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Variational Problems in Intersection Homology Theory and Optimal Transport

by

Qinglan Xia

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APPROVED, THESIS COMMITTEE:

Robert Hardt, Professor, Chair Mathematics

Robin Forman, Professor Mathematics

Nathaniel Dean, Associate Professor Applied Mathematics

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ABSTRACT

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This thesis studies geometric variational problems derived from intersection homology theory of singular varieties as well as from optimal transportation.

Part I: Intersection Homology Theory via Rectifiable Currents. Here is given a rectifiable currents’ version of intersection homology theory on stratified subanalytic pseudomanifolds. This new version enables one to study some variational problems on stratified subanalytic pseudomanifolds. We first achieve an isomorphism between this rectifiable currents’ version and the version using subanalytic chains. Then we define a suitably modified mass on the complex of rectifiable currents to ensure that each sequence of subanalytic chains with bounded modified mass has a convergent subsequence and the limit rectifiable current still satisfies the crucial perversity condition of the approximating chains. The associated mass minimizers turn out to be almost minimal currents and this fact leads to some regularity results.

Part II: Optimal paths related to transport problems. In transport problems of Monge’s types, the total cost of a transport map is usually an integral of some function
of the distance, such as $|x - y|^p$. In many real applications, the actual cost may naturally be determined by a transport path. For shipping two items to one location, a "Y shaped" path may be preferable to a "V shaped" path. Here, we show that any probability measure can be transported to another probability measure through a general optimal transport path, which is given by a normal 1-current in our setting. Moreover, we define a new distance on the space of probability measures which in fact metrizes the usual weak * topology of measures. When we take into account the time consumption, we get a Lipschitz flow of probability measures, which helps us to visualize the actual flow of measures as well as the new distance between measures. Relations as well as related problems about transport paths and transport plans are also discussed in the end.
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Finally, I dedicate this thesis to my parents and to my wife Yingjuan Wang, whose love, support and encouragement made this thesis possible.
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CHAPTER 1

PRELIMINARIES: THE PLATEAU PROBLEM IN GEOMETRIC MEASURE THEORY

Here we survey some of the definitions, notations and results of Geometric Measure Theory developed for studying generalized $k$ dimensional surfaces of minimal $k$-dimensional area. This information will be freely used later on. There are many good books on geometric measure theory [10] [27] [23] [15]. Most of the material in this chapter can be found in any of these books.

1.1 Plateau’s Problem

Suppose $C$ is a closed curve in space. What is the surface $S$ of smallest area having boundary $C$? There have been many answers depending on ones definition of surface, area, and boundary. This general problem is called Plateau’s problem in honor of the Belgian physicist and professor, J. Plateau (1801-1883). He made many interesting experiments with soap films, and determined a number of the geometric properties of soap films and soap bubbles. Attracted by the beauty and geometric appeal of the surfaces formed by soap films, mathematicians have attempted to find a mathematical theory that would describe the essential geometric properties of soap films and soap bubbles. This was classically studied by mappings of 2 dimensional surfaces of a fixed topology ([25]). Unfortunately, this not only does not describe all soap films but also leads to solutions which do not occur as soap films [1]. One may
also consider the general $m$ dimensional Plateau problem which is roughly speaking as follows:

Given an $m - 1$ dimensional boundary $B$, find an $m$ dimensional surface $S$ spanning $B$ of least $m$ dimensional area.

Now, geometric measure theory provides for general $m$ several precise formulations and definitions of these terms. The most popular involves the rectifiable currents of Federer and Fleming which we review in the following sections.

1.2 Currents

Let $U \subset \mathbb{R}^N$ be an open subset and $\mathcal{D}^m(U)$ be the set of all $C^\infty$ differential $m$-forms in $U$ with compact support with the usual Frechet topology [10].

An $m$-dimensional current $T$ on $U$ is a continuous linear functional on $\mathcal{D}^m(U)$. Let $\mathcal{D}_m(U)$ denote the set of all $m$-dimensional currents on $U$. Motivated by Stokes’ theorem, the boundary of a current $T \in \mathcal{D}_m(U)$ is the current $\partial T \in \mathcal{D}_{m-1}(U)$ defined by

$$\partial T (\psi) = T (d\psi)$$

for any $\psi \in \mathcal{D}^{m-1}(U)$.

The mass function on $\mathcal{D}_m(U)$ is defined by

$$M(T) = \sup \left\{ T(\omega) : \omega \in \mathcal{D}^m(U) \text{ with } <e_1 \wedge \cdots \wedge e_m, \omega(x)> \leq 1 \text{ for each } x \in U \text{ and } e_1, \cdots, e_m \in S^{m-1} \right\}$$
Given a sequence \( \{T_j\} \in \mathcal{D}_m(U) \), we write \( T_j \rightharpoonup T \) in \( U \) if \( \{T_j\} \) converges weakly to \( T \in \mathcal{D}_m(U) \) in the usual sense of distributions:

\[
T_j \rightharpoonup T \iff \lim T_j(\omega) = T(\omega), \forall \omega \in \mathcal{D}^m(U).
\]

Clearly, the mass function \( M \) is lower semicontinuous under the weak convergence of currents.

Given \( T \in \mathcal{D}_m(U) \), the support of \( T \) is defined by

\[
spt(T) = U \setminus \bigcup \{ \text{open } V \subset U : spt(\omega) \subset V \Rightarrow T(\omega) = 0 \}.
\]

### 1.3 Rectifiable currents

A subset \( M \subset U \) is (countably) \( m \)-rectifiable if \( M = \bigcup_{i=0}^{\infty} M_i \), where \( \mathcal{H}^m(M_0) = 0 \) and each \( M_i \), for \( i = 1, 2, \cdots \), is a subset of an \( m \)-dimensional \( C^1 \) submanifold in \( U \).

An (integer multiplicity) rectifiable current \( T \) is a current coming from an oriented rectifiable set with integer multiplicities. More precisely, \( T \in \mathcal{D}_m(U) \) is a rectifiable current if it can be expressed as

\[
T(\omega) = \int_M \langle \omega(x), \xi(x) \rangle > \theta(x) \, d\mathcal{H}^m(x), \forall \omega \in \mathcal{D}^m(U)
\]

where

- \( M \) is an \( \mathcal{H}^m \) measurable and \( m \)-rectifiable subset of \( U \),
- \( \theta \) is an \( \mathcal{H}^m \)-integrable positive integer-valued function, called the multiplicity of \( T \),
- \( \xi : M \to \Lambda_m(\mathbb{R}^N) \) is \( \mathcal{H}^m \) measurable and \( ||\xi(x)|| = 1 \) for \( \mathcal{H}^m |M \) a.e.
The rectifiable current $T$ describes as above is often denoted by

$$T = <M, \theta, \xi>.$$ 

For a rectifiable $m-$current $T = <M, \theta, \xi>$, its mass has a simple form

$$M(<M, \theta, \xi>) = \int_M \theta d\mathcal{H}^m(x) < +\infty.$$ 

For any compact subset $K \subset U$, we denote

$$\mathcal{R}_{m,K}(U) = \{T \in \mathcal{D}_m(U) : T \text{ is rectifiable with spt } T \subset K\}.$$ 

### 1.4 Flat seminorm and flat chains

Let $K \subset U$ be a compact set.

The **flat seminorm** of a $T \in \mathcal{D}_m(U)$ with spt $T \subset K$ is defined by

$$F_K(T) := \sup \left\{ T(\omega) | \omega \in \mathcal{D}^m(U), \sup_{x \in K} ||\omega(x)|| \leq 1, \sup_{x \in K} ||d\omega(x)|| \leq 1 \right\}.$$ 

The space of **normal currents** in $U$ relatively to $K$ is defined by

$$\mathcal{N}_{m,K}(U) := \{T \in \mathcal{D}_m(U) | M(T) + M(\partial T) < +\infty, \text{ support } (T) \subset K\}$$

and the space of **flat chains** in $U$ relative to $K$

$$\mathcal{F}_{m,K}(U) = \text{the } F_K \text{ closure of } \mathcal{N}_{m,K}(U) \text{ in } \mathcal{D}_m(U).$$

There is another possibly more convenient version of flat current which we state as follows.

For each current $T \in \mathcal{D}_p(U)$, $T$ can be viewed as a $p$ vector field with distribution coefficients, i.e.

$$T = \sum_{|I|=p} T_I \frac{\partial}{\partial x^I}$$
with each $T_i$ is a distribution in $U$.

Let $L_p(U)$ denote the space of currents

$$T = \sum_{|I|=p} (h_I dx) \frac{\partial}{\partial x^I}$$

of dimension $p$, with each $h_I$ is a locally Lebesgue integrable function in $U$. Similarly, let $L^q(U)$ denote the space of forms

$$g = \sum_{|I|=q} g_I dx^I$$

of degree $q = \dim(U) - p$ with each coefficient $g_I$ being a locally Lebesgue integrable function on $U$. One may identify the space $L_p(U)$ with the space $L^q(U)$ by the identification

$$\sum (h_I dx) \frac{\partial}{\partial x^I} \sim \sum h_I \left( dx^I \frac{\partial}{\partial x^I} \right).$$

A current in $L_p(U) = L^q(U)$ is called a locally integrable current of dimension $p$ and degree $q = \dim(U) - p$.

The space of flat currents of dimension $p$, denoted $\mathcal{F}_p(U)$, consists of those currents $T \in \mathcal{D}_p(U)$ which can be expressed as

$$T = g + \partial h,$$

with $g \in L_p(U)$ and $h \in L_{p+1}(U)$. Equivalently, $T$ may be viewed as a flat current of degree $q$ with an expression

$$T = g + dh$$

for some $g \in L^q(U)$ and $h \in L^{q-1}(U)$. 
Let

\[ \mathcal{L}_K^q(U) := \{T \in \mathcal{L}^q(U) : \text{spt}T \subset K\} \]

which is a Banach space under the mass norm. We next define

\[ \mathcal{F}_K^q(U) := \{g + dh : g \in \mathcal{L}_K^q(U) \text{ and } h \in \mathcal{L}_K^{q-1}(U)\} \]

and consider the map

\[ \pi : \mathcal{L}_K^q(U) \oplus \mathcal{L}_K^{q-1}(U) \to \mathcal{F}_K^q(U) \]

by \( \pi(g, h) = g + dh \). Then,

\[ 0 \to \ker \pi \to \mathcal{L}_K^q(U) \oplus \mathcal{L}_K^{q-1}(U) \to \mathcal{F}_K^q(U) \to 0 \]

is exact. Therefore, \( \mathcal{F}_K^q(U) \) is naturally a quotient Banach space. Thus, we have a new version of flat norm on \( \mathcal{F}_K^q(U) \) given by

\[ \mathbf{F}_K(T) \equiv \inf \{\mathbf{M}_K(g) + \mathbf{M}_K(h) : T = g + dh \text{ with } g \in \mathcal{L}_K^q(U) \text{ and } h \in \mathcal{L}_K^{q-1}(U)\} \]

\[ = \inf \{\mathbf{M}(T - \partial S) + \mathbf{M}(S) \mid S \in \mathcal{D}_{m+1}(U) \text{ with spt } S \subset K\}. \]

**Proposition 1** The vector space \( \mathcal{F}_K^q(U) \) with the flat norm \( \mathbf{F}_K \) is a Banach space.

1.5 **Compactness Theorem**

Federer and Fleming proved the fundamental compactness theorem in [12]:

**Theorem 2 (Compactness)** If \( K \) is a compact Lipschitz neighborhood retract in \( \mathbb{R}^N \).

For \( m \in \{0, 1, \cdots, N\} \), and \( 0 < c < +\infty \), we have
1. \[ \{ T \in \mathcal{R}_{m,K} (\mathbb{R}^N) : M(T) + M(\partial T) < c \} \text{ is weakly sequentially compact.} \]

2. \[ \{ T \in \mathcal{N}_{m,K} (\mathbb{R}^N) : M(T) + M(\partial T) < c \} \text{ is } F_K \text{ compact.} \]

With the lower semicontinuity of \( M \), this theorem ensures, for a given \( T_0 \in \mathcal{R}_m (\mathbb{R}^N) \), the existence of a minimizer for the basic Plateau problem

\[
\text{minimize } M(T) \text{ among } T \in \mathcal{R}_m (\mathbb{R}^N), \partial T = \partial T_0
\]

When \( m = 2, n = 3 \), a mass minimizing rectifiable current \( T \) provides a good model for some but not all soap films. To use currents in a better model for soap films, Almgren introduced the notion of size for a rectifiable current. Here for \( T = \langle M, \theta, \xi \rangle \) as above, the size is defined by ignoring the multiplicity; that is,

\[
\text{Size}(T) := \int_M d\mathcal{H}^m (x) = \mathcal{H}^m (M).
\]

However, size minimization problems are generally quite difficult to solve. In [6], De Pauw and Hardt use the notion of scans to deal with such problems.

1.6 Motivation of the following chapters

As we have seen, the ideas of both rectifiable currents and flat currents have many beautiful applications. These work in \( \mathbb{R}^N \) or more generally a smooth Riemannian manifold. In algebraic geometry, one is interested in the behavior of objects inside varieties which are not necessarily smooth manifolds. Intersection homology theory was invented in [13][14] to treat the algebraic topology of such singular varieties. In chapter 2, we will use the ideas of rectifiable currents to consider Plateau-type variational
problems inside singular projective varieties, or more generally on stratified subanalytic pseudomanifolds.

One dimensional currents of least mass are simply certain sums of oriented geodesic paths. Size minimizers are more complicated as in the following example:

\[ - \quad + \quad - \quad + \]

\[ \text{Mass-Minimizer} \quad \text{Size-Minimizer} \]

In chapter 3, we'll use 1 dimensional flat currents to study a new class of optimal transport problems, which are related to both mass and size minimization. These provide good models in some situations where combined transport is more cost efficient than individual transport.
CHAPTER 2

INTERSECTION HOMOLOGY THEORY VIA RECTIFIABLE CURRENTS

2.1 Introduction

The goal of this chapter is to develop a setting for treating variational problems on stratified pseudomanifolds with singularities, such as complex projective varieties. Rather than using the ordinary homology theory on the base space, we will instead use a generalized “homology theory” —the intersection homology theory, introduced by MacPherson and Goresky in the early 1980’s ([13], [14]). Such a theory turns out to be more suitable than ordinary homology theory for pseudomanifolds with singularities (see [21] or [5] for details).

In variational problems, one needs to take various limits of e.g. minimizing sequences, but a basic problem is that a limit of geometric intersection chains [13] may fail to be a geometric chain; and even if it is, it may not satisfy the important perversity conditions of the approximating chains concerning intersection with singular set. This motivates our use of rectifiable currents with a suitably modified mass norm.

In section 1, we present some necessary preliminaries on the categories of intersection homology theory, geometric measure theory, and subanalytic sets and chains.
In section 2, for a compact stratified subanalytic pseudomanifold, we show how to express the intersection homology groups in terms of integer multiplicity rectifiable currents. These are then isomorphic to the usual intersection homology groups defined by geometric or subanalytic chains with the corresponding perversity conditions. The key idea here involves a technical modification of the proof of the Federer-Fleming's Deformation Theorem [27, section 29] to accommodate the perversity condition of intersection homology theory. We study properties of a "safety function" used to quantify the perversity condition for each simplex of the singular locus.

In section 3, we give a suitably modified mass on rectifiable currents such that all rectifiable currents with finite mass and finite boundary mass satisfy automatically the given perversity conditions. Also, by using the Lojasiewicz' inequality of subanalytic sets, we're able to show that all allowable subanalytic chains have finite (modified) mass and finite boundary mass. This fact ensures that our category of rectifiable currents with finite modified mass is still rich enough to contain all the "nice" chains one may consider. Moreover, this modified mass satisfies an important theorem—an analogue of the compactness theorem of Geometric Measure theory (See [10], [27]), which implies that each sequence of rectifiable currents with bounded modified mass and boundary mass will have a convergent subsequence and that the limit is a rectifiable current satisfying the perversity conditions of the approximating chains. This property of rectifiable currents overcomes the weakness of geometric chains stated above in the basic problem. The support of the currents we consider may intersect (in a controlled fashion) the singular locus of the pseudomanifold. A related problem with currents avoiding the vertex of a regular cone was studied by [22].
In section 4, we first achieve an existence theorem for modified mass minimizers. Moreover, we show that these mass minimizers are in fact almost minimizing currents [4]. Thus, by a lemma of Almgren, we achieve a partial regularity theorem for these suitable mass minimizers.

We will, for notational convenience and clarity, restrict to manifolds and pseudomanifolds in $\mathbb{R}^N$, although most of our results carry over to more general contexts.

2.2 Backgrounds

**Intersection Homology groups $IH^p(X)$** As in [5, section 1.1], let $X$ be a topological stratified pseudomanifold of dimension $n$ with singular locus $\Sigma$ and a given stratification

$$X = X_n \supset X_{n-2}(= \Sigma) \supset X_{n-3} \supset \cdots \supset X_0.$$  

For a triangulation $T$ of $X$, compatible with the stratification, let $C^T_i(X)$ be the complex of simplicial chains of $T$. Then an element $\xi \in C^T_i(X)$ is a linear combination

$$\xi = \sum_{\sigma \in T^{(i)}} \xi_\sigma \sigma, \quad \xi_\sigma \in \mathbb{Z}.$$  

For $\xi \in C^T_i(X)$, define $|\xi|$ (the support of $\xi$) to be the union of the closures of those $i$-simplices $\sigma$ for which the coefficient of $\sigma$ in $\xi$ is non-zero. The complex $C_i(X)$ of all geometric chains of $X$ with integer coefficients is the direct limit of $C^T_i(X)$ under refinement over all such triangulations of $X$.

Let $\bar{p} = (p_2, p_3, \cdots, p_n)$, called the perversity, be any fixed nonnegative integers satisfying $p_2 = 0$, and $p_k \leq p_{k+1} \leq p_k + 1$ for all $2 \leq k \leq n$. A geometric chain $\xi$ is said
to be \((\tilde{p}, i)\) allowable if

\[\dim_{\mathbb{R}}(\xi| \cap X_{n-k}) \leq i - k + p_k, \text{ for all } k \geq 2.\]

The group \(IC^p_i(X)\) of intersection chains of dimension \(i\) and perversity \(\tilde{p}\) is the subgroup of geometric chain \(\xi \in C_i(X)\) such that \(\xi\) is \((\tilde{p}, i)\) allowable and \(\partial \xi\) is \((\tilde{p}, i-1)\) allowable.

**Definition 3** The intersection homology groups \(IH^p_i(X)\) are defined to be the homology groups of the chain complex \(IC^p_i(X)\).

This definition is independent of the choice of the stratification. See [14]

**Geometric Measure Theory** Let \(U \subset \mathbb{R}^N\) be an open subset and \(D^i(U)\) be the set of all differential \(i\)-forms in \(U\) with compact support.

A \(i\)-dim current \(T\) on \(U\) is a continuous linear functional on \(D^i(U)\). Let \(D_i(U)\) denote the set of all \(i\)-dim currents on \(U\) (See [10] or [27] for more details).

Given a sequence \(\{T_j\} \subset D_i(U)\), we write \(T_j \rightharpoonup T\) in \(U\) if \(\{T_j\}\) converges weakly to \(T \in D_i(U)\) in the usual sense of distributions:

\[T_j \rightharpoonup T \iff \lim T_j(\omega) = T(\omega), \forall \omega \in D^i(U).\]

Given \(T \in D_i(U)\), the support of \(T\) is defined by

\[spt T = U \setminus \cup \{\text{open } V \subset U : spt \omega \subset V \Rightarrow T(\omega) = 0\}.\]

The mass function on \(D_i(U)\) is defined by

\[M(T) = \sup_{\|\omega\| \leq 1, \omega \in D^i(U)} T(\omega).\]
More generally for any open $W \subset U$, we define

$$M_W(T) = \sup_{\|\omega\| \leq 1, \omega \in D^*(U), \text{spt} \omega \subset W} T(\omega).$$

Clearly, $M(T)$ is lower semicontinuous under the weak convergence of currents.

An integer multiplicity rectifiable current $T$ is a current coming from an oriented rectifiable set with integer multiplicities (See [10] or [27]). Let $\mathcal{R}_i(\mathbb{R}^N)$ be the set of all $i$ dimensional integer multiplicity rectifiable currents in $\mathbb{R}^N$ and for any subset $X \subset \mathbb{R}^N$,

$$\mathcal{R}_i(X) = \{ T \in \mathcal{R}_i(\mathbb{R}^N) \mid \text{spt} (T) \subset X \}.$$

**Subanalytic sets and chains** According to [19, proposition 6.11], a subset $A$ of a real-analytic space $X$ is subanalytic if at any point $a \in A$, there exists an open neighborhood $U$ of $a$ in $X$, a real analytic manifold $Y$ and a finite system of proper real-analytic maps $g_{ij} : Y \to U$, $1 \leq i \leq p$ and $j = 1, 2$, such that $A \cap U = \bigcup_{i=1}^{p}(\text{Im}(g_{i1}) - \text{Im}(g_{i2}))$.

Examples of subanalytic subsets of $\mathbb{R}^N$:

1. analytic varieties;
2. polyhedrons;
3. finite unions, intersections, proper projections of subanalytic subsets.

A really important inequality about a subanalytic set is the following (See [19, 9.5]):

**Proposition 4 (Lojasiewicz' inequality)** Let $f$ be a function on $\mathbb{R}^n$ with subanalytic graph. Then for each compact subset $K$ of $\mathbb{R}^n$, we can find $N \in \mathbb{Z}_+$ and $C \in \mathbb{R}_+$ such that for all $x \in K$,

$$C|f(x)| \geq \text{dist}_{\mathbb{R}^n}(x, f^{-1}(0))^N.$$
Definition 5  An integer multiplicity rectifiable current \( T \in \mathcal{R}_i(\mathbb{R}^N) \) is a subanalytic chain if \( \text{spt}(T) \) and \( \text{spt}(\partial T) \) are \( i \) and \( i - 1 \) dimensional subanalytic subsets of \( \mathbb{R}^N \). That is, \( T \) is a geometric chain where all the supporting simplicies are subanalytic sets.

Definition 6  A stratified subanalytic pseudomanifold \( X \) of dimension \( n \) in \( \mathbb{R}^N \) is a subanalytic pseudomanifold with a stratification

\[
X = X_n \supset X_{n-2} (= \Sigma) \supset X_{n-3} \supset \cdots \supset X_0 
\]

with closed subanalytic subsets \( X_{n-k} \) of \( \mathbb{R}^N \) such that \( X_{n-k} \setminus X_{n-k-1} \) is empty or a subanalytic manifold of dimension \( n - k \) and such that the local normal triviality holds in the subanalytic category. See [17] for details.

General Setup  For the rest of the chapter, we let \( X \subset \mathbb{R}^N \) be a compact stratified subanalytic pseudomanifold with singular set \( \Sigma \) and a given stratification

\[
X = X_n \supset X_{n-2} (= \Sigma) \supset X_{n-3} \supset \cdots \supset X_0 .
\]

Also, let \( \tilde{p} \) be a fixed perversity function (as in section 1.1). A rectifiable current \( T \in \mathcal{R}_i(X) \) is said to be \((\tilde{p}, i)\) allowable if

\[
dim (\text{spt}(T) \cap X_{n-k}) \leq i - k + p_k, \text{ for all } k \geq 2.
\]

Finally, we fix an integer \( i \in \{0, 1, \cdots, n\} \) and consider various dimensional chains in \( X \). Let

\[
\mathcal{R}_i(X) = \{ T \in \mathcal{R}_i(\mathbb{R}^N) | \text{spt} T \subset X \};
\]

\[
\mathcal{P}_i(X) = \{ T \in \mathcal{R}_i(X) | T \text{ is } (\tilde{p}, i) \text{ allowable and } \partial T \text{ is } (\tilde{p}, i - 1) \text{ allowable} \};
\]

\[
\mathcal{S}_i(X) = \{ T \in \mathcal{P}_i(X) \text{ subanalytic chain} \}.
\]
2.3 Intersection homology theory in rectifiable currents' version

Here we prove that intersection homology theory defined via rectifiable currents coincides with the usual definition [13] involving geometric chains or subanalytic chains.

**Lemma 7** There exists a triangulation $T$ of $(X, \Sigma)$, compatible with the given stratification, such that:

For all open simplices $\sigma \subset \Sigma$ of $T$, if $\sigma \subset X_{n-j} \setminus X_{n-j-1}$ for some $2 \leq j \leq n$, then $\partial \sigma \cap X_{n-j-1}$ is either empty or a face of $\partial \sigma$.

**Proof.** By [18], there exists a subanalytic triangulation of $(X, \Sigma)$, compatible with the given stratification. Subanalytically subdividing this triangulation, one gets a triangulation with the desired properties. $\blacksquare$

From now on, we fix this triangulation $T$ and denote

$$T_\Sigma = \{\text{open simplices } \sigma \in T : \sigma \subset \Sigma\}.$$  

For notational convenience, we rephrase the perversity condition using the following

**Definition 8** The $i$-th safety function $s : T_\Sigma \to \mathbb{R}$ is given by:

$$s(\sigma) := i - j + p_j - 1, \text{ if } \sigma \subset X_{n-j} \setminus X_{n-j-1}.$$  

**Remark 9** Now, $T \in \mathcal{P}_i$ if and only if for any $\sigma \in T_\Sigma$,

$$\dim spt(T) \cap \sigma \leq s(\sigma) + 1 \text{ and } \dim spt(\partial T) \cap \sigma \leq s(\sigma)$$  

(2.1)

**Proposition 10** If $\sigma_1 \prec \sigma_2$, then $s(\sigma_1) \leq s(\sigma_2)$, i.e. the interior is safer than the boundary.
Proof. One may assume that $\sigma_1 \subset X_{n-j-1} \setminus X_{n-j-2}$ and $\sigma_2 \subset X_{n-j} \setminus X_{n-j-1}$ for some $2 \leq j \leq n$. Then

$$s(\sigma_1) - s(\sigma_2) = i - (j + 1) + p_{j+1} - 1 - (i - j + p_j - 1) = p_{j+1} - p_j - 1 \leq 0. \qed$$

**Lemma 11** Suppose $\sigma_1, \sigma_2 \in \mathcal{T}_2$. If $\tau \in \mathcal{T}_E$ is the open simplex of minimum dimension such that $\sigma_1 \prec \bar{\tau}, \sigma_2 \prec \bar{\tau}$. Then $s(\tau) = \max(s(\sigma_1), s(\sigma_2))$.

**Proof.** Otherwise, by the proposition 10, $s(\sigma_1) < s(\tau)$ and $s(\sigma_2) < s(\tau)$. This implies that if $\tau \subset X_{n-j} \setminus X_{n-j-1}$, then $\bar{\sigma}_1, \bar{\sigma}_2 \subset X_{n-j-1} \cap \partial \tau$. From the property 7, $\partial \tau \cap X_{n-j-1}$ is a unique closed simplex which is denoted by $\bar{\sigma}$. Therefore, we have $\sigma_1 \prec \bar{\sigma}$ and $\sigma_2 \prec \bar{\sigma}$, which contradicts to the minimum dimension property of $\tau$. \qed

For an open simplex $\sigma \in \mathcal{T}$, let

$$st(\sigma) = \cup \{\tau \in \mathcal{T} : \sigma \prec \tau\}$$

be the open star of $\sigma$ and $St(\sigma) = \overline{st(\sigma)}$ denotes the closed star of $\sigma$.

For the rest of this section, we fix one rectifiable current $T \in \mathcal{P}_x(X)$. Our goal is to deform $T$ to an allowable subanalytic chain $S \in \mathcal{S}_x(X)$ using allowable currents. To achieve this, we make the following technical definitions.

**Definition 12** Given an open simplex $\sigma \in \mathcal{T}_E$.

1. $\sigma$ is absolutely good if $\dim \sigma \leq s(\sigma)$;

2. $\sigma$ is good w.r.t. $T$ of type (I) if $\sigma$ is absolutely good;

   $\sigma$ is good w.r.t. $T$ of type (II) if $\dim(\sigma) = s(\sigma) + 1$ and $spt(\partial T) \cap st(\sigma) = \emptyset$;

   $\sigma$ is good w.r.t. $T$ of type (III) if $spt(T) \cap st(\sigma) = \emptyset$. 

3. $\sigma$ is bad w.r.t. $T$ if $\sigma$ is not good w.r.t. $T$, i.e.

$$\dim \sigma > s(\sigma), \ \text{spt}(T) \cap st(\sigma) \neq \emptyset$$

and also $\text{spt}(\partial T) \cap st(\sigma) \neq \emptyset$ in case $\dim(\sigma) = s(\sigma) + 1$.

Note that $\sigma \in T$ being absolutely good trivially gives the perversity condition (2.1) for any $\tilde{T} \in R_{i}(X)$.

To make the inductive argument in our deformation theorem 18, we will first fix any open simplex $\sigma_{0}$ of minimum dimension in the family

$$\{\tau : s(\tau) = \min \{s(\sigma) : \sigma \text{ is bad w.r.t. } T\}\}.$$

**Lemma 13** Any face $\sigma_{1}$ of $\partial \sigma_{0}$ is good w.r.t. $T$ of types I or II.

**Proof.** By the proposition 10, $s(\sigma_{1}) \leq s(\sigma_{0})$. Since $\dim \sigma_{1} < \dim \sigma_{0}$, by the minimum of $\sigma_{0}$, $\sigma_{1}$ is good w.r.t. $T$. On the other hand, the fact $st(\sigma_{0}) \subset st(\sigma_{1})$ implies $\text{spt}(T) \cap st(\sigma_{1}) \supseteq \text{spt}(T) \cap st(\sigma_{0}) \neq \emptyset$. i.e. $\sigma_{1}$ is not good w.r.t. $T$ of types III. Thus, $\sigma_{1}$ is good w.r.t. $T$ of types either I or II. $\blacksquare$

**Proposition 14** For $T$ and $\sigma_{0}$ as above, there exists a $T_{1} \in \mathcal{P}_{i}$ and $R \in \mathcal{P}_{i+1}$, $L \in \mathcal{P}_{i}$ such that

(a) $T = T_{1} + \partial R + L$;

(b) $\{\sigma : \sigma \text{ is bad w.r.t. } T_{1}\} \subseteq \{\sigma : \sigma \text{ is bad w.r.t. } T\}$.

(c) $L = 0$ if $\partial T = 0$;

(d) $L \in S_{i}$ if $\partial T \in S_{i-1}$.
Proof. We'll obtain $T_1$ as $p#T$ for a suitable map $p$ constructed differently in the two possible cases:

Case 1: $\dim \sigma_0 > s(\sigma_0) + 1$.

Since

$$\dim \sigma_0 > s(\sigma_0) + 1 \geq \dim \text{spt}(T) \cap \sigma_0,$$

there exists a point $a_0 \in \sigma_0 \setminus \text{spt}(T)$. Let

$$p : \text{St}(\sigma_0) \setminus a_0 \rightarrow \partial(\text{st}(\sigma_0))$$

be the “radial retraction” of $\text{St}(\sigma_0)$ with $a_0$ as origin (See [27, page 166]). Outside the closed star $\text{St}(\sigma_0)$, one may extend it to be the identical map.

Now, let’s show that $p#T \in \mathcal{P}_i$. It’s sufficient to show that for any $\sigma \in \mathcal{T}_\Sigma \cap \text{St}(\sigma_0)$ which is not absolutely good,

$$\dim \text{spt}(p#T) \cap \sigma \leq s(\sigma) + 1 \text{ and } \dim \text{spt}(\partial p#T) \cap \sigma \leq s(\sigma). \quad (2.2)$$

In fact, if $\sigma$ is a face of $\partial \sigma_0$, then by the lemma 13, $\sigma$ is good w.r.t. $T$ of type III; i.e. $\dim \sigma = s(\sigma) + 1$ and $\text{spt}(\partial T) \cap \text{st}(\sigma) = \emptyset$. Thus, $\dim \text{spt}(p#T) \cap \sigma \leq \dim \sigma \leq s(\sigma) + 1$ and $\text{spt}(\partial p#T) \cap \sigma = \text{spt}(p#\partial T) \cap \sigma = \text{spt}(\partial T) \cap \sigma = \emptyset$. Therefore, $\sigma$ satisfies the identity (2.2).

If $\sigma$ belongs to the open star $\text{st}(\sigma_0)$, then by the definition of radial retraction, $\sigma \cap \text{spt}(p#T) = \emptyset$ and hence $\sigma$ satisfies (2.2).

If $\sigma$ belongs to $\partial \text{st}(\sigma_0) \setminus \partial \sigma_0$, then there are three subcases:

Subcase 1: $\sigma$ is good w.r.t. $T$ of type III . i.e. $\text{st}(\sigma) \cap \text{spt}(T) = \emptyset$. This is a trivial case, because $\text{st}(\sigma) \cap \text{spt}(p#T) = \emptyset$ is still true and hence $\sigma$ satisfies (2.2) and
is good w.r.t. \( p\#T \) of type II.

**Subcase 2:** \( \sigma \) is good w.r.t. \( T \) of type II, i.e. \( \dim \sigma = s(\sigma) + 1 \) and \( \text{spt}(\partial T) \cap st(\sigma) = \emptyset \). This is also a trivial case, because one still has \( \dim \sigma = s(\sigma) + 1 \) and \( st(\sigma) \cap \text{spt}(p\#\partial T) = \emptyset \). Hence \( \sigma \) satisfies (2.2) and is good w.r.t. \( p\#T \) of type II.

**Subcase 3:** \( \sigma \) is bad w.r.t. \( T \). Then, \( s(\sigma) \geq s(\sigma_0) \) and we choose \( \tau \in T_\Sigma \) to be the open simplex of minimum dimension such that \( \sigma_0 \prec \tau \) and \( \sigma \prec \tau \). Since \( s(\sigma_0) \leq s(\sigma) \leq s(\tau) \), the lemma 11 implies \( s(\sigma) = s(\tau) \). Now, note that

\[
\dim \text{spt}(p\#T) \cap \sigma \leq \max(\dim \text{spt}(T) \cap \tau, \dim \text{spt}(T) \cap \sigma) \leq s(\tau) + 1 = s(\sigma) + 1;
\]

\[
\dim \text{spt}(\partial (p\#T)) \cap \sigma = \dim \text{spt}((p\#\partial T)) \cap \sigma \leq \max(\dim \text{spt}(\partial T) \cap \tau, \dim \text{spt}(\partial T) \cap \sigma) \leq s(\tau) = s(\sigma).
\]

Therefore, \( \sigma \) satisfies (2.2).

This shows that \( p\#T \in \mathcal{P}_i \). 

**Case 2:** \( \dim \sigma_0 = s(\sigma_0) + 1 \). In this case, \( \text{spt}(\partial T) \cap st(\sigma_0) \neq \emptyset \).

Here we will use a different formula for \( p \). Since \( T \in \mathcal{P}_i \), by the identity (2.1), \( \dim (\text{spt}(\partial T) \cap \sigma_0) \leq s(\sigma_0) < \dim \sigma_0 \). We choose a point \( a_0 \in \sigma_0 \setminus \text{spt}(\partial T) \) with \( \text{dist}(a_0, \sigma_0 \setminus \text{spt}(\partial T)) > \epsilon \) for some small positive constant \( \epsilon \). (It is possible that \( a_0 \in \text{spt}(T) \)). Define a help function \( f : [0, +\infty) \rightarrow [0, +\infty) \) by

\[
f(t) = \begin{cases} 
\frac{t}{\epsilon}, & 0 \leq t < \epsilon \\
1, & \epsilon \leq t \leq 1 \\
t, & t > 1
\end{cases}
\]
Now, for any \( x \in \text{st}(\sigma_0) \), let \( \gamma_x : [0, 1] \to X \) be the “radial arc” starting at \( a_0 \), passing through \( x \) and ending at a point \( \gamma_x(1) \) on \( \partial \text{st}(\sigma_0) \). Now we define \( p : X \to X \) by

\[
p(x) = \begin{cases} 
\gamma_x \left( f \left( \frac{\text{dist}(x, a)}{\text{dist}(\gamma_x(1), a)} \right) \right), & \text{for } x \in \text{St}(\sigma_0); \\
x & \text{for } x \in X \setminus \text{St}(\sigma_0).
\end{cases}
\]

To describe this map geometrically, let \( U_\varepsilon \) be the small star neighborhood of the point \( a \), similar to \( \text{st}(\sigma_0) \) but scaled down by a factor \( \varepsilon \). Then the map \( p \) fixes all points outside \( \text{st}(\sigma_0) \), maps \( U_\varepsilon \) homothetically to \( \text{st}(\sigma_0) \) and projects the remaining “annular region” radially to \( \partial (\text{St}\sigma_0) \). Clearly, this map is Lipschitz (with Lipschitz constant \( \frac{\varepsilon}{\varepsilon} \)). We need to show \( p_* T \in P_1 \).

As in case 1, it’s sufficient to show (2.2) for any \( \sigma \in \mathcal{S}_2 \cap \text{St}(\sigma_0) \) which is not absolutely good. We treat all the possible subcases as before:

If \( \sigma \) belongs to the open star \( \text{st}(\sigma_0) \) i.e. \( \sigma_0 \prec \sigma \), then since \( \text{spt}(\partial T) \subset X \setminus U_\varepsilon \), we have \( \text{spt}(\partial p_* T) = \text{spt}(p_* \partial T) \subset X \setminus \text{st}(\sigma_0) \). Thus, the fact \( \text{st}(\sigma) \subset \text{st}(\sigma_0) \) implies \( \text{spt}(\partial p_* T) \cap \text{st}(\sigma) = \emptyset \). On the other hand,

\[
\dim \text{spt}(p_* T) \cap \sigma = \dim \text{spt}(T) \cap \sigma \leq s(\sigma) + 1.
\]

Therefore, \( \sigma \) also satisfies (2.2).

If \( \sigma \) belongs to the boundary \( \partial \sigma_0 \), then by the lemma 13, \( \sigma \) is good w.r.t. \( T \) of type II; i.e. \( \dim \sigma = s(\sigma) + 1 \) and \( \text{spt}(\partial T) \cap \text{st}(\sigma) = \emptyset \). Now, \( \dim \text{spt}(p_* T) \cap \sigma \leq \dim \sigma = s(\sigma) + 1 \) and \( \text{spt}(\partial p_* T) \cap \sigma = \emptyset \). Hence, \( \sigma \) satisfies (2.2).

If \( \sigma \) belongs to \( \partial \text{st}(\sigma_0) \setminus \partial \sigma_0 \), then we choose \( \tau \in \mathcal{S}_2 \) to be the open simplex of minimum dimension such that \( \sigma_0 \prec \tau \) and \( \sigma \prec \tau \). Since \( s(\sigma_0) \leq s(\sigma) \leq s(\tau) \), the
lemma 11 implies \( s(\sigma) = s(\tau) \). Thus,

\[
\dim spt(p_#T) \cap \sigma \leq \max (\dim spt(T) \cap \tau, \dim spt(T) \cap \sigma) \leq s(\tau) + 1 = s(\sigma) + 1
\]

and

\[
\dim spt(\partial p_#T) \cap \sigma = \dim spt(p_#\partial T) \cap \sigma \\
\leq \max (\dim spt(\partial T) \cap \tau, \dim spt(\partial T) \cap \sigma) \leq s(\tau) = s(\sigma).
\]

Therefore, \( \sigma \) satisfies (2.2).

This shows that also in case 2, \( p_#T \in \mathcal{P}_i \). 

Now, let \( T_1 = p_#T \) and let \( h \) be an “affine homotopy” from the identity to \( p \), \( R = h_t([0,1] \times T) \), and \( L = h_t([0,1] \times \partial T) \). Then for any \( \sigma \in T_\Sigma \),

\[
\dim(R \cap \sigma) \leq \dim(T \cap \sigma) + 1 \leq i - j + p_j + 1,
\]

and

\[
\dim(\partial R \cap \sigma) \leq \dim(\partial T \cap \sigma) + 1 \leq i - j + p_j,
\]

so \( R \in \mathcal{P}_{i+1} \). Also,

\[
\dim L \leq \dim(\partial T \cap \sigma) + 1 \leq (i - 1) - j + p_j + 1 = i - j + p_j,
\]

so \( L \in \mathcal{P}_i \).

(a) now follows from the homotopy formula [27, pg 139].

To prove (b), note that if \( \sigma \) is good w.r.t. \( T \), then \( \sigma \) is still good w.r.t. \( p_#T \).

Also, there is one open simplex, namely \( \sigma_0 \), which is good w.r.t. \( p_#T \) but bad w.r.t. \( T \).
Thus, we have

\[ \{ \sigma : \sigma \text{ is bad w.r.t. } p_\# T \} \subseteq \{ \sigma : \sigma \text{ is bad w.r.t. } T \} . \]

(c) and (d) readily follow from the definition of \( L \). ■

**Corollary 15** For any \( T \in \mathcal{P}_i \), there exists a \( T_1 \in \mathcal{P}_i \) and \( R \in \mathcal{P}_{i+1} \), \( L \in \mathcal{P}_i \) such that

(a) \( T = T_1 + \partial R + L \);

(b) All open simplicies \( \sigma \in \mathcal{I}_T \) are good w.r.t. \( T_1 \);

(c) \( L = 0 \) if \( \partial T = 0 \);

(d) \( L \in S_i \) if \( \partial T \in S_{i-1} \).

**Proof.** Since \( X \) is compact, the \( \# \{ \sigma : \sigma \text{ is bad w.r.t. } T \} \) is finite. After using proposition 14, we may apply it a second time with \( T \) replaced by \( T_1 \) (and new choice of \( \sigma_0 \)). Continuing inductively a finite number of times, one will get the desired results. ■

**Lemma 16** Suppose \( T \in \mathcal{P}_i \) with \( spt T \subset \mathcal{T}_k \), the \( k \)-skeleton of \( T \), for some \( k \geq i+1 \). If all open simplicies \( \sigma \in \mathcal{I}_T \) are good w.r.t. \( T \), then there exists \( T_1 \in \mathcal{P}_i \) with \( spt T_1 \subset \mathcal{T}_{k-1} \), \( R \in \mathcal{P}_{i+1} \) and \( L \in \mathcal{P}_i \) such that

(a) all open simplicies \( \sigma \in \mathcal{I}_T \) are good w.r.t. \( T_1 \),

(b) \( T = T_1 + \partial R + L \)

(c) \( L = 0 \) if \( \partial T = 0 \)

(d) \( L \in S_i \) if \( \partial T \in S_{i-1} \)

**Proof.** As in [27, lemma 29.4], one can choose, for each \( k \)-simplex \( \tau \) of \( \mathcal{T}_k \), a suitable point \( a_\tau \in \tau \) so that the radial retraction away from \( a_\tau \) gives a locally Lipschitz map \( \psi : \mathcal{T}_k \setminus \cup \tau \{ a_\tau \} \rightarrow \mathcal{T}_{k-1} \) along with a mass bound for \( \psi_\# T \).
For any $\sigma \in T_\Sigma$, we'll show that $\sigma$ is also good w.r.t. $T_1 = \psi_\#T$. A basic fact about the map $\psi$ is that $\psi^{-1}(x) \subset st(\sigma)$ for any $x \in \sigma$ and hence $\psi^{-1}(st(\sigma)) \subset st(\sigma)$.

Therefore, if $\sigma$ is absolutely good, then $\sigma$ is automatically good w.r.t. $T_1$ of type I. If $\sigma$ is good w.r.t. $T$ of type II, i.e. $\dim(\sigma) = s(\sigma) + 1$ and $spt(\partial T) \cap st(\sigma) = \emptyset$, then $\dim T_1 \cap \sigma \leq \dim \sigma = s(\sigma) + 1$ and $spt(\partial T_1) \cap st(\sigma) \subset \psi(spt(\partial T) \cap st(\sigma)) = \emptyset$.

Hence $\sigma$ is good w.r.t. $T_1$ of type II. Finally, if $\sigma$ is good w.r.t. $T$ of type III, i.e. $spt(T) \cap st(\sigma) = \emptyset$, then $spt(T_1) \cap st(\sigma) \subset \psi(spt(T) \cap st(\sigma)) = \emptyset$. Hence $\sigma$ is also good w.r.t. $T_1$ of type III. This shows that all open simplicies $\sigma \in T_\Sigma$ are still good w.r.t. $T_1$.

Now as usual, let $h$ be an "affine homotopy" from the identity to $\psi$, $R = h_t([0,1] \times T)$, and $L = h_t([0,1] \times \partial T)$. One readily checks that $R \in P_{i+1}$ and $L \in P_i$ have the desired properties. ■

By applying this lemma repeatedly, we will eventually get $spt(T) \subset T_i$, then we'll use the following:

**Lemma 17** Suppose $T \in P_i$ with $sptT \subset T_i$. If all open simplicies $\sigma \in T_\Sigma$ are good w.r.t. $T$, then there exists $S = \Sigma_{F \in T_i} \beta_F [[F]] \in S_i$ for some integers $\beta_F$ such that $M(T - S) + M(\partial(T - S)) \leq cM(\partial T)$ for some constant $c$.

**Proof.** As in [27, page 175-176], for any $i$-dimensional face $F$, one can find an integer $\beta_F$ such that $M(T \downarrow F - F) + M(\partial(T \downarrow F - F)) \leq cM(\partial T \downarrow F)$.

Now, we'll show that $S = \Sigma_{F \in T_i} \beta_F [[F]] \in P_i$. In fact, for any $\sigma \in T_\Sigma$ we know that $\sigma$ is good w.r.t. $T$. We'll show that $\sigma$ is also good w.r.t. $S$. If $\sigma$ is absolutely good, then $\sigma$ is automatically good w.r.t. $S$. If $\sigma$ is good w.r.t. $T$ of type II, i.e. $\dim(\sigma) = s(\sigma) + 1$.
and \( spt(\partial T) \cap st(\sigma) = \emptyset \). This implies that for any \( F \in T_i \cap st(\sigma) \), we have \( \partial T \cap F = 0 \). Therefore, \( T \cap F = \beta_F[F] \) and hence \( spt(\partial S) \cap st(\sigma) = spt(\partial T) \cap st(\sigma) = \emptyset \). Thus, \( \sigma \) is also good w.r.t. \( S \) of type II. Finally, if \( \sigma \) is good w.r.t. \( T \) of type III, i.e. \( spt(T) \cap st(\sigma) = \emptyset \), then for any \( F \in T_i \cap st(\sigma) \), \( T \cap F = 0 \) and hence \( \beta_F = 0 \). This also shows \( spt(S) \cap st(\sigma) = \emptyset \) and \( \sigma \) is good w.r.t. \( S \) of type III. Therefore, all open simplicies \( \sigma \in T_i \) are good w.r.t. \( S \), which automatically implies \( S \in \mathcal{P}_i \). By the construction of \( S \), we know that \( S \) is a subanalytic chain. ■

**Theorem 18 (Deformation Theorem)** For any \( T \in \mathcal{P}_i \), there exists a \( S \in \mathcal{S}_i \) and \( R \in \mathcal{P}_{i+1}, \ L \in \mathcal{P}_i \) such that

(a) \( T = S + \partial R + L \)

(b) \( L = 0 \) if \( \partial T = 0 \)

(c) \( L \in \mathcal{S}_i \) if \( \partial T \in \mathcal{S}_{i-1} \)

**Proof.** First apply corollary 15 to change \( T \) so that all the open simplicies are good w.r.t. \( T \), then apply lemma 16 inductively, we may assume \( sptT \subset T_i \). At last, apply lemma 17. ■

**Definition 19** Let \( IH_{\text{subanalytic}}^*(X) \), \( IH_*^*(X) \) denote the homology groups of the chain complexes \( \{ \mathcal{S}_i \}, \{ \mathcal{P}_i \} \) defined above respectively.

Then, we have the following isomorphism theorem:

**Theorem 20 (Isomorphism Theorem)** The inclusion map \( j : IH_{\text{subanalytic}}^*(X) \hookrightarrow IH_*^*(X) \) is an isomorphism.
Proof. (i) \( j \) is injective.

If \([S] = 0 \in IH_i(X)\), i.e. \( S = \partial T \) for some \( T \in P_{i+1}(X) \), then by the deformation theorem 18,

\[
T = S' + \partial R' + L'
\]

with \( L' \) subanalytic. So, \( S = \partial(S' + L') \) and \( S' + L' \in S_{i+1} \). Hence, \( j \) is injective.

(ii) \( j \) is onto.

For any \([T] \in IH_i(X)\), by the deformation theorem 18, \( T = S + \partial R \). Hence, \([T] = [S] = j([S])\), i.e. \( j \) is onto. ■

2.4 A modified mass on the complex of rectifiable currents

The limit of a sequence of rectifiable currents with bounded mass and bounded boundary mass is a rectifiable current [27, theorem 27.3]. However, the limit rectifiable current may fail to satisfy the allowability conditions of the approximating chains. This motivates us to modify the usual mass by adding some suitable mass modifiers. For any rectifiable current \( T \in R_i(X) \) and each singular stratum, we’ll add a mass modifier for \( T \) corresponding to that stratum. To control the amount of mass modified, we choose and fix a small tolerance \( \delta > 0 \).

As in section 1.4, we have a fixed perversity \( \vec{p} = (p_2, \cdots, p_n) \) and a fixed dimension \( i \in \{0, \cdots, n\} \). Now, for each singular stratum \( X_{n-k} \) with some integer \( k \in \{2, \cdots, n\} \) and any rectifiable current \( T \in R_i(X) \), we’ll define a \( k \)-th mass modifier \( m_k^\delta(T) \) in the 3 possible cases as follows:

**Case 1:** \( i - k + p_k \geq n - k \), i.e. \( i + p_k \geq n \) (e.g. \( i = n \)).
In this case,

\[ \dim(spt(T) \cap X_{n-k}) \leq \dim(X_{n-k}) \leq n - k \leq i - k + p_k. \]

i.e. the perversity condition is automatically satisfied. Since it’s unnecessary to make any modification on the mass, we here set \( m_k^\delta(T) := 0 \).

**Case 2:** \( i - k + p_k < 0 \), i.e. \( i < k - p_k \), e.g. \( i = 0 \) or \( 1 \).

In this case, \( \dim(spt(T) \cap X_{n-k}) \leq i - k + p_k \) iff \( spt(T) \cap X_{n-k} = \emptyset \).

Here we define the \( k \)-mass modifier

\[ m_k^\delta(T) := \ln \frac{\delta}{\text{dist}(spt(T), X_{n-k})} M(T \llcorner B(X_{n-k}, \delta)). \]

Before considering the remaining case, we first make two easy but important observations:

**Lemma 21** \( \dim(spt(T) \cap X_{n-k}) \leq i - k + p_k \) iff \( m_k^\delta(T) < +\infty \).

**Proof.**

\[
\begin{align*}
m_k^\delta(T) &< +\infty \iff \text{dist}(spt(T), X_{n-k}) > 0 \\
&\iff spt(T) \cap X_{n-k} = \emptyset \iff \dim(spt(T) \cap X_{n-k}) \leq i - k + p_k.
\end{align*}
\]

**Lemma 22** \( m_k^\delta \) is lower-semi continuous with respect to the weak convergence of currents.

**Proof.** If \( T_j \rightharpoonup T \), then \( M(T \llcorner B(X_{n-k}, \delta)) \leq \liminf_{j \to \infty} M(T_j \llcorner B(X_{n-k}, \delta)) \) and

\[ \text{dist}(spt(T), X_{n-k}) \geq \limsup_{j \to \infty} \text{dist}(spt(T_j), X_{n-k}). \]
Therefore,

\[
\ln \frac{\delta}{\text{dist}(\text{spt}(T), X_{n-k})} \leq \liminf_{j \to \infty} \ln \frac{\delta}{\text{dist}(\text{spt}(T_j), X_{n-k})}
\]

and

\[
m_k^\delta(T) \leq \liminf_{j \to \infty} \ln \frac{\delta}{\text{dist}(\text{spt}(T_j), X_{n-k})} \liminf_{j \to \infty} M(T_j \cap B(X_{n-k}, \delta)) \leq \liminf_{j \to \infty} \ln \frac{\delta}{\text{dist}(\text{spt}(T_j), X_{n-k})} M(T_j \cap B(X_{n-k}, \delta)) = \liminf_{j \to \infty} m_k^\delta(T_j).
\]

\[
\therefore m_k^\delta \text{ is lower-semi continuous.} \quad \blacksquare
\]

**Case 3:** \(0 \leq i - k + p_k < n - k\) i.e. \(k - p_k \leq i < n - p_k\)

**Remark 23** If \(X\) has isolated singularities, i.e. \(\dim(\Sigma) = 0\), then case 3 will not happen.

Set

\[G_k = \{\text{all } N - (i - k + p_k) - 1 \text{ dimensional planes in } \mathbb{R}^N\}\]

with the standard measure \(\mu\), induced from \(G_k\) being an \((i - k + p_k) + 1\) dimensional vector bundle over the grassmannian manifold \(G(N, N - (i - k + p_k) - 1)\) with its invariant measure.

Define \(d_T : G_k \to [0, +\infty]\) by

\[
d_T(H) = \text{dist}(\text{spt}(T) \cap H, X_{n-k})/\delta.
\]

and define

\[
u_k^T(t) := \frac{1}{t} \mu(d_T^{-1}[0, t]) = \frac{1}{t} \mu(\{H \in G_k : T \cap B(X_{n-k}, t\delta) \cap H \neq \emptyset\}),
\]
so \( tu_k^T(t) \) is increasing and hence differentiable for a.e. \( t \in [0, 1] \).

The function \( u_k^T \) gives a normalized count (without multiplicity) of the number of planes intersecting the current \( T \) near the singular stratum \( X_{n-k} \). It's similar to the quermassintegrale [26, 13.8] of convex sets. We'll use its \( L^1 \) norm to define the mass modifier in the definition 26. Note that \( \| u_k^T \|_{L^1([0,1])} \) may be infinite for some allowable rectifiable current \( T \) having infinite order contact with \( X_{n-k} \), but it will be finite for allowable subanalytic chain by theorem 30.

To obtain the lower semicontinuity of our modified mass (defined in 26), we need the following lemmas:

**Lemma 24** If \( \text{sup } M(T_j) + M(\partial T_j) < \infty \) and \( T_j \to T \), then \( u_k^T(t) \leq \liminf u_k^{T_j}(t) \) for all \( t \in (0, 1] \).

**Proof.** The hypotheses imply, by [10, 4.3.2], that for a.e. \( H \in G_k, T_j \cap H \to T \cap H \). Thus,

\[
tu_k^T(t) = \mu(\{H \in G_k : M(T \cap B(X_{n-k}, t\delta) \cap H) > 0\})
\]

\[
\leq \mu(\{H \in G_k : \liminf_{j \to \infty} M(T_j \cap B(X_{n-k}, t\delta) \cap H) > 0\})
\]

\[
= \int \chi_{\{H \in G_k : \liminf_{j \to \infty} M(T_j \cap B(X_{n-k}, t\delta) \cap H) > 0\}} \, d\mu
\]

\[
\leq \int \liminf_{j \to \infty} \chi_{\{H \in G_k : M(T_j \cap B(X_{n-k}, t\delta) \cap H) > 0\}} \, d\mu \quad \text{(by Fatou's lemma)}
\]

\[
= \liminf_{j \to \infty} \mu(\{H \in G_k : M(T_j \cap B(X_{n-k}, t\delta) \cap H) > 0\})
\]

\[
= t \liminf_{j \to \infty} u_k^{T_j}(t).
\]
Lemma 25. For any $T \in \mathcal{R}_t(X)$, if $u^T_k(t) \in L^1([0,1])$, then

1. $\mu(d_T^{-1}(0)) = 0$;
2. $\dim(spt(T) \cap X_{n-k}) \leq i - k + p_k$;
3. $\|u^T_k(t)\|_{L^1([0,1])} = \int_{d_T^{-1}([0,1])} \ln \left(\frac{1}{d_T}\right) d\mu$

Proof. Since $tu^T_k(t) = \mu(d_T^{-1}([0,t]))$ is increasing in $t$ and $u^T_k(t) \in L^1([0,1])$, we have

$$\mu(d_T^{-1}(0)) = \mu(\{H : H \cap spt(T) \cap X_{n-k} \neq \emptyset\}) = 0.$$ 

This implies that $spt(T) \cap X_{n-k}$ is a set of $(i - k + p_k) + 1$ dimensional integral geometric Favard measure zero (see [11] or [26, III. 14.7.1] for details). By [11, theorem 9],

$\dim(spt(T) \cap X_{n-k}) \leq i - k + p_k$.

As for (3), a classical application of Fubini’s theorem is the formula

$$\int_E f(x) \, dx = \int_0^\infty m \{x \mid f(x) > t\} \, dt$$

where $f$ is a nonnegative measurable function on a $\mu$ measurable set $E$.

Since $d_T^{-1}(0)$ has $\mu$ measure 0, we can apply the above formula to the function $ln \left(\frac{1}{d_T}\right) : d_T^{-1}([0,1]) \to [0, \infty)$ as follows:

$$\int_{d_T^{-1}([0,1])} \ln \left(\frac{1}{d_T}\right) d\mu = \int_0^\infty \mu \left(\left\{H \mid \ln \left(\frac{1}{d_T(H)}\right) > t\right\}\right) dt = \int_0^\infty \mu \left(\left\{H \mid 0 < d_T(H) < e^{-t}\right\}\right) dt = \int_0^1 \mu \left(\left\{H \mid 0 < d_T(H) < \frac{s}{s}\right\}\right) ds = \int_0^1 u_k^T(t) \, dt.$$ 

All our above discussion leads to the following:
Definition 26 For fixed $\delta > 0$ and any $T \in \mathcal{R}_i(X)$, we define the $k$-mass modifier of $T$ to be

$$ m_k^i(T) \equiv \begin{cases} 
0, & \text{if } p_k \geq n - i \\
\|u_k^T(t)\|_{L^1([0,1])} M(T \llcorner B(X_{n-k}, \delta)), & \text{if } k - i \leq p_k < n - i \\
\ln_{\text{dist}(x_0(T), x_{n-k})} M(T \llcorner B(X_{n-k}, \delta)) & \text{if } p_k < k - i
\end{cases} $$

A new modified mass on $\mathcal{R}_i(X)$ is given by $\tilde{M}(T) = \tilde{M}^i(T) \equiv \sum_{k=2}^n m_k^i(T) + M(T)$ for any $T \in \mathcal{R}_i(X)$. Also, we set

$$ \mathcal{I}_i(X) = \{ T \in \mathcal{R}_i(X) : \tilde{M}(T) + \tilde{M}(\partial T) < +\infty \} $$

to be the set of all rectifiable currents with finite modified mass and finite modified boundary mass.

Proposition 27 If $\tilde{M}^{\delta_0}(T) < +\infty$ for some $\delta_0 > 0$, then $\lim_{\delta \to \delta_+} \tilde{M}^i(T) = M(T)$.

Proof. For each $k : k - i \leq p_k < n - i$, whenever $\delta < \delta_0$,

$$ m_k^i(T) = \|u_k^{T,\delta}(t)\|_{L^1([0,1])} M(T \llcorner B(X_{n-k}, \delta)) $$

$$ = \|u_k^{T,\delta_0}(t)\|_{L^1([0,1])} \frac{\delta}{\delta_0} M(T \llcorner B(X_{n-k}, \delta)) $$

$$ \leq \|u_k^{T,\delta_0}(t)\|_{L^1([0,1])} M(T \llcorner B(X_{n-k}, \delta_0)) $$

$$ \to 0, \text{ as } \delta \to 0 +. $$

Similarly, if $p_k < k - i$, we also have $m_k^i(T) \to 0$ as $\delta \to 0$. Thus, $\lim_{\delta \to \delta_+} \tilde{M}^i(T) = M(T)$.

Proposition 28 $\tilde{M}$ mass is lower semi-continuous. i.e. if $T_j \rightharpoonup T$, then $\tilde{M}(T) \leq \liminf \tilde{M}(T_j)$. 

**Proof.** For $k - i \leq p_k < n - i$, if $T_j \rightarrow T$, then by the lemma 24, $u_k^T(t) \leq \liminf_j T_j^j(t)$. Therefore, by the Fatou's lemma,

$$
m_k^\delta(t) = \left\| u_k^T(t) \right\|_{L^1([0,1])} M(T \llcorner B(X_{n-k}, \delta)) \leq \liminf_{j \rightarrow \infty} \left\| u_k^{T_j}(t) \right\|_{L^1([0,1])} M(T \llcorner B(X_{n-k}, \delta)) \leq \liminf_{j \rightarrow \infty} \left[ \left\| u_k^{T_j}(t) \right\|_{L^1([0,1])} \liminf_{j \rightarrow \infty} M(T_j \llcorner B(X_{n-k}, \delta)) \right] = \liminf_{j \rightarrow \infty} m_k^\delta(T_j).
$$

This, along with lemma 22 and the lower semi-continuity of the usual mass, implies the lower semi-continuity of $M$. ■

The following proposition says that a rectifiable current with finite modified mass and finite modified boundary mass automatically satisfies the perversity condition:

**Proposition 29** \( I_i(X) \subset P_i(X) \)

**Proof.** For any \( T \in I_i(X) \), by the lemma 21 and the lemma 25, we have \( \dim(spt(T) \cap X_{n-k}) \leq i - k + p_k \) and \( \dim(spt(\partial T) \cap X_{n-k}) \leq i - k + p_k - 1 \) for each \( k \). Thus, \( T \in P_i(X) \). ■

**Theorem 30** \( S_i(X) \subset I_i(X) \subset P_i(X) \)

**Proof.** It is sufficient to show \( S_i(X) \subset I_i(X) \).

For any subanalytic chain \( T \in S_i(X) \) and \( k \) with \( k - i \leq p_k < n - i \), the graph of the function \( d_T \) defined in (2.3) is a subanalytic set because both \( spt(T) \) and \( X_{n-k} \) are subanalytic. By the Lojasiewicz's inequality (See Proposition 4), there exists a constant
\[ C > 0 \text{ and } N > 0 \text{ such that } C d_T(H) \geq \operatorname{dist}_{G_0}(H, d_T^{-1}(0))^N. \] Also, \( T \in \mathcal{P}_i(X) \) implies \( \mu(d_T^{-1}(0)) = 0 \). Thus,

\[ u_k^T(t) = \frac{1}{t} \mu(d_T^{-1}(0, t]) \leq \frac{1}{t} \mu \left( \left\{ H : \operatorname{dist}_{G_0}(H, d_T^{-1}(0)) \leq (Ct)^{-1} \right\} \right) \leq C_1 t^{\beta - 1} \]

for some \( \beta > 0 \). Therefore, \( u_k^T(t) \in L^1[0, 1] \) and hence \( m_k^\xi(T) < +\infty \) for each \( k \) with \( k - i \leq p_k < n - i \). Similar arguments yield \( m_k^\xi(T) < +\infty \) for all other \( k \)'s. This shows \( \tilde{M}(T) < +\infty \). Also, one has \( \tilde{M}(\partial T) < +\infty \) because \( \partial T \in S_{i-1} \). Thus we have \( T \in I_i \).

\[ \blacksquare \]

**Theorem 31 (Compactness theorem)**  
Any sequence \( \{T_j\} \) in \( I_i \) with

\[ \liminf \tilde{M}(T_j) + \tilde{M}(\partial T_j) < +\infty, \]

contains a subsequence \( \{T_{j_k}\} \) weakly convergent to some \( T \in I_i \).

**Proof.** Since \( \liminf M(T_j) + M(\partial T_j) < +\infty \), by the usual compactness theorem of integer multiplicity rectifiable currents (c.f. [27, Theorem 27.3]), there exists a subsequence \( \{T_{j_k}\} \) of \( \{T_j\} \) such that \( T_{j_k} \rightharpoonup T \) for some \( T \in R_i(X) \). By the lower semi-continuity of \( \tilde{M} \), we have \( \tilde{M}(T) + \tilde{M}(\partial T) \leq \liminf \tilde{M}(T_{j_k}) + \tilde{M}(\partial T_{j_k}) < +\infty \), and hence \( T \in I_i(X) \). \( \blacksquare \)

A direct corollary of the previous two theorems is:

**Corollary 32**  
Any sequence of subanalytic chains \( \{T_j\} \subset S_i \) with

\[ \liminf \tilde{M}(T_j) + \tilde{M}(\partial T_j) < +\infty, \]

contains a subsequence weakly convergent to some rectifiable current \( T \in I_i \).
2.5 $\tilde{M}$-Mass Minimizing currents

**Definition 33** We say that $T \in \mathcal{P}_i(X)$ is $\tilde{M}$-mass minimizing if

$$\tilde{M}(T) \leq \tilde{M}(S)$$

whenever $S \in \mathcal{P}_i(X)$ and $\partial T = \partial S$.

The following theorem says that there is a $\tilde{M}$-mass minimizer in each intersection homology class of $IH_i(X)$:

**Theorem 34 (Existence theorem)** 1. Suppose $E = \partial S$ for some $S \in \mathcal{I}_i(X)$, then there exists a rectifiable current $T \in \mathcal{I}_i(X)$ such that $\partial T = E$ and $\tilde{M}(T) \leq \tilde{M}(R)$ for any $R \in \mathcal{I}_i(X)$ with $\partial R = E$.

2. For each homology class $\alpha \in IH_i(X)$, there exists $T \in \alpha \cap \mathcal{I}_i(X)$ such that $\tilde{M}(T) \leq \tilde{M}(T')$ for any $T' \in \alpha$.

**Proof.** (1) follows from the direct method [23, 1.3] and the compactness theorem 31.

To obtain a minimizing sequence in (2), we note, by the isomorphism theorem 20, that each class $\alpha \in IH_i(X)$ contains at least one subanalytic representative $S$ and $\tilde{M}(S) < +\infty$, by theorem 30. Now, the compactness theorem 31 ensures the existence of an $\tilde{M}$-mass minimizing in the nonempty subset $\alpha \cap \mathcal{I}_i(X)$. ■

Now, let's consider the regularity of $\tilde{M}$-Mass Minimizing currents.

For any $T \in \mathcal{I}_i(X)$, let $a \in spt(T) \setminus (spt(\partial T) \cup \Sigma)$. Recall that $a$ is a regular point of $spt(T)$ if there exists an open neighborhood $U$ of $a$ in $X \setminus \Sigma$ such that $U \cap sptT$ is an open $i$-manifold of class $C^1$. 
Definition 35  Let $\omega(t)$ be defined for $0 < t \leq \delta$, with $\lim_{t \to 0} \omega(t) = 0$. Let $\Psi$ be a positive integrand and let $T$ be a rectifiable current with compact support in $\Omega$, an open subset of some Euclidean space $\mathbb{R}^N$. Recall that $T$ is $(\Psi, \omega, \delta)$-minimal if

$$\Psi[T \cdot K] \leq \Psi[T \cdot K + X] + \omega(r)M(T \cdot K + X)$$

for all rectifiable $X$ with compact support in $K \subset \Omega$ and with

$$\partial X = 0, \text{ diam(spt } X) \leq r \leq \delta.$$  

Remark 36  By an argument similar to [16], the above inequality can be replaced by

$$\Psi[T \cdot K] \leq \Psi[T \cdot K + X] + \omega(r)C \text{ for some constant } C > 0.$$ 

Let's first recall a lemma of Almgren in [4]:

Lemma 37  Let $\Psi$ be $\lambda$-elliptic and of class $C^2$, and let $T$ be a $(\Psi, \omega, \delta)$-minimal current for some $\lambda < \infty, \delta > 0$ and some $\omega(t)$ with

$$\int_0^\delta \frac{\omega(t)^{1/2}}{t} dt < \infty.$$ 

Then regular points are dense in $\text{spt}(T) \setminus \text{spt}(\partial T)$.

Such current $T$ is often called an almost minimizing current. Note that this notion is very general. An oriented compact $C^2$ submanifold $T$ is automatically locally almost minimizing where $(\Psi, \omega, \delta)$ depend on $T$ and the distance to $\text{spt}(\partial T)$.

Proposition 38  A $\tilde{M}$-mass minimizing current $T$ locally is an almost minimizing current (with $(\Psi, \omega, \delta)$ depending on $T$ and distance to $\text{spt}(\partial T) \cup \Sigma$).
Proof. For any \( a \in \text{spt} T \setminus (\text{spt} (\partial T) \cup \Sigma) \), let \( U_a \) be a very small open neighborhood of \( a \) of radius \( r_a < \frac{1}{2} \text{dist}(a, \Sigma) \). Since \( T \) is a \( \tilde{M} \)-mass minimizing current, for any \( i \)-current \( S \in \mathcal{R}_i(U_a) \) with \( \partial S = 0 \) and \( \text{diam}(\text{spt} S) \leq r \leq r_a \), we have

\[
\tilde{M}(T) \leq \tilde{M}(T + S).
\]

For each \( k = 2, \ldots, n \), let \( \Phi_k \) be the nonnegative integrand whose total integral over a rectifiable current \( R \) is

\[
\Phi_k(R) \equiv h_k M (R \setminus B (X_{n-k}, \delta)),
\]

where

\[
h_k = \begin{cases} 
||u^T_k||_{L^1([0,1])}, & \text{if } \text{dist}(a, X_{n-k}) \leq \delta \text{ and } k - i \leq p_k < n - i \\
\ln \frac{\delta}{\text{dist}(\text{spt}(T), X_{n-k})}, & \text{if } \text{dist}(a, X_{n-k}) \leq \delta \text{ and } p_k \geq n - i \\
0, & \text{otherwise}
\end{cases} \quad (2.4)
\]

Lemma 39 For each \( k = 2, \ldots, n \) and \( r \) small enough,

\[
[\Phi_k(T) - \Phi_k(T + S) - [m_k^\delta(T) - m_k^\delta(T + S)]] \\
\leq C_k \frac{r}{\text{dist}(a, X_{n-k})} M ((T + S) \setminus B(X_{n-k}, \delta))
\]

for some positive constant \( C_k \) independent of \( T, S \) and \( a \).

Proof. If \( \text{dist}(a, X_{n-k}) > \delta \), we may choose \( U_a \) small enough such that

\[
\text{dist}(U_a, X_{n-k}) > \delta, \quad (2.5)
\]

then \( m_k^\delta(T) = m_k^\delta(T + S) \). Thus,

\[
[\Phi_k(T) - \Phi_k(T + S) - [m_k^\delta(T) - m_k^\delta(T + S)]] = [0 - 0] - 0 = 0.
\]
Now, we assume $\text{dist}(a, X_{n-k}) \leq \delta$. In this case, $\Phi_k(T) = m_k^\delta(T)$, so it is sufficient to show

$$m_k^\delta(T + S) - \Phi_k(T + S) \leq C_k \frac{r}{\text{dist}(a, X_{n-k})} M((T + S) \cap B(X_{n-k}, \delta))$$

1. If $p_k \geq n - i$, then $m_k^\delta(T + S) - \Phi_k(T + S) = 0 - 0 = 0$.

2. If $k - i \leq p_k < n - i$, then for any $t \in (0, 1]$, we consider

$$t \left( u_k^{T+S}(t) - u_k^T(t) \right)$$

$$= \mu \{ H \in G_k : \text{spt}(T + S) \cap B(X_{n-k}, t\delta) \cap H \neq \emptyset \}$$

$$-\mu \{ H \in G_k : \text{spt}(T) \cap B(X_{n-k}, t\delta) \cap H \neq \emptyset \}$$

$$\leq \mu \{ H \in G_k : \text{spt}(T + S) \cap B(X_{n-k}, t\delta) \cap H \neq \emptyset \}
\quad \text{and spt}(T) \cap B(X_{n-k}, t\delta) \cap H = \emptyset \}$$

$$= \mu \{ H \in G_k : \text{spt}(S) \cap B(X_{n-k}, t\delta) \cap H \neq \emptyset \}
\quad \text{and spt}(T) \cap B(X_{n-k}, t\delta) \cap H = \emptyset \}$$

$$\leq \mu \{ H \in G_k : U_r \cap B(X_{n-k}, t\delta) \cap H \neq \emptyset \}$$

$$\leq \begin{cases} 
0, & \text{if } t\delta \leq \text{dist}(U_a, X_{n-k}) \\
C_k r^{(i-k+p_k)+1} & \text{if } t\delta \geq \text{dist}(U_a, X_{n-k})
\end{cases}$$

where $U_r \subset U_{r_a}$ is an Euclidean ball of radius $r$ that contains $S$ and $C_k$ is a constant depends only on $N$ and $(i - k + p_k) + 1$, the dimension of the moving planes. Note the last inequality follows from the formula ([26, 13.46]) for the
quermassintegrale of a ball. Thus, we have

\[
||u_k^{T+S}||_{L^1[0,1]} - ||u_k^T||_{L^1[0,1]}
= \int_0^1 u_k^{T+S}(t) - u_k^T(t) \, dt \\
\leq \int_{\text{dist}(U_a,X_{n-k})/\delta}^1 \frac{C_k r^{(i-k+p_k)+1}}{t} \, dt \\
\leq C_k r^{(i-k+p_k)+1} \ln \frac{\delta}{\text{dist}(U_a,X_{n-k})} \\
\leq C_k \frac{2\delta}{\text{dist}(a,X_{n-k})}
\]

Hence,

\[
m_k^\Phi(T+S) - \Phi_k(T+S)
= \left[||u_k^{T+S}||_{L^1[0,1]} - ||u_k^T||_{L^1[0,1]}\right] M((T+S) \cup B(X_{n-k},\delta)) \\
\leq 2\delta C_k \frac{r}{\text{dist}(a,X_{n-k})} M((T+S) \cup B(X_{n-k},\delta)).
\]

3. If \( p_k < k - i \), then when \( r \) small enough,

\[
\ln \frac{\delta}{\text{dist}(\text{spt}(T+S),X_{n-k})} - \ln \frac{\delta}{\text{dist}(\text{spt}(T),X_{n-k})}
= \begin{cases} 
0, & \text{if } \text{dist}(a,X_{n-k}) > \text{dist}(\text{spt}(T),X_{n-k}) \\
\ln \frac{\text{dist}(\text{spt}(T),X_{n-k})}{\text{dist}(\text{spt}(T+S),X_{n-k})}, & \text{if } \text{dist}(a,X_{n-k}) = \text{dist}(\text{spt}(T),X_{n-k}) \\
0, & \text{if } \text{dist}(a,X_{n-k}) > \text{dist}(\text{spt}(T),X_{n-k}) \\
\ln \frac{\text{dist}(\text{spt}(T),X_{n-k})}{\text{dist}(\text{spt}(T),X_{n-k}) - r}, & \text{if } \text{dist}(a,X_{n-k}) = \text{dist}(\text{spt}(T),X_{n-k}) \\
\end{cases}
\leq 2r \frac{\delta}{\text{dist}(a,X_{n-k})}.
\]
Therefore, by the definition of $h_k$ in (2.4),

$$ m_k^\delta (T + S) - \Phi_k (T + S) $$

$$ = \left( \ln \frac{\delta}{\text{dist} (\text{spt} (T + S), X_{n-k})} - h_k \right) M ((T + S) \cap B (X_{n-k}, \delta)) $$

$$ \leq \frac{2r}{\text{dist} (a, X_{n-k})} M ((T + S) \cap B (X_{n-k}, \delta)). $$

\[
\]

Now, consider the positive integrand $\Psi = \sum_{k=2}^n \Phi_k + \Phi$, where $\Phi$ is the area integrand. Since $T$ is $\tilde{M}$-mass minimizer, by the above lemma 39,

$$ \Psi(T) - \Psi(T + S) $$

$$ \leq \Psi(T) - \Psi(T + S) + \tilde{M} (T + S) - \tilde{M} (T) $$

$$ = \sum_{k=2}^n [\Phi_k (T) - \Phi_k (T + S)] - [m_k^\delta (T) - m_k^\delta (T + S)] $$

$$ \leq \sum_{k=2}^n C_k \frac{r}{\text{dist} (a, X_{n-k})} M ((T + S) \cap B (X_{n-k}, \delta)) $$

$$ \leq \sum_{k=2}^n C_k \frac{r}{\text{dist} (a, \Sigma)} M ((T + S) \cap B (\Sigma, \delta)) $$

$$ = \frac{C}{\text{dist} (a, \Sigma)} r M (T \cap B (\Sigma, \delta) + S) $$

where $C = \sum_{k=2}^n C_k$ is a constant. Thus,

$$ \Psi(T \cap U_a) \leq \Psi(T \cap U_a + S) + \frac{C}{\text{dist} (a, \Sigma)} r M (T \cap B (\Sigma, \delta) + S). $$

This implies $T$ is $(\Psi, \frac{C}{\text{dist} (a, \Sigma)} r, r_a)$-minimal at $a$. \[
\]

**Corollary 40 (Regularity Theorem)** Let $T$ be a $\tilde{M}$-mass minimizing current of $X$, then the regular points are dense in $\text{spt} (T) \setminus (\text{spt} (\partial T) \cup \Sigma)$. 

2.6 Some simple examples

Example 41 Consider the stratified space $\mathbb{R}^2 \supset \{0\}$ with a single singular point $\{0\}$. Suppose $p, q \in \mathbb{R}^2 \setminus \{0\}$ such that 0 is on the line segment $pq$. Then under the usual mass of $\mathbb{R}^2$, the minimal path from $p$ to $q$ is the line segment $pq$ which passes through the singular point $\{0\}$ and hence does not satisfy the allowability condition. Given any small number $\delta > 0$, one can easily check that the minimum path from $p$ to $q$ under the modified mass is the minimum path from $p$ to $q$ in the domain $\mathbb{R}^2 \setminus B_\delta(0)$. A typical example looks like the following graph:

![Graph](image)

Figure 2.1: $p=(-3,0), q=(2,0)$ and $\delta = 0.5$

Example 42 Consider Whitney umbrella $W : x^2 - yz^2 = 0$ in $\mathbb{C}^3$ with its Whitney stratification: $W \supset W_2 \supset W_0$, where $W_2$ is the complex $z$-axis defined by $x = y = 0$ and $W_4$ is the point $\{(0,0,0)\}$.
Note that for any nonzero complex number $z_0$, there are two associated real 2-dimensional planes $x^2 = z_0y^2$. These two planes coincide at 0 to the plane $x = z = 0$.

Now, let's consider some simple variational problems on this famous stratified space.

Case 1: $i = 1$. Since $W \setminus W_2$ is path connected, any two points $p, q \in W \setminus W_2$ can be joined by some path inside $W \setminus W_2$. Let $\gamma_{pq} \subset W$ be a length minimizer from $p$ to $q$ under the usual mass. If $\gamma_{pq}$ does not intersect the singular part $W_2$, i.e. $\gamma_{pq}$ is $(\bar{p}, 1)$-allowable, then when $\delta$ is small enough, $\gamma_{pq}$ will also be a minimizer from $p$ to $q$ under the modified mass. If $\gamma_{pq}$ intersects $W_2$, then there exists a modified mass minimizer $\bar{\gamma}_{pq}$ from $p$ to $q$ which does not intersect the singular part $W_2$. The shape of $\bar{\gamma}_{pq}$ looks like the graph in the previous example.

Case 2: $i = 2$. In this case, the crucial number $i - j + p_j$ in the perversity condition is given by

$$i - j + p_j = \begin{cases} 
0, & j = 2 \text{ and } p_2 = 0 \\
-2, & j = 4 \text{ and } p_4 = 0 \\
-1, & j = 4 \text{ and } p_4 = 1 \\
0, & j = 4 \text{ and } p_4 = 2 
\end{cases}.$$

This means that $(\bar{p}, 2)$-allowable chains are allowed to intersect $W_2$ with 0 dimensional set but are not allowed to intersect $W_4$ unless $p_4 = 2$.

Now, let's consider two $(\bar{p}, 1)$-allowable circles in $W$:

$$A = \{(0, e^{i\theta}, 0) : \theta \in [0, 2\pi]\},$$

the unit circle on the plane $x = z = 0$ (i.e. on the complex $y$-axis);

$$B = \{(z_0 e^{i\theta}, e^{i\theta}, z_0^2) : \theta \in [0, 2\pi]\},$$

a circle around $(0, 0, z_0^2)$ on one of the two
planes $x^2 = z_0^2 y^2$ associated to some $z_0 \neq 0$.

Under the usual mass, the minimal surface having boundary $A - B$ will be either a catenoid or union of two disjoint unit disks, depending on the location of $z_0$. In the later situation, the centers of these two unit disks are $(0,0,0) \in W_4$ and $(0,0,z_0^2) \in W_2 \setminus W_4$, lying on the singular sets. Therefore, when $p_4 = 0$ or 1, the union of two unit disks does not satisfy the desired perversity conditions.

Under the modified mass, the corresponding minimal surface with boundary $A - B$ will still be either a catenoid or union of two disjoint slightly curved disks. However, in the later situation, the centers of the disks are no longer touching $W_4$, they all lie on $W_2 \setminus W_4$ now. This makes the minimal surface to be $(\tilde{p}, 2)$-allowable as desired.

Their graphs are shown in the following diagram:

<table>
<thead>
<tr>
<th>$i = 2$</th>
<th>$p_4 = 0$ or $1$</th>
<th>$p_4 = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>minimizer from $A$ to $B$</td>
<td><img src="image1" alt="Graph 1" /> or <img src="image2" alt="Graph 2" /></td>
<td><img src="image3" alt="Graph 3" /> or <img src="image4" alt="Graph 4" /></td>
</tr>
</tbody>
</table>

**Example 43** Consider the stratified space $X \supseteq X_2 \supseteq X_0$ with $X = \mathbb{R}^4 = \{(x, y, z, t)\}$, $X_2 = \{(x, y, z, t) | x = y = 0\}$ and $X_0 = \{(0,0,0,0)\}$. Let

$$T = \left\{ \left(2 + \cos \theta \left(2 + \frac{\cos \alpha}{2}\right), \sin \theta \left(2 + \frac{\cos \alpha}{2}\right), \frac{\sin \alpha}{2}, 0 \right) : \theta, \alpha \in [0, 2\pi] \right\}$$

be a torus inside $X$. 
Note that $T$ is $(\bar{p}, 2)$-allowable for any perversity $\bar{p}$. By using the convex hull property of mass minimizers, one easily see that the 3-dimensional volume minimizer having boundary $T$ (under the usual mass) is simply the solid torus $S$. Now, $S \cap X_2 = \{(0, 0, z, 0) : z \in [-\frac{1}{2}, \frac{1}{2}]\}$ has dimension 1 and passes through $X_0$. Thus, when $p_4 = 1$ or 2, $S$ is $(\bar{p}, 3)$-allowable. However, when $p_4 = 0$, $S$ becomes not allowable. In this case, one can use our modified mass to get a modified mass minimizer $S'$ in $X$, having boundary $T$ and is $(\bar{p}, 3)$-allowable. In other words, this modified mass minimizer doesn't intersect the singular point $\{(0, 0, 0, 0)\}$. 
CHAPTER 3

OPTIMAL PATHS RELATED TO TRANSPORT PROBLEMS

3.1 Introduction

The transport problem introduced by Monge in 1781 [24] has been studied in many interesting works in the last 10 years [28] [7] [8] [9] [2]. In these works, the cost of a transport mapping or a transport plan is usually an integral of some convex (or concave [9]) function of the distance, such as $|x - y|^p$. However, in some real applications, the actual cost of the transport procedures is not necessarily determined by just knowing an optimal mapping from the starting position to the target position. For example in shipping two items from nearby cities to the same far away city, it may be less expensive to first bring them to a common location and put them on a single truck for most of the transport. In this case, a “Y shaped” path is preferable to a “V shaped” path. In both cases, the transport mapping is trivially the same, but the actual transport path naturally gives the total cost. In this chapter, we will consider the following general problem:

**Problem 44** Given two general probability measures $\mu^+$ and $\mu^-$, find an optimal (“Y shaped”) path for transporting $\mu^+$ to $\mu^-$. 

To solve this problem, one needs to find a suitable category of transport paths as well as a cost functional acting on these paths. Such a category should be broad enough to give existence of an optimal transport path. Also, an optimal transport
path should allow the possibility that some parts overlap in a cost efficient (maybe complicated) fashion but still enjoy some nice regularity properties. If possible, one may hope to visualize such an optimal transport path using numerical analysis and computer graphics.

The family of paths we choose in this chapter is a convex subset of the space of 1-dim flat currents ([10]) with boundary being the difference of the given two measures, viewed as 0-dim currents. For example, a directed graph, with balanced weighting at interior vertices, defines a 1 dimensional flat current joining two atomic measures. In fact each of our transport paths is a limit of some weighted directed graphs. The cost on each transport path is a suitably modified weighted mass of the flat current, similar to the $H$ mass of integral currents in [6]. Unlike in [6], we work with currents whose multiplicities are not necessarily integer valued. With this category and cost functional, the original optimal transport path problem becomes a Plateau-type problem as in the study of minimal surfaces. Luckily, we have the existence theorem of an optimal transport path joining any given probability measure to another.

In section 3, we consider a new distance on the space of probability measures on a fixed convex set. Such a distance is different from any of the Wasserstein distances [7], but still metrizes the weak * topology of the space of probability measures.

In section 4, we take into account the cost on time consumption. This approach gives a natural view of any transport path as a Lipschitz flow of probability measures from the starting probability measure to the targeting probability measure, parameterized by time. Thus, it tells us the exact locations of the transporting measures at a certain time. Moreover, when the cost of time consumption approaches zero, the
corresponding limit flow will naturally give our new distance between measures.

In section 5, we use numerical analysis to visualize optimal transport paths.

In the last section, we discuss the relationship between transport paths and transport plans. A compatible pair of a transport plan and a transport path contains necessary information about the actual transportation such as how, where and when the original measure is decomposed into the targeting measure on the road.

3.2 General setup

In this chapter, we fix and use the following notations as in [10].

Notations: Let

- $U \subset \mathbb{R}^m$ be an open subset of $\mathbb{R}^m$;

- $X \subset U$ be a compact convex subset;

- $\mathcal{D}_1(U)$ be the vector space of 1 dimensional currents in $U$;

- $\mathcal{P}_{1,X}(U)$ be the additive subgroup of $\mathcal{D}_1(U)$ generated by all 1-dimensional oriented simplexes $[u_0, u_1]$ such that $X$ contains the line segment $\{u_0, u_1\}$;

- $\mathcal{P}_{1,X}(U)$ be the real vector space generated by $\mathcal{P}_{1,X}(U)$;

- $P(X)$ be the space of probability measures on $X$;

- $\alpha \in (1 - \frac{1}{m}, 1]$.

We first describe the class of our transport paths in the following
Definition 45 Let

$$G_{1,x}(U) = \{G \in P_{1,x}(U) : \partial G = b - a \text{ with } a, b \in P(X) \text{ atomic}\}.$$ 

Then $G_{1,x}(U)$ is a convex subset of $P_{1,x}(U)$. Also, let

$$T_{1,x}(U) = \text{the } F_1 \text{ closure of } G_{1,x}(U) \text{ in } F_{1,x}(U),$$ 

it is also a convex subset. Here $F_1$ is the flat metric defined on $F_{1,x}(U)$ ([10, 4.1.12]) by

$$F_1(T) = \inf \{M(R) + M(S) : T = R + \partial S, \ R \in F_{1,x}(U), S \in F_{2,x}(U)\}.$$ 

Any element of $T_{1,x}(U)$ will be called a "transport path".

The cost functional on each transport path is given as follows.

Definition 46 Since each $G \in G_{1,x}(U)$ is in fact a weighted directed graph $G = \sum w(e)[[e]]$ with weights $w(e) \in (0, +\infty)$ and $[[e]]$ being an oriented segment viewed as an one dimensional current. Define its $M^\alpha$ mass to be

$$M^\alpha(G) = \sum_{e \in E(G)} w(e)^\alpha \text{ length}(e);$$ 

For each transport path $T \in T_{1,x}(U)$ we define

$$M^\alpha(T) := \inf_{\{G_i\} \in G_{1,x}(U)} \lim_{\substack{i \to \infty \\ F_1(G_i - T) \to 0}} \inf M^\alpha(G_i).$$ 

Also, for each atomic probability measure $\alpha = \sum_{i=1}^n n_i[[x_i]]$, define

$$M^\alpha(\alpha) = \sum_{i=1}^n n_i^\alpha.$$
This definition is motivated by the $H$ mass of integral currents in [6] or [29].

**Example 47** Let $a = m_1 [[x_1]] + m_2 [[x_2]]$ and $b = m_3 [[x_3]]$ with $m_3 = m_1 + m_2$. Then the optimal transport path from $a$ to $b$ looks like the following "Y shaped" graph

Here $x$ is determined by a balance formula:

$$m_1^2 \overline{n}_1^2 + m_2^2 \overline{n}_2^2 = m_3^2 \overline{n}_3^2,$$

where $\overline{n}_i = \frac{x-x_i}{|x-x_i|}$ is the unit vector from $x$ to $x_i$, $i = 1, 2, 3$. Let $\theta_i$ be the angle between $\overline{n}_i$ and $- \overline{n}_3$ for $i = 1, 2$ and $k_1 = \frac{m_1}{m_1+m_2}$, $k_2 = \frac{m_3}{m_1+m_2} = 1 - k_1$. Then the above formula implies that the angles satisfy

$$\cos \theta_1 = \frac{k_1^\alpha + 1 - k_2^\alpha}{2k_1^\alpha},$$

$$\cos \theta_2 = \frac{k_2^\alpha + 1 - k_1^\alpha}{2k_2^\alpha}$$

and

$$\cos (\theta_1 + \theta_2) = \frac{1 - k_1^{2\alpha} - k_2^{2\alpha}}{2k_1^{2\alpha}k_2^{2\alpha}}.$$
In particular, if \( m_1 = m_2 \), then

\[
\theta_1 + \theta_2 = \arccos \left( 2^{2^{\alpha - 1}} - 1 \right).
\]

Also, if \( \alpha = 1/2 \), then \( \theta_1 + \theta_2 = \pi/2 \) for any \( m_1 \) and \( m_2 \).

**Example 48** For \( a = m_1 \,[x_1] + m_2 \,[x_2] \) and \( b = m_3 \,[x_3] + m_4 \,[x_4] \) with \( m_1 + m_2 = m_3 + m_4 \), there are three possible types of optimal transport path, depending on the positions of \( x_i \) as well as the ratios of \( m_i \):

Using the above definitions, the problem (44) is rewritten as the following Plateau-problem as in minimal surfaces:

**Problem 49** Given \( \mu^+ \) and \( \mu^- \) in \( P(X) \), find a minimizer in the family

\[
Path(\mu^+, \mu^-) := \{ T \in T_{1,X}(U) : \partial T = \mu^+ - \mu^- \}
\]

of transport paths of least \( M^\alpha \) cost.

We first see the lower semicontinuity of \( M^\alpha \):
Proposition 50  (lower semicontinuity of $M^\alpha$) Suppose $\{S_i\} \subset T_{1,X}(U)$, if $S_i$ converges to $S \in F_{1,X}(U)$ in $F_1$ metric, then $S \in T_{1,X}(U)$ and

$$M^\alpha(S) \leq \liminf M^\alpha(S_i).$$

Proof. For each $i$, there exist a sequence $G_{ij} \in G_{1,X}(U)$ such that as $j \to \infty$,

$$F_1(G_{ij} - S_i) \to 0$$

and

$$\lim_j M^\alpha(G_{ij}) \leq M^\alpha(S_i) + \frac{1}{2^{i+1}}.$$

Choose $j_i$ large enough so that

$$M^\alpha(G_{ij_i}) \leq M^\alpha(S_i) + \frac{1}{2^i}$$

and

$$F_1(G_{ij_i} - S_i) \leq \frac{1}{2^i}.$$

Thus $S \in T_{1,X}(U)$ because

$$F_1(G_{ij_i} - S) \leq F_1(G_{ij_i} - S_i) + F_1(S_i - S) \leq \frac{1}{2^i} + F_1(S_i - S) \xrightarrow{i \to \infty} 0.$$

Also,

$$M^\alpha(S) \leq \liminf M^\alpha(G_{ij_i}) \leq \liminf \left[M^\alpha(S_i) + \frac{1}{2^i}\right] = \liminf M^\alpha(S_i).$$

Now, we point out a trivial but important fact:

Lemma 51  If $G \in G_{1,X}(U)$ contains no loops, then $0 < w(e) < 1$ for any edge $e$ of $G = \sum w(e) [e]$. Thus,

$$M^\alpha(G) \geq M(G).$$
The following proposition says any element of $G_{1,X}(U)$ can be modified to be another element of $G_{1,X}(U)$ containing no loops and having less $M^\alpha$ cost.

**Proposition 52** For any $G \in G_{1,X}(U)$, there exists a $\tilde{G} \in G_{1,X}(U)$ such that $\tilde{G}$ contains no loops, $\partial \tilde{G} = \partial G$ and $M^\alpha(\tilde{G}) \leq M^\alpha(G)$.

**Proof.** Suppose $G$ contains a loop $L$. For each edge $e$ of $L$, define

$$m(e) = \frac{\alpha \text{length}(e)}{w(e)^{1-\alpha}}.$$ 

Arbitrarily pick an orientation for $L$ and let

$$L_1 = \sum \{[[e]] : \text{edge } e \text{ of } G \text{ has same orientation as } L\},$$

$$L_2 = \sum \{[[e]] : \text{edge } e \text{ of } G \text{ has reverse orientation as } L\}.$$ 

Note $L_1$ or $L_2$ is possibly to be empty and $\partial L_1 = \partial L_2$. By changing orientation on $L$ if necessary, we may assume that

$$\sum_{e \in L_1} m(e) \leq \sum_{e \in L_2} m(e).$$

Now take $G' = G + w(L_1 - L_2)$ with $w = \min \{w(e) : e \in L_2\}$, then $G' \in G_{1,X}(U)$, $\partial G' = \partial G$ and has less loops than $G$ does. Moreover, $M^\alpha(G') \leq M^\alpha(G)$. To see this, we consider the function on $[0,w]$ defined by

$$f(\lambda) := M^\alpha(G + \lambda(L_1 - L_2)) - M^\alpha(G)$$

$$= \sum_{e \in L_1} \text{length}(e) [(w(e) + \lambda)^\alpha - w(e)^\alpha] + \sum_{e \in L_2} \text{length}(e) [(w(e) - \lambda)^\alpha - w(e)^\alpha].$$
Then, since $\alpha \leq 1$, trivial calculations imply that $f''(\lambda) \leq 0$, $f'(\lambda) \leq f'(0) = \sum_{e \in L_1} m(e) - \sum_{e \in L_2} m(e) \leq 0$ and $f(\lambda) \leq f(0) = 0$. Therefore, $M^\alpha(G') \leq M^\alpha(G)$. Repeat the above procedure, we get the desired $\tilde{G}$ with no loops. ■

For any probability measure $\mu \in P(X)$, we'll construct a transport path of finite $M^\alpha$ cost from $\mu$ to a Dirac measure.

**Proposition 53** For any $\mu \in P(X)$, suppose the support of $\mu$ is contained in a cube in $\mathbb{R}^m$ with center $c$ and edge length $d$, then there exists a $T \in \text{Path}(\mu, [[c]])$ and $M^\alpha(T) \leq \frac{1}{21^{m(1-\alpha)-1}} \frac{\sqrt{md}}{2}$.

**Proof.** We may assume $X \subset [0,d]^m \subset \mathbb{R}^m$.

For each $i$, let $Q_i = \{Q_i^h : h \in \mathbb{Z}^m \cap [0,2^i)^m\}$ be a partition of $[0,d]^m$ into cubes of edge length $\frac{d}{2^i}$. For each $i$ and $h$, let $c_i^h$ be the center of $Q_i^h$ and $m_i^h = \mu(Q_i^h)$ be the measure of $\mu$ on the cube $Q_i^h$.

For each $n \geq 0$, by dyadic subdivision, each cube $Q_i^h$ of level $n$ corresponds to $2^m$ cubes $\{Q_{n+1}^{2^m h+h'} : h' = 0, 1, 2, \ldots, 2^m - 1\}$ of level $n+1$. We define $G_{n+1} \in \mathcal{G}_{1,X}(U)$ inductively by setting

$$G_{n+1} = G_n + \sum_{h \in \mathbb{Z}^m \cap [0,2^{m+1})^m} \sum_{h'=1}^{2^m-1} m_{n+1}^{2^m h+h'} \left[\left(\frac{c_{n+1}^{2^m h+h'}}{c^h_n}\right)\right]$$

$$= \sum_{i=0}^{n} \sum_{h \in \mathbb{Z}^m \cap [0,2^i)^m} \sum_{h'=1}^{2^m-1} m_{i+1}^{2^m h+h'} \left[\left(\frac{c_{i+1}^{2^m h+h'}}{c_i^h}\right)\right]$$

where $G_0 = 0$. Since $\sum_{h'=1}^{2^m-1} m_{n+1}^{2^m h+h'} = \sum_{h'=1}^{2^m-1} \mu(Q_{n+1}^{2^m h+h'}) = \mu(Q_n^h)$, we have

$$\partial G_n = [[c]] - \sum_{h \in \mathbb{Z}^m \cap [0,2^m)^m} m_n^h \left[[\frac{c_n^h}{c_n^h}]\right] - [[c]] - \mu.$$

Thus, $G_n \in \mathcal{G}_{1,X}(U)$ for any $n$ and $F(\partial G_n - [[c]] + \mu) \to 0$ as $n \to \infty$. 

```
Since

\[ M^\alpha (G_{n+1}) = \sum_{i=0}^{n} \sum_{h \in \mathbb{Z}^n \cap [0,2^i)^m} \sum_{h'=1}^{2^{m-1}} \left( m_{i+1}^{2^m h + h'} \right)^\alpha \text{length} \left( c_{i+1}^{2^m h + h'}, c_i^h \right) \]

\[ = \sum_{i=0}^{n} \sum_{h \in \mathbb{Z}^n \cap [0,2^i)^m} \sum_{h'=1}^{2^{m-1}} \left( m_{i+1}^{2^m h + h'} \right)^\alpha \frac{\sqrt{md}}{2^{i+2}} \]

\[ \leq \sum_{i=0}^{n} \sum_{h \in \mathbb{Z}^n \cap [0,2^i)^m} \sum_{h'=1}^{2^{m-1}} \left( \frac{1}{2^{m(i+1)}} \right)^\alpha \frac{\sqrt{md}}{2^{i+2}} \]

\[ = \sum_{i=0}^{n} \left( 2^{i+1} \right)^{m(1-\alpha) - 1} \frac{\sqrt{md}}{2} \]

\[ \leq \frac{1}{2^{1-m(1-\alpha) - 1}} \frac{\sqrt{md}}{2}, \text{ if } \alpha > 1 - \frac{1}{m}. \]

Since \( M (G_n) \leq M^\alpha (G_n) \) is uniformly bounded above, by the compactness of 1-dimensional flat currents, it has a limit \( S \) with

\[ M^\alpha (S) \leq \lim \inf M^\alpha (G_i) \leq \frac{1}{2^{1-m(1-\alpha) - 1}} \frac{\sqrt{md}}{2} \]

and \( \partial S = \llbracket [c] \rrbracket - \mu. \)

Now, the Problem 49 is solved in the following existence theorem:

**Theorem 54 (Existence theorem)** Given \( \mu^+ \) and \( \mu^- \) in \( P(X) \), there exists a minimizer \( S \) among the family \( \text{Path}(\mu^+, \mu^-) \) with least \( M^\alpha \) cost. Moreover

\[ M^\alpha (S) \leq \frac{\sqrt{mdiam(X)}}{2^{1-m(1-\alpha) - 1}}. \]

**Proof.** Let \( \{T_i\} \) be a \( M^\alpha \) minimizing sequence in \( \text{Path}(\mu^+, \mu^-) \). For each \( T_i \), there exists a \( G_i \in G_{1,X} (U) \) such that

\[ M^\alpha (G_i) \leq M^\alpha (T_i) + \frac{1}{2^i} \] and \( \mathbf{F} (\partial G_i - \partial T_i) < \frac{1}{2^i}. \)
By proposition 52, we may assume $G_i$ has no loops, thus

$$M(G_i) \leq M^\alpha(G_i) \leq M^\alpha(T_i) + \frac{1}{2^i}$$

is bounded. Therefore, by [10, 4.2.17], there is a subsequence $\{G_{ij}\}$ of $\{G_i\}$ convergent to $S \in \mathbf{F}_{1,X}(U)$. Since $\partial S = \mu^\sim - \mu^+$ and

$$M^\alpha(S) \leq \lim \inf M^\alpha(G_{ij})$$

$$\leq \lim \inf M^\alpha(T_{ij}) + \frac{1}{2^i}$$

$$= \lim M^\alpha(T_i)$$

$$= \inf \left\{ M^\alpha(T) : T \in \text{Path}(\mu^+, \mu^-) \right\},$$

we know $S$ is an $M^\alpha$ minimizer in $\text{Path}(\mu^+, \mu^-)$. Also, by the proposition 53, we know

$$M^\alpha(S) \leq \frac{\sqrt{md}}{2^{1-m(1-\alpha)-1}}$$

with $d$ being the diameter of $X$. □

### 3.3 A new distance $d_\alpha$ on the space of probability measures

**Definition 55**  For any $\mu^+, \mu^- \in P(X)$, define

$$d_\alpha(\mu^+, \mu^-) := \min \left\{ M^\alpha(T) : T \in \text{Path}(\mu^+, \mu^-) \right\}.$$  

$d_\alpha$ is a distance  To show that $d_\alpha$ is in fact a distance on $P(X)$, we need the following lemma:

**Lemma 56**  Suppose $\{a_i\}, \{b_i\}$ are two sequences of atomic probability measures on $X$. If $a_i \rightarrow \mu$ and $b_i \rightarrow \mu$, then $d_\alpha(a_i, b_i) \rightarrow 0$. 
Proof. As before, we may assume support($\mu$) $\subset [0, d]^m \subset \mathbb{R}^m$. Given $\varepsilon > 0$, since $m(1 - \alpha) - 1 < 0$, there exists a natural number $n$ large enough so that

$$n^{m(1-\alpha)-1} \frac{1}{2^{1-m(1-\alpha)}-1} \frac{\sqrt{md}}{2} < \frac{\varepsilon}{3}$$

For any small number $\beta > 0$, we can find a partition $Q_n = \{Q_n^h : h \in \mathbb{Z}^m \cap [0, 2^n]^m\}$ of $[0, d]^m$ consist of cubes of edge length between $[(1 - \beta) \frac{d}{n}, (1 + \beta) \frac{d}{n}]$ such that for all $i$, the finite set $\text{spt}(a_i) \cup \text{spt}(b_i)$ doesn’t intersect the boundary of those cubes. For each $h$, let $c^h$ be the center of $Q_n^h$, $p_i^h = a_i(Q_n^h)$ and $q_i^h = b_i(Q_n^h)$. Since $a_i - b_i \to 0$, we have $p_i^h - q_i^h = (a_i - b_i)(\chi(\text{interior of } Q_n^h)) \to 0$ as $i \to \infty$ for all $h$.

By proposition 53, if $p_i^h \neq 0$, then there exists an $S_i^h \in \text{Path} \left(\frac{1}{n} a_i | Q_n^h, [[c^h]]\right)$ with

$$M^\alpha (S_i^h) \leq \frac{1}{2^{1-m(1-\alpha)}-1} \frac{\sqrt{md}}{2n}$$

Therefore, $S_i = \sum_{h \in \mathbb{Z}^m \cap [0, 2^n]^m} p_i^h S_i^h \in T_{1, \chi}(U)$ because it’s a convex combination of $S_i^h$.

Let $p_i = \sum_{h \in \mathbb{Z}^m \cap [0, 2^n]^m} p_i^h [[c^h]]$, then $S_i \in \text{Path} (a_i, p_i)$, and

$$M^\alpha (S_i) \leq \sum_{h \in \mathbb{Z}^m \cap [0, 2^n]^m} (p_i^h)^\alpha M^\alpha (S_i^h)$$

$$\leq \sum_{h \in \mathbb{Z}^m \cap [0, 2^n]^m} \left( \frac{1}{n^m} \right)^\alpha \frac{1}{2^{1-m(1-\alpha)}-1} \frac{\sqrt{md}}{2n}$$

$$\leq n^m \left( \frac{1}{n^m} \right)^\alpha \frac{1}{2^{1-m(1-\alpha)}-1} \frac{\sqrt{md}}{2n}$$

$$< \frac{\varepsilon}{3}$$

Similarly, for $q_i = \sum_{h \in \mathbb{Z}^m \cap [0, 2^n]^m} q_i^h [[c^h]]$, we may find some $S_i' \in \text{Path} (b_i, q_i)$ with $M^\alpha (S_i') < \frac{\varepsilon}{3}$. 
Finally, let $G_i \in G_{1, X} (U)$ be a cone over $p_i - q_i$, then $G_i \in \text{Path} (p_i, q_i)$ and

$$M^\alpha (G_i) \leq \sum_{h \in (0, 2^n)^n} (|p_i^h - q_i^h|) \alpha d < \varepsilon \over 3$$
when $i$ large enough. Therefore, we have $T_i = S_i + G_i + S'_i \in \text{Path} (a_i, b_i)$ with $M^\alpha (T_i) < \varepsilon$ when $i$ large enough. Thus, $d_\alpha (a_i, b_i) \to 0$. ■

Lemma 57. For any $\mu_1, \mu_2 \in P (X)$, we have $F (\mu_1 - \mu_2) \leq d_\alpha (\mu_1, \mu_2)$.

Proof. Let $\{G_i\} \subset G_{1, X} (U)$ be an $M^\alpha$ minimizing sequence with $\partial G_i \to \mu_1 - \mu_2$ and $M^\alpha (G_i) \to d_\alpha (\mu_1, \mu_2)$. We may assume that each $G_i$ contains no loops. Thus,

$$F (\mu_1 - \mu_2) \leq \lim \inf F (\partial G_i)$$

$$\leq \lim \inf M (G_i)$$

$$\leq \lim \inf M^\alpha (G_i) = d_\alpha (\mu_1, \mu_2).$$

■

Corollary 58. If $d_\alpha (\mu_i, \mu) \to 0$, then $\mu_i \to \mu$.

Corollary 59. If $d_\alpha (\mu_1, \mu_2) = 0$, then $\mu_1 = \mu_2$.

Theorem 60. $d_\alpha$ is a distance on $P (X)$.

Proof. It's sufficient to show that $d_\alpha (\mu_1, \mu_3) \leq d_\alpha (\mu_1, \mu_2) + d_\alpha (\mu_2, \mu_3)$ for any $\mu_1, \mu_2, \mu_3 \in P (X)$.

In fact, given $\varepsilon > 0$, there exists $G_i, P_i \in G_{1, X} (U)$ such that $\partial G_i = b_i - a_i$, $\partial P_i = d_i - c_i$ and

$$\lim M^\alpha (G_i) \leq d_\alpha (\mu_1, \mu_2) + \varepsilon \over 3$$
and

$$\lim M^\alpha (P_i) \leq d_\alpha (\mu_2, \mu_3) + \varepsilon \over 3.$$
where \( \{a_i\}, \{b_i\}, \{c_i\}, \{d_i\} \) are atomic probability measures on \( X \) and \( F(a_i - \mu_1) + F(b_i - \mu_2) + F(c_i - \mu_2) + F(d_i - \mu_3) \to 0 \) as \( i \to \infty \). As in the proof of the lemma 56, we find a \( T_i \in \text{Path}(b_i, c_i) \) with \( \lim M^\alpha(T_i) < \epsilon/3 \). Thus, \( G_i + T_i + P_i \in T_{1,X}(U) \) and \( \delta(G_i + T_i + P_i) = a_i - d_i \to \mu_1 - \mu_3 \). Now,

\[
d_\alpha(\mu_1, \mu_3) \leq \liminf M^\alpha(G_i + T_i + P_i) \\
\leq d_\alpha(\mu_1, \mu_2) + d_\alpha(\mu_2, \mu_3) + \epsilon
\]

Therefore, \( d_\alpha(\mu_1, \mu_3) \leq d_\alpha(\mu_1, \mu_2) + d_\alpha(\mu_2, \mu_3) \).

**Corollary 61** For any \( \mu^+, \mu^- \in P(X) \), let \( \{a_i^+\} \) and \( \{a_i^-\} \) be any two sequences of atomic probability measures such that

\[
a_i^+ \to \mu^+ \text{ and } a_i^- \to \mu^-.
\]

Also for each \( i \), suppose \( G_i \in \text{Path}(a_i^+, a_i^-) \) is an weighted directed graph from \( a_i^+ \) to \( a_i^- \). Suppose the sequence \( \{G_i\} \) is subsequently convergent to some \( T \in \text{Path}(\mu^+, \mu^-) \) under the flat metric of current. If \( G_i \) is optimal for each \( i \), then \( T \) is also optimal.

**Proof.** Let \( S \in \text{Path}(\mu^+, \mu^-) \) be an optimal transport path, thus there exists a sequence of weighted directed graphs \( \{F_i\} \) with \( F_i \in \text{Path}(b_i^+, b_i^-) \) for some atomic measures \( b_i^+, b_i^- \in P(X) \) such that

\[
d_\alpha(\mu^+, \mu^-) = M^\alpha(S) = \lim_{i \to \infty} M^\alpha(F_i)
\]

and

\[
b_i^+ \to \mu^+ \text{ and } b_i^- \to \mu^-.
\]
Now,

\[ M^\alpha (T) \leq \liminf_{i \to \infty} M^\alpha (G_i) \]

\[ = \liminf_{i \to \infty} d_\alpha (a_i^+ , a_i^-) \]

\[ \leq \liminf_{i \to \infty} d_\alpha (b_i^+ , b_i^-) + d_\alpha (a_i^+ , b_i^+) + d_\alpha (a_i^- , b_i^-) \]

\[ = \liminf_{i \to \infty} M^\alpha (F_i) = d_\alpha (\mu^+ , \mu^-) . \]

Therefore, \( M^\alpha (T) = d_\alpha (\mu^+ , \mu^-) \) and \( T \) is also optimal. ■

**Topology on \( P(\mathcal{X}) \) induced by \( d_\alpha \)** In general, the \( d_\alpha \) distance is different from any of the Wasserstein distances because the optimal transport path for \( d_\alpha \) will be "Y shaped" rather than "V shaped" as in Wasserstein distances. However, we'll show that they induced the same topology on \( P(\mathcal{X}) \), namely the weak * topology of \( P(\mathcal{X}) \). We first show that atomic measures are dense in \( (P(\mathcal{X}) , d_\alpha) \):

**Lemma 62** For each \( \mu \in P(\mathcal{X}) \), there exists a sequence of atomic probability measures \( \{\mu_n\} \in P(\mathcal{X}) \), such that \( d_\alpha (\mu , \mu_n) \to 0 \) as \( n \to \infty \). In other words, atomic probability measures are dense in \( (P(\mathcal{X}) , d_\alpha) \).

**Proof.** For any \( n \geq 1 \), let

\[ \mathcal{Q}_n = \{Q^h_n : h \in \mathbb{Z}^m \cap [0, 2^n)^m\} \]

be a partition of \([0, d]^m\) into cubes of edge length \( \frac{d}{n} \), and let \( c^h \) be the center of \( Q^h_n \). Let

\[ a_n = \sum_h \mu (Q^h_n) [[c^h]] , \]
then

\[ d_\alpha (\mu, a_n) \leq \sum_h \mu (Q^h_n) ^\alpha d_\alpha \left( \frac{\mu [Q^h_n]}{\mu (Q^h_n)}, [[c^h]] \right) \]

\[ \leq \sum_h \mu (Q^h_n) ^\alpha \frac{1}{2^{1-m(1-\alpha)} - 1} \frac{\sqrt{md}}{2n} \]

\[ \leq \sum_h \left( \frac{1}{n^m} \right) ^\alpha \frac{1}{2^{1-m(1-\alpha)} - 1} \frac{\sqrt{md}}{2n} \]

\[ = \frac{\sqrt{md}}{2^{1-m(1-\alpha)} - 1} n^{m(1-\alpha)-1} \to 0 \text{ as } n \to \infty. \]

\[ \blacksquare \]

**Theorem 63** \( d_\alpha \) metrizes the weak * topology of \( P(X) \).

**Proof.** By lemma 57, it’s sufficient to show that if \( \mu_i \rightharpoonup \mu \), then \( d_\alpha (\mu_i, \mu) \to 0 \).

In fact, for each \( i \), by the lemma 62, we can find an atomic probability measure \( a_i \) such that \( d_\alpha (a_i, \mu_i) \leq \frac{1}{2^i} \). So, \( a_i \rightharpoonup \mu \) also. On the other hand, by the lemma 62 again, we may find a sequence of atomic probability measures \( b_i \rightharpoonup \mu \) and \( d_\alpha (b_i, \mu) \to 0 \).

By lemma 56, \( d_\alpha (a_i, b_i) \to 0 \) and thus

\[ d_\alpha (\mu_i, \mu) \leq d_\alpha (\mu_i, a_i) + d_\alpha (a_i, b_i) + d_\alpha (b_i, \mu) \to 0. \]

\[ \blacksquare \]

### 3.4 Time parametrization of optimal transport path

Now, we know how to transport a general probability measure \( \mu^+ \) to another general probability measure \( \mu^- \) through an optimal transport path. At this moment, we want to find a time parametrization of an optimal transport path. To achieve
this goal, we first take into account of the cost for time consumption. Therefore, we introduce a time parameter and set a coordinate on the time axis by setting

\[
\text{cost of consumption on per unit time} = \text{cost of per unit distance}.
\]

Let

\[
H = \frac{\text{cost of time consumption on per hour}}{\text{cost of per unit distance}}
\]

i.e. \( H \) unit time=1 hour. Suppose the maximal transporting speed for transporting a unit mass is \( V \) unit distance/hour. Note that, in real applications, the transport speed may be proportional to the weights of goods, thus we use \( \frac{V}{w(e)^a} \) to denote the speed for transporting goods of weight \( w(e) \). Therefore, any meaningful line segment \( e \) with weight \( w(e) \) in the space-time plane has slope at least \( \frac{H}{V}w(e)^a \). Hence we consider the following category:

**Definition 64** Let \( H \) and \( V \) be two positive numbers defined as above. Given any time parameter \( t > 0 \) (in hours), we consider a convex subset of \( P_{1,X \times [0,tH]}(U \times \mathbb{R}) \):

\[
G^H_{1,X \times [0,tH]}(U \times \mathbb{R}) = \left\{ G \in P_{1,X \times [0,tH]}(U \times \mathbb{R}) : \partial G = b \times \{ tH \} - a \times \{ 0 \} \right\}
\]

with atomic \( a,b \in P(X) \) and for each edge \( e \) of \( G \),

\[
\text{slope}(\overrightarrow{e}) \geq w(e)^a \frac{H}{V}.
\]

Also, we let

\[
T^H_{1,X \times [0,tH]}(U \times \mathbb{R})
\]

be the \( F_1 \) closure of \( G^H_{1,X \times [0,tH]}(U \times \mathbb{R}) \) in \( F_{1,X \times [0,tH]}(U \times \mathbb{R}) \), it is also a convex subset.
**Notations:** Let

- \( \pi : X \times [0, tH] \to X \) be the projection to the first coordinate;

- \( p : X \times [0, tH] \to [0, tH] \) be the projection to the second coordinate;

- \( \pi_u^w : X \times [0, u] \to X \times [0, v] \) be the map sending \((x, \lambda)\) to \((x, \lambda v/u)\) for any \( u, v \in (0, +\infty) \).

**Lemma 65** Suppose \( T \in T_{1, X \times [0, tH]}^H (U \times \mathbb{R}) \).

1. For any \( s \geq t \) and any \( h > 0 \), \( (\pi_{1H}^h)_# T \in T_{1, X \times [0, sh]}^h (U \times \mathbb{R}) \).

2. If \( sh \leq tH \), then

\[
M^\alpha (T) \geq M^\alpha ( (\pi_{1H}^h)_# T) \geq M^\alpha (\pi_{#} T).
\]

3. If \( h \geq H \), then

\[
M^\alpha ( (\pi_{1H}^h)_# T) \geq M^\alpha (T) \geq M^\alpha (\pi_{#} T).
\]

4. For any \( G \in G_{1, X \times [0, tH]}^H (U \times \mathbb{R}) \),

\[
M^\alpha (G) \leq M (p_{#} G) \frac{\sqrt{H^2 + V^2}}{H}
\]

**Proof.** Suppose \( s \geq t, h \in (0, +\infty) \) and \( G \in G_{1, X \times [0, tH]}^H (U \times \mathbb{R}) \). For each edge e of G, we have

- the slope of \((\pi_{1H}^h)_# e\)

\[
= \frac{sh}{tH} \times \text{(the slope of } e) \geq \frac{sh}{tH} \times \frac{H}{V} w(e)^\alpha \geq \frac{h}{V} w(e)^\alpha.
\]
Thus, \( (\pi_{tH}^h)_\# G \in \mathcal{G}_{1,X \times [0, tH]}^h (U \times \mathbb{R}) \) and hence \( (\pi_{tH}^h)_\# T \in \mathcal{T}_{1,X \times [0, tH]}^h (U \times \mathbb{R}) \) for any \( T \in \mathcal{T}_{1,X \times [0, tH]}^h (U \times \mathbb{R}) \).

If \( sh \leq tH \), then for each edge \( e \) of \( G \),

\[
l(e) \geq l \left( (\pi_{tH}^h)_\# e \right) \geq l \left( \pi_\# e \right).
\]

Thus,

\[
M^\alpha (G) \geq M^\alpha \left( (\pi_{tH}^h)_\# G \right) \geq M^\alpha (\pi_\# G).
\]

By the lower semicontinuity of \( M^\alpha \), we have for any \( T \in \mathcal{T}_{1,X \times [0, tH]}^h (U \times \mathbb{R}) \)

\[
M^\alpha (T) \geq M^\alpha \left( (\pi_{tH}^h)_\# T \right) \geq M^\alpha (\pi_\# T).
\]

If \( h \geq H \), then since \( \pi_{tH}^h \circ \pi_{tH}^h = id \),

\[
M^\alpha \left( (\pi_{tH}^h)_\# T \right) \geq M^\alpha (T) \geq M^\alpha (\pi_\# T).
\]

Moreover, for any edge \( e \) of \( G \in \mathcal{G}_{1,X \times [0, tH]}^H (U \times \mathbb{R}) \),

\[
\frac{l(p_\# e)}{l(e)} \geq \frac{H w(e)^\alpha}{\sqrt{H^2 w(e)^{2\alpha} + V^2}} \geq \frac{H w(e)^\alpha}{\sqrt{H^2 + V^2}}.
\]

Thus,

\[
M^\alpha (G) = \sum_{e \in E(G)} l(e) w(e)^\alpha \leq \sum_{e \in E(G)} l(p_\# e) \frac{\sqrt{H^2 + V^2}}{H} = M (p_\# G) \frac{\sqrt{H^2 + V^2}}{H}.
\]

The following proposition says that an element in \( \mathcal{T}_{1,X \times [0, tH]}^H (U \times \mathbb{R}) \) is a Lipschitz flow of probability measures from \( \mu^+ \) to \( \mu^- \).
Proposition 66  Any $S \in T_{1, X \times [0, tH]}(U \times \mathbb{R})$ corresponds to a Lipschitz map $S_f : [0, tH] \to (P(X), d_\alpha)$ with Lipschitz constant $\frac{\sqrt{H^2 + V^2}}{H}$. Hence, $S_f : [0, tH] \to (P(X), d_\alpha)$ is weak *-differentiable for $\mathcal{L}^1$ a.e. $s \in [0, tH]$.

Remark 67  The Lipschitz map $S_f : [0, tH] \to (P(X), d_\alpha)$ is also called the time parametrization of $S \in T_{1, X \times [0, tH]}(U \times \mathbb{R})$.

Proof. For any $G \in G_{1, X \times [0, tH]}(U \times \mathbb{R})$ and any $s_1, s_2 \in [0, tH]$,

$$d_\alpha(<G, p, s_1>, <G, p, s_2>)$$

$$\leq M^\alpha(G|p^{-1}(s_1, s_2)) \quad \text{(by lemma 65)}$$

$$\leq M(p_#(G|p^{-1}(s_1, s_2))) \frac{\sqrt{H^2 + V^2}}{H}$$

$$= \frac{\sqrt{H^2 + V^2}}{H} |s_1 - s_2|.$$

Thus, $G$ corresponds to a Lipschitz map $G_f : [0, tH] \to (P(X), d_\alpha)$ with Lipschitz constant $\frac{\sqrt{H^2 + V^2}}{H}$. As a limit of Lipschitz maps with same Lipschitz constant, each element $S \in T_{1, X \times [0, tH]}(U \times \mathbb{R})$ also corresponds to a Lipschitz map $S_f$ from $[0, tH]$ to $(P(X), d_\alpha)$ with the same Lipschitz constant.

By a general result proved in [3], $S_f$ is weakly* differentiable for $\mathcal{L}^1$ a.e. $s$. 

Lemma 68  Suppose $G \in G_{1, X}(U)$ contains no loops, then for any $t > \frac{M^\alpha(G)}{V}$, there exists a lift $\tilde{G} \in G_{1, X \times [0, tH]}(U \times \mathbb{R})$ of $G$ such that

$$M^\alpha(\tilde{G}) \leq M^\alpha(G) \left(1 + \frac{H}{V}\right) + tHM^\alpha(\partial G).$$

Proof. We may suppose that $G$ is connected. Let $c$ be a connected 1-chain in $G$ with maximal $M^\alpha$ cost among all possible connected chains in $G$, where the $M^\alpha$ cost
of each 1-chain $c'$ in $G$ is defined by

$$M^\alpha(c') = \sum_{e \in E(c')} l(e) w(e)^\alpha.$$  

Step 1: Fix the initial vertex of $c$ in $X \times \{0\}$ and then lift $c$ continuously to a
chain in $X \times [0, tH]$ such that each edge $e$ of $c$ lifts to an edge $\tilde{e}$ with slope $\frac{H}{V} w(e)^\alpha$.

Note that since

$$\sum_{e \in E(c)} l(e) = \sum_{e \in E(c)} l(e) \frac{H}{V} w(e)^\alpha$$

$$= M^\alpha(c) \frac{H}{V} \leq M^\alpha(G) \frac{H}{V} \leq tH,$$

the lift of $c$ is well defined in $X \times [0, tH]$.

Step 2: Fix this lift of $c$, one may lift other edges $e$ of $G$ continuously to edges
in $X \times [0, tH]$ with slope $\frac{H}{V} w(e)^\alpha$. Since $c$ is a maximal chain, these lifted edges will
still be contained in $X \times [0, tH]$.

Step 3: Each vertex of degree 1 in the lifted graph corresponds to a single point
in $\partial G$. Connect them by a vertical weighted directed line segment.

Now, we get a lift $\tilde{G} \in G_{1,X \times [0,tH]}^H(U \times \mathbb{R})$ of $G$ with

$$M^\alpha(\tilde{G}) = \sum_{e \in E(G)} l(e) w(e)^\alpha + \sum_{\text{vertical line}} l(\tilde{e}) w(\tilde{e})^\alpha$$

$$\leq \sum_{e \in E(G)} l(e) \left(1 + \frac{H}{V}\right) w(e)^\alpha + tHM^\alpha(\partial G)$$

$$= M^\alpha(G) \left(1 + \frac{H}{V}\right) + tHM^\alpha(\partial G).$$
For any given two probability measure $\mu^+, \mu^- \in P(X)$, we denote

$$\mathcal{F}_i^H := \{ S \in \mathcal{T}_{1, X \times [0, t_i]}^H (U \times \mathbb{R}) : \partial S = \mu^- \times \{ tH \} - \mu^+ \times \{ 0 \} \}.$$

**Theorem 69** Suppose $\text{spt}(\mu^+) \cup \text{spt}(\mu^-) \subseteq$ a cube in $U$ with edge length $d$. Then, for any $t \geq \frac{\sqrt{md}}{\sqrt{2}}$,

1. there exists an element of $S \in \mathcal{F}_i^H$ such that

$$M^\alpha (S) \leq \frac{1}{2^{1-m(1-\alpha)} - 1} \frac{\sqrt{md^2 + t^2H^2}}{2}.$$

2. there exists a $M^\alpha$ minimizer $R_i^H$ in the family $\mathcal{F}_i^H$. Denote

$$f(t, H) = M^\alpha (R_i^H) = \min \{ M^\alpha (S) : S \in \mathcal{F}_i^H \}.$$

3. for any $t > \frac{d_\alpha(\mu^+, \mu^-)}{\sqrt{2}}$, $f(t, H)$ is an increasing function of $H$ and

$$\lim_{H \to 0^+} f(t, H) = d_\alpha(\mu^+, \mu^-). \quad (3.1)$$

4. In the set of the lift of each $R_i^H$ from $X \times [0, t_i]$ to $X \times [0, \hat{t}]$

$$\left\{ (\pi_{t_i}^1)_\# R_i^H \right\} \subseteq \mathcal{F}_{i/t}^{1/t} = \left\{ S \in \mathcal{T}_{1, X \times [0, 1]}^1 (U \times \mathbb{R}) : \partial S = \mu^- \times \{ 1 \} - \mu^+ \times \{ 0 \} \right\},$$

there exists a subsequence $\left\{ (\pi_{t_i}^1)_\# R_i^{H_i} \right\}$ convergent to some $R \in \mathcal{F}_{i/t}^{1/t}$ as $H_i \to 0$. Moreover, $\pi_\# R$ is an optimal transport path among $\text{Path}(\mu^+, \mu^-)$, i.e.

$$M^\alpha (\pi_\# R) = d_\alpha(\mu^+, \mu^-).$$

5. In particular, $\pi_\# R$ has a time parametrization induced by the time parametrization of $R \in \mathcal{T}_{1, X \times [0, 1]}^1 (U \times \mathbb{R})$. 

Proof. For any $t \geq \frac{\sqrt{md}}{V}$, we may construct two graphs as in the proof of the proposition 53

$$G^+_i \in G_{1, X \times [0, tH/2]} (U \times \mathbb{R}) \text{ and } G^-_i \in G_{1, X \times [tH/2, tH]} (U \times \mathbb{R})$$

such that $\partial G^+_i = \left[ [c \times \{tH/2\}] - \mu^+ \times \{0\} \right]$, $\partial G^-_i = \mu^- \times \{tH\} - \left[ [c \times \{tH/2\}] \right]$ and each slope of each edge of $G^+_i$ or $G^-_i$ is at least $\frac{H}{\sqrt{md}} \geq \frac{H}{V}$, where $c$ is the center of the cube. Therefore, we have $G_i = G^+_i + G^-_i \in G^H_{1, X \times [0, tH]} (U \times \mathbb{R})$, $\partial G_i = \mu^- \times \{tH\} - \mu^+ \times \{0\}$ and

$$M^\alpha (G_i) \leq \frac{1}{2^{1-m(1-\alpha)} - 1} \frac{\sqrt{md^2 + t^2H^2}}{2}.$$

Suppose a subsequence of $\{G_i\}$ converges to some $S$ under flat metric of flat currents, then $S \in \mathcal{F}_t^H$ and $M^\alpha (S) \leq \frac{1}{2^{1-m(1-\alpha)} - 1} \frac{\sqrt{md^2 + t^2H^2}}{2}$.

Through similar arguments as in the section 2, we get an optimal $M^\alpha$ minimizer $R^H_i$ in the family $\mathcal{F}_t^H$.

Now, we begin to prove 3.1.

By lemma 65, we have $f(t, H)$ is increasing in $H$ and the limit

$$\lim_{H \to 0^+} f(t, H) \geq d_\alpha (\mu^+, \mu^-).$$

Thus, it's sufficient to prove $\lim_{H \to 0^+} f(t, H) \leq d_\alpha (\mu^+, \mu^-)$. We follow the following steps to prove this fact.

Step 1: Given $t > \frac{d_\alpha (\mu^+, \mu^-)}{V}$ and for any $0 < \epsilon < \frac{tV - d_\alpha (\mu^+, \mu^-)}{3}$, we choose $n$ large enough so that it satisfies the following constraints:
\[ d_\alpha (\mu^+, c^+_n) \leq \epsilon \text{ and } d_\alpha (\mu^-, c^-_n) \leq \epsilon \text{ where} \]
\[ c^+_n := \sum_h \mu^+ (Q^h_n) [[c^h]] \text{ and } c^-_n := \sum_h \mu^- (Q^h_n) [[c^h]] \]
as in the proof of lemma 62.

\[ \frac{\sqrt{md^2 + H^2 n^{2\beta}}}{21-m(1-\alpha) - 1} n^{-2\beta} \leq \epsilon \]

where \( \beta = \frac{1-m(1-\alpha)}{2} \in (0, 1) \).

\[ 2n^{\beta-1} < t - \frac{d_\alpha (\mu^+, \mu^-) + 3\epsilon}{V} \]

\[ n^{\beta} \geq \frac{\sqrt{md}}{V} \]

For any \( a_i \to \mu^+ \), \( b_i \to \mu^- \), define
\[ c^+_i = \sum_h a_i (Q^h_n) [[c^h]] - c^+_n \]
and
\[ c^-_i = \sum_h b_i (Q^h_n) [[c^h]] - c^-_n \]
as \( i \to \infty \). Choose \( G_i \in G_{1,\chi} (U) \) with no loops, \( \partial G_i = c^+_i - c^-_i \) and
\[ M^\alpha (G_i) \leq d_\alpha (c^+_i, c^-_i) + \epsilon \to d_\alpha (c^+_n, c^-_n) + \epsilon \]
\[ \leq d_\alpha (\mu^+, \mu^-) + 3\epsilon < tV. \]

Step 2: Lift \( G_i \) to \( \tilde{G}_i \in G^{H}_{1,\chi \times [Hn^{\beta-1}, (I-n^{\beta-1})H]} (U \times \mathbb{R}) \) such that
\[ M^\alpha (\tilde{G}_i) \leq M^\alpha (G_i) \left( 1 + \frac{H}{V} \right) + tHM^\alpha (\partial G_i) \]
\[ \leq (d_\alpha (\mu^+, \mu^-) + 3\epsilon) \left( 1 + \frac{H}{V} \right) + tH (M^\alpha (c^+_n) + M^\alpha (c^-_n)) \]
Step 3: Connecting $a_i$ with $c_i^a$ by $g_i^a \in \mathcal{G}_{1,X \times [0,Hn^{\alpha-1}]}^H (U \times \mathbb{R})$ such that

$$M^\alpha (g_i^a) \leq \sum_h a_i \left( Q^h \right)^{\alpha} \frac{1}{2^{1-m(1-\alpha)} - 1} \frac{\sqrt{md^2 + H^2n^{2\beta}}}{2n}$$

$$\leq \frac{\sqrt{md^2 + H^2n^{2\beta}}}{2^{1-m(1-\alpha)} - 1} n^{(1-\alpha) - 1} \leq \epsilon$$

and each edge of $g_i^a$ has slope $\geq \frac{Hn^{\alpha}}{\sqrt{md}} \geq \frac{H}{V}$.

Similarly, find $g_i^b \in \mathcal{G}_{1,X \times ((t-n^{\alpha-1})H,tH]}^H (U \times \mathbb{R})$ such that $M^\alpha (g_i^b) \leq \epsilon$.

Step 4: Let $\tilde{G}_i = g_i^a + \tilde{G}_i + g_i^b \in \mathcal{G}_{1,X \times [0,tH]}^H (U \times \mathbb{R})$, then $\partial \tilde{G}_i = b_i \times \{tH\} - a_i \times \{0\}$ and

$$M^\alpha (\tilde{G}_i) \leq 2\varepsilon + M^\alpha (\tilde{G}_i)$$

$$\leq 2\varepsilon + \left( d_\alpha (\mu^+, \mu^-) + 3\varepsilon \right) \left( 1 + \frac{H}{V} \right) + tH \left( M^\alpha (c_n^+) + M^\alpha (c_n^-) \right)$$

Therefore

$$f(t, H) \leq 2\varepsilon + \left( d_\alpha (\mu^+, \mu^-) + 3\varepsilon \right) \left( 1 + \frac{H}{V} \right) + tH \left( M^\alpha (c_n^+) + M^\alpha (c_n^-) \right)$$

$$\rightarrow d_\alpha (\mu^+, \mu^-) + 5\varepsilon \text{ as } H \rightarrow 0^+.$$ 

Hence

$$\lim_{H \rightarrow 0^+} f(t_0, H) = d_\alpha (\mu^+, \mu^-).$$

As for (4), by lemma 65,

$$M \left( (\pi_{tH})_# R_t^H \right) \leq M^\alpha \left( (\pi_{tH})_# R_t^H \right)$$

$$\leq M \left( p_{\#} \left( (\pi_{tH})_# R_t^H \right) \right) \sqrt{H^2 + V^2}$$

$$\leq \frac{\sqrt{H^2 + V^2}}{H}$$
is uniformly bounded. Thus, by the compactness of $F_{1, X}$ with respect to the usual mass $M$, $\{(\pi^1_{tH_i})_# R^H_i \}$ has a convergent subsequence. Suppose $\left( \pi^1_{tH_i} \right)_# R^H_i \rightharpoonup R$ as $H_i \rightarrow 0$, then $\pi_# R \in \text{Path} (\mu^+, \mu^-)$. By 3.1,

$$M^\alpha(\pi_# R) \leq \liminf_{i \to \infty} M^\alpha(\pi_# \left( \left( \pi^1_{tH_i} \right)_# R^H_i \right))$$

$$= \liminf_{i \to \infty} M^\alpha(\pi_# R^H_i)$$

$$\leq \liminf_{i \to \infty} M^\alpha(R^H_i) = d_\alpha(\mu_1, \mu_0)$$

Therefore, $\pi_# R$ is an optimal transport path among $\text{Path} (\mu^+, \mu^-)$ and

$$M^\alpha(\pi_# R) = d_\alpha(\mu^+, \mu^-).$$

3.5 Computer Visualizations

In this section, we'd like to use the ideas of the previous sections to give some computer visualizations about optimal transport paths.

General setup: Given two probability measure $\mu^+$ and $\mu^-$, we fix a maximal speed $V$ and a time $t$. Also, we always keep in our mind the crucial slope constraints.

**Flows from any probability measure to a Dirac measure** Given any probability measure $\mu$ supported in a cube $C$ in $\mathbb{R}^m$ with edge length $d$, we'd like to flow it into a Dirac measure $\delta$ with an optimal transport path.

1. With a given approximating depth $n$, dyadically subdivide $C$ into $2^{nm}$ small cubes with edge length $\frac{d}{2^n}$. Then we have an atomic measure $\alpha_n$ supported at the
centers of those cubes and with weights equal to $\mu$ (each cube). By lemma 62, the
difference between $a_n$ and $\mu$ is $d_\alpha(a_n, \mu^+) = O\left((2^{m(1-\alpha)}1)^n\right)$.

2. For each cube $C_{n-1}$ of level $n-1$ consisting of $2^m n$-level cubes, find an optimal
point in $X \times [0,T]$ such that the transport path flow $a_n[I \times \{T\}, \{C_{n-1} \times \{T\}\}]$ to this
point then to the given point $p$ is optimal in the family of all such paths. Denote
$a_{n-1}$ to be the atomic measure supported at those optimal points with associated
measures.

3. For each $k = n-1, \cdots, 1$, repeatedly doing step 2 to get $a_{k-1}$. In the end we get
a transport path $G$ from $a_n$ to $p$ with finite $M^\alpha$ mass.

4. Repeatedly doing upward optimization and downward optimization (see example
1, for instance) to the transport path $G$.

5. Increase depth $n$ to get better approximation.

Example 70 When taking $\mu$ = Lebesgue measure on $[0,1]$ and $p = \{[1/2]\}$, $\alpha =
0.95, t = H = V = 1$ and take the depth $n = 6$, the above algorithm gives the fol-
lowing graph.
As we increase the approximating depth $n$, the $M^\alpha$ mass of approximating paths may also be increasing. However, by theorem 69, they will be convergent to a finite number i.e. the cost of an optimal transport path. This phenomenon may be illustrated by the following example:

**Example 71** Take $\mu^+$ to be the Lebesgue measure on $[0, 1]$ and $\mu^- = [[1/2]]$ to be the Dirac measure at $1/2$. $t = H = V = 1$. Then the above algorithm gives approximating transporting flows from $\mu^+$ to $\mu^-$ with different alphas:
**General measure to general measure** Now, we want to flow a general measure $\mu^+$ to another general measure $\mu^-$. 

1. Use lemma 62 to pick two atomic measures $a_n$ and $b_n$ to approximate $\mu^+$ and $\mu^-$ respectively within given tolerance.

2. Flow both $a_n$ and $b_n$ to a common Dirac measure at time $\frac{\mu H}{2}$.

3. Simplify the graphs by getting rid of unnecessary vertices (e.g. some vertex may have only one child and one parent).

4. Optimize the locations of each vertex as before.
5. If a vertex has two parents and two children (or has two parent and one child but the child has two children), use example 2 to optimize the positions. This step may change the topology of the graphs.

6. Repeat steps 3-5 until it converges to an optimal path.

**Example 72** Use the above algorithm, we flow the Lebesgue measure (with depth $n = 6$, $t = H = V = 1$) into an atomic measure $\frac{1}{10} \left[ \left[ \frac{1}{3} \right] \right] + \frac{2}{10} \left[ \left[ \frac{2}{3} \right] \right] + \frac{3}{10} \left[ \left[ \frac{5}{6} \right] \right] + \frac{4}{10} \left[ \left[ \frac{7}{6} \right] \right]$ as follows:

![Diagram](image)

**Distance between two probability measures** Now, we use theorem 69(4) to see the distance between two arbitrary probability measures $\mu^+$ and $\mu^-$

1. For each $H$, find an approximate optimal transport path $R^H_t$ from $\mu^+$ to $\mu^-$ within $X \times [0, tH]$;
2. Lifting $R_t^H$ to $(\pi_{th})_{#} R_t^H$ supported in $X \times [0,1]$;

3. Let $H$, the cost of time consumption, approach 0, we get a limit $R$ for some subsequence of $\{(\pi_{th})_{#} R_t^H\}$. This $R$ will be the desired (approximated) optimal flow from $\mu^+$ to $\mu^-$ within $t$ hours and $M^\alpha(\pi_{#}R) = d_\alpha (\mu^+, \mu^-)$.

**Example 73** Let $\mu^+ = \frac{1}{10} [[1/8]] + \frac{2}{10} [[3/8]] + \frac{3}{10} [[5/8]] + \frac{4}{10} [[7/8]]$ and $\mu^- = \frac{2}{10} [[1/8]] + \frac{1}{10} [[3/8]] + \frac{4}{10} [[5/8]] + \frac{3}{10} [[7/8]]$, $t = 1$, the above algorithm gives the following graphs:

Here, the left column shows the transporting flows as $H$ approaches 0 before lifting, while the right column shows the associated flows lifted to time $t = 1$. The last figure tells us “exactly” the locations of the transported path at any given time.
3.6 Transport path versus transport plan

When splitting a vertex on a transport path, information about source and target may become unclear. However, we'll see very soon that information about source and target can be traced by a transport path together with a compatible transport plan.

**Atomic case** In this subsection, we fix two given atomic probability measures

\[ a = \sum_{i=1}^{m} m_i \delta_{x_i} \text{ and } b = \sum_{j=1}^{n} n_j \delta_{y_j} \]

in \( P(X) \).

**Definition 74** Define

\[ \text{Path}(a, b) = \{ G \in \text{G}_{1,X}(U) : \partial G = a - b \text{ and } G \text{ contains no loops} \} \text{ and} \]

\[ \text{Plan}(a, b) = \{ \gamma \in P(X \times X) : \pi_0 \# \gamma = a, \pi_1 \# \gamma = b \} \]

Note that for any \( G \in \text{Path}(a, b) \), each \( i \) and \( j \), there exists at most one connected oriented piecewise linear curve \( g_{ij} \) from \( x_i \) to \( y_j \), supported in \( G \). If such curve doesn't exist, we set \( g_{ij} = 0 \). Thus, we may associate each \( G \in \text{Path}(a, b) \) with an \( m \times n \) 1-dimensional current valued matrix

\[ g(G) = (g_{ij})_{m \times n} \quad (3.2) \]

with

each entry \( g_{ij} \) being either zero or an oriented piecewise linear curve \( g_{ij} \) from \( x_i \) to \( y_j \) and the union of support \( (g_{ij}) \) contains no loops. \( \quad (3.3) \)
Similarly, any transport plan \( \gamma \in \text{Plan} (a, b) \) can be expressed as

\[
\gamma = \sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij} \delta_{(x_i, y_j)} \in P(X \times X) \tag{3.4}
\]

Thus, associated with each \( \gamma \), there is an \( m \times n \) real matrix

\[
u(\gamma) = (u_{ij})
\]

with

\[
u_{ij} \geq 0, \sum_{j=1}^{n} u_{ij} = m_i, \sum_{i=1}^{m} u_{ij} = n_j \text{ and } \sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij} = 1. \tag{3.5}
\]

**Definition 75** Any pair \((G, \gamma) \in \text{Path} (a, b) \times \text{Plan} (a, b)\) is said to be compatible if

\[
G = \sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij} g_{ij}
\]

i.e. \( G = u(\gamma) \cdot g(G) \), where \( u_{ij} \) and \( g_{ij} \) are given in (3.4) and (3.2) respectively.

Note that for any compatible pair \((G, \gamma)\) and any \( i, j \)

\[
g_{ij} = 0 \implies u_{ij} = 0. \tag{3.6}
\]

**Proposition 76** Any pair \((u, g)\) satisfying (3.5), (3.3), and (3.6) provides a compatible pair \((G, \gamma)\) by

\[
G = u \cdot g \text{ and } \gamma = u \cdot \text{bdry} (g)
\]

where the matrix

\[
\text{bdry} (g) = (\delta_{(x_i, y_j)}).
\]

Moreover, if \( g = g(G_0) \) for some \( G_0 \in \text{Path} (a, b) \), then \( G = G_0 \).

**Proof.** Suppose \( g = g(G_0) \), then \( G - G_0 \) is closed and supported in a contractible 1-dimensional set \text{support}(G_0). Thus \( G - G_0 = 0 \). ■
**Corollary 77**  If the matrix $g = (g_{ij})$ has no zero entries, then $g$ is compatible with $u(\gamma)$ for any $\gamma \in \text{Plan}(a,b)$.

**Corollary 78**  There exists $G \in \text{Path}(a,b)$ compatible with all $\gamma \in \text{Plan}(a,b)$.

**Proof.** Pick $c \in X$ outside the join of $a$ and $b$, i.e. $c$ is not on any line from a point of $a$ to a point of $b$. Then let $G$ be a difference of two cones:

$$G = [[c]] \# b - [[c]] \# a.$$

Since each entry of $g(G)$ is nonzero, by the previous corollary, $G$ is compatible with any $\gamma \in \text{Plan}(a,b)$. ■

**Lemma 79**  For any $G \in \text{Path}(a,b)$, there exists a $\gamma \in \text{Plan}(a,b)$ compatible with $G$.

**Proof.** Starting with each beginning vertex $x_i$ of $a$ having weight $m_i$, we move down vertex by vertex. At each vertex $v \in V(G)$, we consider the total amount received from the "ancestors" of $v$ and decompose this amount "fairly" to the children of $v$, i.e. proportional to the weights on the edges connecting $v$ and its children. For each beginning vertex $x_i$ on $a$ and each ending vertex $y_j$ on $b$, the amount of measure $u_{ij}$ on the "descendant" $y_j$ inherited from the "ancestor" $x_i$ is then well defined. Then $\gamma = \sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij} \delta_{(x_i,y_j)} \in \text{Plan}(a,b)$ is compatible with $G$. ■

**Example 80**  Suppose $G \in \text{Path}(a,b)$. There exists a unique plan $\gamma \in \text{Plan}(a,b)$ compatible with $G$ if and only if $G$ and $\gamma$ have the following forms:

- Each connected component of $G$ has either a single beginning point or a single ending point (or both).
• For each $i, j$, either the whole $i^{th}$ row or the whole $j^{th}$ column of the matrix representation $u(\gamma)$ of $\gamma$, except the entry $u_{ij}$, consists of zero.

In fact, each nonzero column or row of $u(\gamma)$ corresponds to precisely one connected component of $G$.

**Definition 81** For any compatible pair $(G, \gamma)$, the pair $(u(\gamma), g(G))$ given by (3.4), (3.2) is called the matrix representation of the pair $(G, \gamma)$.

**Remark 82** Note that each pair $(u, g)$ provides exactly the transporting information about source and targets. Each $u_{ij}$ tells us the amount of transported measure from $x_i$ to $y_j$, while each $g_{ij}$ provides the actual transport path for this transportation. Moreover, the pair $(u, g)$ also tells us how to split the measures at each possible splitting points of the transport path. Thus, each transport path, together with a compatible transport plan, provides the necessary transporting information by its unique matrix representation $(u(\gamma), g(G))$.

**Remark 83** If we restrict each $g_{ij}$ in (3.3) to be the line segment $[x_i, y_j]$ from $x_i$ to $y_j$, then each transport plan $\gamma$ may be identified with the transport path $u(\gamma) \cdot g = \sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij}[x_i, y_j]$. In this sense, a transport path is a more general notion than a transport plan.

**General case** Now, given two general probability measures $\mu^+, \mu^- \in P(X)$. Denote

$$\text{Path} (\mu^+, \mu^-) = \{ T \in T_{i,X}(U) : \partial T = \mu^+ - \mu^- \}$$

as before and

$$\text{Plan} (\mu^+, \mu^-) = \{ \gamma \in P(X \times X) : \pi_x \# \gamma = \mu^+, \pi_y \# \gamma = \mu^- \}$$
Definition 84 The pair \((T, \gamma) \in \text{Path}(\mu^+, \mu^-) \times \text{Plan}(\mu^+, \mu^-)\) is said to be compatible if

1. there exist two sequences of atomic probability measures \(\{a_i\}, \{b_i\} \in P(X)\) such that \(a_i \rightarrow \mu^+\) and \(b_i \rightarrow \mu^-\).

2. there exists a compatible pair \((G_i, \gamma_i) \in \text{Path}(a_i, b_i) \times \text{Plan}(a_i, b_i)\) such that

\[G_i \xrightarrow{\text{flat metric}} T \text{ and } \gamma_i \xrightarrow{w^*} \gamma.\]

We want to consider two optimization problems related to compatible transport paths and transport plans.

Proposition 85 There exists a transport path \(T \in \text{Path}(\mu^+, \mu^-)\) such that \(T\) is compatible with every transport plan \(\gamma \in \text{Plan}(\mu^+, \mu^-)\), and \(M^\alpha(T) < +\infty\).

Proof. As in the proof of proposition 53, for any \(n \geq 1\), we let

\[Q_n = \{Q^h_n : h \in \mathbb{Z}^m \cap [0, 2^n)^m\}\]

be a partition of \(X \subset [0, d]^m\) into cubes of edge length \(2^{-n}d\), and let \(c^h\) be the center of \(Q^h_n\). Also let

\[a_n = \sum_h \mu^+ (Q^h_n) [[c^h]] \rightarrow \mu^+\]

and

\[b_n = \sum_h \mu^- (Q^h_n) [[c^h]] \rightarrow \mu^-\]

Let \(G_n\) be the union of two directed weighted graphs constructed as in the proof of proposition 53. One of them is from the atomic measure \(a_n\) to a Dirac measure \([[c]]\),.
while the other is from \([c]\) to the atomic measure \(b\). Note that \(G_n \in \text{Path} (a_n, b_n)\) and \(g (G_n)\) has no zero entries. Since \(M^\alpha (G_n)\) is uniformly bounded, it has a convergent subsequence converges to some \(T \in \text{Path} (\mu^+, \mu^-)\) and \(M^\alpha (T) < +\infty\).

Now, for any transport plan \(\gamma \in \text{Plan} (\mu^+, \mu^-)\), let

\[
\gamma_n = \sum_{h, h'} \gamma \left( Q^h_n \times Q^{h'}_n \right) \delta_{(\cdot^h, \cdot^{h'})} \rightarrow \gamma
\]

Since \(G_n \in \text{Path} (a_n, b_n)\) and \(g (G_n)\) has no zero entries, by corollary 77, \((G_n, \gamma_n)\) is compatible. This implies \((T, \gamma)\) is compatible for any \(\gamma\). ■

**Proposition 86** Given a transport plan \(\gamma \in \text{Plan} (\mu^+, \mu^-)\), there exists an optimal transport path \(T \in \text{Path} (\mu^+, \mu^-)\) with least finite \(M^\alpha\) cost among all compatible pairs \((T, \gamma)\).

**Proof.** Follows from proposition 85 and an analogous proof of theorem 54. ■

A typical realistic application for this proposition is in the mailing problem. Everyday, the headquarters of the Post Service has a given “transport plan” for sending out letters and packages, given by the addresses of the recipients. The best way to transport those items to their destinations is given by the optimal transport path compatible with the given plan as stated above.

Conversely, by lemma 79 and a simple compactness argument about probability measures, one has the existence result for the following converse problem:

**Proposition 87** Given a transport path \(T \in \text{Path} (\mu^+, \mu^-)\), there exists an optimal transport plan \(\gamma \in \text{Plan} (\mu^+, \mu^-)\) with least \(I (\gamma)\) cost among all compatible pairs \((T, \gamma)\),
where

$$I(\gamma) := \int_{spt(\mu^+) \times spt(\mu^-)} c(x, y) \, d\gamma(x, y)$$

for any given Borel cost density function $c : spt(\mu^+) \times spt(\mu^-) \rightarrow [0, +\infty)$ as in the Monge-Kantorovich problem.

This problem also has many applications. In shipping a given material of fixed amounts from several suppliers to be received in fixed amounts by several customers (along a fixed, possibly $M^\alpha$ optimal transport path), there may be additional preferences of each customer $y$ concerning the relative amounts from each supplier $x$. These preferences can be handled using the "cost" density function $c(x, y)$.
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