INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps.
NOTE TO USERS

This reproduction is the best copy available.

UMI
RICE UNIVERSITY

DIFFUSION IN PULSATING FLOW IN A CONDUIT

by

Estrella B. Fagela-Alabastro

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY
IN CHEMICAL ENGINEERING

Thesis Director's signature:

[Signature]

Houston, Texas

April 1967
TO MY MOTHER
ACKNOWLEDGMENTS

I am deeply indebted to the following people and organizations without whose assistance this paper would never have been written.

Dr. J. D. Hellums, for his able guidance of this research and for his invaluable assistance in the preparation of this manuscript.

Drs. D. C. Dyson and J. D. Ingram, for serving as members of the oral examination committee.

My husband, for reasons that are too numerous and personal to detail.

My mother, for the encouragement and support which she so willingly gave.

Miss C. G. Alabastro, for the care and affection which she lavished on my daughter while I was in school.

Mr. Z. M. Bartolome, for having persuaded me into pursuing a career in chemical engineering.

The National Science Foundation and the Public Health Service through the Artificial Heart Project (PHS Grant HE 09251), for the financial support of this study.
TABLE OF CONTENTS

Page

LIST OF FIGURES .................................................... vi
LIST OF TABLES .................................................... vii

I. INTRODUCTION .................................................... 1

II. BACKGROUND INFORMATION ..................................... 4
    A. Prior Work .................................................... 4
    B. Review of the Concepts of Convective Diffusion ............ 12

III. LAMINAR FLOW IN A RIGID CONDUIT WITH A
     PERIODIC PRESSURE GRADIENT ................................ 16
    A. Velocity Distribution ....................................... 16
    B. Concentration Distribution .................................. 20
       1. Zeroth Order Term ......................................... 22
       2. First Order Term .......................................... 24
          a. Region of High Frequency ............................... 25
          b. Region of Low Frequency ............................... 34
       3. Second Order Term ......................................... 53
          a. Time-independent Term ................................. 54
             (1.) Region of High Frequency ......................... 54
             (2.) Region of Low Frequency ......................... 60
          b. Transient Term ......................................... 62
             (1.) Region of High Frequency ......................... 62
             (2.) Region of Low Frequency ......................... 68
    C. Parameters Used in Calculations ........................... 82
    D. Numerical Solution ......................................... 84
    E. Discussion of Results ...................................... 93
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>IV</td>
<td>LAMINAR FLOW IN A DISTENSIBLE CONDUIT WITH A PROGRESSING PULSE WAVE</td>
<td>101</td>
</tr>
<tr>
<td>A</td>
<td>Velocity Distribution</td>
<td>101</td>
</tr>
<tr>
<td>B</td>
<td>Concentration Distribution</td>
<td>107</td>
</tr>
<tr>
<td>1</td>
<td>First Order Term</td>
<td>108</td>
</tr>
<tr>
<td>2</td>
<td>Second Order Term</td>
<td>114</td>
</tr>
<tr>
<td>a</td>
<td>Steady Component</td>
<td>115</td>
</tr>
<tr>
<td>b</td>
<td>Transient Component</td>
<td>117</td>
</tr>
<tr>
<td>D</td>
<td>Discussion of Results</td>
<td>125</td>
</tr>
<tr>
<td>V</td>
<td>CONCLUSIONS</td>
<td>127</td>
</tr>
<tr>
<td></td>
<td>NOMENCLATURE</td>
<td>129</td>
</tr>
<tr>
<td></td>
<td>REFERENCES</td>
<td>132</td>
</tr>
<tr>
<td></td>
<td>APPENDICES</td>
<td>134</td>
</tr>
<tr>
<td>A</td>
<td>The Inverse of the Product</td>
<td>A-1</td>
</tr>
<tr>
<td></td>
<td>[ \text{Ai}\left(\frac{4U_0\rho s}{\rho D}y^{1/3}\right) L(G(x)) ]</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>Solution of a Differential Equation Containing a Large Parameter</td>
<td>B-1</td>
</tr>
<tr>
<td>C</td>
<td>Tabulated Results for the Case of Flow in a Rigid Conduit</td>
<td>C-1</td>
</tr>
<tr>
<td>D</td>
<td>Tabulated Results for the Case of Flow in a Distensible Conduit</td>
<td>D-1</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

FLOW IN A RIGID CONDUIT

1. Frequency Dependence of the Phase of Nu, First Harmonic
2. Frequency Dependence of the Amplitude of Nu, First Harmonic
3. Frequency Dependence of the Phase of Nu, Second Harmonic
4. Frequency Dependence of the Amplitude of Nu, Second Harmonic
5. Frequency Dependence of the Increase in Nu over the Steady Flow Value
6. The Effect of Axial Position on the Increase in Nu

FLOW IN A DISTENSIBLE CONDUIT

7. Frequency Dependence of the Phase of Nu, First Harmonic
8. Frequency Dependence of the Amplitude of Nu, First Harmonic
9. Frequency Dependence of the Phase of Nu, Second Harmonic
10. Frequency Dependence of the Amplitude of Nu, Second Harmonic
11. Frequency Dependence of the Increase in Nu over the Steady Flow Value
<table>
<thead>
<tr>
<th>Table Number</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Auxiliary Data for High Frequency Solution of ( \text{Nu}_1 ) at ( \xi = 0.20 \times 10^{-2} )</td>
<td>C-2</td>
</tr>
<tr>
<td>2</td>
<td>Phase and Amplitude of ( \text{Nu}_1 ) at ( \xi = 0.20 \times 10^{-2} ), High Frequency Solution</td>
<td>C-3</td>
</tr>
<tr>
<td>3</td>
<td>Phase and Amplitude of ( \text{Nu}_1 ) at ( \xi = 0.20 \times 10^{-2} ), Low Frequency Solution</td>
<td>C-4</td>
</tr>
<tr>
<td>4</td>
<td>Phase and Amplitude of ( \text{Nu}_2 ) at ( \xi = 0.20 \times 10^{-2} ), High Frequency Solution</td>
<td>C-5</td>
</tr>
<tr>
<td>5</td>
<td>Phase and Amplitude of ( \text{Nu}_2 ) at ( \xi = 0.20 \times 10^{-2} ), Low Frequency Solution</td>
<td>C-6</td>
</tr>
<tr>
<td>6</td>
<td>Increase in ( \bar{\text{Nu}} ) over the Steady Flow Value, High Frequency Solution</td>
<td>C-7</td>
</tr>
<tr>
<td></td>
<td>A. at ( \xi = 0.20 \times 10^{-3} )</td>
<td>C-8</td>
</tr>
<tr>
<td></td>
<td>B. at ( \xi = 0.10 \times 10^{-2} )</td>
<td>C-9</td>
</tr>
<tr>
<td></td>
<td>C. at ( \xi = 0.20 \times 10^{-2} )</td>
<td>C-10</td>
</tr>
<tr>
<td></td>
<td>D. at ( \xi = 0.20 \times 10^{-1} )</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Increase in ( \bar{\text{Nu}} ) over the Steady Flow Value, Low Frequency Solution for all ( \xi )</td>
<td>C-11</td>
</tr>
<tr>
<td>8</td>
<td>Numerical Solution at ( \xi = 0.20 \times 10^{-2} )</td>
<td>C-12</td>
</tr>
<tr>
<td>9</td>
<td>Increase in ( \bar{\text{Nu}} ) over the Steady Flow Value, Numerical Solution</td>
<td>C-13</td>
</tr>
<tr>
<td>10</td>
<td>Analytical Solution of ( f_1, M ) and ( f_2 ) at ( \xi = 0.20 \times 10^{-2} )</td>
<td>C-14</td>
</tr>
</tbody>
</table>
FLOW IN A DISTENSIBLE CONDUIT

<table>
<thead>
<tr>
<th>Table Number</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>Phase and Amplitude of $\text{Nu}_1$ at $\xi = 0.20 \times 10^{-2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A. $U_0/C = 10^{-4}$ and $U_0/C = 10^{-1}$</td>
<td>D-2</td>
</tr>
<tr>
<td></td>
<td>B. $U_0/C = 10^{-2}$ and $U_0/C = 10^{-1}$</td>
<td>D-3</td>
</tr>
<tr>
<td>12</td>
<td>Phase and Amplitude of $\text{Nu}_2$ at $\xi = 0.20 \times 10^{-2}$</td>
<td>D-4</td>
</tr>
<tr>
<td>13</td>
<td>Increase in $\overline{\text{Nu}}$ over the Steady Flow Value at $\xi = 0.20 \times 10^{-2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A. $U_0/C = 10^{-4}$ and $U_0/C = 10^{-1}$</td>
<td>D-6</td>
</tr>
<tr>
<td></td>
<td>B. $U_0/C = 10^{-2}$ and $U_0/C = 10^{-1}$</td>
<td>D-7</td>
</tr>
</tbody>
</table>
I. INTRODUCTION

The purpose of this study is to determine the effect of oscillating flow in a conduit in the absence of secondary flow on the rate of mass transfer to the wall, and the parameters that influence the magnitude of the change in the flux.

Flow in an artery is pulsatile in nature and it will be shown in a later section that the model discussed in this paper, that of pulsating flow in a distensible tube is applicable to such flows. Nutrition of the arterial wall depends on the rate of diffusion of luminal plasma to and through the wall. This research is of importance in understanding the functioning of the cardiovascular system. It also finds application in the study of certain disorders of the artery. For example, one school of thought on the genesis of atherosclerosis emphasizes the role of the reaction of subendothelial cells to excesses of lipids in their environment. There is a large body of clinical and experimental evidence which indicates that plasma lipids separate out in the endothelial spaces of the artery wall during the diffusion of lipoproteins and hence the diffusion of luminal plasma to and through the arterial wall is of major importance in the genesis of this disease.

Another area of interest is in the study of diffusion controlled heterogeneous reactions in tubular reactors.
The term 'heterogeneous' refers to chemical and physico-chemical reactions and transformations that take place on certain surfaces. Under this classification fall such processes as catalytic reactions, adsorption, and desorption at solid and liquid surfaces, evaporation, sublimation, and condensation. The majority of the heterogeneous reactions and especially those of industrial importance are diffusion controlled. If pulsations bring about a change in the rate of diffusion as has been postulated, then a change in the rate of the diffusion controlled reaction will result.

Two physical models are discussed in this paper. The first refers to flow in a rigid conduit with a periodic disturbance superimposed on a steady mean flow. The second case applies to flow in a distensible tube which is longitudinally constrained, and the steady flow is disturbed by a pulse wave progressing in the direction of flow. It will be shown that the second model is equivalent to the first model when the wave velocity is infinite. The method of solution is applicable to any form of periodic disturbance of small amplitude. The momentum equation is linear and a solution is found by superposition. The boundary layer equation for the concentration distribution is solved by a perturbation technique in which are retained up to the second order terms of the dimensionless amplitude of the pressure gradient fluctuation. The individual terms in the perturbation expansion of the concentration variable
are functions only of the two independent variables of position, \( n \) and \( \xi \), a frequency parameter \( \omega \) and the ratio \( U_0/C \) where \( U_0 \) is the mean flow in the conduit and \( C \) is the wave velocity.
II. BACKGROUND INFORMATION

A. PRIOR WORK

When pulsations are imposed on a flowing fluid it may normally be expected that mass or heat transfer would be changed. The pulsations alter the thickness of the boundary layer and hence the resistance. Several experimental and theoretical studies of such processes have appeared during the past ten years. The results of these studies have oftentimes been conflicting. The published works are broadly subdivided into those pertaining to (1) flow along flat plates, cylinders, and spheres; and (2) flow in conduits.

Literature belonging to the first category will be mentioned first since research in this area is the largest in volume. One of the earliest experiments was conducted by Martinelli and Boelter\textsuperscript{18} who studied the effects of vertical vibrations upon the rate of heat transfer from a horizontal tube immersed in a tube of water. The amplitude of the sinusoidal vibration ranged from 0 to 0.10 inch and the frequency range was from 0 to 40 cps. The results of their experiments showed that at low Reynolds number the coefficient of heat transfer was unaffected. This result was attributed to the dominating influence of free convection in slow flow. For sufficiently intense
vibrations, however, the coefficient of heat transfer was observed to increase by as much as 400% of its value without vibrations.

Other ensuing experimental investigations include those of Lemlich\textsuperscript{14} who determined the effect of transverse vibrations on the heat transfer rates from horizontal heated wires to air, and Tsui\textsuperscript{29} who performed experimental determinations on the effect of transverse vibrations on the heat transfer rate from a vertical heated plane surface. The latter investigator noted increases in the heat transfer coefficients up to 25% under the influence of vibrations.

Employing methods similar to Tsui's, Shine\textsuperscript{25} observed increases in the rate of heat transfer up to approximately 50% when he vibrated a vertical heated plate normal to itself over a frequency range of 11 to 315 cps and an amplitude range from 0 to 0.061 inch. The results of these tests showed that as the vibration intensity, defined as the product of amplitude and frequency, was increased, the heat transfer was unaffected until a certain critical intensity was reached; above this critical intensity the heat transfer coefficient increased appreciably. The critical intensity for the vertical heated plate was found to be equal to 0.083, where the amplitude is in feet and the frequency in cycles per second.

One of the first analytical studies in this field was made by Lighthill\textsuperscript{17} who used an approximate method to
examine the effect of small periodic oscillations superimposed on a steady flow about a solid body. He demonstrated that the maxima of the oscillations in the shearing stress at the solid surface anticipate the maxima of the velocity oscillations; this is due to the fact that the boundary layer responds faster to changes in the pressure gradient than the external flow. Although Lighthill did not consider the existence of secondary flow, he was able to demonstrate very clearly the phase shifts of the skin friction with respect to the fluctuations in the stream velocity. Fluctuations in the rate of heat transfer were also determined. However, the solution derived did not give any net change in the heat flux due to the oscillations since only linear terms were considered.

A combined theoretical and experimental approach to the effect of fluctuations on the laminar boundary layer on a flat plate was performed by Nickerson\textsuperscript{20}. His analysis predicted a negligible change in the steady rate of heat transfer from the plate surface and his experimental data corroborated this prediction. However, his data were limited to operating frequencies below 20 cps.

Kestin, Maeder, and Wang\textsuperscript{12} sought to bring to light the effects of a traveling harmonic wave superimposed on a uniform flow along a flat plate. Their analysis was confined to the region of very small frequencies. The results obtained indicated an increase in skin friction over the steady state case but a decrease in the heat transfer rate.
Recently, Mori and Tokuda\textsuperscript{19} have studied heat transfer from an isothermal cylinder vibrating in simple harmonic motion in the direction of an oncoming flow. This problem is equivalent to that of a cylinder in an oscillating stream of incompressible fluid. That is, the velocity and temperature fields for an incompressible fluid is the same in a fixed frame of reference as in a coordinate system which moves with the surface. The fluctuation of local heat transfer coefficient was measured instantaneously by an optical method and theoretically calculated by a power series expansion in both ranges of low and high frequencies. The expansions were carried out up to the first order term in $\varepsilon$ where $\varepsilon$ is the dimensionless amplitude of oscillating velocity. As in the case of Lighthill's solution, the analytical solution of Mori and Tokuda did not show any net change in the Nusselt number. The theoretical calculations of the amplitude and phase lag of the unsteady component of the Nusselt number agreed very well with the experimental results obtained by the optical method except in the immediate vicinity of the separation point.

An exact solution of the Navier-Stokes equation for the case of fluctuating flow past an infinite plane wall with constant suction was derived by Stuart\textsuperscript{27} and compared with Lighthill's approximate solution. His results suggest that generally, low and high frequency solutions should be regarded with a certain amount of reserve in the overlap region near to the critical frequency. The critical
frequency was suitably defined by Lighthill to be the fre-
quency at which the phase lead of the fluctuating component
as given by the solution for very small frequencies rises
to the final value of \( \pi/4 \), the phase lead for very large
frequencies. For this same model Stuart also derived a
solution for the energy equation for an incompressible
fluid with constant physical properties under the condition
of no heat transfer between the fluid and the wall.

An extension of Stuart’s work was done by K. Chandra-
sekhar Reddy by changing Stuart’s no slip flow conditions
to slip at the wall. Comparison of the results for slip
and no slip flow reveals that in general slip boundary
conditions exhibit a subduing influence on the response of
the skin friction and heat transfer at the wall to the main
stream fluctuations.

In a recent paper, Hsu-Chieh Yeh and Wen-Jei Yang\(^{10}\)
presented a mathematical treatment of the influence of
small, harmonic, radial vibrations of a sphere placed in a
free stream of an incompressible fluid. Theoretical re-
sults for first order perturbations of the vibrations of
the spherical surface were obtained for both high and low
frequencies.

Another analytical work by Schoenhals and Clark\(^{24}\) em-
ployed a perturbation method to obtain the velocity and
temperature oscillations in a laminar free convection
boundary layer on a plane wall vibrating transversely.
Separate solutions were derived for the limiting regions
of high and low values of the parameter $\omega \sqrt{4x}$ where $\omega$ and $x$ are, respectively, the dimensionless frequency and the distance from the leading edge. The solution in the intermediate region of $\omega \sqrt{4x}$ is then estimated by extrapolation of the solutions for the limiting regions.

Experiments were conducted by Bayley et. al.\textsuperscript{3} to determine the effect of artificially imposed flow pulsations upon the rate of heat transfer by forced convection from a flat plate. It was found that the heat transfer observations in pulsating flow could be correlated by the use of a dimensionless parameter defined as the ratio of the half pressure amplitude over the dynamic head corresponding to the average flow rate. No effect upon the heat transfer rates was found until this parameter attained a value of 12. Above this value it was found that the heat transfer rate increased as the 0.23 power of the pressure amplitude when the frequency was constant. The effect of frequency was found to be small and over the range of 10 to 100 cps the heat transfer rate varied for a given amplitude above the critical with about the (-0.1) power of the frequency.

To the second category of published works belong those pertaining to flow in liquid-liquid extractors, pipes and ducts. The effect of flow pulsations on the performance of liquid-liquid extractors has been extensively investigated\textsuperscript{21,28,31}. In general these studies indicate that the mass transfer rates increased as a result of pulsation and the improvement is most pronounced at low average flow
rates. The primary cause of the increase in the mass transfer coefficient has been pinpointed to be the large increase in contact area between the phases. This effect is of course absent in mass or heat transfer between a liquid and the surface of a conduit. Not much research has been done on this latter problem and the results of work that has been done do not appear to agree with each other. Darling\textsuperscript{6} conducted experiments to find out if heat transfer to liquids can be increased by using intermittent flow. His system consisted of water and 50\% glycerol solution in a 0.379 inch diameter heated tube. He observed no increase in heat transfer when the flow was laminar. Similar results were obtained by West and Taylor\textsuperscript{33}. They varied the intensity of the pulsation by altering the degree of damping provided by an air vessel on the pump outlet. With a pump speed of 100 rpm they obtained a 70\% increase in the film coefficient in both the heater and the cooler at optimum conditions in turbulent flow. No increases were obtained for streamline flow. Another experimental study on heat transfer in pulsating flow in a pipe was made by Havemann and Narayan Rao\textsuperscript{9}. Their set-up consisted of air flowing in a horizontal pipe of 1 inch inside diameter and 6 feet 10 inches effective heating length. The Reynolds number was varied from 5,000 to 35,000 and the frequency of pressure fluctuation from 5 to 33 cps. The experiments were repeated with different wave forms and amplitudes. It was found that the Nusselt number
changed up to about 30% in a rather uneven but defined manner under different conditions of frequency, amplitude, wave form, and Reynolds number. In general, the change was negative below a certain frequency and positive above it in the range of frequencies investigated. This critical frequency was a function of the wave form and to a lesser extent of the Reynolds number.

A related problem that of heat transfer to a fluid in a pulsating laminar flow between two parallel plates was treated analytically by Siegel and Perlmutter. The flow was assumed to be two-dimensional and the plates were maintained at a constant temperature different from that of the entering fluid. The pulsations were caused by having sinusoidal oscillations superimposed on the mean pumping pressure. Numerical results obtained demonstrate the effect of the amplitude and frequency of the pulsating pressure gradient, and the influence of the fluid Prandtl number. A comparison with the quasi-steady values was made which showed that when the frequency was sufficiently small, the transient results can be predicted from the quasi-steady theory. From this study the conclusion was made that the heat transfer is increased or decreased at a given Prandtl number and amplitude of pulsation depending on the axial location. It was noted that the curves are generally more negative near the beginning of a channel. Hence if the heated section is short, the heat transfer would be decreased slightly but for a sufficiently long duct the net
effect is found to be very small.

From known solutions to problems involving oscillations it can be stated that oscillations have two distinct effects. First, the various layers in the boundary layer acquire oscillations which experience phase shifts with respect to the forcing stimulus and to each other and whose amplitudes decrease away from the wall. The phase shifts and amplitude decays are brought out with the simplest mathematical models and are present even in cases when the problem is linearized. Thus phase shifts and amplitude decays do not depend on the existence of convective terms and are merely due to fluid viscosity. The second effect whose presence was shown by Schlichting in the case of a circular cylinder oscillating in a fluid at rest, is the appearance of a secondary, steady streaming motion which is superimposed on the oscillating field of flow. This effect is shown by mathematical analysis only if the convective terms in the equation of motion are taken into account. Its occurrence can therefore be attributed to the interplay between the inertia terms and the viscous forces in the boundary layer.

B. REVIEW OF THE CONCEPTS OF CONVECTIVE DIFFUSION

For a better understanding of the physical picture the basic concepts of the theory of convective diffusion in liquids will be briefly discussed. The transport of solute in a moving liquid is governed by two different mechanisms. First, there is molecular diffusion as a result of
concentration differences; second, solute particles are entrained by the moving liquid and are transported with it. The combination of these two processes make up what is called convective diffusion of solute in a liquid.

The behaviour of a liquid containing a diffusing solute will now be examined. By taking a mass balance on an element of liquid volume, a form of the equation for convective diffusion in two-dimensional flow is derived in Cartesian coordinates.

\[
\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right)
\]

It is important to realize what restrictions must be placed on the system in order that the above form of the diffusion equation will be valid.

(1) The concentration of the diffusing component is low so that both density and the diffusion coefficient D can be considered constant, and Fick's binary law can be applied.

(2) The solution is isothermal and diffusion due to external forces such as gravitational, magnetic or electrical fields is negligible.

(3) There are no sources or sinks in the bulk of the fluid which may occur as a result of processes which may cause the disappearance or the generation of the diffusing species. This limitation does not preclude the presence of sources and sinks on the bounding surface since the differential equation represents the material balance in the bulk of the solution.
The relationship between the convective and the diffusional transfer of matter is described by a dimensionless parameter, the Peclet number, a product of the Reynolds number and the Schmidt number. It plays the same role in convective diffusion as the Reynolds number does in fluid flow.

Liquids, in general, have high Schmidt numbers, hence the Peclet number is greater than unity even for Reynolds number as low as $10^{-2}$. For Reynolds number usually met in practice the Peclet numbers are very high, in which case, the liquid may be nominally divided into two regions; the first, a region of constant concentration far from the reaction surface and the second, a region of rapidly varying concentration in the immediate vicinity of the surface. In the first region molecular diffusion is zero and only convective diffusion exists. The second region represents a very thin liquid layer, the boundary layer, wherein the derivatives of concentration with respect to distance normal to the boundary are very large. The right hand side of the differential equation which stands for molecular diffusion becomes comparable to the left side despite the small values of the diffusion coefficient.

Since the main concern of this study is the diffusional flux to the wall, only the concentration boundary layer is of importance. By considering the relative orders of magnitude of the second order differential terms, analogous to Prandtl's approach in deriving the thermal boundary layer
equation, it can be shown that within the concentration boundary layer $\frac{\partial^2 c}{\partial y^2} \gg \frac{\partial^2 c}{\partial x^2}$. Thus the boundary layer approximation for two dimensional flow is

$$\frac{3c}{\partial G} + \frac{u^3 c}{\partial x} + \frac{v^3 c}{\partial y} = D \frac{\partial^2 c}{\partial y^2}$$
III. LAMINAR FLOW IN A RIGID CONDUIT WITH A PERIODIC PRESSURE GRADIENT

A. VELOCITY DISTRIBUTION

For the first test case, an examination will be made of fully developed laminar flow inside a rigid tube on which is superimposed a periodic fluctuation of small amplitude. The fluid inside the conduit is incompressible, of Newtonian behaviour, and with a large Schmidt number.

Consider a steady pressure gradient on which is superimposed a periodic disturbance of an arbitrary form. Any periodic function may be represented by a Fourier series provided it satisfies very general conditions of regularity, namely, the Dirichlet conditions. The series describes a periodic waveform as a sum of sinusoidal harmonic oscillations with frequencies that are integral multiples of the fundamental frequency of the wave.

\[- \frac{1}{\rho} \frac{\partial p}{\partial x} = \rho \left[ 1 + \sum_{n=1}^{\infty} \left( a_n \cos n \beta t + b_n \sin n \beta t \right) \right]\]

and can be written in the equivalent form

\[- \frac{1}{\rho} \frac{\partial p}{\partial x} = \rho \left[ 1 + \sum_{n=-\infty}^{\infty} \lambda_n e^{i n \beta t} \right]\]

In this paper the disturbance is assumed to a sinusoidal wave of frequency $\beta$ and an amplitude of $\rho \lambda$. However, the method of solution which will be presented may be used
for any kind of periodic disturbance which has a Fourier expansion. The pressure gradient is written in the complex notation with the understanding that only the real part has any physical significance.

\[
\frac{-1}{\rho} \frac{\partial p}{\partial x} = p(1 + \lambda e^{i\beta t})
\]

The continuity equation and the equation of motion with the boundary conditions for the components of the velocity are given below. In these equations as in all other equations in this paper, \( u \) shall refer to the velocity component in the axial direction and \( v \) is used to designate the velocity in the transverse direction.

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{v}{r} \frac{\partial}{\partial r} r \frac{\partial u}{\partial r}
\]

\[
\frac{\partial u}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} r u = 0
\]

\[
u (r = R) = 0
\]

\[
\frac{\partial u}{\partial r} \bigg|_{r=0} = 0
\]

Since the flow is assumed to be fully developed, \( u \) is independent of the axial distance, and therefore, there is no velocity component in the radial direction as can be verified directly from the continuity relationship. The
nonlinear terms of the equation of motion are therefore zero.

A periodic driving force, in this particular case the pressure gradient, gives rise to a periodic response of the same frequency or integral multiples of that frequency. Hence \( u(r,t) \) may be expanded in a trigonometric series which is written in the complex form

\[
(3-3) 
\quad u(r,t) = u_0(r) + \sum_{n=1}^{\infty} u_n(r)e^{i\omega nt}
\]

Substituting the assumed form of the solution into the equation of motion, collecting like terms and then equating coefficients of each periodic term to zero, the separate equations for the steady state term \( u_0 \) and for each of the periodic terms are derived.

\[
(3-4) 
\quad \frac{\nu}{r} \frac{d}{dr} r \frac{du_0}{dr} + p = 0
\]

\[
 u_0(r=R) = 0
\]

\[
 \frac{du_0}{dr} \bigg|_{r=0} = 0
\]

\[
(3-5) 
\quad i\beta u_n = \rho \lambda + \frac{\nu}{r} \frac{d}{dr} r \frac{du_n}{dr}
\]

\[
 u_n(r=R) = 0
\]

\[
 \frac{du_n}{dr} \bigg|_{r=0} = 0
\]
The solution to (3-4) is the well known Poiseuille flow

\[(3-6) \quad u_0(r) = \frac{PR^2}{\nu} \left(1 - \frac{r^2}{R^2}\right) = 2U_0 \left(1 - \frac{r^2}{R^2}\right)\]

where \(U_0\) is the mean velocity in the tube. Equation (3-5) is solved for \(n = 1\) and the solution for the first harmonic fluctuation of the velocity \(u_1\) is obtained.

\[(3-7) \quad u_1(r) = -\frac{8iU_0}{\omega^2} \lambda \left[1 - \frac{J_0(\omega R i^{1/2})}{J_0(\omega l^{1/2})}\right]\]

The term \(J_0(\omega R i^{1/2})\) is the Bessel function of order zero and \(\omega\) refers to a dimensionless parameter defined by

\[\omega = \left(\frac{PR^2}{\nu}\right)^{1/2}\]

All terms beyond the first harmonic are zero. Thus the exact solution to the flow problem is

\[(3-8) \quad u(r, t) = 2U_0 \left(1 - \frac{r^2}{R^2}\right) - \frac{8\lambda iu_0}{\omega^2} e^{ist} \lambda \left[1 - \frac{J_0(\omega R i^{1/2})}{J_0(\omega l^{1/2})}\right]\]

\[v(r, t) = 0\]

The velocity distribution is given in complex form but of course only the real part has any physical meaning.
B. CONCENTRATION DISTRIBUTION

When a liquid flows through a tube, the tube can be divided into three distinct sections.

(1) the flow inlet region
(2) the diffusion inlet region
(3) the zone of developed regions

For liquids the Schmidt number $\frac{\nu}{D}$ is usually very large with an order of magnitude of $10^3$ to $10^4$. Hence the momentum boundary layer is much thicker than the mass boundary layer and even in the region of fully developed flow the mass boundary layer still exists.

The analysis will therefore be restricted to the diffusion inlet region where rapid changes in concentration are confined to a very thin layer adjacent to the wall. In this layer, the velocity can be considered to be a linear function of the distance from the wall $y$ ($y = R-r$) with a slope equal to the actual slope at the wall.

$$u(y, t) = y \frac{3u}{\partial y} \bigg|_{y=0}$$

The derivative of $u$ with respect to $y$ at the tube wall is given by

$$\frac{\partial u}{\partial y} \bigg|_{y=0} = \frac{U_2}{R} \left( 4 + i f_n e^{i\beta t} \right)$$

Hence, near the wall the velocity profile is adequately described by the expression
\( u(y_1, t) = 4 \frac{U_0 y}{R} + \lambda \frac{U_0 y}{R} f_n e^{i \beta t} \)

where \( f_n = \frac{-8i \frac{\lambda}{\omega}}{\omega} \frac{J_1(\omega_1^{\frac{3}{2}})}{J_0(\omega_1^{\frac{3}{2}})} \)

With this simplified velocity profile, the mass boundary layer equation with the appropriate boundary conditions,

\[ \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial y^2} \]

\( c(x, y=0, t) = c_w \)
\( c(x, y=\infty, t) = c_0 \)
\( c(x=0, y, t) = c_0 \)

is solved by a perturbation method. But first, \( c \) is reduced to a dimensionless form \( \phi = (c-c_w)/(c_0-c_w) \). The diffusion equation is then written in terms of this dimensionless variable.

\[ \frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = D \frac{\partial^2 \phi}{\partial y^2} \]

\( \phi(x, y=0, t) = 0 \)
\( \phi(x, y=\infty, t) = 1 \)
\( \phi(x=0, y, t) = 1 \)

The symbol \( \phi_0 (x, y) \) denotes the solution in the absence of fluctuations. The solution to (3-11) is expressed as a perturbation of \( \phi_0 (x, y) \) with \( \lambda \) as the perturbation parameter.

\[ \phi(x, y, t) = \phi_0(x, y) + \lambda \phi_1(x, y, t) + \lambda^2 \phi_2(x, y, t) + \ldots + \lambda^N \phi_N(x, y, t) \]
In subsequent sections in this paper, $\phi_n(x,y,t)$ shall be referred to as the $n^{th}$ order term.

The above series representation of the dimensionless concentration variable can be expected to be valid provided that for some $n = k$ less than $N$

$$\lambda \phi_n(x,y,t) \ll \phi_{n-1}(x,y,t)$$

for all $n$ greater than or equal to $k$ and for any set of values of $x$, $y$ and $t$. This method therefore places some restrictions on the magnitude of $\lambda$ although no definite limits can be set at this point since the behavior of $\phi_n(x,y,t)$ is still unknown. In cases where $\phi_n(x,y,t)$ is very small, $\lambda$ may be considerably increased without violating the perturbation limitations. Besides the perturbation restriction on the magnitude of $\lambda$, an added condition is imposed. The flow oscillation must not be so large that flow reversals take place which carry fluid from within the conduit back out past $x = 0$. Otherwise, the boundary condition at $x = 0$ no longer holds.

The series expansion of $\phi(x,y,t)$ is substituted into the diffusion equation and each term in the resulting series is individually equated to zero giving rise to $(N+1)$ differential equations. The number $N$ has been arbitrarily chosen to be 2, that is, only terms up to the second order in $\lambda$ will be considered.

1. Zeroth Order Term

The steady state problem or the equation for the zeroth order term is
(3-12) $4 \frac{U_b Y}{R} \frac{\partial \phi}{\partial x} = D \frac{\partial^2 \phi}{\partial y^2}$

$\phi_0(x, y=0) = 0$

$\phi_0(x, y=\infty) = 1$

$\phi_0(x=0, y) = 1$

The preceding equation is reduced to an ordinary differential equation involving only the dimensionless variable $\eta$ by performing a similarity transformation, where $\eta$ is defined as $y \left( \frac{u_\infty}{RD_x} \right)^{1/3}$.

(3-13) $\phi_0'' + \frac{4}{3} \eta^2 \phi_0' = 0$

$\phi_0(\eta=0) = 0$

$\phi_0(\eta=1) = 1$

An exact solution can now be found for the steady state problem.

(3-14) $\phi_0 = \frac{(12)^{1/3}}{\Gamma(7/3)} \int_0^\eta \exp\left(-\frac{4}{3} \eta^3\right) d\eta$

and its contribution to the diffusional flux to the wall is

(3-15) $j_0 = (c_l - c_w) D \frac{\partial \phi_0}{\partial y} |_{y=0} = (c_l - c_w) D \frac{(12)^{1/3}}{\Gamma(7/3)} \left( \frac{u_\infty}{RD_x} \right)^{1/3}$

The thickness of the unperturbed diffusion layer is approximately
\[
(3-16) \quad \delta_0 \equiv \frac{D(c_0 - c_\infty)}{J_0} = \frac{\nu}{(1/3) \left( \frac{U_R}{RDx} \right)^{1/3} (12)^{1/3}}
\]

and the dimensionless mass flux to the wall defined as \( \text{Nu}_0 = \frac{\partial \phi}{\partial \eta} \mid_{\eta=0} \) is equal to \( \frac{(12)^{1/3}}{(1/3)} \).

2. First Order Term

The differential equation for the first order term with the pertinent boundary conditions is

\[
(3-17) \quad \frac{\partial \phi_1}{\partial t} + \frac{4U_0 y}{R} \frac{\partial \phi_1}{\partial x} + \frac{U_0 y}{R} \frac{f_n}{c} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \phi_1 = \frac{\partial}{\partial y^2} \phi_1 = \frac{\partial^2 \phi_1}{\partial y^2}
\]

The time dependence of \( \phi_1(x,y,t) \) is eliminated by expressing it as a periodic function

\[ \phi_1(x,y,t) = f_1(x,y) \text{ e } i \beta t \]

with \( f_1(x,y) \) as the first harmonic fluctuation. Substituting the above form of \( \phi_1(x,y,t) \) into (3-17) an equation is derived for \( f_1(x,y) \).

\[
(3-18) \quad \frac{\partial^2 f_1}{\partial y^2} - \frac{4U_0 y}{RD} \frac{\partial f_1}{\partial x} - \frac{i \omega^2}{R^2} f_1 = \frac{U_0 y}{RD} f_n \frac{\partial \phi_1}{\partial x}
\]

where \( \bar{\omega} = \omega \sqrt{Sc} \)

The first harmonic fluctuation is then determined for the ranges of very high frequencies and very low frequencies. Lighthill was the first to make use of high and low frequency approximations to describe the temperature and
velocity distributions in boundary layers. Several others have used the same method and have found their results to be satisfactory when compared with experimental data except for frequencies which are very close to a 'critical' frequency.

(a) Region of High Frequency

It can be shown that when the value of the frequency parameter $\bar{w}$ is very large, the dominant terms in the asymptotic solution of $f_1(x,y)$ will come from the terms in the differential equation which contain the large parameter and the highest order derivative. For details of the proof, reference may be made to Appendix B of this paper. Equation (3-18) then has a solution which is asymptotic with respect to the parameter $\bar{w}$ and this solution can be found by considering the differential equation

\[
\frac{\partial^2 f_1}{\partial y^2} - i \frac{\bar{w}^2 f_1}{R^2} = \frac{U_0 y f_1}{RD} \frac{\partial \phi_0}{\partial x} = \frac{-(12)^{1/3}}{f(1/3)} y^2 \left( \frac{U_0}{RDx} \right)^{4/3} \exp \left( \frac{4}{9} y^3 \left( \frac{U_0}{RDx} \right) \right)
\]

\[
f_1 = a_0 \exp \left( i \frac{\bar{w}y}{R} \right) + a_1 \exp \left( -i \frac{\bar{w}y}{R} \right)
\]

and a particular integral is derived by the method of variation of parameters.
\[ f_1 = \frac{(12)^{1/3}}{3\Gamma(1/3)} f_n \left( \frac{U_0}{R \Delta x} \right)^{1/3} \frac{R}{2\pi i i y} \left\{ -\exp \left( \frac{i \omega y}{R} \right) \right\} y^2 \times \]
\[ \exp \left[ -\frac{i \omega y}{R} - \frac{4}{9} y^3 \left( \frac{U_0}{R \Delta x} \right) \right] \, dy + \exp \left( -\frac{i \omega y}{R} \right) \right\} y^2 \times \]
\[ \exp \left\{ \frac{i \omega y}{R} - \frac{4}{9} y^3 \left( \frac{U_0}{R \Delta x} \right) \right\} \, dy \}
\]

which can be more conveniently written as

\[ f_1 = \frac{(12)^{1/3}}{6 \Gamma(1/3)} f_n \left( \frac{U_0}{R \Delta x} \right)^{1/3} y \exp \left( \frac{-i \omega y}{R} \right) \int_0^\infty y^2 \exp \left( -\frac{i \omega y}{R} \right) \]
\[ \left[ \left( -\frac{\omega y}{R} - \frac{4}{9} y^3 \left( \frac{U_0}{R \Delta x} \right) \right) \, dy \right] \, dy + \int_0^\infty y^2 \exp \left( -\frac{\omega y}{R} \right) \]
\[ \left( \int_0^\infty y^2 \exp \left( -\frac{\omega y}{R} \right) \right) dy \right\} + a_2 + a_1 \exp \left( -\frac{\omega y}{R} \right) \]
\[ + a_6 \exp \left( -\frac{\omega y}{R} \right) \]

However, the exponential terms outside the integral signs can be incorporated with the complementary solution. By retaining only those terms which are exponentially decaying functions of \( y \), the general solution to equation (3-19) can be written as
\[
\begin{align*}
  f_1 &= \frac{(12)^{1/3}}{6^{1/3}} \int_0^\infty \frac{U_0}{RDX} Y \exp \left( \frac{-i k y}{R} \right) \left( \int_0^\infty y^2 \exp \left( \frac{-i k y}{R} \right) - \frac{4}{9} y^3 \frac{U_0}{RDX} \right) dy \right) dy + \\
  \int_0^\infty y \exp \left( \frac{-i k y}{R} \right) \left( \int_0^\infty y^2 \exp \left( \frac{-i k y}{R} \right) - \frac{4}{9} y^3 \frac{U_0}{RDX} \right) dy \right) dy + \\
  a_1 \exp \left( \frac{-i k y}{R} \right) + a_2
\end{align*}
\]

The functions of integration \( a_1 \) and \( a_2 \) are determined by making the solution satisfy the imposed boundary conditions.

\[
\begin{align*}
  a_1 &= \frac{(12)^{1/3}}{6^{1/3}} \int_0^\infty \frac{U_0}{RDX} Y \exp \left( \frac{-i k y}{R} \right) \left( \int_0^\infty y^2 \exp \left( \frac{-i k y}{R} \right) - \frac{4}{9} y^3 \frac{U_0}{RDX} \right) dy \right) dy + \\
  \int_0^\infty y \exp \left( \frac{-i k y}{R} \right) \left( \int_0^\infty y^2 \exp \left( \frac{-i k y}{R} \right) - \frac{4}{9} y^3 \frac{U_0}{RDX} \right) dy \right) dy
\end{align*}
\]

\[
a_2 = -a_1
\]

The first harmonic fluctuation of \( \phi(x, y, t) \) is therefore given by

\[
(3-20) \quad f_1 = \frac{(12)^{1/3}}{6^{1/3}} \int_0^\infty \frac{U_0}{RDX} Y \exp \left( \frac{-i k y}{R} \right) - \frac{4}{9} y^3 \frac{U_0}{RDX} \right) dy \right) dy + \\
\int_0^\infty y \exp \left( \frac{-i k y}{R} \right) \left( \int_0^\infty y^2 \exp \left( \frac{-i k y}{R} \right) - \frac{4}{9} y^3 \frac{U_0}{RDX} \right) dy \right) dy + \\
\int_0^\infty y \exp \left( \frac{-i k y}{R} \right) \left( \int_0^\infty y^2 \exp \left( \frac{-i k y}{R} \right) - \frac{4}{9} y^3 \frac{U_0}{RDX} \right) dy \right) dy + \\
\int_0^\infty y \exp \left( \frac{-i k y}{R} \right) \left( \int_0^\infty y^2 \exp \left( \frac{-i k y}{R} \right) - \frac{4}{9} y^3 \frac{U_0}{RDX} \right) dy \right) dy
\]
A check on the above solution can be made by deriving two approximate solutions of (3-19): one for the outer region far from the solid surface and another for the inner region inside the mass boundary layer. It has been demonstrated, using a simple mathematical method devised by C. C. Lin, that for a high frequency oscillation, the effect of the oscillation is most pronounced near the wall and that it decays towards the outer edge of the boundary layer. In the outer region then the molecular diffusion effect is small and the term \( \frac{3}{3} \frac{\partial}{\partial y^2} \) can be neglected.

\[
(3-21) \quad f_1 = \frac{i(12)^{1/3}}{3!^{(1/3)}} \frac{\partial n}{\partial y} y^2 R^2 \left( \frac{U_0}{R D x} \right)^{1/3} \exp \left( -\frac{4}{9} \frac{y^3}{\left( \frac{U_0}{R D x} \right)} \right)
\]

In the inner region where \( u_1 \) changes rapidly the above approximation is no longer applicable. Instead a linear profile for the steady state concentration near the surface is assumed. Then (3-19) has the solution

\[
(3-22) \quad f_1 = \frac{-i(12)^{1/3}}{3!^{(1/3)}} \frac{\partial n}{\partial y} R^2 \left( \frac{U_0}{R D x} \right)^{1/3} \times
\]

\[
\left[ y^2 + \frac{2}{(\frac{\omega R}{\mu})^2} \right] - \frac{2 \exp \left( \frac{-\omega R y}{\mu} \right)}{(\frac{\omega R}{\mu})^2}
\]

It shall be shown presently that the solution as given by equation (3-20) converges to (3-21) for very large values of the variable \( y \) and to (3-22) at points very close to the wall. An asymptotic expansion of the integrals
with respect to the parameter \( \bar{\omega} \) is obtained by repeated integration by parts. The asymptotic expansions of the integrals contained in the definition of \( f_1(x, y) \) are given below.

Let \( I = \exp\left(\frac{\omega_1 \bar{k}_y}{R}\right) \int_0^\infty \exp\left(-\frac{\omega_1 \bar{k}_y}{R} \frac{4y^3}{9} \left(\frac{U_o}{\text{RD}x}\right)\right) dy + \exp\left(\frac{-\omega_1 \bar{k}_y}{R}\right) \frac{4}{9} \int_0^\infty y^3 \left(\frac{U_o}{\text{RD}x}\right) dy \)

Then \( \int_0^\infty dy = \frac{2}{(\frac{\omega_1 \bar{k}_y}{R})^2} \left(\frac{y^2}{2} + \frac{2}{(\frac{\omega_1 \bar{k}_y}{R})^2} \exp\left(-\frac{4y^3}{9} \left(\frac{U_o}{\text{RD}x}\right)\right) + O\left(\frac{1}{\omega_n^6}\right) \) 

An asymptotic expansion for \( f_1(x, y) \) can now be presented

\[
(3-23) \quad f_1 \approx \frac{-12^{1/3}}{3\Gamma(1/3)} \frac{1}{\omega^2} \frac{f_1}{\omega^2} \frac{R^2}{2} \left(\frac{U_o}{\text{RD}x}\right)^{1/3} \frac{1}{2} \left[y^2 \exp\left(-\frac{4y^3}{9} \left(\frac{U_o}{\text{RD}x}\right)\right) + O\left(\frac{1}{\omega_n^6}\right) \right] + 2 \exp\left(-\frac{\omega_1 \bar{k}_y}{R}\right) \frac{\omega_1 \bar{k}_y}{R} \frac{2}{(\frac{\omega_1 \bar{k}_y}{R})^2} - \frac{2}{(\frac{\omega_1 \bar{k}_y}{R})^2} + O\left(\frac{1}{\omega_n^6}\right)
\]

the first term being the dominant term in the expansion. When \( y \) is large the second and third terms are small compared to the first and hence may be neglected. Thus
equation (3-21) adequately describes the behaviour of the first harmonic fluctuation far from the surface. When y is small, however, no terms can be discarded since they are of the same order of magnitude. However, the first and second terms can be simplified by expanding \( \exp\left(-\frac{4}{9}y^3 \left(\frac{U_0}{Rdx}\right)\right) \) in a Taylor series about \( y=0 \), retaining only first term in the series and then substituting into (3-23). The equation which results from these operations is identical to equation (3-22).

The first harmonic fluctuation of the mass flux is determined from the full solution of \( f_1(x,y) \) as given by (3-20).

\[
(3-24a) \quad j_1 = D \frac{3f_1}{3y} \bigg|_{y=0} = \frac{D(12)^{1/3}}{2\Gamma(\nu/3)} f_n \left(\frac{U_0}{Rdx}\right)^{\nu/3} y^\nu \exp \\
\left(-\frac{\omega_1^2y}{R} - \frac{4y^3}{9} \left(\frac{U_0}{Rdx}\right)\right) dy
\]

which will be compared to the flux obtained from the approximate solution.

\[
(3-24b) \quad j_1^* = \frac{D(12)^{1/3}}{3\Gamma(\nu/3)} f_n \left(\frac{U_0}{Rdx}\right)^{\nu/3} \left[\frac{2R^3}{\nu^3 \bar{a}_1^{3/2}}\right]
\]

The dimensionless forms of \( j_1 \) and \( j_1^* \) which shall be denoted as \( Nu_1 \) and \( Nu_1^* \) respectively are expressed in the following relations. In these equations, \( f_n \) has been expressed in terms of the amplitudes and phase angles of
Bessel functions. \( m_1(\omega) \) and \( \theta_1(\omega) \), \( M_0(\omega) \) and \( \theta_0(\omega) \), are the amplitudes and phase angles of \( J_1(\omega y^3/2) \) and \( J_0(\omega y^3/2) \) respectively.

\[
(3-25a) \quad Nu = \frac{8 (12)^{1/3}}{3 \Gamma (1/3)} \frac{M_1(\omega)}{1^{1/2} \omega M_0(\omega)} \left( \frac{U_0}{RDX} \right) \exp \left[ \frac{\omega y^3}{2} - \theta_0(\omega) \right] \\
\int_0^\infty y^2 \exp \left( -\frac{\omega y}{R} - \frac{4}{9} y^3 \left( \frac{U_0}{RDX} \right) \right) dy
\]

\[
(3-25b) \quad Nu_1^* = \frac{8 (12)^{1/3}}{3 \Gamma (1/3)} \frac{M_1(\omega)}{\omega M_0(\omega)} \left[ \frac{2 R^3 (\frac{\omega y}{RDX})}{\omega^3} \right] \exp \left( -\frac{3 \pi}{2} + \theta_1(\omega) - \theta_0(\omega) \right)
\]

In this section reference shall be made to (3-25a) as the 'exact' solution and to (3-25b) as the 'approximate' solution of the first harmonic fluctuation of the diffusional mass flux to the wall for the region of high frequency.

The integrals contained in equation (3-25a)

\[
S_1 = \int_0^\infty y^2 \exp \left( -\frac{4}{9} y^3 \left( \frac{U_0}{RDX} \right) - \frac{\omega y}{R/2} \right) \cos \frac{\omega y}{R/2} dy
\]

\[
S_2 = \int_0^\infty y^2 \exp \left( -\frac{4}{9} y^3 \left( \frac{U_0}{RDX} \right) - \frac{\omega y}{R/2} \right) \sin \frac{\omega y}{R/2} dy
\]

were numerically evaluated by the Gauss quadrature method using 32 quadrature points and two subintervals. The numerical values of the integrals are presented in Table
1, page C-2. The values of the parameters used in the calculations are discussed in section C of this chapter.

Numerical calculations were then carried out to determine the phase and amplitude of the first harmonic fluctuation of the mass flux as given by the 'exact' solution, \( \text{Nu}_1 \).

\[
\psi = \tan^{-1}\left(\frac{S \cdot \sin \theta - S_2 \cos \theta}{\frac{S_1 \cdot \cos \theta + S_2 \cdot \sin \theta}{S_1^2 + S_2^2}}\right)
\]

\[
|\text{Nu}_1| = 8(12)^{1/3} \frac{M_\text{r}(\omega)}{M_0 \cdot \omega} \left(\frac{\mathrm{d} \omega}{\rho \cdot \mathrm{d} x}\right) \sqrt{S_1^2 + S_2^2}
\]

where \( \theta = \theta_1(\omega) - \theta_0(\omega) - \frac{3\pi}{4} \)

The results of the computations are presented in a tabular form in Table 2, page C-3. As can be seen from the table, the phase of the first harmonic fluctuation lags behind that of the pressure gradient \( -\frac{1}{\rho} \frac{\mathrm{d}P}{\mathrm{d}x} \). The phase lag approaches \( (+\pi) \) as the frequency becomes infinite and the amplitude of the fluctuation decreases to zero for very large frequencies.

At this point it is important to determine how well the 'approximate' solution agrees with the 'exact' solution. This is especially useful for problems with very complicated flow fields. For such problems it may be impossible to derive the 'exact' solution purely by analytical means although the 'approximate' solution would still be relatively simple to find.
In comparing \( j_1 \) to \( j_1^* \) it is noted that in the 'approximate' solution the term \(-\sqrt{2}R^3/\omega^3\) which is designated as \( S_1^* \) is equivalent to the term \( S_1 \) in the 'exact' solution and the term \( S_2^* = \sqrt{2}R^3/\omega^3 \) takes the place of \( S_2 \). Therefore it can be seen how well the 'approximate' solution approaches the 'exact' solution by comparing the values of \( S_1^* \) to \( S_1 \) and \( S_2^* \) to \( S_2 \). The numerical values of \( S_1^* \) and \( S_2^* \) are shown on Table 1 alongside those of \( S_1 \) and \( S_2 \). Table 1 proves that the 'approximate' solution agrees very well with the 'exact' solution for values of \( \omega \) greater than 2 but differs significantly from it for \( \omega \) much less than 2.

To complete the comparison between the 'exact' and 'approximate' solutions, the phase and amplitude of \( Nu_1^* \)

\[
\psi_1^* = \theta_1(\omega) - \theta_0(\omega) - \frac{3\pi}{2}
\]

\[|Nu_1^*| = \frac{8(12)^{1/3}}{3\Gamma(1/3)} \left( \frac{U_{RB}}{RDx} \frac{M_0(\omega)}{M_0(\omega)} \right)^{1/3} \sqrt{Sc} \left[ \frac{2R^3}{\omega^3} \right]
\]

were likewise calculated. Even without going through the results of the calculations the asymptotic values of the phase and amplitude of the fluctuations can be deduced. As may be verified from any standard table of Bessel functions, the ratio of the amplitude of the Bessel function of first order to the amplitude of the Bessel function of zero order, \( \frac{M_1(\omega)}{M_0(\omega)} \), becomes unity as the value of the argument \( \omega \) approaches infinity. Hence, the amplitude of the fluctuation
is inversely proportional to $\bar{w}$ and is zero for infinite $\bar{w}$. The angle $(\theta_1(\omega) - \theta_0(\omega))$ is $(\pi/2)$ for infinite $\omega$ and therefore the phase of the fluctuation must be asymptotic to $(-\pi)$.

The results of the computations on the phase and amplitude of $Nu_1^{*}$ are likewise shown on Table 2.

(b) Region of Low Frequency

For small frequencies a solution to equation (3-18) is sought by a perturbation technique. The equation for $f_1(x, y)$ is first made dimensionless by introducing the dimensionless variables $\eta$ and $\xi$; $\eta$ has already been defined while

$$
\xi = \frac{Dx}{U_0 R^*}.
$$

(3-28) \quad \frac{\partial^2 f}{\partial \eta^2} + \frac{4}{3} \eta \frac{\partial f}{\partial \eta} - 4n \xi \frac{\partial f}{\partial \xi} - i \bar{w}^2 \xi^{2/3} \xi' =
\frac{- (12)^{1/3}}{3 \pi (1/3)} f_n \eta^2 \exp(-\frac{4}{3} \eta^3)

The solution of $f_1(\eta, \xi)$ is then expressed as a perturbation of the quasi-steady solution $f_{10}$, the solution when the frequency of oscillation is zero.

(3-29) \quad f_1(\eta, \xi) = f_{10}(\eta, \xi) + \omega^2 f_{12}(\eta, \xi) + \omega^* f_{11}(\eta, \xi)

Here $\omega^2$ is used as the perturbation parameter. Before proceeding with the derivation of the solution, an approximate expression for the term $f_n$ in the velocity distribution must first be obtained for very small values of $\omega$. The
series expansion of the Bessel functions

\[ J_1(\omega^{1/2}) = i^{1/2} \left[ \sum_{j=0}^{\infty} \frac{(-1)^j \omega^{j+1}}{2^{j+1} (2j)! \Gamma(2j+2)} + \right. \]

\[ \left. i \sum_{j=0}^{\infty} \frac{(-1)^j \omega^{j+3}}{2^{j+3} (2j+1)! (2j+3)} \right] \]

\[ J_0(\omega^{1/2}) = \sum_{j=0}^{\infty} \frac{(-1)^j \omega^{j+1}}{2^j ((2j)!)^2} + i \sum_{j=0}^{\infty} \frac{(-1)^j \omega^{j+2}}{2^{j+2} ((2j+1)!)^2} \]

are substituted in the expression for \( f_n \), retaining only the first few terms in the series which are significant.

\[ (3-30) \quad f_n = 4 \left[ 1 + \frac{i \omega^2}{8} - \frac{\omega^3}{192} + O(\omega^6) \right] \]

\[ \frac{1}{1 + \frac{i \omega^2}{4} - \frac{\omega^3}{64} + O(\omega^6)} \]

This can be further simplified by making use of the binomial expansion of

\[ \frac{1}{1 + \frac{i \omega^2}{4} - \frac{\omega^3}{64}} = 1 - \left( \frac{i \omega^2}{4} - \frac{\omega^3}{64} \right) + \left( \frac{i \omega^2}{4} - \frac{\omega^3}{64} \right)^2 + \]

\[ O\left( \frac{i \omega^2}{4} - \frac{\omega^3}{64} \right)^3 \]

in (3-30). Thus for small values of \( \omega \), \( f_n \) can be approximated by the expression

\[ (3-31) \quad f_n \approx 4 \left[ 1 - \frac{i \omega^2}{8} - \frac{\omega^3}{48} \right] \]
To obtain the differential equations for $f_{10}$, $f_{11}$, and $f_{12}$, equations (3-28), (3-29), and (3-31) are combined. Terms of the same order in $\omega$ are then grouped together and then equated to zero, giving the following equations.

\[
(3-32) \quad \frac{3^2 f_{12}}{\partial \eta^2} + \frac{4}{3} \eta^2 \frac{\partial f_{12}}{\partial \eta} - 4 \eta \xi \frac{\partial f_{12}}{\partial \xi} = -\frac{(12) .1/3}{3! (1/3)} \cdot 4 \eta^2 \exp \left(-\frac{4}{9} \eta^3 \right)
\]

\[
(3-33) \quad \frac{3^2 f_{11}}{\partial \eta^2} + \frac{4}{3} \eta^2 \frac{\partial f_{11}}{\partial \eta} - 4 \eta \xi \frac{\partial f_{11}}{\partial \xi} = -\frac{(12) .1/3}{3! (1/3)} \cdot \frac{4}{8} \eta^2 \exp \left(-\frac{4}{9} \eta^3 \right) + 1S \xi^{2/3} f_0
\]

\[
(3-34) \quad \frac{3^2 f_{12}}{\partial \eta^2} + \frac{4}{3} \eta^2 \frac{\partial f_{12}}{\partial \eta} - 4 \eta \xi \frac{\partial f_{12}}{\partial \xi} = -\frac{(12) .1/3}{3! (1/3)} \cdot \frac{4}{8} \eta^2 \exp \left(-\frac{4}{9} \eta^3 \right) + 1S \xi^{2/3} \xi_{1/3}
\]

Solutions to equations (3-32), (3-33), and (3-34) will be sought of the form:

\[
f_{1n}(\eta, \xi) = \sum_{j=0}^{\infty} (\xi^{2/3})_j \int F_{nj}(\eta) \quad n=0,1,2
\]

with the conditions

\[
F_{nj}(\eta=0) = 0
\]

\[
F_{nj}(\eta=\infty) = 0
\]
for $j = 0, 1, 2 \ldots$.

The dimensionless variable $\xi$ is proportional to the ratio $x/H$ where $H$ is the length of the mass boundary layer region.

$$\xi = \frac{Dx}{h^2U_0} = \frac{2x}{R} \left( \frac{\nu}{2hU_0} \right) \left( \frac{D}{\nu} \right) = 2x \frac{1}{R N_{RE} S_c}$$

and since $H \leq R N_{RE} S_c$ as will be demonstrated in section C

$\xi \propto x/H$

the value of $x/H$ is always less than unity and is usually very small when $Sc$ is large. Hence, it will suffice to take only a few terms in the series and solutions for $F_{n0}$ and $F_{n1}$ are derived only for $n = 0, 1, \text{and} 2$.

Solution of $f_{10}$:

The leading term of the quasi-steady solution is obtained from the equation

$$(3-35) \quad F_{\alpha0}'' + \frac{4}{3} \eta^2 F_{\alpha0}' = \frac{-4 (12)^{1/3}}{3 \Gamma(1/3)} \eta^2 \exp(-\frac{4}{9} \eta^3)$$

The solution to this equation which satisfies the zero conditions at $\eta = 0$ and $\eta = \infty$ is

$$(3-36) \quad F_{\alpha0} = \frac{4 (12)^{1/3}}{12 \Gamma(1/3)} \left[ \frac{\eta}{\Gamma} \exp(-\frac{4}{9} \eta^3) d\eta - \frac{4}{3} \int_0^\eta \eta^3 \exp(-\frac{4}{9} \eta^3) d\eta \right]$$

The terms of higher order in $\xi$ are derived from the homogeneous equations.
(3-37) \[ F_{0j}'' + \frac{4}{3} \eta^2 F_{0j}' - \frac{8}{3} \eta j F_{0j} = 0 \quad j=1,2,3,\ldots,\infty \]

These equations and their boundary conditions are satisfied by

\[ F_{0j} = 0 \]

Hence the quasi-steady term of the first harmonic fluctuation of the concentration is given by

(3-38) \[ f_{10} = \frac{4(12)^{1/3}}{12! (1/3)} \left[ \frac{7}{9} \exp\left(-\frac{4}{9}\eta^3\right)dn - \frac{4}{3} \left(\frac{7}{9}\eta^3\right) \exp\left(-\frac{4}{9}\eta^3\right)dn \right] \]

Solution of \( f_{11} \):

The equation for \( F_{10} \) is very similar to that for \( F_{00} \).

(3-39) \[ F_{10}'' + \frac{4}{3} F_{10}' = -\frac{4(12)^{1/3}}{3! (1/3)} \left(-\frac{i}{8}\right) \eta^2 \exp\left(-\frac{4}{9}\eta^3\right) \]

The two equations, (3-35) and (3-39), differ only in their inhomogeneous terms. The inhomogeneous term of (3-39) is \((-i/8)\) that of equation (3-35). And since both \( F_{00} \) and \( F_{10} \) are subject to the same boundary conditions,

\[ F_{10} = \left(-\frac{i}{8}\right) F_{00} \]

The next term in the series solution of \( F_{11} \) is the solution of the equation

(3-40) \[ F_{11}'' + \frac{4}{3} \eta^2 F_{11}' - \frac{8}{3} \eta F_{11} = \frac{4(12)^{1/3}}{12! (1/3)} i\eta \left[ \frac{7}{9} \exp\left(-\frac{4}{9}\eta^3\right)dn - \frac{4}{3} \left(\frac{7}{9}\eta^3\right) \exp\left(-\frac{4}{9}\eta^3\right)dn \right] \]
To put (3-40) in a more convenient form, the transformation

\[ F_{i1}(\eta) = \eta^2 G_{i1}(\eta) \]

is made and the equation for \( G_{i1} \) obtained by substituting the above expression into (3-40)

\[
(3-41) \quad G_{i1}'' + \left( \frac{4}{\eta} + \frac{4\eta^2}{3} \right) G_{i1}' + \frac{2}{\eta^2} G_{i1} = \frac{4i(12)^{1/3}}{12^{1/3}} \left( \frac{\text{Sc}}{\eta^2} \right)
\]

\[
\left[ \eta \exp\left(-\frac{4}{9}\eta^3\right) d\eta - \frac{4}{3} \eta^3 \exp\left(-\frac{4}{9}\eta^3\right) d\eta \right]
\]

The solution for \( G_{i1} \) cannot be given in closed form. Instead an outer expansion (or solution for large \( \eta \)) and an inner expansion (or solution for small \( \eta \)) are derived and the two solutions joined by the method of numerical 'patching'. This method consists in joining the two expansions by forcing their values and one or more of their derivatives to agree at some arbitrary intermediate boundary. This arbitrary point was chosen to be the width of the unperturbed mass boundary layer which has a dimensionless value of \( \eta_1 = \frac{\Gamma(1/3)}{(12)^{1/3}} \).

For small \( \eta \), the method which has been adopted in solving the linear differential equation (3-41) makes use of a power series. The point \( \eta = 0 \) is a regular singular point of the equation and it can be proved that the complementary solution of the equation is an infinite series which takes one of two forms

\[
(3-42a) \quad A \sum_{n=0}^{\infty} a_n \eta^{n+\nu_1} + B \sum_{n=0}^{\infty} b_n \eta^{n+\nu_2}
\]
or

\[(3-42b) \quad (A + B \ln \eta) \sum_{n=0}^{\infty} a_n \eta^n + B \sum_{n=0}^{\infty} b_n \eta^n\]

where A and B are arbitrary constants. Further it can be proved that the infinite series which occur in the above forms converge in at least the annular region bounded by two circles centered at \( \eta = 0 \). One circle has an arbitrarily small radius and the other extends up to the singular point nearest \( \eta = 0 \). In this case the equation in question has no other singular point in the finite plane and the solution is expected to converge for all finite values of \( \eta \).

Assuming that there is a solution of the form,

\[G_{11} = \sum_{n=0}^{\infty} a_n \eta^n\]

direct substitution of the \( G_{11} \) into the left hand side of equation (3-41) yields

\[L(G_{11}) = \sum_{n=0}^{\infty} a_n [(n+\nu)(n+\nu-1) + 4(n+\nu)+2] \eta^n + \sum_{n=3}^{\infty} 4 a_{n-3}(n+\nu-3) \eta^{n-2} = 0\]

Since it may further be assumed without loss of generality that \( a_0 \) is not equal to zero, then the equation which determines \( \nu \), called the indicial equation is

\[(\nu+2)(\nu+1) = 0\]
The roots of the indicial equation are
\[ \nu_1 = -2 \]
\[ \nu_2 = -1 \]
and the difference of the roots \( s \) is 1. Because the roots differ by a positive integer it is not possible to get two linearly independent solutions of the form (3-42a), one with each choice of \( \nu \). However, if the smaller root \( \nu = -2 \) is used then a trial solution of the form
\[
\sum_{n=0}^{\infty} a_n \frac{n-2}{n!} x^n
\]
has a chance of picking up both solutions. If \( a_0 \) and \( a_s \) turn out to be both arbitrary, as is true in this case, then two independent solutions are derived with this technique. Recurrence formulae will give the other \( a_n \) in terms of \( a_0 \) and \( a_s \).

The series solution derived in the manner described above, after some manipulations, can be written as

\[
G_1 = A_0 x^{-2} \left( 1 + \sum_{n=1}^{\infty} \frac{(-4n^3)^n}{n!} \frac{(3n-5)(3(n-1)-5) \cdots (-2)}{(3n)(3n-1)(3(n-1)-1) \cdots (3 \times 2)} \right)
\]

\[
+ B_0 x^{-1} \left( 1 + \sum_{n=1}^{\infty} \frac{(-4n^3)^n}{n!} \frac{(3n-4)(3(n-1)-4) \cdots (-1)}{(3n+1)(3n)(3(n-1)+1)(3(n-1)) \cdots (4 \times 3)} \right)
\]

which has the equivalent expression
\[ G_{1,1} = A_0 \, n^{-2} \sum_{n=0}^{\infty} \frac{(-4n^3)^n}{n!} \frac{\Gamma(n-2/3)}{\Gamma(n+2/3) \Gamma(-2/3)} \]

\[ + B_0 \, n^{-1} \sum_{n=0}^{\infty} \frac{(-4n^3)^n}{n!} \frac{\Gamma(n-1/3)}{\Gamma(n+1/3) \Gamma(-1/3)} \]

To simplify the notations the following symbols are defined:

\((a)_k = a(a+1)(a+2) \ldots (a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}\)

\[ \mathcal{T}_{pq}(a_1, a_2, \ldots, a_p, l+b_1, l+b_2, \ldots, l+b_q) = \sum_{k=0}^{\infty} \frac{(a)_k (a_1)_k \cdots (a_p)_k z^k}{k! (l+b_1)_k (l+b_2)_k \cdots (l+b_q)_k} \]

The important properties of \( \mathcal{T}_{pq} \) are:

1. It is not defined if any \( b_q \) is a negative integer.
2. It terminates if any \( a_p \) is a negative integer or zero.
3. If \( \mathcal{T}_{pq} \) does not terminate it converges for all finite \( z \) if \( p < q \), converges for \( |z| < 1 \) if \( p = q+1 \), converges for \( z=1 \) if \( p = q+1 \) and \( \Re \left( \sum_{k=1}^{q+1} (l+b_k) - \sum_{k=1}^{q} a_k \right) > 0 \), and diverges for all \( z \neq 0 \) if \( p > q+1 \).

In terms of these symbols then, the complementary solution of equation (3-41) takes the very simple form:

\[ (3-43) \, G_{1,1} = A_0 \, n^{-2} \mathcal{T}_{i_1 i_1}(-\frac{2}{3}; \frac{2}{3}; -\frac{4}{9} n^3) + B_1 \, n^{-1} \mathcal{T}_{i_1 i_1}(-\frac{1}{3}; \frac{4}{3}; -\frac{4}{9} n^3) \]
A particular solution of (3-41) will now be derived. The right hand side of the equation which can be equivalently written as

$$\frac{4i(12)^{i/3}}{12\Gamma(1/3)} \frac{SC}{\eta^2} \eta \exp\left(-\frac{4}{9} \eta \right)$$

can be expanded in a power series about $\eta = 0$.

$$\frac{4i(12)^{i/3}}{12\Gamma(1/3)} \frac{SC}{\eta} \exp\left(-\frac{4}{9} \eta \right) = \frac{4i(12)^{i/3}}{12\Gamma(1/3)} \frac{SC}{\eta} \sum_{n=0}^{\infty} \frac{(-\frac{4}{9} \eta \right)^n}{n!}$$

The differential equation can then be rewritten as

$$(3-44) \quad G_{\eta\eta}'' + \left(\frac{4}{n^2} + \frac{4}{3} \eta^2\right) G_{\eta\eta} + \frac{2}{n^2} G_{\eta\eta} + \frac{4i(12)^{i/3}}{12\Gamma(1/3)} \frac{SC}{\eta} x$$

$$= \sum_{n=0}^{\infty} \frac{(-\frac{4}{9} \eta \right)^n \eta^{3n-1}}{n!}$$

and hence it would be natural to seek a solution of the form

$$(3-45) \quad G_{\eta\eta} = \sum_{n=0}^{\infty} a_n \eta^{3n}$$

where the $a_n$ and $\nu$ are yet to be determined. If the $G_{\eta\eta}$ of (3-45) is to satisfy equation (3-44), then the equation

$$\sum_{n=0}^{\infty} a_n \left[(3n+\nu)(3n+\nu-1)+4(3n+\nu)+2\right] \eta^{3n+\nu-2} + \frac{4}{3} \sum_{n=1}^{\infty} a_{n-1} (3(n-1)+\nu)$$

$$= \frac{4i(12)^{i/3}}{12\Gamma(1/3)} \frac{SC}{\eta} \sum_{n=0}^{\infty} \frac{(-\frac{4}{9} \eta \right)^n \eta^{3n-1}}{n!}$$
must hold. The lowest degree term on the left is \((v-2)\) and the lowest degree term on the right is \((-1)\). It would be logical therefore to set \(v\) to 1.

For two power series to be identical in a region on the corresponding coefficients must be equal. Then the \(a_n\) are defined in terms of the coefficients on the right hand side of (3-44). And a particular solution of (3-44) is given by

\[
G_{11} = \frac{4i(12)^{1/3}}{12\Gamma(1/3)} \text{Scn}\left\{\frac{n}{6} + \frac{1}{6} \sum_{n=1}^{\infty} \frac{(-4)^n (3n-2) (3n-1) - 2 \cdots (1) \eta^{3n+1}}{(3n+3)(3n+2)!(3n+1+3)} \right\}
\]

\[
= \sum_{n=1}^{\infty} \frac{(-4)^n n \eta^{3n+1}}{n! (3n+3)(3n+2)} + \sum_{n=2}^{\infty} \frac{n-1}{j=1} \eta^{3n+1}
\]

\[
= \frac{(-4)^n n-j}{(n-j)! (3n+3)(3n+2)} \cdot \frac{(3n-2) (3n-1) - 2 \cdots (3n-j+1) - 2}{(3n-j+3)(3n-j+2)} \cdot \frac{3n+1}{(n-j)! (3n+3)(3n+2)(3n+1+3) \cdots (3n-j+3)(3n-j+2)}
\]

which can be written in a more compact form as

\[(3-46) \quad G_{11} = \frac{4i(12)^{1/3}}{12\Gamma(1/3)} \text{scn}\left\{\frac{n}{6} + \frac{1}{6} \sum_{n=1}^{\infty} \frac{(-4)^n (\eta^{3})^n (\frac{1}{3})^n (\frac{2}{3})^n - j}{n! (n+1)! (\frac{3}{3})^n (\frac{1}{3})^{n-j}} \right\} + \]

\[
= \frac{1}{9} \sum_{n=1}^{\infty} \frac{(-4)^n n}{n(n+1)! (n+2/3)}
\]

The general solution of \(F_{11}\) is then expressed as the sum of the complementary solution (3-43) and the particular solution (3-46) of \(G_{11}\) multiplied by the factor \(\eta^2\).

\[(3-47) \quad F_{11} = A_{11} \int_{\frac{2}{3}}^{\frac{2}{3}} \left( -\frac{2}{3}; \frac{2}{3}; -\frac{4}{9} \right) + B_{11} \int_{\frac{1}{3}}^{\frac{4}{3}} \left( -\frac{1}{3}; \frac{4}{3}; -\frac{4}{9} \right)
\]

\[
+ \frac{4i(12)^{1/3}}{12\Gamma(1/3)} \text{scn}\left\{\frac{n}{6} + \frac{1}{6} \sum_{n=1}^{\infty} \frac{(-4)^n (\eta^{3})^n (\frac{1}{3})^n (\frac{2}{3})^n - j}{n! (n+1)! (\frac{3}{3})^n (\frac{1}{3})^{n-j}} \right\} + \]

\[
= \frac{1}{9} \sum_{n=1}^{\infty} \frac{(-4)^n n}{n(n+1)! (n+2/3)}
\]
Applying the zero condition at the wall, it is evident that the constant $A_1$ must be zero. The other constant $B_1$ is left undetermined for the moment.

While the infinite series (3-47) converges even for large values of $n$, it is apt to do so with discouraging slowness. For this reason, the problem of obtaining a solution which is particularly applicable for large $n$ shall be investigated. In order to get a solution for $G_{11}$ which is asymptotic with respect to the independent variable $n$, some transformation is first performed on the dependent variable.

$$G_{11}(n) = Z_1(n) \exp\left[\frac{-1}{2} \left(\frac{4}{n} + \frac{4}{3}n^2\right)dn\right]$$

A differential equation in terms of the new variable $Z_1$ is then obtained by substituting the above equation in place of $G_{11}$ in equation (3-41).

$$(3-48) \quad Z_1'' - (4n + \frac{4}{9}n^n)Z_1 = \frac{4i(12)^{1/3}Sc}{12I_{1/3}'(1/3)} n^{\exp(-\frac{4}{9}n^{3/2})}$$

The complementary solution of (3-48) will now be found. Let $q = 4n + \frac{4}{9}n^n$ and put

$$(3-49) \quad Z_1(n) = u(n) \exp (v(n))$$

where $v(n)$ is a polynomial in positive powers of $n$ and $u(n)$ is a series in inverse powers of $n$. By combining the above expression with the left hand side of (3-48) and equating
to zero, the homogeneous equation

\[(3-50) \quad (v')^2 u + 2v'u' + v''u + u'' - qu = 0\]

is obtained. From the terms of first order of magnitude

\[(v')^2 u - qu = 0\]

\(v(n)\) can be determined

\[v = \tau \int q^{\frac{1}{2}} \, dn\]

Since \(v\) is now known \(u\) can be found from the terms of next order of magnitude.

\[2v'u' + v''u = 0\]

\[u = q^{-\frac{1}{2}}\]

Thus the dominant term in the complementary solution of \(Z_1\) is

\[(3-51) \quad z \equiv \frac{\exp[\tau / q^{\frac{1}{2}} dn]}{q^{\frac{1}{2}}} = \]

\[\exp[t\left\{\frac{2n^{\frac{1}{2}}}{\sqrt{1+n^{\frac{1}{3}}} + 2\ln(\sqrt{1+n^{\frac{1}{3}}} + n^{1/2})}{3} \right\} (4n^{\frac{1}{3}} + \frac{4}{9}n^{\frac{1}{3}})^{\frac{1}{2}}\}

A particular solution of \((3-48)\) can be found easily merely by inspection.

\[(3-52) \quad Z_1 = \frac{-4i(12)^{\frac{1}{2}}}{64\Gamma(\frac{1}{2})} \text{ Sc } \exp\left(-\frac{2}{9}n^3\right)\]
Combining (3-51) and (3-52) and transforming back to the original function \( F_{11} \)

\[
F_{11}(\eta) = Z(\eta) \exp\left(-\frac{2}{9} \eta^3\right)
\]

a general solution for \( F_{11} \) which is asymptotic with respect to \( \eta \) is derived.

\[
(3-53) \quad F_{11} = C \frac{\left(\frac{\eta^{1+\frac{1}{3}}}{3} + \frac{\eta^{3/2}}{2}\right)^2 \exp\left(\frac{2\eta^{3/2}}{3} \sqrt{1+\eta^3} - \frac{2}{9} \eta^3\right)}{(4\eta + \frac{4}{9} \eta^*)^{\frac{1}{3}}}
\]

\[
+ D \frac{\exp\left(-\frac{2\eta^{3/2}}{3} \sqrt{1+\eta^3} - \frac{2}{9} \eta^3\right)}{\left(\frac{\eta^{1+\frac{1}{3}}}{3} + \frac{\eta^{3/2}}{2}\right)^2 (4\eta + \frac{4}{9} \eta^*)^{\frac{1}{3}}}
\]

\[
- \frac{4i(12)^{1/3}}{64\Gamma(1/3)} \text{Sc} \exp\left(-\frac{4}{9} \eta^3\right)
\]

Since the condition at \( \eta = \infty \),

\[
\lim_{\eta \to \infty} F_{11}(\eta) = 0
\]

must be satisfied, the constant \( C_1 \) must vanish. The other constant \( D_1 \) and the constant \( B_1 \) in equation (3-47) are evaluated by 'patching'. Two equations are needed in order to solve for these two constants. The first equation is obtained by equating the value of \( F_{11} \) at \( \eta_1 = \frac{\Gamma(i/3)}{(12)^{1/3}} \) as given by (3-47) and the value of \( F_{11} \) from (3-53) at the same point. The second equation comes from equating their
derivatives.

Numerical calculations performed led to the following values of the constants.

\[
B_c = \frac{-0.2255}{3 \Gamma(\cdot/3)} i Sc
\]

\[
D_c = \frac{0.4764}{3 \Gamma(\cdot/3)} i Sc
\]

The solution of \( f_{11} \) can now be given in its two forms.

For small \( \eta \):

\[(3-54a) \quad f_{11} = \frac{i (12)^{\cdot/3}}{3 \Gamma(\cdot/3)} \left\{ \frac{-n \exp \left( -\frac{4n^3}{9} \right)}{8} - \frac{2255}{8} \frac{Sc \xi^2}{3} \frac{1}{\eta} \right\}
\]

\[
\sum_{j=1}^{\infty} \frac{n^j}{(n+1)!} \frac{n^j}{(n+2/3)^j}
\]

\[
+ \frac{1}{9} \sum_{n=1}^{\infty} \frac{(-4n^3)^n}{(n+1)! (n+2/3)^j}
\]

For large \( \eta \):

\[(3-54b) \quad f_{11} = \frac{i (12)^{\cdot/3}}{3 \Gamma(\cdot/3)} \left\{ \frac{-n \exp \left( -\frac{4n^3}{9} \right)}{8} + \frac{4764}{16} \frac{Sc \xi^2}{3} \right\}
\]

\[
\exp \left( \frac{-2n^3/3}{9} \right) \frac{-2n^3}{9} \left( 4n^3 + \frac{4n^3}{9} \right)^{1/2}
\]

\[
\left( \frac{\sqrt{1+\frac{n^3}{9}} + \frac{n^{3/2}}{3}}{2} \right)^2 (4n+\frac{4n^3}{9})^{1/2}
\]

\[
\frac{-Sc \xi^2/3}{16} \exp \left( -\frac{4n^3}{9} \right)
\]
Solution of $f_{12}$:

The solution of $f_{12}$ is immediately evident by comparing the equations of its components with preceding equations and their respective solutions.

\[
(3-55) \quad F_{20}'' + \frac{4}{3} \eta^2 F_{20}' = \frac{4(12)^{1/3}}{3 \Gamma(1/3)} \left( \frac{1}{48} \right) \eta^2 \exp(-\frac{4}{9} \eta^3)
\]

\[
(3-56) \quad F_{21}'' + \frac{4}{3} \eta^2 F_{21}' = \frac{8\eta F_{21}}{8 \times 3 \Gamma(1/3)} \left( \frac{12}{3 \times 48 \Gamma(1/3)} \right) \eta \exp(-\frac{4}{9} \eta^3)
\]

Equation (3-55) is identical to equation (3-35) except for the factor $(-\frac{1}{48})$ on the right hand side of (3-55). Hence its solution must be proportional to $F_{00}$.

\[
(3-57) \quad F_{20} = \frac{-F_{00}}{48} = \frac{-(12)^{1/3}}{3 \times 48 \Gamma(1/3)} \eta \exp(-\frac{4}{9} \eta^3)
\]

By making a similar comparison between equations (3-56) and (3-35) the conclusion is formed that equation (3-56) has the solution

For small $\eta$:

\[
(3-58a) \quad F_{21} = \frac{(12)^{1/3}}{8 \times 3 \Gamma(1/3)} \{ -0.2255 \ \text{Sc} \ \eta \ \tau_{1/3} \ (-1, 4/3, -4/9 \eta^3) \\
+ \text{Sc} \eta \ (\frac{1}{6} + \frac{1}{6} \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{(-4/9 \eta^3)^n (1/3) n (2/3) n-j}{(n+1)!} (\frac{2}{3})^{n-j} n-j) \\
+ \frac{1}{9} \sum_{n=1}^{\infty} (-\frac{4}{9} \eta^3)^n (n+\frac{2}{3}) \frac{1}{(n+1)! (n+\frac{2}{3})} \}
\]
For large \( n \):

\[
(3-58b) \quad F_{2,1} = \frac{(12)^{1/3}}{8 \times 3^{1/3}(1/3)} \left\{ 0.4764 \, \text{Sc} \, \exp \left( \frac{-2n^{3/2}\sqrt{1+n^{3/9}}}{3} - \frac{2n^{3}}{9} \right) \frac{\sqrt{1+n^{3/9}+n^{1/2}}}{3} \left( \frac{4}{(4n+9n^{4})^{1/4}} \right) \right\} - \frac{\text{Sc}}{16} \exp \left( \frac{-4}{9}n^{3} \right) \}
\]

The solution of \( f_{12} \) can therefore be written as

For small \( n \):

\[
(3-59a) \quad f_{12} = \frac{(12)^{1/3}}{3^{1/3}(1/3)} \left\{ - \frac{n \exp \left( \frac{-4}{9}n^{3} \right)}{48} - \frac{0.2255}{8} \, \xi^{2/3}n \right\} + \frac{\text{Sc} \xi^{2/3}n^{3}}{8} \left\{ \frac{1}{6} + \frac{1}{6} \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{\left( -\frac{4}{9}n^{3} \right)^{n} \left( \frac{1}{3}n \left( \frac{2}{3} \right)^{n-j} \right)}{(n+1)! \left( n+\frac{2}{3} \right)} \right\}
\]

For large \( n \):

\[
(3-59b) \quad f_{2,2} = \frac{(12)^{1/3}}{3^{1/3}(1/3)} \left\{ -n \exp \left( \frac{-4}{9}n^{3} \right) \right\} + \frac{\text{Sc} \xi^{2/3}}{8} \left\{ 0.4764 \, \exp \left( \frac{-2n^{3/2}\sqrt{1+n^{3/9}}}{3} - \frac{2n^{3}}{9} \right) \frac{\sqrt{1+n^{3/9}+n^{1/2}}}{3} \left( \frac{4}{(4n+9n^{4})^{1/4}} \right) \right\} - \frac{\text{Sc}}{16} \exp \left( \frac{-4}{9}n^{3} \right) \}
\]
The solutions for \( f_{10}, f_{11} \) and \( f_{12} \) have all been found and thus an approximation for the first harmonic fluctuation which is valid for small values of the frequency of oscillation can be presented.

For small \( \eta \):

\[
(3-60a) \quad f_1 = \frac{(12)^{1/3}}{3!(\cdot)}(\eta \exp(-\frac{4}{9}\eta^3) + \omega^2 i \eta \exp(-\frac{4}{9}\eta^3))
\]

\[-\cdot2255 \cdot \text{Sc} \xi^2/\eta \left\{ \frac{1}{\frac{1}{3}}; \frac{4}{3}; -\frac{4}{9}\eta^3 \right\}
\]

\[+ \text{Sc} \xi^2/\eta^3 \left( \frac{1}{6} + \frac{1}{6} \sum_{n=1}^{\infty} \sum_{j=1}^{n+1} \left( \frac{4}{9}\eta^3 \right)^n \frac{n(\frac{1}{3})}{n(\frac{2}{3})} \frac{n-j}{(n+1)!(\frac{5}{3})^n(\frac{1}{3})^n-j} \right)
\]

\[+ \frac{\eta \exp(-\frac{4}{9}\eta^3)}{48} - \cdot2255 \cdot \text{Sc} \xi^2/\eta \left\{ \frac{1}{\frac{1}{3}}; \frac{4}{3}; -\frac{4}{9}\eta^3 \right\}
\]

\[+ \frac{\text{Sc} \xi^2/\eta^3}{8} \left( \frac{1}{6} + \frac{1}{6} \sum_{n=1}^{\infty} \sum_{j=1}^{n+1} \left( \frac{4}{9}\eta^3 \right)^n \frac{n(\frac{1}{3})}{n(\frac{2}{3})} \frac{n-j}{(n+1)!(\frac{5}{3})^n(\frac{1}{3})^n-j} \right)
\]

\[+ \frac{1}{9} \sum_{n=1}^{\infty} \left( \frac{\eta \exp(-\frac{4}{9}\eta^3)}{n(\frac{2}{3})} \right) \left\{ \frac{1}{\frac{1}{3}}; \frac{4}{3}; -\frac{4}{9}\eta^3 \right\}
\]

For large \( \eta \):
\[(3-60b) \quad f_i = \frac{(12)^{1/3}}{3\Gamma(1/3)} \left( \eta \exp \left( -\frac{4}{9} \eta \right) + \omega^4 \exp \left( -\frac{4}{9} \eta \right) \right) + \frac{\csc^{2/3}}{16} \left( -\exp \left( -\frac{4}{9} \eta \right) \right) + \frac{4.764 \exp \left( -\frac{2}{3} \sqrt{1+\frac{3}{3} \eta^2} \right) - \frac{2}{9} \eta \right)}{\left( 1+\frac{3}{3} \eta^2 \right)^2 (4\eta + \frac{4}{9} \eta^3)^{1/2}} \]

\[+ \omega^4 \left[ \exp \left( -\frac{4}{9} \eta \right) + \frac{4.764 \exp \left( -\frac{2}{3} \sqrt{1+\frac{3}{3} \eta^2} \right) - \frac{2}{9} \eta \right)}{8 \left( 1+\frac{3}{3} \eta^2 \right)^2 (4\eta + \frac{4}{9} \eta^3)^{1/2}} \right] \}

From the first form of the solution the dimensionless mass flux at the wall is determined.

\[(3-61) \quad \frac{3f_{i1}}{\eta} \bigg|_{\eta=0} = \text{Nu}_i \]

\[= \frac{(12)^{1/3}}{3\Gamma(1/3)} \left( 1 + \omega^2 \left( -\frac{1}{8} - \frac{2255}{8} \csc^{2/3} \right) + \omega^6 \left( -\frac{1}{48} - \frac{2255}{8} \csc^{2/3} \right) \right) \]

Its phase and amplitude are

\[(3-62) \quad \psi = \tan^{-1} \left( \frac{\omega^2 \left( -\frac{1}{8} - \frac{2255}{8} \csc^{2/3} \right)}{1 + \omega^6 \left( -\frac{1}{48} - \frac{2255}{8} \csc^{2/3} \right)} \right) \]

\[|\text{Nu}_i| = \frac{(12)^{1/3}}{3\Gamma(1/3)} \left( (1 + \omega^6 \left( -\frac{1}{48} - \frac{2255}{8} \csc^{2/3} \right))^2 \right) \]

\[+ \left( -\frac{1}{8} - \frac{2255}{8} \csc^{2/3} \right)^2 \omega^6)^{1/2} \]
It is observed from Figure 1 that the phase lag of the first harmonic fluctuation approaches zero as the frequency becomes zero but increases as the frequency increases. For very small values of $\omega$ the amplitude approaches the quasi-steady value of $\frac{(12)^{1/3}}{3^{1/3}}$.

3. Second Order Term

The second order term is made up of a transient part which is a harmonic function of time and a time-independent component. An analysis of the inhomogeneous term in the differential equation for $\phi_2(x,y,t)$ will bear this out.

$$\frac{U_0 Y}{RD} \Re (f_n e^{i\beta t}) \Re \left( \frac{\partial f_n}{\partial x} \right) = U_0 Y \left( \Re (f_n) \Re \left( \frac{\partial f_n}{\partial x} \right) + \Im (f_n) \Im \left( \frac{\partial f_n}{\partial x} \right) \right)$$

$$+ \Re \{ e^{2i\beta t} (\Re (f_n) \Re \left( \frac{\partial f_n}{\partial x} \right) - \Im (f_n) \Im \left( \frac{\partial f_n}{\partial x} \right) \right)$$

$$+ i \Im (f_n) \Re \left( \frac{\partial f_n}{\partial x} \right) + i \Re (f_n) \Im \left( \frac{\partial f_n}{\partial x} \right) \right)$$

Hence $\phi_2(x,y,t)$ has the form

$$(3-63a) \quad \phi_2(x,y,t) = M(x,y) + \Re \{ f_2(x,y) e^{2i\beta t} \}$$

$$(3-63b) \quad \frac{\partial \phi_2}{\partial t} + \frac{4U_0 Y}{R} \frac{\partial \phi_2}{\partial x} + \frac{U_0 Y}{R} \Re (f_n e^{i\beta t}) \Re \left( \frac{\partial f_n}{\partial x} e^{i\beta t} \right) = \frac{D \partial^2 \phi_2}{\partial y^2}$$

By substituting (3-63a) into (3-63b) and separating the transient terms from the time-independent terms, two partial
differential equations are obtained.

\[ \frac{\partial^2 M}{\partial y^2} - \frac{4U_0y}{RD} \frac{\partial M}{\partial x} = \frac{U_0y}{2RD} \left[ \text{Re}(f_n) \text{Re}\left(\frac{\partial f_1}{\partial x}\right) + \text{Im}(f_n) \text{Im}\left(\frac{\partial f_1}{\partial x}\right) \right] \]

\[ \frac{\partial^2 f_2}{\partial y^2} - \frac{4U_0y}{RD} \frac{\partial f_2}{\partial x} - \frac{2i\omega^2 f_2}{R} = \frac{U_0y}{2RD} \frac{f_n}{\partial x} \frac{\partial f_1}{\partial x} \]

a. Steady Component

(1.) Region of High Frequency

A Laplace transformation with respect to the variable \( x \) is performed on each term of equation (3-64) giving an ordinary differential equation in \( y \) and involving the parameter \( s \).

\[ \frac{d^2 L[M]}{dy^2} - \frac{4U_0y}{RD} s L[M] = L[G] \]

The symbol \( G \) stands for

\[ G = \frac{U_0y}{2RD} \left[ \text{Re}(f_n) \text{Re}\left(\frac{\partial f_1}{\partial x}\right) + \text{Im}(f_n) \text{Im}\left(\frac{\partial f_1}{\partial x}\right) \right] \]

and the Laplace transforms \( L[M] \) and \( L[G] \) are defined as

\[ L[M] = \int_0^\infty e^{-sx} M \, dx \]
\[ L[G] = \int_0^\infty e^{-sx} G \, dx \]

Equation (3-66) is subject to the conditions
\[ [L]M = 0 \quad \text{at} \quad y=0 \]
\[ [L]M = 0 \quad \text{at} \quad y=\infty \]

The complementary solution of (3-66) is a linear combination of the Airy functions.

\[ [L]M = A \, \text{Ai}(\zeta) + B \, \text{Bi}(\zeta) \]

where

\[ \zeta = \epsilon^{2/3} y = y \left( \frac{4U_\pi s}{RD} \right)^{1/3} \]
\[ \epsilon = \frac{4U_\pi s}{RD} \]

By employing the method of variation of parameters the particular solution

\[ L[M] = \frac{\pi}{\epsilon^{2/3}} \text{Bi}(\zeta) \int_0^y \text{Ai}(\zeta) \, L[G] \, dy - \frac{\pi}{\epsilon^{2/3}} \text{Ai}(\zeta) \int_0^y \text{Bi}(\zeta) L[G] \, dy \]

is obtained. Hence the general solution is

\[ (3-67) \quad L[M] = A \, \text{Ai}(\zeta) + B \, \text{Bi}(\zeta) + \frac{\pi}{\epsilon^{2/3}} \text{Bi}(\zeta) \int_0^y \text{Ai}(\zeta) L[G] \, dy \]

\[- \frac{\pi}{\epsilon^{2/3}} \text{Ai}(\zeta) \int_0^y \text{Bi}(\zeta) L[G] \, dy \]

and the arbitrary constants A and B are determined by the zero boundary conditions.

\[ A = \frac{\text{Bi}(0)}{\text{Ai}(0)} \frac{\pi}{\epsilon^{2/3}} \int_0^\infty \text{Ai}(\zeta) \, L[G] \, dy \]
\[ B = - \frac{\pi}{\epsilon^{2/3}} \int_0^\infty \text{Ai}(\zeta) \, L[G] \, dy \]
To get the solution for $M$ the inverse of (3-67) must be obtained. However this would involve too many complicated mathematical operations. Instead an approximate solution for small $\eta$ and another for large $\eta$ are derived independently of each other. In order to determine the solution for small $\eta$ it is necessary to know the derivative of $L[M]$ at the wall.

$$\frac{dL[M]}{dy} = \frac{\pi}{\varepsilon^{2/3}} \left[ \frac{Bi(0)}{Ai(0)} \right] \frac{dAi(\zeta)}{dy} \quad y=0$$

$$-\frac{dBi(\zeta)}{dy} \bigg|_{y=0} = \int_{0}^{\infty} Ai(\xi) \ L[C] \, dy$$

The above expression can be further simplified by evaluating the Airy functions and their derivatives at $y = 0$.

$$\frac{d(Ai(\zeta))}{dy} = -\frac{\xi \varepsilon^{2/3}}{\pi^{1/3}} R_{2/3} \left( \frac{2 \xi^{3/2}}{3} \right)$$

$$\frac{dBi(\zeta)}{dy} = \frac{\xi \varepsilon^{2/3}}{\sqrt{3}} I_{-2/3} \left( \frac{2 \xi^{3/2}}{3} \right) + I_{2/3} \left( \frac{2 \xi^{1/2}}{3} \right)$$

$$Ai(0) = \frac{1}{3^{2/3} \Gamma(2/3)} \quad Bi(0) = \frac{1}{3^{1/6} \Gamma(2/3)}$$

$$\frac{dAi(\zeta)}{dy} \bigg|_{y=0} = -\frac{\varepsilon^{2/3}}{3^{0/3} \Gamma(1/3)}$$

$$\frac{dBi(\zeta)}{dy} \bigg|_{y=0} = \frac{\varepsilon^{2/3}}{3^{-1/6} \Gamma(1/3)}$$
A compact expression for \( \frac{dL[M]}{dy} \) results after the series of manipulations

\[
(3-68) \quad \frac{dL[M]}{dy} \bigg|_{y=0} = -\frac{2\pi x 3^{1/6}}{\Gamma(1/3)} \int_0^\infty Ai(\xi) L[G] dy
\]

whose inverse will be obtained by an application of the convolution theorem. The inverse of the product \( Ai(\xi)L[G] \) is given by the formula

\[
L^{-1} Ai(\xi) a G = \int_0^x L^{-1} Ai(\eta) s \rightarrow x-t G(\tau,y) dt
\]

The subscript \( s \rightarrow x-t \) indicates that the parameter \( s \) is transformed back to the variable \( (x-t) \).

Consequently, the derivative of \( M \) at the wall is

\[
\frac{dM}{dy} \bigg|_{y=0} = \frac{2\pi x 3^{1/6}}{\Gamma(1/3)} \int_0^\infty \int_0^x Ai(\xi) s \rightarrow x-t G(\tau,y) dt \, dy
\]

The details of the inversion operation are presented in detail in Appendix A. The result

\[
\frac{dM}{dy} \bigg|_{y=0} = -2(3)^{1/6} (\frac{4}{3})^{1/3} \cos \frac{\pi}{6} \int_0^\infty \int_0^x G(\tau,y) y \left( \frac{U_0}{R \cdot D} \right)^{1/3} \, \frac{d\tau}{\frac{U_0}{R \cdot D}}
\]

\[
\exp \left( \frac{4}{9} y^3 \left( \frac{U_0}{R \cdot D(x-t)} \right) \right) dt \, dy
\]

gives the time averaged second order term of the mass flux at the wall. To make the evaluation of the integral less
difficult, the function $G$ is expanded in a Taylor series about $y = 0$, and only the linear term in $y$ is retained. The expansion will be of order 3 in $y$ since the second derivative of $G$ at $y = 0$ is zero.

$$G(\tau, y) \approx \frac{U_0 y^2}{2\mathcal{R}D} \left[ \text{Re}(f_n) \text{Re}(\frac{3^2 f_{i}}{\partial \tau \partial y \mid y=0}) + \text{Im}(f_n) \text{Im}(\frac{3^2 f_{i}}{\partial \tau \partial y \mid y=0}) \right]$$

The order of integration is reversed and the integral with respect to $y$ is evaluated first.

$$\frac{3M}{\partial y \mid y=0} = \frac{1}{8} \left[ \text{Re}(f_n) \text{Re}(\frac{3 f_{i}}{\partial y \mid y=0}) + \text{Im}(f_n) \text{Im}(\frac{3 f_{i}}{\partial y \mid y=0}) \right]$$

+ $\text{Im}(f_n) \text{Im}(\frac{3 f_{i}}{\partial \tau \partial y \mid y=0}) d\tau$

The integral in $\tau$ is then very easy to find.

$$(3-69) \quad \frac{3M}{\partial y \mid y=0} = -\frac{1}{8} \left[ \text{Re}(f_n) \text{Re}(\frac{3 f_{i}}{\partial y \mid y=0}) + \text{Im}(f_n) \text{Im}(\frac{3 f_{i}}{\partial y \mid y=0}) \right]$$

$$= -\frac{8}{3} \frac{(12)^{1/3}}{\Gamma(1/3)} \frac{M_1(\omega)}{M_0(\omega) \omega^{1/3}} \left( \frac{U_0}{\mathcal{R} \mathcal{D} \chi} \right)^{1/3} \int_0^\infty y^2 \exp \left( -\frac{4}{9} \mathcal{Y} \frac{1}{\mathcal{R} \mathcal{D} \chi} \right) \cos \left( \frac{\omega y}{\sqrt{2}} \right) dy$$

The same result would have been arrived at by considering only the particular solution of the untransformed equation (3-64). For small values of $y$, $M$ can be expressed as a linear function of $y$. 
\begin{align*}
M & \approx y(-8 \frac{12}{3} \Gamma(1/3) (\frac{M_1(\omega)}{\omega M_0(\omega)})^2 \left(\frac{U_0}{RDX}\right)^4)^{1/3} \\
\int_0^\infty y^2 \exp\left(\frac{4}{9} y^3 \left(\frac{U_0}{RDX}\right)^2 \frac{\omega y}{R}\right) \frac{\omega y}{R^2} \, dy \\
\end{align*}

For points which are far from the solid surface an approximate solution can be found by dropping the second derivative of \( M \) with respect to \( y \) in equation (3-64).

\[ M = -\frac{1}{8} \left[ \text{Re}(f_n) \text{Re}(f_1) + \text{Im}(f_n) \text{Im}(f_1) \right] \]

A more explicit form of the solution is obtained by replacing \( f_1 \) in the above expression by the approximate outer expansion for \( f_1 \) in the region of high frequency.

\begin{align*}
(3-71) & \quad M = \frac{-16(12)^{1/3}}{3 \Gamma(1/3)} \left[ \frac{M_1(\omega)}{\omega M_0(\omega)} \right]^2 \left( \frac{U_0}{RDX} \right)^4 \frac{R^4}{\omega} \times \\
& \quad \left\{ \exp\left(-\frac{\omega y}{R^2}\right) \cos\left(\frac{\omega y}{R}\right) - \exp\left(-\frac{4}{9} y^3 \left(\frac{U_0}{RDX}\right)^2 \frac{\omega y}{R^2}\right) \right\} \\
\end{align*}

The time averaged mass flux to the wall is made up of the zeroth order term and the steady component of the second order term. Hence the increase (or decrease) of the mass flux over the steady state problem is proportional to \( \lambda^2 \).
\[
\bar{N_u} = \frac{(12)^{1/3}}{\Gamma(1/3)} \left( 1 - \lambda^2 \left( \frac{\theta}{3} \right) \left( \frac{U_0}{RDX} \right) \frac{M_1(\omega)}{\omega M_0(\omega)} \right)^{1/2} \times
\]

\[
\int_0^\infty y^2 \exp \left( -\frac{4y^3}{9\theta} \right) \left( \frac{U_0}{RDX} \right) \cos \left( \frac{\omega y}{R \sqrt{2}} \right) dy
\]

The \( \theta \) increase (or decrease as the case may be) in the di-
mensionless mass flux to the wall resulting from the fluctua-
tions in the flow was calculated for several values of \( \omega \)
at three values of the variable \( \xi \). The results of the com-
putations are presented in Table 6 and also in Figure 6. As

can be observed from Figure 5, there is an increase

in mass flux over the steady state case provided the fre-
quency of fluctuation is greater than a 'critical' frequen-
cy whose value depends on the value of the variable \( \xi \). For

\( \xi = 0.2 \times 10^{-2} \) the 'critical' frequency is in the vicinity

of \( \omega = 0.6 \). It can also be noted that for this value of \( \xi \)

the increase goes through a maximum at about \( \omega = 0.9 \), de-
creases and then approaches zero as the frequency becomes

infinitely large.

(2.) Region of Low Frequency

For small frequencies the inhomogeneous term of the dif-
ferential equation for \( M \) is

\[
-8 \frac{(12)^{1/3}}{3 \times 12 \Gamma(1/3)} y^2 \left( \frac{U_0}{RDX} \right)^{3/2} (1 - \frac{4\eta^3}{3}) \exp \left( -\frac{4\eta^3}{9} \right) [1 - \frac{11\omega^2}{192}] + O(\omega^6)
\]
and the differential equation (3-64) becomes

\[
\frac{\partial^2 M}{\partial y^2} - \frac{4U_0 y}{RD} \frac{\partial M}{\partial x} = -\frac{8(12)^{1/3}}{3\times 12\Gamma(1/3)} y^2 \left( \frac{U_0}{RXD} \right)^{4/3} \\
(1 - \frac{4\eta^3}{3})(1 - \frac{11\omega^*}{192}) \exp \left( -\frac{4}{9\eta^3} \right)
\]

It has the general solution

\[
M = \frac{8(12)^{1/3}}{3\Gamma(1/3)} (1 - \frac{11\omega^*}{192}) \left[ c_1 + c_2 \int_0^n \exp \left( -\frac{4}{9\eta^3} \right) d\eta \right]
\]

and application of the zero boundary conditions at \( \eta = 0 \)

and at \( \eta = \infty \) gives

\[
c_1 = 0
\]

\[
c_2 = -\frac{1}{48}
\]

The dimensionless mass flux at the wall averaged over one period can now be determined.

\[
\overline{Nu} = \frac{(12)^{1/3}}{\Gamma(1/3)} (1 - \frac{\lambda^2}{18} (1 - \frac{11\omega^*}{192}))
\]

The above expression shows that small fluctuations in the flow with a frequency that approaches zero actually cause a decrease in the mass flux; this decrease becomes smaller as the frequency rises.
b. Transient Term

(1.) Region of High Frequency

As was previously done with the equation for the first harmonic fluctuation the term involving the first partial of \( f_2 \) is thrown out in accordance with the theory of differential equations with a large parameter. The equation for \( f_2 \), equation (3-65) then becomes

\[
(3-76) \quad \frac{\partial^2 f_2}{\partial y^2} - \frac{2i\omega^2}{R^2} f_2 = \frac{U_0 Y}{2RD} \sin \frac{2(12)}{3} \left( \frac{U_0}{PD} \right)^{\gamma/3} \left( \frac{R}{2\omega i k} \right)^{Y} \exp \left( \frac{-\omega_1^2 y}{R} \right) \left\{ I_1 \, dy + \exp \left( \frac{-\omega_1^2 y}{R} \right) \left[ \int_0^Y I_2 \, dy - I_2 \, dy \right] \right\}
\]

where the symbols \( I_1 \) and \( I_2 \) stand for

\[
I_1 = \left( -\frac{4}{3} Y^2 + \frac{4}{9} Y^5 \right) \left( \frac{U_0}{RD} \right) \exp \left( \frac{-\omega_1^2 y}{R} \right) - \frac{4}{9} Y^4 \left( \frac{U_0}{RD} \right) \exp \left( \frac{-\omega_1^2 y}{R} \right)
\]

\[
I_2 = \left( -\frac{4}{3} Y^2 + \frac{4}{9} Y^5 \right) \left( \frac{U_0}{RD} \right) \exp \left( \frac{-\omega_1^2 y}{R} \right) - \frac{4}{9} Y^4 \left( \frac{U_0}{RD} \right) \exp \left( \frac{-\omega_1^2 y}{R} \right)
\]

The complementary solution of the above equation is

\[
f_2 = a_1 \exp \left( \frac{-\omega \sqrt{2} i k Y}{R} \right) + a_2 \exp \left( \frac{-\omega \sqrt{2} i k Y}{R} \right)
\]

and a particular solution is found by the method of variation of parameters.
\[ f_2 = \frac{(12)^{1/3}}{24\Gamma(1/3)} \frac{f_n^2}{\sqrt{2}} \left( \frac{U_0}{RDx} \right)^{7/3} \left( \frac{R}{\omega F_2} \right)^2 \exp\left( -\frac{\omega F_2}{R} \right) \times \]

\[ \int_0^y \exp(\frac{\omega F_2}{R} (1 + \sqrt{2})) \left( \int_0^{I_2} dy \right) dy \]

\[ + \int_0^y \exp(\frac{\omega F_2}{R} (-1 + \sqrt{2})) \left( \int_0^{I_1} dy - \int_0^{I_2} dy \right) dy \]

\[ \frac{(12)^{1/3}}{24\Gamma(1/3)} \frac{f_n^2}{\sqrt{2}} \left( \frac{U_0}{RDx} \right)^{7/3} \left( \frac{R}{\omega F_2} \right)^2 \exp\left( -\frac{\omega F_2}{R} \right) \times \]

\[ \int_0^y \exp(\frac{\omega F_2}{R} (1 - \sqrt{2})) \left( \int_0^{I_1} dy - \int_0^{I_2} dy \right) dy \]

\[ + \int_0^y \exp(\frac{\omega F_2}{R} (-1 - \sqrt{2})) \left( \int_0^{I_1} dy - \int_0^{I_2} dy \right) dy \]

and the functions \( a_1 \) and \( a_2 \) are determined by imposing the zero conditions at \( y = 0 \) and at \( y = \infty \).

\[ a_1 = 0 \]

\[ a = \frac{(12)^{1/3}}{24\Gamma(1/3)} \frac{f_n^2}{\sqrt{2}} \left( \frac{U_0}{RDx} \right)^{7/3} \left( \frac{R}{\omega F_2} \right)^2 \left( -\int_0^y \exp(\frac{\omega F_2}{R} (1 + \sqrt{2})) \times \right) \]

\[ \int_0^{I_2} dy \right) dy - \int_0^y \exp(\frac{\omega F_2}{R} (-1 + \sqrt{2})) \left( \int_0^{I_1} dy - \int_0^{I_2} dy \right) dy \]

\[ \int_0^y \exp(\frac{\omega F_2}{R} (1 - \sqrt{2})) \left( \int_0^{I_2} dy \right) dy + \int_0^y \exp(\frac{\omega F_2}{R} (-1 - \sqrt{2})) \times \]

\[ \int_0^{I_1} dy - \int_0^{I_2} dy \right) dy \]
Thus the so called 'exact' solution of (3-76) is

\[
(11-77) \quad f_2 = - \frac{(12)^{1/3}}{24\Gamma(1/3)} \frac{fn^2}{\sqrt{2}} (\frac{U_0}{RDx})^{7/3} (\frac{R}{\omega_1^2})^2 \exp(-\frac{\omega_1y\sqrt{2}y}{R}) \\
\int_0^Y y \exp\left(\frac{-\omega_1y}{R}(1+\sqrt{2})\right) \left(\int_0^Y I_2 dy\right) dy \\
+ \int_0^Y y \exp\left(\frac{-\omega_1y}{R}(-1+\sqrt{2})\right) \left(\int_0^Y I_1 dy - \int_0^Y I_2 dy\right) dy \\
- \int_0^Y y \exp\left(\frac{-\omega_1y}{R}(1-\sqrt{2})\right) \left(\int_0^Y I_2 dy\right) dy \\
- \int_0^Y y \exp\left(\frac{-\omega_1y}{R}(-1-\sqrt{2})\right) \left(\int_0^Y I_1 dy - \int_0^Y I_2 dy\right) dy \}
\]

\[
+ \frac{(12)^{1/3}}{24\Gamma(1/3)} \frac{fn^2}{\sqrt{2}} (\frac{U_0}{RDx})^{7/3} (\frac{R}{\omega_1^2})^2 \exp\left(\frac{-\omega_1y\sqrt{2}y}{R}\right) \times \\
\int_0^Y y \exp\left(\frac{-\omega_1y}{R}(1-\sqrt{2})\right) \left(\int_0^Y I_2 dy\right) dy \\
- \int_0^Y y \exp\left(\frac{-\omega_1y}{R}(-1-\sqrt{2})\right) \left(\int_0^Y I_1 dy - \int_0^Y I dy\right) dy \}
\]

For the sake of comparison the 'approximate' solution consisting of a solution for small \( n \) and another solution for large \( n \) will be presented. The procedure in deriving the 'approximate' solution is the same as that for the first harmonic fluctuation hence only the final result will be given.
For small $\eta$:

\begin{equation}
(3-78a) \quad f_2 = \frac{(12)^{1/3}}{6\pi^{1/3}} \frac{f_n^2}{(U_0/RDx)^{7/3}} \left( \frac{R^2}{2\omega^2} \right) \times 
\end{equation}

\begin{equation}
(iy^2 + \frac{R^2}{\omega^2} - \frac{R^2}{\omega^2} \exp(\frac{-\omega y^1/\sqrt{2}}{R}) \int_0^\infty I_2 dy 
\end{equation}

For large $\eta$:

\begin{equation}
(3-78b) \quad f_2 = \frac{-(12)^{1/3}}{24\pi^{1/3}} \left( \frac{R}{\omega^{1/3}} \right)^2 \frac{f_n^2}{(U_0/RDx)^{7/3}} [y \exp(\frac{-\omega y^1/\sqrt{2}}{R}) - y] \times 
\end{equation}

\begin{equation}
\int_0^\infty \exp(\frac{-\omega y^1/\sqrt{2}}{R}) (\int_0^\infty I_2 dy) dy 
\end{equation}

\begin{equation}
+ \int_0^\infty \exp(\frac{-\omega y^1/\sqrt{2}}{R}) (\int_0^\infty I_1 dy) dy 
\end{equation}

\begin{equation}
\left\{ \int_0^\infty \exp(\frac{-\omega y^1/\sqrt{2}}{R}) (\int_0^\infty I_2 dy) dy + \int_0^\infty \exp(\frac{-\omega y^1/\sqrt{2}}{R}) (\int_0^\infty I_2 dy) dy \right\} 
\end{equation}

From equations (3-77) and (3-78a) the 'exact' and 'approximate' solutions of the second harmonic fluctuation of the dimensionless mass flux are determined.
Exact solution:

\[(3-79a) \quad \text{Nu}_2 = \frac{(12)^{1/3} \sqrt{2}}{12 \Gamma(1/3)} \left[ \frac{8M_1(\omega)}{\omega M_0(\omega)} \right]^2 \frac{(U_0)_{RDX}^2 R^3}{\omega^3} \exp \{ i(2(\theta_i(\omega)) - \theta_0(\omega) + \frac{3\pi}{4} \} \times \left\{ \int_0^\infty \left( -\frac{4}{3} y^2 + \frac{4}{9} y^5 (U_0)_{RDX}^{-1} \right) \exp \left\{ -\frac{\omega i}{R} \sqrt{2} y - \frac{4}{9} y^3 (U_0)_{RDX}^{-1} \right\} dy \right\} \]

Approximate solution:

\[(3-79b) \quad \text{Nu}_2^* = \frac{(12)^{1/3} \sqrt{2}}{12 \Gamma(1/3)} \left[ \frac{8M_1(\omega)}{\omega M_0(\omega)} \right]^2 \frac{(U_0)_{RDX}^2 R^3}{\omega^3} \exp \{ i(2(\theta_i(\omega)) - \theta_0(\omega) + \frac{3\pi}{4} \} \times \left\{ \int_0^\infty \left( -\frac{4}{3} y^2 + \frac{4}{9} y^5 (U_0)_{RDX}^{-1} \right) \exp \left\{ -\frac{\omega i}{R} \sqrt{2} y \right\} dy \right\} \]

The phase and amplitude of the second harmonic fluctuation are given by

Exact solution:

\[(3-80a) \quad \psi_2 = P/Q \]

\[|\text{Nu}_2| = \frac{(12)^{1/3} \sqrt{2}}{12 \Gamma(1/3)} \left[ \frac{8M_1(\omega)}{\omega M_0(\omega)} \right]^2 \frac{(U_0)_{RDX}^2 R^3}{\omega^3} \left( P^2 + Q^2 \right)^{1/2} \]
where \( Q = \text{Re} \left\{ \exp \left[ 2\left( \theta_1(\omega) - \theta_0(\omega) \right) + \frac{3\pi}{4} \right] \times \right\} \)

\[
\left[ 4 \int_0^\infty \left( -\frac{4}{3} y^2 + \frac{4}{9} y^5 \right) \left( \frac{U_0}{R \text{DX}} \right) \exp \left( \frac{-\omega i k y}{R} - \frac{4}{9} y^3 \left( \frac{U_0}{R \text{DX}} \right) \right) dy \right.
\]

\[
-\int_0^\infty \frac{\sqrt{2} \omega i k y}{R} + 4 \left( -\frac{4}{3} y^2 + \frac{4}{9} y^5 \right) \left( \frac{U_0}{R \text{DX}} \right) \exp \left( -\frac{\omega \sqrt{2} i k y}{R} - \frac{4}{9} y^3 \left( \frac{U_0}{R \text{DX}} \right) \right) dy \right] \}
\]

and \( P = \text{Im} \left\{ \exp \left[ 2\left( \theta_1(\omega) - \theta_0(\omega) \right) + \frac{3\pi}{4} \right] \times \right\} \)

\[
\left[ 4 \int_0^\infty \left( -\frac{4}{3} y^2 + \frac{4}{9} y^5 \right) \left( \frac{U_0}{R \text{DX}} \right) \exp \left( -\frac{\omega i k y}{R} - \frac{4}{9} y^3 \left( \frac{U_0}{R \text{DX}} \right) \right) dy \right.
\]

\[
-\int_0^\infty \frac{\sqrt{2} \omega i k y}{R} + 4 \left( -\frac{4}{3} y^2 + \frac{4}{9} y^5 \right) \left( \frac{U_0}{R \text{DX}} \right) \exp \left( -\frac{\omega \sqrt{2} i k y}{R} - \frac{4}{9} y^3 \left( \frac{U_0}{R \text{DX}} \right) \right) dy \right] \}
\]

Approximate solution:

\[(3-80b) \quad \psi^*_z = \frac{S}{T}\]

\[
\text{Nu}^*_z = \frac{(12) i}{\sqrt{2}} \left( \frac{\omega M_1(\omega)}{\omega M_0(\omega)} \right)^2 \left( \frac{U_0}{R \text{DX}} \right)^2 \frac{R^3}{\omega^3} (S^2 + T^2)^{1/2}
\]

where \( S = \text{Im} \left\{ \exp \left[ 2\left( \theta_1(\omega) - \theta_0(\omega) \right) + \frac{3\pi}{4} \right] \times \right\} \)

\[
\int_0^\infty \left( -\frac{4}{3} y^2 + \frac{4}{9} y^5 \right) \left( \frac{U_0}{R \text{DX}} \right) \exp \left( -\frac{\omega i k y}{R} - \frac{4}{9} y^3 \left( \frac{U_0}{R \text{DX}} \right) \right) dy \}
\]

and \( T = \text{Re} \left\{ \exp \left[ 2\left( \theta_1(\omega) - \theta_0(\omega) \right) + \frac{3\pi}{4} \right] \times \right\} \)

\[
\left( -\frac{4}{3} y^2 + \frac{4}{9} y^5 \right) \left( \frac{U_0}{R \text{DX}} \right) \exp \left( -\frac{\omega i k y}{R} - \frac{4}{9} y^3 \left( \frac{U_0}{R \text{DX}} \right) \right) dy \} \]
Both solutions are presented in tabular form in Table 4. From the table it is observed that for a value of $\omega$ greater than 2 the phase and amplitude of the second harmonic fluctuation from the 'approximate' solution compare favorably with those of the 'exact' solution. In Figures 3 and 4 the phase and amplitude of the second harmonic fluctuation of the diffusional mass flux as defined by the 'exact' solution are drawn as functions of the frequency parameter $\omega$. It is readily seen that as the frequency becomes very large the phase approaches zero degree from the positive side and the amplitude approaches zero asymptotically.

(2.) Region of Low Frequency

The procedure is the same as that followed in obtaining the low frequency solution of the first harmonic fluctuation. The equation for $f_2$ is first transformed to the dimensionless form:

\[
\frac{\partial^2 f_2}{\partial \eta^2} + \frac{4}{3} \eta^2 \frac{\partial f_2}{\partial \eta} = 4 \eta \xi \frac{\partial f_2}{\partial \xi} - 2i \omega^2 \xi^2 / 3 f_2
\]

\[
= \frac{1}{2} f_n \left[ -\eta^2 \frac{\partial f_1}{\partial \eta} - \eta \xi \frac{\partial f_1}{\partial \xi} \right]
\]

By making use of the low frequency solutions for $f_1$ (one for small $\eta$ and another for large $\eta$) the right hand side of the equation can be expressed as
For small \( n \):

\[
\frac{1}{2} \frac{\varphi}{\varphi} \left(-\frac{n^2}{3} - \frac{\partial f}{\partial n} + \varphi \frac{\partial f}{\partial \varphi}\right) = 2 \left(1 - \frac{1}{8} \frac{\omega^2}{4^8} \right) \left(\frac{12}{31} \frac{1}{\varphi} \right)
\]

\[
\left(-\frac{n^2}{3} \exp\left(-\frac{4}{9} \varphi \frac{1}{12} \left(1 - \frac{4}{3} \varphi^3\right)\right) + \omega^2 \frac{n^2}{24} \exp\left(-\frac{4}{9} \varphi \frac{1}{12} \left(1 - \frac{4}{3} \varphi^3\right)\right)\right)
\]

\[
- 0.2255 \frac{\varphi^3}{3} \times \frac{2}{9} \frac{n^2}{\varphi} \exp\left(\frac{2}{3} \frac{4}{3} \varphi \frac{1}{12} \right)
\]

\[
+ \frac{\varphi^3}{3} \frac{n^2}{\varphi} \exp\left(\frac{2}{3} \frac{4}{3} \varphi \frac{1}{12} \right)
\]

\[
- \frac{n\varphi}{3} \exp\left(-\frac{4}{9} \varphi \frac{1}{12} \left(1 - \frac{4}{3} \varphi^3\right)\right)\right)\}
\]

\[
+ \frac{\varphi^3}{3} \frac{n^2}{\varphi} \exp\left(\frac{2}{3} \frac{4}{3} \varphi \frac{1}{12} \right)
\]

\[
- \frac{0.2255 \varphi^3}{3} \times \frac{2}{9} \frac{n^2}{\varphi} \exp\left(\frac{2}{3} \frac{4}{3} \varphi \frac{1}{12} \right)
\]

\[
+ \frac{\varphi^3}{3} \frac{n^2}{\varphi} \exp\left(\frac{2}{3} \frac{4}{3} \varphi \frac{1}{12} \right)
\]

\[
- \frac{\varphi^3}{3} \frac{n^2}{\varphi} \exp\left(\frac{2}{3} \frac{4}{3} \varphi \frac{1}{12} \right)\}
\]
For large \( n \):

\[
\frac{1}{2} \bar{f}_n \left[ \frac{n^2}{3} \frac{3}{\eta} + n \xi \frac{3}{\eta} \right] = 2 \left( 1 - \frac{i \omega^2}{8} - \frac{\omega^6}{48} \right) \frac{(12)^{1/3}}{3 \Gamma(1/3)}
\]

\[
- \frac{n^2}{3} (1 - \frac{4}{3} n^3) \exp \left( -\frac{4}{9} n^3 \right) + \omega^2 i \left( \frac{n^2}{24} (1 - \frac{4}{3} n^3) \exp \left( -\frac{4}{9} n^3 \right) \right)
\]

\[
+ \frac{4764 \, Sc \xi^2 / 3 n^2 \exp \left( -\frac{2 n^{1/2}}{3} \sqrt{1 + \frac{n^3}{9}} - \frac{2}{9} n^3 \right)}{3 \left( \sqrt{1 + \frac{n^3}{9}} + \frac{n^{1/2}}{3} \right)^2 \left( 4 n + \frac{4}{9} n^4 \right)^{1/4}} - \left( \frac{\sqrt{4 n + \frac{4}{9} n^4}}{9 n^2} - \frac{2}{3} n^2 \right)
\]

\[
\frac{(9 + \frac{12}{9} n^3)}{3 \times 16} (2 n + \frac{4}{9} n^4) \exp \left( -\frac{4}{9} n^4 \right)
\]

\[
+ \omega^6 \left( \frac{n^2}{3 \times 48} (1 - \frac{4}{3} n^3) \exp \left( -\frac{4}{9} n^3 \right) \right) + \frac{4764 \, Sc \xi^2 / 3 n^2 \exp \left( -\frac{2 n^{1/2}}{3} \sqrt{1 + \frac{n^3}{9}} - \frac{2}{9} n^3 \right)}{24 \left( \sqrt{1 + \frac{n^3}{9}} + \frac{n^{1/2}}{3} \right)^2 \left( 4 + \frac{4}{9} n^4 \right)^{1/4}} - \frac{2 n^{1/2}}{9 n^2} - \frac{(9 + \frac{12}{9} n^3)}{3 \times 16} (2 n + \frac{4}{9} n^4) \exp \left( -\frac{4}{9} n^4 \right)
\]

The quasi-steady solution \( f_{20} \) of equation (3-81) which is the same for both small and large \( n \) is derived by making \( \omega = 0 \).

\[
(3-82) \quad f_{20} = \frac{8 (12)^{1/3}}{3 \Gamma(1/3)} \int_0^n \left( \frac{1}{48} + \frac{n^6 - n^3}{54} \right) \exp \left( -\frac{4}{9} n^3 \right) d n
\]

\[
= \frac{(12)^{1/3}}{3 \Gamma(1/3)} \left( \frac{n}{6} - \frac{n^8}{9} \right) \exp \left( -\frac{4}{9} n^3 \right)
\]
It is then perturbed

\[ (3-83) \quad f_2 = f_{20} + \omega^2 f_{21} + \omega^4 f_{22} \]

and separate equations for \( f_{21} \) and \( f_{22} \) are obtained by plugging \((3-82)\) into \((3-81)\), gathering terms of the same power in \( \omega \) and equating each one to zero.

For small \( \eta \):

\[ (3-84a) \quad \frac{\partial^2 f_2}{\partial \eta^2} + \frac{4}{3} \eta^2 \frac{\partial f_2}{\partial \eta} - 4 \eta \xi \frac{\partial^2 f_2}{\partial \xi^2} = 2i \text{Sc} \xi^{2/3} f_{20} + \frac{2(12)}{3} \eta \frac{\partial f_{21}}{\partial \eta} \left\{(\frac{i}{4})\frac{\eta^2}{3} \left(1 - \frac{4}{9} \eta^3\right) \text{exp} \left(-\frac{4}{9} \eta^3\right) \right\} \]

\[ -0.225 \times \frac{2}{9} \eta^{2/3} x \left(\frac{2}{3}, \frac{4}{3}; -\frac{4}{9} \eta^3\right) + \frac{i \text{Sc} \xi^{2/3} \eta^4}{3} \times \]

\[ \left\{ \frac{1}{6} - \frac{1}{6} \right\} \sum_{n=1}^{\infty} \sum_{j=1}^{n+1} \frac{(\frac{4}{9} \eta^3)^n (\frac{2}{3}) n-j}{(n+1)! (\frac{5}{3}) n (\frac{1}{3}) n-j} - \frac{1}{3} \sum_{n=1}^{\infty} \frac{(\frac{4}{9} \eta^3)^n (n+\frac{1}{3})}{(n+1)! (n+\frac{2}{3})} \right\} \]

\[ (3-84b) \quad \frac{\partial^2 f_{22}}{\partial \eta^2} + \frac{4}{3} \eta^2 \frac{\partial f_{22}}{\partial \eta} - 4 \eta \xi \frac{\partial^2 f_{22}}{\partial \xi^2} = 2i \text{Sc} \xi^{2/3} f_{21} + \frac{2(12)}{3} \eta \frac{\partial f_{21}}{\partial \eta} \left\{(\frac{i}{64})\frac{\eta^2}{3} \left(1 - \frac{4}{9} \eta^3\right) \text{exp} \left(-\frac{4}{9} \eta^3\right) \right\} \]

\[ -0.225 \times \frac{2}{4} \text{Sc} \xi^{2/3} \eta^{2/3} x \left(\frac{2}{3}, \frac{4}{3}; -\frac{4}{9} \eta^3\right) + \frac{\text{Sc} \xi^{2/3} \eta^4}{4 \times 3} \times \]

\[ \left\{ \frac{1}{6} - \frac{1}{6} \right\} \sum_{n=1}^{\infty} \sum_{j=1}^{n+1} \frac{(\frac{4}{9} \eta^3)^n (\frac{2}{3}) n-j}{(n+1)! (\frac{5}{3}) n (\frac{1}{3}) n-j} - \frac{1}{3} \sum_{n=1}^{\infty} \frac{(\frac{4}{9} \eta^3)^n (n+\frac{1}{3})}{(n+1)! (n+\frac{2}{3})} \right\} \]
For large $\eta$:

\[
(3-85a) \quad \frac{\Delta^2 f_{22}}{\Delta \eta^2} + \frac{4}{3\eta^2} \frac{\Delta^2 f_{21}}{\Delta \eta^2} = 4\eta \frac{\Delta^2 f_{21}}{\Delta \xi^2}
\]

\[
= 2iSc\xi^{2/3}f_{20} + \frac{2(12)^{1/3}}{3\Gamma(1/3)} \left( \frac{1}{12} \eta^2 (1-\frac{4}{3} \eta^3) \exp \left( -\frac{4}{9} \eta^3 \right) \right)
\]

\[
+ \frac{.4764iSc\xi^{2/3}}{3} \eta^2 \exp \left( -\frac{2n^{3/2}}{3} \sqrt{1+\eta^3} - \frac{2}{9} \eta^3 \right) \left( \sqrt{1+\eta^3} + n^{3/2} \right)^2 \left( 4n + \frac{4}{9} \eta^3 \right)^{1/3} \left( -\sqrt{4 + \frac{4}{9} \eta^3} \right)
\]

\[
- \frac{2}{3} \eta^2 - \frac{9+12n^3}{9} \left( 2n + \frac{4}{9} \eta^3 \right) \exp \left( -\frac{4}{9} \eta^3 \right) \}
\]

\[
(3-85b) \quad \frac{\Delta^2 f_{22}}{\Delta \eta^2} + \frac{4}{3n^2} \frac{\Delta f_{22}}{\Delta \eta} - 4\eta \frac{\Delta f_{22}}{\Delta \xi}
\]

\[
= 2iSc\xi^{2/3}f_{21} + \frac{2(12)^{1/3}}{3\Gamma(1/3)} \left( \frac{n^2}{3} \left( \frac{1}{24} + \frac{1}{64} \right) (1-\frac{4}{3} \eta^3) \exp \left( -\frac{4}{9} \eta^3 \right) \right)
\]

\[
+ \frac{.4764}{12} \frac{Sc\xi^{2/3}}{3} \eta^2 \exp \left( -\frac{2n^{3/2}}{3} \sqrt{1+\eta^3} - \frac{2}{9} \eta^3 \right) \left( \sqrt{1+\eta^3} + n^{3/2} \right)^2 \left( 4n + \frac{4}{9} \eta^3 \right)^{1/3} \left( -\sqrt{4 + \frac{4}{9} \eta^3} \right) - \frac{9+12n^3}{9} \left( 2n + \frac{4}{9} \eta^3 \right) \exp \left( -\frac{4}{9} \eta^3 \right) \}
\]

The solutions to these equations are assumed to take the form

\[
f_{22} = H_{20}(\eta) + \xi^{2/3} H_{21}(\eta) + O(\xi^0)
\]

\[
f_{21} = H_{10}(\eta) + \xi^{2/3} H_{11}(\eta) + O(\xi^0)
\]
These forms of the solutions are combined with equations (3-84) and (3-85) to get

\[(3-86a) \quad H_{10}'' + \frac{4}{3} \eta^2 H_{20}' = \frac{2}{3} \frac{(12)^{1/3}}{\Gamma(1/3)} \frac{(i)}{3} \eta^2 (1-\frac{4}{9} \eta^3) \exp(-\frac{4}{9} \eta^3)\]

\[(3-86b) \quad H_{20}'' + \frac{4}{3} \eta^2 H_{20}' = \frac{2}{3} \frac{(12)^{1/3}}{\Gamma(1/3)} \frac{1}{24+64} \frac{(i)}{3} \eta^2 (1-\frac{4}{9} \eta^3) \exp(-\frac{4}{9} \eta^3)\]

For small \( \eta \):

\[(3-87a) \quad H_{10}'' + \frac{4}{3} \eta^2 H_{10}' - \frac{8}{3} \eta H_{10} = 2iScf_{20} + \frac{2(12)^{1/3}}{3\Gamma(1/3)} \]

\[-\frac{2255iSc}{9} \times \frac{2}{9} \frac{\eta^2}{\Gamma(1/3)} \left(\frac{2}{3}; \frac{4}{3}; -\frac{4}{9} \eta^3\right)\]

\[+ \frac{iScn^2}{3} \left( \begin{array}{c} \frac{1}{6} - \frac{1}{6} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{n} \frac{(\frac{4}{3} \eta^3)^n (\frac{4}{3})^n (\frac{2}{3})^n}{n-j} \end{array} \right)\]

\[= \left( \begin{array}{c} \frac{1}{3} \sum_{n=1}^{\infty} \frac{(-\frac{4}{9} \eta^3)^n (n+1)}{(n+1)! (\frac{5}{3} n (\frac{1}{3})^{n-j})} \end{array} \right)\]

\[- \frac{3}{3} \sum_{n=1}^{\infty} \frac{(-\frac{4}{9} \eta^3)^n (n+\frac{2}{3})}{(n+\frac{3}{4}) (n+\frac{2}{3})}\]

\[(3-87b) \quad H_{21}'' + \frac{4}{3} \eta^2 H_{21}' - \frac{8}{3} \eta H_{21} = 2iScf_{10} + \frac{2(12)^{1/3}}{3\Gamma(1/3)} \]

\[-\frac{2255}{4} \frac{Sc}{9} \times \frac{2}{9} \frac{\eta^2}{\Gamma(1/3)} \left(\frac{2}{3}; \frac{4}{3}; -\frac{4}{9} \eta^3\right) + \frac{Sc}{3 \times 4} \eta^2\]

\[\left( \begin{array}{c} \frac{1}{6} - \frac{1}{6} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{n} \frac{(-\frac{4}{9} \eta^3)^n (\frac{4}{3})^n (\frac{2}{3})^n}{n-j} \end{array} \right)\]

\[= \left( \begin{array}{c} \frac{1}{3} \sum_{n=1}^{\infty} \frac{(-\frac{4}{9} \eta^3)^n (n+1)}{(n+1)! (\frac{5}{3} n (\frac{1}{3})^{n-j})} \end{array} \right)\]

\[= \left( \begin{array}{c} \frac{1}{3} \sum_{n=1}^{\infty} \frac{(-\frac{4}{9} \eta^3)^n (n+\frac{2}{3})}{(n+\frac{3}{4}) (n+\frac{2}{3})}\end{array} \right)\]
For large \( n \):

\[
(3-88a) \quad H_{1*}^{-} + \frac{4}{3} n^2 H_{1*}^* - \frac{8}{3} n H_{1*} = 2iSc f_{20} + \frac{2(12)^{1/3}}{3\Gamma(1/3)} \times
\]

\[
\left( \frac{4764}{3} Sc \sqrt{\frac{n^2 \exp\left(-\frac{2n^{1/2}}{3} \sqrt{1 + \frac{n^3}{9}}\right) - \frac{2n^3}{9}}}{(1 + \frac{n^3}{9})^{3/2} (4n + \frac{4}{9}n^3)^{1/4}} \times
\]

\[
(-\sqrt{4n + \frac{4}{9}n^3} - \frac{2n^2 - (\frac{9 + 12n^3}{9})}{(4n + \frac{4}{9}n^3)} - \frac{1i Sc}{3 \times 16} (2n + \frac{4}{3}n^3) \exp\left(-\frac{4}{9}n^3\right))
\]

\[
(3-88b) \quad H_{2*}^{-} + \frac{4}{3} n^2 H_{2*}^* - \frac{8}{3} n H_{2*} = 2iSc f_{10} + \frac{2(12)^{1/3}}{3\Gamma(1/3)} \times
\]

\[
\left( \frac{4764}{12} Sc \sqrt{\frac{n^2 \exp\left(-\frac{2n^{1/2}}{3} \sqrt{1 + \frac{n^3}{9}}\right) - \frac{2n^3}{9}}}{(1 + \frac{n^3}{9})^{3/2} (4n + \frac{4}{9}n^3)^{1/4}} \times
\]

\[
(-\sqrt{4n + \frac{4}{9}n^3} - \frac{2n^2 - (\frac{9 + 12n^3}{9})}{(4n + \frac{4}{9}n^3)} - \frac{1i Sc}{3 \times 64} (2n + \frac{4}{3}n^3) \exp\left(-\frac{4}{9}n^3\right))
\]

Equations (3-86a) and (3-86b) differ from the equation for the quasi-steady state term \( f_{20} \) only by some constant factors, hence their solutions must differ from that of \( f_{20} \) by these same factors.

\[
(3-90a) \quad H_{1*} = \frac{8(12)^{1/3}}{3\Gamma(1/3)} \left(-\frac{1}{4}\right) \int_0^n \left(-\frac{1}{48} - \frac{n^3}{36} + \frac{n^6}{54}\right) \exp\left(-\frac{4}{9}n^3\right) dn
\]

\[
(3-90b) \quad H_{2*} = -\frac{8(12)^{1/3}}{3\Gamma(1/3)} \left(\frac{1}{24} + \frac{1}{64}\right) \int_0^n \left(-\frac{1}{48} - \frac{n^3}{36} + \frac{n^6}{54}\right) \exp\left(-\frac{4}{9}n^3\right) dn
\]
The solutions for \( H_{11} \) and \( H_{21} \) are divided into the cases of small \( \eta \) and large \( \eta \). For small \( \eta \) a series expansion about \( \eta = 0 \) is sought and for large \( \eta \) a solution which is asymptotic with respect to \( \eta \) is derived. The two solutions are then joined numerically by 'patching' at the point 
\[ \eta_1 = \frac{\Gamma(\frac{1}{3})}{(12)^{1/3}}, \]

Solutions of \( H_{11} \) and \( H_{21} \) for Small \( \eta \):

The complementary solution which is the same for both \( H_{11} \) and \( H_{21} \) is a linear combination of two infinite series

\[ H_{11} = H_{21} = A_2 \mathcal{F}_1 \left( \frac{2}{3}; \frac{2}{3}; -\frac{4}{9} \eta^3 \right) + B_2 \mathcal{F}_1 \left( \frac{1}{3}; \frac{4}{3}; -\frac{4}{9} \eta^3 \right) \eta \]

with arbitrary constants \( A_2 \) and \( B_2 \) whose values will be determined by the boundary conditions.

A particular solution of equation (3-87a) will now be found. But first the right hand side of the equation will be put in an appropriate form by putting \( f_{20} \) in a series expansion. Equation (3-87a) then becomes

\[ H_{11}'' + \frac{4}{3} \eta^2 H_{11}'' - \frac{8}{3} \eta H_{11} = \frac{2iScx8(12)^{1/3}}{3 \Gamma(1/3)} \left[ -\frac{n}{48} - \frac{\eta^n}{216} + \eta \Sigma_{n=2}^{\infty} \frac{(-\frac{4}{3} \eta^3)^n}{n! (3n+1)^x} \right] \]

\[ \text{where } (3n-2)(3n-1) + \frac{2(12)^{1/3}}{3 \Gamma(1/3)} \left[ -\frac{2255iScx}{9} - \frac{2}{9} \eta^2 \mathcal{F}_1 \left( \frac{2}{3}; \frac{4}{3}; -\frac{4}{9} \eta^3 \right) \right] \]

\[ + \frac{iScn^n}{3} \left( \frac{-1}{6} \Sigma_{n=1}^{\infty} \frac{\eta n(4)^n(4)^{n-j}}{(5)^n(1)^{n-j}} - \frac{\frac{(-\frac{4}{3} \eta^3)^n(n+1)^2}{3}}{(n+1)(n+\frac{2}{3})} \right) \]
and it has a particular solution

\[ H_1 := -\frac{(12)^{1/3}}{81\Gamma(1/3)} \times 2255iSc \left\{ \sum_{n=1}^{\infty} \frac{(-4n^3)n(3n)}{3n+1} \frac{\Gamma(2, \frac{7}{3}; -\frac{4}{9}n^3)}{\Gamma(1/3)} \right\} \]

\[ + \frac{2(12)^{1/3}iSc}{9\Gamma(1/3)} \]

\[ \left\{ \sum_{n=2}^{\infty} \frac{(-4n^3)n(\frac{2}{3})n}{n+1} \frac{\Gamma(2, \frac{7}{3}; -\frac{4}{9}n^3)}{\Gamma(1/3)} \right\} \]

\[ \left\{ \sum_{n=2}^{\infty} \frac{n-2}{n+1} \frac{(-4n^3)n(\frac{2}{3})n}{n+1} \frac{\Gamma(2, \frac{7}{3}; -\frac{4}{9}n^3)}{\Gamma(1/3)} \right\} \]

Therefore the general solution of \( H_{11} \) is

\[ (3-91) \quad H_{11} = A_2 \left\{ \int \left( -\frac{2}{3}; \frac{7}{3}; -\frac{4}{9}n^3 \right) + B_2 \left\{ \int \left( -\frac{1}{3}; \frac{4}{3}; -\frac{4}{9}n^3 \right) \right\} \]

\[-\frac{(12)^{1/3}}{81\Gamma(1/3)} \times 2255iSc \left\{ \sum_{n=1}^{\infty} \frac{(-4n^3)n(3n)}{3n+1} \frac{\Gamma(2, \frac{7}{3}; -\frac{4}{9}n^3)}{\Gamma(1/3)} \right\} \]

\[ + \frac{2(12)^{1/3}iSc}{9\Gamma(1/3)} \]

\[ \left\{ \sum_{n=2}^{\infty} \frac{n-2}{n+1} \frac{(-4n^3)n(\frac{2}{3})n}{n+1} \frac{\Gamma(2, \frac{7}{3}; -\frac{4}{9}n^3)}{\Gamma(1/3)} \right\} \]

Now that a solution for \( H_{11} \) has been found, a solution for \( H_{21} \) can be easily obtained. Comparison of the differential equations for \( H_{11} \) and \( H_{21} \) shows that the solutions
should differ only by a constant factor. The inhomogeneous term of the equation for $H_{21}$ is $(-i/4)$ that of the inhomogeneous term of $H_{21}$. Hence,

\[(3-92) \quad H_{21} = (-\frac{i}{4})H_{11}\]

The constant $A_2$ has to vanish in order to satisfy the zero boundary conditions at the tube wall. The other constant $B_2$ will be determined later.

**Solutions of $H_{11}$ and $H_{21}$ for large $\eta$:**

Equations (3-88a) and (3-88b) are made more amenable to algebraic operations by transforming the dependent variables. Since it is already known that $H_{21}$ has a solution which is proportional to $H_{11}$ it is only necessary to find $H_{11}$.

Let $H_{11} = Z_2 \exp(\frac{2}{9} \eta^3)$

The differential equation in terms of the new variable $Z_2$ is

\[
Z_2'' - (4\eta + \frac{4}{9} \eta^3)Z_2 = \frac{i\text{Sc}(12)^{1/3}}{3\Gamma(1/3)} \left\{-\frac{5}{24} (\eta^3 + \frac{2}{3} \eta^4) \exp(-\frac{2}{9} \eta^3) \right. \\
+ \left. 0.4764 \eta^2 \exp(-2\eta^3/2) \frac{\sqrt{1+\eta^3}}{9} \frac{1}{(\sqrt{\eta^3/2} - \frac{2}{3} \eta^3 - \frac{9}{8} \eta^3)} \frac{1}{(4\eta + \frac{4}{9} \eta^3)^{1/2}} \frac{1}{(4\eta + \frac{4}{9} \eta^3)^{1/2}} \frac{1}{(4\eta + \frac{4}{9} \eta^3)^{1/2}} \right\}
\]
The asymptotic solution is found in the manner described on page 45. The dominant terms in the asymptotic expansion are:

\[
Z_2 = \exp\left(\frac{2}{3\eta^{1/2}}\sqrt{1+\eta^3} - \frac{2}{9}\frac{\eta^{1/2}}{\eta^3} \right)^2 \left\{ c_2 + \frac{i\text{Sc}(12)^{1/3}}{3\Gamma(1/3)} \right\} 
\]

\[
\left[ \frac{5}{24} \int \left( \frac{n+2\eta^*}{\eta} \right) \exp\left(\frac{2}{3}\eta^{1/2}\sqrt{1+\eta^3} - \frac{2}{9}\eta^{1/2} \right) d\eta \right]
\]

\[
\int \left( \frac{4\eta^{1/2}}{\eta^*} \right)^{1/2} \left( \frac{1}{\eta^3} + \frac{2}{3} \eta^{1/2} \right) \left( \frac{9+12\eta}{4\eta} \right) \exp\left(\frac{4}{3}\eta^{1/2}\sqrt{1+\eta^3} \right) d\eta
\]

\[
+ \frac{4764}{3} \int \left( \frac{4\eta^{1/2}}{\eta^*} \right)^{1/2} \left( \frac{1}{\eta^3} + \frac{2}{3} \eta^{1/2} \right) \left( \frac{9+12\eta}{4\eta} \right) \exp\left(\frac{4}{3}\eta^{1/2}\sqrt{1+\eta^3} \right) d\eta
\]

\[
+ \exp\left(\frac{2}{3\eta^{1/2}}\sqrt{1+\eta^3} \right) \left\{ D_2 - \frac{i\text{Sc}(12)^{1/3}}{3\Gamma(1/3)} \right\}
\]

\[
\left[ \frac{-5}{24} \int \left( \frac{n+2\eta^*}{\eta} \right) \exp\left(\frac{2}{3}\eta^{1/2}\sqrt{1+\eta^3} - \frac{2}{9}\eta^{1/2} \right) \left( \frac{1}{\eta^3} + \frac{2}{3} \eta^{1/2} \right) d\eta \right]
\]

\[
- \frac{4764}{3} \left( \frac{n^3}{3} + \eta^{1/2} \sqrt{1+\eta^3} - \frac{\eta^{1/2}}{4\sqrt{1+\eta^3}} \right)
\]

Transforming back to the original variable, the general solution to equation (3-88) is thereby obtained.
(3-93) \[ H_{11} = \frac{\exp\left(\frac{2}{3}n^3/2\sqrt{1+n^2/9} - \frac{2}{9}n^3\right) - \frac{n^{1/2}}{3}}{(4n + \frac{4}{9}n^*)^{1/3}} \]

\[
\{c_2 + \frac{i\text{Sc}(12)^{1/3}}{3\Gamma(1/3)} \sum_{n=24}^{\infty} \int_{\frac{2}{3}n^3/2\sqrt{1+n^2/9} - \frac{2}{9}n^3}^{\frac{9}{3}n^3} \exp\left(-\frac{2}{3}n^3/2\sqrt{1+n^2/9} - \frac{2}{9}n^3\right) \text{dn} \}
\]

\[
+ \frac{4.764}{3} \sum_{n=3}^{\infty} \int_{\frac{4}{9}n^*}^{\infty} \left(\sqrt{\frac{4}{9}n^*} + \frac{2}{3}n^2 + \frac{9+\frac{12}{3}n^3}{(4n+\frac{4}{9}n^*)^{1/3}}\right) \exp\left(-\frac{2}{3}n^3/2\sqrt{1+n^2/9}\right) \text{dn} \]

\[
+ \frac{\exp\left(-\frac{2}{3}n^3/2\sqrt{1+n^2/9} - \frac{2}{9}n^3\right)}{(4n+\frac{4}{9}n^*)^{1/3}(\sqrt{1+n^2/9} + \frac{n^{3/2}}{3})^2} \times \frac{i\text{Sc}(12)^{1/3}}{3\Gamma(1/3)} \times D_2 \]

\[
- \frac{5}{24} \int_{0}^{n} \left(\frac{2}{3}n^*\right) \left(\frac{n^{3/2}}{3} + \frac{n^3/2}{3\sqrt{1+n^2/9}} - \frac{n^{3/2}}{3\sqrt{1+n^2/9}}\right) \text{dn} \]

\[
- \frac{4.764}{3} \left(\frac{n^3}{3} + \frac{n^{3/2}}{3\sqrt{1+n^2/9}} - \frac{n^{3/2}}{3\sqrt{1+n^2/9}}\right) \}
\]

Since the solution should meet the zero condition at \(\eta = \infty\), the constant \(c_2\) must vanish. To determine the constants \(B_2\) and \(D_2\) the two solutions of \(H_{11}\) for large \(\eta\) and small \(\eta\) are joined at the point \(\eta_1 = \frac{\Gamma(1/3)}{(12)^{1/3}}\). The results of the calculations give the following values for the constants.

\[ B_2 = \frac{2}{9} i\text{Sc} \left(\frac{12}{1/3}\right) \times 0.20554 \]
\[ D_2 = \frac{-2}{9} i Sc \frac{(12)^1}{1^3} \frac{1}{\Gamma(1/3)} x.28618 \]

To summarize, the second harmonic fluctuation of the dimensionless concentration in the region of low frequency is given by the following formulae.

For small \( \eta \):

\[(3-94a) \quad f_2 = \frac{(12)^1}{3 \Gamma(1/3)} \left(1 - \frac{i}{4} \omega^2 - \omega^2 \left(\frac{1}{24} + \frac{1}{64}\right)\right) \times\]

\[\left(-\frac{\eta}{6} - \frac{\eta^3}{9}\right) \exp\left(-\frac{4 \eta^3}{9}\right) + \frac{(12)^1}{\Gamma(1/3)} i Sc \omega^2 \xi^2/3 \left(\frac{1}{9} (1 -\frac{1}{4} \omega^2) \right) \times\]

\[- \frac{n^1}{6} + \frac{n^1}{4} \sum_{n=1}^{\infty} \frac{(-4 \eta^3)^n}{(n+1)! \left(\frac{3}{5}\right)^n} + \frac{n^1}{2} \sum_{n=2}^{\infty} \sum_{j=0}^{n-2} \sum_{k=0}^{n-j-1} \frac{(-4 \eta^3)^n}{(n+1)! \left(\frac{3}{5}\right)^n (-\frac{3}{2})^{n-j-k}}\]

\[- \frac{n^1}{6} \sum_{n=2}^{\infty} \sum_{j=0}^{n-2} (-\frac{4 \eta^3}{9})^n \left(\frac{1}{3}\right)^{n-j} \left(\frac{1}{3}\right)^{n-j-1} \left(\frac{2}{3}\right)^{n-j} \left(\frac{5}{3}\right)^{n-j-\frac{1}{3}}\]

\[= \frac{.2255}{81} \left(1 -\frac{i}{4} \omega^2\right) \eta^4 \int_{1} \left(\frac{2}{3}, \frac{7}{3}; -\frac{4 \eta^3}{9}\right) + \frac{2}{9} x .20554 \left(1 -\frac{i}{4} \omega^2\right) \eta^4 \int_{1} \left(-\frac{1}{3}, \frac{4}{3}; -\frac{4 \eta^3}{9}\right)\]
For large \( n \):

\[
(3-94b) \quad f_2 = \frac{(12)^{1/3}}{3 \Gamma(1/3)} \left(1 - \frac{i\omega^2}{4} - \omega^2 \left(\frac{1}{24} + \frac{1}{64}\right) \left(-n + \frac{n^3}{9}\right) \exp\left(-\frac{4n^3}{9}\right)\right) \\
+ \frac{(12)^{1/3}}{3 \Gamma(1/3)} i \text{Sc} \omega^2 \left(1 - \frac{i\omega^2}{4}\right) \left\{ \exp\left(\frac{2}{3} \sqrt{1+\frac{n^3}{9}} - \frac{2}{9} n^3 \right) \left(1 + \frac{n^3}{9} \right) \right\} \times \\
\left[ \frac{5}{24} \int \frac{(\frac{2}{3} n^3 + n) \exp\left(-\frac{2}{3} n^{3/2} \sqrt{1+\frac{n^3}{9}} - \frac{2}{9} n^3 \right)}{(4n + \frac{4}{9} n^9)^{1/4} \left(\sqrt{1 + \frac{n^3}{9}} + \frac{n^{3/2}}{3}\right)} dn \right] \\
+ \frac{4764}{3} \int \frac{\alpha n^2 \left(\sqrt{4n + \frac{4}{9} n^9} + \frac{2}{3} n^2 + \frac{g + \frac{12}{9} n^3}{4n + \frac{4}{9} n^9} \right) \exp\left(-\frac{4}{3} n^{3/2} \sqrt{1+\frac{n^3}{9}} \right)}{(4n + \frac{4}{9} n^9)^{1/4} \left(\sqrt{1 + \frac{n^3}{9}} + \frac{n^{3/2}}{3}\right)} dn \\
+ \frac{\exp\left(-\frac{2}{3} n^{3/2} \sqrt{1+\frac{n^3}{9}} - \frac{2}{9} n^3 \right)}{(4n + \frac{4}{9} n^9)^{1/4} \left(\sqrt{1 + \frac{n^3}{9}} + \frac{n^{3/2}}{3}\right)^2} \left[-\frac{2}{3} \times .28618 \right] \\
- \frac{5}{24} \int \frac{(n^3 + \frac{2}{3} n^9) \left(\sqrt{1 + \frac{n^3}{9}} + \frac{n^{3/2}}{3}\right)^2 \exp\left(\frac{2}{3} n^{3/2} \sqrt{1+\frac{n^3}{9}} - \frac{2}{9} n^3 \right)}{(4n + \frac{4}{9} n^9)^{1/4}} dn \\
- \frac{4764}{3} \left(\frac{n^3}{3} + n^{3/2} \sqrt{1 + \frac{n^3}{9}} - \frac{n^{3/2}}{4 \sqrt{1+\frac{n^3}{9}}} \right) \zeta^{2/3}
\]

The second harmonic fluctuation of the dimensionless mass flux can now be found by taking the derivative of (3-94a) at the wall.

\[
(3-95) \quad Nu_2 = \frac{(12)^{1/3}}{18 \Gamma(1/3)} \left[-1 + \text{Sc} \xi^{2/3} \times .82216 \times i\omega^2 \left(1 - \frac{i\omega^2}{4}\right) \right]
\]
Its phase and amplitude

\[ \psi_2 = \tan^{-1} \left( \frac{\frac{Sc\xi^2 + 82216}{1 + \omega^3 Sc\xi^2 / 3 \times 20554}}{\omega} \right) \]

\[ |Nu_2| = \left( \frac{(12)^{1/3}}{18\Gamma(1/3)} \right) \left( \frac{0.82216 Sc\xi^2 / 3 \omega^2}{1 + Sc\xi^2 / 3 \omega^3 x \times 20554} \right)^{\frac{1}{2}} \]

are shown on Figures 3 and 4 as functions of \( \omega \). The numerical results are also listed down on Table 5. From the figures it can be observed that as the frequency becomes zero, the phase approaches its limiting value of \( \pi \) and the amplitude approaches the quasi-steady value of \( \frac{(12)^{1/3}}{18\Gamma(1/3)} \).

C. Parameters Used in Calculations

All the preceding equations were derived assuming a fully developed flow and utilizing boundary layer approximations for the mass distribution, and therefore the analyses are restricted to a definite region along the tube. Hence, it is necessary to specify the portion of the tube to which the equations would apply.

The flow in the inlet length of a circular pipe has been extensively studied\(^{23}\) and it has been established that a parabolic velocity profile is attained at a distance of

\[ h = 0.08 R N_{Re} \]

from the inlet of the tube. The results of the present analyses are applicable to lengths of the order of \( h \) or greater.
At a point farther down the tube where the thickness of the concentration boundary layer becomes comparable to the radius of the tube the dimensionless concentration, $\phi$, no longer changes with the distance in the axial direction and the concentration distribution is then said to be fully established.

The first task is to find the order of magnitude of the length of the conduit which marks the end of the concentration boundary layer and the beginning of the fully developed concentration distribution region. This length will be denoted by $H$. Assuming that the fluctuations in the flow are not large enough to cause drastic alterations in the width of the unperturbed boundary layer, it is possible to get an estimate of the actual size of the boundary layer from the magnitude of the unperturbed boundary layer. $H$ can be approximated by replacing $\delta_0$ with $R$ and $x$ with $H$ in the expression for $\delta_0$ given by equation (3-16).

$$H \approx 0.3R \text{ Sc } N_{RE}$$

The values of the independent variable $x$ must therefore be limited to

$$0.08 R N_{RE} < x < 0.3R \text{ Sc } N_{RE}$$

or

$$x = K R \text{ Sc } N_{RE}$$

where

$$\frac{0.08}{\text{Sc}} < K < 0.3$$
For large $Sc$, $H \gg h$ and in practice the ratio $H/h$ is usually so large that the entire length of the tube does not suffice for the establishment of a steady diffusion regime. The Schmidt number has been arbitrarily set to $10^3$.

A few typical values of the frequency parameter for arteries in the human cardiovascular system are shown below. The parameter was calculated for a pulse rate of 72 beats per minute (7.34 radians/sec) and for a kinematic viscosity of 0.0381 cm$^2$/sec.

<table>
<thead>
<tr>
<th>Radius, cm</th>
<th>$\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thoracic aorta</td>
<td>0.6</td>
</tr>
<tr>
<td>Abdominal aorta</td>
<td>0.45</td>
</tr>
<tr>
<td>Mesenteric artery</td>
<td>0.50</td>
</tr>
<tr>
<td>Inferior mesenteric artery</td>
<td>0.20</td>
</tr>
<tr>
<td>Iliac artery</td>
<td>0.30</td>
</tr>
<tr>
<td>Femoral artery</td>
<td>0.20</td>
</tr>
<tr>
<td>Renal artery</td>
<td>0.275</td>
</tr>
<tr>
<td>Arterioles</td>
<td>$0.125\times10^{-2}$</td>
</tr>
</tbody>
</table>

D. Numerical Solution

As a check on the analytical solution, numerical solutions were obtained for $f_1$, $f_2$ and $M$. The following equations were solved numerically for a few values of $\omega$.

\[
\frac{\partial^2 f_1}{\partial \eta^2} + \frac{4}{3} \eta^2 \frac{\partial f_1}{\partial \eta} - 4n\frac{\partial f_1}{\partial \xi} - i\omega^2 \xi^{1/3} f_1 = \frac{-(12)^{1/3}}{3^{(1/3)}} \frac{\varepsilon n^2}{\eta} \exp\left(-\frac{4}{9\eta}\right)
\]
\[ \frac{\partial^2 f_2}{\partial \eta^2} + \frac{4}{3} \eta^2 \frac{\partial f_2}{\partial \eta} - 4n \xi \frac{\partial f_1}{\partial \xi} - 2i \omega^2 \xi^2 f_2 = \frac{f_n}{2} \left( -\frac{n^2}{3} \frac{\partial f_1}{\partial \eta} + n \xi \frac{\partial f_1}{\partial \xi} \right) \]

\[ \frac{\partial^2 M}{\partial \eta^2} + \frac{4}{3} \eta^2 \frac{\partial M}{\partial \eta} - 4n \xi \frac{\partial M}{\partial \xi} = \frac{1}{2} \left( \text{Re}(f_n) \left( -\frac{n^2}{3} \text{Re} \left( \frac{\partial f_1}{\partial \eta} \right) + n \xi \text{Re} \left( \frac{\partial f_1}{\partial \xi} \right) \right) \right) + \text{Im}(f_n) \left( -\frac{n^2}{3} \text{Im} \left( \frac{\partial f_1}{\partial \eta} \right) + n \xi \text{Im} \left( \frac{\partial f_1}{\partial \xi} \right) \right) \]

The Crank Nicholson analogue was used for partials in \( \eta \) and a simple backwards difference analogue for partials in \( \xi \).

The grid spacings

\[ \nabla \xi = 0.25 \times 10^{-2} \]
\[ \nabla \eta = 0.22 \times 10^{-1} \]

were found to be sufficiently small such that a further reduction in grid size did not bring about a discernible change in the numerical solution for the range of values of \( \omega \) from 0 to 3.

The results for \( \text{Nu}_1 \), \( \text{Nu}_2 \) and \( \overline{\text{Nu}_2} \) for \( \xi = 0.002 \) are plotted on Figures 1 to 5 alongside their analytical counterparts and they are seen to agree well with them. An important point which is brought out by the favorable comparison between the analytical and numerical solutions is that high and low frequency approximations cover almost the entire frequency range.
In the frequency range $0 < \omega \leq 3$, the computer time needed to get a solution for one value of the frequency is about 25 minutes. A much larger amount of time, approximately one hour when $\omega = 7$, is required to determine the behaviour of the solution in the range of high frequencies. To be able to draw a curve that will cover a large portion of the frequency range such that the asymptotic behaviour is determined, the total computer time which would be used up is almost prohibitive. In this respect the analytical solution is far superior to any numerical method since it takes less than 10 minutes to compute the solutions for 20 points covering the frequency range up to $\omega = 10$. Another advantage of the analytical solution is that the asymptotic behaviour of the flux can be readily discerned from the mathematical expression of the solution.

All calculations were carried out on the IBM 7040.
\( \psi_1 \), PHASE OF THE FIRST HARMONIC OF \( \text{Nu} \)

\[ \Xi = 0.20 \times 10^{-2} \]

**Figure 1 - Frequency Dependence of the Phase of Nu, First Harmonic**
\[ |N_u|, \text{ Amplitude of the first harmonic of } N_u \]

Low Frequency Solution

Numerical Solution

High Frequency Solution

\[ \xi = 0.20 \times 10^{-2} \]

\( \omega, \text{ Dimensionless Frequency Parameter} \)

Figure 2 - Frequency dependence of the amplitude of \( N_u \), first harmonic
\( \psi_2 \), PHASE OF THE SECOND HARMONIC OF \( \nu \)

\[ \xi = 0.20 \times 10^{-2} \]

FIGURE 3 - FREQUENCY DEPENDENCE OF THE PHASE OF \( \nu \), SECOND HARMONIC
\[ |\nu_2|, \text{ amplitude of the second harmonic of \( \nu_2 \)} \]

\[ \omega, \text{ dimensionless frequency parameter} \]

\[ \xi = 0.20 \times 10^{-2} \]

**Figure 4 - Frequency dependence of the amplitude of \( \nu_2 \), second harmonic**
Figure 5 - Frequency dependence of the increase in $\overline{Nu}$ over the steady flow value

$\xi = 0.20 \times 10^{-2}$
Figure 6 - Effect of Axial Position on the Increase in $\bar{N}_u$
E. Discussion of Results

The results of the calculations show that the time-averaged local dimensionless mass flux, $\text{Nu}$, is influenced by several factors, namely, the frequency of the fluctuation, its amplitude and the position along the conduit. The effect of each of these factors will be discussed in turn.

To avoid any unnecessary confusion, it should be stressed that $\text{Nu}$ as defined in this text is not the conventional Nusselt number $\text{N}_\text{Nu}$ but is related to it in a simple manner.

$$\text{N}_\text{Nu} = \frac{2R}{(C_0 - C_w)} \frac{\partial C}{\partial y} \bigg|_{y=0} = \frac{2}{\xi} \text{Nu}$$

Figures 5 and 6 would likewise be applicable to $\text{N}_\text{Nu}$.

Effect of Frequency

Both the analytical and numerical solutions give a rather surprising result, namely, a decrease in the time-averaged mass flux at the wall in the region of very low frequency. In view of this unexpected behaviour, an attempt to explain this decrease will be made.

At very low values of the frequency parameter the boundary layer at any instant will be, to a good approximation, that corresponding to steady flow at the instantaneous value of the velocity. This mode of fluctuation is called quasi-steady. An independent check on the results of the present analyses can therefore be made by solving for the quasi-steady solution of the mass flux.
For very small frequencies, the dimensionless concentration variable $\phi(x,y,t)$ will take the form

$$\phi(x,y,t) \cong \phi_0(x,y) + \lambda f_{10}(x,y)e^{i\beta t} + \lambda^2 [M_0(x,y) + f_{20}(x,y)e^{2i\beta t}]$$

and specifically when $\omega = 0$.

$$\phi(x,y,t) = \phi_0(x,y) + \lambda f_{10}(x,y) + \lambda^2 [M_0(x,y) + f_{20}(x,y)]$$

The velocity distribution for a zero frequency of fluctuation is obtained by letting the frequency approach zero in the expression for the axial velocity (3-8).

$$u(r) = 2U_0 \left(1 - \frac{r^2}{R^2}\right)(1+\lambda)$$

The velocity $u(r)$ is then simplified by assuming a linear profile near the wall with slope equal to $\frac{\partial u}{\partial y} \bigg|_{y=0}$.

$$u \cong \frac{4U_0}{R} \nu(1 + \lambda)$$

A general solution to the diffusion problem is then derived with the above expression for the velocity.

$$\phi = a_0 \int_0^\eta \exp\left(-\frac{4}{9}\eta^3(1+\lambda)\right) d\eta + a.$$ 

To satisfy the zero condition at the wall, $a_1$ must be zero. The other constant $a_0$ is determined by applying the
condition that \( \phi = 1 \) at \( \eta = \infty \).

\[
a_0 = \frac{(12)^{1/3}}{\Gamma(1/3)} \frac{1}{(1 + \lambda)^{1/3}}
\]

The term \((1 + \lambda)^{1/3}\) is then expanded in a binomial expansion for \(\lambda \ll 1\).

\[
(1 + \lambda)^{1/3} \approx 1 + \frac{\lambda}{3} - \frac{\lambda^2}{9}
\]

and the flux evaluated by taking the derivative of \(\phi\) with respect to \(\eta\) at the wall.

\[
Nu = \frac{(12)^{1/3}}{\Gamma(1/3)} (1 + \frac{\lambda}{3} - \frac{\lambda^2}{9})
\]

By comparing the coefficients of \(\lambda\) and \(\lambda^2\) in the derived expression for \(\phi\) and in the assumed form, it can be shown that the components of the quasi-steady solution are given by:

\[
\frac{\partial \phi | \eta=0}{\partial \eta} = \frac{(12)^{1/3}}{3 \Gamma(1/3)}
\]

\[
\frac{\partial (M_0 + f_2| \eta=0)}{\partial \eta} = -\frac{(12)^{1/3}}{9 \Gamma(1/3)}
\]

Looking at the equations for \(M\) and \(f_2\) in section C it is noted that for \(\omega = 0\), the equations for \(M\) and \(f_2\) are identical. And therefore

\[
\frac{\partial M_0}{\partial \eta} | \eta=0 = \frac{\partial f_2 | \eta=0}{\partial \eta} | \eta=0 = -\frac{(12)^{1/3}}{18 \Gamma(1/3)}
\]
Thus the dimensionless mass flux to the wall as \( \beta \) approaches zero is given by:

\[
\text{Nu} = \frac{3\phi}{n_0} = \frac{(12)^{1/3}}{\Gamma(1/3)} \left[ (1 - \frac{\lambda^2}{18}) + \frac{\lambda}{3} e^{i\beta t} - \frac{\lambda^2}{18} e^{2i\beta t} \right]
\]

This is exactly the same result obtained by the analytical method described herein.

In order to arrive at a general conclusion concerning slow oscillations, the quasi-steady solution for an arbitrary waveform will be determined. Consider the pressure gradient to be

\[
- \frac{1}{\rho} \frac{\partial p}{\partial x} = p(1 + |\lambda| \sum_{n=1}^{\infty} (a_n \cos n\beta t + b_n \sin n\beta t))
\]

where \(|\lambda|\) is a norm of the amplitude of the fluctuation. The flux \(\text{Nu}\) with this pressure gradient expression for \(\beta\) approaching zero is obtained by again assuming quasi-steady behaviour.

\[
\text{Nu} = \frac{(12)^{1/3}}{\Gamma(1/3)} \left[ 1 + \frac{|\lambda|}{3} \sum_{n=1}^{\infty} (a_n \cos n\beta t + b_n \sin n\beta t) \right]
\]

\[
- \frac{|\lambda|^2}{9} \left( \sum_{n=1}^{\infty} a_n \cos n\beta t + b_n \sin n\beta t \right)^2
\]

At any time, the coefficient of \(|\lambda|^2\) is less than or equal to zero. Hence, the time-averaged \(\text{Nu}\) is always less than or equal to the steady state value for a frequency of
oscillation approaching zero.

An extensive survey of prior work has revealed that most of the work on oscillating flow, both experimental and theoretical, was done on flow along cylinders and flat plates with oscillating free streams or the equivalent problem of flow along an oscillating cylinder or flat plate immersed in an incompressible fluid. In general, the researchers report large increases in heat flux or mass flux due to oscillations in flow for frequencies beyond a certain critical frequency. Below this frequency, however, no observable increase is reported. In contrast, results of this study indicate a decrease in flux below some frequency. These findings are consistent with experimental data of Havemann and Rao\(^9\) on heat transfer in pulsating flow in a pipe with a range of Reynolds number in the turbulent zone. It should be pointed out that while previous studies on a developing boundary layer provide valuable material for qualitative comparison, no quantitative comparison can be made since the characteristics of flow along a cylinder or flat plate are different from those of fully developed flow in a conduit.

At this point it might be interesting to mention the results of an experimental study conducted by Van der Hegge Zijnen\(^{30}\) on heat transfer in turbulent flow. His experimental results show that the heat transfer form a wire vibrating in the direction of a uniform flow is somewhat lower than the heat transfer of a stationary wire in the
same flow. A rough analysis made by Zijnen shows that this negative change is expected when the velocity in the transverse direction is zero.

When the time of oscillation is small compared to the characteristic diffusion time of the system which will be taken as $R^2/D$, the dominating mechanism of heat or mass transfer is that of transport to or from the steady streaming flow. This theory is supported by experimental data obtained by Jameson\textsuperscript{11}. Therefore, for flow in which high frequency oscillations have been superimposed on a steady flow and in the absence of secondary flow, the solution to the mass transfer problem will approach that of the steady state case. That is, the concentration boundary layer will behave just as if there were no oscillations in the flow. The analytical solution presented in this paper shows this tendency to approach the steady state solution for very high frequencies.

**Effect of Amplitude**

The change in $\overline{Nu}$ over the steady flow value is proportional to $\lambda^2$. An increase in the value of $\lambda$ will bring about an increase in the net mass flux for frequencies beyond a certain critical frequency but will cause a decrease for frequencies below it.

The accuracy of a perturbation expansion is dependent on the magnitude of the perturbation parameter as well as on the number of terms in the expansion. To get a rough
estimate of the values of \( \lambda \) for which a second-order expansion will hold, a tabulation of \( \phi_0, |f_1|, M \) and \( |f_2| \) versus \( n \) at \( \xi = 0.2 \times 10^{-2} \) for some values of \( \omega \) (Table 10, page C-14) was made. It is noted that

\[
\phi_0 > |f_1| > |M| + |f_2|
\]

for all \( n \) and that the ratios \( \frac{|f_1|}{\phi_0} \) and \( \frac{|M| + |f_2|}{|f_1|} \) decrease as the frequency increases. In the high frequency range, therefore, a larger value of \( \lambda \) may be used. A value of \( \lambda \) greater than one is feasible for a sufficiently high frequency of pulsation.

**Effect of Axial Position**

In the region of very small frequencies, the time-averaged dimensionless mass flux is unaffected by the value of the dimensionless x variable \( \xi \) as shown by equation (3-75). For high frequencies, however, the increase in the net rate of mass transfer is clearly a function of distance along the conduit. The position of the maximum with respect to frequency as well as the value of the critical frequency tends to move towards the direction of decreasing frequency as \( \xi \) becomes larger. In the limit of very high frequencies, the relationship between the net increase in flux and the axial position becomes very simple, that is, the increase in \( \overline{Nu} \) is inversely proportional to \( \xi \). This is clearly shown in Figure 6, where the function \( \left( \frac{\overline{Nu} - \overline{Nu}_0}{\lambda^2 \overline{Nu}_0} \right) \times 100 \)
is plotted against the frequency parameter $\omega$. The curves for different values of $\xi$ are observed to approach a limiting curve. On semi-log graph this asymptote is very nearly a straight line.
IV. DIFFUSION IN FLOW WITH A PROGRESSING PULSE WAVE

For the second test case the following physical model was chosen.

An incompressible Newtonian fluid is flowing in a distensible long tube which is longitudinally constrained so that there is no motion of the tube wall in this direction. A traveling wave of small amplitude is impressed on the steady flow. It is assumed that the elastic tube retains a circular cross section at all times.

To simplify the flow problem the following assumptions are adopted:

(1) The wave velocity \( c \) is large compared to the mean axial velocity \( U_0 \).

(2) The wavelength, \( \frac{C}{\beta} \), is long compared to the radius of the conduit.

(3) The frequency of disturbance is high.

(4) There are no reflected waves.

A. Velocity Distribution

Let the pressure gradient along the tube be given by:

\[
-\frac{1}{\rho} \frac{\partial p}{\partial x} = P(1 + \lambda e^{i\beta(t-x/c)})
\]

The momentum and continuity relations for this problem are:

\[
(4-1) \ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = \nu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial x^2} \right] - \frac{1}{\rho} \frac{\partial p}{\partial x}
\]

\[
\frac{\partial u}{\partial x} + \frac{1}{r} \frac{\partial rv}{\partial r} = 0
\]
It can be shown by a non-rigorous method, the order of magnitude method initially employed by Prandtl, that

\[
\left| \frac{u}{\partial x} + v \frac{\partial u}{\partial r} \right| \ll \left| \frac{\partial u}{\partial t} \right|
\]

\[
\left| \frac{\partial^2 u}{\partial x^2} \right| \ll \left| \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial u}{\partial r} \right|
\]

provided

\[
\frac{U_0}{C} \ll 1
\]

\[
\frac{BR}{C} \ll 1
\]

\[
\frac{NRF}{2\omega^2} \ll 1
\]

For this particular problem, then, a linearized equation describes the axial velocity component.

\[
(4-2) \quad \frac{\partial u}{\partial t} = \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{v}{r} \frac{\partial}{\partial r} r \frac{\partial u}{\partial r}
\]

The solution to the flow problem is assumed to have the form

\[
(4-3) \quad u(r,x,t) = u_0(r) + \lambda u_1(r)e^{i\beta(t-x/C)}
\]

\[
v(r,x,t) = v_0(r) + \lambda v_1(r)e^{i\beta(t-x/C)}
\]
Substituting into (4-1) and equating like powers of $\lambda$ gives the differential equations

\[(4-4) \quad p + \frac{\nu}{r} \frac{d}{dr} r \frac{du}{dr} = 0\]

\[v_0(r) = 0\]

\[(4-5) \quad iB u_1 = p + \frac{\nu}{r} \frac{d}{dr} r \frac{du}{dr} - \frac{i\beta u}{C} + \frac{1}{r} \frac{d}{dr} rv_1 = 0\]

The condition of symmetry at the center of the conduit is

\[(4-6) \quad \frac{du_1}{dr} \bigg|_{r=0} = 0\]

\[\frac{du_1}{dr} \bigg|_{r=0} = 0\]

To derive the conditions at $r = R$, it is necessary to know the radial expansion of the wall which will be denoted as $\alpha$. By imposing the condition that the rate of radial expansion of the tube wall is equal to the radial velocity at the wall

\[\frac{\partial \alpha}{\partial t} = v(R+\alpha, x, t) = \lambda v_1(R+\alpha)e^{i\beta(t-x/C)}\]

a relationship between $\alpha$ and the mean fluctuating axial velocity can be obtained. The function $v_1$ is then expanded
in a Taylor series about \( r = R \) giving

\[
\frac{\partial \alpha}{\partial t} = \lambda e^{i\beta(t-x/C)} \left[ v_1(R) + \alpha \frac{dv_1}{dr} \bigg|_{r=R} \right]
\]

The value of \( v_1 \) at \( r = R \) is obtained from the continuity relationship in (4-5)

\[
v_1(R) = \frac{i\beta}{2C} \left[ \frac{2\pi ru_1 \, dr}{\pi R^2} \right] = \frac{i\beta R}{2C} \bar{u}_1
\]

Thus an expression for \( \alpha \) is given by

\[
(4-7) \quad \alpha = \lambda \frac{R}{2C} \bar{u}_1 e^{i\beta(t-x/C)} + O(\lambda^2)
\]

The condition of no tangential flow at the surface gives

\[
u(R + \alpha, x, t) = 0
\]

This is rewritten in the form of (4-3), giving

\[
u_0(R + \alpha) + \lambda u_1(R + \alpha) e^{i\beta(t-x/C)} = 0
\]

Here the perturbation parameter appears explicitly as well as implicitly in the argument of the functions \( u_0 \) and \( u_1 \). Therefore, in this present form it is not possible directly to equate like powers of \( \lambda \) to zero. However, if it is assumed that \( u_1 \) and \( u_0 \) are both analytic in their dependence on \( r \), both functions can be expanded in a Taylor series. The condition at the surface is then expressed as
\[ u_0(R) + \alpha \frac{du_0}{dr} \bigg|_{r=R} + \lambda e^{i\beta(t-x/C)} \left( u_1(R) + \alpha \frac{du_1}{dr} \right) = 0 \]

By substituting (4-7) in place of \( \alpha \) in the above equation a series expansion in \( \lambda \) is derived.

\[ u_0(R) + \lambda e^{i\beta(t-x/C)} \left( \frac{R}{2C} u_1 \frac{du_0}{dr} \bigg|_{r=R} + u_1(R) + O(\lambda^2) \right) = 0 \]

In this form, like powers of \( \lambda \) can be equated, giving

\[(4-8a) \quad u_0(R) = 0 \]

\[(4-8b) \quad u_1(R) + \frac{ru_1}{2C} \frac{du_1}{dr} \bigg|_{r=R} = 0 \]

By looking at the differential equation (4-4) and the boundary conditions (4-6) and (4-8a), it can be noted that the basic term \( u_0 \) is the Poiseuille flow solution.

The first order term \( u_1 \) has a solution

\[(4-9) \quad u_1(r) = \frac{8U_0}{I\omega^2} - a \frac{J_0(\frac{\omega ri^{3/2}}{R})}{J_0(\frac{\omega i^{3/2}}{R})} \]

where 'a' is a constant which will be determined so as to satisfy the no-slip condition at the wall. The value of \( u_1 \) at \( r = R \) is

\[ u_1(R) = \frac{8U_0}{I\omega^2} - a \]
and its mean over a cross section of radius $R$ is given by

$$
\overline{u}_1 = \frac{8U_0}{i\omega^2} - \frac{2a}{\omega i^{3/2}} \frac{J_1(\omega i^{3/2})}{J_0(\omega i^{3/2})}
$$

The constant 'a' can then be evaluated with the use of (4-8b)

$$
a = \frac{8U_0^2}{i\omega^2} \left( 1 - \frac{U_0}{C} \right) \frac{2\left(\frac{U_0}{C}\right) J_1(\omega i^{3/2})}{\left[ 1 - \frac{\omega i^{3/2}}{J_0(\omega i^{3/2})} \right]} \]

For small values of $U_0/C$, 'a' is very nearly equal to $(8U_0/i\omega^2)$. This approximate value of 'a' will be adopted in this analysis.

The continuity equation can then be employed to give

(4-10) $\nu_1(r) = \frac{8\nu}{R} \left( \frac{U_0}{C} \right) \left[ \frac{r}{2R} - a \frac{J_1(\omega i^{3/2})}{\omega i^{3/2} J_0(\omega i^{3/2})} \right]$}

It may be observed that while the tangential velocity at the wall is zero, the transverse velocity there is not, in general, zero. It may be positive, negative or zero depending on the position along the tube. For the case of a distensible tube, the transverse velocity at the wall gives the rate of expansion or contraction of the tube in the radial direction.

As in the rigid conduit case, the velocity components are defined in terms of a new variable $y$, the distance from
the tube wall.

\[
\begin{align*}
u &= 2U_0 [1 - (1 - \frac{V}{R})^2] + \frac{8\lambda U_0}{1\omega^2} e^{i\beta(t - \frac{X}{C})} \left[ 1 - \frac{J_0(\omega i^{3/2}(1 - \frac{V}{R}))}{J_0(\omega i^{3/2})} \right] \\
v &= \frac{-8\lambda v}{R} \left( \frac{U_0}{C} \right) e^{i\beta(t - \frac{X}{C})} \left[ \frac{1 - \frac{V}{R}}{2} - \frac{J_1(\omega i^{3/2}(1 - \frac{V}{R}))}{\omega i^{3/2} J_0(\omega i^{3/2})} \right]
\end{align*}
\]

Replacing the Bessel functions by their series expansions and retaining only the linear terms in \( v \), the linear approximations to the velocity distributions are thereby derived.

\[\text{(4-11)} \quad u = \frac{4u_s v}{R} + \frac{U_0 v}{R} f_n e^{i\beta(t - \frac{X}{C})} \]

\[v = \frac{\lambda v}{R} \left( \frac{U_0}{C} \right) \left( 1 + \frac{V}{R} \right) (f_n - 4) e^{i\beta(t - \frac{X}{C})} \]

The symbol \( f_n \) has been previously defined in Chapter III.

The velocity relations derived in this section may very well describe the flow in an artery which is longitudinally constrained due to the tethering effect of the connective tissues and the mass of the adjacent organs.\textsuperscript{35}

Patel's clinical studies show that there is in fact very little longitudinal movement of the arterial wall so that analysis of flow in a tethered tube is likely to be valid.\textsuperscript{2}

B. Concentration Distribution

Since the radial expansion is negligible compared to the radius of the tube for small \( U_0/C \), its effect will not
be considered in deriving the equation for the dimensionless concentration variable \( \phi \). The equation for \( \phi \) as well as its boundary conditions will then be the same as in the case of flow in a rigid tube.

The solution to the concentration boundary layer equation is again expressed as a perturbation of the steady flow solution. The steady state solution has been previously derived in Chapter III and is given by (3-14). The differential equations for the first and second order terms are

\[
(4-12) \quad \frac{\partial \phi_1}{\partial t} + 4 \frac{U_0}{R} \frac{\partial \phi_1}{\partial x} + \frac{U_0}{R} f_n e^{i\beta \frac{(t-x)}{C}} \frac{\partial \phi}{\partial x} + 
\]

\[
+ \frac{\nu}{R} \left( \frac{U_0}{C} \right) (1 + \frac{V}{R}) (f_n - 4) e^{i\beta \frac{(t-x)}{C}} \frac{\partial \phi}{\partial y} = D \frac{\partial^2 \phi_1}{\partial y^2}
\]

\[
(4-13) \quad \frac{\partial \phi_2}{\partial t} + 4 \frac{U_0}{R} \frac{\partial \phi_1}{\partial x} + \frac{U_0}{R} f_n e^{i\beta \frac{(t-x)}{C}} \frac{\partial \phi_1}{\partial x} + 
\]

\[
+ \frac{\nu}{R} \left( \frac{U_0}{C} \right) (1 + \frac{V}{R}) (f_n - 4) e^{i\beta \frac{(t-x)}{C}} \frac{\partial \phi_1}{\partial y} = D \frac{\partial^2 \phi_2}{\partial y^2}
\]

1. First Order Term

The first order term \( \phi_1(x,y,t) \) is expressed as

\[
\phi_1(x,y,t) = f_1(x,y) e^{i\beta \frac{(t-x)}{C}}
\]

where \( f_1(x,y) \), the first harmonic fluctuation, is the solution to the differential equation
\[ \frac{\partial^2 f_1}{\partial y^2} - 4 \frac{U_0 \nu}{RD} \frac{\partial f_1}{\partial x} - \frac{i \omega^2}{R^2} \left[ 1 - \frac{4 \nu}{R} \left( \frac{U_0}{C} \right) \right] f_1 = \frac{U_0 \nu}{RD} f_n \frac{\partial \phi_n}{\partial y} + \]

\[ \frac{SC}{R} \left( \frac{U_0}{C} \right) (1 + \frac{\nu}{R}) (f_n - 4) \frac{\partial \phi_n}{\partial y} \]

with the boundary conditions

\[(4-14) \quad f_1(x, y=0) = 0 \]
\[ f_1(x, y = \infty) = 0 \]

The analyses will be limited to the region of very high frequency since this was one of the assumptions in obtaining the velocity distribution.

In keeping with the theory of differential equations containing a large parameter, only the terms with the large frequency parameter and the highest order derivative of \( f_1 \) will be retained.

\[(4-15) \quad \frac{\partial^2 f_1}{\partial y^2} - \frac{i \omega^2}{R^2} \left( 1 - \frac{4 \nu}{R} \left( \frac{U_0}{C} \right) \right) f_1 = Q_1 \]

with \( Q_1 \) standing for the group of terms

\[ Q_1 = \frac{U_0 \nu}{RD} f_n \frac{\partial \phi_n}{\partial x} + \frac{SC}{R} \left( \frac{U_0}{C} \right) (1 + \frac{\nu}{R}) (f_n - 4) \frac{\partial \phi_n}{\partial y} \]

The complementary solution of equation (8-15) is a linear combination of the Airy functions.
\[ f_1 = A_1 \text{Ai} \left( \epsilon_1^{2/3} \zeta \right) + B_1 \text{Ai} \left( \epsilon_1^{2/3} \zeta \right) \]

The symbols \( \epsilon_1 \) and \( \zeta \) are defined as

\[ \epsilon_1 = \frac{-w_{1/2}}{4U_0/C} \quad \zeta = \left( 1 - \frac{4v(U_0)}{R - C} \right) \]

A particular solution is derived by variation of parameters.

\[ f_1 = \frac{\pi R}{4\epsilon_1^{2/3}(U_0/C)} \left\{ \text{Bi}(\epsilon_1^{2/3}\zeta) \int_0^\infty \frac{\text{Ai}(\epsilon_1^{2/3}y)}{y} \right\} \]

\[ \text{Ai}(\epsilon_1^{2/3}\zeta) \int_0^\infty \frac{\text{Bi}(\epsilon_1^{2/3}\zeta)}{\text{Ai}(\epsilon_1^{2/3}y)} \right\} \]

The complementary solution and the particular solution are then added together and the constants \( A_1 \) and \( B_1 \) determined by applying the boundary conditions.

\[ B_1 = 0 \]

\[ A_1 = \frac{-\text{Bi}(\epsilon_1^{2/3}y)}{\text{Ai}(\epsilon_1^{2/3}y)} \int_0^\infty \frac{\text{Ai}(\epsilon_1^{2/3}\zeta)}{y} \right\} \]

Therefore the solution to equation (4-15) which satisfies the conditions listed in (4-14) is

\[ f_1 = \frac{\pi R}{4\epsilon_1^{2/3}(U_0/C)} \left\{ \text{Bi}(\epsilon_1^{2/3}\zeta) \int_0^\infty \frac{\text{Ai}(\epsilon_1^{2/3}y)}{y} \right\} \]

\[ \text{Ai}(\epsilon_1^{2/3}\zeta) \left[ \int_0^\infty \frac{\text{Bi}(\epsilon_1^{2/3}y)}{\text{Ai}(\epsilon_1^{2/3}y)} \right] \]

\[ \text{Ai}(\epsilon_1^{2/3}\zeta) \left[ \int_0^\infty \frac{\text{Bi}(\epsilon_1^{2/3}y)}{\text{Ai}(\epsilon_1^{2/3}y)} \right] \]
The first harmonic fluctuation of the mass flux comes from the above expression.

\[(4-17) \quad j_1 = \frac{\pi RD}{4\varepsilon_1^{2/3}(U_0/C)} \left[ \frac{dB_i(\varepsilon_1^{2/3}\zeta)}{dy} \right]_{y=0}^{\infty} - \frac{Bi(\varepsilon_1^{2/3})}{Ai(\varepsilon_1^{2/3})} \frac{dAi(\varepsilon_1^{2/3}\zeta)}{dy} \right]_0^\infty \int_0^\infty Ai(\varepsilon_1^{2/3}\zeta)Q \, dy
\]

\[= \frac{D}{Ai(\varepsilon_1^{2/3})} \int_0^\infty Ai(\varepsilon_1^{2/3}\zeta)Q \, dy
\]

An approximate value of the integral is obtained by replacing \(Q_1\) in (4-17) by

\[Q_1 \approx \frac{U_0 y^2}{RD} \frac{f_n}{3\varepsilon_1} \frac{\partial^2 \phi_0}{\partial x \partial y} 
+ \frac{Sc(U_0/C)}{R} \left( \frac{1}{R} + \frac{y}{R} \right) \left( f_n - 4 \right) \frac{3\phi_0}{\partial y} \mid_{y=0}
\]

and the resulting definite integral

\[\int_0^\infty Ai(\varepsilon_1^{2/3}\zeta) \left( \frac{U_0 y^2}{RD} \frac{f_n}{3\varepsilon_1} \frac{\partial^2 \phi_0}{\partial x \partial y} \right) \mid_{y=0}^{\infty} \frac{Sc(U_0/C)}{R} \left( \frac{1}{R} + \frac{y}{R} \right) \left( f_n - 4 \right) \frac{3\phi_0}{\partial y} \mid_{y=0}^\infty \, dy
\]

has a value

\[\frac{1}{\pi} \left( \frac{\varepsilon_1^{2/3}}{\sqrt{3}} \varepsilon_1 \right) - \frac{R}{4U_0/C} \exp -\frac{2}{3\varepsilon_1} \left( \frac{\pi}{4} \right) \frac{k}{k} \right]_{k=0}^{\infty} \left( -1 \right)^k \left( \frac{2}{3\varepsilon_1} \right)^{-k} \chi
\]

\[\left\{ \left( \frac{R}{4U_0/C} \right)^2 U_0 f_n \frac{\partial^2 \phi_0}{\partial x \partial y} \right\} \mid_{y=0}^{\infty} \left[ a_k(0,1/3) - 2a_k(2/3,1/3) + a_k(4/3,1/3) \right]
\]

\[+ \frac{Sc(U_0/C)}{R} (f_n-4) \frac{\partial \phi_0}{\partial y} \right\} \mid_{y=0}^{\infty} \left( 1 + \frac{1}{4U_0/C} a_k(0,1/3) - \frac{1}{4U_0/C} a_k(2/3,1/3) \right) \]
The symbol $a_k(\mu, \nu)$ stands for

$$a_k(\mu, \nu) = \frac{(\frac{1}{2} - \frac{1}{3}) k (\frac{1}{2} + \frac{1}{3}) k}{2^k k!} \gamma_2(-k, 1, \mu-k+1/2; \frac{1}{2} + \nu - k, \frac{1}{2} - \nu - k; 2)$$

hence, an approximate expression for $Nu_1$ is given by

$$(4-18) \quad Nu_1 = \frac{-(12)^{1/3}}{2 \Gamma(1/3)} \exp\left(\frac{2}{3} \epsilon_1\right) \frac{\epsilon^{-7/6}}{\sqrt{\pi \text{Ai}(\epsilon_1^{2/3})}} \frac{R}{4U_0/C} x$$

$$\sum_{k=0}^{\infty} (-1)^k \left(\frac{2}{3} \epsilon_1\right)^{-k} \left(-\frac{R}{4U_0/C}\right)^{2} \frac{R}{\text{RD}x} \left(\frac{U_0}{RDX}\right) \times$$

$$[a_k(0,1/3) - 2a_k(2,1/3) + a_k(4,1/3)] + \text{Sc}(\frac{U_0}{C})(f_n-4) \times$$

$$[(1 + \frac{1}{4U_0/C}) a_k(0,1/3) - \frac{1}{4U_0/C} a_k(2,1/3)]$$}

As a check, it is going to be demonstrated that the solution when the ratio $U_0/C$ is zero is the same as that for the first test case.

If $|\epsilon_1| \gg 1$ which would be the case when the frequency is very large for a finite value of $U_0/C$ or the ratio $U_0/C$ is zero for any non-zero value of the frequency, then the Airy function $\text{Ai}(\epsilon_1^{2/3})$ can be replaced by its asymptotic expansion

$$\text{Ai}(\epsilon_1^{2/3}) = \frac{-\frac{5}{2}}{2} (\epsilon_1^{2/3})^{-k} \exp\left(\frac{2}{3} \epsilon_1\right) \gamma_2\left(\frac{1}{6}, \frac{1}{3} \epsilon_1; \frac{1}{4}\right)$$
This expression is substituted in place of \( \text{Ai}(\varepsilon_1^{1/3}) \) in equation (4-18) giving

\[
(4-19) \quad \text{Nu}_1 = \frac{- (12)^{1/3}}{\Gamma (1/3)} \frac{1}{\omega_1^{1/3}} \sum_{k=0}^{\infty} (-1)^k \left( \frac{2}{3} \varepsilon_1 \right)^{-k} \times \\
\left\{ -\left( \frac{R}{4U_0/C} \right)^2 \frac{\beta}{3} \right\} \times \\
\left[ a_k(0,1/3) - 2a_k(\frac{2}{3},1/3) + a_k(\frac{4}{3},1/3) \right] + \\
\text{Sc} \left( \frac{U_0}{C} \right) (\varepsilon_-4) \left[ 1 + \frac{1}{4U_0/C} a_k(0,1/3) - \frac{1}{4U_0/C} a_k(\frac{2}{3},1/3) \right] 
\]

When \( U_0/C \) is zero the sum

\[
\text{Sc} \left( \frac{U_0}{C} \right) (\varepsilon_-4) \sum_{k=0}^{\infty} (-1)^k \left( \frac{2}{3} \varepsilon_1 \right)^{-k} \left[ (1 + \frac{1}{4U_0/C}) a_k(0,1/3) - \frac{1}{4U_0/C} a_k(\frac{2}{3},1/3) \right] 
\]

vanishes and the term

\[
(-1)^k \left( \frac{2}{3} \varepsilon_1 \right)^{-k} \left\{ -\left( \frac{R}{4U_0/C} \right)^2 \frac{\beta}{3} \right\} \times \\
\left[ a_k(0,1/3) - 2a_k(\frac{2}{3},1/3) + a_k(\frac{4}{3},1/3) \right] 
\]

is zero for \( k > 2 \). Hence the remaining terms in (4-15) are:

\[
\text{Nu}_1 = \frac{- (12)^{1/3}}{\Gamma (1/3)} \frac{1}{\omega_1^{1/3}} \sum_{k=0}^{2} (-1)^k \left( \frac{2}{3} \varepsilon_1 \right)^{-k} \times \\
\left\{ -\left( \frac{R}{4U_0/C} \right)^2 \frac{\beta}{3} \right\} \times \\
\left[ a_k(0,1/3) - 2a_k(\frac{2}{3},1/3) + a_k(\frac{4}{3},1/3) \right] 
\]
Further simplification yields

\[ (4-20) \quad \text{Nu}_1 = \frac{-(12)^{1/3}}{\Gamma(1/3) \omega^{1/2}} \int_0^{\infty} \frac{1}{6} \left( \frac{1}{6} ; \frac{1}{2} \right) \frac{1}{3^2} \left( \frac{2\pi R}{RDX} \right) \frac{2f_n R^3 (U_0)}{RDx} \left[ -\frac{1}{3\omega^2} \right] \]

which is identical to the 'approximate' solution of the first harmonic fluctuation of the mass flux for the first test case.

Curves of the phase and amplitude of \( \text{Nu}_1 \) versus \( \omega \) are shown in Figures 7 and 8 for a few values of \( U_0/C \).

2. Second Order Term

The second order term \( \phi_2(x,y,t) \) is decomposed into a steady component and a transient term which is a periodic function of time.

\[ \phi_2(x,y,t) = M(x,y) + f_2(x,y)e^{i\beta(t-x/C)} \]

The differential equations for these components are

\[ (4-21) \quad \frac{\partial^2 M}{\partial y^2} - \frac{4U_0 Y}{RD} \frac{\partial M}{\partial x} = \frac{U_0 Y}{2RD} \left[ \text{Re}(f_n) \text{Re}\left( \frac{\partial f_1}{\partial x} \right) + \text{Im}(f_n) \text{Im}\left( \frac{\partial f_1}{\partial x} \right) \right] \]

\[ + \frac{SC}{2R} \left( \frac{U_0}{C} \right) (1 + \frac{Y}{R}) \left[ \text{Re}(f_n - 4) \text{Re}\left( \frac{\partial f_1}{\partial Y} \right) + \text{Im}(f_n - 4) \text{Im}\left( \frac{\partial f_1}{\partial Y} \right) \right] \]

\[ (4-22) \quad \frac{\partial^2 f_2}{\partial y^2} - \frac{4U_0 Y}{RD} \frac{\partial f_2}{\partial x} = \frac{2i\omega^2}{R^2} \left[ 1 - \frac{4y}{R} \left( \frac{U_0}{C} \right) \right] f_2 = \frac{U_0 Y}{2RD} f_n \frac{\partial f_1}{\partial x} \]

\[ + \frac{SC}{2R} \left( \frac{U_0}{C} \right) (1 + \frac{Y}{R}) (f_n - 4) \frac{\partial f_1}{\partial Y} \]
a. Steady Component

The equation for $M$ is seen to be identical to that for the first test case except for the inhomogeneous term. By comparison, then, the mass flux for this case is given by the expression

\[
J_2 = \frac{2(3)^{1/6} (4^{1/3}) \cos \pi}{3 \Gamma(1/3)} \int_0^\infty \int_0^\infty G(\tau, y) y \left( \frac{U_0}{RD(x-\tau)} \right)^{"/3} x 
\exp \left( -\frac{4}{9} y^3 \left( \frac{U_0}{RD(x-\tau)} \right) \right) d\tau \ dy
\]

where $G(x, y)$ denotes

\[
G(x, y) = \frac{U_0}{2RD} \left[ \Re(f_n) \Re\left( \frac{\partial^2 f_1}{\partial x^2} \right) + \Im(f_n) \Im\left( \frac{\partial^2 f_1}{\partial x^2} \right) \right] + \frac{Sc}{2R C} \left( 1 + \frac{y}{R} \right) \left[ \Re(f_n - 4) \Re\left( \frac{\partial f_1}{\partial y} \right) + \Im(f_n - 4) \Im\left( \frac{\partial f_1}{\partial y} \right) \right]
\]

The flux $J_2$ as given by (4-23) cannot be given in closed form but an approximation can be derived as was done in the case of flow in a rigid tube by replacing $f_1$ by its linear expansion about $y = 0$. Thus the function $G(x, y)$ takes the simpler form

\[
G(x, y) \approx \frac{U_0}{2RD} \left[ \Re(f_n) \Re\left( \frac{\partial^2 f_1}{\partial x^2} \right) y = 0 + \Im(f_n) \Im\left( \frac{\partial^2 f_1}{\partial x^2} \right) y = 0 \right] + \frac{Sc}{2R C} \left( 1 + \frac{y}{R} \right) \left[ \Re(f_n - 4) \Re\left( \frac{\partial f_1}{\partial y} \right) y = 0 + \Im(f_n - 4) \Im\left( \frac{\partial f_1}{\partial y} \right) y = 0 \right]
\]
This expression is substituted in place of $G$ in (4-23) and integration is carried out with respect to $y$ giving

\[
\mathbf{j}_2 = -\frac{D(3)^{1/3}(3)^{1/6}\cos \frac{\pi}{6}}{12} \int_0^x [\text{Re}(f_n) \text{Re} \left( \frac{\partial^2 f_L}{\partial \tau \partial y} \right)_{y=0} + \text{Im}(f_n) \text{Im} \left( \frac{\partial^2 f_L}{\partial \tau \partial y} \right)_{y=0}] \, dt - \frac{D(2)^{1/3}(3)^{1/6}\cos \frac{\pi}{6}}{12} \int_0^x \left[ \text{Re}(f_n-4) \text{Re} \left( \frac{\partial f_L}{\partial y} \right)_{y=0} + \text{Im}(f_n-4) \text{Im} \left( \frac{\partial f_L}{\partial y} \right)_{y=0} \right] \, dt
\]

The first integral in the above expression is then evaluated yielding

\[
(4-24) \quad \mathbf{j}_2 = -\frac{D}{8} [\text{Re}(f_n) \text{Re} \left( \frac{\partial f_L}{\partial y} \right)_{y=0} + \text{Im}(f_n) \text{Im} \left( \frac{\partial f_L}{\partial y} \right)_{y=0}]
\]

\[
-\frac{(2)^{1/3}}{4\Gamma(1/3)} \frac{SCD}{R} \frac{U_0}{C} \int_0^x \left( \frac{U_0}{RD(x-\tau)} \right)^{2/3} \left[ \Gamma \left( \frac{2}{3} \right) + \frac{1}{R} \left( \frac{U_0}{RD(x-\tau)} \right)^{-1/3} \right] \, \mathbf{x}
\]

\[
[\text{Re}(f_n-4) \text{Re} \left( \frac{\partial f_L}{\partial y} \right)_{y=0} + \text{Im}(f_n-4) \text{Im} \left( \frac{\partial f_L}{\partial y} \right)_{y=0}] \, dt
\]

Thus the dimensionless mass flux $\overline{Nu_2}$ is given by
\[ \bar{N}u_2 = \frac{1}{6} \left[ \text{Re}(f_n) \text{Re}(Nu_1) + \text{Im}(f_n) \text{Im}(Nu_1) \right] \]

\[- \frac{U_0^{-1/3}}{4 \Gamma(1/3)} \left( \frac{U_0}{R} \right)^{2/3} \frac{\mathcal{K}^2}{2} \left( \frac{U_0}{R} \right) \int_0^x \left( \frac{U_0}{RD(x-\tau)} \right)^{2/3} [R(2/3) + \frac{1}{R} \left( \frac{U_0}{RD(x-\tau)} \right)^{-1/3}] \]

\[\left[ \text{Re}(f_n - 4) \text{Re}(\frac{3f_1}{\partial y})_{y=0} + \text{Im}(f_n - 4) \text{Im}(\frac{3f_1}{\partial y})_{y=0} \right] \, d\tau \]

The increase in the \( \bar{N}u \) due to pulsation is shown as a function of \( \omega \) at \( \zeta = 0.20 \, 10^{-2} \) on Figure 11.

b. Transient Component

For very high frequencies the first partial terms are again dropped leaving

\[ \frac{3}{U_0} \frac{f_2}{y^2} - 2i \omega \frac{f_2}{R_2} \left[ 1 - \frac{4y}{R} \left( \frac{U_0}{C} \right) \right] f_2 = \frac{U_0 y}{2RD} f_n \frac{3f_1}{\partial x} \]

\[+ \frac{SC}{2R} \left( \frac{U_0}{C} \right) \left( 1 + \frac{y}{R} \right) (f_n - 4) \frac{3f_1}{\partial y} \]

By comparison with the solution for \( f_1 \), the solution for \( f_2 \) can be written as

\[ f_2 = \frac{\pi R}{4 \varepsilon_2^{2/3} U_0 / C} \left( \text{Bi}(\varepsilon_2^{2/3} \zeta) \int_0^\infty \text{Ai}(\varepsilon_2^{2/3} \zeta) Q_2 \, dy \right) \]

\[+ \text{Ai}(\varepsilon_2^{2/3} \zeta) \left( \int_0^Y \text{Bi}(\varepsilon_2^{2/3} \zeta) Q_2 \, dy \right) \]

\[- \frac{\text{Bi}(\varepsilon_2^{2/3} \zeta)}{\text{Ai}(\varepsilon_2^{2/3} \zeta)} \int_0^\infty \text{Ai}(\varepsilon_2^{2/3} \zeta) Q_2 \, dy \} \]
where the symbols $Q_2$ and $\varepsilon_2$ stand for

$$Q_2 = \frac{U_0 Y}{2RD} f_n \frac{3f_1}{3x} + \frac{SC}{2R} \left( \frac{U_0}{C} \right) (1 + \frac{V}{R}) (f_n - 4) \frac{3f_1}{3y}$$

$$\varepsilon_2 = \frac{\sqrt{2 \omega_i k_i}}{4U_0/C}$$

The second harmonic fluctuation of the mass flux is then obtained by differentiating the solution for $f_2$ and taking its value at $y = 0$.

$$j_2 = \frac{D}{Ai(\varepsilon_2^{2/3})} \int_0^\infty Ai(\varepsilon_2^{2/3} \zeta) Q_2 d\zeta$$

The integral $\int_0^\infty Ai(\varepsilon_2^{2/3} \zeta) Q_2 d\zeta$ would be very difficult to evaluate as it is. However, a good approximation can be found by using the expression

$$\frac{U_0 y^2}{2RD} f_n \frac{3^2 f_1}{3x 3y} \bigg|_{y=0} + \frac{SC}{2R} \left( \frac{U_0}{C} \right) (1 + \frac{V}{R}) (f_n - 4) \frac{3f_1}{3y} \bigg|_{y=0}$$

in place of $Q_2$ in the integral. The resulting integral can be evaluated in close form and the approximate solution of the dimensionless mass flux $Nu_2$ is given by

...
\[ \text{(4-27)} \quad \text{Nu}_2 = \frac{-\exp(-\frac{2}{3} \varepsilon_2) R}{2\sqrt{\pi} \text{ Ai}(\varepsilon_2^{2/3})} \left( \frac{U_0}{RDx} \right)^{-1/3} \frac{\varepsilon_2^{-7/6}}{4U_0/C} \times \]

\[ \sum_{k=0}^{\infty} (-1)^k \left( \frac{2}{3} \varepsilon_2 \right)^{-k} \left\{ \left( \frac{R}{4U_0/C} \right)^2 \frac{f_{n+1}}{2RD} \frac{\partial^2 f_{n+1}}{\partial x \partial y} \right\} y=0 \times \]

\[ \left[ a_k(0,1/3) - 2a_k\left( \frac{2}{3}, 1/3 \right) + a_k\left( \frac{4}{3}, 1/3 \right) \right] + \frac{SC}{2R(\frac{U_0}{C})} (f_{n+4}) \frac{\partial f_{n+1}}{\partial y} y=0 \times \]

\[ \left[ (1 + \frac{1}{4U_0/C}) a_k(0,1/3) - \frac{1}{4U_0/C} a_k\left( \frac{2}{3}, 1/3 \right) \right] \]

The phase and amplitude of Nu$_2$ for several values of U$_0$/C are shown in Figures 9 and 10 and also on Table 12.
Figure 7 - Frequency dependence of the phase of Nu, first harmonic
| Figure 8 - Frequency Dependence of the Amplitude of \( N_u \), First Harmonic

- \( |N_u| \) vs. \( \Omega \), Dimensionless Frequency Parameter
- Curves for \( \frac{U_0}{c} = 10^{-2} \), \( 10^{-3} \), \( 10^{-4} \), \( 0 \)
- \( \xi = 0.20 \times 10^{-2} \)
Figure 9 - Frequency dependence of the phase of Nu, second harmonic
Figure 10 - Frequency Dependence of the Amplitude of $\nu$, Second Harmonic

$\xi = 0.20 \times 10^{-2}$

$\frac{U_0}{c} = 10^{-2}$

$\frac{U_0}{c} = 10^{-3}$

$\frac{U_0}{c} = 10^{-4}$

$\frac{U_0}{c} = 0$

Curves indistinguishable
Figure 11 - Frequency dependence of the increase in $\bar{Nu}$ over the steady flow value.
C. Discussion of Results

The results for the distensible conduit case for $\xi = 0.20$ are presented on Figures 7 to 11 and on Tables 11 to 13 in Appendix D. The results for flow in a rigid tube are also shown on the graphs since it has been demonstrated that this is merely a special case of the second model with $U_0/C$ equal to zero, i.e., the wave velocity is infinitely large. All the calculations were carried out for a dimensionless $x$ variable $\xi$ equal to $0.20 \times 10^{-2}$.

The results for the first harmonic fluctuation of the dimensionless mass flux, $Nu$, show a phase lag or a phase advance relative to the pressure gradient $(-\frac{1}{\rho} \frac{\delta P}{\delta x})$ depending on the value of $\omega$ and $U_0/C$. For a non-zero value of $U_0/C$ a phase advance which approaches $\pi/2$ as the frequency becomes infinite is observed. But for a zero value of $U_0/C$ there is always a phase lag which goes to $\pi$ at an infinite frequency. To understand the difference in behaviour at infinite frequency between the case of zero $U_0/C$ and the case of a non-zero $U_0/C$, it must be realized that although the amplitude of the axial velocity fluctuation dies down at infinite frequency, the amplitude of the transverse velocity does not for the case of a non-zero $U_0/C$. For the case of zero $U_0/C$, of course, the transverse velocity is zero at all frequencies. At infinite frequency therefore, there is transport of materials both by convection and by molecular diffusion when $U_0/C$ is not zero but when $U_0/C$ is zero only the molecular diffusion effect is present.
The amplitude of the first harmonic fluctuation increases with the ratio \( U_0/C \) at a given frequency and decreases to zero at infinite frequency for all values of \( U_0/C \).

It is also observed that the steady component of the second order term, \( \bar{Nu}_2 \), has a positive value in the frequency range \( 2 \leq \omega \leq 20 \) in which the calculations were done. For all values of \( U_0/C \) the increase in the Nusselt number vanishes at an infinite frequency of fluctuation. Furthermore, \( \bar{Nu}_2 \) is seen to increase with increasing \( U_0/C \) at any given frequency. It should be stressed, however, that \( U_0/C \) cannot be raised indefinitely to obtain an even larger increase in the Nusselt number since this analysis has been restricted to small values of \( U_0/C \).

The phase of the second harmonic fluctuation, \( Nu_2 \), approaches the value \( -\pi \) as the frequency becomes infinitely large except when \( U_0/C \) is zero in which case the asymptotic value of the phase is zero degree. The amplitude increases for increasing values of \( U_0/C \) at any frequency and it vanishes at infinite frequency for all values of \( U_0/C \).
V. CONCLUSIONS

The following conclusions have been drawn from this investigation.

(1) Pulsations in flow can cause an increase in the time-averaged local mass flux, \( \overline{Nu} \), when the frequency range is above some critical frequency. The increase in \( \overline{Nu} \) over that of Poiseuille flow with mean velocity \( U_0 \) is a function of the wave velocity, the frequency and amplitude of the pulse and the position along the conduit.

(2) Below the critical frequency, pulsations can cause the time-averaged \( Nu \) to be less than that of the steady flow case. This was clearly shown by the study of diffusion in flow in a rigid conduit.

(3) The first harmonic fluctuation of \( Nu \) exhibits a phase lag or a phase advance with respect to the phase of the negative pressure gradient depending on the value of the ratio \( U_0/C \). When \( U_0/C \) is zero, the first harmonic fluctuation experiences a phase lag for all frequencies which approach \( \pi \) at infinite frequency. For non-zero values of \( U_0/C \) there is a phase advance which approaches \( \pi/2 \) asymptotically as the frequency becomes infinite.

(4) The amplitudes of the first and second harmonic fluctuations of \( Nu \) both vanish at infinite frequency.

(5) High and low frequency approximations cover almost the entire frequency range as was shown by a comparison of the analytical solution with a numerical solution for the special case of \( U_0/C \) equal to zero.
(6) The method of solution described herein presumably can be used in the analysis of more complicated models.
### NOMENCLATURE

<table>
<thead>
<tr>
<th>Roman Letters</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
<td>Concentration</td>
</tr>
<tr>
<td>( c_0 )</td>
<td>Uniform concentration at the inlet</td>
</tr>
<tr>
<td>( c_w )</td>
<td>Constant concentration at the wall</td>
</tr>
<tr>
<td>C</td>
<td>Wave velocity</td>
</tr>
<tr>
<td>D</td>
<td>Diffusion coefficient</td>
</tr>
<tr>
<td>( f_1 )</td>
<td>Function related to the first order term of the dimensionless concentration by: ( \phi_1 = f_1 e^{i\beta t} )</td>
</tr>
<tr>
<td>( f_2 )</td>
<td>Function related to the transient component of the second order term of the dimensionless concentration by: ( \phi_2 = M + f_2 e^{i\beta t} )</td>
</tr>
<tr>
<td>j</td>
<td>Mass flux to the wall</td>
</tr>
<tr>
<td>M</td>
<td>Steady component of the second order term of the dimensionless concentration</td>
</tr>
<tr>
<td>( N_{Nu} )</td>
<td>Nusselt number, ( \frac{2R}{y} \frac{\partial \phi}{\partial y} ) ( y=0 )</td>
</tr>
<tr>
<td>( N_{Re} )</td>
<td>Reynolds number</td>
</tr>
<tr>
<td>Nu</td>
<td>Dimensionless mass flux, ( \frac{\partial \phi}{\partial \eta} = 0 )</td>
</tr>
<tr>
<td>( \overline{Nu} )</td>
<td>Dimensionless mass flux averaged over one period</td>
</tr>
<tr>
<td>Nu(_0)</td>
<td>Dimensionless flux for steady flow, ( \frac{\partial \phi_0}{\partial \eta} \right</td>
</tr>
<tr>
<td>Nu(_1)</td>
<td>First harmonic of Nu, defined as ( \frac{\partial f_1}{\partial \eta} \right</td>
</tr>
<tr>
<td>(</td>
<td>Nu_1</td>
</tr>
<tr>
<td>Nu(_2)</td>
<td>Second harmonic of Nu, defined as ( \frac{\partial f_2}{\partial \eta} \right</td>
</tr>
<tr>
<td>Roman Letters</td>
<td>Definition</td>
</tr>
<tr>
<td>---------------</td>
<td>------------</td>
</tr>
<tr>
<td>(</td>
<td>Nu_2</td>
</tr>
<tr>
<td>(Nu_2)</td>
<td>Steady component of the second order term of (Nu),</td>
</tr>
<tr>
<td>(R)</td>
<td>Radius of the conduit</td>
</tr>
<tr>
<td>(Sc)</td>
<td>Schmidt number</td>
</tr>
<tr>
<td>(U_0)</td>
<td>Time-averaged mean axial velocity</td>
</tr>
<tr>
<td>(\text{Im}(z))</td>
<td>Imaginary part of the complex variable (z)</td>
</tr>
<tr>
<td>(\text{Re}(z))</td>
<td>Real part of the complex variable (z)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Greek Letters</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha)</td>
<td>Radial expansion of the tube wall</td>
</tr>
<tr>
<td>(\lambda)</td>
<td>Dimensionless amplitude of the fluctuating component of ((-\frac{1}{\rho} \frac{\partial P}{\partial x}))</td>
</tr>
<tr>
<td>(\xi)</td>
<td>Dimensionless (x) variable, (\frac{\text{Dx}}{U_0 R^2})</td>
</tr>
<tr>
<td>(\eta)</td>
<td>Dimensionless variable (\frac{U_0}{\text{RDx}}) (^{1/3})</td>
</tr>
<tr>
<td>(\omega)</td>
<td>Dimensionless frequency parameter, (\frac{BR^2}{\nu\xi})</td>
</tr>
<tr>
<td>(\omega)</td>
<td>Dimensionless frequency parameter, (\frac{BR^2}{D})</td>
</tr>
<tr>
<td>(\beta)</td>
<td>Frequency in radians per unit time</td>
</tr>
<tr>
<td>(\psi_1)</td>
<td>Phase of (Nu_1)</td>
</tr>
<tr>
<td>(\psi_2)</td>
<td>Phase of (Nu_2)</td>
</tr>
<tr>
<td>(\phi)</td>
<td>Dimensionless concentration variable, (\frac{C - \text{cw}_1}{C_0 - \text{cw}})</td>
</tr>
<tr>
<td>Greek Letters</td>
<td>Definition</td>
</tr>
<tr>
<td>---------------</td>
<td>------------</td>
</tr>
<tr>
<td>$\phi_0$</td>
<td>Zeroth order term in the perturbation expansion of $\phi$</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>First order term in the perturbation expansion of $\phi$</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>Second order term in the perturbation expansion of $\phi$</td>
</tr>
</tbody>
</table>
REFERENCES


APPENDIX A

THE INVERSE TRANSFORM OF THE PRODUCT

\[ \{ \text{Ai}(\sqrt{\frac{4ux}{\pi RD}})^{1/3} \} \text{L}[G(x)] \]

By the convolution operation the inverse of the product \( \{ f(s) \text{L}[G(x)] \} \) is expressed as

\[
L^{-1}\{ f(s) \text{L}[G(x)] \} = \int_{0}^{x} F(x-\tau)G(\tau)d\tau
\]

where \( F(x) \) is the inverse transform of \( f(s) \), provided \( F(x) \) and \( G(x) \) are both sectionally continuous on each interval \( 0 \leq x \leq X \) and are of exponential order as \( x \) tends to infinity.

To obtain the inverse of \( f(s) \) the following theorem (Churchill, 'Operational Mathematics') will be used. It gives the sufficient conditions on any function \( f(s) \) such that a function \( F(x) \) exists whose transform is \( f(s) \) and that \( F(x) \) is given by the inversion integral

\[
F(x) = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma-i\beta}^{\gamma+i\beta} e^{st} f(s)ds
\]

THEOREM. If \( f(s) \) is any function of the complex variable \( s \), \( s = u + iv \), that is

1. analytic for all \( s \) in a half plane \( u \geq u_0 \)
2. of order \( O(s^{-k}) \) for all \( s \) in a half plane \( u \geq u_0 \)
3. real for real \( s \) and \( u \geq u_0 \)
then the inversion integral of \( f(s) \) along any line \( u = \gamma \), where \( \gamma \geq u_0 \), converges to a real-valued function \( F(x) \) that is independent of \( \gamma \).

It shall be presently demonstrated that the function

\[
f(s) = \text{Ai}((\frac{4u^3}{RD})^{1/3}y)
\]

meets all the conditions of the above theorem. The function \( \text{Ai}((\frac{4u^3}{RD})^{1/3}y) \) is analytic everywhere except on the branch cut \( \theta = \pi, r \geq 0 \). In the half plane \( \text{Re}(s) \geq \gamma \) where \( \gamma \) is a positive constant and \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\), the function \( s^k f(s) \) is continuous when \( r \geq \gamma \), and

\[
\lim_{r \to \infty} |s^k f(s)| = 0
\]

Therefore, \( |f(s)| \) is of order \( O(s^{-k}) \) in the half plane for every value of the constant \( k \). The third condition of the theorem is likewise satisfied since the Airy function \( \text{Ai} \) takes on real values for all real, positive arguments. Hence the inverse transform \( F(x) \) is defined by the inversion integral.

\[
(A-1) \quad F(x) = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma-i\beta}^{\gamma+i\beta} \text{Ai}((\frac{4u^3}{RD})^{1/3}y) \, e^{sx} \, ds
\]

Since the singular points of the function \( f(s) \) are not isolated the method of residues cannot be conveniently employed to evaluate the complex inversion integral. However, the integral can be reduced to a more desirable form by deforming the path of integration. Consider the path of integration ABCDD'C'B'A'A shown in the figure below.
From Cauchy's theorem, it is known that the integral around the loop in the direction indicated by the arrow is zero. The integral in (A-1) can therefore be written as

\[
(A-2) \quad - \int_{\gamma-i\beta}^{\gamma+i\beta} e^{st} \text{Ai}((\frac{4Ugs}{RD})^{1/3}y) \, ds = I_{AC} + I_{CD} + I_{DD'} + I_{D'C'} + I_{C'A'}
\]

where \(I_{AC}\) denotes the value of the integral over the arc AC. The other subscripts on I have a similar meaning.

Let \(R_0\) and \(r_0\) be the radii of the large and the small circular arcs respectively, and \(\theta_0\) the angle between DC or D'C' and the negative real axis. For any fixed \(R_0\) the limit of the integrals \(I_{AC}\) and \(I_{A'C'}\) as \(\theta_0\to 0\) exists. Also for any fixed positive \(r_0\) the limits of the integrals over the other parts of the path exist. Since (A-2) holds for every
positive $\theta_0$ and the integral on the left is independent of $\theta_0$, it follows that each of the integrals on the right hand side of (A-2) can take their limiting values as $\theta_0 \to 0$. The symbol $J_{AC}$ will be used to denote the $\lim_{\theta \to 0} I_{AC}$ and similar expressions will be used for the integrals over the other arcs and lines. The integrals $J_{AC}$, $J_{CD}$, etc. are then evaluated for $r_0 \to 0$ and $R_0 \to \infty$.

Along the arc $DD'$:

When $s$ is on the arc $DD'$, $r = r_0$ and $s = r_0 \exp(i\theta)$. The integral over the circle can then be written as

$$J_{DD'} = i \int_{\pi}^{-\pi} \text{Ai}((\frac{r_0e^{i\theta}}{RD})^{1/3}y)r_0e^{i\theta} \exp(r_0e^{i\theta}x) d\theta$$

The integrand is a continuous function of $\theta$ and $r_0$ when $r_0 \geq 0$. Therefore

$$\lim_{r_0 \to 0} J_{DD'} = 0$$

Along the lines $CD$ and $D'C'$:

On $CD$ and $D'C'$, the $\lim s = -r$, and therefore $\theta_0 \to C$

$$J_{CD} + J_{D'C'} = -\int_{CD} \text{Ai}((\frac{-4Us}{RD})^{1/3}y)e^{-rx} dr$$

$$-\int_{D'C'} \text{Ai}((\frac{-4Us}{RD})^{1/3}y) e^{-rx} dr$$

The function $\text{Ai}$ is then expanded in a power series taking note that $s^{1/3} = r^{1/3}i^{2/3}$ on $CD$ and $s^{1/3} = r^{1/3}i^{-2/3}$ on $D'C'$. 
\[ J_{CD} + J_{D',C'} = \int_{0}^{\infty} \sum_{k=0}^{\infty} \frac{(4U_s)k+1/3(-r)^k r^{1/3}i^{2/3}y^{3k+1} e^{-rx}}{(3)^{2k+1/3} k! \Gamma(1/3 + k + 1)} dr + \]

\[ \int_{r_0}^{\infty} \sum_{k=0}^{C} \frac{(4U_s)k+1/3(-r)^k r^{1/3}i^{-2/3}y^{3k+1} e^{-rx}}{(3)^{2k+1} k! \Gamma(1/3 + k + 1)} dr \]

The limit of the sum is now taken as \( r_0 \to 0 \) and \( R_0 \to \infty \).

\[ \lim_{r_0 \to 0} \lim_{R_0 \to \infty} [J_{CD} + J_{D',C'}] = \left(\frac{i^{2/3} + i^{-2/3}}{3x}\right)^{\infty} \sum_{k=0}^{\infty} \frac{(4U_s)k+1/3(-1)^k y^{3k+1}}{(3)^{2k+1/3} k!} \]

This can also be written as

\[ \lim_{r_0 \to 0} \lim_{R_0 \to \infty} [J_{CD} + J_{D',C'}] = \frac{\left(\frac{i^{2/3} + i^{-2/3}}{3x}\right)^{1/3}}{3\left(\frac{U_s}{RD}\right)} y^{\frac{U_s}{RDx}^{1/3}} e^{\frac{4U_s}{9y^3} \left(\frac{U_s}{RD}\right)} \]

Along the arcs ABC and C'B'A':

When \( |s| \) is large, the asymptotic expansion of \( \text{Ai} \) can be used to give

\[ J_{ABC} < \frac{1}{2\sqrt{\pi}} \int_{\text{ABC}} \exp\left(xs - \frac{2}{3}\left(\frac{4U_s}{RD}\right)^{1/3} y^{3/2}\right) ds \]

\[ \frac{\exp(xs - \frac{2}{3}\left(\frac{4U_s}{RD}\right)^{1/3} y^{3/2})}{\left[\left(\frac{4U_s}{RD}\right)^{1/3} y\right]^k} \]
On the arc AB,

\[ |J_{AB}| < \frac{R_0^{11/12} e^{YX}}{2\sqrt{\pi} \left[ \left( \frac{4U_0}{RD} \right)^{1/3} \right]^2} \int_{\theta_A}^{\pi/2} e^{\frac{11}{12} \theta} \, d\theta = \frac{R_0^{11/12} e^{YX}}{2\sqrt{\pi} \left[ \left( \frac{4U_0}{RD} \right)^{1/3} \right]^2} \left( \frac{\pi}{2} - \theta_A \right) \]

and by applying L'Hospital's rule it is seen that the limit of \( |J_{AB}| \) tends to zero as \( R_0 \to \infty \) and \( \theta_A \to \frac{\pi}{2} \). In a similar manner the limit of \( J_{B'C'} \) can be proved to vanish as \( R_0 \to \infty \).

When \( s \) is on the arc BC,

\[ |J_{BC}| < \frac{R_0^{1/2} e^{YX}}{2\sqrt{\pi} \left[ \left( \frac{4U_0}{RD} \right)^{1/3} \right]^2} \int_{\pi/2}^{\pi} e^{sR_0} \cos \theta \, d\theta = \frac{\pi}{4\sqrt{\pi} \cdot xR_0^{1/2} \left[ \left( \frac{4U_0}{RD} \right)^{1/3} \right]^2} (1 - e^{-R_0X}) \]

Hence, \( |J_{BC}| \) vanishes as \( R_0 \) becomes infinite provided \( x > 0 \). Likewise the limit of \( J_{C'B'} \) is zero.

It therefore follows from equation (A-1) that the inverse function of \( f(s) \) is

(A-3) \[ L^{-1} \{ f(s) \} = \frac{\left( \frac{4}{3} \right)^{1/3} \cos \theta}{3\pi \left( \frac{U_0}{RD} \right)} \int \left( \frac{U_0}{RD} \right)^{-3/3} \exp \left( -\frac{4}{3}Y \left( \frac{U_0}{RD} \right) \right) \]

and the inverse of the product \( f(s) L[G(x)] \) is given by

(A-4) \[ L^{-1} \{ f(s) L[G(x)] \} = \frac{\left( \frac{4}{3} \right)^{1/3} \cos \theta}{3\pi \left( \frac{U_0}{RD} \right)} \int \frac{\theta}{G(\tau)} \left( \frac{U_0}{RD} \right)^{-3/3} \exp \left( -\frac{4}{3}Y \left( \frac{U_0}{RD} \right) \right) \]

\[ \exp \left[ -\frac{4}{3}Y \left( \frac{U_0}{RD} (x-\tau) \right) \right] \, d\tau \]
APPENDIX B

SOLUTION OF A DIFFERENTIAL EQUATION CONTAINING A LARGE PARAMETER

The equation in question is

\[ \frac{\partial^2 z}{\partial y^2} + p(y) \frac{\partial z}{\partial x} + q(y, \lambda) = 0 \]  

where the absolute value of \( \lambda \) is very large and \( q(y, \lambda) \) does not vanish in the interval where \( y \) varies.

Assuming that \( q(y, \lambda) = \lambda^2 q_0(y) \), we obtain a formal solution to (B-1) which has the form

\[ \sum_{n=0}^{\infty} a_n(x,y) \lambda^{-n} \exp \{ \beta_0(y) \lambda \} \]  

Substituting (B-2) into (B-1) we get

\[ \sum_{n=2}^{\infty} \frac{\partial^2 a_{n-2}}{\partial y^2} \lambda^{-n+2} + 2 \sum_{n=1}^{\infty} \beta_0 \frac{\partial a_{n-1}}{\partial y} \lambda^{-n+2} 
+ \sum_{n=1}^{\infty} a_{n-1} \beta_0'' \lambda^{-n+2} + \sum_{n=0}^{\infty} a_n (\beta_0')^2 \lambda^{-n+2} 
+ p(y) \sum_{n=2}^{\infty} \frac{\partial a_{n-2}}{\partial x} \lambda^{-n+2} + q_0(y) \sum_{n=0}^{\infty} a_n \lambda^{-n+2} = 0 \]

The following equations are then derived from (B-3) by equating the two lowest powers of \( \lambda \) to zero.
(B-4) \[ (\beta_0')^2 + q_0 = 0 \]

(B-5) \[ 2\beta_0 \frac{3a_0}{\beta y} + a_0 \beta_0'' = 0 \]

Equation (B-4) defines the term \( \beta_0 \)

\[ \beta_0 = \int (-q_0)^{\frac{1}{4}} \, dy \]

and (B-5) gives

\[ a_0 = \frac{1}{(-q_0)^{\frac{1}{4}}} \]

Therefore the dominant term in the formal solution is

\[ a_0(x,y)\exp(\beta_0\lambda) = \left[ \exp(\lambda \int (-q_0)^{\frac{1}{4}} \, dy) \right] / (-q_0)^{\frac{1}{4}} \]

It can be seen by examining the terms in (B-4) and (B-5) that the dominant term comes only from the second order differential term and the term containing the large parameter \( \lambda \). Corresponding to the two branches of \((-q_0)^{\frac{1}{4}}\) two formal solutions of the form (B-2) are obtained.

It has been shown by Erdelyi\(^7\) that the formal solutions of the form (B-2) are asymptotic with respect to the parameter \( \lambda \).
APPENDIX C

TABULATED RESULTS FOR THE CASE OF FLOW IN A RIGID CONDUIT
<table>
<thead>
<tr>
<th>ω</th>
<th>S₁</th>
<th>S₂</th>
<th>S₁*</th>
<th>S₂*</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>0.10256499E-05</td>
<td>0.33466061E-06</td>
<td>-0.44721359E-04</td>
<td>0.44721359E-04</td>
</tr>
<tr>
<td>0.20</td>
<td>0.63971521E-06</td>
<td>0.44016537E-06</td>
<td>-0.55901699E-05</td>
<td>0.55901699E-05</td>
</tr>
<tr>
<td>0.30</td>
<td>0.36420850E-06</td>
<td>0.42512067E-06</td>
<td>-0.16563425E-05</td>
<td>0.16563425E-05</td>
</tr>
<tr>
<td>0.40</td>
<td>0.18524954E-06</td>
<td>0.35976356E-06</td>
<td>-0.69877123E-06</td>
<td>0.69877123E-06</td>
</tr>
<tr>
<td>0.50</td>
<td>0.77986569E-07</td>
<td>0.28313318E-06</td>
<td>-0.35770878E-06</td>
<td>0.35770878E-06</td>
</tr>
<tr>
<td>0.60</td>
<td>0.18681261E-07</td>
<td>0.21347426E-06</td>
<td>-0.20704333E-06</td>
<td>0.20704333E-06</td>
</tr>
<tr>
<td>0.70</td>
<td>-0.11017763E-07</td>
<td>0.15706549E-06</td>
<td>-0.13038297E-06</td>
<td>0.13038297E-06</td>
</tr>
<tr>
<td>0.80</td>
<td>-0.23695606E-07</td>
<td>0.11424489E-06</td>
<td>-0.87346400E-07</td>
<td>0.87346400E-07</td>
</tr>
<tr>
<td>0.90</td>
<td>-0.27284108E-07</td>
<td>0.82958987E-07</td>
<td>-0.61346171E-07</td>
<td>0.61346171E-07</td>
</tr>
<tr>
<td>1.00</td>
<td>-0.26449921E-07</td>
<td>0.60590710E-07</td>
<td>-0.44721359E-07</td>
<td>0.44721359E-07</td>
</tr>
<tr>
<td>1.50</td>
<td>-0.12363696E-07</td>
<td>0.15711529E-07</td>
<td>-0.13250773E-07</td>
<td>0.13250773E-07</td>
</tr>
<tr>
<td>2.00</td>
<td>-0.55282266E-08</td>
<td>0.60405070E-08</td>
<td>-0.55901699E-08</td>
<td>0.55901699E-08</td>
</tr>
<tr>
<td>2.50</td>
<td>-0.28549616E-08</td>
<td>0.29769008E-08</td>
<td>-0.28621669E-08</td>
<td>0.28621669E-08</td>
</tr>
<tr>
<td>3.00</td>
<td>-0.16550658E-08</td>
<td>0.16940387E-08</td>
<td>-0.16563425E-08</td>
<td>0.16563425E-08</td>
</tr>
<tr>
<td>4.00</td>
<td>-0.69868176E-09</td>
<td>0.70536628E-09</td>
<td>-0.69877123E-09</td>
<td>0.69877123E-09</td>
</tr>
<tr>
<td>5.00</td>
<td>-0.35775916E-09</td>
<td>0.35948886E-09</td>
<td>-0.35777087E-09</td>
<td>0.35777087E-09</td>
</tr>
<tr>
<td>6.00</td>
<td>-0.20704107E-09</td>
<td>0.20761706E-09</td>
<td>-0.20704333E-09</td>
<td>0.20704333E-09</td>
</tr>
<tr>
<td>7.00</td>
<td>-0.13038240E-09</td>
<td>0.13061017E-09</td>
<td>-0.13038297E-09</td>
<td>0.13038297E-09</td>
</tr>
<tr>
<td>10.00</td>
<td>-0.44721330E-10</td>
<td>0.44748046E-10</td>
<td>-0.44721359E-10</td>
<td>0.44721359E-10</td>
</tr>
</tbody>
</table>
TABLE 2
PHASE AND AMPLITUDE OF Nu₁ AT ζ = 0.20×10⁻²,
HIGH FREQUENCY SOLUTION

<table>
<thead>
<tr>
<th>'Exact Solution'</th>
<th>'Approximate Solution'</th>
</tr>
</thead>
<tbody>
<tr>
<td>ψ₁</td>
<td></td>
</tr>
<tr>
<td>.05</td>
<td>-9°16'</td>
</tr>
<tr>
<td>.10</td>
<td>-18°09'</td>
</tr>
<tr>
<td>.20</td>
<td>-34°49'</td>
</tr>
<tr>
<td>.30</td>
<td>-50°04'</td>
</tr>
<tr>
<td>.40</td>
<td>-63°54'</td>
</tr>
<tr>
<td>.50</td>
<td>-76°03'</td>
</tr>
<tr>
<td>.60</td>
<td>-87°34'</td>
</tr>
<tr>
<td>.70</td>
<td>-97°31'</td>
</tr>
<tr>
<td>.80</td>
<td>-106°16'</td>
</tr>
<tr>
<td>.90</td>
<td>-113°57'</td>
</tr>
<tr>
<td>1.00</td>
<td>-120°39'</td>
</tr>
<tr>
<td>1.50</td>
<td>-143°21'</td>
</tr>
<tr>
<td>2.00</td>
<td>-156°29'</td>
</tr>
<tr>
<td>2.50</td>
<td>-165°04'</td>
</tr>
<tr>
<td>3.00</td>
<td>-170°09'</td>
</tr>
<tr>
<td>4.00</td>
<td>-174°01'</td>
</tr>
<tr>
<td>5.00</td>
<td>-175°14'</td>
</tr>
<tr>
<td>6.00</td>
<td>-176°05'</td>
</tr>
<tr>
<td>7.00</td>
<td>-176°44'</td>
</tr>
<tr>
<td>10.00</td>
<td>-177°48'</td>
</tr>
<tr>
<td>( \psi_1 )</td>
<td>(</td>
</tr>
<tr>
<td>------</td>
<td>----------------</td>
</tr>
<tr>
<td>0.00</td>
<td>0°</td>
</tr>
<tr>
<td>0.05</td>
<td>-0°30'</td>
</tr>
<tr>
<td>0.10</td>
<td>-2°03'</td>
</tr>
<tr>
<td>0.20</td>
<td>-8°09'</td>
</tr>
<tr>
<td>0.30</td>
<td>-17°55'</td>
</tr>
<tr>
<td>0.40</td>
<td>-30°05'</td>
</tr>
<tr>
<td>0.50</td>
<td>-42°38'</td>
</tr>
<tr>
<td>0.60</td>
<td>-53°38'</td>
</tr>
<tr>
<td>0.70</td>
<td>-63°01'</td>
</tr>
<tr>
<td>0.80</td>
<td>-70°22'</td>
</tr>
<tr>
<td>0.90</td>
<td>-76°18'</td>
</tr>
<tr>
<td>1.00</td>
<td>-81°13'</td>
</tr>
</tbody>
</table>
TABLE 4

PHASE AND AMPLITUDE OF $\text{Nu}_2$ AT $\xi = 0.20 \times 10^{-2}$,

HIGH FREQUENCY SOLUTION

'Exact Solution'           'Approximate Solution'

<p>| $\psi_2$ | $|\text{Nu}_2|$ | $\psi_2$ | $|\text{Nu}_2|$ |
|---------|----------------|---------|----------------|
| 0.05    | 344°09'        | 0.18179325E 00 | 221°26'        | 0.50780605E 02 |
| 0.10    | 328°50'        | 0.14256571E 00 | 219°33'        | 0.62427533E 01 |
| 0.20    | 299°46'        | 0.81310672E-01 | 208°11'        | 0.71877205E 00 |
| 0.30    | 272°43'        | 0.46479004E-01 | 194°46'        | 0.18585990E 00 |
| 0.40    | 247°43'        | 0.26665357E-01 | 180°57'        | 0.65863606E-01 |
| 0.50    | 224°35'        | 0.15345598E-01 | 167°20'        | 0.27532959E-01 |
| 0.60    | 203°20'        | 0.88964314E-02 | 154°20'        | 0.12785137E-01 |
| 0.70    | 180°52'        | 0.51896007E-02 | 142°02'        | 0.63704040E-02 |
| 0.80    | 166°07'        | 0.30591548E-02 | 130°34'        | 0.33542884E-02 |
| 0.90    | 150°02'        | 0.18612542E-02 | 119°56'        | 0.18397925E-02 |
| 1.00    | 135°26'        | 0.10919468E-02 | 110°07'        | 0.10434631E-02 |
| 1.50    | 81°55'         | 0.10489263E-03 | 72°54'         | 0.91483476E-04 |
| 2.00    | 50°30'         | 0.14399345E-04 | 46°37'         | 0.12765606E-04 |
| 2.50    | 31°26'         | 0.26616780E-05 | 29°39'         | 0.24050175E-05 |
| 3.00    | 20°32'         | 0.55209048E-06 | 19°34'         | 0.56870231E-06 |
| 4.00    | 12°17'         | 0.61068822E-07 | 11°54'         | 0.56096075E-07 |
| 5.00    | 9°41'          | 0.10385860E-07 | 9°29'          | 0.95981242E-08 |
| 6.00    | 7°54'          | 0.24659957E-08 | 7°48'          | 0.22821786E-08 |
| 7.00    | 6°36'          | 0.72923543E-09 | 6°32'          | 0.67536156E-09 |
| 10.00   | 4°25'          | 0.43263974E-10 | 4°23'          | 0.40103929E-10 |
| $\omega$ | $\psi_2$ | $|\text{Nu}_2|$ |
|-------|--------|---------------|
| 0.00  | 180°   | .47456024E-01|
| 0.05  | 178°08' | .47480148E-01|
| 0.10  | 172°34' | .47830817E-01|
| 0.20  | 157°01' | .51281407E-01|
| 0.30  | 129°39' | .72399514E-01|
| 0.40  | 113°41' | .10824809E 00|
| 0.50  | 103°21' | .16320772E 00|
| 0.60  | 97°00'  | .22565869E 00|
| 0.70  | 91°56'  | .30346008E 00|
| 0.80  | 87°42'  | .39626111E 00|
| 0.90  | 83°51'  | .50465157E 00|
| 1.00  | 80°10'  | .62880756E 00|</p>
<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$\overline{\text{Nu}}_2$</th>
<th>$\frac{(\text{Nu}-\text{Nu}_0) \times 100}{\chi^2 \text{Nu}_0}$</th>
<th>$\frac{(\text{Nu}-\text{Nu}_0)}{\chi^2 \text{Nu}_0} \times 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.05</td>
<td>- .39451807E 00</td>
<td>- .4616E 02</td>
<td>- .9232E-02</td>
</tr>
<tr>
<td>.10</td>
<td>- .36240252E 00</td>
<td>- .4240E 02</td>
<td>- .8480E 02</td>
</tr>
<tr>
<td>.20</td>
<td>- .30106396E 00</td>
<td>- .3522E 02</td>
<td>- .7044E-02</td>
</tr>
<tr>
<td>.30</td>
<td>- .24522402E 00</td>
<td>- .2869E 02</td>
<td>- .5738E 02</td>
</tr>
<tr>
<td>.40</td>
<td>- .19596774E 00</td>
<td>- .2293E 02</td>
<td>- .4586E-02</td>
</tr>
<tr>
<td>.50</td>
<td>- .15332310E 00</td>
<td>- .1794E 02</td>
<td>- .3588E-02</td>
</tr>
<tr>
<td>.60</td>
<td>- .11764490E 00</td>
<td>- .1376E 02</td>
<td>- .2752E-02</td>
</tr>
<tr>
<td>.70</td>
<td>- .88093908E-01</td>
<td>- .1031E 02</td>
<td>- .2062E-02</td>
</tr>
<tr>
<td>.80</td>
<td>- .64304279E-01</td>
<td>- .7523E 01</td>
<td>- .1505E-02</td>
</tr>
<tr>
<td>.90</td>
<td>- .45301397E-01</td>
<td>- .5300E 01</td>
<td>- .1060E-02</td>
</tr>
<tr>
<td>1.00</td>
<td>- .30438827E-01</td>
<td>- .3561E 01</td>
<td>- .7122E-03</td>
</tr>
<tr>
<td>1.50</td>
<td>+ .25877555E-02</td>
<td>+ .3028E 00</td>
<td>+ .6056E-04</td>
</tr>
<tr>
<td>2.00</td>
<td>+ .55882256E-02</td>
<td>+ .6538E 00</td>
<td>+ .1308E-03</td>
</tr>
<tr>
<td>2.50</td>
<td>+ .32916874E-02</td>
<td>+ .3851E 00</td>
<td>+ .7702E-04</td>
</tr>
<tr>
<td>3.00</td>
<td>+ .16064845E-02</td>
<td>+ .1880E 00</td>
<td>+ .3760E-04</td>
</tr>
<tr>
<td>4.00</td>
<td>+ .41579709E-03</td>
<td>+ .4865E-01</td>
<td>+ .9730E-05</td>
</tr>
<tr>
<td>5.00</td>
<td>+ .14108115E-03</td>
<td>+ .1651E-01</td>
<td>+ .3302E-05</td>
</tr>
<tr>
<td>6.00</td>
<td>+ .58235716E-04</td>
<td>+ .6814E-02</td>
<td>+ .1363E-05</td>
</tr>
<tr>
<td>7.00</td>
<td>+ .27412180E-04</td>
<td>+ .3207E-02</td>
<td>+ .6414E-06</td>
</tr>
<tr>
<td>10.00</td>
<td>+ .47524153E-05</td>
<td>+ .5560E-03</td>
<td>+ .1112E-06</td>
</tr>
</tbody>
</table>
### TABLE 6-B

**INCREASE IN \( \bar{\nu}_s \) OVER THE STEADY FLOW VALUE,**

**HIGH FREQUENCY SOLUTION AT \( \xi = 0.10 \times 10^{-2} \)**

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \frac{\bar{\nu}_s}{\bar{\nu}_0} )</th>
<th>( \frac{(\bar{\nu}_s - \bar{\nu}_0) \times 100}{\lambda^2 \bar{\nu}_0} )</th>
<th>( \frac{(\bar{\nu}_s - \bar{\nu}_0) \xi \times 100}{\lambda^2 \bar{\nu}_0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>-.37164290E 00</td>
<td>-.4348E 02</td>
<td>-.4348E-01</td>
</tr>
<tr>
<td>0.10</td>
<td>-.31835886E 00</td>
<td>-.3725E 02</td>
<td>-.3725E-01</td>
</tr>
<tr>
<td>0.20</td>
<td>-.22370298E 00</td>
<td>-.2617E 02</td>
<td>-.2617E-01</td>
</tr>
<tr>
<td>0.30</td>
<td>-.14862185E 00</td>
<td>-.1739E 02</td>
<td>-.1739E-01</td>
</tr>
<tr>
<td>0.40</td>
<td>-.93001184E-01</td>
<td>-.1088E 02</td>
<td>-.1088E-01</td>
</tr>
<tr>
<td>0.50</td>
<td>-.53895170E-01</td>
<td>-.6306E 01</td>
<td>-.6306E-02</td>
</tr>
<tr>
<td>0.60</td>
<td>-.27906772E-01</td>
<td>-.3265E 01</td>
<td>-.3265E-02</td>
</tr>
<tr>
<td>0.70</td>
<td>-.11428986E-01</td>
<td>-.1337E 01</td>
<td>-.1337E-02</td>
</tr>
<tr>
<td>0.80</td>
<td>-.16040780E-02</td>
<td>-.1877E 00</td>
<td>-.1877E-03</td>
</tr>
<tr>
<td>0.90</td>
<td>+.38236452E-02</td>
<td>+.4474E 00</td>
<td>+.4474E-03</td>
</tr>
<tr>
<td>1.00</td>
<td>+.64408590E-02</td>
<td>+.7536E 00</td>
<td>+.7536E-03</td>
</tr>
<tr>
<td>1.50</td>
<td>+.52576378E-02</td>
<td>+.6151E 00</td>
<td>+.6151E-03</td>
</tr>
<tr>
<td>2.00</td>
<td>+.21740206E-02</td>
<td>+.2544E 00</td>
<td>+.2544E-03</td>
</tr>
<tr>
<td>2.50</td>
<td>+.85250400E-03</td>
<td>+.9974E-01</td>
<td>+.9974E-04</td>
</tr>
<tr>
<td>3.00</td>
<td>+.35596822E-03</td>
<td>+.4165E-01</td>
<td>+.4165E-04</td>
</tr>
<tr>
<td>4.00</td>
<td>+.84643584E-04</td>
<td>+.9903E-02</td>
<td>+.9903E-05</td>
</tr>
<tr>
<td>5.00</td>
<td>+.28329012E-04</td>
<td>+.3314E-02</td>
<td>+.3314E-05</td>
</tr>
<tr>
<td>6.00</td>
<td>+.11661126E-04</td>
<td>+.1364E-02</td>
<td>+.1364E-05</td>
</tr>
<tr>
<td>7.00</td>
<td>+.54848814E-05</td>
<td>+.6417E-03</td>
<td>+.6417E-06</td>
</tr>
<tr>
<td>10.00</td>
<td>+.95051952E-06</td>
<td>+.1112E-03</td>
<td>+.1112E-06</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$\bar{\text{Nu}}_2$</td>
<td>$\frac{(\text{Nu}-\text{Nu}_0) \times 100}{\lambda^2 \text{Nu}}$</td>
<td>$\frac{(\text{Nu}-\text{Nu}_0) \xi \times 100}{\lambda^2 \text{Nu}}$</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>0.05</td>
<td>-3.5751090E-00</td>
<td>-4.183E-02</td>
<td>-8.366E-01</td>
</tr>
<tr>
<td>0.10</td>
<td>-2.9204325E-00</td>
<td>-3.417E-02</td>
<td>-6.834E-01</td>
</tr>
<tr>
<td>0.20</td>
<td>-1.8215235E-00</td>
<td>-2.131E-02</td>
<td>-4.262E-01</td>
</tr>
<tr>
<td>0.30</td>
<td>-1.0370464E-00</td>
<td>-1.213E-02</td>
<td>-2.426E-01</td>
</tr>
<tr>
<td>0.40</td>
<td>-5.2747906E-01</td>
<td>-6.172E-01</td>
<td>-1.234E-01</td>
</tr>
<tr>
<td>0.50</td>
<td>-2.2161512E-01</td>
<td>-2.593E-01</td>
<td>-5.186E-02</td>
</tr>
<tr>
<td>0.60</td>
<td>-5.2747896E-02</td>
<td>-6.172E-00</td>
<td>-1.234E-02</td>
</tr>
<tr>
<td>0.70</td>
<td>+3.1058127E-02</td>
<td>+3.634E-00</td>
<td>+7.268E-03</td>
</tr>
<tr>
<td>0.80</td>
<td>+6.6801862E-02</td>
<td>+7.816E-00</td>
<td>+1.563E-02</td>
</tr>
<tr>
<td>0.90</td>
<td>+7.6428947E-02</td>
<td>+8.939E-00</td>
<td>+1.788E-02</td>
</tr>
<tr>
<td>1.00</td>
<td>+7.3339917E-02</td>
<td>+8.576E-00</td>
<td>+1.715E-02</td>
</tr>
<tr>
<td>1.50</td>
<td>+3.1148744E-02</td>
<td>+3.644E-00</td>
<td>+7.288E-03</td>
</tr>
<tr>
<td>2.00</td>
<td>+1.1293579E-02</td>
<td>+1.321E-00</td>
<td>+2.642E-03</td>
</tr>
<tr>
<td>2.50</td>
<td>+4.3020230E-03</td>
<td>+5.031E-01</td>
<td>+1.006E-03</td>
</tr>
<tr>
<td>3.00</td>
<td>+1.7846623E-03</td>
<td>+2.088E-01</td>
<td>+4.176E-04</td>
</tr>
<tr>
<td>4.00</td>
<td>+4.2339247E-04</td>
<td>+4.954E-02</td>
<td>+9.908E-05</td>
</tr>
<tr>
<td>5.00</td>
<td>+1.4165854E-04</td>
<td>+1.658E-02</td>
<td>+3.316E-05</td>
</tr>
<tr>
<td>6.00</td>
<td>+5.8307590E-05</td>
<td>+6.821E-03</td>
<td>+1.364E-05</td>
</tr>
<tr>
<td>7.00</td>
<td>+2.7424776E-05</td>
<td>+3.206E-03</td>
<td>+6.412E-06</td>
</tr>
<tr>
<td>10.00</td>
<td>+4.7526079E-06</td>
<td>+5.558E-04</td>
<td>+1.112E-06</td>
</tr>
</tbody>
</table>
TABLE 6-D

INCREASE IN \( \bar{\text{Nu}} \) OVER THE STEADY FLOW VALUE,

HIGH FREQUENCY SOLUTION AT \( \xi = 0.20 \times 10^{-1} \)

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \bar{\text{Nu}} )</th>
<th>( \frac{(\bar{\text{Nu}}-\text{Nu}_0) \times 100}{\lambda^2 \text{Nu}_0} )</th>
<th>( \frac{(\bar{\text{Nu}}-\text{Nu}_0) \xi \times 100}{\lambda^2 \text{Nu}_0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>-0.22970751E 00</td>
<td>-0.2689E 02</td>
<td>-0.5378E 00</td>
</tr>
<tr>
<td>0.10</td>
<td>-0.14967201E 00</td>
<td>-0.1751E 02</td>
<td>-0.3502E 00</td>
</tr>
<tr>
<td>0.20</td>
<td>-0.47239359E-01</td>
<td>-0.5527E 01</td>
<td>-0.1105E 00</td>
</tr>
<tr>
<td>0.30</td>
<td>-0.50434116E-02</td>
<td>-0.5901E 00</td>
<td>-0.1180E-01</td>
</tr>
<tr>
<td>0.40</td>
<td>+0.65374621E-02</td>
<td>+0.7649E 00</td>
<td>+0.1530E-01</td>
</tr>
<tr>
<td>0.50</td>
<td>+0.70830990E-02</td>
<td>+0.8287E 00</td>
<td>+0.1657E-01</td>
</tr>
<tr>
<td>0.60</td>
<td>+0.52080135E-02</td>
<td>+0.6093E 00</td>
<td>+0.1219E-01</td>
</tr>
<tr>
<td>0.70</td>
<td>+0.35130486E-02</td>
<td>+0.4110E 00</td>
<td>+0.8220E-02</td>
</tr>
<tr>
<td>0.80</td>
<td>+0.23940167E-02</td>
<td>+0.2801E 00</td>
<td>+0.5602E-02</td>
</tr>
<tr>
<td>0.90</td>
<td>+0.16876536E-02</td>
<td>+0.1973E 00</td>
<td>+0.3946E-02</td>
</tr>
<tr>
<td>1.00</td>
<td>+0.12278585E-02</td>
<td>+0.1437E 00</td>
<td>+0.2874E-02</td>
</tr>
<tr>
<td>1.50</td>
<td>+0.33367673E-03</td>
<td>+0.3904E-01</td>
<td>+0.7808E-03</td>
</tr>
<tr>
<td>2.00</td>
<td>+0.11419223E-03</td>
<td>+0.1336E-01</td>
<td>+0.2672E-03</td>
</tr>
<tr>
<td>2.50</td>
<td>+0.43127947E-04</td>
<td>+0.5046E-02</td>
<td>+0.1009E-03</td>
</tr>
<tr>
<td>3.00</td>
<td>+0.17860344E-04</td>
<td>+0.2090E-02</td>
<td>+0.4180E-04</td>
</tr>
<tr>
<td>4.00</td>
<td>+0.42344657E-05</td>
<td>+0.4954E-03</td>
<td>+0.9908E-05</td>
</tr>
<tr>
<td>5.00</td>
<td>+0.14166430E-05</td>
<td>+0.1657E-03</td>
<td>+0.3314E-05</td>
</tr>
<tr>
<td>6.00</td>
<td>+0.58308355E-06</td>
<td>+0.6822E-04</td>
<td>+0.1364E-05</td>
</tr>
<tr>
<td>7.00</td>
<td>+0.27424975E-06</td>
<td>+0.3209E-04</td>
<td>+0.6418E-06</td>
</tr>
<tr>
<td>10.00</td>
<td>+0.47526596E-07</td>
<td>+0.5561E-05</td>
<td>+0.1112E-06</td>
</tr>
</tbody>
</table>
### TABLE 7

**INCREASE IN \( \bar{Nu} \) OVER THE STEADY FLOW VALVE, LOW FREQUENCY SOLUTION FOR ALL \( \xi \)**

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \bar{Nu}_2 )</th>
<th>( \frac{(Nu - Nu_0) \times 100}{\chi^2 Nu_0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.00</td>
<td>-.47456420E-01</td>
<td>-.5552E 01</td>
</tr>
<tr>
<td>.05</td>
<td>-.47456420E-01</td>
<td>-.5552E 01</td>
</tr>
<tr>
<td>.10</td>
<td>-.47456135E-01</td>
<td>-.5552E 01</td>
</tr>
<tr>
<td>.20</td>
<td>-.47451674E-01</td>
<td>-.5552E 01</td>
</tr>
<tr>
<td>.30</td>
<td>-.47434590E-01</td>
<td>-.5549E 01</td>
</tr>
<tr>
<td>.40</td>
<td>-.47386659E-01</td>
<td>-.5544E 01</td>
</tr>
<tr>
<td>.50</td>
<td>-.47286526E-01</td>
<td>-.5533E 01</td>
</tr>
<tr>
<td>.60</td>
<td>-.47109988E-01</td>
<td>-.5512E 01</td>
</tr>
<tr>
<td>.70</td>
<td>-.46803894E-01</td>
<td>-.5476E 01</td>
</tr>
<tr>
<td>.80</td>
<td>-.46342618E-01</td>
<td>-.5422E 01</td>
</tr>
<tr>
<td>.90</td>
<td>-.45672059E-01</td>
<td>-.5343E 01</td>
</tr>
<tr>
<td>1.00</td>
<td>-.44837167E-01</td>
<td>-.5246E 01</td>
</tr>
</tbody>
</table>
### TABLE 8
NUMERICAL SOLUTION AT $\xi = 0.20 \times 10^{-2}$

#### A. Phase and Amplitude of $\text{Nu}_1$

| $\psi_1$  | $|\text{Nu}_1|$        |
|----------|------------------------|
| .05      | - 31'                  |
| .10      | -2°04'                 |
| .50      | -51°17'                |
| .80      | -125°10'               |
| 1.00     | -134°25'               |
| 2.00     | -156°22'               |
| 3.00     | -169°50'               |
|          | .28461120E 00           |
|          | .28455808E 00           |
|          | .25437608E 00           |
|          | .13038376E 00           |
|          | .34288486E-01           |
|          | .38533904E-02           |
|          | .14323336E-02           |

#### B. Phase and Amplitude of $\text{Nu}_2$

<p>| $\psi_2$  | $|\text{Nu}_2|$        |
|----------|------------------------|
| .10      | 171°52'                |
| .40      | 129°19'                |
| .50      | 120°10'                |
| .70      | 103°54'                |
| .80      | 96°05'                 |
| .90      | 89°57'                 |
| 1.00     | 84°34'                 |
| 1.50     | 61°09'                 |
| 2.00     | 50°59'                 |
|          | .47228672E-01           |
|          | .57897203E-01           |
|          | .64340544E-01           |
|          | .54787270E-01           |
|          | .37218310E-01           |
|          | .20163846E-01           |
|          | .87995840E-02           |
|          | .14816282E-03           |
|          | .70450048E-04           |</p>
<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$\overline{Nu}_2$</th>
<th>$\frac{(\overline{Nu} - \overline{Nu}_2) \times 100}{\lambda^2 \overline{Nu}_0}$</th>
<th>$\overline{Nu}_2$</th>
<th>$\frac{(\overline{Nu} - \overline{Nu}_2) \times 100}{\lambda^2 \overline{Nu}_0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.10</td>
<td>-.47124474E-01</td>
<td>-.5513E 01</td>
<td>-.47647027E-01</td>
<td>-.5575E 01</td>
</tr>
<tr>
<td>.50</td>
<td>-.23286509E-01</td>
<td>-.2724E 01</td>
<td>-.46283654E-01</td>
<td>-.5415E 01</td>
</tr>
<tr>
<td>.70</td>
<td>+.11447245E-01</td>
<td>+.1339E 01</td>
<td>-.42563661E-01</td>
<td>-.4980E 01</td>
</tr>
<tr>
<td>.90</td>
<td>+.16103539E-01</td>
<td>+.1894E 01</td>
<td>-.34577606E-01</td>
<td>-.4046E 01</td>
</tr>
<tr>
<td>1.00</td>
<td>+.71767296E-02</td>
<td>+.8397E 00</td>
<td>-.28545146E-01</td>
<td>-.3340E 01</td>
</tr>
<tr>
<td>1.50</td>
<td>+.67617856E-03</td>
<td>+.7911E-01</td>
<td>+.10406285E-01</td>
<td>+.1218E 01</td>
</tr>
<tr>
<td>2.00</td>
<td>+.22436928E-03</td>
<td>+.2625E-01</td>
<td>+.11597696E-01</td>
<td>+.1357E 01</td>
</tr>
</tbody>
</table>
TABLE 10
ANALYTICAL SOLUTION OF \( f_1 \), \( M \) AND \( f_2 \) AT \( \xi = 0.20 \times 10^{-2} \)

| \( \eta \)  | \( \phi_0 \)  | \( |f_1| \) | \( M \)           | \( |f_2| \) |
|-----------|-------------|-----------|------------------|-----------|
| .44E00    | .3723372E00 | .1507053E-01 | -.3297092E-03   | .3303294E-03 |
| .88E00    | .6993910E00 | .2312364E-01 | -.6967744E-03   | .6973135E-03 |
| .132E01   | .9068758E00 | .1688578E-01 | -.8868276E-03   | .8869705E-03 |
| .176E01   | .9844534E00 | .5529781E-02 | -.5303299E-03   | .5303428E-03 |
| .220E01   | .9985111E00 | .6661504E-03 | -.1101210E-03   | .1101212E-03 |
| .264E01   | .9995269E00 | .4703607E-05 | -.1164767E-05   | .1164760E-05 |

\( \omega = 1.0 \)

| \( \eta \)  | \( \phi_0 \)  | \( |f_1| \) | \( M \)           | \( |f_2| \) |
|-----------|-------------|-----------|------------------|-----------|
| .440E00   |              | .2130594E-02 | .5324776E-04    | .6706948E-04 |
| .880E00   |              | .5497998E-02 | .1183885E-03    | .1648229E-03 |
| .132E01   |              | .6292559E-02 | .7052430E-04    | .1534617E-03 |
| .176E01   |              | .2777269E-02 | -.4778860E-04   | .8342290E-04 |
| .220E01   |              | .4040522E-03 | -.2848070E-04   | .3051809E-04 |
| .264E01   |              | .3089670E-05 | -.4170053E-06   | .4144526E-06 |
TABLE 10 - CONTINUED

ANALYTICAL SOLUTION OF $f_1$, M AND $f_2$ AT $\xi = 0.20 \times 10^{-2}$

| $n$     | $|f_1|$       | $M$            | $|f_2|$       |
|---------|---------------|----------------|---------------|
| $\omega = 0.5$ |
| .44E00 | .1352502E-01  | -.1713125E-03 | .4263140E-03 |
| .88E00 | .2116025E-01  | -.4480034E-03 | .7111610E-03 |
| .132E01 | .1582967E-01  | -.6954186E-03 | .7743321E-03 |
| .176E01 | .5284021E-02  | -.4624416E-03 | .4697623E-03 |
| .220E01 | .6446636E-03  | -.1010603E-03 | .1011365E-03 |
| .264E01 | .4575409E-05  | -.1090131E-05 | .1085934E-05 |
| $\omega = 2.0$ |
| .44E00 | .3631737E-03  | .1675190E-05  | .3595407E-06 |
| .88E00 | .1086437E-02  | .3292472E-05  | .1685313E-05 |
| .132E01 | .1195994E-02  | .1743647E-05  | .1170052E-05 |
| .176E01 | .5241746E-03  | -.1610307E-05 | .1876832E-05 |
| .220E01 | .8134635E-04  | -.8690726E-06 | .1112046E-05 |
| .264E01 | .8266233E-06  | -.2328361E-07 | .3290038E-07 |
APPENDIX D

TABULATED RESULTS FOR THE CASE OF FLOW

IN A DISTENSIBLE CONDUIT
TABLE 11-A

PHASE AND AMPLITUDE OF \( N_u_1 \) AT \( \xi = 0.20 \times 10^{-2} \)

\[
\frac{U_a}{C} = 10^{-4} \quad \quad \frac{U_a}{C} = 10^{-3}
\]

| \( \omega \) | \( \psi_1 \) | \( |N_u_1| \) | \( \psi_1 \) | \( |N_u_1| \) |
|---|---|---|---|---|
| 1.0 | -142°06' | 0.356205E-01 | -142°34' | 0.358767E-01 |
| 1.5 | -150°20' | 0.100617E-01 | -151°58' | 0.103272E-01 |
| 2.0 | -159°28' | 0.383333E-02 | -163°58' | 0.403590E-02 |
| 3.0 | -172°16' | 0.831909E-03 | -183°27' | 0.951472E-03 |
| 4.0 | -177°42' | 0.269231E-03 | -199°16' | 0.342628E-03 |
| 5.0 | -181°26' | 0.108410E-03 | -213°30' | 0.166051E-03 |
| 7.0 | -187°38' | 0.299145E-04 | -232°42' | 0.679400E-04 |
| 10.0 | -199°48' | 0.789402E-05 | -248°30' | 0.312393E-04 |
| 14.0 | -217°16' | 0.252402E-05 | -257°22' | 0.160009E-04 |
| 20.0 | -237°15' | 0.924570E-06 | -262°48' | 0.812179E-05 |
TABLE 11-B

PHASE AND AMPLITUDE OF $\text{Nu}_1$ AT $\xi = 0.20 \times 10^{-2}$

| $r$ | $\psi_1$ (deg) | $|\text{Nu}_1|$ | $\psi_1$ (deg) | $|\text{Nu}_2|$ |
|-----|----------------|-----------------|----------------|----------------|
| 1.0 | -146.49        | 0.391870E-01    | -170.46        | 0.945540E-01  |
| 1.5 | -164.17        | 0.130309E-01    | -192.53        | 0.579916E-01  |
| 2.0 | -185.25        | 0.663248E-02    | -209.35        | 0.478502E-01  |
| 3.0 | -219.27        | 0.296201E-02    | -232.57        | 0.307514E-01  |
| 4.0 | -238.40        | 0.175160E-02    | -245.17        | 0.296010E-01  |
| 5.0 | -246.14        | 0.110667E-02    | -251.08        | 0.135704E-01  |
| 7.0 | -254.29        | 0.627597E-03    | -257.00        | 0.741448E-02  |
| 10.0| -260.01        | 0.306581E-03    | -261.27        | 0.385777E-02  |
| 14.0| -263.16        | 0.162568E-03    | -263.52        | 0.204940E-02  |
| 20.0| -265.31        | 0.820921E-04    | -265.47        | 0.103506E-02  |
### TABLE 12

**PHASE AND AMPLITUDE OF Nu₂ AT ξ = 0.20×10⁻²**

| ω   | ψ₂    | |Nu₂|   | ψ₂    | |Nu₂|   |
|-----|-------|-----|-----|-------|-----|-----|
| 1.0 | 73°14' | .217999E-02 | 75°45' | .210028E-02 |
| 1.5 | 49°35' | .185290E-03 | 58°48' | .168429E-03 |
| 2.0 | 13°43' | .320868E-04 | 39°44' | .238728E-04 |
| 3.0 | -74°54' | .341194E-05 | 10°51' | .117761E-05 |
| 4.0 | -110°37' | .109181E-05 | -6°22' | .119551E-06 |
| 5.0 | -130°44' | .484915E-06 | -23°02' | .205851E-07 |
| 7.0 | -148°24' | .144677E-06 | -74°18' | .179612E-08 |
| 10.0 | -159°58' | .387855E-07 | -139°39' | .307372E-09 |
| 14.0 | -166°08' | .106309E-07 | -153°52' | .102242E-09 |
| 20.0 | -170°59' | .278504E-08 | -165°09' | .254326E-10 |
**TABLE 12 - CONTINUED**

PHASE AND AMPLITUDE OF Nu₂ AT \( \xi = 0.20 \times 10^{-2} \)

\[
\frac{U_0}{C} = 10^{-6}
\]

| \( \omega \) | \( \psi_2 \) | \( |Nu_2| \) |
|--------------|--------------|-------------|
| 1.0          | 75°51'       | .209320E-02 |
| 1.5          | 59°34'       | .166605E-03 |
| 2.0          | 42°56'       | .233207E-04 |
| 3.0          | 17°37'       | .111194E-05 |
| 4.00         | 9°50'        | .111879E-06 |
| 5.00         | 6°30'        | .193714E-07 |
| 7.00         | 44'          | .110114E-08 |
| 10.0         | -7°40'       | .804834E-10 |
| 14.0         | -20°05'      | .676850E-11 |
| 20.0         | -74°56'      | .328258E-12 |
TABLE 13-A

INCREASE IN $\overline{\text{Nu}}$ OVER THE STEADY FLOW VALUE AT $\zeta = 0.20 \times 10^{-2}$

$$\frac{U_0}{C} = 10^{-4} \quad \quad \quad \quad \quad \frac{U_0}{C} = 10^{-3}$$

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$\overline{\text{Nu}}$</th>
<th>$[\frac{\overline{\text{Nu}} - \text{Nu}_0}{\lambda^2 \text{Nu}_0}] \times 100$</th>
<th>$\overline{\text{Nu}}$</th>
<th>$[\frac{\overline{\text{Nu}} - \text{Nu}_0}{\lambda^2 \text{Nu}_0}] \times 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.124384E-01</td>
<td>0.1455E-01</td>
<td>0.126198E-01</td>
<td>0.1477E-01</td>
</tr>
<tr>
<td>1.5</td>
<td>0.335465E-02</td>
<td>0.3925E-00</td>
<td>0.353563E-02</td>
<td>0.4137E-00</td>
</tr>
<tr>
<td>2.0</td>
<td>0.115864E-02</td>
<td>0.1356E-00</td>
<td>0.129090E-02</td>
<td>0.1510E-00</td>
</tr>
<tr>
<td>3.00</td>
<td>0.185790E-03</td>
<td>0.2174E-01</td>
<td>0.247051E-03</td>
<td>0.2890E-01</td>
</tr>
<tr>
<td>4.0</td>
<td>0.459906E-04</td>
<td>0.5381E-02</td>
<td>0.742526E-04</td>
<td>0.8688E-02</td>
</tr>
<tr>
<td>5.0</td>
<td>0.159541E-04</td>
<td>0.1867E-02</td>
<td>0.311981E-04</td>
<td>0.3650E-02</td>
</tr>
<tr>
<td>7.0</td>
<td>0.337648E-05</td>
<td>0.3950E-03</td>
<td>0.918637E-05</td>
<td>0.1075E-02</td>
</tr>
<tr>
<td>10.0</td>
<td>0.704275E-06</td>
<td>0.8240E-04</td>
<td>0.272650E-05</td>
<td>0.3190E-03</td>
</tr>
<tr>
<td>14.0</td>
<td>0.173705E-06</td>
<td>0.2032E-04</td>
<td>0.827453E-06</td>
<td>0.9681E-04</td>
</tr>
<tr>
<td>20.0</td>
<td>0.442391E-07</td>
<td>0.5176E-05</td>
<td>0.306786E-06</td>
<td>0.3589E-04</td>
</tr>
</tbody>
</table>
### TABLE 13-B

INCREASE IN $\bar{\text{Nu}}$ OVER THE STEADY FLOW VALUE AT $\xi = 0.20 \times 10^{-2}$

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$\bar{\text{Nu}}$</th>
<th>$\frac{\bar{\text{Nu}} - \text{Nu}_0}{\lambda^2 \text{Nu}_0} \times 100$</th>
<th>$\bar{\text{Nu}}$</th>
<th>$\frac{\bar{\text{Nu}} - \text{Nu}_0}{\lambda^2 \text{Nu}_0} \times 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.145549E-01</td>
<td>0.1703E 01</td>
<td>0.382710E-01</td>
<td>0.4478E 01</td>
</tr>
<tr>
<td>1.5</td>
<td>0.525159E-02</td>
<td>0.6144E 00</td>
<td>0.271800E-01</td>
<td>0.3180E 01</td>
</tr>
<tr>
<td>2.0</td>
<td>0.266091E-02</td>
<td>0.3113E 00</td>
<td>0.201691E-01</td>
<td>0.2360E 01</td>
</tr>
<tr>
<td>3.0</td>
<td>0.875281E-03</td>
<td>0.1024E 00</td>
<td>0.904432E-02</td>
<td>0.1058E 01</td>
</tr>
<tr>
<td>4.0</td>
<td>0.384216E-03</td>
<td>0.4495E-01</td>
<td>0.406389E-02</td>
<td>0.4755E 00</td>
</tr>
<tr>
<td>5.0</td>
<td>0.187891E-03</td>
<td>0.2199E-01</td>
<td>0.217258E-02</td>
<td>0.2542E 00</td>
</tr>
<tr>
<td>7.0</td>
<td>0.678005E-04</td>
<td>0.7933E-02</td>
<td>0.823075E-03</td>
<td>0.9640E-01</td>
</tr>
<tr>
<td>10.0</td>
<td>0.236261E-04</td>
<td>0.2764E-02</td>
<td>0.292476E-03</td>
<td>0.3422E-01</td>
</tr>
<tr>
<td>14.0</td>
<td>0.869790E-05</td>
<td>0.1018E-02</td>
<td>0.108684E-03</td>
<td>0.1272E-01</td>
</tr>
<tr>
<td>20.0</td>
<td>0.301740E-05</td>
<td>0.3530E-03</td>
<td>0.378872E-04</td>
<td>0.4433E-02</td>
</tr>
</tbody>
</table>