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GENERALIZATIONS OF CONVEXITY FOR
FUNCTIONS OF ONE VARIABLE

by

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A THESIS SUBMITTED
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DEDICATION

This thesis is respectfully dedicated to the memory of my very dear friend and roommate,

John Kenneth Jeanes

whose sudden, tragic death, at the young age of twenty-two, dealt the progress of science a cruel blow.
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The author would like to acknowledge his indebtedness to his thesis adviser, Dr. Guy Johnson, Jr., for his careful guidance and encouragement, to his parents, for their financial and moral support, and to his professors, for their patience and perseverance, especially Dr. Jim Douglas, Jr., whose words of encouragement, at a difficult time, changed the course of the author's life. The author would like also to express his gratitude to Mrs. C. E. Dildy for her careful typing of the thesis from the manuscript.
INTRODUCTION

The organization of this thesis is particularly simple. The thesis itself is divided into three chapters, each of which is divided into at most four sections. Principal theorems, definitions and equations are numbered consecutively. Numbers in brackets refer to the bibliography.

In Chapter I, the author presents an integral representation theorem for functions convex with respect to a pair of functions which is his principal contribution to the study of generalized convex functions. The proof of the representation theorem, though complicated in its details, makes use of the simple fact that a function convex with respect to a pair of functions may be transformed to an ordinary convex function. For an ordinary convex function, the representation theorem amounts to little more than stating that such a function is the integral of its derivative. In the proof, we make use of well-known properties of the Riemann-Stieltjes integral, an excellent summary of which may be found in the first chapter of [25].

There are several integral representation theorems in the literature, both for ordinary convex functions and for functions convex with respect to a pair of functions. Those representation theorems for functions convex with respect to a pair of functions, however, involve the hypotheses which are more restrictive than those imposed by the author.

In 1916, Blaschke and Pick [7] published an integral representation theorem for ordinary convex functions, continuous on the closed interval [0,1]. This paper also contains (pp. 282-285) a good summary of the elementary properties of the Riemann-Stieltjes integral. Blaschke and Pick also remarked (pp. 293-294) that their results also hold for functions convex with respect to a basis \( \phi_1, \phi_2 \) for

\[
(*) \quad L(y) = \frac{d}{dx} [p(x) y'] + q(x) y,
\]
provided (*) has a solution which is strictly positive throughout the closed interval [0,1]. They seem to have been the first to give a determinantal characterization for functions convex with respect to a pair of functions.

In 1917, Winternitz [26] extended the results of Blaschke and Pick by presenting an integral representation theorem for functions convex with respect to a pair of functions $\phi_1$ and $\phi_2$ which he assumed to be positive on the open interval (0,1) and of "bounded rotation", i.e., he assumed that $\phi_1$ and $\phi_2$ were integrals of functions of bounded variation. He also essentially assumed that the Wronskian of the functions $\phi_1$ and $\phi_2$ was strictly positive and made other assumptions regarding the behavior of the functions $\phi$ at the boundary of the interval.

In 1932, Valiron [24] defined the notion of functions convex with respect to a pair of functions in terms of the determinantal characterization presented earlier by Blaschke and Pick, but made no reference to their work. Valiron concerned himself with the regularity properties of functions convex with respect to a pair of functions. In §4 of Chapter I, the author points out the relation between his own work and that of Valiron.

Bonsall [9] and Reid [22] have considered functions convex with respect to an operator

\[(**)
L(y) = y'' + p_1(x) y' + p_2(x) y,
\]

where it is assumed that given any two values $y_1, y_2$ and $x_1 < x_2$, there is a unique solution $\phi$ of $L(\phi) = 0$ such that $\phi(x_j) = y_j$, $j = 1,2$. A function $y(x)$ is said to be "sub-$L$" if

\[y(x) \leq \phi(x)\]

for $x_1 \leq x \leq x_2$, $\phi(x_j) = y(x_j)$, $j = 1,2$. A $C^2$ function $y$ is sub-$L$ if and only if $L(y) \geq 0$. Reid shows that a sub-$L$ function may be transformed to an ordinary convex function. If one considers a basis $\phi_1, \phi_2$ for (**), with positive Wronskian, it is
easily seen that a function is sub-L if and only if it is convex with respect to the pair of functions \( \phi_1 \) and \( \phi_2 \). Bonsall and Reid made no reference to the earlier work of Blaschke and Pick. It is comparatively simple to show that the author's integral representation theorem holds for sub-L functions.

Chapter II is included only to motivate the results presented in Chapter III. These chapters deal with functions convex with respect to a system of functions \( \phi_1, \ldots, \phi_n \), \( n \geq 2 \), which were defined by Popoviciu [19], [20], [21]. He has concerned himself primarily with the regularity properties of such functions. The author seems to have discovered an interesting identity for the derivative of a quotient of two determinants which has enabled him to "turn-around" a theorem proved earlier by Popoviciu (Theorem 8 of Chapter III). However, this is done only for a completely regular system of functions \( \phi \). Thus, to the best of the author's knowledge, Theorems 9, 10 and 11 are original. Corollary 3 to Theorem 9 gives us an integral representation theorem if all the functions concerned are sufficiently regular.

Theorem 12, which gives a very useful sufficient condition that a system of functions \( \phi \) be completely regular, seems also to be new. Using Theorem 12, the author has pointed out, in \S 3 of Chapter III, a connection between the polyharmonic functions of Almansi [1] and Nicolesco [16] and functions convex with respect to a completely regular system of functions. This connection does not seem to have been made previously and was suggested to the author by his thesis adviser, Dr. Guy Johnson, Jr. This fact could be used to simplify many of the arguments employed in studying polyharmonic functions. For example, Almansi ([1], pp. 20-21) goes through quite a bit of work to show that the determinant

\[
\begin{vmatrix}
1 & \log r & r^2 & r^2 \log r \\
0 & \frac{1}{r} & 2r & r + 2r \log r \\
1 & \log r' & r^{'2} & r'^{2} \log r' \\
0 & \frac{1}{r'} & 2r' & r' + 2r' \log r'
\end{vmatrix} \neq 0
\]
for $r \neq r'$. But this is an immediate consequence of the fact that the system $\phi_1 = 1$, $\phi_2 = \log r$, $\phi_3 = r^2$, $\phi_4 = r^2 \log r$
is completely regular.

Boas and Widder [8] have re-proved a theorem by Popoviciu concerning the regularity properties of functions convex with respect to the system of functions $1, x, \ldots, x^k$, $k \geq 1$. However, their treatment of the problem is somewhat more readable than that of Popoviciu. Using the properties of the Bernstein polynomials and the results of Boas and Widder, Temple [23] has obtained an integral representation theorem for such functions. (The same result was obtained a good deal earlier by P. Montel, *Sur les fonctions convexes et les fonctions sousharmoniques*, J. de Math. vol. 7 (1928) pp. 29-60.) It is probable that a similar theorem holds for functions convex with respect to a completely regular system of $n$ functions, but the author has not been able to show this except, of course, for $n = 2$.

Our bibliography, though fairly extensive, is by no means complete. The interested reader can obtain a fairly complete bibliography by combining the bibliographies in the expository papers [5] and [21] by Beckenbach and Peixoto, respectively. A summary of more recent results may be found in a paper by Green [11]. We close the introduction by referring the reader to the beautiful and elegant paper by J.L.W.V. Jensen [14], in which he introduced the notion of convex functions.
CHAPTER I

An Integral Representation Theorem for Functions
Convex with Respect to a Pair of Functions

§1. Preliminaries and Definitions.

In his master's thesis [2]*, the author defined the notion of functions convex with respect to a pair of functions. For the sake of completeness, the principal definitions will be repeated here. The functions which appear in what follows are all real-valued functions of the real variable $x$. All values of $x$ which appear will be understood to be in an open interval $(a,b)$ which may be finite, semi-infinite or infinite. Accordingly, any statements made regarding the continuity and differentiability of the functions considered are made relative to $(a,b)$. In particular, the functions $\varphi_1$ and $\varphi_2$, with respect to which convexity is defined, are assumed to be continuous on $(a,b)$.

**Definition 1.**

$$F(x_1, x_2) = \begin{vmatrix} \varphi_1(x_1) & \varphi_2(x_1) \\ \varphi_1(x_2) & \varphi_2(x_2) \end{vmatrix}.$$ 

**Definition 2.** The functions $\varphi_1$ and $\varphi_2$ are said to satisfy the unique solvability condition U if and only if $F(x_1, x_2) > 0$ for arbitrary $x_1 < x_2$.

From Definition 2, it follows immediately that if condition U holds, then $\varphi_1$ and $\varphi_2$ have no common zero in $(a,b)$ and $\varphi_j$, $j = 1, 2$, has at most one zero in $(a,b)$.

* Numbers in brackets refer to the bibliography at the end of this paper.
Definition 3. Let condition U hold. If \( y(x) \) is a real-valued function defined on \((a,b)\), then \( y(x) \) is said to be convex with respect to \( \phi_1 \) and \( \phi_2 \) on \((a,b)\), i.e., \( (\phi_1, \phi_2, y) \), if and only if

\[
D(\phi_1, \phi_2, y; x_1, x_2, x_3) \geq 0
\]

for arbitrary \( x_1 < x_2 < x_3 \), where \( D(\phi_1, \phi_2, y; x_1, x_2, x_3) \) is the three by three determinant whose \( i \)'th row is

\[
\phi_1(x_i) \quad \phi_2(x_i) \quad y(x_i)
\]

\( i = 1, 2, 3 \).

Fig. 1.

Take \( x_1 < x_2 \) and let

\[
\psi(x) = A_1 \phi_1(x) + A_2 \phi_2(x)
\]

where \( A_1 \) and \( A_2 \) are solutions of the linear system

(1) \[
y(x_j) = A_1 \phi_1(x_j) + A_2 \phi_2(x_j)
\]

\( j = 1, 2 \). The unique solvability condition U implies that the linear system (1) is solvable uniquely for \( A_1 \) and \( A_2 \). The geometric significance of Definition 3 is that if \( (\phi_1, \phi_2, y) \), then the graph of \( y(x) \) interlaces with the graph of \( \psi(x) \) in the manner indicated in Figure 1.
Our definition of convexity with respect to a pair of functions is easily seen to be a special case of a more general definition of convexity given by E. F. Beckenbach [3].

The author has shown in Theorem II of [2] that if $\phi_1, \phi_2$ and $y$ are $C^2$ functions, condition $U$ holds, and $W(x) > 0$, where

$$W(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix}$$

is the Wronskian of $\phi_1$ and $\phi_2$, then $(\phi_1, \phi_2, y)$ if and only if

$$y(x) = \int_{x_0}^{x} F(t, x) \frac{g(t)}{W^2(t)} \, dt + A_1 \phi_1(x) + A_2 \phi_2(x),$$

where $A_1$ and $A_2$ are constants depending on $x_0$, and

$$g(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1'(x) & \phi_2'(x) & y'(x) \\ \phi_1''(x) & \phi_2''(x) & y''(x) \end{vmatrix}$$

is a non-negative continuous function.

We will use the notations $G(x) \uparrow$ to indicate that $G(x)$ is monotone increasing and $G(x) \uparrow \uparrow$ to indicate that $G(x)$ is monotone strictly increasing. If we write

$$G(x) = \int_{x_0}^{x} \frac{g(t)}{W^2(t)} \, dt$$

then $G(x) \uparrow$ and $G(x) \in C'$. Equation (2) may then be written in the form

$$(3) \quad y(x) = \int_{x_0}^{x} F(t, x) \, dG(t) + A_1 \phi_1(x) + A_2 \phi_2(x).$$
The purpose of the next section will be to obtain a representation of the form (3) for the convex function \( y(x) \) assuming only that \((\phi_1, \phi_2, y)\). This, of course, implies that the unique solvability condition U holds.

\[ \text{§2. The Integral Representation Theorem.} \]

**Theorem 1.**

1° If \((\phi_1, \phi_2, y)\) and \( \alpha \) is a non-negative constant, then \((\phi_1, \phi_2, \alpha y)\).

2° If \((\phi_1, \phi_2, y_1)\) and \((\phi_1, \phi_2, y_2)\), then \((\phi_1, \phi_2, y_1 + y_2)\).

3° If \(\{y_n\}_{n=1}^{\infty}\) is a pointwise convergent sequence with limit function \(y(x)\) and \((\phi_1, \phi_2, y_n)\), \(n = 1, 2, \ldots\), then \((\phi_1, \phi_2, y)\).

4° If \((\phi_1, \phi_2, y)\), then \(y\) is continuous.

5° If \(p(x)\) is a positive continuous function, then \((\phi_1, \phi_2, y)\) if and only if \(\left(\frac{\phi_1}{p}, \frac{\phi_2}{p}, \frac{y}{p}\right)\).

6° If \(p(x) = \sqrt{\phi_1^2(x) + \phi_2^2(x)}\), then \(p(x)\) is a positive continuous function.

**Proof.** The proofs of 1°, 2° and 3° are an immediate consequence of Definition 3. The statement 4° is Beckenbach's principal conclusion for his more general "sub-F" functions. However, a simple direct proof of 4° is also possible for the special case with which we are concerned. The proof of 5° is also immediate from Definition 3. We need only note that since \(p(x)\) is a positive continuous function, the functions \(\frac{\phi_1(x)}{p(x)}\) and \(\frac{\phi_2(x)}{p(x)}\) are a pair of continuous functions which satisfy the unique solvability condition U. Statement 6° holds since \(\phi_1\) and \(\phi_2\) are continuous and have no common zero.
We should note here that the openness of the interval \((a, b)\) is important in the proof of statement 4°. In fact, if the interval were closed or even half-open, statement 4° would be false. For example, the function

\[
y(x) = \begin{cases} 
1, & x = 0, \\
0, & x > 0,
\end{cases}
\]

is an ordinary convex function on the semi-infinite half-open interval \([0, \infty)\), but it is certainly not continuous on this interval.

**Theorem 2.** If \(G(x) \uparrow\), \(A_1\) and \(A_2\) are arbitrary constants, and

\[
Y(x) = \int_{x_0}^{x} F(t, x) \, dJ(t) + A_1 \phi_1(x) + A_2 \phi_2(x),
\]

then \((\phi_1, \phi_2, Y)\).

**Proof.** We must show that for the function \(Y(x)\) defined in the statement of the theorem,

\[
D(\phi_1, \phi_2, Y; x_1, x_2, x_3) \geq 0
\]

for arbitrary \(x_1 < x_2 < x_3\).

If we note that

\[
Y(x) = \int_{x_1}^{x} F(t, x) \, dG(t) + \int_{x_0}^{x_1} F(t, x) \, dG(t)
\]

\[
+ A_1 \phi_1(x) + A_2 \phi_2(x)
\]

\[
= \int_{x_1}^{x} F(t, x) \, dG(t) + A_1^* \phi_1(x) + A_2^* \phi_2(x),
\]

where
\[ A_1^* = A_1 - \int_{x_0}^{x_1} \phi_2(t) \, dG(t) , \]
\[ A_2^* = A_2 + \int_{x_0}^{x_1} \phi_1(t) \, dG(t) , \]

we see that

\[
D(\phi_1, \phi_2^T; x_1, x_2, x_3) = \begin{vmatrix}
\phi_1(x_1) & \phi_2(x_1) & 0 \\
\phi_1(x_2) & \phi_2(x_2) & \int_{x_1}^{x_2} F(t, x_2) \, dG(t) \\
\phi_1(x_3) & \phi_2(x_3) & \int_{x_1}^{x_3} F(t, x_3) \, dG(t)
\end{vmatrix}
\]

\[ = \int_{x_1}^{x_3} F(x_1, x_2) F(t, x_3) \, dG(t) - \int_{x_1}^{x_2} F(x_1, x_3) F(t, x_2) \, dG(t) \]

\[ = \int_{x_1}^{x_2} \left[ F(x_1, x_2) F(t, x_3) - F(x_1, x_3) F(t, x_2) \right] \, dG(t) \]

\[ + \int_{x_2}^{x_3} F(x_1, x_2) F(t, x_3) \, dG(t) . \]

If we now make use of the identity

\[
\begin{vmatrix}
\phi_1(x_1) & \phi_2(x_1) & F(t, x_1) \\
\phi_1(x_2) & \phi_2(x_2) & F(t, x_2) \\
\phi_1(x_3) & \phi_2(x_3) & F(t, x_3)
\end{vmatrix} = 0 ,
\]
which holds for arbitrary $x_1, x_2, x_3$ and $t$, we obtain

$$D(\phi_1, \phi_2, y; x_1, x_2, x_3) = \int_{x_1}^{x_2} F(x_2, x_2) F(x_2, t) \, dt$$

$$+ \int_{x_1}^{x_3} F(x_1, x_2) F(t, t) \, dt$$

But by the unique solvability condition and the monoticity of $G(x)$, both these integrals are non-negative for $x_1 < x_2 < x_3$.

Hence

$$D(\phi_1, \phi_2, y; x_1, x_2, x_3) \geq 0$$

for arbitrary $x_1 < x_2 < x_3$, and the proof of the theorem is complete.

**Theorem 3.** If $(\phi_1, \phi_2, y)$ and $x_0 \in (x, t)$, then there exist constants $A_1, A_2$ (depending on $x_0$), and a monotone increasing function $G(x)$ such that

$$y(x) = \int_{x_0}^{x} F(t, t) \, dt + A_1 \phi_1(x) + A_2 \phi_2(x).$$

The function $G(x)$ is unique to within an additive constant except possibly on a countable set of points. Moreover, we may assume that $G(x)$ is continuous on the right (or left) in which case $y(x)$ is unique to within an additive constant.

**Lemma 1°.** If $\phi_1 = 1, \phi_2 = x$, then Theorem 3 holds.

**Proof.** Since $\phi_1 = 1, \phi_2 = x$, a function $y(x)$ such that $(\phi_1, \phi_2, y)$ is an ordinary convex function. The proof of Lemma 1° depends on certain well-known properties of ordinary convex functions such as may be found in [12]. For the sake of completeness, we will list those properties of ordinary convex functions which we will use to prove Lemma 1°. Consider the difference quotient
\[ \frac{y(x_2) - y(x_1)}{x_2 - x_1} \]

for \( x_2 \neq x_1 \). Since \((1, x, y(x))\), the expression (4) is a monotone increasing function of \( x_2 \) for \( x_2 > x_1 \) and is a monotone increasing function of \( x_1 \) for \( x_1 < x_2 \). From this property of expression (4), it follows that the right- and left-hand derivatives

\[ y'_n(x) = \lim_{x' \downarrow x} \frac{y(x') - y(x)}{x' - x}, \quad y'_l(x) = \lim_{x' \uparrow x} \frac{y(x') - y(x)}{x' - x}, \]

exist and are monotone increasing functions of \( x \). Moreover, \( y'_n(x) \) is continuous on the right, \( y'_l(x) \) is continuous on the left, and \( y'_l(x) \leq y'_n(x) \). Also, the functions \( y'_n(x) \) and \( y'_l(x) \) have the same points of discontinuity.

Let \( S \) denote the set of points of discontinuity of \( y'_n(x) \) (or \( y'_l(x) \)). Since \( y'_n(x) \uparrow \), \( S \) is a countable subset of the open interval \((a, b)\). Moreover,

\[ y'_l(x) < y'_n(x) \]

if and only if \( x \in S \). Hence the derivative \( y'(x) \) exists if and only if \( x \notin S \).

Since \( y'_n(x) \uparrow \) on \((a, b)\), \( y'_n(x) \) is Riemann integrable on any closed subinterval of \((a, b)\). Moreover, for \( x_0, x \in (a, b) \)

\[ y(x) - y(x_0) = \int_{x_0}^{x} y'_n(t) \, dt. \]

The expression (7) also holds if \( y'_n(t) \) is replaced by \( y'_l(t) \).

Suppose now that \( G(x) \uparrow \) and that

\[ y(x) - y(x_0) = \int_{x_0}^{x} G(t) \, dt. \]
It follows from (8) that
\begin{equation}
G(x) \leq \frac{y(x_2) - y(x_1)}{x_2 - x_1} \leq G(x_2).
\end{equation}

From the properties of expression (4) and from (5) and (9), it then follows that
\[ y_l(x) \leq G(x) \leq y_u(x) \]
for all \( x \in (a,b) \). Hence for \( x \notin S \),
\begin{equation}
G(x) = y'(x) = y_n(x),
\end{equation}
so that \( G(x) \) may differ from \( y_n(x) \) only on the countable set \( S \). If we further assume that \( G(x) \) is continuous on the right, it will follow from (9) and the properties of expression (4) that
\begin{equation}
G(x) = y_n(x)
\end{equation}
for all \( x \in (a,b) \).

Performing an integration by parts in equation (7), we obtain
\begin{equation}
y(x) = \int_{x_0}^{x} (x - t) \, d \lambda_n(t) + A_1 + A_2 \, x
\end{equation}
where
\begin{align}
\begin{cases}
\lambda_n(x) = y_n(x), \\
A_1 = y(x_0) - y_n(x_0) \, x_0, \\
A_2 = y_n(x_0).
\end{cases}
\end{align}

From (12), we see that equality will still hold if we replace \( \lambda_n(t) \) by \( \lambda_n(t) + C \), \( C \) a constant, in the integral portion of (11). Moreover if \( G(x) \uparrow \) and
\begin{equation}
y(x) = \int_{x_0}^{x} (x - t) \, d G(t) + A_1' + A_2' \, x
\end{equation}
it follows easily from (10) that $G(x)$ differs from $G_n(x)$ by a constant except possibly on the set $S$. Moreover, if we further assume that $G(x)$ is continuous on the right, it follows easily from (11) that $G(x)$ differs from $G_n(x)$ by a constant, and the proof of Lemma 1 is complete.

Let us note here that we may obtain expressions analogous to (12) and (13) by replacing $G_n(x)$ by $G_i(x) = y_i(x)$. We see then that

$$G_i(x) \leq G_n(x)$$

with equality holding if and only if $x \notin S$.

Lemma 2. If $\phi_1 = 1$ and $\phi_2 = h(x)$, then Theorem 3 holds.

Proof. Since the functions $\phi_1$ and $\phi_2$ are always assumed to be continuous, $h(x) \in C$. The unique solvability condition $U$ implies that $h(x) \uparrow \uparrow$.

Let

$$X = h(x), \quad x = h^{-1}(X), \quad X' = h(x'),$$

$$X_0 = h(x_0), \quad x_0 = h^{-1}(X_0), \quad x' = h^{-1}(X'),$$

$$Y(X) = y[h^{-1}(X)] = y(x).$$

It follows easily from these definitions that $(1, h(x), y(x))$ if and only if $(1, X, Y(X))$. But by Lemma 1,

$$Y(X) = \int_{X_0}^X (X - T) \, d \, G_n^*(T) + A_1 + A_2 \, X,$$

where

$$G_n^*(X) = \lim_{X' \downarrow X} \frac{Y(X') - Y(X)}{X' - X} = \lim_{x' \downarrow x} \frac{y(x') - y(x)}{h(x') - h(x)} \equiv \frac{i_n^f(x)}{h(x)},$$

$$A_1 = Y(X_0) - G_n^*(X_0)X_0,$$

$$A_2 = G_n^*(X_0).$$
If we define

\[ G_n(x) = G_n^*[h(x)] = \frac{d_n y(x)}{d h(x)}, \]

then \( G_n(x) \uparrow \) and \( G_n(x) \) is continuous on the right. Similarly, if

\[ G_t(x) = \frac{d_t y(x)}{d h(x)}, \]

then \( G_t(x) \uparrow \) and \( G_t(x) \) is continuous on the left. Moreover,

\[ G_t(x) \leq G_n(x), \]

with equality holding if and only if \( x \neq S \), where \( S \) is the set of points of discontinuity of \( G_n(x) \) (or \( G_t(x) \)).

If in the Riemann-Stieltjes integral (14) we make the change of variable \( T = h(t) \), then

\[ y(x) = \int_{x_0}^{x} \left[ h(x) - h(t) \right] d G_n(t) + A_1 + A_2 h(x), \]

where

\[ \begin{cases} G_n(x) = \frac{d_n y(x)}{d h(x)}, \\ A_1 = y(x_0) - G_n(x_0) h(x_0), \\ A_2 = G_n(x_0). \end{cases} \]

Since \( G_n(x) \) is unique to within an additive constant except possibly on a countable set of points, the same holds true for \( G_n(x) \), and the proof of Lemma 2 is complete.

We note again that we may obtain expressions analogous to the preceding simply by replacing \( G_n(x) \) by \( G_t(x) \).

**Lemma 3.** If \( \phi_1 = \cos x, \phi_2 = \sin x, \alpha < x < \beta, \beta - \alpha \leq x \), then Theorem 3 holds.

**Proof:** The reason for restricting the interval \((\alpha, \beta)\) to length less than or equal to \( x \) is that for this particular choice of \( \phi_1 \) and \( \phi_2 \),
\[ P(x_1, x_2) = \sin(x_2 - x_1) > 0 \]

for \( x_2 > x_1 \), provided \( x_2 - x_1 < \pi \). Hence for an open interval \((\alpha, \beta)\) of the type stated in the lemma, the unique solvability condition \( U \) holds. We will now reduce our present problem to the problem solved in Lemma 2.

Let \( x^* = \frac{1}{2}(\alpha + \beta) \). Certainly

\[
D(\cos x, \sin x, y(x); x_1, x_2, x_3) = \frac{1}{2}(\alpha + \beta) \]

(16)

\[
D(\cos (x - x^*), \sin (x - x^*), y(x); x_1, x_2, x_3) \]

for arbitrary \( x_i \in (\alpha, \beta), \ i = 1, 2, 3 \). But \( \cos(x - x^*) > 0 \) for \( x \in (\alpha, \beta) \). Applying equation (16) and Theorem 1, 5°,

\[(\cos x, \sin x, y(x))\]

if and only if

\[
\left(1, \tan(x - x^*), \frac{y(x)}{\cos(x - x^*)} \right).
\]

Applying Lemma 2,

\[
\frac{y(x)}{\cos(x - x^*)} = \int_{x_0}^{x} \left[ \tan(x - x^*) - \tan(t - x^*) \right] dG^*_n(t) + A^*_1 + A^*_2 \tan(x - x^*)
\]

where

\[
G^*_n(x) = \frac{d_n \left[ \frac{y(x)}{\cos(x - x^*)} \right]}{d[\tan(x - x^*)]},
\]

(17)

\[
A^*_1 = \frac{y(x_0)}{\cos(x_0 - x^*)} - G^*_n(x_0) \tan(x_0 - x^*),
\]

\[
A^*_2 = G^*_n(x_0).
\]
Hence,

\[ y(x) = \int_{x_0}^{x} \frac{\sin(x-t)}{\cos(t-x^*)} \, d \, G_n^*(t) + A_1^* \cos(x-x^*) + A_2^* \sin(x-x^*) \]

\[ = \int_{x_0}^{x} \frac{\sin(x-t)}{\cos(t-x^*)} \, d \, G_n^*(t) + A_1 \cos x + A_2 \sin x \]

where

\[ \begin{align*}
& \begin{cases} 
A_1 = A_1^* \cos x^* - A_2^* \sin x^*, \\
A_2 = A_2^* \sin x^* + A_1^* \cos x^*.
\end{cases} \\
& \text{(13)}
\end{align*} \]

If we define

\[ G_n(x) = \int_{x_0}^{x} \frac{1}{\cos(t-x^*)} \, d \, G_n^*(t), \]

then \( G_n(x) \uparrow \) and continuous on the right. Moreover,

\[ y(x) = \int_{x_0}^{x} \sin(x-t) \, d \, G_n(t) + A_1 \cos x + A_2 \sin x \]

where \( A_1 \) and \( A_2 \) are given by (17) and (18). Since \( G_n^*(x) \) is unique to within an additive constant, except possibly on a countable set, the same holds true for \( G_n(x) \) and the proof of Lemma 3 is complete.

We again remark that expressions analogous to (17), (18) and (19) may be obtained by replacing \( G_n^*(x) \) by

\[ G^*_n(x) = \frac{d \left[ \frac{y(x)}{\cos(x-x^*)} \right]}{d[\tan(x-x^*)]} \]

and defining

\[ G_n(x) = \int_{x_0}^{x} \frac{1}{\cos(t-x^*)} \, d \, G_n^*(t). \]
We recall from the proof of Lemma 2 that

\[(21)\quad G^*_\xi(x) \leq G^*_\eta(x)\]

with equality holding if and only if \(x \notin S\), where \(S\) is the set of points of discontinuity of \(G^*_\eta(x)\) (or \(G^*_\xi(x)\)). We remark that an inequality of the form \((21)\) does not necessarily hold for \(G^*_\eta(x)\) and \(G^*_\xi(x)\). However, if we take \(x_0\) to be a point of continuity of \(G^*_\eta(x)\) (or \(G^*_\xi(x)\)), then

\[(22)\quad G^*_\xi(x) \leq G^*_\eta(x)\]

with equality holding if and only if \(x \notin S\). Note that the functions \(G^*_\xi(x), G^*_\eta(x), G^*_\xi(x)\) and \(G^*_\eta(x)\) all have the same set of points of discontinuity \(S\).

We will now obtain more convenient expressions for the functions \(G^*_\eta(x), G^*_\xi(x)\) and for the constants \(A_1\) and \(A_2\) given by \((17)\) and \((18)\).

The existence of the right-hand derivative

\[
\frac{d}{dx} \left[ \frac{v(x)}{\cos(x - x^*)} \right] = \frac{d}{\tan(x - x^*)} \left[ \frac{v(x)}{\cos(x - x^*)} \right]
\]

implies the existence of the right-hand derivative \(y^*_\eta(x)\). Moreover,

\[
G^*_\eta(x) = \frac{d}{\tan(x - x^*)} \left[ \frac{v(x)}{\cos(x - x^*)} \right]
\]

so that

\[(23)\quad G^*_\eta(x) = \cos(x - x^*) y^*_\eta(x) + \sin(x - x^*) y(x) .\]

Since \(G^*_\eta(x)\) is Riemann integrable on any closed subinterval of \((\alpha, \beta)\). Moreover,
\( y(x) - y(x_0) = \int_{x_0}^{x} y_\infty(t) \, dt. \)

But (24) informs us that \( y(x) \) is of bounded variation on any closed subinterval of \((\alpha, \beta)\). Applying this fact in (23), we see that \( y_\infty(x) \) is also of bounded variation on any closed subinterval of \((\alpha, \beta)\).

Similarly, \( y_\infty(x) \) exists and

\[ G_\infty^*(x) = \cos(x - x^*) \, y_\infty(x) + \sin(x - x^*) \, y(x). \]

From (21), (23) and (25), we obtain

\[ y_\infty(x) \leq y_\infty^*(x) \]

with equality holding if and only if \( x \notin S \). Also, \( y_\infty(x) \) is continuous on the left, \( y_\infty^*(x) \) is continuous on the right, and both are of bounded variation on any closed subinterval of \((\alpha, \beta)\).

Hence, for any function \( y(x) \) such that \((\cos x, \sin x, y(x))\), we have obtained the result that \( y_\infty(x) \) and \( y_\infty^*(x) \) exist, \( y_\infty(x) \leq y_\infty^*(x) \), and that \( y'(x) \) exists if and only if \( x \notin S \). A function convex with respect to \( \cos x \) and \( \sin x \) is therefore differentiable except possibly on a countable set of points. This is in conformity with a result previously obtained by Polya in [18].

Performing an integration by parts in the expression

\[ G_\infty^*(x) = \int_{x_0}^{x} \frac{1}{\cos(t - x^*)} \, d J_\infty^*(t), \]

we see that

\[ G_\infty(x) = - \frac{G_\infty^*(x_0)}{\cos(x_0 - x^*)} + \frac{G_\infty^*(x)}{\cos(x - x^*)} \]

\[ - \int_{x_0}^{x} G_\infty^*(t) \sec(t - x^*) \tan(t - x^*) \, dt. \]
Making use of (23),

\[ G_n(x) = -y_n(x_o) - \tan(x_o - x^*) \ y(x_o) + y_n(x) + \tan(x - x^*) \ y(x) \]

\[ - \int_{x_o}^{x} y_n(t) \tan(t - x^*) \ dt - \int_{x_o}^{x} y(t) \tan^2(t - x^*) \ dt \]

\[ = -y_n(x_o) - \tan(x_o - x^*) \ y(x_o) + y_n(x) - \tan(x - x^*) \ y(x) \]

\[ \int_{x_o}^{x} y_n(t) \tan(t - x^*) \ dt + \int_{x_o}^{x} y(t) \ dt \]

\[ - \int_{x_o}^{x} y(t) \sec^2(t - x^*) \ dt \]

\[ = -y_n(x_o) + y_n(x) + \int_{x_o}^{x} y(t) \ dt \]

\[ - \int_{x_o}^{x} y_n(t) \tan(t - x^*) \ dt + \int_{x_o}^{x} \tan(t - x^*) \ dy(t) \]

But since (24) holds,

\[ \int_{x_o}^{x} y_n(t) \tan(t - x^*) \ dt = \int_{x_o}^{x} \tan(t - x^*) \ dy(t) \]

so that

\[ G_n(x) = -y_n(x_o) + y_n(x) + \int_{x_o}^{x} y(t) \ dt. \]
Since \( G_n(x) \) may be altered by an additive constant without changing (19), we may take

\[
(27) \quad G_n(x) = y_n(x) + \int_{x_0}^{x} y(t) \, dt.
\]

From (17), (18) and (23), it follows that

\[
(28) \quad \begin{cases} 
A_1 = y(x_0) \cos x_0 - y_n(x_0) \sin x_0 \\
A_2 = y_n(x_0) \cos x_0 + y(x_0) \sin x_0.
\end{cases}
\]

A rather interesting fact which follows immediately from (27) is that the function

\[
y_n(x) + \int_{x_0}^{x} y(t) \, dt
\]

is a monotone increasing function of \( x \).

Similarly, we may take

\[
(29) \quad G_\varphi(x) = y_\varphi(x) + \int_{x_0}^{x} y(t) \, dt.
\]

Let us note that if \( G_n(x) \) and \( G_\varphi(x) \) are given by (27) and (29) then

\[
(30) \quad G_\varphi(x) \leq G_n(x)
\]

with equality holding if and only if \( x \notin S \).

We are now in a position to prove Theorem 3. From Theorem 1, 5°, 6°,

\[
(\varphi_1, \varphi_2, y)
\]

if and only if

\[
\left( \frac{\varphi_1}{p}, \frac{\varphi_2}{p}, \frac{y}{p} \right),
\]

where \( p(x) = \sqrt{\varphi_1^2(x) + \varphi_2^2(x)} \).
If we let $\tau(x)$ be defined by the equations

\[
\begin{align*}
\frac{\phi_1(x)}{p(x)} &= \cos \tau(x), \\
\frac{\phi_2(x)}{p(x)} &= \sin \tau(x),
\end{align*}
\]

(30)

then $\tau(x)$ is defined mod $2\pi$ and $\tau(x) \in C$. Moreover, since

\[
0 < \begin{vmatrix}
\frac{\phi_1(x_1)}{p(x_1)} & \frac{\phi_2(x_1)}{p(x_1)} \\
\frac{\phi_1(x_2)}{p(x_2)} & \frac{\phi_2(x_2)}{p(x_2)}
\end{vmatrix} = \sin[\tau(x_2) - \tau(x_1)]
\]

for $x_1 < x_2$, we may assume that the range of $\tau(x)$ is an open interval $(a, b)$ of length less than or equal to $\pi$, and that $\tau(x) \uparrow$. It follows then that if

(31)

\[
\eta(x) = \frac{\psi(x)}{p(x)},
\]

then

\[
(\phi_1, \phi_2, \eta)
\]

if and only if

\[
(\cos \tau(x), \sin \tau(x), \eta(x)).
\]

Letting

\[
X = \tau(x), \quad x = \tau^{-1}(X), \\
X_0 = \tau(x_0), \quad x_0 = \tau^{-1}(X_0),
\]

\[
Y(X) = \eta[\tau^{-1}(X)] = \eta(x),
\]

we have, by Lemma 3,

(32)

\[
Y(X) = \int_{X_0}^{X} \sin(X - T) \, dG_T^* + A_1 \cos X + A_2 \sin X,
\]
where

\[
\begin{align*}
G_n^*(x) &= Y_n(x) + \int_{X_0}^{x} Y(t) \, dt, \\
A_1 &= Y(X_0) \cos(X_0) - Y_n(X_0) \sin X_0, \\
A_2 &= Y_n(X_0) \cos X_0 + Y(X_0) \sin X_0,
\end{align*}
\]

by (19), (27) and (28).

Transforming back to the original variables,

\[
y(x) = \int_{X_0}^{x} F(t,x) \frac{1}{p(t)} \, d \, G_n^{**}(t) + A_1 \phi_1(x) + A_2 \phi_2(x),
\]

where

\[
\begin{align*}
G_n^{**}(x) &= G_n^*[\tau(x)] = \frac{d_n[ \eta(x)]}{d \tau(x)} + \int_{X_0}^{x} \eta(t) \, d \tau(t), \\
A_1 &= \frac{1}{p(x_0)} \left[ \eta(x_0) \phi_1(x_0) - \int_{X_0}^{x_0} \phi_1(x_0) \, d \eta(x_0) \right], \\
A_2 &= \frac{1}{p(x_0)} \left[ \int_{X_0}^{x_0} \phi_1(x_0) + \eta(x_0) \phi_2(x_0) \right],
\end{align*}
\]

and

\[
\int_{X_0}^{x} \phi_1(x) = \frac{d_n[ \eta(x)]}{d \tau(x)} = \frac{d_n \left[ \frac{\eta(x)}{p(x)} \right]}{d \tau(x)}
\]

If we now let

\[
G_n(x) = \int_{X_0}^{x} \frac{1}{p(t)} \, d \, G_n^{**}(t),
\]

then \(G_n(x)\) is increasing and continuous on the right. Moreover,

\[
y(x) = \int_{X_0}^{x} F(t,x) \, d \, G_n(t) + A_1 \phi_1(x) + A_2 \phi_2(x).
\]
It is evident from the transformation employed that $G_\kappa(x)$ is unique to within an additive constant except possibly on its countable set of points of discontinuity $S$, and the proof of Theorem 3 is complete.

If we let

\begin{equation}
G^{**}_\kappa(x) = \frac{d}{d\tau(x)} \left[ \eta(x) \right] + \int_{x_0}^{x} \eta(t) \, d\tau(t) ,
\end{equation}

\begin{equation}
\mathcal{P}_\kappa(x) = \frac{d}{d\tau(x)} \left[ \eta(x) \right] ,
\end{equation}

\begin{equation}
\mathcal{S}_\kappa(x) = \int_{x_0}^{x} \frac{1}{p(t)} \, dG^{**}_\kappa(t) ,
\end{equation}

then an expression analogous to (38) holds if we replace $G_\kappa(x)$ and $\mathcal{P}_\kappa(x)$ by $G_\kappa(x)$ and $\mathcal{P}_\kappa(x)$ respectively in equations (35) and (37).

We will call the quantities $\mathcal{P}_\kappa(x)$ and $\mathcal{S}_\kappa(x)$ the normalized right- and left-hand $\tau$-derivatives of $y(x)$. From the proof of Lemma 3 and equations (35), (36), (39) and (40), $\mathcal{P}_\kappa(x)$ is continuous on the right, $\mathcal{S}_\kappa(x)$ is continuous on the left, $\mathcal{P}_\kappa(x)$ and $\mathcal{S}_\kappa(x)$ have the same points of discontinuity $S$, and

\begin{equation}
\mathcal{P}_\kappa(x) \leq \mathcal{P}_\kappa(x)
\end{equation}

with equality holding if and only if $x \notin S$. Hence a function $y(x)$ such that $(\phi_1, \phi_2, y)$ has a normalized $\tau$-derivative

\begin{equation}
\mathcal{P}(x) = \frac{d}{d\tau(x)} \left[ \eta(x) \right] = \lim_{x' \to x} \frac{\eta(x') - \eta(x)}{\tau(x') - \tau(x)}
\end{equation}

except possibly on a countable set of points.

Since $\eta(x)$, $\tau(x) \in C$ and $\tau(x) \uparrow \downarrow$,

\begin{equation}
\int_{x_0}^{x} \eta(t) \, d\tau(t)
\end{equation}
is a function of bounded variation on any closed subinterval of \((a, b)\). It therefore follows from (35) and (39) that the normalized right- and left-hand \(\tau\)-derivatives of \(y(x)\) are of bounded variation on any closed subinterval of \((a, b)\).

Combining the statements of Theorems 2 and 3, we now have an integral representation theorem for functions convex with respect to a pair of functions.

§3. A Simple Generalization.

Let \((\alpha, \beta)\) be an arbitrary open interval. A real-valued function \(y(x)\), defined on \((\alpha, \beta)\), is said to be locally convex with respect to \(\cos x\) and \(\sin x\) on \((\alpha, \beta)\), i.e.,

\[
\text{loc}(\cos x, \sin x, y(x)),
\]

if and only if

\[
D(\cos x, \sin x, y(x) ; x_1, x_2, x_3) \geq 0
\]

for arbitrary \(x_1 < x_2 < x_3\) such that \(x_3 - x_1 < \epsilon\).

Simple modifications of the arguments of the preceding section, particularly the proof of Lemma 3, will show that the representation theorem holds for functions locally convex with respect to \(\cos x\) and \(\sin x\). Indeed, it is immediately obvious that the representation theorem holds on any open subinterval of \((\alpha, \beta)\) of length less than or equal to \(\epsilon\). We will merely state that the representation (19), with \(G_n(x)\) given by (27), \(A_1\) and \(A_2\) given by (28), holds for a function \(y(x)\) locally convex with respect to \(\cos x\) and \(\sin x\). The uniqueness of \(G_n(x)\) to within an additive constant, except possibly on the set of points of discontinuity of \(G_n(x)\), also carries over.

**Definition 4.** Let \((a, \omega)\) be an arbitrary open interval, \(p(x), \tau(x) \in C, p(x) > 0, \tau(x) \uparrow \uparrow\) on \((a, \omega)\) and let \(\phi_1\) and \(\phi_2\) be defined by the equations

\[
\begin{align*}
\phi_1(x) &= p(x) \cos \tau(x), \\
\phi_2(x) &= p(x) \sin \tau(x).
\end{align*}
\]
A real-valued function $y(x)$ defined on $(a, b)$ is said to be locally convex with respect to $\phi_1$ and $\phi_2$ on $(a, b)$, i.e., $\text{loc}(\phi_1, \phi_2, y)$, if and only if

$$D(\phi_1, \phi_2, y; x_1, x_2, x_3) \geq 0$$

for arbitrary $x_1 < x_2 < x_3$ such that $\tau(x_3) - \tau(x_1) < x$.

There are several remarks to be made about functions locally convex with respect to a pair of functions. First of all, the geometric significance of Definition 4 is that the situation in Figure 1 holds locally. Theorems 1, 2 and 3 hold with the statement $(\phi_1, \phi_2, y)$ replaced by $\text{loc}(\phi_1, \phi_2, y)$. Theorem 3 holds since by (43) and Theorem 1, 5°, 6° a function $y(x)$ such that $\text{loc}(\phi_1, \phi_2, y)$ may be transformed into a function $\eta(x)$ such that $\text{loc}(\cos X, \sin X, \eta(X))$, as in the proof of Theorem 3. But the remarks preceding Definition 4, the representation theorem holds for $\eta(x)$. Transforming to the original variables, we therefore have a representation theorem for functions $y(x)$ such that $\text{loc}(\phi_1, \phi_2, y)$. We will list several properties of functions locally convex with respect to a pair of functions which follow by simple modifications of the arguments in the preceding section.

If $\text{loc}(\phi_1, \phi_2, y)$ then the right- and left-hand normalized $\tau$-derivatives of $y(x)$, $\mathcal{F}_n(x)$ and $\mathcal{F}_l(x)$, given by (36) and (46), exist and are continuous on the right and left respectively. Moreover, $\mathcal{F}_n(x)$ and $\mathcal{F}_l(x)$ have the same points of discontinuity $S$ and $\mathcal{F}_l(x) \leq \mathcal{F}_n(x)$ with equality holding if and only if $x \notin S$. The functions $G^*(x)$ and $G^*(x)$, given by (35) and (39), are monotone increasing on the interval $(a, b)$. The normalized $\tau$-derivative of $y(x)$, $\mathcal{F}(x)$, given by (42), exists except on the countable set $S$. The functions $G_n(x)$ and $G_l(x)$ given by (37) and (41) are monotone increasing and are continuous on the right and left respectively. The functions $\mathcal{F}_n(x)$ and $\mathcal{F}_l(x)$ are of bounded variation on any closed subinterval of $(a, b)$. Finally,

$$y(x) = \int_{x_0}^{x} \mathcal{F}(t, x) \, d G_n(t) + A_{\phi_1}(x) + A_{\phi_2}(x)$$
where
\[
\begin{align*}
A_1 &= \frac{1}{p(x_0)} \left[ \eta(x_0) \phi_1(x_0) - f_\alpha(x_0) \phi_2(x_0) \right], \\
A_2 &= \frac{1}{p(x_0)} \left[ f_\alpha(x_0) \phi_1(x_0) + \eta(x_0) \phi_2(x_0) \right].
\end{align*}
\]
(45)

Expressions similar to (44) and (45) hold if \( G_\alpha(x) \) and \( f_\alpha(x) \) are replaced by \( G_\gamma(x) \) and \( f_\gamma(x) \) respectively.

§4. Some Applications.

M. J. Valiron, in [24], has given the following definition of functions convex with respect to a pair of functions.

Let \( \phi_1 \) and \( \phi_2 \) be \( C' \) functions defined for all values of \( x \). Assume also that \( \phi_1 \) and \( \phi_2 \) are of bounded variation, that \( \phi_1 \) and \( \phi_2 \) have isolated simple zeros, that the zeros of \( \phi_1 \) and \( \phi_2 \) interlace, and that the Wronskian of \( \phi_1 \) and \( \phi_2 \) is strictly positive. A continuous real-valued function \( y(x) \) defined on an open interval \( (\alpha, \beta) \) is said to be convex with respect to \( \phi_1 \) and \( \phi_2 \) on the open interval \( (\alpha, \beta) \) in the sense of Valiron, i.e., \( V(\phi_1, \phi_2, y) \), if and only if for every \( x \in (\alpha, \beta) \) and for all \( x' \) and \( x'' \) sufficiently close to \( x \), with \( \alpha < x' < x < x'' < \beta \),

\[
D(\phi_1, \phi_2, y; x', x, x'') \geq 0.
\]

Valiron shows that the above inequality still holds if \( x' \), \( x \) and \( x'' \) are in the same "canonical interval" and \( \alpha < x' < x < x'' < \beta \).
The canonical interval with left-hand end point \( x_0 \) is the open interval \((x_0, x_1)\) where \( x_1 \) is the first zero of

\[
P(x_0, x) = \phi_1(x_0) \phi_2(x) - \phi_2(x_0) \phi_1(x)
\]

to the right of \( x_0 \). Valiron points out that in any canonical interval, a function of the form

\[
y(x) = A_1 \phi_1(x) - A_2 \phi_2(x)
\]

may be transformed to an ordinary convex function.
We will now show that a function \( y(x) \) such that \( V(\phi_1, \phi_2, y) \) is locally-convex with respect to \( \phi_1 \) and \( \phi_2 \) in the sense of Definition 4.

Let \( p(x) = \sqrt{\phi_1^2(x) + \phi_2^2(x)} \). Then \( p(x) > 0 \) since \( \phi_1 \) and \( \phi_2 \) have no common zero. Let \( \tau(x) \) be defined by the equations

\[
\begin{align*}
\frac{\phi_1(x)}{p(x)} & = \cos \tau(x), \\
\frac{\phi_2(x)}{p(x)} & = \sin \tau(x).
\end{align*}
\]

(46)

Since \( \phi_1 \) and \( \phi_2 \in C' \), \( \tau(x) \in C' \) and \( \tau(x) \) is defined mod \( 2\pi \). Moreover, it follows from (46) that

\[
W(x) = \begin{vmatrix}
\phi_1(x) & \phi_2(x) \\
\phi_1'(x) & \phi_2'(x)
\end{vmatrix} = p^2(x) \tau'(x)
\]

(47)

If we choose a particular determination of \( \tau(x_0) \), then \( \tau(x) \) is uniquely determined by

\[
\tau(x) - \tau(x_0) = \int_{x_0}^{x} \frac{W(t)}{p^2(t)} \, dt.
\]

(48)

Since \( W(x) > 0 \) and \( p(x) > 0 \), (48) implies that \( \tau(x) \uparrow \uparrow \). It therefore follows that \( V(\phi_1, \phi_2, y) \) implies \( \text{loc}(\phi_1, \phi_2, y) \). The reverse implication is certainly not true, since we have assumed much less for the functions \( \phi_1 \) and \( \phi_2 \) than did Valiron. The representation theorem for functions \( y(x) \) such that \( V(\phi_1, \phi_2, y) \) therefore holds.

Let us note that

\[
F(x_0, x) = p(x_0) \, p(x) \sin [\tau(x) - \tau(x_0)],
\]

so that for the canonical interval \( (x_0, x_1) \), \( x_1 \) is determined by the equation

\[
\tau(x_1) - \tau(x_0) = \pi, \quad x_1 > x_0.
\]
From the proof of Theorem 3, it therefore follows that a function of the form
\[ y(x) - A_1 \phi_1(x) - A_2 \phi_2(x) \]
may be transformed to an ordinary convex function in any canonical interval.

From the discussion following Definition 4, the normalized right- and left-hand \( \tau \)-derivatives of \( y(x) \) exist, are functions of bounded variation on any closed subinterval of \( (\alpha, \beta) \), and \( \tilde{f}_L(x) \leq \tilde{f}_R(x) \), with equality holding if and only if \( x \in S \), where \( S \) is the countable set of points of discontinuity of \( \tilde{f}_R(x) \) (or \( \tilde{f}_L(x) \)). But since \( \phi_1, \phi_2 \in C' \) and \( \tau'(x) > 0 \), \( y_\alpha(x) \) and \( y_\tilde{L}(x) \) exist and
\[
\begin{align*}
\tilde{f}_L(x) &= \frac{p(x) y_1(x) - y(x) p'(x)}{\tilde{w}(x)}, \\
\tilde{f}_R(x) &= \frac{p(x) y_\alpha(x) - y(x) p'(x)}{\tilde{w}(x)}.
\end{align*}
\]

Since \( \phi_1 \) and \( \phi_2 \) are integrals of continuous functions of bounded variation, it follows from (49) that \( y_1(x) \) and \( y_\alpha(x) \), continuous on the left and right respectively, are of bounded variation on any closed subinterval of \( (\alpha, \beta) \) and
\[ y_1(x) \leq y_\alpha(x) \]
with equality holding if and only if \( x \in S \). This is in conformity with Valiron's Theorem II.

The notion of functions locally convex with respect to \( \cos \Theta \) and \( \sin \Theta \), \( -\infty < \Theta < \infty \), may be used to characterize bounded, closed, plane convex sets in terms of their boundaries. Let \( T \) be a bounded, closed, plane convex set containing the origin in its interior, and let \( r = r(\Theta) \), \( 0 \leq \Theta < 2\pi \), be the equation of the boundary of \( T \) in polar coordinates. Extend the definition of \( r(\Theta) \) to all values of \( \Theta \) by periodicity, i.e., by writing \( r(\Theta + 2\pi) = r(\Theta) \). Since \( T \)
contains the origin in its interior, \( r(\theta) > 0 \). If we let
\[ y(\theta) = \frac{1}{r(\theta)} \], then \( y(\theta) \) is periodic of period \( 2\pi \), \( y(\theta) > 0 \) and
\( \operatorname{loc}(\cos \theta, \sin \theta, y(\theta)) \). Conversely, if \( y(\theta) \) is periodic of period
\( 2\pi \), \( y(\theta) > 0 \) and \( \operatorname{loc}(\cos \theta, \sin \theta, y(\theta)) \), then
\[ r = r(\theta) = \frac{1}{y(\theta)} \]
is the equation of the boundary of a bounded, closed, plane convex
set containing the origin in its interior.

Fig. 2

We will first show that \( \operatorname{loc}(\cos \theta, \sin \theta, y(\theta)) \). Take \( \theta_1 < \theta_2 \),
\( \theta_2 - \theta_1 > \pi \). (See Fig. 2). The equation of the line segment
\( r = r^*(\theta), \theta_1 \leq \theta \leq \theta_2 \), joining the points \((\theta_1, r(\theta_1))\),
\((\theta_2, r(\theta_2))\) is given by

\[
\begin{vmatrix}
\cos \theta_1 & \sin \theta_1 & \frac{1}{r(\theta_1)} \\
\cos \theta & \sin \theta & \frac{1}{r^*(\theta)} \\
\cos \theta_2 & \sin \theta_2 & \frac{1}{r(\theta_2)} \\
\end{vmatrix} = 0,
\]

\( \theta_1 \leq \theta \leq \theta_2 \), i.e.,

\[
\frac{1}{r^*(\theta)} = \frac{\sin(\theta_2 - \theta)}{\sin(\theta_2 - \theta_1)} \frac{1}{r(\theta_1)} - \frac{\sin(\theta - \theta_1)}{\sin(\theta_2 - \theta_1)} \frac{1}{r(\theta_2)}.
\]
But if \( \theta_1 < \theta < \theta_2 \),

\[
\frac{1}{r^*(\theta)} \geq \frac{1}{r(\theta)} = y(\theta),
\]

since \( T \) is convex, which is equivalent to stating that

\[
\begin{vmatrix}
\cos \theta_1 & \sin \theta_1 & y(\theta_1) \\
\cos \theta & \sin \theta & y(\theta) \\
\cos \theta_2 & \sin \theta_2 & y(\theta_2)
\end{vmatrix} \geq 0
\]

for arbitrary \( \theta_1 < \theta < \theta_2 \), \( \theta_2 - \theta_1 < \pi \). Hence, \( \text{loc}(\cos \theta, \sin \theta, y(\theta)) \).

To show that the converse statement is true, we need only note that the set

\[
T = \{ (\theta, \rho) \mid 0 \leq \rho \leq r(\theta) = \frac{1}{y(\theta)}, \ 0 \leq \theta < 2\pi \}
\]

is a bounded, closed, plane convex set containing the origin in its interior with boundary \( r = r(\theta) = \frac{1}{y(\theta)} \).

By the integral representation theorem, the boundary \( r = r(\theta) \) of a bounded, closed, plane convex set containing the origin in its interior may be written in the form

\[
\frac{1}{r(\theta)} = \int_{0}^{\theta} \sin(\theta - t) \, dG_\kappa(t) + A_1 \cos \theta + A_2 \sin \theta
\]

(51)

where

\[
\begin{align*}
G_\kappa(\theta) &= y_\kappa(\theta) + \int_{0}^{\theta} y(\tau) \, d\tau \\
A_1 &= y(0) \\
A_2 &= y_\kappa(0) \\
y(\theta) &= \frac{1}{r(\theta)}
\end{align*}
\]

(52)
In particular, if \( r(\Theta) = 1 \), in which case \( T \) is the closed unit disc,

\[
1 = \int_{0}^{\Theta} \sin(\Theta - t) \, dt + \cos \Theta.
\]

The notion of functions locally convex with respect to a pair of functions may be used to characterize \( C^2 \) functions \( y \) which satisfy second order differential inequalities of the form

\[
y'' + r(x)y' + q(x)y \geq 0.
\]

Let \( r(x), q(x) \in \mathbb{C} \) on an open interval \((a,b)\). If \( \phi_1 \) and \( \phi_2 \) are a basis for the ordinary second order differential equation

\[
(53) \quad \phi'' + r(x)\phi' + q(x)\phi = 0
\]

such that the Wronskian of \( \phi_1 \) and \( \phi_2 \) is strictly positive, then we may employ the transformation given by equations (46), (47) and (48), and thus consider functions locally convex with respect to \( \phi_1 \) and \( \phi_2 \). Slight modifications of the methods employed by the author in the proof of Theorem II of \([2]\) will show that for a \( C^2 \) function \( y \), defined on \((a,b), \text{loc}(\phi_1, \phi_2, y)\) if and only if

\[
\begin{vmatrix}
\phi_1 & \phi_2 & y \\
\phi_1' & \phi_2' & y' \\
\phi_1'' & \phi_2'' & y''
\end{vmatrix} \geq 0.
\]

But since \( \phi_1 \) and \( \phi_2 \) are solutions of (53),

\[
\begin{vmatrix}
\phi_1 & \phi_2 & y \\
\phi_1' & \phi_2' & y' \\
\phi_1'' & \phi_2'' & y''
\end{vmatrix} = W(x)[y'' + r(x)y' + q(x)y],
\]

where \( W(x) \) is the Wronskian of \( \phi_1 \) and \( \phi_2 \). Hence, \( \text{loc}(\phi_1, \phi_2, y) \) if and only if

\[
y'' + r(x)y' + q(x)y \geq 0.
\]
§1. Definitions and Notation.

The primary purpose of this chapter is to provide motivation for the results presented in Chapter III. The notion of functions convex with respect to \( n \) functions was first considered by T. Popoviciu [26]. It is therefore appropriate to name them functions convex in the sense of Popoviciu. The theorems which follow may be considered as generalizations of Theorems I and II of [26]. As usual, the functions \( \phi \), with respect to which convexity is defined, are assumed to be continuous on an open interval \((a, b)\).

**Definition 5.** The functions \( \phi_i, i = 1, 2, 3 \), are said to satisfy the unique solvability condition \( U \) if and only if

\[
D(\phi; x_1, x_2, x_3) > 0
\]

for arbitrary \( x_1 < x_2 < x_3 \), where \( D(\phi; x_1, x_2, x_3) \) is the three by three determinant whose \( i \)'th row is

\[
\phi_1(x_i) \phi_2(x_i) \phi_3(x_i) x_{i+1}
\]

\( i = 1, 2, 3 \).

**Definition 6.** Let condition \( U \) hold. If \( y(x) \) is a real-valued function defined on \((a, b)\), then \( y(x) \) is said to be convex with respect to \( \phi_1, \phi_2 \) and \( \phi_3 \) on \((a, b)\), i.e., \((\phi, y)\), if and only if

\[
D(\phi, y; x_1, x_2, x_3, x_4) \geq 0
\]

for arbitrary \( x_1 < x_2 < x_3 < x_4 \), where \( D(\phi, y; x_1, x_2, x_3, x_4) \) is the four by four determinant whose \( i \)'th row is

\[
\phi_1(x_i) \phi_2(x_i) \phi_3(x_i) y(x_i)
\]

\( i = 1, 2, 3, 4 \).
Take $x_1 < x_2 < x_3$ and let

$$
\psi(x) = A_1 \phi_1(x) + A_2 \phi_2(x) + A_3 \phi_3(x),
$$

where the $A_i$ are solutions of the linear system

$$
\psi(x_j) = A_1 \phi_1(x_j) + A_2 \phi_2(x_j) + A_3 \phi_3(x_j),
$$

$j = 1, 2, 3$. The geometric significance of Definition 6 is that if $(\phi, y)$, then the graph of $y(x)$ interlaces with the graph of $\psi(x)$ in the manner indicated in Figure 3.

We mention here that Theorem 1 holds for functions $y$ such that $(\phi, y)$, with obvious modifications.

Let $\alpha_i$ be a non-negative integer. We will employ the notations

$$
D(\alpha_1, \alpha_2, \alpha_3, \alpha_4)(\phi; x_1, x_2, x_3, x_4),
$$

$$
D(\alpha_1, \alpha_2, \alpha_3)(\phi; x_1, x_2, x_3),
$$

for the determinants whose $i$'th rows are respectively

$$
\phi_1(\alpha_i)(x_i) \phi_2(\alpha_i)(x_i) \phi_3(\alpha_i)(x_i) \psi^{(\alpha_i)}(x_i)
$$

and

$$
\phi_1^{(\alpha_i)}(x_i) \phi_2^{(\alpha_i)}(x_i) \phi_3^{(\alpha_i)}(x_i).
$$

It is easily seen that the unique solvability condition $U$ implies that

1° $D(0, 0, 1)(\phi; x_1, x_2, x_2) \geq 0$,

2° $D(0, 1, 0)(\phi; x_1, x_1, x_2) \geq 0$,

3° $D(0, 1, 2)(\phi; x_1, x_1, x_1) \geq 0$. 
for arbitrary $x_1 < x_2$, if the functions $\phi$ are sufficiently regular. We will say that the functions $\phi$ satisfy the unique solvability conditions $U_1, U_2$ and $U_3$ respectively, if and only if the inequalities $1^o, 2^o$ and $3^o$ are strict. Then, for example, the unique solvability condition $U_3$ is nothing more than the statement that the Wronskian of the functions $\phi$ is strictly positive throughout $(a, b)$.

§ 2. Some Useful Identities.

Let $\phi$ and $y$ be sufficiently regular and assume that all the unique solvability conditions defined in the preceding section hold. We claim that

\[ 1^o \quad \frac{d}{dt} \left[ \frac{D(\phi, y; x_1, x_2, x_3, t)}{D(\phi; x_1, x_2, t)} \right] = \frac{D(\phi; x_1, x_2, x_3)}{D^2(\phi; x_1, x_2, t)} \ D(0, 0, 0, 1)(\phi, y; x_1, x_2, t, t), \]

\[ 2^o \quad \frac{d}{dt} \left[ \frac{D(\phi, y; x_1, x_2, t, x_3)}{D(\phi; x_1, t, x_3)} \right] = \frac{D(\phi; x_1, x_2, x_3)}{D^2(\phi; x_1, t, x_3)} \ D(0, 0, 1, 0)(\phi, y; x_1, t, t, x_3), \]

\[ 3^o \quad \frac{d}{dt} \left[ \frac{D(\phi, y; x_1, t, x_2, x_3)}{D(\phi; t, x_2, x_3)} \right] = \frac{D(\phi; x_1, x_2, x_3)}{D^2(\phi; t, x_2, x_3)} \ D(0, 1, 0, 0)(\phi, y; t, t, x_2, x_3), \]

\[ 4^o \quad \frac{d}{dt} \left[ \frac{D(0, 1, 0, 0)(\phi, y; x_1, x_1, x_2, t)}{D(0, 1, 0)(\phi; x_1, x_1, t)} \right] = \frac{D(0, 1, 0)(\phi; x_1, x_1, x_2)}{D^2(0, 1, 0)(\phi; x_1, x_1, t)} \ D(0, 1, 0, 1)(\phi, y; x_1, x_1, t, t), \]
5° \quad \frac{d}{dt} \left[ \frac{D(0,0,0,1)(\theta,y;x_1,x_2,t,t)}{D(0,0,1)(\theta;x_1,t,t)} \right]

= \frac{D(\theta;x_1,x_2,t)}{D^2(0,0,1)(\theta;x_1,t,t)} D(0,0,1,2)(\theta,y;x_1,t,t,t),

6° \quad \frac{d}{dt} \left[ \frac{D(0,0,1,0)(\theta,y;x_1,t,t,x_2)}{D(0,1,0)(\theta;t,t,x_2)} \right]

= \frac{D(\theta;x_1,t,x_2)}{D^2(0,1,0)(\theta;t,t,x_2)} D(0,1,2,0)(\theta,y;t,t,t,x_2),

7° \quad \frac{d}{dt} \left[ \frac{D(0,0,1,2)(\theta,y;x_1,t,t,t)}{D(0,1,2)(\theta;t,t,t)} \right]

= \frac{D(\theta,0,1,t)}{D^2(0,1,2)(t,t,t)} D(0,1,2,3)(\theta,y;t,t,t,t),

for arbitrarily, mutually distinct \( x_1,x_2,x_3,t \).

The proofs of equations 1°-7° are merely exercises in differentiation. They fall naturally into the four groups 1°-3°; 4°; 5°, 6°; and 7°. We will prove 1°, 4° and 7°.

For 1°,

\[
\frac{d}{dt} \left[ \frac{D(\theta,y;x_1,x_2,x_3,t)}{D(\theta;x_1,x_2,t)} \right] = \frac{1}{D^2(\theta;x_1,x_2,t)} \begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & \phi_3(x_1) & y(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \phi_3(x_2) & y(x_2) \\ \phi_1(x_3) & \phi_2(x_3) & \phi_3(x_3) & y(x_3) \\ f_1(t) & f_2(t) & f_3(t) & Y(t) \end{vmatrix}
\]

where

\[ f_j(t) = D(\theta;x_1,x_2,t) \phi_j(t) - D_{(0,0,1)}(\theta;x_1,x_2,t) \phi_j(t) \]

\[ Y(t) = D(\theta;x_1,x_2,t) y'(t) - D_{(0,0,1)}(\theta;x_1,x_2,t) y(t), \]
\( j = 1, 2, 3 \). If we now employ the identities
\[
D(0, 0, 0, 1)(\varphi, \varphi_j; x_1, x_2, t, t) = 0,
\]
\( j = 1, 2, 3 \), we see that
\[
f_j(t) = -D(0, 0, 1)(\varphi; x_1, t, t) \varphi_1(x_2) + D(0, 0, 1)(\varphi; x_2, t, t) \varphi_1(x_1)
\]
\( j = 1, 2, 3 \). Substituting this last set of expressions into the determinant involving the \( f' \)'s and adding appropriate multiples of the first two rows to the last row, we obtain
\[
\frac{d}{dt} \left[ \begin{array}{c}
\frac{D(\varphi; y; x_1, x_2, x_3, t)}{D(\varphi; x_1, x_2, t)}
\end{array} \right]
\]
\[
= \frac{D(\varphi; x_1, x_2, x_3)}{D^2(\varphi; x_1, x_2, t)} \left[ Y(t) + D(0, 0, 1)(\varphi; x_1, t, t) y(x_2)
\right.
\]
\[
- \left. D(0, 0, 1)(\varphi; x_2, t, t) y(x_1) \right]
\]
\[
= \frac{D(\varphi; x_1, x_2, x_3)}{D^2(\varphi; x_1, x_2, t)} D(0, 0, 0, 1)(\varphi; y; x_1, x_2, t, t)
\]

For \( 4^{th} \)
\[
\frac{d}{dt} \left[ \begin{array}{c}
\frac{D(0, 1, 0, 0)(\varphi; y; x_1, x_1, x_2, t)}{D(0, 1, 0)(\varphi; x_1, x_1, t)}
\end{array} \right]
\]
\[
= \frac{1}{D^2(0, 1, 0)(\varphi; x_1, x_1, t)}
\]
\[
\begin{vmatrix}
\varphi_1(x_1) & \varphi_2(x_1) & \varphi_3(x_1) & y(x_1) \\
\varphi_1(x_1) & \varphi_2(x_1) & \varphi_3(x_1) & y(x_1) \\
\varphi_1(x_2) & \varphi_2(x_2) & \varphi_3(x_2) & y(x_2) \\
f_1(t) & f_2(t) & f_3(t) & Y(t)
\end{vmatrix}
\]

where
\[
f_j(t) = D(0, 1, 0)(\varphi; x_1, x_1, t) \varphi''(t) - D(0, 1, 1)(\varphi; x_1, x_1, t) \varphi'(t),
\]
\[
Y(t) = D(0, 1, 0)(\varphi; x_1, x_1, t) y''(t) - D(0, 1, 1)(\varphi; x_1, x_1, t) y'(t),
\]
\( j = 1, 2, 3 \). If we now employ the identities
\[
D(0,1,0,1)(\phi, \phi_j; x_1, x_1, t, t) = 0
\]

and proceed as in the proof of 1°, we obtain

\[
\frac{d}{dt} \left[ \frac{D(0,1,0,0)(\phi, y; x_1, x_1, x_2, t)}{D(0,1,0)(\phi; x_1, x_1, t)} \right] = \frac{D(0,1,0)(\phi; x_1, x_1, x_2)}{D^2(0,1,0)(\phi; x_1, x_1, t)} D(0,1,0,1)(\phi, y; x_1, x_1, t, t).
\]

To prove 7°, we carry out the indicated differentiation, as in the proofs of 1° and 4°, and then make use of the identities

\[
D(0,1,2,3)(\phi, \phi_j; t, t, t, t) = 0,
\]

\[j = 1, 2, 3,\] to obtain the result.

§3. Principal Theorems.

**Theorem 4.** Let \( \phi \) and \( y \) be sufficiently regular. Then \((\phi, y)\) implies that

1° \quad D(0,0,0,1)(\phi, y; x_1, x_2, x_3, x_3) \geq 0,

2° \quad D(0,0,1,0)(\phi, y; x_1, x_2, x_2, x_3) \geq 0,

3° \quad D(0,1,0,0)(\phi, y; x_1, x_1, x_2, x_3) \geq 0,

4° \quad D(0,1,0,1)(\phi, y; x_1, x_1, x_2, x_2) \geq 0,

5° \quad D(0,0,1,2)(\phi, y; x_1, x_2, x_2, x_2) \geq 0,

6° \quad D(0,1,2,0)(\phi, y; x_1, x_1, x_1, x_2) \geq 0,

7° \quad D(0,1,2,3)(\phi, y; x_1, x_1, x_1, x_1) \geq 0,

for arbitrary \( x_1 < x_2 < x_3 \).
Proof. For the inequalities $1^o-4^o$, it is sufficient to assume that $\phi, y \in C^1$, for $5^o$ and $6^o$, that $\phi, y \in C^2$, for $7^o$ that $\phi, y \in C^3$. We will prove $1^o$ and $5^o$ to indicate the method of proof. Since $(\phi, y)$,

$$D(\phi, y; x_1, x_2, x_3, x_4) \geq 0$$

for $x_1 < x_2 < x_3 < x_4$. But

$$D(\phi, y; x_1, x_2, x_3, x_4) = 0.$$ 

Hence $\exists \ x^*, \ x_3 < x^* < x_4$ such that

$$D(\phi, y; x_1, x_2, x_3, x_4) = (x_4 - x_3) D(0, 0, 0, 1)(\phi, y; x_1, x_2, x_3, x^*),$$

by the mean-value theorem. Therefore,

$$D(0, 0, 0, 1)(\phi, y; x_1, x_2, x_3, x^*) \geq 0.$$

In the limit as $x_4 \to x_3$, we obtain $1^o$.

To prove $5^o$,

$$D(0, 0, 0, 1)(\phi, y; x_1, x_2, x_3, x_4)$$

$$= (x_3 - x_2) D(0, 0, 1, 1)(\phi, y; x_1, x_2, x_3, x_4)$$

where $x_2 < x_1, x_2 < x_3$. Applying the mean-value theorem again,

$$D(0, 0, 1, 1)(\phi, y; x_1, x_2, x_3, x_4)$$

$$= (x_3 - x_1) D(0, 0, 1, 1)(\phi, y; x_1, x_2, x_3, x_1),$$

where $x_1 < x_2 < x_3$. Hence,

$$D(0, 0, 1, 1)(\phi, y; x_1, x_2, x_3, x_1, x_2) \geq 0.$$

In the limit as $x_3 \to x_2$, we obtain $5^o$.

Theorem 5. Let $\phi, y \in C^1$ and let the unique solvability condition $U$ hold. Then $(\phi, y)$ if and only if

$$1^o \quad D(0, 0, 0, 1)(\phi, y; x_1, x_2, x_3, x_4) \geq 0 \text{ or}$$

$$2^o \quad D(0, 0, 1, 0)(\phi, y; x_1, x_2, x_3, x_4) \geq 0 \text{ or}$$

$$3^o \quad D(0, 1, 0, 0)(\phi, y; x_1, x_2, x_3, x_4) \geq 0$$

for arbitrary $x_1 < x_2 < x_3$, i.e., each of the determinantal inequalities $1^o$, $2^o$ and $3^o$ is equivalent to the statement $(\phi, y)$. 
Proof. That the statement \((\phi, y)\) implies the determinantal inequalities 1°, 2° and 3°, follows from Theorem 4. That each of the determinantal inequalities 1°, 2° and 3° implies that \((\phi, y)\), follows from the determinantal identities 1°, 2° and 3° of §2. We will carry out the details of the proof for the determinantal inequality 1°.

Given 1°, we must show that
\[
D(\phi, y; x_1, x_2, x_3, x_4) \geq 0
\]
for arbitrary \(x_1 < x_2 < x_3 < x_4\). But by the determinantal identity 1° of §2 and the unique solvability condition \(U\),
\[
\int_{x_3}^{x_4} \frac{1}{x} \left[ \frac{D(\phi, y; x_1, x_2, x_3, t)}{D(\phi; x_1, x_2, t)} \right] dt \geq 0
\]
and therefore
\[
D(\phi, y; x_1, x_2, x_3, x_4) \geq 0.
\]

Theorem 6. Let \(\phi, y \in C^2\) and let the unique solvability conditions \(U, U_1\) and \(U_2\) hold. Then \((\phi, y)\) if and only if

4° \(D(0, 1, 0, 1)(\phi, y; x_1, x_1, x_2, x_2) \geq 0\) or

5° \(D(0, 0, 1, 2)(\phi, y; x_1, x_2, x_2, x_2) \geq 0\) or

6° \(D(0, 1, 2, 0)(\phi, y; x_1, x_1, x_1, x_2) \geq 0\) ,

for arbitrary \(x_1 < x_2\).

Proof. That the statement \((\phi, y)\) implies 4°, 5° and 6°, follows from Theorem 4. The remainder of the theorem follows from the determinantal identities 4°, 5° and 6° of §2 and from Theorem 5.
Theorem 7. Let \( \phi, y \in C^3 \) and let the unique solvability conditions \( U, U_1, U_2 \) and \( U_3 \) hold. Then \((\phi, y)\) if and only if

\[
D(0,1,2,3)(\phi, y; x, x, x, x) \geq 0.
\]

Proof. We make use of Theorem 4, the determinantal identity 7° of §2 and Theorems 5 and 6 to obtain the result.

Corollary. Under the hypotheses of Theorem 7, \((\phi, y)\) if and only if \( y(x) \) may be written in the form

\[
y(x) = \int_{x_0}^x D(0,1,0)(\phi; t, t, x) \frac{g(t)}{w^2(t)} \, dt + \sum_{i=1}^{3} A_i \phi_i(x),
\]

where \( g(x) \in C, \ g(x) \geq 0, \) and

\[
w(x) = D(0,1,2)(\phi; x, x, x)
\]

is the Wronskian of the functions \( \phi \).

Proof. From Theorem 7, \((\phi, y)\) if and only if

\[
D(0,1,2,3)(\phi, y; x, x, x, x) = g(x) \geq 0.
\]

Solving the third order differential equation (55) by the method of variation of parameters, we obtain (54). The coefficients \( A_i \) are the solutions of the linear system

\[
y^{(k)}(x_0) = \sum_{i=1}^{3} A_i \phi_i^{(k)}(x_0)
\]

\( k = 0, 1, 2. \)
CHAPTER III

Function Convex with Respect to n Functions

§1. Definitions and Notation.

We will now define convexity with respect to a set of functions \( \phi_1, \ldots, \phi_n \), \( n \geq 2 \), continuous on an open interval \((a, b)\).

Definition 7. The functions \( \phi_1, \ldots, \phi_n \) are said to satisfy the unique solvability condition \( U \) if and only if

\[
D(\phi; x_1, \ldots, x_n) > 0
\]

for arbitrary \( x_1 < x_2 < \ldots < x_{n-1} < x_n \), where \( D(\phi; x_1, \ldots, x_n) \) is the \( n \) by \( n \) determinant whose \( i \)'th row is

\[
\phi_1(x_i) \ldots \phi_n(x_i)
\]

\( i = 1, \ldots, n \).

Definition 8. Let condition \( U \) hold. If \( y(x) \) is a real-valued function defined on \((a, b)\), then \( y(x) \) is said to be convex with respect to \( \phi_1, \ldots, \phi_n \) on \((a, b)\), i.e., \((\phi, y)\), if and only if

\[
D(\phi, y; x_1, \ldots, x_{n+1}) \geq 0
\]

for arbitrary \( x_1 < x_2 < \ldots < x_n < x_{n+1} \), where \( D(\phi, y; x_1, \ldots, x_{n+1}) \) is the \( n+1 \) by \( n+1 \) determinant whose \( i \)'th row is

\[
\phi_1(x_i) \ldots \phi_n(x_i) \, y(x_i)
\]

\( i = 1, \ldots, n+1 \).

The geometric significance of Definition 8 is evident from the geometric significance of Definition 6. We again remark that Theorem 1 holds for functions \( y(x) \) such that \((\phi, y)\), with obvious modifications.
We will now develop a system of notation which will enable us to state and prove theorems which are generalizations of the theorems proved in Chapter II.

Let $\alpha$ be an $n+1$-dimensional vector of the form

$$\alpha = (\alpha_1, \ldots, \alpha_{n+1})$$

where the $\alpha_i$ are non-negative integers. The norm of $\alpha$, denoted $|\alpha|$ is defined to be the non-negative integer

$$|\alpha| = \sum_{i=1}^{n+1} \alpha_i .$$

Let $x = (x_1, \ldots, x_{n+1})$ and assume that $\rho, y \in C^{|\alpha|}$. Then

$$D_{\alpha}(\rho, y; x)$$

is defined to be the $n+1$ by $n+1$ determinant whose $i$'th row is

$$\phi_i^1(x_i) \ldots \phi_i^n(x_i) y_i^1(x_i)$$

$i = 1, \ldots, n+1$.

By a segment $A$ of a vector $\alpha$ we mean any vector of the form

$$A = (\alpha_k, \alpha_{k+1}, \ldots, \alpha_{k+p-1})$$

where $1 \leq k \leq x+p-1 \leq n+1$. A segment $A$ is therefore of dimension $p$, $1 \leq p \leq n+1$. We will use the notation $\text{dim } V$ for the dimension of a vector $V$.

A segment $A$ of $\alpha$ is a proper segment if and only if

1° $1 \leq \text{dim } A \leq n+1$ and $A = (0, \ldots, 0)$ or

2° $2 \leq \text{dim } A \leq n+1$ and $A = (0, 1, \ldots, p-1)$, where $p = \text{dim } A$.

If case 1° holds, the proper segment $A$ is called a trivial proper segment. If case 2° holds, the proper segment $A$ is called a non-trivial proper segment. A non-trivial proper segment is necessarily of dimension greater than or equal to 2.
Evidently, an arbitrary vector \( \alpha \) may have no proper segments. For example, the vector \( \alpha = (1,1, \ldots, 1) \) has no proper segments.

A vector \( \alpha \) is proper if and only if \( \alpha = (A_1, A_2, \ldots, A_\nu) \), where the \( A_i \) are proper segments such that no two consecutive \( A_i \) are both trivial. For example, the eight-dimensional vector

\[
\begin{array}{cccc}
A_1 & A_2 & A_3 & A_4 \\
0 & 0 & 1 & 2 & 0 & 1 & 0 & 0
\end{array}
\]

is proper, the trivial proper segments being underlined once; the non-trivial proper segments being underlined twice.

Let \( x = (x_1, \ldots, x_{n+1}) \) be a vector consisting of points \( x_i \) in the open interval \((a,b)\). The symbol \( x \) will be used both as a scalar and a vector. It will be clear from context, which meaning is intended. A segment \( x \) of \( x \) is defined in the same manner as segment \( A \) of \( \alpha \). We will now define vectors \( x_\alpha \) associated with proper vectors \( \alpha \).

Let \( \alpha \) be proper, i.e., \( \alpha = (A_1, \ldots, A_\nu) \), where the \( A_i \) are proper segments. A vector \( x_\alpha \) is associated with \( \alpha \) if and only if

\[
x_\alpha = (X_1, \ldots, X_\nu)
\]

where the segments \( X_i \) satisfy the requirements:

1° \( \dim X_i = \dim A_i, \; i = 1, \ldots, \nu \).

2° If \( A_i \) is trivial, then \( X_i = (x_1^{(i)}, x_2^{(i)}, \ldots, x_{p_i}^{(i)}) \),

where \( p_i = \dim X_i \) and \( x_1^{(i)} < x_2^{(i)} < \ldots < x_{p_i}^{(i)} \).

3° If \( A_i \) is non-trivial, then \( X_i = (x_1^{(i)}, x_1^{(i)}, \ldots, x_1^{(i)}) \),

i.e., all the elements of the segment \( X_i \) are the same element \( x_1^{(i)} \in (a,b) \).

4° The last element of \( X_i \) is strictly less than the first element of \( X_{i+1} \).

5° Otherwise, \( x_\alpha \) is arbitrary in the sense that its elements may be arbitrary points in \((a,b)\) satisfying the requirements 1°–4°.
For example if 
\[ \alpha = (0,0,1,2,0,1,0,0), \]
then any vector of the form 
\[ x_\alpha = (x_1, x_2, x_2, x_3, x_3, x_4, x_5), \]
where \( a < x_1 < \ldots < x_5 < b \), is associated with \( \alpha \) and any 
vector \( x_\alpha \), associated with \( \alpha \), may be written in this form.

For an arbitrary vector \( \alpha \), whose elements are non-negative 
integers, we define the non-negative integer 
\[ \lambda(\alpha) = \max_i \alpha_i, \]
where \( \alpha = (\alpha_1, \ldots, \alpha_{n+1}) \). For proper \( n+1 \)-dimensional vectors \( \alpha \), 
\[ 0 \leq \lambda(\alpha) \leq n, \quad 0 \leq |\alpha| \leq \frac{n(n+1)}{2}. \] Also \( \lambda(\alpha) = 0 \) if and only if 
\( \alpha = (0, \ldots, 0) \) and \( \lambda(\alpha) = n \) if and only if \( \alpha = (0,1, \ldots, n) \). 
A similar statements holds for \( |\alpha| \).

Similarly, we may define \( n \)-dimensional proper vectors \( \gamma \), 
associated \( n \)-dimensional vectors \( x_\gamma \), and derivatives of \( n \) by \( n \) 
determinants. In particular, if \( \gamma = (\gamma_1, \ldots, \gamma_n) \), 
\( x = (x_1, \ldots, x_n) \), then \( D_\gamma(\phi; x) \) is defined to be the \( n \) by \( n \) 
determinant whose \( i \)'th row is 
\[ \phi_1^{(\gamma_1)}(x_1), \ldots, \phi_n^{(\gamma_1)}(x_1) \]
\[ i = 1, \ldots, n. \]

**Definition 9.** Let \( \gamma \) be proper and let \( \phi \in C^{\lambda(\gamma)}. \) The 
functions \( \phi \) are said to satisfy the unique solvability condition 
\( U_\gamma \) if and only if 
\[ D_\gamma(\phi; x_\gamma) > 0 \]
for arbitrary associated \( x_\gamma \).

For example, if \( |\gamma| = 0 \), then \( \lambda(\gamma) = 0 \), i.e., \( \phi \in C \), \( \gamma \) is 
itself a trivial proper segment, and we therefore have the unique 
solvability condition of Definition 7. If \( |\gamma| = \frac{(n-1)n}{2} \), then 
\( \lambda(\gamma) = n - 1 \), \( \gamma \) is itself a non-trivial proper segment, and we
therefore have the statement that the Wronskian of the functions \( \phi \) is strictly positive throughout the open interval \((a,b)\). It is obvious that there are no \( n \)-dimensional proper vectors \( \gamma \) whose norm is greater than \( \frac{(n-1)n}{2} \).

\section{Principal Theorems}

\textbf{Theorem 8.} Let \( \phi, y \in \mathbb{C}^n \). Then \((\phi, y)\) implies that

\[ D_\alpha(\phi, y; x) \geq 0 \]

for all proper \( n+1 \)-dimensional vectors \( \alpha \) and arbitrary associated \( x_\alpha \).

\textbf{Proof.} The proof is by finite induction. Since \((\phi, y)\), the theorem is certainly true for \(|\alpha| = 0\). Let \( \alpha^* \) be proper and such that \( 1 \leq |\alpha^*| \leq \frac{n(n+1)}{2} \). Assuming the theorem true for all proper \( \alpha \) such that \( 0 \leq |\alpha| < |\alpha^*| \), we will show that the theorem is also true for \( \alpha^* \).

Since \( \alpha^* \) is proper, it may be written in the form

\[ \alpha^* = (A_1^*, A_2^*, \ldots, A_{\nu-1}^*, A_{\nu}^*) \]

where the \( A_i^* \) are proper segments. Since \( |\alpha^*| \geq 1 \) and no two consecutive segments are both trivial, one of the two proper segments \( A_{\nu-1}^*, A_{\nu}^* \) is non-trivial. It will be sufficient to consider the case where \( A_{\nu}^* \) is non-trivial.

Since \( A_{\nu}^* \) is non-trivial, it may be written in the form

\[ A_{\nu}^* = (0, 1, \ldots, p-1) \]

where \( p = \dim A_{\nu}^* \). Let an arbitrary associated \( x_{\alpha^*} \) be given. \( x_{\alpha^*} \) may be written in the form

\[ x_{\alpha^*} = (x_1^*, x_2^*, \ldots, x_{\nu-1}^*, x_{\nu}^*) \]

and the segment \( x_{\nu}^* \) of \( x_{\alpha^*} \) may be written

\[ x_{\nu}^* = (x_1(\nu), x_1(\nu), \ldots, x_1(\nu)) \]
since $A^*_j$ is non-trivial.

Now consider the vector

$$\alpha = (A^*_1, A^*_2, \ldots, A^*_j, A^*_v, A^*_{v+1})$$

and the vector

$$x_\alpha = (x^*_1, x^*_2, \ldots, x^*_j, x^*_v, x^*_{v+1}),$$

where

$$A = (0), \quad A^*_{v+1} = (0, 1, \ldots, p-2),$$

$$X = (x_1, \ldots, x^*_j), \quad x^*_{v+1} = (t, t, \ldots, t),$$

for $t > x_1^{(v)}$ then $\dim X_{v+1} = \dim A^*_{v+1} = p - 1$.

Certainly $\alpha$ is proper, $0 \leq |\alpha| < |x^*|$, and $x_\alpha$ is associated with $\alpha$. Employing the method of proof of Theorem 4, i.e., by successive applications of the mean-value theorem to the determinantal inequality

$$D_\alpha(\phi, y; x_\alpha) \geq 0,$$

we have the result. It is easily seen that the same approach works if $A^*_{v-1}$ is non-trivial and $A^*_v$ is trivial.

**Corollary.** Let $\phi, y \in C^k$, $0 \leq k \leq n$. Then $(\phi, y)$ implies that

$$D_\alpha(\phi, y; x_\alpha) \geq 0$$

for all proper $\alpha$ and arbitrary associated $x_\alpha$ for which $0 \leq \lambda(\alpha) \leq k$.

**Proof.** This corollary is proved in precisely the same manner as the preceding theorem. It is sufficient to note that if

$$\lambda(\alpha^*) \leq k,$$

then for the vector $\alpha^*$, constructed in the proof of Theorem 8, $\lambda(\alpha) \leq \lambda(\alpha^*)$. (The induction proof is still carried out using $|\alpha|.$)
Before proving the next theorem, it will be convenient to make the following definition. A proper vector $\alpha$ is **completely proper** if and only if

1° $\alpha$ is itself a proper segment or

2° $\alpha = (A_1, A_2)$, where $A_1$ is a trivial proper segment and $A_2$ is a non-trivial proper segment.

**Theorem 9.** Let $\phi, y \in C^n$ and let the unique solvability conditions $U_\gamma$ hold for all completely proper $\gamma$. If

$$
\begin{bmatrix}
\phi_1(x) & \ldots & \phi_n(x) & y(x) \\
\phi'_1(x) & \ldots & \phi'_n(x) & y'(x) \\
\vdots & \ddots & \vdots & \vdots \\
\phi_1^{(n)}(x) & \ldots & \phi_n^{(n)}(x) & y^{(n)}(x)
\end{bmatrix} \geq 0
$$

for all $x \in (a, b)$, i.e., if

$$D_{\alpha^*}(\phi, y; x_{\alpha^*}) \geq 0$$

for $\alpha^*$ proper, $\ell(\alpha^*) = n$, and arbitrary associated $x_{\alpha^*}$, then $(\phi, y)$.

**Proof.** We are given that

$$D_{\alpha^*}(\phi, y; x_{\alpha^*}) \geq 0$$

where $\alpha^* = (0, 1, \ldots, n)$. We may therefore write $x_{\alpha^*} = (t, t, \ldots, t)$ for $t \in (a, b)$.

Let

$\gamma = (0, 1, \ldots, n-1)$, \hspace{1cm} $\gamma^* = (0, 0, 1, \ldots, n-2)$,

$x_\gamma = (t, t, \ldots, t)$, \hspace{1cm} $x_{\gamma^*} = (x_1, t, t, \ldots, t)$,

where $\dim \gamma = \dim x_\gamma$, $\dim \gamma^* = \dim x_{\gamma^*}$, $t > x_1$. Certainly $\gamma$ and $\gamma^*$ are completely proper $n$-dimensional vectors and $x_\gamma$ and $x_{\gamma^*}$ are associated with $\gamma$ and $\gamma^*$ respectively. Also,
\[ D_\gamma(\phi;x_\gamma) > 0, \]

and
\[ D_{\gamma^*}(\phi;x_{\gamma^*}) > 0. \]

Now let
\[ \alpha = (0,0,1, \ldots, n-1), \]
\[ x_\alpha = (x_1, t, t, \ldots, t), \]

where \( \dim \alpha = \dim x_\alpha \). Then \( \alpha \) is completely proper and \( x_\alpha \) is associated with \( \alpha \). We claim that for \( t \geq x_1 \),

\[ \frac{d}{dt} \left[ \frac{D_\alpha(\phi,y;x_\alpha)}{D_\gamma(\phi;x_\gamma)} \right] = \frac{D_{\gamma^*}(\phi,x_{\gamma^*}) D_{\alpha^*}(\phi,y;x_{\alpha^*})}{D_\gamma^2(\phi;x_\gamma)}. \]

This follows by carrying out the differentiation, as in the proof of the identities of §2 of Chapter II, and making use of the identities \( D_{\alpha^*}(\phi,\phi_j;x_{\alpha^*}) = 0 \), \( j = 1, \ldots, n \). Hence for \( x_2 > x_1 \),

\[ \int_{x_1}^{x_2} \frac{d}{dt} \left[ \frac{D_\alpha(\phi,y;x_\alpha)}{D_\gamma(\phi;x_\gamma)} \right] dt \geq 0, \]

i.e.,
\[ D_\alpha(\phi,y;x_\alpha) \geq 0 \]

for \( \alpha = (0,0,1, \ldots, n-1), \)
\[ x_\alpha = (x_1,x_2,x_2, \ldots, x_2), \]

and hence
\[ D_\alpha(\phi,y;x_\alpha) \geq 0, \]

for arbitrary associated \( x_\alpha \). By applying the above approach \( n - 1 \) more times, for a total of \( n \) times, we obtain the result. We need only mention that in the successive applications of the above approach, we use the unique solvability conditions \( U_{\gamma} \) only with \( \gamma \) completely proper.
Corollary 1. If the requirements of the theorem hold and
\[ D_\alpha(\phi, y; x_\alpha) > 0 \]
for \( \alpha = (0, 1, \ldots, n), x_\alpha = (x, x, \ldots, x), \) then
\[ D_\alpha(\phi, y; x_\alpha) > 0 \]
for all completely proper \( \alpha \) and arbitrary associated \( x_\alpha \).

Corollary 2. If the requirements of the theorem hold, then
\[ D_\alpha(\phi, y; x_\alpha) \geq 0 \]
for all proper \( \alpha \) and arbitrary associated \( x_\alpha \).

The proof of Corollary 1 involves only slight modifications of the proof of Theorem 9. The proof of Corollary 2 is immediate from Theorems 8 and 9.

Corollary 3. If the requirements of the theorem hold, then \((\phi, y)\) if and only if \(y(x)\) may be written in the form

\[
(56) \quad y(x) = \int_{x_0}^{x} D(0, 1, \ldots, n-2, 0)(\phi; t, \ldots, t, x) \frac{g(t)}{W^2(t)} \, dt + \sum_{i=1}^{n} A_i \phi_i(x)
\]

where \(g(x) \in C, g(x) \geq 0\), and

\[
W(x) = D(0, 1, \ldots, n-1)(\phi; x, \ldots, x)
\]
is the Wronskian of the functions \(\phi\).

Proof. We obtain the representation (56) by solving the \(n\)-th order differential equation
\[ D(0, 1, \ldots, n)(\phi, y; x, \ldots, x) = g(x) \geq 0 \]

by the method of variation of parameters. The coefficients \(A_i\) are the solutions of the linear system
\[
y^{(k)}(x_0) = \sum_{i=1}^{n} A_i \phi_i^{(k)}(x_0),
\]
k = 0, \ldots, n-1.
We will say that a system of functions \( \phi \) is **completely regular** if and only if the unique solvability conditions \( U_\gamma \) hold for all proper \( \gamma \). For completely regular systems, we may state a theorem, somewhat stronger than Theorem 9, which is a generalization of Theorems 5, 6 and 7.

**Theorem 10.** Let \( \phi, \gamma \in C^n \) and let the system of functions \( \phi \) be completely regular. Then \((\phi, \gamma)\) if and only if

\[
D_\alpha(\phi, \gamma; x_\alpha) \geq 0
\]

for any particular proper \( \alpha \) and arbitrary associated \( x_\alpha \). Hence, any of the determinantal inequalities

\[
D_\alpha(\phi, \gamma; x_\alpha) \geq 0
\]

is equivalent to the statement \((\phi, \gamma)\).

**Proof.** That the statement \((\phi, \gamma)\) implies \(D_\alpha(\phi, \gamma; x_\alpha) \geq 0\) for all proper \( \alpha \) and arbitrary associated \( x_\alpha \), follows from Theorem 8. The remainder of the proof is by finite induction. We wish to show that, for any proper \( \alpha \), \(D_\alpha(\phi, \gamma; x_\alpha) \geq 0\) implies that \((\phi, \gamma)\). This is obviously true for \( |\alpha| = 0 \). Let \( \alpha^* \) be proper and such that

\[1 \leq |\alpha^*| \leq \frac{n(n+1)}{2} \]

Assuming the statement true for all proper \( \alpha \) such that \( 0 \leq |\alpha| < |\alpha^*| \), we will show that it is also true for \( \alpha^* \).

Write \( \alpha^* \) in the form

\[\alpha^* = (A_1^*, A_2^*, \ldots, A_{\gamma-1}^*, A_\gamma^*)\]

where the \( A_i^* \) are proper segments. As in the proof of Theorem 8, it will be sufficient to consider the case where \( A_\gamma^* \) is a non-trivial segment.

Since \( A_\gamma^* \) is non-trivial

\[A_\gamma^* = (0,1,\ldots,p-1)\]

where \( p = \dim A_\gamma^* \). Write the associated vector \( x_{\alpha^*} \) in the form

\[x_{\alpha^*} = (X_1^*, X_2^*, \ldots, X_{\gamma-1}^*, X_\gamma^*)\]

where

\[X_\gamma^* = (t, \ldots, t)\]

since \( A_\gamma^* \) is non-trivial.
Now consider the vectors

\[ \alpha = (A_1^*, A_2^*, \ldots, A_{\nu-1}^*, A_\nu, A_1^{(1)}, A_2^{(1)}, A_\nu^{(1)}), \]
\[ x_\alpha = (X_1^*, X_2^*, \ldots, X_{\nu-1}^*, X_\nu^{(1)}, X_\nu^{(2)}), \]

where

\[ A_\nu^{(1)} = (0), \quad A_\nu^{(2)} = (0, 1, \ldots, p-2), \]
\[ X_\nu^{(1)} = (x_1^{(\nu)}), \quad X_\nu^{(2)} = (t, t, \ldots, t), \]

the element \( x_1^{(\nu)} \) being strictly greater than the last element of the segment \( X_{\nu-1}^* \) and \( \dim X_\nu^{(2)} = \dim A_\nu^{(2)} = p-1 \).

If we write

\[ \gamma = (A_1^*, A_2^*, \ldots, A_{\nu-1}^*, A_\nu, A_1^{(3)}, A_2^{(3)}), \]
\[ x_\gamma = (X_1^*, X_2^*, \ldots, X_{\nu-1}^*, X_\nu^{(3)}), \]
\[ \gamma^* = (A_1^*, A_2^*, \ldots, A_{\nu-1}^*, A_\nu^{(1)}, A_\nu^{(3)}), \]
\[ x_{\gamma^*} = (X_1^*, X_2^*, \ldots, X_{\nu-1}^*, X_\nu^{(1)}, X_\nu^{(3)}), \]

where

\[ A_\nu^{(3)} = (0, 1, \ldots, p-3), \]
\[ X_\nu^{(3)} = (t, t, \ldots, t), \]

\( \dim X_\nu^{(3)} = \dim A_\nu^{(3)} = p-2 \) and if, for simplicity, we adopt the convention that if \( p = 2 \), then the segments \( A_\nu^{(3)} \) and \( X_\nu^{(3)} \) simply do not appear in the expressions for \( \gamma^* \) and \( x_{\gamma^*} \), we see that \( \gamma \) and \( \gamma^* \) are \( n \)-dimensional proper vectors and that \( x_\gamma \) and \( x_{\gamma^*} \) are associated with \( \gamma \) and \( \gamma^* \) respectively. Moreover, for \( t \) greater

then the last element of the segment \( X_{\nu-1}^* \).
\[
\frac{d}{dt} \left[ \frac{D_{\alpha}(\phi, y; x_{\alpha})}{D_{\alpha}(\phi; x_{\alpha})} \right] = \frac{D_{\alpha}(\phi; x_{\alpha})}{D_{\alpha}(\phi; x_{\alpha})} \quad D_{\alpha}(\phi, y; x_{\alpha}),
\]

which is verified in the same manner as the identities of §2 of Chapter 2. We need only carry out the differentiation and then make use of the identities

\[
D_{\alpha}(\phi, \phi_j; x_{\alpha}) = 0,
\]

\( j = 1, \ldots, n \) to obtain the result. It now follows that

\[
\int_{x_1(\nu)}^{x_2(\nu)} \frac{d}{dt} \left[ \frac{D_{\alpha}(\phi, y; x_{\alpha})}{D_{\alpha}(\phi; x_{\alpha})} \right] dt \geq 0,
\]

for \( x_2(\nu) > x_1(\nu) \), so that

\[
D_{\alpha}(\phi, y; x_{\alpha}) \geq 0
\]

for \( \alpha = (A_1^*, A_2^*, \ldots, A_{y-1}^*, A_y^{(1)}, A_y^{(2)}) \),

\[
x_{\alpha} = (x_1^*, x_2^*, \ldots, x_{y-1}^*, x_y^{(1)}, x_y^{(2)}),
\]

\[
x_y^{(2)} = (x_2(\nu), \ldots, x_2(\nu)),
\]

and therefore

\[
D_{\alpha}(\phi, y; x_{\alpha}) \geq 0
\]

for arbitrary associated \( x_{\alpha} \).

But by the induction hypothesis, \( D_{\alpha}(\phi, y; x_{\alpha}) \geq 0 \) implies that \((\phi, y)\). Hence \( D_{\alpha}(\phi, y; x_{\alpha}) \geq 0 \) implies that \((\phi, y)\) and the proof of the theorem is complete. We remark that an identity similar to (57) will give the result if \( A_{y-1}^* \) is non-trivial and \( A_y^* \) is trivial.
Corollary. Let \((\phi, y) \in C^k, \ 0 \leq k \leq n\), and let the unique solvability conditions \(U_\gamma\) hold for all proper \(\gamma\) such that \(0 \leq \xi(\gamma) \leq k\). Then \((\phi, y)\) if and only if

\[D_\alpha(\phi, y; x_\alpha) \geq 0\]

for any proper \(\alpha\) such that \(0 \leq \xi(\alpha) \leq k\) and arbitrary associated \(x_\alpha\).

The proof of this corollary makes use of the corollary to Theorem 8 and slight modifications of the proof of Theorem 9.

The following theorem, an improvement of Corollary 1 to Theorem 9, will enable us to state a rather interesting sufficient condition that a system \(\phi\) be completely regular.

**Theorem 11.** Let \(\phi, y \in C^N\) and let the system \(\phi\) be completely regular. If

\[D_\alpha(\phi, y; x_\alpha) > 0\]

for \(\alpha = (0, 1, \ldots, n)\), \(x_\alpha = (x, x, \ldots, x)\), then

\[D_\alpha(\phi, y; x_\alpha) > 0\]

for all proper \(\alpha\) and arbitrary associated \(x_\alpha\).

**Proof.** The proof is by finite induction. The theorem is certainly true if \(|\alpha| = \frac{n(n+1)}{2}\). It is also true for all completely proper \(\alpha\), and hence for \(|\alpha| = 0\), by Corollary 1 to Theorem 9.

We may therefore assume that \(0 < |\alpha| < \frac{n(n+1)}{2}\).

Let \(\alpha\) be proper, fixed, and such that \(0 < |\alpha| < \frac{n(n+1)}{2}\).

Assuming the theorem true for all proper \(\alpha^*\) such that

\(|\alpha| < |\alpha^*| \leq \frac{n(n+1)}{2}\),

we wish to show that it is also true for \(\alpha\). Let us note that Corollary 2 to Theorem 9 informs us that

\[D_\alpha(\phi, y; x_\alpha) \geq 0\].
Since \( \alpha \) is proper,

\[
\alpha = (A_1, A_2, \ldots, A_{v-1}, A_v).
\]

Since \(|\alpha| > 0\), not all of the numbers \(l(A_i)\) are zero, and since

\[|\alpha| < \frac{n(n+1)}{2}, \quad l(A_i) < n, \quad i = 1, \ldots, v.\]

Let \( A_{\mu} \) be a segment on which the number

\[q = \max_i l(A_i)\]

is assumed. Then \( 0 < q < n \), and if we let \( p = q + 1, \ p = \dim A_{\mu}. \)

We now have four cases to consider.

**Case 1.** \( A_{\mu} = A_1 \) and \( A_2 \) is trivial.

Write the associated vector \( x_{\alpha} \) in the form

\[x_{\alpha} = (x_1, x_2, \ldots, x_{v-1}, x_v),\]

where

\[x_1 = (t, t, \ldots, t), \]
\[x_2 = (x_1^{(2)}, x_2^{(2)}, \ldots, x_{p_2}^{(2)}),\]

and \( \dim x_1 = \dim A_1 = p, \ \dim x_2 = \dim A_2 = p_2. \) From the definition of an associated \( x_{\alpha} \),

\[t < x_1^{(2)} < \ldots < x_{p_2}^{(2)},\]

where \( x_{p_2}^{(2)} \) is less than the first element of \( X_3. \)

Consider the vectors

\[\alpha^* = (A_1^*, A_2^*, A_3^*, \ldots, A_{v-1}^*, A_v^*), \]
\[x_{\alpha^*} = (x_1^*, x_2^*, x_3^*, \ldots, x_{v-1}^*, x_v^*),\]

where

\[A_1^* = (0, 1, \ldots, p), \quad A_2^* = (0, \ldots, 0), \]
\[x_1^* = (t, t, \ldots, t), \quad x_2^* = (x_2^{(2)}, \ldots, x_{p_2}^{(2)}),\]
\[x_3^* = (x_3^{(2)}, \ldots, x_{p_2}^{(2)}),\]
and \( \dim X_1^* = \dim A_1^* = p + l \), \( \dim X_2^* = \dim A_2^* = p_2 - l \).

Certainly \( \alpha^* \) is a proper \( n+1 \)-dimensional vector such that \( |\alpha^*| > |\alpha| \). Also, \( |\alpha^*| \leq \frac{n(n+1)}{2} \) since \( |\alpha| < \frac{n(n+1)}{2} \).

Moreover, the vector \( x_{\alpha^*} \) is associated with \( \alpha^* \).

Now let
\[
\gamma = (A_1, A_2, \ldots, A_{\gamma-1}, A_{\gamma}) ,
\]
\[
x_{\gamma} = (X_1, X_2, \ldots, X_{\gamma-1}, X_{\gamma}) ,
\]
\[
\gamma^* = (A_1^{**}, A_2, \ldots, A_{\gamma-1}, A_{\gamma}) ,
\]
\[
x_{\gamma^*} = (X_1^{**}, X_2, \ldots, X_{\gamma-1}, X_{\gamma}) ,
\]

where
\[
A_1^{**} = (0, 1, \ldots, p-2) ,
\]
\[
X_1^{**} = (t, t, \ldots, t) ,
\]

and \( \dim A_1^* = \dim X_1^* = p - l \). Then \( \gamma \) and \( \gamma^* \) are proper \( n \)-dimensional vectors and the vectors \( x_{\gamma} \) and \( x_{\gamma^*} \) are associated with \( \gamma \) and \( \gamma^* \) respectively. Moreover

\[
\frac{d}{dt} \left[ \frac{D_{\alpha}(\phi; x_{\alpha})}{D_{\gamma}(\phi; x_{\gamma})} \right] = - \frac{D_{\gamma^*}(\phi; x_{\gamma^*}) D_{\alpha^*}(\phi; y; x_{\alpha^*})}{D_{\gamma}^2(\phi; x_{\gamma})}
\]

for \( t < x_1^{(2)} \).
But by the induction hypothesis and the complete regularity of the system \( \phi \), the right-hand side of the above equation is strictly negative for \( t < x_1^{(2)} \). Hence, if we take \( x_1^{(1)} < x_2^{(1)} < x_1^{(2)} \),

\[
\int_{x_1^{(1)}}^{x_2^{(1)}} \frac{d}{dt} \left[ \frac{D_\alpha(\phi; y; x_\alpha)}{D_\nu(\phi; X_\nu)} \right] dt < 0
\]

and therefore,

\[
\frac{D_\alpha(\phi; y; x_\alpha^{(1)})}{D_\nu(\phi; x_\nu^{(1)})} > \frac{D_\alpha(\phi; y; x_\alpha^{(2)})}{D_\nu(\phi; x_\nu^{(2)})},
\]

where \( x_\nu^{(i)} \) and \( x_\alpha^{(i)} \) are the vectors \( x_\nu \) and \( x_\alpha \) respectively, with \( t \) replaced by \( x_1^{(1)} \), \( i = 1,2 \).

But by Corollary 2 to Theorem 9,

\[
\frac{D_\alpha(\phi; y; x_\alpha^{(2)})}{D_\nu(\phi; x_\nu^{(2)})} \geq 0,
\]

so that

\[
D_\alpha(\phi; y; x_\alpha^{(1)}) > 0.
\]

Since \( x_1^{(1)} \) and \( x_2^{(2)} \) were essentially arbitrary, we conclude that

\[
D_\alpha(\phi; y; x_\alpha) > 0
\]

for arbitrary associated \( x_\alpha \).
Case 2. \( A_\mu = A_1 \) and \( A_2 \) is non-trivial.

We will use much of the notation of Case I. The essential difference is that

\[
A_2 = (0, 1, \ldots, p_2 - 1),
\]

\[
X_2 = (x_1^{(2)}, x_1^{(2)}, \ldots, x_1^{(2)}),
\]

where \( \dim A_2 = \dim X_2 = p_2 \) and \( p_2 \leq p \), since \( \lambda(A_2) \leq \lambda(A_1) \).

Defining

\[
\alpha^* = (A_1^*, A_2^*, \ldots, A_{\nu-1}^*, A_\nu)
\]

where \( A_2^* = (0, 1, \ldots, p_2 - 2) \), we see that the norm of \( \alpha \) has been increased by \( p \) and decreased by \( p_2 - 1 \), i.e.,

\[
|\alpha^*| - |\alpha| = p - (p_2 - 1) \geq 1,
\]

since \( p \geq p_2 \). Hence, \( |\alpha^*| > |\alpha| \).

If we now define

\[
x_{\alpha^*} = (X_1^*, X_2^*, \ldots, X_{\nu-1}^*, X_\nu)
\]

where

\[
X_2^* = (x_1^{(2)}, x_1^{(2)}, \ldots, x_1^{(2)}),
\]

with \( \dim X_2^* = \dim A_2^* = p_2 - 1 \), and if we define

\[
\gamma = (A_1, A_2^*, \ldots, A_{\nu-1}^*, A_\nu),
\]

\[
x_\gamma = (X_1^*, X_2^*, \ldots, X_{\nu-1}^*, X_\nu),
\]

\[
\gamma^* = (A_1^*, A_2, \ldots, A_{\nu-1}^*, A_\nu),
\]

\[
x_{\gamma^*} = (X_1^*, X_2, \ldots, X_{\nu-1}^*, X_\nu),
\]

\[
x_{\gamma^*} = (X_1^*, X_2, \ldots, X_{\nu-1}^*, X_\nu),
\]

where
then $\nu$ and $\nu^*$ are proper n-dimensional vectors and $x_\nu$ and $x_{\nu}^*$ are associated with $\nu$ and $\nu^*$ respectively. Moreover,

$$
\frac{d}{dt} \left[ \frac{D_\alpha(\phi,y;x_\alpha)}{D_\nu(\phi;x_\nu)} \right] = - \frac{D_{\nu^*}(\phi;x_{\nu^*})}{D_\nu(\phi;x_\nu)} \cdot \frac{D_{\alpha^*}(\phi,y;x_{\alpha^*})}{D_\nu(\phi;x_\nu)}.
$$

The remainder of the proof for Case 2 follows as in the proof of Case 1.

**Case 2.** $A_\mu \neq A_1$ and $A_{\mu-1}$ is trivial.

For this case, the vector $\alpha$ may be written in the form

$$
\alpha = (A_1, \ldots, A_{\mu-1}, A_\mu, \ldots, A_\nu),
$$

where $\dim A_\mu = p$. We may write the associated vector in the form

$$
x_\alpha = (X_1, \ldots, X_{\mu-1}, X_\mu, \ldots, X_\nu),
$$

where

$$
X_{\mu-1} = (x_1(\mu-1), x_2(\mu-1), \ldots, x_p(\mu-1)),
$$

$$
X_\mu = (t, t, \ldots, t),
$$

and $\dim X_{\mu-1} = \dim A_{\mu-1} = p_{\mu-1}$, $\dim X_\mu = \dim A_\mu = p$.

$x_1(\mu-1) < x_2(\mu-1) < \ldots < x_p(\mu-1) < t$, $t$ being less than the first element of $X_{\mu+1}$.
Consider the vector
\[ \alpha^* = (A_1, A_2, \ldots, A_{\mu-2}, A_{\mu-1}^*, A_{\mu}^*, A_{\mu+1}, \ldots, A_N) , \]
where
\[ A_{\mu-1}^* = (0, \ldots, 0) , \]
\[ A_{\mu}^* = (0, 1, \ldots, p) , \]
and \( \dim A_{\mu-1}^* = p_{\mu-1} - 1 \). The vector \( x_{\alpha^*} \) is a proper \( n+1 \)-dimensional vector such that \( |\alpha^*| > |\alpha| \). Also, \( |\alpha^*| \leq \frac{n(n+1)}{2} \).

Consider also the vector
\[ x_{\alpha^*} = (X_1, X_2, \ldots, X_{\mu-2}, X_{\mu-1}^*, X_{\mu}^*, X_{\mu+1}, \ldots, X_N) , \]
where
\[ X_{\mu-1}^* = (x_1^{(\mu-1)}, x_2^{(\mu-1)}, \ldots, x_{p_{\mu-1}-1}^{(\mu-1)}) , \]
\[ X_{\mu}^* = (t, t, \ldots, t) , \]
and \( \dim X_{\mu}^* = \dim A_{\mu}^* = p + 1 \). Certainly \( x_{\alpha^*} \) is associated with \( \alpha^* \).

Now let
\[ \gamma = (A_1, A_2, \ldots, A_{\mu-2}, A_{\mu-1}^*, A_{\mu}^*, \ldots, A_N) , \]
\[ x_{\gamma} = (X_1, X_2, \ldots, X_{\mu-2}, X_{\mu-1}^*, X_{\mu}^*, \ldots, X_N) , \]
\[ \gamma^* = (A_1, A_2, \ldots, A_{\mu-2}, A_{\mu-1}^*, A_{\mu}^*, \ldots, A_N) , \]
\[ x_{\gamma^*} = (X_1, X_2, \ldots, X_{\mu-2}, X_{\mu-1}^*, X_{\mu}^*, \ldots, X_N) , \]
where
\[ A_{\mu}^{**} = (0, 1, \ldots, p-2) , \]
\[ X_{\mu}^{**} = (t, t, \ldots, t) , \]
and \( \dim X_{\mu}^{**} = \dim A_{\mu}^{**} = p - 1 \).
The vectors $\mathbf{\nu}$ and $\mathbf{\nu}^*$ are proper n-dimensional vectors and the vectors $x_{\mathbf{\nu}}$ and $x_{\mathbf{\nu}^*}$ are associated with $\mathbf{\nu}$ and $\mathbf{\nu}^*$ respectively. Moreover,
\[
\frac{d}{dt} \left[ \frac{D_\alpha(\phi; y; x_\alpha)}{D_\nu(\phi; x_\nu)} \right] = \frac{D_{\nu^*}(\phi; x_{\nu^*}) \; D_\alpha(\phi; y; x_\alpha)}{D_\nu(\phi; x_\nu)}
\]
for the values of $t$ under consideration.

But by the unique solvability conditions and the induction hypothesis, the right side of the above equation is strictly positive for the values of $t$ under consideration. Hence if we take
\[
x_{p_{\mu-1}}^{(\mu)} < x_1^{(\mu)} < x_2^{(\mu)},
\]
where $x_2^{(\mu)}$ is less than the first element of $X_{\nu+1}$,
\[
\int \frac{d}{dt} \left[ \frac{D_\alpha(\phi; y; x_\alpha)}{D_\nu(\phi; x_\nu)} \right] dt \geq 0,
\]
and therefore
\[
\frac{D_\alpha(\phi; y; x_\alpha^{(2)})}{D_\nu(\phi; x_\nu^{(2)})} > \frac{D_\alpha(\phi; y; x_\alpha^{(1)})}{D_\nu(\phi; x_\nu^{(1)})} \geq 0,
\]
where $x_\nu^{(1)}$ and $x_\nu^{(2)}$ are the vectors $x_\nu$ and $x_\alpha$ respectively, with $t$ replaced by $x_1^{(\mu)}$, $i = 1, 2$. We therefore have
\[
D_\alpha(\phi; y; x_\alpha^{(2)}) > 0,
\]
so that
\[
D_\alpha(\phi; y; x_\alpha) > 0
\]
for arbitrary associated $x_\alpha$. 
Case 4. \( A_\mu \neq A_1 \) and \( A_{\mu-1} \) is non-trivial

We will use much of the notation of Case 3. The essential difference is that

\[
A_{\mu-1} = (0, 1, \ldots, p_{\mu-1} - 1),
X_{\mu-1} = (x_1(\mu-1), \ldots, x_1(\mu-1)),
\]

where \( \dim X_{\mu-1} = \dim A_{\mu-1} = p_{\mu-1} \).

Defining

\[
\alpha^* = (A_1, A_2, \ldots, A_{\mu-2}, A_{\mu-1}^*, A_\mu, A_{\mu+1}, \ldots, A_\nu),
\]

where

\[
A_{\mu-1}^* = (0, 1, \ldots, p_{\mu-1} - 2),
\]

we see that the norm of \( \alpha \) has been increased by \( p \) and decreased by \( p_{\mu-1} - 1 \), i.e.,

\[
|\alpha^*| - |\alpha| = p + 1 - p_{\mu-1} \geq 1,
\]

since \( p \geq p_{\mu-1} \). Hence \( |\alpha^*| > |\alpha| \).

If we now define

\[
x_{\alpha^*} = (X_1, X_2, \ldots, X_{\mu-2}, X_{\mu-1}^*, X_\mu, X_{\mu+1}, \ldots, X_\nu),
\]

where

\[
X_{\mu-1}^* = (x_1(\mu-1), \ldots, x_1(\mu-1)),
\]

and \( \dim X_{\mu-1}^* = \dim A_{\mu-1}^* = p_{\mu-1} - 1 \), and if we define

\[
\gamma = (A_1, A_2, \ldots, A_{\mu-2}, A_{\mu-1}^*, A_\mu, \ldots, A_\nu),
\]

\[
x_\gamma = (X_1, X_2, \ldots, X_{\mu-2}, X_{\mu-1}^*, X_\mu, \ldots, X_\nu),
\]

\[
\gamma^* = (A_1, A_2, \ldots, A_{\mu-2}, A_{\mu-1}^*, A_\mu^*, \ldots, A_\nu),
\]

\[
x_{\gamma^*} = (X_1, X_2, \ldots, X_{\mu-2}, X_{\mu-1}^*, X_\mu^*, \ldots, X_\nu),
\]

we again obtain
\[
\frac{d}{dt} \left[ \frac{D_\alpha(\phi, y; x_\alpha)}{D_\gamma(\phi; x_\gamma)} \right] = \frac{D_{\gamma*}(\phi; x_{\gamma*}) \, D_{\alpha*}(\phi, y; x_{\alpha*})}{D^2(\phi; x_\gamma)}
\]
which gives us the conclusion exactly as in Case 3.


One of the main purposes of this section will be to obtain a useful sufficient condition that a system of functions \( \phi \) be completely regular.

**Theorem 12.** Let \( \phi, \ldots, \phi_n \in C^n \) and let

\[ W(\phi, \ldots, \phi_k) > 0, \]

\( k = 1, \ldots, n \). Then the system of functions \( \phi \) is completely regular.

**Proof.** We have adopted the usual convention that the Wronskian of a single function is the function itself. The proof of the theorem is by finite induction. Since \( \phi_1 > 0 \) and since

\[ W(\phi_1, \phi_2) > 0, \]

the subsystem \( \phi_1, \phi_2 \) is completely regular. But the induction step is an immediate consequence of Theorem 12. The theorem therefore holds.

**Corollary.** If the requirements of the theorem hold, each of the subsystems \( \phi_1, \ldots, \phi_k, \ k = 2, \ldots, n \), is completely regular.

It follows from Theorem 12 that the system \( \phi_k = x^{k-1}, \ k = 1, \ldots, n \), is completely regular. Functions convex with respect to this system were introduced by E. Hopf [13] and developed by T. Popoviciu [14].
Let
\[ \Delta^0 = 1, \quad \Delta^k = \Delta(\Delta^{k-1}), \quad k = 1, 2, \ldots, \]
where
\[ \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \]
is the Laplace operator in polar coordinates in \( \mathbb{E}^2 \).

A function \( u(r, \phi) \) which satisfies
\[ \Delta^n u(r, \phi) = 0 \]
is said to be polyharmonic of order \( n \). Polyharmonic functions were introduced by E. Almansi [1] and developed by M. Nicolesco [16].

It is easily seen that the \( n \)-th iterated Laplace operator in \( \mathbb{E}^2 \) may be written in the form
\[ \Delta^n = \frac{1}{r^{2n}} \sum_{k=0}^{n-1} \left[ (\Delta_r - 2k)^2 + \Delta_\phi^2 \right] \]
where
\[ \Delta_r = \frac{\partial}{\partial \log r}, \quad \Delta_\phi = \frac{\partial}{\partial \phi}. \]

To obtain the solutions of \( \Delta^n u = 0 \) which are independent of \( \phi \), we must solve the ordinary differential equation
\[ (58) \quad \left[ \sum_{k=0}^{n-1} (\Delta_r - 2k)^2 \right] \phi(r) = 0, \]
for which we have the basis
\[ (59) \quad \begin{cases} 
\phi_{2k-1} = r^{2(k-1)} \\
\phi_{2k} = r^{2(k-1)} \log r,
\end{cases} \]
\[ k = 1 \ldots, n. \]
We claim that for the functions $\phi$ defined by (59)

\[
\begin{align*}
W(\phi_1, \ldots, \phi_{2m}) &= \frac{4(m-1)m}{r^m} \sum_{k=0}^{m-1} \binom{m-1}{k} (k!)^4, \\
W(\phi_1, \ldots, \phi_{2m+1}) &= \frac{4m^2}{r^m(m!)^2} \sum_{k=0}^{m-1} \binom{m-1}{k} (k!)^4.
\end{align*}
\]

(60)

Direct computation shows that (60) holds for $m = 1$.

Since the functions $\phi_1, \ldots, \phi_{2m}$ satisfy

\[
\frac{1}{r^{2m}} \left[ \sum_{k=0}^{m-1} (\Delta_r - 2k)^2 \right] \phi(r) = 0,
\]

it follows that

\[
W(\phi_1, \ldots, \phi_{2m+1}) = W(\phi_1, \ldots, \phi_{2m}) \frac{1}{r^{2m}} \left[ \sum_{k=0}^{m-1} (\Delta_r - 2k)^2 \right] \phi_{2m+1}
\]

\[
= 4^m(m!)^2 W(\phi_1, \ldots, \phi_{2m})
\]

Similarly,

\[
W(\phi_1, \ldots, \phi_{2m+2}) = W(\phi_1, \ldots, \phi_{2m+1}) \times
\]

\[
\times \frac{1}{r^{2m+1}} \left[ \sum_{k=0}^{m-1} (\Delta_r - 2k)^2 \right] (\Delta_r - 2m) \phi_{2m+2}
\]

\[
= \frac{1}{r} 4^m(m!)^2 W(\phi_1, \ldots, \phi_{2m+1})
\]

Hence,
\[
\left\{ \begin{array}{l}
W(\varphi_1, \ldots, \varphi_{2m+2}) = \frac{4^{2m}}{r} (m!)^4 W(\varphi_1, \ldots, \varphi_{2m}) , \\
W(\varphi_1, \ldots, \varphi_{2m+3}) = \frac{4^{2m+1}}{r(m+1)^2} (m!)^4 W(\varphi_1, \ldots, \varphi_{2m+1}) , 
\end{array} \right.
\]

(61)

from which the induction step for (60) follows easily.

Since (60) holds, it follows from Theorem 12 and its corollary that the system \( \Phi \) given by (59) is completely regular along with all its subsystems of the form \( \varphi_1, \ldots, \varphi_m \). One can therefore consider functions convex with respect to the system \( \Phi \) for \( r > 0 \).

Let \( M(r) \in C^{2n} \) for \( 0 < a < r < b \). It follows easily from Theorem 10 that \( (\varphi, M(r)) \) if and only if

\[
\left[ \prod_{k=0}^{n-1} (\Delta_r - 2k)^2 \right] M(r) \geq 0 .
\]

(62)

Let \( u(r, \Theta) \in C^{2n} \) in the annulus \( 0 < a < r < b \) and let

\[
\Delta^m u(r, \Theta) \geq 0
\]

(63)

there. If we also let

\[
M(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r, \Theta) \, d\Theta ,
\]

then \( M(r) \in C^{2n} \) for \( 0 < a < r < b \). Moreover, it follows from (63) that (62) holds and therefore \( (\varphi, M(r)) \).


