INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

ProQuest Information and Learning
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
800-521-0600

UMI®
RICE UNIVERSITY

Minimizers in Polyhedral Space

by

Simon Peter Howell Morgan

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

Doctor of Philosophy

APPROVED, THESIS COMMITTEE:

Robert M. Hardt
Robert Hardt, Moody Professor, Chair Mathematics

Michael Wolf, Professor Mathematics

Steven Cox, Professor Computational and Applied Mathematics

HOUSTON, TEXAS

MAY, 2002
MINIMIZERS IN POLYHEDRAL SPACE

By

SIMON P. H. MORGAN

Area minimizing surfaces and energy minimizing maps from surfaces into piecewise Euclidean pseudo 3-manifolds are studied. These spaces considered contain geometric singularities, having curvature concentrated at vertices and straight-line segments, and topological singularities, non-manifold points.

We examine how these types of singularity affect area minimizing surfaces using calibration forms which also give information about foliations of those spaces by area minimizing surfaces. We extend the notion of foliation to allow for these singularities in the ambient space by defining a codimension one pseudo-foliation. Also the Poincaré-Hopf index theorem tells us that a foliation of any compact connected surface other than a genus one torus will contain singularities. If we allow for this type of singularity (singular pseudo-foliation), we can place a piecewise Euclidean structure on the cone of
any compact connected orientable boundaryless topological surface so that there is a space of such area minimizing piecewise planar pseudo-foliations containing $S^1$ union a point. Conversely cones of non-connected compact surfaces will not admit pseudo-foliations.

Dirichlet energy minimizing maps from one two-dimensional cone to any other two dimensional cone can map the cone point to the cone point. Thus a geometric singularity in the domain is mapped onto a geometric singularity in the range. However in an energy minimizing map from a Moebius band to a disc, we can show that the topological singularity of the map, where the image of a curve is a point, will not coincide with a positive curvature geometric singularity in the range.

Topological collapsing is also demonstrated in limits of images under energy minimizing maps of planar domain surfaces into $\mathbb{R}^3$ with Dirichlet boundary data. The limit of images of energy minimizing maps can be surface components connected by straight line segments. As the limit is approached the metric on the domain degenerates as its conformal class approaches the boundary of its moduli space. The straight-line segments are achieved in the Hausdorff set topology and in the general varifold topology. Also varifold compactness theorems were developed for this proof.
Acknowledgements

I would like to thank Robert Hardt, my thesis advisor, Thierry De Pauw, Michael Wolf, Robin Forman, Penny Smith for suggesting the study of bundles in chapter 2, John Hempel, Frank Jones, Reese Harvey, Frank Morgan, Diane Hoffoss, Richard Evans, Qinglan Xia, Chun-chi Lin and Eva Knoll for helpful discussions.
# Table of Contents

1 Introduction. .................................................. 1

Thesis Outline ................................................. 1

Approaches to studying minimal and area minimizing surfaces .... 3

Extrinsic curvature ........................................... 4

First Variation .................................................. 6

Qualitative geometric arguments with first variation: application to tangent planes of soap films with polyhedral free boundary I. .... 7

Direct comparison vs. calculus of variations: application to tangent planes of soap films with polyhedral free boundary II. .... 10

Calibration ...................................................... 13

Definitions of foliations and background for chapter 2. .............. 16

Expressing minimal surfaces as currents and varifolds ............. 17

Radon Measures ............................................... 19

Rectifiable sets ............................................... 19

Rectifiable varifolds ....................................... 20

Rectifiable currents ........................................ 20

General varifolds ........................................... 21

Current compactness and Allard's varifold compactness theorems ... 22

Energy minimization and background for chapter 3 ................. 24
Energy minimization and calculus of variations

Area, energy and conformal mappings between surfaces and Douglas' result for minimal surfaces as images of harmonic maps

Radial calculations from annulus to $\mathbb{R}^3$

Limits under domain deformation

Calculating energy of these maps

2 Foliations by minimal surfaces of $S^1$ bundles and cones of boundaryless topological surfaces and their connect sums.

Introduction

Defining pseudo-foliations and singular pseudo-foliations

Proposition 2.1 (Piecewise Euclidean structures on compact surfaces)

Theorem 2.2 (Foliating these compact surfaces)

Foliating products of $S^1$ with compact surfaces

Theorem 2.3: We can foliate products of $S^1$ with compact surfaces

Theorem 2.4 (foliations exists)

Theorem 2.5: these foliations are area minimizing

Theorem 2.6 The space of piecewise planar foliations of the above products of surfaces with $S^1$ is $\mathbb{R} P^2$, a full Grassmannian worth

Other bundles

Extending foliations across connect sum operations

Theorem 2.7
Construction of piecewise Euclidean structures for cones of compact connected surfaces that admit pseudo-foliations. 40

Theorem 2.8 (Construction of cones) 40

The short cut lemma 44

Lemma 2.9 (Short cut lemma) 45

Searching for the space of our pseudo-foliations by piecewise planar area minimizing surfaces 46

Theorem 2.10 (orthogonal piecewise planar pseudo-foliations) 47

Theorem 2.11 (the space of piecewise planar pseudo-foliations) 48

Piecewise flat cones of non-connected surfaces have no area minimizing pseudo-foliations. 50

Theorem 2.12 The cone of any non-connected surface with a piecewise Euclidean structure cannot admit a pseudo-foliation such that every leaf is a manifold. 50

3 Can geometric singularities in the range coincide with topological singularities of harmonic maps? 51

Introduction 51

Conjecture 3.1 (geometric range singularities and topological map singularities do not coincide in energy minimizers) 52

Examples of maps with topological singularities 53

An energy comparison with a positive curvature cone point in the range 56

4 Limits of images of energy minimizing sequences of harmonic maps from planar domains into $\mathbb{R}^3$ 61

Motivating examples 61
Theorem 4.1

Proof of 4.1

Lemma 4.2 The identity map from a minimal surface in \( \mathbb{R}^3 \) to itself in \( \mathbb{R}^3 \) is harmonic

Construction of \( F_n \)'s surfaces

Constructing the \( h_n \)'s

Relaxing the \( F_n \)'s to \( G_n \)'s by harmonic maps

Definition of surface parts:

Lemma 4.3 (Surface components stay close to \( S \))

Lemma 4.4 Tube length in the image of \( F_n \) is linearly bounded by energy available.

Lemma 4.5 Energy for pre-image in tubes of \( F_n \) is at least a linear function of length independent of \( n \).

Lemma 4.6 Energy for pre-image in surface part of \( F_n \) is at least a linear function of length independent of \( n \).

Lemma 4.7 (Tube-surface joints stay small and close)

Lemma 4.8 (Y joints stay small)

Lemma 4.9 Tubes stay thin and straight within the convex hull of their ends

Lemma 4.10 Ensuring angles at Y joints remain close to 120 degrees

Lemma 4.11 Tetrahedral joints stay small

Reducing deformations to zero

Limits of images and graphs of maps

Generalizing to polyhedral space
5 Lifted varifold compactness theorems 84
Examples of general varifold compactness 85
Non-constructive version of lifted varifold compactness 88
Theorem 5.1: 88
Theorem 5.2: The 1-dimensional lifted varifold compactness theorem (easy version) 90
Theorem 5.3: The 2-dimensional lifted varifold compactness theorem (easy version) 91
Cones satisfy the hypotheses of theorem 5.3 93
Proof of theorems 5.2 and 5.3 94
Lemma 5.4: 95
Lemma 5.5: Relationship between principal curvature bounds and mean and Gaussian curvature bounds. 98
Finite first variation of lifts 100
Lemma 5.6: The lifts, H, have finite first variation. 100
The harder version of the lifted varifold compactness theorem. 103
Conjecture 5.7 103
Conjecture 5.7a 105
Example sequence of varifolds whose lifts only converge as currents. 106
Steps toward proof of conjecture 5.7 107
Finite mass and filling in. 108
Lemma 5.8: The filled in lift of the V, s has uniformly bounded finite mass. 108
Lemma 5.9: The lift of the V, s without filling in has uniformly bounded finite mass. 108
Lemma 5.10: Extra mass from filling in lifts of dihedral angles edges is uniformly bounded.

Lemma 5.11: Distributional Gaussian curvature in $V_i$'s bounds extra boundary mass in lifts of $V_i$'s due to filling-in polyhedral vertices and cones of smooth curves.

Choosing assignation of orientation to make the lift a current.

Lemma 5.12 Almost all choices of orientation vector $x$ give well defined orientations.

Lemma 5.13: Almost all choices of orientation vector $x$ avoid mass cancellation in the limit as currents converge due to submanifolds coming together with canceling orientations.

Proving uniformly finite boundary mass of currents.

Lemma 5.14: The boundary mass of the currents produced by filling in and assigning orientation to the lifts of the $V_i$'s is uniformly bounded.

Lemma 5.15: The one-dimensional Hausdorff measure of the lift of boundary of the $V_i$'s is bounded.

Lemma 5.16: The length of $Y$ type singularities is uniformly bounded in the $V_i$'s.

$Y$ singularities in 1 dimension

Lemma 5.17: The $Y$ singularities in one dimensional $V_i$'s cannot accumulate at a point.

Conjecture 5.18 $Y$ singularities in 2-varifolds cannot accumulate with infinite one-dimensional Hausdorff measure without infinite mass or infinite first variation.

Lemma 5.19: Extra current boundary at places where assigned orientations do not match up is uniformly bounded.

Lemma 5.20: Almost all choices of $x$ induce uniformly bounded mass in $\mathbb{R}^3$ of projections down of new boundary components to the $V_i$ in $\mathbb{R}^3$. 
Theorem 5.21 (One dimensional version of conjecture 5.7) 118
Theorem 5.22 (Weaker version of conjecture 5.7) 119
Discussion of conjecture 5.7a 121
Can the lifted varifold compactness theorem work for rectifiable varifolds of bounded mass and bounded first variation? 122
Generalizing to polyhedral space. 123
References 124
1 Introduction.

Thesis Outline

The general theme of the thesis is to examine the local qualitative effects of geometric and topological singularities in an ambient space on area minimizers and images of energy minimizing maps from surfaces. The kind of ambient spaces we are considering are polyhedral in that the geometric and topological singularities are those created by identifying a finite number of solid polyhedra as subsets of $\mathbb{R}^3$ isometrically on their faces. This creates singularities on edges and at vertices where the metric is no longer smooth. Neighborhoods of edges can have the metric structure of a product of a surface cone with an interval. The cone is flat everywhere except the cone point where there will be a cone angle not equal to $2\pi$. Also it is possible to identify solid polyhedra to create vertices whose links are not spheres. These vertices are not manifold points as they do not have neighborhoods homeomorphic to 3-balls.

In chapter 2 we observe that area minimizing surfaces can pass through edge singularities if locally they have the product structure of an interval with a cone. If the cone angle is $2n\pi$, then the surface can pass through with a unit normal in any direction. Consideration of the case of geodesics on cone surfaces shows that positive curvature acts as an
obstruction to a geodesic passing through a cone point. By considering a product of a
cone with the unit interval this shows that an area minimizing surface $S$ will not pass
through a positively curved edge singularity with the edge in the tangent plane of $S$.

Chapter 2 defines codimension one pseudo-foliations and singular pseudo-foliations.
These are foliations of pseudo-manifolds. It also examines spaces of pseudo-foliations by
area minimizing piecewise planar surfaces of pseudo manifolds.

In chapter 3 we also see that positive curvature cone points can act as obstructions to
certain energy minimizing maps. A two-dimensional cone of any curvature can be
mapped to another cone of any curvature with minimal energy using the conformal map
$z \rightarrow z^\alpha$, for suitable $\alpha$. However we show an example in chapter 3 where a map has the
topological singularity of collapsing a curve down to a point, which in the image will not
be a positively curved cone point under an energy minimizing map.

Chapter 4 extends the theme of maps changing the topology from the domain to the
image by collpasing. A sequence of images of energy minimizing maps from surfaces
into $\mathbb{R}^3$ is set up as the metric on the domain is deformed. In the limit the conformal class
of the metric on the domain approaches the boundary of its moduli space; that is in
certain directions, in the limit, distance between distinct points tends to zero. Then, in the
limit, the image under an energy minimizing map collapses down to a union of surface
and straight-line segments. To capture this information we use the topology associated with general varifolds. We develop compactness theorems in chapter 5 for this purpose.

In this first chapter we will now review the techniques used in the thesis. In area minimizing surfaces these are: extrinsic curvatures, first variation, calibration and the use of currents and varifolds to model surfaces and we also review varifold and current compactness theorems. In energy minimizing surfaces these are the basic definitions of Dirichlet energy and the relationship between minimal surfaces, conformality and harmonic maps.

**Approaches to studying minimal and area minimizing surfaces**

Minimal surfaces can be defined in two ways which are equivalent at interior points in smooth ambient spaces. They can considered as area stationary with respect to smooth deformations, or they can be consider as zero mean curvature surfaces. The techniques of calculus of variations and differential geometry accompany this definition. Recently complex geometry has given ways of representing minimal surfaces in terms of functions. This has been applied to the problem of finding minimal surfaces in space with polyhedral boundary [Hildebrandt and Sauvigny].
Area minimizers with respect to various boundary conditions have also been studied classically using harmonic maps [Douglas], and in general dimensions the use of geometric measure theory techniques of currents, varifolds, minimal sets and calibration. See [Morgan] for a review. Even in two dimensions these new notions differ from minimal surfaces in at least two respects. Firstly, problems exhibiting boundary and interior singularities such non-manifold points in ‘Y junctions’ between soap films can be considered [Taylor]. A ‘Y junction’ here refers to a soap film structure locally consisting of three half planes in \( \mathbb{R}^3 \) coming together at their boundaries at a common edge with dihedral angles of 120 degrees. Secondly they are used not just to look at general area stationary surfaces [Allard]. They are used to examine global area minimizers among all surfaces that satisfy given boundary conditions or homology conditions [Federer-Flemming].

**Extrinsic curvature**

Consider a smoothly embedded surface in \( \mathbb{R}^3 \). An extrinsic curvature of the surface at a point \( p \), is given by the curvature of the intersection of the surface with a plane containing the \( p \) and the normal to the surface at \( p \). See figure 1.1. The curvature is represented by a vector \( h \), parallel to the normal pointing toward the center of the circle tangent to the intersection with the surface at \( p \) and with the same curvature at \( p \). The magnitude of \( h \) is
curvature of the intersection curve which equals the reciprocal of the radius of the tangent circle.

![Diagram](image)

**Figure 1.1: The extrinsic curvature vector $h$**

The choice of plane through $p$ gives a range of values of curvatures. The two extremal values correspond to perpendicular choices of planes and these are called the principal curvatures. The absolute value of the vector sum of these is called the mean curvature and the dot product of the vectors is called the Gaussian curvature. Minimal surfaces have zero mean curvature. Gaussian curvature is useful because, for among many other
reasons, its integral relates to the global topology of a surface via the Gauss-Bonnet formula.

**First Variation**

We shall describe first variation of surfaces from the point of view of rectifiable 2-varifolds [Simon 1983]. A surface $U$ with boundary $\partial U$ defines a 2-recitifiable varifold $V$ whose underlying set $M_V = U$ and whose underlying measure $\mu_V$ is two dimensional Hausdorff measure $H^2$ restricted to $U$ (see sections later in this chapter on rectifiable varifolds). The first variation of $U$ associated with a smooth compactly supported vector field $g$ is:

$$\delta V(g) = \left. \frac{d}{dt} M(\phi_t(V)) \right|_{t=0} = -\int_U g \cdot \vec{H} d\mu_v - \int_{\partial U} g \cdot \vec{n} ds$$

where $\vec{H}$ is the mean curvature vector on $U$, vector $\vec{n}$ is the inward pointing normal vector on the boundary, and $\phi_t$ is any smooth deformation or isotopy whose initial velocity is $g$. The rectifiable varifold $V$ is called stationary when the first variation vanishes for all admissible vector fields $\phi$. This happens when $U$ has no boundary and has zero mean curvature. That is when $U$ is a minimal surface with no boundary.
Qualitative geometric arguments with first variation: application to tangent planes of soap films with polyhedral free boundary I.

We will discuss the behavior of a soap film having free boundary in the form of a polyhedral surface of a solid body. In this discussion we will concern ourselves with soap films which can be represented by a connected embedded minimal surface $V$ in a polyhedral subset $K$ of $\mathbb{R}^3$. The minimal surface has boundary that lies the polyhedral boundary of $K$, i.e.: $\partial V \subset \partial K$.

Using one basic first variation argument about $V$ near $\partial K$ we will derive some geometric obstructions to $V$ having certain tangent cones at vertices of $\partial K$ where $\partial K \cap \partial V \neq \emptyset$.

We will use the fact that if a soap film intersects a face of a solid object at any angle other than at 90 degrees, then the soap film will move and reduce area until it is meeting the face perpendicularly, or until it meets an edge. So we can say that where $\partial V$ intersects the interior of a face of $\partial K$ then the normal to $\partial V$ will be parallel to $\partial K$.

Figure 1.2. shows a polyhedral free boundary $\partial K$ with only one possible tangent plane for $V$ if $V$ passes through the vertex of $\partial K$ at the origin. Note that in figure 1.2
Figure 1.2 A vertex of $\partial K$ with a unique permissible tangent cone to $V$ at the origin, contained in the plane $X=0$

$V \in K,$ the region $(x,y,z)$ where $z \geq Z,$ and $(x,y,Z) \in \partial K,$ and

$$\partial K = \{(x,y,z): y=z, x>0\} \cup \{(x,y,z): y = -z, x<0\} \cup \{(x,y,z): x=0, z<|y|\}$$

Where $V$ meets a vertex $\partial K$ it will have a tangent cone. This follows from a monotonicity argument for stationary varifolds [Simon]. Also we know this tangent cone will be contained in a plane, by first variation considerations. This tangent cone will have boundary passing through the vertex. On each side of the vertex, the tangent cone boundary will have a line segment which either runs along an edge of the polyhedral free boundary or across a face. The tangent cone will lie in a plane which will therefore
contain one of the following three possibilities: two normals to boundary faces touching the vertex, one such normal and one free boundary polyhedral edge or two polyhedral edges.

This means that if there are n faces touching a vertex with n edges touching the vertex, then combinatorially there are potentially n(n-1) possible planes that could contain the tangent plane if physically possible. If a plane lies entirely outside the interior of K, then it is physically impossible for that plane to contain the tangent cone for V. Another reason for physical impossibility follows studies of soap films near edges of polyhedral free boundary [Hildebrant and Sauvigny], and see the next section. That is we can also say that V will not intersect a concave edge of the free boundary in an interval.

In the example in figure 1.2 there are four edges and four faces touching the vertex at the origin. However the only physically possible tangent plane for the soap film passing through the vertex is the plane x=0. A physically impossible plane, for example, is z=0. It contains two normal vectors to surfaces, but as it intersects \( \partial K \) on the x axis V would be split into two components if its tangent cone were \{z=0\} \( \cap \) K.
Direct comparison vs. calculus of variations: application to tangent planes of soap films with polyhedral free boundary II.

Even if first variation is zero it is possible that second variation is positive or negative. If it is positive then the candidate will be a minimizer compared to nearby deformations. If second variation is negative then the candidate will be a local maximum compared to nearby deformations. The following example shows how alternatively we can use direct comparisons to determine if something is a local minimizer or not.

First we need to see what the free boundary obstacle, the ramp in the valley, looks like.

See figure 1.3.

Figure 1.3, Three views of the ramp in the valley free boundary obstacle.
In the notation of the previous section, a surface $V$ is contained in $K$, the region above the set $\partial K = \{ (x,y,z) : z = \max (x, -x, (1/n) y) \}$, for some $n > 0$.

Now we can compare two candidate surfaces $V$ and $V'$. See figure 1.4. Note both candidates have the same fixed boundary, the transverse horizontal line going across the top of the valley, like a bridge, shown in bold in figure 1.4. Where the candidates $V$ and $V'$ touch the valley walls and the ramp is the free boundary ($\partial K$). The candidate $V$, on the left is stationary in the conventional sense. On the interior it is planar, mean curvature zero, and along its free boundary edges it meets the obstacle perpendicularly. The candidate on the right, $V'$, is in fact not stationary, as it does not meet the obstacle perpendicularly on its edges. The fact that the surface on the left $V$ meets the ramp at one point non-perpendicularly does not contribute to first variation because a point has codimension 2 relative to a surface and a codimension 2 set cannot contribute to first variation.
Figure 1.4, The stationary candidate $V$ on left and new candidate $V'$ on right.

Now we can calculate the areas of the two candidates (figure 1.5).

Area of $V = h^2$
Area of projected end view of $V' = h^2(1 - \sin^2 a)$
Area of $V' = h^2(1 - \sin^2 a) \left( \frac{1}{\cos a} \right) = h^2 \cos a$
So the area of the non-stationary candidate on the right in figure 1.4, is less than that of the stationary candidate on the left. Note that for any ramp angle \( \alpha > 0 \), the area of the new candidate is less. So we have shown that the stationary candidate is not a local minimizer. See figure 1.6 that shows how an arbitrarily small deformation of the stationary plane away from vertical causes reduced area inside a ball of small radius \( \epsilon \).

Ball of radius \( \epsilon \)

**Figure 1.6, small deformation that reduces area of the stationary candidate.**

This example gives an alternative to calculus of variations. However we should note in this example that second variation would be negative for non-negative values of \( t \).

**Calibration**

Calibration is a technique [Morgan] that shows that a given candidate for which a calibration form exists is a minimizer. Three versions of this can be considered; boundariless, fixed boundary and free boundary. In the case of no boundary or fixed boundary the standard definitions below suffice. In the case of free boundary the calibration form must evaluate to zero on the free boundary. That means a minimizer is
free to lie in the free boundary without contributing to the integral with respect to the calibration form. So for example if free boundary consists of the line \( x=0 \) and \( x=1 \), \( dx \) can be a calibration form for a candidate minimizer \( T \). See figure 1.7.

![Figure 1.7](image)

The proof that \( T \) is a minimizer is as follows:

First check that we have the conditions for \( dx \) to be a calibration form for \( T \):

(i) \( dx \) is exact

(ii) \( |dx| = 1 \)

(iii) the orientation of \( T \), which is a current, can be represented as a unit vector in the positive \( x \) direction. This unit vector is parallel to \( dx \).

Notice also that \( dx \) integrates to zero on the free boundary. call this the free boundary condition. Now we have:
Length of $T = \int_T dx = \int_S dx \leq \text{Length of } S$

as $\int_T dx - \int_S dx = 0$ by Stokes Theorem

Where $S$ is another candidate minimizer represented by a current such that $T - S$ union part of the free boundary (with orientations) is the boundary of a region. The free boundary condition and (i) enables us to use Stokes theorem to get the second equation. (ii) and (iii) give us the first equality in the first equation and the inequality is automatic given the orientation on $S$.

Relative homology classes can be used to define classes of candidates for a given free boundary problem (Federer). We take the union of all boundary components, free and fixed. Then homology classes relative to this union can describe classes of candidates from which we seek minimizers.
Definitions of foliations and background for chapter 2.

An n-manifold has a foliation structure of codimension 1 leaves when there exists a neighborhood of each point that is homeomorphic to \( I \times B^{n-1} \) and where this homeomorphism maps each leaf of the foliation to an n-1 ball, \( x \times B^{n-1} \), with \( x \in I \).

Here are some pertinent references to previous work done in the problems covered by chapter 2. These studies came about as an extension of work on minimal surfaces in subsets of \( \mathbb{R}^3 \) with free polyhedral boundary. These polyhedral boundary problems have also been studied using complex analytic methods [Hildebrandt and Sauvigny]. Rather than study problems with boundary, chapter 2 focuses on spaces with piecewise Euclidean structure. These spaces have been studied by topologists such as Kerry Jones (1992) who found Euclidean structures on universal links and their branched covers.

Branched (1974) studied foliations of knot compliments. Sullivan (1978) shows that foliations of compact manifolds have leaves which are minimal surfaces under some metric if and only they satisfy a homological condition. Dippolito (1978) studied codimension one foliations of closed manifolds. These are just a few examples of the broad literature on foliations that has developed. For a classical text for geometric
measure theory see Federer's book. Frank Morgan's book gives an up to date introductory
review of methods and results in geometric measure theory [Morgan].

Expressing minimal surfaces as currents and varifolds

The above section on calibration shows how an oriented finite volume submanifold can
be regarded as a dual to smooth forms of degree equal to the dimension of the
submanifold. By the Riesz representation theorem this means that the submanifolds can
be regarded as measures. Integrating a form over the submanifold is integrating the form
over the measure that represents the submanifold. In [Morgan] many examples of
surfaces and sequences of surfaces are given showing the advantages of regarding
surfaces as measures. One such advantage is that as sets of zero measure can be ignored,
so a topology is provided in which sequences of surfaces become compact which are not
compact as surfaces under Hausdorff set convergence or as maps from discs. An
additional advantage is that minimizers can be taken over a wider range of candidate
surface topologies. A simple example of a self intersecting disc shows this in [Morgan].
See also figure 4.4. As a measure this self-intersecting disc is replaced by an embedded
surface with a handle.

Rectifiable n-currents and rectifiable n-varifolds [Simon] can both be regarded as
representable by an n-dimensional rectifiable set in $\mathbb{R}^{n+k}$ together with a density function
and an orientation, or normal vector, almost everywhere on the set. Rectifiable $n$-currents have an orientation in the sense of an oriented submanifold. Rectifiable $n$-varifolds are locally the same as currents but without the orientation information. Thus varifolds can represent more submanifolds than currents according to this standard definition.

We shall introduce some definitions from geometric measure theory.
Radon Measures

Radon measures on a space $X$ can be regarded as non-negative linear functionals on the space $\mathcal{K}(X)$ of continuous real valued functions of compact support on $X$.

There is a compactness theorem for Radon measures [Simon p22]: Suppose $\{\mu_k\}$ is a sequence of Radon measures on $X$ with $\sup_{x \in X} \mu_k(U) < \infty$ for each open $U$ in $X$ with \overline{U} compact. Then there is a subsequence $\{\mu_{k_j}\}$ which converges to a Radon measure $\mu$ on $X$ in the sense that

$$\lim \mu_k(f) = \mu(f), \forall f \in \mathcal{K}(X)$$

Rectifiable sets

An $n$-dimensional set $M \subset \mathbb{R}^{n-k}$ is countably rectifiable if

$$M \subset M_0 \cup \bigcup_{j=1}^{\infty} F_j(\mathbb{R}^n), H^n(M_0) = 0, \text{ and } F_j: \mathbb{R}^n \to \mathbb{R}^{n-k}. \text{ are Lipschitz functions for } j=1, 2, \ldots \text{ [Simon p 58]}$$

This leads to the equivalent definition:

$$M \subset \bigcup_{j=0}^{\infty} N_j \text{ where } H^n(N_0) = 0 \text{ and each } N_j, j > 1 \text{ is an embedded } C^1 \text{ submanifold of } \mathbb{R}^{n-k}.$$
Rectifiable varifolds

Let $M$ be a countably rectifiable $H^n$-measurable subset of $\mathbb{R}^{n+k}$, and let $\theta$ be a positive locally $H^n$ integrable function on $M$. We can set up an equivalence class, up to a set of measure zero, corresponding to the pair $(M, \theta)$ which we call the rectifiable $n$-varifold $v(M, \theta)$. The function $\theta$ is called the multiplicity function of $v(M, \theta)$. If $\theta$ is integer valued then $v(M, \theta)$ is called an integer multiplicity rectifiable $n$-varifold.

Associated to each rectifiable $n$-varifold $v(M, \theta)$ is a radon measure $\mu$, called the weight measure of $v$. Adopting the convention that $\theta$ is zero on the complement of $M$, we can write for a measurable set $A$ in $\mathbb{R}^{n+k}$

$$\mu_*(A) = \int_{A \cap M} \theta \, dH^n$$

and we define the mass $M(v)$ of $v$ as

$$M(v) = \mu_*(\mathbb{R}^{n+k})$$

Rectifiable currents

An $n$-dimensional current in $U$ is a continuous linear functional on $D^n(U)$, the space of smooth $n$-forms with compact support in $U$. These currents are denoted $D_n(U)$. For $n>0$ the currents can be viewed as generalizations of the $n$-dimensional oriented submanifolds
M having locally finite $H^0$-measure in $U$. Given such an $M$ with an orientation $\xi$, we can write $\xi(x) = \pm \tau_1 \wedge \tau_2 \wedge ... \wedge \tau_n \ \forall x \in M$ where $\tau_1 \wedge \tau_2 \wedge ... \wedge \tau_n$ is an orthogonal basis for $T_xM$. Additionally, just as with rectifiable n-varifolds we can endow $M$ with a density function $\theta$, a positive locally $H^n$ integrable function on $M$. Then we have the corresponding n-dimensional current $\|M\|$ where:

$$\|M\|(\omega) = \int_M (\omega(x) \cdot \xi(x)) \theta(x) \ dH^n, \ \omega \in D^o(U)$$

**General varifolds**

Another kind of varifold is a general varifold. It is a Radon measure $\nu$ in the Grassman bundle $G_n(U)$, $U \subset \mathbb{R}^{n+1}$ where:

$$G_n(U) = \{(x, S) : x \in U, \ S \text{ is an } n \text{-dimensional linear subspace } (0 \in S) \text{ of } \mathbb{R}^{n+1}\}$$

General varifolds can represent objects more general than submanifolds and we will use them in chapter 5.

If $\nu$ is a rectifiable varifold represented by a rectifiable $n$-submanifold $M$ with a density function $\theta$, a positive locally $H^n$ integrable function on $M$, then the corresponding general varifold $\nu_g$ is defined by:
\( \nu_e(A) = \mu_e(\pi(A)), \) where \( \pi(A) = \{ x \in U : (x, T_m(x)) \in A \} \) for any subset \( A \) of \( G_n(U) \).

Here \( \langle T_m(x) \rangle \) denotes the \( n \)-dimensional linear subspace determined by \( T_m(x) \). Radon measure compactness means that a sequence of general varifolds on \( U \) will have a subsequence converging to a Radon measure in the Grassman bundle.

**Current compactness and Allard's varifold compactness theorem**

There are two compactness theorems which we will use and extend in chapter 5. in proving lifted varifold compactness theorems.

Firstly, the statement of the Federer-Flemming current compactness theorem [Simon],[Federer-Flemming] is as follows:

Suppose \( T_j \) and \( \partial T_j \) are integer multiplicity rectifiable \( n \)-currents in \( \mathbb{R}^{n+k} \) and

\[
\sup_{j \in \mathbb{N}} (M_w(T_j) + M_w(\partial T_j)) < \infty
\]

then there will be a subsequence \( T_i \) that converges weakly in \( W \) to a rectifiable current \( T \), which is also an integer multiplicity rectifiable \( n \)-dimensional current.

Here \( \partial T \) is defined as \( \partial T(\omega) = T(d\omega) \). In the case of \( T \) representing an orientable submanifold with a smooth finite volume boundary, then \( \partial T \) will be the current representing the boundary in the sense of boundary in homology.
$M_w(T)$ is the mass of the current $T$ in a compact set $W$ is given by

$$M_w(T) = \sup_{\omega} |T(\omega)|, \omega \in D^a(W), |w(x)| \leq 1 \forall x \in W$$

Weak convergence is in the obvious sense. $T_i \rightarrow T \iff \lim_{j \rightarrow \infty} (T_j(\omega)) = T(\omega), \forall \omega \in D^a.$

Secondly, there is an analogous compactness theorem [Allard] for rectifiable $n$-dimensional varifolds. Instead of having a uniform bound on mass and boundary mass, there is a uniform bound on mass and first variation.

Let $v_i$ be a sequence of integer multiplicity rectifiable $n$-dimensional varifolds in $\mathbb{R}^{n-k}$ such that

$$\sup_{j \geq 1}(\text{Mass}_w(v_j) + \delta_w(v_j)) \leq \infty,$$

where $\delta_w(v_j)$ is first variation as defined earlier in the section on first variation where the support of deformations $g$ are restricted to $W$. $W$ is a compact subset of $\mathbb{R}^{n-k}$.

Then there will be a convergent subsequence $v_j$ that converges to $v$ as Radon measures in the Grassman bundle, $G_n(W).$ As in the current compactness case, mass and first variation are evaluated only in a compact set $W.$

**Energy minimization and background for chapter 3**
Here are some pertinent references to previous work done in the area of chapter 3. The result in chapter 3 concerns singular harmonic maps into positively curved polyhedral spaces. In particular we find by direct calculation of the energy of an alternative map that the image of a certain type of ‘collapsing’ singularity will not lie in certain positively curved cone points. This leads us to conjecture that the result in this example can be generalized. For a review of related work on singular harmonic maps see [Hardt]. In particular see [Lin, 1987] for a study of the map \( x \to \frac{x}{|x|} \) and [Lin, 1989] for harmonic maps into a special three dimensional cone. For a treatise on harmonic maps into polyhedral spaces see [Eells and Fuglede, 2001]. See [Kuwert, 1996] for a study of pre-images under harmonic maps of vertices in non-positively curved polyhedral ranges. Kuwert uses results from the approach used by [Gromov and Schoen, 1992] and [Korevaar and Schoen, 1993] for a study of regularity of harmonic maps into singular non-positively curved spaces.

Energy minimization and calculus of variations

The Dirichlet energy of a \( C^1 \) map \( h \) from an \( m \)-dimensional domain to an \( n \)-dimensional range is given by:

\[
\text{Dirichlet Energy} = \int_{\text{Domain}} \sum_{i,j} |a_{ij}|^2 \, dv \quad \text{where the } a_{ij} \text{s are the differentials of } h.
\]
Maps from a Riemannian domain to a Riemannian range that minimize Dirichlet energy provide important examples of harmonic maps. They can be considered to be higher dimensional generalizations of harmonic functions or of geodesics [Eells and Fuglede, 2001].

**Area, energy and conformal mappings between surfaces and Douglas’ result for minimal surfaces as images of harmonic maps**

We will use the material in this section in chapter 4. There is an inequality concerning area of images of harmonic maps from surfaces to surfaces and energy [Jost, lemma 1.3.1]:

$$\text{Energy of map} \geq 2(\text{Area of image})$$

Equality is only achieved when the map is conformal and the image is a minimal surface.

Douglas showed that a simply connected minimal surface bounded by a simple closed curve can be achieved as the image of a harmonic map from a disc mapping the boundary circle monotonically onto the curve [Douglas].

We will show an example of the use of calculus of variations to find energy minimizing maps within the class of radially symmetric maps from annuli into $\mathbb{R}^3$. This motivates the content in chapter 4 on limits of images of harmonic maps.
Radial calculations from annulus to $\mathbb{R}^3$

Figure 1.8: Annular domain being mapped to $\mathbb{R}^3$ with fixed boundary data.

Figure 1.8 shows the cylindrical domain and range in $\mathbb{R}^3$ with coordinates $f(r,\theta) \rightarrow (R, \Theta, Z)$.

Let us write the Jacobean of the radially symmetric $f$ which can written as $R(r), \Theta(\theta)$ and $Z(r)$.

$$\begin{pmatrix} R' & 0 & Z' \\ 0 & R\Theta' & 0 \end{pmatrix} = \begin{pmatrix} R' & 0 & Z \\ 0 & R & 0 \end{pmatrix} \text{ when } \Theta = \theta.$$

This gives energy as $\int_{0}^{\pi/2} \int_{-1/2}^{1/2} (R')^2 + (R)^2 + (Z')^2 \, dr \, d\theta$. Now we want to find $R$ so that energy is stationary so we write

$$\frac{d}{dt} \left( \int_{0}^{\pi/2} \int_{-1/2}^{1/2} ((R + \zeta')^2 + (R + \zeta)^2 + (Z')^2 \, dr \, d\theta \right) \bigg|_{t=0} = 0$$
On multiplication and differentiation this gives \( \int_0^{2\pi} \int_{-1/2}^{1/2} 2R' \zeta'' + 2R \zeta dr d\theta = 0 \). So we set the integrand to zero to obtain a solution and integrating by parts yields

\[
-2R'' \zeta + 2R \zeta = 0
\]

\[
R'' = R
\]

\[
R = C_1 \cosh(r) + C_2 \sinh(r)
\]

Now boundary conditions \( R(-1/2) = 1 \) and \( R(1/2) = 1 \) give us that \( R = \frac{\cosh(r)}{\cosh(1/2)} \).

Similarly we obtain that \( Z'' = 0 \) and so \( Z = mr + c \). If \( Z(-1/2) = -1/2 \) and \( Z(1/2) = 1/2 \) we have \( z = r \) and consequently in the range we have a formula.

**Limits under domain deformation**

Under a domain variation where the annulus becomes wider or narrower say \( r \) ranges from \(-h\) to \(+h\) we now obtain \( z = r/h \) and so \( R = \frac{\cosh(r)}{\cosh(h)} = \frac{\cosh(hz)}{\cosh(h)} \) describes the image of \( f \). Notice that the points \((z=1, R=1)\) and \((z=-1, R=1)\) are contained in this image of \( f \).

Now we can see how this image varies as \( h \) tends to zero and to infinity.
Figure 1.9 Graphs of $R = \cosh(hz)/\cosh(h)$ which approach but never touch the Z axis as h gets larger.

Figure 1.9 shows that as h tends to infinity the graph of $R(z)$ tends closer and closer, but never reaches a limit of three straight line segments which when rotated about the z axis will become two discs connected by a line segment.

**Calculating energy of these maps**

Now we can use the explicit equations for the radially symmetric energy minimizers we can calculate their energies. For the given boundary data on the domain of two discs of radius 1, and using h to parameterize the variable domain, Dirichlet energy is given by
\[ E = \int_{0}^{\frac{2\pi}{h}} \int_{-h}^{h} \left( (R')^2 + (R)^2 + (Z')^2 \right) dr d\theta \]

\[ E = \int_{0}^{\frac{2\pi}{h}} \int_{-h}^{h} \left( \frac{1}{\cosh^2(h)} \sinh^2(r) + \frac{1}{\cosh^2(h)} \cosh^2(r) + \frac{1}{h^2} \right) dr d\theta \]

\[ = \frac{2\pi}{\cosh^2(h)} \int_{-h}^{h} \cosh(2r) + \frac{1}{h^2} dr \]

\[ = \frac{2\pi}{\cosh^2(h)} \left[ \sinh(2h) + \frac{2}{h^2} \right] \]

Thus as \( h \to 0 \), \( E \to \infty \)

and as \( h \to \infty \), \( E \to 4\pi \)

In harmonic maps the energy is bounded below by twice the area, so as \( E \to 4\pi \). This is approaching twice the area of two discs. Also at the same time the image is approaching the two discs joined by a curve. As \( h \to 0 \) energy tends to infinity. This corresponds to the image approaching a cylinder, \( \{(R, \Theta, Z) : R=1, -1 \leq Z \leq 1, 0 \leq \Theta < 2\pi \} \).
2 Foliations by minimal surfaces of $S^1$ bundles and cones of boundaryless topological surfaces and their connect sums.

Introduction

This chapter defines and examines pseudo-foliations and singular pseudo-foliations, by area minimizing surfaces, which are piecewise planar, of 3 dimensional pseudo-manifolds given a piecewise Euclidean structure. Firstly we examine $S^1$ bundles of compact surfaces as the ambient space for such foliations. This involves leaves passing through singularities in the metric structure on a topological 3 manifold.

Secondly we move on to extend the definition of foliation to enable topological non-manifold points to be foliated. We call these pseudo-foliations as they are foliations of pseudo-manifolds. This is applied to cones of compact surfaces. We also give each topological cone a piecewise Euclidean metric that enables us to demonstrate the existence of pseudo-foliations and to prove that they are area minimizing and piecewise hyperplanes.

We also examine the space of pseudo-foliations by piecewise planar area minimizing surfaces in these piecewise Euclidean ambient spaces. Just as the space of foliations of $\mathbb{R}^3$ by hyperplanes is $\mathbb{R}P^2$, we determine that $S^1$ bundles of compact surfaces with a
piecewise Euclidean structure has an \( \mathbb{RP}^2 \) space of foliations by piecewise planar surfaces. We go on to determine that cones of orientable compact topological surfaces can be given a piecewise Euclidean structure which admits a space of pseudo-foliations or singular pseudo-foliations by area minimizing piecewise planar surfaces which contains \( S^1 \).

Finally we provide a converse theorem, the cone of any non-connected surface will not admit a pseudo-foliation with area minimizing leaves according to our definition of pseudo-foliation without the metric blowing up at the cone point.

**Defining pseudo-foliations and singular pseudo-foliations**

We will be concerned with foliation structures whose leaves are area minimizing within local neighborhoods and which are hyperplanes on Euclidean neighborhoods. The spaces we will foliate have geometric singularities in the form of distributional Gaussian curvature and topological singularities in the form of non-manifolds points, such as the cone of \( \mathbb{RP}^2 \).

We must extend the general definition of foliation to cover interior points in the foliated space which are not manifold points. Before stating the definition we will illustrate how it describes neighborhoods in pseudo-foliations. See figure 2.1.
Figure 2.1 Definition of pseudo-foliation

A point $x$ and its neighborhood $N(x)$ is shown together with some of the leaves of the pseudo-foliation and a tubular neighborhood $T_i(B_i)$, where $B$ is a leaf and $B_i$ is a component of $(B \cap N(x))$. Note also that for $A_i \in (A \cap N(x))$ and $C_i \in (C \cap N(x))$

$N(x) \cap \partial(T_i(B_i)) = A_i \cup C_i$. We will now state the definition of pseudo-foliation.

We define a codimension 1 pseudo-foliation of a space $X$ as a set of submanifolds of $X$ called leaves satisfying the following conditions:

(i) Each interior point in $X$ intersects a leaf, and no two leaves intersect.

(ii) Each interior point in $X$ has a compact neighborhood $N(x)$ where:
a) if \( L_i \) is a connected component of \( L \cap N(\mathbf{x}) \) where \( L \) is a leaf of the foliation, then \( L_i \) has a one parameter family of tubular neighborhoods \( T_t(L_i) \) parameterized by \( t \) so that \( \bigcup_t T_t(L_i) = L_i \). Also each component of \( N(\mathbf{x}) \cap \partial(T_t(L_i)) \) is a component \( L' \cap N(\mathbf{x}) \) where \( L' \) is a leaf.

b) each component \( L_i \) of \( L \cap N(\mathbf{x}) \) is a component of \( N(\mathbf{x}) \cap \partial(T_t(L'_i)) \) for some \( t \) and some nearby component \( L'_i \) of \( L' \cap N(\mathbf{x}) \) where \( L' \) is a leaf.

(iii) If \( \mathbf{x} \in \partial X \) then the pseudo foliation structure at \( \mathbf{x} \) satisfies the standard definition of a foliation in a neighborhood of a boundary point.

Next we define **singular pseudo-foliations** of a space \( X \) whereby regular points of a foliation in \( X \) have an open neighborhood where the above conditions hold, and on singular sets conditions (i) and (ii) are relaxed to allow leaves to intersect. Note that this definition allows for the singularities created when a surface of genus 2 or more is foliated by lines as required by the Poincaré-Hopf index theorem.

**Proposition 2.1** (Piecewise Euclidean structures on compact surfaces):
It is well known that compact surfaces of non-positive Euler characteristic can be given a piecewise Euclidean metric with one singular point of negative curvature equal to an integer multiple of $2\pi$.

This can be verified by taking the regular $2n$-gon and identifying its edges to make the desired compact surface with one vertex class. By Gauss-Bonnet, the solid angle will be $2\pi$ plus an angle deficit of $2\pi$ times the Euler characteristic of the surface. Hence and integer multiple of $2\pi$. We have verified proposition 2.1.

We can contrast this special case of negative curvature where the solid angle is an integer multiple of $2\pi$ with the general case. In the case where we have an integer multiple of $2\pi$ we can partition a neighborhood of the vertex into half planes. These can each be foliated by geodesic leaves parallel to their boundaries. This partitioning into half planes also demonstrates by construction that cone points with a piecewise Euclidean neighborhood that can be foliated by geodesics will have a cone angle of an integer multiple of $\pi$.

**Theorem 2.2 (Foliating these compact surfaces):**

Let $S$ be an orientable surface. The piecewise Euclidean structure in theorem 2.1 enables foliations by straight lines on the surface in any direction. There is one singular point of
the foliation at the single identified vertex where leaves intersect for surfaces of genus greater than 1. There are no singularities for genus one tori.

Proof

Start with a foliation of a regular 2n-gon in the plane by parallel lines. When pairs of opposite sides of the 2n-gon are identified (by translations), the foliation extends over the edges because parallel edges are identified and the foliation can extend around the vertex with a singularity of the required index because the cone angle is an integer multiple of $2\pi$. This can be verified by examining how foliations of neighborhoods of vertices in the 2n-gon match up as the vertices are identified. The neighborhoods become connected in a cycle and at each transition the foliations match up. We have proven theorem 2.2

**Foliating products of $S^1$ with compact surfaces**

**Theorem 2.3** : We can foliate products of $S^1$ with compact surfaces.

To prove this, consider the product of $S^1$ with a topological surface. One foliation is just to take each copy of the surface as a leaf. Taking the metric to be the product metric this foliation will be minimal with no singular points. We also know, above, that we can foliate an orientable surface with straight lines and in the product this will give leaves that are products of $S^1$ and the straight line leaves. These foliations will, in general, be
singular on a set $S^1 \times \{p\}$, where $p$ is the cone point. Leaves of the foliation will intersect on this set.

**Theorem 2.4 (foliations exists):**

When the cone angle in a cone is an integer multiple of $2\pi$ the product of the cone with an interval will admit a foliation by hyperplanes in any direction.

This can be seen by taking $n$ copies of a unit cube in $\mathbb{R}^3$, where $n$ is the integer multiple of $2\pi$ giving the curvature. Put the same foliation on each copy. Then cut each cube in the same half plane passing through the three midpoints of the top face the bottom face and the same side face. Now identify the cubes in a cycle creating the desired product structure. To do this for the example of three cubes, see figure 2.2.

![Figure 2.2 End view of three cubes to be identified](image)

Three cubes are shown end on with the line to the center indicating the removed rectangle intersecting the square face shown. So A, B, C, D, E and F each represent a side of a
removed plane. The identity map would identify A with B, C with D and E with F. It would also, in the process, replace the missing rectangles. To connect the cubes we will add three rectangles, but this time with a different identification pattern. A will be identified with D, as one rectangle is added, C with F as another rectangle is added and E with B for the final rectangle. This process can be extended to n cubes.

We have proven theorem 2.4.

**Theorem 2.5** (these foliations are area minimizing):

Each leaf of the foliation is an area minimizer as described in chapter 1.

Proof: A calibration form orthogonal to the leaves extends across the new product structure. The singular set has area measure zero, so can be ignored. The calibration form proves that all leaves are minimizers. We have proven theorem 2.5 and can observe:

**Theorem 2.6** The space of piecewise planar foliations of the above products of surfaces with $S^1$ is $RP^2$, a full Grassmannian worth.
Proof: Take the polygon cross the unit interval as a subset of \( \mathbb{R}^3 \). Choose any unit vector in \( \mathbb{R}^3 \) and foliate the polygon cross the interval with its normal planes. The space of such piecewise planar foliations is \( \mathbb{R}\mathbb{P}^2 \). The identifications on faces and the edges match up as in theorem 2.4. Thus we have proven theorem 2.5.

Other bundles.

If we take a regular polygon then we can apply a 180 rotation about the center. This will preserve all foliations when it becomes the monodromy map in the bundle. For a circle bundle over a torus, the monodromy map can be a translation. The space of piecewise planar foliations of all these spaces and their connect sums, see below, is also \( \mathbb{R}\mathbb{P}^2 \), a full Grassmannian worth.

Extending foliations across connect sum operations.

Theorem 2.7

Suppose that, for \( n = 2 \) or \( 3 \), two \( n \)-manifolds \( M \) and \( N \) each have a piecewise Euclidean structure. Also assume that \( M \) has a foliation by area minimizing surfaces which on Euclidean neighborhoods, as subsets of \( \mathbb{R}^3 \), have leaves which are parallel hyperplanes. Then a connect sum can be constructed between \( M \) and \( N \) so that the foliation \( M \) extends
uniquely to M#N. This foliation on M#N will also have the property of having leaves which are area minimizing surfaces which on Euclidean neighborhoods, as subsets of R^3, are parallel hyperplanes.

We prove this using the construction of the connect sum. Our technique is to remove topological spheres, which in the metric are either geodesic line segments in two dimensions or discs in three dimensions. A minimal foliation and its calibration form can be extended across this by finding two corresponding directions on each side of the connect sum. The correspondence of directions ensures that if the foliation is deformed on the line or disc, any reduction of length or area respectively on one of the components will be compensated for by an equivalent increase on the other. See figure 2.3. The diagonal leaf switches from the left component above the thick line to the right component below the thick line with the identifications as shown by the arrows. Moving the contact point with the thick line from above to the left will shorten the leaf above the line but will move the leaf to the right and lengthen it below the line.
Figure 2.3 extending a foliation across the connect sum operation in 2 dimensions

Thus we have proven theorem 2.7.

Construction of piecewise Euclidean structures for cones of compact connected surfaces that admit pseudo-foliations.

Theorem 2.8 (Construction of cones):

A cone of any compact connected boundaryless surface can be given a piecewise Euclidean structure that admits a pseudo-foliation by piecewise planar area minimizing surfaces.

We shall now give a general construction for the cone of any compact surface with one connected component. To start with a simple example, take a cube as in the upper left of figure 2.4 equipped with a Euclidean metric structure. Remove triangle LMN, where L is in the interior and M and N are on the bottom boundary of the cube. Then consider opening the slit made by removing the triangle. Now take the closure of the space so that we get more boundary than before, in effect two pushed apart copies of the triangle LMN. This opening or pushing apart is done topologically without changing the intrinsic metric of the cube.
The next stage is to place identifications on the two copies of the triangle so that the new boundary is removed. The identifications are made according to the arrows labeled ‘a’ and ‘b’ at each horizontal level, but not between horizontal levels. Note that as the triangle gets thinner nearer \(L\) the lengths of ‘a’ and ‘b’ get shorter, but the identification pattern remains the same. This creates the cone of a surface with \(L\) being the cone point.

Figure 2.4 shows the cone of a sphere (bigon with boundary identification \(aa^{-1}\)) on the left and the cone of a projective plane (bigon with boundary identification \(aa\)) on the right. To see the projective plane look at the entire boundary of the cube. Also below in figure 2.4 is the cone of a torus (rectangle with boundary identification \(aba^{-1}b^{-1}\)).
Figure 2.4: Cones of the sphere, $\mathbb{RP}^3$ and a torus

One pseudo-foliation is simply where all horizontal planes in the cube are leaves. The proof that this is minimizing involves a calibration argument using the form $dz$, where $x$, $y$ and $z$ are orthogonal directions with $x$ and $y$ horizontal and $z$ vertical. See introduction for a full explanation of calibration and free boundary. Locally at every point away from triangle LMN the calibration form $dz$ proves that the horizontal surfaces are minimal. If we treat the triangle LMN as free boundary, see figure 2.5, then we can use the calibration form $dz$ to prove minimality of the leaves in that free boundary case. The horizontal leaf shown in figure 2.5 has boundary shown in thick lines. The outside is fixed boundary and the thick line inside is where the leaf touches the free boundary formed by triangle LMN. When we say the leaf is a minimizer by calibration we mean
that any surface with the same fixed boundary and free boundary will have areas greater than or equal to our leaf.

![Diagram showing fixed and free boundary](image)

**Figure 2.5, A leaf with fixed and free boundary.**

We now apply the short cut lemma (lemma 2.9) below to prove our theorem by saying that as our leaves are minimizers after free boundary has been added, then they are minimizers in the original cone space.

**The short cut lemma**

We now need a definition of adding free boundary so we may introduce the short cut lemma. Say we have a length, area or volume minimizing problem for a submanifold with fixed boundary in an ambient manifold. We can introduce a set into the ambient
space on which our minimizer can also have boundary. We shall call any boundary components of this set free boundary components. For example see figure 2.6. Two pictures are shown of two boats, A and B. A swimmer has to get from one boat to the other and there is no current in the water. On the left there is only clear water to swim through in a straight line. Thus A and B are the fixed boundaries of the path. Once the island is introduced, the swimmer can swim to one side of the island, walk to the other side and swim the rest of the way. The island perimeter is a free boundary for the swimmer as the swimmer is free to chose where to land and where leave the island from. Altogether, in this case, the swimming sections have four boundary points, A, B, and the two points on the island.

![Figure 2.6](image)

We shall consider a second example which we can regard as a one dimensional prototype for the subsequent lemma.
Say instead of having a circular island, the swimmer can land and walk along a pier between the boats shown in bold (Figure 2.7). This pier represented by the thick line is added free boundary, although it has no interior points as we are regarding it as a line.

Now we can state and prove the lemma.

**Lemma 2.9 (Short cut lemma):**

If a surface $S$ with fixed boundary in a space $X$ becomes a minimizer in a space $Y$ obtained by defining a subset of $X$ to become free boundary, then it was already a minimizer in the original space $X$.

For example see figure 2.3. The space $X$ was the space before the triangle was determined to be free boundary. The space $Y$ is the space as shown in figure 2.3 with the triangle defined as free boundary. The horizontal leaf is a minimizer in the space $Y$ by calibration, and hence also in $X$ by the short cut lemma.

Proof:
If a surface $S$ is an area minimizer among candidates for a space $Y$, and all candidate surfaces for space $X$ are candidates for space $Y$, and the area forms in $Y$ and $X$ are equal up to a set of measure zero, then $S$ is a minimizer in space $X$.

We have now proven lemma 2.9.

**Searching for the space of our pseudo-foliations by piecewise planar area minimizing surfaces**

Using the short cut lemma we can also prove that two other piecewise planar pseudo-foliations, possibly singular pseudo-foliations, are area minimizing. These have leaves which are perpendicular to each other and to the leaves of the pseudo-foliation we have just found.

**Theorem 2.10 (orthogonal piecewise planar pseudo-foliations):**

There are three piecewise planar pseudo-foliations, singular pseudo-foliations, of the constructed space in theorem 2.8, for the cone of every connected compact surface, whose leaves are mutually orthogonal on every Euclidean neighborhood.
We will use the coordinate axes in figure 2.8 to identify our three piecewise planar, possibly singular, pseudo-foliations. This gives us, possibly singular, pseudo-foliations by surfaces with equations of the form $x=k$, by surfaces with equations of the form $y=k$, and by surfaces with equations of the form $z=k$. For the surfaces of the form $x=k$ and $z=k$, the triangle $LMN$ is made a free boundary, and then the short cut lemma is used. For the $y=k$ surfaces an orthogonal set of free boundaries is used. All these surfaces contain lines parallel to the $y$ axis. These intersect $LMN$ at points where arrows such as ‘a’ and ‘b’ (see figure 2.4) begin and end. This effectively cuts the space into purely Euclidean piecewise flat sections in which the short cut lemma is applied.

![Diagram](image)

**Figure 2.8: The positions of the coordinate axes with respect to triangle $LMN$.**

Note that for equations $x=k$ and $y=k$ the foliation structure is a singular pseudo-foliation on the cone of the heads and tails of identification arrows for cones of surfaces of genus 0
or higher. This corresponds to the way in which vertices are identified in polygons to make surfaces. We have proven theorem 2.10

**Theorem 2.11 (the space of piecewise planar pseudo-foliations):**

The space of area minimizing piecewise planar foliations, possibly singular, near a cone of a compact connected surface, constructed as in theorem 2.8 contains a circle union a disjoint point.

We can also show with a different use of calibration that equations of the form \( ay + bz = k. a \neq 0 \neq b \), describe minimizing surfaces for the constructed cones. For this we need to show that deformations of the surface as shown in figure 2.9 do not decrease area. The deformation shown pushes the surface up on the left in front of the triangle LMN and correspondingly pushes it up on the right at the rear. A calibration form to prove the plane is a minimizer can be represented as \(-adz + bdy\). To use calibration we must fill in the holes by adding surface where the black semi-circles are shown. Fortunately when we integrate using the calibration form the contribution of the black semi-circles cancel as they have opposite orientation. This means that the deformed surface will not have less area than the original. Hence the original is a minimizer.
In the Grassman bundle we now have a circle of values for the directions of minimizing foliation planes with equations $ay + bz = k$, union a point for the plane $x = 0$. This proves that the space of area minimizing piecewise planar pseudo-foliations near a cone of a compact connected surface contains a circle union a point. We have proven theorem 2.11.

**Piecewise flat cones of non-connected surfaces have no area minimizing pseudo-foliations.**

By way of a converse for the particular way we defined a pseudo-foliation we have:
Theorem 2.12, the cone of any non-connected surface with a piecewise Euclidean structure cannot admit a pseudo-foliation such that every leaf is a manifold.

This is a simple proof by contradiction. Call the cone X and call X minus the cone point Y. Then Y will have more than one component. Consider the leaf in X going through the cone point. It can only intersect Y in one component, otherwise the link of the cone point, in the leaf, would not be a circle. In the other component or components of Y the neighborhoods of the leaf will have spherical boundary components. In a piecewise flat space these cannot be area minimizing. We have proven theorem 2.12.
3 Can geometric singularities in the range coincide with topological singularities of harmonic maps?

Introduction

We investigate harmonic maps from surfaces with boundary into polyhedral spaces. By the term 'geometric singularity' we mean cone points in the polyhedral range that may be positively or negatively curved. By the term 'topological singularity' we refer to sets in the domain, and their images if they exist, where the regularity of the map changes in the sense of a point in the domain no having a well defined image point, or in the sense of a change of dimension of pre-images of points in the image. Topological singularities can occur when the domain and image have different topologies. For example take a map from a Moebius band to a disc. Also the map from $B^1$ to $S^2$, $x \rightarrow \frac{x}{|x|}$ is singular at the origin and is an example of many maps studied in the literature.

Dirichlet energy minimizing maps from one two-dimensional cone to any other two-dimensional cone can map the cone point to the cone point. That is, geometric singularities in the range can coincide with images of geometric singularities in the domain of energy minimizing harmonic maps.
We shall investigate a ‘collapsing’ topological singularity in a map which is regular (i.e. Jacobian of full rank) everywhere except on a curve whose image is a single point in the range. We find that in one example of an energy minimizing map this type of collapsing topological singularity will not coincide in the range with the geometric singularity of a cone point of sufficient positive curvature. This is because the energy of the map is not a minimum when the singularity has image in the cone point. Moving the image of a topological singularity away from the cone point in the range reduces the overall energy of the map.

This motivates further research into conjectures concerning how the energy associated with topological singularities is affected by coincidence with geometric singularities in the domain and range. Smooth approximations to cone points in the range and domain can be investigated to see if they also affect energy minimizing maps with topological singularities.

**Conjecture 3.1 (geometric range singularities and topological map singularities do not coincide in energy minimizers):**

Let the domain be a smooth surface and the range be a smooth surface except for a positively curved cone point. Consider a map that is smooth and of full rank everywhere on the domain except for a closed smooth curve where the map is of rank one or less. 
Suppose that this curve is mapped to a point. The minimizer of Dirichlet energy among all these maps will not have the singular point of the range in the image of the closed curve that collapses to a point, as long as for the given domain, the cone in the range is sufficiently positively curved at the cone point.

We will verify conjecture 3.1 for one case and further conjecture that homothety arguments might allow this proof to be extended. However this one case is an important prototype as it gives an approach for examining other situations.

**Examples of maps with topological singularities**

Two examples of maps show collapsing topological singularities. The first is a map from a Moebius band to a disc where the boundary of the Moebius band is identified isometricly with boundary of the disc (see figure 3.1). Continuously extending this boundary data will cause some center circle of the Moebius band to collapse to the center of the disc. Thus we have created a topological singularity in the map by giving the range and the domain different topologies.
Figure 3.1: Map from Moebius band to disc

The second example is a map from a cylinder of unit height in $\mathbb{R}^3$ to $\mathbb{R}^3$ where each boundary circle is given boundary data (see figure 3.2). The lower circle is given the identity values and the upper circle is rotated in position 180 degrees about the axis of the cylinder. In the harmonic extension, this creates a twist in the cylinder where the center circle of the cylinder is mapped to a point.
Figure 3.2 The cylinder map with a half twist

The equation for the image of a radially symmetric energy minimizing map is of the form

\[ R(z) = A \sinh(az), \]

where \( R \) is radius in the image in cylindrical coordinates and \( z \) is height in the domain in cylindrical coordinates and \( a \) is a constant. By harmonicity, height in the range is a linear function of height in the domain. This implies that the point where the image pinches off has a local tangent cone of the form \( R = mz \) with finite \( m = Aa \).

Having given an example of an energy minimizer within the class of radially symmetric maps with topological singularities we may proceed.
An energy comparison with a positive curvature cone point in the range

Figure 3.3: Map from Moebius band to cone

We shall start with a map from the Moebius band to a cone of radius $h$ and circumference $l$ (see figure 3.3). The Dirichlet energy is the integral over the domain of the sum of the squares of the entries in the Jacobian. 

$$\text{Energy} = \int_{\text{Domain}} \left( \sum_{i,j} |a_{ij}|^2 \right) dA,$$

where the $a_{ij}$s are entries in the Jacobian.

Let the map be $f(x,y) = (R(y), \Theta(x))$ where $x \in (-1/2, 1/2]$, $y \in [0,1]$.
\[
R(y) = \max \left| y - 1/2 \right|, \quad \Theta(x) = 2\pi x
\]

Jacobian of \( f \):
\[
\begin{pmatrix}
R' & 0 \\
0 & R\theta'
\end{pmatrix} = \begin{pmatrix}
\pm h & 0 \\
0 & 2\pi \left| y - 1/2 \right|
\end{pmatrix}
\]

To get a lower bound on energy we just consider the first entry:

\[
\text{Energy} > \int_{\text{Domain}} h^2 dA = h^2
\]

This estimate will apply to any radially symmetric map. This radial component of energy is the same as if the band were simply being mapped to an interval, and given that the domain is parameterized in rectangular coordinates the minimum energy map is linear.

By comparison we can choose a non-radially symmetric map \( \tilde{f} \) which maps a small disc of radius \( a \) to the interior of the cone and the rest of the Moebius band to the circumference. The map in the region outside of the disc radius \( a \) has linear stretch factors of bounded by \( \frac{1}{2\pi a} \). The area of this region is bounded by 1, so its contribution to the energy of \( \tilde{f} \) is \( \left( \frac{1}{2\pi a} \right)^2 \).

The energy contribution from the disc can be calculated using a conformal map which in complex coordinates could be written as \( w = \tilde{f}(z) \bigg|_{\text{disc radius}} = \left( \frac{h}{a^{\frac{1}{2\pi}}} \right) z^{\frac{1}{2\pi}} \). If the disc is given
real polar coordinates \((r, \theta)\) we can calculate the energy of the map restricted to the disc as follows:

\[
R = \left( \begin{array}{c}
\frac{h}{a} \\
\frac{1}{2am} \end{array} \right) r \left( \frac{1}{2am} \right), \quad \Theta = \frac{\theta}{2\pi r}
\]

\[
R' = \left( \begin{array}{c}
\frac{1}{2am} \\
\frac{1}{2am} \end{array} \right) r \left( \frac{1}{2am} \right), \quad G' = \frac{1}{2\pi r}
\]

The Jacobian is given by

\[
\begin{pmatrix}
R' \\
0
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2am} & 0 \\
0 & \frac{1}{2am} \\
\end{pmatrix}
\]

\[
\text{Energy} = \iint_{\text{disc radius } a} 2 \left( \frac{r \left( \frac{1}{2am} \right)^{\frac{1}{2am}}}{2am} \right)^2 rdrd\theta
\]

\[
= \frac{1}{2\pi^2 a} \int_{\text{disc radius } a} r \left( \frac{1}{2am} \right)^{\frac{1}{2am}} drd\theta
\]

\[
= \frac{h}{a} \left[ \frac{1}{2am} \right]^\pi \bigg|_0 \equiv h
\]
Now to find a value for $h$ so that the radially symmetric case is not the energy minimizer, we want:

$$h^2 > h + \left( \frac{1}{2\pi a^2} \right)$$

For $\alpha = 1/2$, we get $h^2 - h > 26$, so $h > 6$

This would correspond to a cone with solid angle $1/6$ radians, or 9 degrees.

We can construct a homotopy between the first map and a map satisfying the assumptions of the second energy calculation. This homotopy is given by two isotopies on the range.

The first consists of moving the image of the center of the circle radius $a$ to the cone point in the range by an isotopy of the range preserving boundary. Then another isotopy of the range will expand points within the circle of radius $a$ out to their images under the map $\tilde{f}$ given above.

We have verified conjecture 3.1 for this case, and in this example, for a cone angle of 9 degrees or less in a two-dimensional range the kind of topological singularities shown here do not coincide with the geometric singularity of the cone apex. That is the critical set is a curve and if it is mapped to such a cone point, then the map is not energy minimizing. A homothety argument suggests this finding from the Moebius band example can be extended to others. We conjecture that whenever one has locally a collapsing critical set coinciding with the cone point geometric singularity, then the map
will not be an energy minimizer in its homotopy class. We have showed that such cone points can dominate energy locally by a linear factor \( h \) for a topologically regular map to its neighborhood and quadratically \( h^2 \) when it coincides with this kind of topological singularity.

This result motivates further study of interactions between different kinds of topological singularity of energy minimizing maps and geometric singularities in the range or domains. We can look at cases where smooth approximations replace these geometric singularities in the domain or range. This calculation based on energy comparisons of different maps, rather than calculus of variations, gives a prototypical method for examining other situations specifically as well as suggesting general theorems.
4 Limits of images of energy minimizing sequences of harmonic maps from planar domains into $\mathbb{R}^3$

Motivating examples

We shall start by recalling the example in chapter 1 of harmonic maps from an annulus into $\mathbb{R}^3$. Figure 4.1 shows the annular domain on the left. In the center a minimal surface of least area is shown with two parallel and close circles in $\mathbb{R}^3$ as its boundary. The center minimal surface is a catenoid. It can be achieved as the image of a conformal harmonic map from a suitable planar annulus. On the right we have a minimal surface with two parallel circles as its boundary, and it has the form of two discs. Also shown on the right is a straight line segment connecting the two discs. This is achieved as a limit of images of harmonic maps from the annular domains whose conformal structure changes and approaches some boundary point of its moduli space. Recall from chapter 1 that these images are surfaces of revolution of the graph of a cosh function. Recall also that in the case on the right the infimum of energies from the annulus, over all its conformal structures, to $\mathbb{R}^3$ with that boundary is approached by the maps whose images approach the case on the right. It was equal to twice the area of the two discs.
Figure 4.1: Planar domain and minimal surfaces with 2 boundary components

If we stay in the category of planar domains and increase the number of boundary components we can achieve more configurations for minimal surfaces. Figure 4.2 shows a planar domain with 3 boundary components on the left, and two examples of minimal surfaces connected by straight line segments, center and right. The sets of straight-line segments are the shortest connecting the minimal surfaces.

Figure 4.2: Planar domain and minimal surface with 3 boundary components

We ask, in theorem 4.1, if these combinations of minimal surfaces and straight line segments can also be achieved as limits of images of an energy minimizing sequence of harmonic maps as the conformal structure on the planar domain approaches some
boundary point of its moduli space. In general we wish to allow for less symmetric situations such as figure 4.3. Note that the shortest line segment joining the discs does not meet both on the interior, but meets one at a boundary.

Figure 4.3: Straight-line segment joining a boundary to an interior point

Figure 4.4 Topologically inadmissible minimal surfaces; Moebius band, self intersecting disc, and linked boundaries
Figure 4.4 shows some minimal surfaces that cannot be achieved as images from planar domains or limits of images with the same boundary components. In each case the minimal surfaces are either non-orientable or are have non-zero genus.

**Theorem 4.1**

Let $C$ be a finite set of smooth embedded closed curves in $\mathbb{R}^3$. Furthermore suppose $C = \partial S$, where $S$ is a unique orientable compact surface of minimal area and components of planar topological type. If $S$ has one component then it can be realized as an image of a harmonic map from a smooth connected surface (for example the identity map).

If $S$ has more than one component then $S \cup V$, where $V$ is a set of straight line segments, can be realized as a limit of images of harmonic maps from a smooth connected surface $D$ of planar topological type into $\mathbb{R}^3$. The components of $C$ act as free boundary components. The limit is approached as the conformal structure on $D$ is suitably varied and approaches the boundary of its moduli space. The energy of these harmonic maps will approach the infimum of energies of all maps from that domain into $\mathbb{R}^3$ with the prescribed free boundary data. This infimum will be twice the area of the minimal surface.
Furthermore the straight line segments form a stationary 1-varifold which meet S either on the interior of S perpendicularly or at C forming an angle greater the 90 degrees with all tangent vectors to S from C pointing into S. In fact V is a least length set of line segments connecting the components of S without prescribed intersection points.

Remarks: If this least length set is not unique then we know we can achieve one of them, but we can’t always control which one. It may also be possible for theorem 4.1 to be generalized. The proof techniques below may well apply to domains other planar ones. Also our reliance in the proof on having a unique minimal surface with the given boundary data in \( \mathbb{R}^3 \) does not need to limit the theorem. The theorem can be applied to a sequence of boundary data problems with unique minimal surfaces, and which approach the boundary data problem with the non-unique solution.

**Proof of 4.1**

The outline of the proof is as follows. We explicitly construct a sequence of smooth surfaces \( F_n \) with given boundary, which tend to the desired set. We then take energy minimizing harmonic maps \( g_n \) from each \( F_n \) to \( \mathbb{R}^3 \) which map its boundary components into themselves. We argue that each \( G_n \) is close to the corresponding \( F_n \) using energy and area arguments: Energy is bounded below by twice the area, with equality only when
the map is conformal. We also explicitly construct smooth diffeomorphisms \( h_n : D \to F_n \), so that, as the cylinders in the \( F_n \) shrink to straight lines, the neck pinches in the corresponding \( h_n \) pull-backs of the conformal structures of \( F_n \), and they approach the boundary of the moduli space of \( D \). As energy is invariant under conformal maps on the domain we now have our sequence of maps \( f_n \circ h_n \) from \( D \) into \( \mathbb{R}^3 \) with prescribed free boundary data whose image is as desired.

First we realize each component of \( S \) as an image of a harmonic map of a surface by using the identity map:

**Lemma 4.2:** The identity map from a minimal surface in \( \mathbb{R}^4 \) to itself in \( \mathbb{R}^3 \) is harmonic.

**Proof:** We will use the definition of harmonic as locally energy minimizing. We will consider the action of smooth vector fields with compact support on the surface. A simply connected compact region \( A \) will be deformed to \( B \) by the action of a vector field of compact support within \( A \). The boundary of \( B \) will equal the boundary of \( A \), and will be a closed curve. As \( A \) is an area minimizing surface the area of \( B \) is greater than or equal to that of \( A \). We can parameterize a family of \( B \)s by the time over which a vector field \( \phi \) acts on \( B \): \( B_t = \{ y : y = x + t\phi(x), x \in A \} \). So area of \( B_t \geq \text{Area of } A \) and \( A = B_0 \).
Now we can examine the energy of the map into \( B_t \) as compared with the energy of the map into \( A \). Here we can use the fact that twice the area provides a lower bound on the energy. As the identity map is conformal, the energy of the map into \( A \) is twice the area of \( A \). The energy of the map into \( B_t \) is therefore greater than or equal to twice the area of \( B_t \) for all \( t \). This lower bound is a minimum for \( t=0 \), so the energy of maps into \( B_t \) is greater than or equal to the energy of the identity map into \( B_0 (=A) \). Therefore the energy of the identity map is locally energy minimizing; i.e. harmonic. We have proven lemma 4.2.

**Construction of \( F_n \) surfaces**

Where \( S \) has more than one component we set up a sequence of surfaces \( F_n \) that are close to \((S - \bigcup d_i) \cup (\partial T_n(V) - \bigcup H_i)\) where the \( d_i \) are discs removed from \( S \), \( T_n \) is a tubular neighborhood of \( V \), and the \( H_i \) are hemispheres on the boundary of the tubular neighborhoods. Each surface \( F_n \) is relaxed to be the image \( G_n \) of a harmonic map from \( F_n \) given the prescribed boundary data \( C \). We show this relaxation does not change the surfaces much, so \( F_n \) is close to \( G_n \). In the limit as \( F_n \) approaches \( V \cup S \), the images \( G_n \) of harmonic maps also approach \( V \cup S \). An area and energy argument shows that the
energy of the maps whose images are $G_n$ approaches the infimum of energies from any domain into $\mathbb{R}^3$ with the prescribed boundary.

We have $S$ and $V$ already determined so now we construct the $F_n$'s accordingly. See figure 4.5 for a cross section of an $F_n$ for the problem shown in figure 4.2 (right). If there are $m$ half rays of $V$ coming out of an interior point $x$ of $S$. Choose, for $n$ large, $m$ discs radius $1/n$ which do not intersect close to $x$. The centers of the discs, $d_i$, must be within a distance $r_n$ of $x$. Choose the positions so that the half rays can come from each disc in directions corresponding to the directions of the half rays of $V$ without any intersections of half rays. This can be done for $n$ large enough because $S$ is locally smooth and because no two half rays of $V$ from $x$ go in the same direction. There will only be a finite number of half rays of $V$. Similarly for points on the $\partial S$ use the same method.

![Figure 4.5 Cross section of an $F_n$](image-url)
As above, now that we have the discs, $d_i$, in $S$ we will need the boundaries of the tubular neighborhoods of $V$ to match up with their boundaries. As $S$ is smooth so we can make the construction a very close approximation when $S$ is curved, and $n$ is large.

Finally smooth off the joints of $F_n$.

**Constructing the $h_n$'s**

We may at this stage construct, for later use, reference diffeomorphisms $h_n$ from a fixed planar domain $D$ onto $F_n$. Fix $h_\infty$ for $N$ large, and construct for $n \geq N$ smooth diffeomorphisms $\Phi_n : F_\infty \to F_n$ so that $\Phi_n$ is the identity on $S \setminus \{\text{radius $1/n$ discs}\}$. Then let $h_n = \Phi_n \circ h_\infty$ for $n \geq N$. The contraction of the $1/n$ tubular neighborhoods of $V$ in $F_n$ then leads to neck-pinching the $h_n$ pullbacks of the conformal structures of the $F_n$.

**Relaxing the $F_n$'s to $G_n$'s by harmonic maps ($g_n$'s)**

Take each $F_n$ with boundary $C$ as the domain of a harmonic maps into $\mathbb{R}^3$ with boundary data on $C$ preserving each component of $C$ with its orientation, as in Douglas' Plateau boundary condition [Douglas]. We compare the energy of the identity map, from $F_n$ to $F_n$, with the energy of a map $g_n$ from $F_n$ to $G_n$. 
\( F_n \) has two main parts, the minimal surfaces and the tubes. We see that for the energy of a \( G_n \) to be less than the energy of the identity on \( F_n \), the \( G_n \) must be close to the \( F_n \). The proof proceeds by examining energy associated to the surfaces, the joints and the tubes of the \( G_n \) separately. By energy associated with part of an image of a map we mean the energy of the map restricted to the pre-image, that is the integral of Dirichlet energy over the pre-image with respect to 2-dimensional Hausdorff measure.

Figure 4.6 shows the \( F_n \) in cross section in the thin black line of the example \( F_n \) from figure 4.5. The thick gray outlines in figure 4.6 show regions that are considered separately in the proof. The labels indicate what is proven about each region.

![Diagram](attachment:image.png)

**Figure 4.6: Four regions in proof of Theorem 4.1**
To proceed with the lemmas for each part of the proof shown in figure 4.6 we need to be able to compare the surfaces, joints and tubes in $F_n$ and $G_n$. We need to define the surface part of $F_n$ and $G_n$.

**Definition of surface parts:**

The surface parts of $X$ are those points \{$y \in X : B(y, 1/n) \cap X$ is a disc\} where $X$ is $F_n$ or $G_n$.

The tube parts of $F_n$ and $G_n$ are the simply those points in $F_n$ and $G_n$ which are not surface parts.

We will start by showing that movement in the surface from the image of $F_n$ to $G_n$ is bounded by energy saved by contraction of tubes from the image of $F_n$ under the identity to $G_n$.

**Lemma 4.3 (Surface components stay close to $S$):**

The surface part of each $G_n$ lies within a small tubular neighborhood of radius $N(n)$ of $S$.

The function $N(n)$ depends upon $S$.

If this is not true, then there is a point ‘a’ in the surface part of the $F_n$’s for all $n$ large, so that $\partial = \liminf_{n \to \infty} \text{dist}(g_n(a), S) > 0$. This is because the surface parts of the $F_n$’s form an
increasing family of sets whose limit is $S$. The Courant–Lebesgue lemma shows that, for large $n$, the pre-image of a point $x$ in the surface part of a $G_n$ outside of a small neighborhood $M(n)$ of the tube parts of $G_n$ and a distance $d$ from the boundary of $S$, will not come from within a distance $d'$ of the boundary of $S$. Here $d'$ is a function of $d$ not $n$ for any given $S$. Also for large $n$, a small collar neighborhood of a component of the tubes part of $F_n$ will stay small in the image under $g_n$, because the map $x \rightarrow \frac{x}{|x|}$ has infinite energy in $\mathbb{R}^2$. So we can conclude that for large $n$, the pre-image of the surface part of each $G_n$ outside of a small neighborhood $M(n)$ of the tube parts of $G_n$, is in the interior of the corresponding component of the surface part of the $F_n$.

Then the uniform bounds on the absolute values of the energies of the $g_n$ near ‘a’ would give a full neighborhood $B_r(a)$ in every $F_n'$, for some subsequence $n' \rightarrow \infty$, so that $\text{dist}(g_n'(B_r(a)), S) > 0$ and the $g_n$, with all its derivatives, converging uniformly on $B_r(a)$.

However as in the Douglas proof, we also have convergence to be an area minimizing surface with boundary $C$, which must, by uniqueness be $S$. We have proven lemma 4.3.

**Lemma 4.4:** Tube length in the image of $F_n$ is linearly bounded by energy available.

**Proof:**
We want to calculate the energy associated with the tubes. We must look at two cases, the energy if the pre-images are tubes in the $F_n$, and the energy if the pre-image is in the surface part of $F_n$.

**Lemma 4.5:** Energy for pre-image in tubes of $F_n$ is at least a linear function of length independent of $n$.

For a given $n$, the tube circumference in $F_n$ is $\frac{2\pi}{n}$. So we can calculate a minimum energy associated with a tube of length $L$. Let a small region of length $h$ be mapped into the tube.

$$\text{Energy} \geq \frac{2\pi}{n} h \left( \frac{L}{h} \right)^2 = \frac{2\pi}{n} \frac{L^2}{h}$$

Also as the $F_n$ is compact, $h$ cannot get larger than a maximum value. Notice that making $h$ small increases the energy. Also setting $h = L$ gives us the axial component of the energy in the tubes under the identity map $F_n = G_n$. The total energy corresponding to the tubes under this map is at least twice the area. Also note that,

$$\text{Total energy in tube of length } L \text{ under identity map} = \frac{4\pi L}{n}$$
We have proven lemma 4.5. Now we need to check that allowing domain in the minimal surfaces away from the tubes in $F_n$ to be mapped into tubes in $G_n$ does not decrease energy.

Lemma 4.6: Energy for pre-image in surface part of $F_n$ is at least a linear function of length independent of $n$.

Consider a radially symmetric map from an annulus of inner radius $1/n$ and outer radius $R$ to a unit interval. This gives an estimate on a map from part of the surface to part of the tubes of length $L$.

$$\text{Energy} = 2\pi \int_{1/n}^R (Z' + t\zeta)^2 r\,dr$$

$$\frac{d}{dt} \text{Energy} \big|_{t=0} \Leftrightarrow \int_{1/n}^R 2rZ' \zeta \,dr = 0. \forall \zeta \in C^\infty \text{with support } \in [1/n, R]$$

$$\int_{1/n}^R 2rz' \zeta \,dr = 0 \Leftrightarrow \int_{1/n}^R 2(rz'' + z') \zeta \,dr = 0$$

$$\Rightarrow z' = \frac{A}{r}. \quad z = A \ln(r) + C$$

Now $z(1/n) = 0$, and $z(R) = L$, so $z = L \frac{\ln(r)}{\ln(Rn)}$ and $z' = \frac{L}{r \ln(Rn)}$

So energy $= 2\pi \int_{1/n}^R \left( \frac{L}{r \ln(Rn)} \right)^2 r\,dr = \frac{2\pi L^2}{\ln(Rn)} \int_{1/n}^R [\ln(r)]^2 r\,dr = \frac{2\pi L^2 (\ln(R) + \ln(n))}{\ln(Rn)^2}$
We must examine what happens to \( \frac{\ln(n)}{\ln(R^n)^2} \) as \( n \to \infty \). By L'Hôpital's rule, 

\[
\lim_{n \to \infty} \left( \frac{\ln(n)}{\ln(R^n)^2} \right) = \left( \frac{\frac{1}{n}}{2 \ln(R^n)} \right) = \frac{1}{2 \ln(R^n)}
\]

As \( n \) gets larger the identity map has less energy than the map where parts of the minimal surface are mapped to the tubes, because linear growth is greater than logarithmic growth. Note also that \( R \) is bounded as the minimal surfaces are compact, so \( n \) determines the limiting behavior. Also as minimal surfaces are non-positively curved the general energy we have for mapping from the surface to the tubes is bounded below by our calculation for the zero curvature annulus.

We can conclude that for large \( n \), the tubes require energy at least proportional to their total length for each \( n \). We have proven lemma 4.6, and completed the proof of lemma 4.4.

**Lemma 4.7 (Tube-surface joints stay small and close):** For each tube-surface joint in \( F_s \), there exists a ball of radius \( R(n) \), center \( x \) such that \( B(x,R(n)) \cap G_n \) is an annular region, and \( x \in S \) in the case of \( S \) containing parallel discs. \( x \) may move as \( n \) grows. In most other cases it will not.
Proof:

The expansion of the joint requires extra area in the surface part of \( G_n \) and hence more energy (twice the area increase) which is only available from shortening or shrinking the tubes in \( F_n \). Energy available from radially shrinking tubes goes to zero as \( n \) goes to infinity, see calculation below. Also movement of the joint normal to the surface requires deformation of the surface which is also bounded by energy available from shrinking tubes. Movement of the joint in the minimal surface would lengthen the tube as the tubes already touch the minimal surfaces in the shortest configuration possible. This lengthening will increase the axial energy component of the map linearly with length but in inverse proportion with \( n \). We must check to see if this energy is available.

The total energy available from circumferentially shrinking the tubes is twice the area of the tubes, \( \frac{2nL}{n} \). This is because Jacobian entries for the identity map are either 1 or 0. Potentially this is enough energy to allow an increase in tube length independent of \( n \).

However if instead of comparing candidate \( G_n \)'s with the image of the identity map on \( F_n \), we compare it with the image \( J_n \) of a map \( j_n \). Each \( J_n \) is the identity on \((F_n\text{-cylindrical sections of tubes})\), but with the cylindrical parts and junctions shrunk down to
a radius of $\frac{1}{n^2}$, as compared with $\frac{1}{n}$ for $F_n$, there is no longer sufficient energy to allow
for a tube length increase that does not tend to zero as $n$ tends to infinity.

We have proven lemma 4.7

**Lemma 4.8 (Y joints stay small):** For each tube Y joint in $F_n$ there exists a ball of radius
$R(n)$, center $x$ for each $n$, such that $B(x,R(n)) \cap G_n$ is a planar region with three
boundary components.

**Proof:**

Y joints stay within a ball of radius $R'(n)$. That is $R'(n) \cap G_n$ has a planar region with
three boundary components. (The ball does not have any pre-specified position).

The size of joint part of the proof is the same proof as for the first part of lemma 4.7, thus
we have proven lemma 4.8
Lemma 4.9: Tubes stay thin and straight within the convex hull of their ends.

Proof: We can use the convex hull property of the image of the harmonic functions \( \gamma_n \) to ensure that tubes do not bend so that one side is convex, nor can their cross sectional diameter increase in the middle.

Lemma 4.10 Ensuring angles at Y joints remain close to 120 degrees:

The image of a 1-varifold Y junction with 3 segments (modeling \( F_n \)), with fixed end points, under an energy minimizing map will remain at 120 degrees (modeling \( G_n \)).

We will only model the situation as we approach the limiting case where each tube is concentrated within a small neighborhood of a line away from a small neighborhood of the discs. This can be approximated as a set of lines connecting vertices both for \( F_n \) and \( G_n \). Calculus of variations shows that if the tubes in the image in \( G_n \) is a set of equal radius cylinders of lengths equal to the lengths of their pre-images in \( F_n \) then the stationary 1-varifold is in the image of a harmonic map \( \gamma_n : F_n \rightarrow G_n \). Thus when the angle is 120 degrees in \( F_n \) we show that the minimum energy is when the angle are at 120 degrees in \( G_n \). See figure 4.7.
Using the equal radius of the cylinders in the domain we can ignore the non-axial dimension. Positions \((Ax,Ay)\), \((Bx,By)\) and \((Cx,Cy)\) are fixed and \(A\), \(B\), and \(C\) are distances of the three branches in the domain. We will allow \((X,Y)\) to vary and examine how energy is a function of \(X\) and \(Y\).

\[
\text{Energy} = \frac{(Ax-x)^2}{A} + \frac{(Bx-x)^2}{B} + \frac{(Cx-x)^2}{C}
\]

The graph of this energy is a surface of revolution of a parabola. We simply need to verify that the minimum is when \((X,Y)\) is in the original position. When the angle is 120 degrees we can put on a small variation as follows in figure 4.8, up to first order.
\[
\text{Energy} = \left( \frac{a - t}{a} \right)^2 + \left( \frac{b + t/2}{b} \right)^2 + \left( \frac{c + t/2}{c} \right)^2
\]
\[
\frac{d}{dt} (\text{Energy}) = -2t + t + t = 0
\]

This variation calculation is same for small variations in any of the three directions as shown, contracting a, b, or c. This verifies that the critical point of energy is indeed where the lines all meet at 120 degrees.

This treatment of the tubes is still valid if the tube fans out to a disc because the axial components of the map to the tube are independent of the others. That is a harmonic map of \( F_n \) to \( \mathbb{R}^3 \) can be seen, restricting to the tubes, as three harmonic maps to \( \mathbb{R} \), one in each dimension.
Lemma 4.11 Tetrahedral joints stay small: For each tube tetrahedral joint in $F_n$, there exists a ball of radius $R(n)$, center $x$ such that $B(x,R(n)) \cap G_n$ is an planar region with four boundary components.

Proof: Using the same proofs as for Y joints we can say that tetrahedral joints (the cone of vertices of a regular tetrahedron to its center) stay small.

Reducing deformations to zero

When we shorten the tubes and deform the surfaces we can trade off an energy increase in the surfaces for an energy decrease in the tubes. However for large $n$ the area of the tubes in the $F_n$ tends to zero. The largest Jacobian entry in the map from $F_n$ to $G_n$ is about 1, so the energy in the tubes goes to zero. Note that we did not consider the small discs removed from the minimal surfaces in the minimal surface part of the energy calculations. As $n$ tends to infinity the effect of these discs on area and energy will go to zero.

As a result of the energy of the identity map on $F_n$ tending to the minimum, all the locally energy increasing deformations, $N(n)$, $R(n)$, $R'(n)$, $R''(n)$, in the above lemmas are sent to zero. This means that the surface part of the $G_n$ tends to the minimal surface conformally with minimum energy and the rest of the image tends to a 1-varifold. For
each $n$, minimizing energy ensures that the $1$ varifold is made up of straight line segments and $Y$ degree joints and 'tetrahedral' joints. We have showed that in the limit the lengths of the tubes will not increase. Therefore the $1$-varifold in $G_n$ is length minimizing given the need to interconnect components $S$ with no specified points of intersection.

Now we can pull back the conformal structure on the $F_n$'s to a planar domain $D$ (e.g., figure 4.1 and figure 4.2) for all $G_n$'s. As energy is invariant under conformal maps on the domain we now have our sequence of harmonic maps $g_n \circ h_n$ from $D$ into $\mathbb{R}^3$ with prescribed free boundary data whose image is as desired.

We have proven theorem 4.1.

**Limits of images and graphs of maps**

In the limit the images of the $G_n$'s and the $F_n$'s can be seen most simply as sets. Convergence of a subsequence in the Hausdorff set topology can be proven by the fact that we are taking a sequence of compact sets in a compact subset of $\mathbb{R}^3$.

We can also regard the graphs and images from $D$ to $G_n$'s as rectifiable 2-varifolds. A compactness theorem for varifolds, discussed in the next chapter, can also be use to prove them 4.1. In the case of the images, $G_n$'s, this convergence gives a general varifold
which has a 2 dimensional measure on the rectifiable 1-varifold segments, representing more information than is given by the Hausdorff set convergence.

**Generalizing to polyhedral space**

Following chapter 3 we can ask, as an open problem, what might happen if the problem were changed to one where the symmetric solution to the calculation in the introduction following Fomenko and White were carried out in a radially symmetric situation with a positive curvature singularity along the axis of symmetry. Would the limit of energy minimizers tend to avoid the singularity in some sense?
5 Lifted varifold compactness theorems

The example problem of the previous chapter motivates the theorems in this chapter. The purpose of this chapter is to provide a measure that represents n dimensional submanifolds and their limits which may have lower dimensional sets. This is done by representing lifts of the submanifolds into the appropriate Grassman bundle as measures. These lifts have limits, as measures, which are also n-dimensional submanifolds in the Grassman bundle, but which can project down to lower dimensional sets in the ambient space.

The general category of measures that we are working in are a subset of general varifolds. This subset consists of general varifolds whose measures in the Grassman bundle can be seen as rectifiable varifolds in the Grassman bundle.

This use of lifted varifold compactness theorems provides a measure theoretic way to strengthen Hausdorff set convergence, at least in the cases where the theorem is applicable. It also can cover non-compact or non locally-compact submanifolds: i.e. is not restricted to sequences of compact sets in compact spaces.
Additionally the compactness theorems we use to prove lifted varifold compactness also result in regularity information about the limit, and lower semi-continuity of mass properties of the limit. This is not given by use of set compactness.

Examples of general varifold compactness

Below is an example of a sequence of rectifiable 1-varifolds in $\mathbb{R}^2$ representing a semicircle as it shrinks down to a point along with the sequence of lifts in the Grassman bundle, $G(1, \mathbb{R}^2)$, see figure 5.1.

![Figure 5.1 A sequence of semicircles in $\mathbb{R}^2$ shrinking to a point (below) and its lift in the unoriented Grassman bundle (above)](image)

The semicircles in $\mathbb{R}^2$ are shown below in figure 5.1 and their lifts above them. The vertical direction in the lifts represents angle in $\mathbb{R}^2$ below. On the upper left a small line segments showing which angle below is represented at which vertical position above.
Note that the limit, as a measure in the Grassman bundle, of the lifts is a straight line in the fiber above the point to which the semi circles contract. The line means that there is some measure spread over all directions in the Grassman bundle above. From the point of view of mass, the mass (length) of the semicircles goes to zero, however the mass of their lifts, which has a component for length of the semicircle and a component for curvature, stays bounded away from zero. Note for the sake of comparison that the integral of curvature with respect to length in the sequence of semicircles remains constant.

If we regard the semicircles as measures in $\mathbb{R}^2$, either as rectifiable 1-varifolds, or once given an orientation, as 1 rectifiable currents, the limit will vanish, in whatever topology we choose for those measures. If we regard their lifts as measures in the Grassman bundle (general varifolds) then the limit does not vanish with the weak topology in the Grassman bundle. In this example we can take the measure to be the characteristic function of the lift to be integrated against smooth test functions in the Grassman bundle with respect to 1-dimensional Hausdorff measure. In the case of the examples in the last chapter we would do the same except integrate with respect to 2-dimensional Hausdorff measure.

Note by contrast that the limit of the semicircles in $\mathbb{R}^2$ vanishes with respect to all dimensions of Hausdorff measure in $\mathbb{R}^2$ except zero. We cannot take limits of the semicircles in $\mathbb{R}^2$ with respect to 0-dimensional Hausdorff measure because all
semicircles in the sequence have infinite zero-dimensional Hausdorff measure. Working in the lift enables to get around this problem.

Note by way of warning that Hausdorff set topology and measure topologies considered in this chapter will not agree in some cases. Take the sequence of closed intervals $[-1/n, 1/n]$. In the Hausdorff set topology $\{0\}$ is the limit set, not $\emptyset$. However as measures this sequence converges to the zero measure. Also notice that $\{0\}$ is a set of one-dimensional Hausdorff measure zero, and sets of measure zero have no effect in topologies based on measure. They are equivalent to $\emptyset$.

We will go on to compare different topologies for the measures in the Grassman bundle, one corresponding to treating them as rectifiable varifolds in the Grassman bundle, and the other will correspond to treating them as rectifiable currents in the Grassman bundle, after assigning them orientations.

**Non-constructive version of lifted varifold compactness**

We will start by stating Allard’s compactness theorem for varifolds with integer multiplicity:
A sequence of rectifiable integer multiplicity n-varifolds with uniformly finite mass and first variation will contain a convergent subsequence on compacta with a rectifiable n-varifold as a limit with integer multiplicity, finite mass and finite first variation. [Allard].

Theorem 5.1:

Let $V_i$ be a sequence of n-submanifolds in $\mathbb{R}^{n+k}$ that lifts to the rectifiable n-varifolds obtained by lifting to the Grassman bundle $G_n(\mathbb{R}^{n+k})$. Let $GV_i$ denote the lifts of this sequence, and suppose they satisfy the hypotheses of Allard's integral varifold compactness theorem applied in the Grassman bundle. Then a subsequence of these lifts will converge to an integer multiplicity, finite first variation. finite mass limit varifold in the Grassman bundle. Furthermore the projection down of this limit as a measure push-forward to the base space $\mathbb{R}^{n+k}$ will be supported in the Hausdorff set topology limit of the $V_i$ up to a set of measure zero. Also the n-dimensional part of the projection down to the base space will give a rectifiable n-varifold that is equivalent as a Radon measure in the Grassman bundle to the rectifiable n-varifold limit of the $V_i$, according to Allard's compactness theorem.

Proof of 5.1.
We need to show that the Radon measure caused by applying Allard's theorem to the $V_i$s is the same Radon measure as obtained by taking the lift of the $V_i$s to $G_n(R^{n+k})$ applying Allard's compactness there, and pushing forward the resulting Radon measure back down to represent a rectifiable varifold in $R^{n+k}$. Let $\mu_{V_i}$ denote the Radon measure in $R^{n+k}$ for the $V_i$s as rectifiable varifolds in $R^{n+k}$. So we can take a continuous compactly supported test function for weak convergence $f:R^{n+k} \to R$. So giving us weak convergence as follows

$$\mu_{V_i} \to \mu \iff \lim_{i \to \infty} (\mu_{V_i}(f)) \to \mu(f), \forall f$$

Using Allard's compactness theorem on the $V_i$s gives us measure convergence in $G_n(R^{n+k})$.

Similarly applying Allard's compactness theorem to the $GV_i$s gives a weak convergence in the Grassman bundle so that

$$\lim_{i \to \infty} \mu_{GV_i}(\tilde{f}) = \mu_{GV}(\tilde{f}), \text{ for any continuous, compactly supported } \tilde{f}:G_n(R^{n+k}) \to R.$$ 

In particular, taking $\tilde{f} = f \circ \pi$, we conclude:

$$\pi_*(\mu_{GV})(\tilde{f}) = \mu_{GV}(f \circ \pi) = \lim_{i \to \infty} \mu_{GV_i}(f \circ \pi) = \lim_{i \to \infty} \mu_{V_i}(f) = \mu_V(f).$$
So the two Radon measures obtained from the different applications of Allard's compactness theorem agree. Thus we have proven theorem 5.1.

The limitation of theorem 5.1 is that we do not have a description of which $V_i$ satisfy the hypotheses without examining their lifts. We shall now present two theorems whose hypothesis only explicitly refer to the submanifolds $V_i$, and not to their lifts. We are able to get a stronger result, in terms of weaker hypothesis for the same conclusions, in the harder version of the varifold compactness theorem. This is by using current compactness instead of varifold compactness for the lifts in the Grassman bundle.

**Theorem 5.2**

The 1-dimensional lifted varifold compactness theorem (easy version)

Suppose a sequence of 1-varifolds $V_i$ in $\mathbb{R}^{n+k}$ satisfies the following:

(i) Each $V_i$ is the union of a finite number ($< N$ for all $V_i$) of $C^1$ images of compact intervals in $\mathbb{R}^{n+k}$.

(ii) Each $V_i$ has bounded first variation ($< P$ for all $V_i$)

(iii) Each $V_i$ has bounded 1-dimensional Hausdorff measure ($< M$ for all $V_i$)
(iv) Each $V_i$ has bounded integral of the absolute value of the derivative of curvature ($< Q$ for all $V_i$)

$V_i$ has a corresponding sequence of lifts, $H_i$, in the oriented Grassman bundle which have a subsequence converging as rectifiable 1-varifolds in $G_i(\mathbb{R}^{n+k})$ to a rectifiable 1-varifold $H$. Also $H$ is finite mass, finite first variation and integer multiplicity. The projection of $H$ down to the base space, $\mathbb{R}^{n+k}$ is the union of a rectifiable 1-varifold and zero-dimensional sets. Also the 1-dimensional part of the projection of $H$ agrees with the rectifiable 1-varifold produced as a limit using Allard’s compactness theorem in $\mathbb{R}^{n+k}$ without taking the lift.

**Theorem 5.3**

**The 2-dimensional lifted varifold compactness theorem (easy version)**

A sequence of rectifiable 2-varifolds $V_i$ in $\mathbb{R}^{n+k}$ satisfies the following

(i) Each $V_i$ is the union of a finite number ($< N$ for all $V_i$) of $C^2$ images of compact polygons and discs. Where there is a uniform bound on the total curvature of the image of interiors of edges of polygons.

(ii) Each $V_i$ has bounded first variation ($< P$ for all $V_i$)
(iii) Each \( V_i \) has bounded integral of absolute value of Gaussian curvature (\(< P \) for all \( V_i \)).

(iv) Each \( V_i \) has bounded mass (\(< M \) for all \( V_i \)).

(v) Each \( V_i \) has bounded integral of the absolute values of the derivatives of principal curvatures (\(< Q \) for all \( V_i \)).

\( V_i \) has a corresponding sequence of lifts \( H_i \) in the oriented Grassman bundle which have a subsequence converging as varifolds to a varifold \( H \). The projection of \( H \) to the ambient space is the union of a rectifiable varifold and possibly lower dimensional sets. Also the 2-dimensional part of the projection of \( H \) agrees with the rectifiable 2-varifold produced as a limit using Allard's compactness theorem in the ambient space without taking the lift.

Note that we are bounding the integral of absolute values of Gaussian curvature. Gauss-Bonnet gives us the integral of Gaussian curvature, which can only give a lower bound on the integral of absolute values of Gaussian curvature. So if Gauss-Bonnet shows that the integral of Gaussian curvature is infinite, then we know we do not satisfy our hypotheses. This is useful in showing that our hypotheses eliminate some undesirable situations.
Cones satisfy the hypotheses of theorem 5.3

We want to check that the hypotheses are satisfied in some interesting applications of the theorem. Any finite length rectifiable 1-varifold or finite area rectifiable 2-varifold will satisfy the above hypotheses if the principal curvatures are bounded. We can also allow another category of 2-varifold with unbounded curvature. This consists of cones of simple closed finite length curves in the sphere. There is a homothety invariance about the cone point. This means that we can take a principal curvature at a fixed distance from the origin and we know from linear scaling of the object we can say that the curvature increases with the reciprocal of distance from the origin. However as this is being integrated over a linearly smaller region, we obtain a finite total integral of principal curvature. The product of principal curvatures is also zero because cones are ruled surfaces with one principal curvature being zero. So Gaussian curvature is zero and first variation is finite on such cones. The cone construction is central to varifold theory, and so for our varifold theorem to have broad application, it is important that it applies to cones.

Proof of theorems 5.2 and 5.3
As in the proof of theorem 5.1 we use Allard's integral compactness theorem. The mass of each $H_i$ is bounded by the mass, principal curvatures and their products and first variation of the corresponding $V_i$. We now prove a lemma in general dimensions.

The idea of the lemma is that the $n$-varifold mass of the Grassman bundle lift of an $n$-varifold is bounded above by the sum of the masses of its local push-forwards under projections onto a set of a sufficient number of orthogonal $n$-planes. These masses of push-forwards are given in terms of (a) integrals of the mass of the rectifiable $n$-varifold in $\mathbb{R}^{n-k}$, (b) integrals of sums of absolute values of principal of the rectifiable $n$-varifold in $\mathbb{R}^{n-k}$, and (c) integrals of products of absolute values of principal of the rectifiable $n$-varifold in $\mathbb{R}^{n-k}$. Thus by bounding these quantities on the rectifiable $n$-varifold in $\mathbb{R}^{n-k}$, we can bound the mass of the lift.

**Lemma 5.4:**

The $n$-volume of the lift of a smooth submanifold $V$ is bounded by the mass of $V$ and integrals of absolute values of principal curvatures. $P_s$, and products of those absolute values:
\[ \int_{\text{Volume of } V} \left( 1 + \sum_{i=1}^{\delta} |P_i| + \sum_{i<j} |PP_j| + \ldots \prod_{i=1}^{\delta} |P_i| \right) dv < N \]

\[ \Rightarrow n - \text{volume of lift of } V < f(N) \]

Proof of lemma 5.4

Part 1

Mass of a smooth n-submanifold in Euclidean space is locally bounded above by the sum of masses of its orthogonal projections onto coordinate n-planes.

Proof of part 1

Using a Reimannian integration style approximation based on the fact that the submanifold is smooth we simply prove the theorem for V being an n-hyperplane.

Take an n-hyperplane, S, in \( \mathbb{R}^{n+k} \) and an arbitrary set of orthogonal directions none of which are parallel to S. Choose an origin not on the hyperplane. Look at the simplex enclosed by the origin, the hyperplane S, and the orthogonal n-planes of the form \( x_{i_1} = 0, \ldots, x_{i_k} = 0 \) in \( \mathbb{R}^{n+k} \). The simplex is bounded by a region of S, call it \( S' \) and its orthogonal projections. Because \( S' \) is planar it is the minimum n-volume submanifold with its boundary (by calibration). The union of its projections also have the same boundary, but their n-volume will be greater.
In the case of one of the orthogonal hyperplanes being parallel to $S$, we know trivially that the projection of a compact region of $S$ onto that plane will have the same $n$-volume as the region. Thus the inequality is trivially true. We have shown part 1.

Part 2

The area forms and projections for the lift are given by the sums and products of absolute values of principal curvature and by orthogonal projections in the ambient Euclidean space below.

Proof: The orthogonal projection directions come from the product structure of the fiber of the Grassman bundle and Euclidean ambient space. Each projection can be characterized in terms of how many of the $n$ directions of the $n$-plane lie in the Euclidean space and how many lie in the fiber. We do not need to be concerned with the directions that lie only in the Euclidean space as we can use the mass of the varifold as a bound for those directions. The projections of interest are the ones that include one or more directions in the fiber. We can examine the $n$-volume in these cases. If we have one direction in the fiber and the rest in Euclidean space, we understand the Euclidean projections, and the other direction in the fiber is given by the curvature in the corresponding direction.
Figure 5.2: The projections of the lift of u

So for example we look at area forms on a cylinder, its lift and the projections of the lift. See figure 5.2. The area forms of the lift (in XYZRS space) can push forward under projection to the cylinder to give the area forms in Euclidean (XYZ) space. However there is also a non-zero projection of area forms to the space (RZ) given by the direction in the fiber corresponding to the curvature and the direction in Euclidean space corresponding to a longitude of the cylinder.
Thus the sums and products of curvatures in the integral in lemma 5.4 cover all possible combinations of dimensions in the fiber and the Euclidean space, where the projections will give a non-zero push forward. We have proven lemma 5.4.

**Lemma 5.5: Relationship between principal curvature bounds and mean and Gaussian curvature bounds.**

Mean curvature has a natural relationship to first variation. In the one dimensional varifold case this is simply the principal curvature and suffices for the proof. In two dimensions one must use more than the mean curvature. The sum of absolute values of principal curvatures is not bounded by mean curvature, as curvatures can cancel in mean curvature. However a combination of mean and Gaussian curvature can be used to bound the sum and product of absolute values of principal curvatures. Let \( a \) and \( b \) be principal curvatures that can take real values. We are interested in

\[ |a||b| \text{ and } |a|+|b| \text{ being bounded by some function of } ab \text{ and } a+b. \]

Let's try \( B = |ab| + |a+b| \).

**Case 1:** \( a \) and \( b \) have the same sign. Clearly \( |a||b| = |ab| \leq B \) and \( |a|+|b| = |a+b| \leq B. \)

**Case 2:** \( a \) and \( b \) have opposite signs. Let \( b = -pa, 0 < p < 1, a > 0 \)
\[ B = \alpha a + |(1-p)a| \]

\[ B = \alpha a + (1-p)a > \alpha a = |ab| = |a||b| \]

Now \(|a|\geq|b|\)

If \(0<p\leq1/2\)

\[ B = \alpha a + (1-p)a > (1-p)a > a/2 > (|a|+|b|)/4 \]

If \(1/2<p\leq1\)

\[ B = \alpha a + (1-p)a > \alpha a/2 > |a||b|/2 \]

\[ |a+b||a+b| = \alpha a + 2|ab|+ bb < 2B + 4B + 2B = 6B \]

\[ |a+b|< \sqrt{6B} \]

Since \(\sqrt{B} \leq B\) for all \(B \geq 1\) a bound for \(B\) gives desired bounds for \(|a| + |b|\) and \(|ab|\).

For all cases the integral of \(B\) and the mass linearly bounds the integral of projections into planes intersecting the fiber. We have proven lemma 5.5

We have now shown that finite mass is achieved in the lifts. We now need to check for finite first variation, rectifiability and integer multiplicities of the lifts.

**Finite first variation of lifts**
Lemma 5.6: The lifts, $H_j$, have finite first variation.

There are two contributions to first variation, curvature on smooth points in the lift and boundary terms on non-smooth points. A curvature in the lift, $H_j$, corresponds to a directional derivative of a curvatures in the $V_i$s. As the $V_i$s are unions of smooth images of compact sets, these curvatures will be bounded, and as the lifts have finite volume, their integrals, and hence their contribution to first variation will be bounded. However we need them to be uniformly bounded over the sequence. This is achieved by the bound on the integrals of derivatives of curvature in our hypotheses. We can now turn our attention to first variation deriving from boundary.

We only need to be concerned with dimension 2 (proof of theorem 5.3) as there are only finitely many boundary points contributing a bounded amount of first variation in the one dimensional case. If we consider the smooth image of a polygon, then there are two types of boundary point. These are the images of the edges of the polygon, and the images of the vertices. If we consider a single image of a single polygon we can examine the edges and vertices on the lift of the image. The edges of the image are $C^2$ curves of finite length. These will lift to edges in the Grassman bundle as the inward pointing normal vector also varies smoothly. These edges in the lift contribute to the first variation of the lifted varifold. The amount by which they contribute will be determined by the length of
the lifted curve. This is bounded by the length of the unlifted edge and the total amount by which the inward pointing normal varies. To quantify the variation of the inward pointing normal we look at the length of its image under the Gauss map counting multiplicity. We shall argue that this variation is bounded by the absolute values of principal curvatures and the bounded curvatures of the edge images. The absolute values of curvatures are bounded on each disc by the fact that each image is a $C^2$ immersed image of a compact set. Therefore there is a maximum absolute value of second derivative, i.e. curvature for each polygon. The edge curvatures are bounded by hypothesis.

There are three ways we could get an infinite length image in the Gauss map of the inward pointing normal on a finite length edge. We will show by contradiction that each way would violate one of three bounds in the hypotheses.

Case (i) There is a uniform bound on the principal curvature normal to the edge, and the infimum of the distance of other edges to the edge in a normal direction is greater than zero, i.e. vertices are obtuse in image (this means the surface is very wavy uniformly over a finite area). This would force the integral of absolute values of principal curvature parallel to the edge to go to infinity in small neighborhood of the edge.
Case (ii) There is no uniform bound on principal curvatures normal to the edge and the principal curvature of a uniform "push off" of the edge into the surface has a set of normal directions with finite total length of image in the Gauss map (this means that the surface is not very wavy away from the edge, but undulates more and more near the edge). Here the Gaussian curvature will be infinite (negative) thus contradicting the bound on absolute values of products of principal curvatures.

Case (iii) The integral with respect to area of principal curvatures and their products is finite, but the Gauss map image of the normal vectors has infinite length. This is only possible if there is another boundary close by making a thin strip that can twist a lot. This contradicts the hypothesis that edges have uniformly bounded integral of absolute value of curvature.

We now need only to comment that first variation is not associated to vertices, for us to conclude that we have achieved finite first variation and proven lemma 5.6. Rectifiability and integer multiplicity of the $H_i$s are inherited from the $V_i$s. We use the same argument as in the proof of 5.1 to get the properties of limit projected back down to the base space $\mathbb{R}^{n+k}$. Thus we have proven Theorems 5.2 and 5.3.

The harder version of the lifted varifold compactness theorem.
We can try to improve on theorem 5.3 by removing the uniform bound on the finite number of smooth images that make up each varifold. This is done by treating the lifts as currents rather than as varifolds. Current compactness does not require bounded first variation on interior points of the lift. This removes the bound on total length of polyhedral edges in the rectifiable n-varifolds in $\mathbb{R}^{n\times k}$. We shall simply state the theorem as a conjecture for rectifiable 2-varifolds.

Conjecture 5.7

Suppose a sequence of rectifiable 2-varifolds $V_i$ in $\mathbb{R}^{n\times k}$ satisfies the following:

(i) Each $V_i$ is the union of a finite number of $C^2$ images of compact polygons and discs.

Where there is a uniform bound on the total curvature of the image of interiors of edges of polygons.

(ii) Each $V_i$ has bounded mass ($< M$ for all $V_i$)

(iii) Each $V_i$ has bounded first variation ($< P$ for all $V_i$)

(iv) Each $V_i$ has bounded integral of absolute value of Gaussian curvature ($< Q$ for all $V_i$)

(v) Each $V_i$ has bounded distributional Gaussian curvature ($< R$ for all $V_i$).
Then $V_i$ has a corresponding sequence of lifts $H_i$ in the oriented Grassman bundle which have a subsequence converging as varifolds to a varifold $H$. The projection of $H$ to the ambient space is the union of a rectifiable varifold and lower dimensional sets.

Note we also removed the bound on derivatives of curvatures and extended the bound on integral of absolute value of Gaussian curvature to include distributional Gaussian curvature, that is Gaussian curvature concentrated at vertices. We have to take the notion of vertex in a general sense. If locally we have a number of surfaces intersecting, then we need to take the distributional Gaussian curvature of each one separately. Also if a boundary component intersects a vertex, in the sense of a point whose approximate tangent cone is the cone of an embedded compact interval on the sphere, then we need to define the absolute value of distributional Gaussian curvature at that point. We can take this as the infimum of absolute curvatures of all vertices constructed by closing the curve in the sphere corresponding to the approximate tangent cone.

It remains as work in progress to prove:

**Conjecture 5.7a**
Also we can say in Theorem 5.7 that the 2-dimensional part of the projection of $H$ agrees with the 2 rectifiable varifold produced as a limit using Allard’s compactness theorem in the ambient space without taking the lift.

A much stronger version of the conjecture 5.7 would allow a nowhere dense countable number of vertices in each $V_i$. We may then be able to use current compactness on an increasing sequence of compact sets that avoid decreasing open neighborhoods of the vertices. So we apply compactness an infinite number of times and use a diagonalization argument to achieve a subsequence.

**Example sequence of varifolds whose lifts only converge as currents.**

![Diagram of varifolds and lifts](image)

*Figure 5.3 A varifold and its disjoint lift (left) and filled in lift (right)*
Consider a set of rectangles of width 1 and height \( \frac{1}{n^2} \). Let \( V_n \) consist of the first \( n \) rectangles placed edge to edge along the long edge. Then place a dihedral angle of \( \frac{1}{n^2} \) radians at each joint. The first variation of each \( V_i \) is uniformly bounded, but the first variation of the lifts, \( H_i \), is not. See figure 5.3 (left).

Figure 5.3 shows a varifold with three segments shown end on and its lifts. On the left the lift is shown as a varifold with disjoint sections. On the right the lift has been filled in with vertical sections to make a connected surface. Both the connected, filled in, and disconnected lifts have approximately a fixed amount of first variation corresponding to each angle in the varifold below. On the left the first variation comes from boundary, and on the right from two 90 degree angles. In the limit as the number of angles go to infinity the first variation of either lift will also go to infinity. This prevents the use of varifold compactness as used in the proof of theorem 5.2 and 5.3.

On the other hand current compactness requires only uniformly bounded mass and boundary mass. The lift on the left shows how a disconnected lift will have infinite boundary as the number of angles goes to infinity. The filled in lift on the right maintains finite mass. and as long as the angles go to zero quickly enough. The extra mass for
filling in corresponds to the dihedral angle at the joint. The uniformly bounded first variation of the $V_i$s is sufficient to ensure this.

Having outlined our strategy to improve on theorem 5.3, here is the approach toward a proof.

**Steps toward proof of conjecture 5.7**

- Filling in lifts of edges with dihedral angle locally, and filing lifts of vertices.

- Proving finite mass of the current made from the lift

- Proving the current has finite boundary mass, given that boundary comes from (i) the lift of the boundary of the $V_i$s (where points have neighborhoods homeomorphic to half planes), (ii) Non-manifold points in the $V_i$s, such as Y singularities. (iii) edges and curves in the lift where orientations of adjacent faces or surface regions do not match up.

- Assigning orientations to make the lift into a current and to avoid unwanted cancellation in the limit.

**Finite mass and filling in.**

**Lemma 5.8:** The filled in lift of the $V_i$s has uniformly bounded finite mass.

**Lemma 5.9:** The lift of the $V_i$s without filling in has uniformly bounded finite mass.
Proof: Without filling in the lift with extra mass for dihedral angle and vertices in the $V_s$, we repeat the proof for theorem 5.3 to show that mass of the lift is bounded by the first variation and mass of the $V_s$. We have proven lemma 5.9.

**Lemma 5.10:** Extra mass from filling in lifts of dihedral angles edges is uniformly bounded.

Proof: Filling in lifts of dihedral angle is as described above. Uniformly bounded first variation in $V_s$ ensures that filling in does not add more than a uniformly fixed amount of mass to the lift. This is because the mass of the lift is the integral over the edge set of angle $a$ (where $a = |180 \text{ degrees-dihedral angle}|$) with respect to 1-dimensional Hausdorff measure. First variation is the integral of $2\sin(a/2)$. So first variation per unit edge length linearly bounds extra mass per unit edge length, and so first variation linearly bounds extra mass from filling in lifts of edges. Thus lemma 5.10 is proven.

**Lemma 5.11:** Distributional Gaussian curvature in $V_s$ bounds extra boundary mass in lifts of $V_s$ due to filling-in polyhedral vertices and cones of smooth curves.
Figure 5.4 A filled in lift of a vertex

Proof: Each vertex in a \( V \) lifts to a region that can be filled in the fiber, or left as boundary. We will simply calculate that filling adds mass equal to the distributional Gaussian curvature at each vertex. See figure 5.4.

Figure 5.4 shows a schematic representation of a lift of the vertex of a cube. Each of the three faces lifts to a face. The three edges have been filled in with quarter cylinder sections, and a spherical triangle fills in the fiber of the lift of vertex \( p \) to remove any boundary. The area of the spherical triangle is the distributional Gaussian curvature at the vertex. This will also be the case for cones of smooth curves. This is because we are
essentially observing that the area of the filled in image of a point under the Gauss map is
the distributional Gaussian curvature at that point. Thus we have proven lemma 5.11, and
together with lemmas 5.9 and 5.10 we have proven lemma 5.8.

Choosing assignation of orientation to make the lift a current.

We make an assignation of orientation of the lift by choosing a vector \( x \) then using it to
determine orientation by means of inner product with the unit normal to surfaces (or to
tangents for \( l \) varifolds).

Lemma 5.12 Almost all choices of orientation vector \( x \) give well defined orientations.

This lemma is saying the directions for choices of \( x \) which lead to a region of positive
measure having no orientation defined has zero Hausdorff measure in the Grassman
bundle of the Grassman bundle.

Proof of lemma 5.12 by contradiction: Suppose there are more than countably infinite
number of bad directions for \( x \). Then each bad direction has a finite positive measure
associated with it with no assigned orientation. As the lifts of the \( V_s \)s are \( C^1 \) almost
everywhere this implies that \( x \) is perpendicular to the normal vectors on a set of positive
measure, i.e. \( x \) is parallel to the tangent space. If we consider the project of \( x \) down to \( \mathbb{R}^3 \),
we can then look at all the vectors in tangent plane directions which have positive mass
associated with them. Firstly we can say that there are only a countably infinite number of such tangent planes, otherwise the mass of the \( V_i \)'s would not be uniformly bounded. Under the oriented Gauss map, the images of these tangent planes are great circles. These have zero area measure. Therefore the set of bad directions is a set of zero measure. Thus we have proven 5.12.

**Lemma 5.13:** Almost all choices of orientation vector \( x \) avoid mass cancellation in the limit as currents converge due to submanifolds coming together with canceling orientations. This proof is very similar to that of lemma 5.12. We associate cancellation of positive mass with \( x \) being perpendicular to the tangent plane where the orientations cancel. See figure 5.5. Vector \( x \) is vertical and two submanifolds with opposite orientations are going to converge to the same horizontal submanifold perpendicular to \( x \). They are approaching the horizontal slope from opposite sides. One with positive slope and the other with negative slope.

These bad choices when projected down to \( \mathbb{R}^3 \) and then mapped to the sphere by the oriented Gauss map give a set of measure zero in the sphere. There can only be a countably finite number of bad directions in \( \mathbb{R}^3 \) and thus only a set of zero measure of bad choices for \( x \). Thus we have proven lemma 5.13.
Below, in lemma 5.19, we will show by similar argument that there are only a countable number of bad directions for $x$ in creating infinite boundary mass.

**Proving uniformly finite boundary mass of currents.**

**Lemma 5.14:** The boundary mass of the currents produced by filling in and assigning orientation to the lifts of the $V_i$'s is uniformly bounded.

**Lemma 5.15:** The one-dimensional Hausdorff measure of the lift of boundary of the $V_i$'s is bounded.
Lemma 5.6 gives us a uniform bound on the one dimensional Hausdorff measure of the lift of boundary of the $V_i$s because otherwise first variation of the lifts would not be uniformly bounded. This proves lemma 5.15.

**Conjecture 5.16:** The length of Y type singularities is uniformly bounded in the $V_i$s.

Y type singularities are locally of the form of three half planes meeting on their boundary. If these are at 120 degrees then they contribute nothing to the first variation of the $V_i$s. However they do contribute finite boundary mass per unit length to the lifts.

Steps toward proof of conjecture 5.16 by contradiction:

Case 1. If all their curves are on average a finite distance apart, then we can use mass of $V_i$s to bound their length. Say they are all $2d$ apart on average. Then mass will be greater than $3d$ (length of the curves of the form of Y singularities). Case 1 is impossible.

Case 2. Now we need to prove that Y singularities cannot accumulate. We will use a one-dimensional version as a first step to prove this.

**Y singularities in 1 dimension**

**Lemma 5.17:** The Y singularities in one dimensional $V_i$s cannot accumulate at a point.
Now we show that if the $Y$ singularities accumulate then we have either infinite mass or infinite first variation.

1) Choose an accumulation point and a sequence of $Y$ singularities approaching it. Do not allow the singularities to accumulate anywhere else, by choosing a subsequence if necessary.

2) Take the radial projection of this sequence onto the unit sphere centered at the accumulation point. If there is more than one accumulation point on the sphere, take a subsequence until there is only one. This accumulation point on the sphere gives us a direction, represented by the vector from the accumulation point to the center.

3) choose a wedge BAP with apex at the accumulation point $P$ (see figure 5.6) and an infinite subsequence of singularity points in the wedge. Note that $AP$ corresponds to the accumulation direction found in 2.

4) choose a further subsequence of $Y$ singularity points which have a 120 sector in a wedge BPA with no other points in.
Figure 5.6

Note that the points shown as small dots each have a clear space (i.e. with no points) in the 120 degree arc beneath them above the line AP. Line PB with line AP forms a wedge in which the remaining nodes must lie in the chosen subsequence. As angle APB can be chosen arbitrarily small, it is less than 30 degrees.

For each such point there is at least one line segment heading toward the line AP. This gives a 'vector mass' of at least 1/2. (vector mass is multiplicity of line segment crossing AP times sine of angle of incidence with AP). We use this infinite quantity in estimating mass and first variation.

5) We now show that either the first variation of this subsequence or the mass is infinite. Suppose mass is finite, then looking at figure 5.6 we consider the projection of varifold onto a vertical line. Call it the Y axis with P at y=0. As the mass is finite, then we can choose a point a≠0 where the projection of the varifold will have finite density. Now we
set up a sequence of smooth vector fields on the $y$ axis that approximates $j\lambda([0, a])$, where $j$ is the unit vector in the positive $y$ direction. As the density of the varifold's projection onto the $y$ axis at $a$ is finite, then we can see that the first variation of the projection is unbounded under the given smooth vector field. We can say the vector mass corresponding to the line $y=0$ is infinite and the vector mass corresponding to the line $y=a$ is finite. The first variation under the vector fields will approach the difference between the vector masses at $y=0$ and $y=a$.

We have proven lemma 5.17.

**Conjecture 5.18** $Y$ singularities in 2-varifolds cannot accumulate with infinite one-dimensional Hasudorff measure without infinite mass or infinite first variation.

The approach to proving conjecture 5.18 is as follows. With 2 dimensional $Y$ singularities we need to choose a point and a direction where infinitely many of the curves accumulate both in space and direction. That is there is almost a product structure. Then take a transverse slice and apply the above argument. We can also adapt the proof to consider $Y$ singularities which are not 120 degrees, 120 degrees, 120 degrees. Angle $APB$ above can be made smaller and smaller to show that the first variation is as large as any number $N$. This enables a broader range of dihedral angles to be brought in. If the
Y's accumulate with angles 180 degrees, 90 degrees, 90 degrees, such a construction is necessary.

Once conjecture 5.18 is proven then the proof of 5.16 follows because there is a uniform bound on mass and first variation of the $V_i$s. However without conjecture 5.18 we can prove theorems 5.21 and 5.22 below.

**Lemma 5.19**: Extra current boundary at places where assigned orientations do not match up is uniformly bounded.

We need to choose our orientation vector $x$ in the Grassman bundle. We will show that when we push $x$ down to the base space, $R^3$, there are only a countable number of bad directions to choose for $x$. We will also project the orientations down to the $V_i$s and treat then as currents. The mass of the projection down of the new boundary can be uniformly bounded.

**Lemma 5.20**: Almost all choices of $x$ induce uniformly bounded mass in $R^3$ of projections down of new boundary components to the $V_i$ in $R^3$.

**Proof**: Each bad direction will induce curves of length not uniformly bounded in the $V_i$s. For first variation to remain finite in each we can look at the amount of bending transverse to the curve. This bending cannot be constant along the lengths of the curves in
the \( V_i \)'s otherwise first variation or the integral of absolute value of Gaussian curvature will be unbounded (The integrals of absolute value of principle curvatures will be unbounded and use lemma 5.5). However there must be a finite amount of bending at each point otherwise the orientation would not be well defined. This means that for each bad direction there is a finite amount of first variation or integral of absolute value of Gaussian curvature. As these quantities are uniformly bounded then there can only be a countable number of bad directions. This we have proven lemma 5.20.

Now we argue that when we pass to the lift, this length remains finite using the proof of lemma 5.6, and the uniform bounds on the \( V_i \)'s in the hypotheses.

Thus if conjecture 5.18 is proven then conjecture 5.7 will be proven.

**Theorem 5.21 (One dimensional version of conjecture 5.7):**

A sequence of rectifiable 1-varifolds \( V_i \) in \( \mathbb{R}^{n+k} \) satisfies the following

(i) Each \( V_i \) is the union of a finite number of \( C^2 \) images of intervals.

(ii) Each \( V_i \) has bounded mass (\(< M \) for all \( V_i \))

(iii) Each \( V_i \) has bounded first variation (\(< P \) for all \( V_i \))
Then $V_i$ has a corresponding sequence of lifts $H_i$ in the oriented Grassman bundle which have a subsequence converging as varifolds to a varifold $H$. The projection of $H$ to the ambient space is the union of a rectifiable 1-varifold and lower dimensional sets.

Proof of theorem 5.21:

We know that the mass of the lift will be uniformly bounded as it is bounded by mass and first variation of the $V_i$s. Just as in the two dimensional case the boundary of the lift is derived from the boundary of the $V_i$, angles in the $V_i$ in curves and $Y$ type singularities.

First variation of the $V_i$s bounds the first two. Lemma 5:17 eliminates the accumulation of $Y$ singularities which could cause infinite boundary mass of the lift for finite mass and first variation of the $V_i$. The above lemmas on choice of assignation of orientation apply in the 1-dimensional case. Current compactness applied in the Grassman bundle completes the proof of theorem 5.21.

**Theorem 5.22 (Weaker version of conjecture 5.7):**

A sequence of rectifiable 2-varifolds $V_i$ in $\mathbb{R}^{n+k}$ satisfies the following

(i) Each $V_i$ is the union of a finite number of $C^2$ images of compact polygons and discs.

Where there is a uniform bound on the total curvature of the image of interiors of edges of polygons.
(ii) Each $V_i$ has bounded mass ($< M$ for all $V_i$)

(iii) Each $V_i$ has bounded first variation ($< P$ for all $V_i$)

(iv) Each $V_i$ has bounded integral of absolute value of Gaussian curvature ($< Q$ for all $V_i$)

(v) Each $V_i$ has bounded distributional Gaussian curvature ($< R$ for all $V_i$).

(vi) The one-dimensional Hausdorff measure, counting multiplicity, of the singular set of $Y$ type singularities is finite.

Proof:

The lemmas toward the proof of conjecture 5.7 together with the additional hypothesis (vi) prove theorem 5.21.

Discussion of conjecture 5.7a

The key issue for conjecture 5.7a is whether current compactness in $G_2({\mathbb R}^3)$ produces a current which when regarded as a Radon measure, by ignoring orientation, agrees with
the Radon measure limit of the Radon measures in $G_2(\mathbb{R}^3)$ that represent the $V_j$s. We eliminated cancellation of submanifolds that converge with opposite orientation, as this would lead to a difference between current convergence and Radon measure convergence of the underlying set with the density function. However there is another situation that can be regarded as cancellation in a sense where current limits and Radon measure limits disagree by a density measure. See figure 5.7.

![Figure 5.7 Counter-example for conjecture 5.7a](image)

Figure 5.7 Counter-example for conjecture 5.7a

The limit of the $T_j$s as a Hausdorff set is the straight line $T$. If we give the $T_j$s a density of 1 and orientation, say from left to right, then the limit as a current is the line $T$ with a density of 1 with a left to right orientation. However as a Radon measure the limit is the set as shown but with a density of $\sqrt{2}$. Say $T$ has length 1, then the mass of each of the $T_j$s is $\sqrt{2}$, and so as a Radon measure limit $T$ also has mass $\sqrt{2}$.

This counter example gives the same underlying set for the current and the Radon measure, as an equivalence class of measurable sets. So we have some good news. It may
be possible either to show that given the hypotheses, or stronger versions of them for theorem 5.7 we can eliminate such counter examples. Also we may be able to give a weaker statement concerning the limits that may still be useful in identifying lower dimensional sets in the limit of the images.

**Can the lifted varifold compactness theorem work for rectifiable varifolds of bounded mass and bounded first variation?**

Can we say that all finite mass, finite first variation polyhedra have this kind of lifted compactness. We need to deal with the fact that vertices add no first variation and no mass but do add mass boundary mass or first variation upstairs. So for this type of proof we need to have either a bound on distributional Gaussian curvature or limit ourselves to a finite number of vertices. The counter example in general is a surface with pyramids placed at rational number points and the linear size of the pyramids decreases quickly enough to prevent unbounded mass being created by the faces of the pyramids or unbounded first variation being created by the edge lengths. However it is an open question as to whether a different style of proof of the above theorem may be extendable to cover the counter example. Some special cases, say where all vertices lie in a finite union of planes can be covered by applying the theorem on an increasing family of compact sets.
Generalizing to polyhedral space.

Essentially the proofs involved in this machinery are local. As singularities are on sets with support of zero measure the techniques will be valid.
References


Fomenko, A.T. The Plateau Problem: Part II the present state of the theory. Chapter 4.

Studies in the Development of Modern Mathematics Volume 1. Gordon and

Breach 1990

Gromov, M., Schoen, R.: Harmonic maps into singular spaces and p-adic superrigidity


(1992), 165-246.

Hardt, R., Singularities of harmonic maps. In Bulletin of the American Mathematical

Society, (1997) volume 34, no. 1, 15-34.

Hildebrandt, Stefan and Sauvigny, Friedrich. Minimal surfaces in a wedge I. Asymptotic


Jones, Kerry N. Geometric structures on branched covers over universal links.


