1) M_{k-1}(\gamma_2) \]

where \[ K(\delta_1^*) = \frac{2(\alpha - \delta_1^*)}{\delta_1^*} < \frac{N}{\delta_1^*}, \] for some positive number \( N \) (for \( \delta_1^* \) less than, say, 1; \( \delta_1^* \) is arbitrary, but need not be the same for each \( k \)).

For given positive integers \( r \) and \( k \), put
\[ \delta_1^* = \delta_{\frac{k}{r}}(\gamma_1), \gamma_1 = \frac{\varepsilon}{k}, \gamma_2 = (k-1) \varepsilon/k. \] Then
\[ M_k(5 \delta_1^{(r)} + \gamma_1^* + \gamma_2) \leq M_k(5 \delta_1^{(r)} + \varepsilon) \]
\[ \leq \frac{L}{\delta_{\frac{k}{r}}^{(r)}} M(\frac{\varepsilon}{k}) M_{k-1} \left( \frac{(k-1) \varepsilon}{k} \right). \]
If \( \varepsilon' \leq 5 \delta_{\frac{k}{r}}^{(r)} + \varepsilon \),
\[ M_k(\varepsilon) \leq \frac{L}{\delta_{\frac{k}{r}}^{(r)}} M(\frac{\varepsilon'}{k}) M_{k-1} \left( \frac{(k-1) \varepsilon'}{k} \right). \]
\[ M_k(\varepsilon) \leq \frac{L}{\delta_{\frac{k}{r}}^{(r)}} M(\frac{\varepsilon'}{k}) M_{k-1} \left( \frac{(k-1) \varepsilon'}{k} \right). \]
\[ M_k(\varepsilon) \leq \frac{L}{\delta_{\frac{k}{r}}^{(r)}} M(\frac{\varepsilon'}{k}) M_{k-1} \left( \frac{(k-1) \varepsilon'}{k} \right). \]
\[ M_k(\varepsilon) \leq \frac{L}{\delta_{\frac{k}{r}}^{(r)}} M(\frac{\varepsilon'}{k}) M_{k-1} \left( \frac{(k-1) \varepsilon'}{k} \right). \]
\[ M_k(\varepsilon) \leq \frac{L}{\delta_{\frac{k}{r}}^{(r)}} M(\frac{\varepsilon'}{k}) M_{k-1} \left( \frac{(k-1) \varepsilon'}{k} \right). \]
\[ M_k(\varepsilon) \leq \frac{L}{\delta_{\frac{k}{r}}^{(r)}} M(\frac{\varepsilon'}{k}) M_{k-1} \left( \frac{(k-1) \varepsilon'}{k} \right). \]
An evident induction yields

\[ M(\varepsilon) \leq \frac{L^F \delta^{(n)}}{\varepsilon} \frac{M(\varepsilon_\infty) - 5 \delta^{(n)}}{\varepsilon} M(\varepsilon_\infty) - 5 \left( \frac{8}{\varepsilon} \cdot \frac{\delta^{(n)}}{\varepsilon} \right) \cdot \]

\[ \ldots \cdot M^{2}(\varepsilon) - 5 \sum_{k=2}^{3} \frac{k^2}{\varepsilon} \delta^{(n)} \cdot \]

Now put \( \varepsilon' = \frac{k}{5r + 2k} \cdot M(\varepsilon') \leq M(\varepsilon) \) for

\[ \varepsilon' < \varepsilon \] hence the largest term in the above expansion of \( M(\varepsilon) \) is the last one. But

\[ \sum_{k=2}^{\infty} \frac{k \varepsilon}{5r + 2k} = \varepsilon \sum_{k=2}^{\infty} \frac{1}{2k} = \frac{\varepsilon}{2r} \cdot \]

Hence \( M(\varepsilon) = 5 \sum_{k=2}^{\infty} \frac{k \varepsilon}{5r + 2k} \leq M(\varepsilon) = 5 \sum_{k=2}^{\infty} \frac{k \varepsilon}{5r + 2k} \leq M(\varepsilon') \).

Also \( \sum_{k=2}^{\infty} \frac{k \varepsilon}{5r + 2k} = \frac{10r}{\varepsilon} \left( \frac{\varepsilon}{5r + 2k} \right) \left( \frac{\varepsilon}{5r + 2k} \right) \left( \frac{\varepsilon}{5r + 2k} \right) \)

\[ \leq \frac{10r}{\varepsilon} \cdot \frac{r^2}{5r + 2r} \gamma_2(2r+1) \leq \frac{r^2}{5r + 2r} \gamma_2(2r+1) \]

\( (if \ \frac{10r}{\varepsilon} > 1) \). Thus

\[ M(\varepsilon) \leq \frac{gF \ r^F}{r!} \left( \frac{\varepsilon}{2r+1} \right)^F M(\varepsilon_\infty), \text{ for} \]

every positive integer \( r \), every positive \( \varepsilon < 10 \).
It has already been seen that, operating formally,

\[ F(s) = \sum_{n} \psi(n) e^{-\lambda n s} = \sum_{n} a_n f_n(s). \]

Lemma 3.2 indicates a region in which all \( f_n(s) \) are holomorphic: the complement of \( S_\lambda + (2S)_\lambda + J(2S)_\lambda + (3S)_\lambda + J(3S)_\lambda + \ldots \). It remains to find conditions on \( f(s) \) and \( \psi(s) \) in order that \( \sum a_n f_n(s) \) may converge uniformly, thus representing a holomorphic function, and that the double series \( \sum a_n e^{-\lambda n s} \) may converge absolutely, thus justifying the formal operations above.

**Lemma 3.3:** Let \( f_k(s) = \sum_{n} e^{\lambda n s} \) \((k = 1, 2, \ldots)\).

Suppose

1. \( f(s) \) is "\( \mathfrak{M} \)" in \( R(s) > \sigma \), and \( \nabla f \leq 0 \), so that \( \mathfrak{M} (\sigma + (k-1)\sigma f) \leq \mathfrak{M} \sigma f_k \) is finite;

2. \( f_k(s) = \sum e_k s^k \) is entire, with \( e_k > 0 \) for each positive \( e \), then \( \sum e_k f_k(s) \) converges uniformly and absolutely to a holomorphic function in every closed region \( R \) interior to \( \mathcal{F} \), that part of the complement in \( R(s) > \sigma \) of \( \mathcal{F} \) which is connected with the half-plane \( R(s) > 0 \); \( \mathcal{F} \) is the closure of the set \( \mathcal{F} = S_\lambda + (2S)_\lambda + (3S)_\lambda + \ldots \).
Note that since, if $H(s) = H(f, q)$, then $a_h < a_q + a_q$, and each of the functions $f_k(s)$ therefore has $a_{f_k} < 0$; for $f_2(s) = H(f, f)$, $a_{f_2} < a_f + a_f = 0$, etc. Hence each of the functions $f_k(s)$ is holomorphic in $R(s) > 0$. By lemma 3.2, each $f_k(s)$ is holomorphic in $S$, defined as the complement in $R(s) > a_{f_k}$ of $R^k + J_k$, where $R^k = (R^k)_{s_k}$. $J_k$ is that part of the complement in $R(s) > a_{f_k}$ of $R^i$ not connected with the half-plane $R(s) > 0$. The set $D$ is the set $R^1 + R^2 + R^3 + \ldots$. It is not then difficult to see that every $f_k(s)$ is holomorphic in $D$. For first, $a_f > a_{f_k}$ for each $k$, also, if $x \in \overline{D}$, then there is a curve $L$ joining $x$ to $R(s) > 0$ which contains no points of $D$, hence no point of any $R_k$. Also, $L$ contains no point of any $J_k$; for if $L$ has a point $y$ of some $J_k$, then every curve in $R(s) > a_{f_k}$ (hence certainly every curve in $R(s) > a_f$) which joins $y$ to $R(s) > 0$, contains a point of $R_k$; in particular $L$ contains a point of $R_k$, a contradiction. Thus $D$ contains no point of any $R^k + J_k$, hence is contained in $S$ for each $k$, and is a region of holomorphism for each $f_k(s)$.

Since $D$ is closed, $D$ is open. Let $R$ be any
closed (connected) region interior to $\Omega$. There then exists a positive number $\delta$ such that the distance between the boundaries of $\Omega$ and $R$ is greater than $\delta$. Let $x$ be any point of $R$. There is then a curve $L_x$ and a positive number $\epsilon < \delta$ such that $L_x$ joins $x$ to $R(s) > 0$, and is then central line of a channel $C_x$ of width greater than $2 \epsilon$ completely interior to $\Omega$. The number $\epsilon$ may be chosen so as to be independent of $x$ in $R$ (the same for all $x$ in $R$). For if $y$ is any other point in $R$, and if $L_{xy}$ is a curve in $R$ joining $x$ and $y$, then the curve $L_y$, the sum of the two curves $L_x$ and $L_{xy}$, joins $y$ to $R(s) > 0$, and is the central line of a channel $C_y$ of width greater than $2 \epsilon$ completely interior to $\Omega$.

The channel $C_x$ contains no point of $R^k$ for any $k$, hence $x \in F^k(\epsilon)$ for every $k$, and for all points $x$ in $R$. Hence for $x$ in $R$, $|f_k(x)| \leq M_k(\epsilon)$, for each $k$. But by the remarks made prior to the statement of this theorem,

$$M_k(\epsilon) \leq \frac{5^F F^F 2^{\frac{1}{2}(k+1)}}{r!} \left(\frac{N}{\epsilon}\right)^F H^F \left(\frac{\epsilon}{\lambda^2}\right).$$
Hence \[ \left| \sum c_k f_k(x) \right| \leq \sum |c_k| |f_k(x)| \]

\[ \leq \sum \frac{\epsilon^k 2^{\frac{k}{2}(k+1)}}{k!} |c_k| k^k m^k \left( \frac{\epsilon}{2^k} \right)^k \cdot \]

Write \[ d_k = \frac{\epsilon^k 2^{\frac{k}{2}(k+1)}}{k!} |c_k| k^k m^k \left( \frac{\epsilon}{2^k} \right)^k \cdot \]

By Stirling's formula,

\[ (1) \quad \frac{1}{n} \log n! = \log n + \frac{1}{n} \left[ \frac{1}{2} \log n + C \right] - 1 + \frac{1}{n} o(1) \]

\[ \Rightarrow \log n - 2 \text{ for } n \text{ sufficiently large}; \]

\[ -\frac{1}{n} \log n! < 2 - \log n, \]

\[ (\frac{1}{n})^{1/n} < \frac{1}{n} e^2 \text{ for } n \text{ sufficiently large, say for } n > n_0. \]

\[ \left| d_0 \right|^k = 5 \cdot 2^{\frac{k+1}{2}} \cdot k \cdot M(\frac{\epsilon}{2^k}) \left( \frac{1}{2^k} \right) \left| c_k \right|^k \]

\[ 5 \epsilon^2 \cdot 2^{\frac{k+1}{2}} \cdot M(\frac{\epsilon}{2^k}) \left| c_k \right|^k \text{ for } k > n_0. \]
But
\[ |\sum e_k f_k(x)| \leq \sum d_k \left( \frac{N}{\varepsilon} \right)^k, \] and by hypothesis

(ii) \( \lim_{k \to \infty} \left| d_k \right|^\frac{1}{k} = 0 \), hence \( \sum d_k s^k \) is entire.

But (2) holds for all \( x \) in \( R \), hence \( \sum e_k f_k(s) \) converges uniformly and absolutely in \( R \), which completes the proof of the lemma.

**Lemma 3.4:** Suppose

(i) \( \max_{s \in R} \frac{1}{\lambda_n} \left| a_n \right| < B < \infty \) for some positive \( B \), for \( n \) sufficiently large;

(ii) \( g(s) = \sum b_n e^{\lambda_n s} \) has \( A_n < \infty \)

\[ \frac{1}{\lambda_n} \log \frac{1}{\log |b_n|} < \infty \); \]

(iii) \( \psi(s) = \sum e_m s^m \) is entire, and

\( \left| e_m \right|^\frac{1}{s} = O(1) \) as \( n \to \infty \);

then \( \sum b_n e_m e^{\lambda_n s} \) converges absolutely for \( R(s) \) sufficiently large, hence

\[ \sum e_m b_n e^{\lambda_n s} \] is entire, and

\[ \sum b_n e_m e^{\lambda_n s} \] for \( R(s) \) sufficiently large.

Note that if \( b_n \equiv 1 \), (ii) becomes \( \frac{1}{\lambda_n} \log n < \infty \).
Proof: Under the hypotheses, there exists a positive number $k_1$ such that $|a_n|^{1/n} < \frac{k_1}{n}$ for all $n$. Then

\[
\sum \left( \sum |a_n|^{1/n} \right) \left( \sum |b_n e^{-\lambda_n s}| \right) \leq \sum |b_n e^{-\lambda_n s}| \sum \frac{|a_n|^{n}}{n^1} \leq \sum |b_n e^{-\lambda_n s}| |e^{k_1 |a_n|} | = H(\sigma),
\]

where

\[h(s) = \sum e^{k_1 |a_n|} |b_n| e^{-\lambda_n s} \quad (s \in \sigma + 1 t).\]

But $A_h = \Im \frac{1}{\lambda_n} \log \sum \frac{e^{k_1 |a_n|} |b_n|}{n^1}$.

Let $A_n = \max_{0 < n < \infty} |a_n|$; then

\[A_h = \Im \left[ \frac{k_1 A_n}{\lambda_n} + \frac{\log \sum \frac{|b_n|}{n^{1/2}}}{\lambda_n} \right] \leq k_1 B + A_2 < \infty; \text{ therefore the double series (1) converges for } R(s) \text{ sufficiently large},\]

which completes the proof of the lemma.
Theorem 3.6: Suppose \( f(s) = \sum a_n e^{-\lambda_n s} \) satisfies the following conditions:

(i) \( f(s) \) is \( \mathcal{M}_\infty \) in \( R(s) > \sigma > 0 \);

(ii) \( \lambda_n \leq 0 \);

(iii) \( \max_{0 \leq m < n} \frac{1}{\lambda_n} |a_m| < B < \infty \) for \( n \) sufficiently large;

(iv) \( \sum \frac{1}{\lambda_n} \log n < \infty \);

(v) \( \psi(s) = \sum c_k s^k \) is entire, with

\[
\lim_{k \to \infty} 2^k \frac{H(\frac{c_k}{2^k})}{|c_k|^{1/2}} = 0 \quad \text{for each positive } \epsilon
\]

then \( f(s) = \sum \psi(a_n) e^{-\lambda_n s} = \sum a_n e^{-\lambda_n s} \) is holomorphic in \( \mathbb{C} \), that part of the complement in \( R(s) > \sigma > 0 \), of \( \mathbb{C} \) which is connected with the half-plane \( R(s) > 0 \), \( \mathcal{C} \) is the closure of the set \( \mathcal{C} = \mathcal{C}^* + (2S_{\sigma_1}^* + (3S_{\sigma_1}^*)_n + \ldots \).

Proof: Formally, \( F(s) = \sum \psi(a_n) e^{-\lambda_n s} \),

\[
F(s) = \sum a_n \sum c_m e^{-\lambda_n s} = \sum a_m f_m(s). \quad \text{Under hypothesis (v), certainly hypothesis (iii) of lemma 3.4 is satisfied. Hence the inversion of the summations is}
\]
justified by lemma 5.4 in view of hypotheses (iii), (iv), and (v), for \( R(s) \) sufficiently large, so that for \( R(s) \) sufficiently large, \( F(s) = \sum_{n=1}^{\infty} c_n f_n(s) \). But by lemma 5.5, \( \sum_{n=1}^{\infty} c_n f_n(s) \) converges to a holomorphic function in every closed region in \( \mathbb{D} \). The region \( \mathbb{D} \) may be assumed to contain points with arbitrarily large part; thus \( F(s) \) may be continued analytically throughout \( \mathbb{D} \). This completes the proof.


