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On the Matrix Cuts of Lovász and Schrijver
and their use in Integer Programming

by

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Abstract

An important approach to solving many discrete optimization problems is to associate the discrete set (over which we wish to optimize) with the 0-1 vectors in a given polyhedron and to derive linear inequalities valid for these 0-1 vectors from a linear inequality system defining the polyhedron. Lovász and Schrijver (1991) described a family of operators, called the matrix-cut operators, which generate strong valid inequalities, called matrix cuts, for the 0-1 vectors in a polyhedron. This family includes the commutative, semidefinite and division operators; each operator can be applied iteratively to obtain, in $n$ iterations for polyhedra in $n$-space, the convex hull of 0-1 vectors.

We study the complexity of matrix-cut based methods for solving 0-1 integer linear programs. We first prove bounds on the (rank) number of iterations required to obtain the integer hull. We show that the upper bound of $n$, mentioned above, can be attained in the case of the semidefinite operator, answering a question of Goemans. We also determine the semidefinite rank of the standard linear relaxation of the traveling salesman polytope up to a constant factor. We study the use of the semidefinite operator in solving numerical instances and present results on some combinatorial examples and also on a few instances from the MIPLIB test set. Finally, we examine the lengths of cutting-plane proofs based on matrix cuts. We answer a question of Pudlák on such proofs, and prove an exponential lower bound on the length of cutting-plane proofs based on one class of matrix cuts.
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Chapter 1

Introduction

Many problems in combinatorial optimization can be modeled as 0-1 integer programs, which are problems of the form $\max\{c^T x \mid Ax \leq b, \ x \in \{0,1\}^n\}$, where $Ax \leq b$ is a system of linear inequalities in $n$ variables. Equivalently, a 0-1 integer program involves optimizing a linear function over the 0-1 vectors in a polytope $P = \{x \mid Ax \leq b\}$ (we assume that $Ax \leq b$ includes the inequalities $0 \leq x \leq 1$).

The usefulness of this modeling approach depends to a large extent, on our ability to derive cutting planes for $P$, or inequalities valid for the 0-1 vectors in $P$. Cutting planes often provide good approximations of $P_I$, the convex hull of 0-1 vectors in $P$, and permit linear programming methods to effectively analyze the underlying structure.

Lovász and Schrijver (1991) introduced the matrix-cut operators for 0-1 integer programs. These operators start with a polytope $P$, and yield better approximations of $P_I$ by adding cutting planes called matrix cuts. Some examples are the non-commutative, commutative, and semidefinite operators (denoted respectively as $N$, $N_0$, and $N_+$). The matrix-cut operators have the important property that if $P$ is a polytope in $\mathbb{R}^n$, then $n$ iterations are sufficient to obtain $P_I$. Thus, 0-1 integer programs can be solved by iteratively generating matrix cuts. Lovász and Schrijver proved that linear functions can be optimized over $N(P), N_0(P)$, and $N_+(P)$ in polynomial time (in the size of an inequality system defining $P$). From this, they derived polynomial-time algorithms to solve the maximum weight stable set problem in perfect graphs and $t$-perfect graphs.
1.1 Overview of thesis

In this thesis we examine the complexity of matrix-cut methods in solving 0-1 integer programs. In particular, we study the rank of polytopes and lengths of cutting-plane proofs. Another goal is to determine if $N_+$ (and other matrix-cut operators) can be used to solve practical instances of 0-1 integer programs. As a first step in this direction, we perform numerical experiments to ascertain the quality of semidefinite relaxations generated by the $N_+$ operator.

Rank of polytopes

The rank of a polytope $P$, with respect to a given matrix-cut operator, is the minimum number of iterations required to obtain $P_I$. Lovász and Schrijver (1991) showed that polytopes in $\mathbb{R}^n$ have rank at most $n$; also, the commutative rank is (about) $n$ in some cases. Goemans (1997) posed the problem of determining whether the upper bound of $n$ is tight for the semidefinite operator. In the context of the matching problem, a lower bound of $\sqrt{n/2}$ on the semidefinite rank was established by Stephen and Tunçel (1999). Goemans and Tunçel (2000) give an example with rank $n/2$.

We present two examples where the upper bound of $n$ is attained for the semidefinite operator (the first example has also been found by Goemans and Tunçel 2000). Even if we combine $N_+$ with the Gomory-Chvátal cutting-plane procedure, the first example still requires $n$ iterations (Corollary 4.6). We use this result and show that the rank of the standard relaxation of the traveling salesman problem, with respect to the combined operator, is at least $\lceil k/8 \rceil$ and at most $k + 1$, where $k$ is the number of cities (Theorem 4.8). Therefore, for the matching problem and the traveling salesman problem, which differ greatly in ease of solvability, $N_+$ behaves similarly (in that we get similar values of rank).

The first example above and the Goemans and Tunçel example with rank $n/2$ have
exponentially many inequalities; this forces rank to be high (we can use connections between depth and length of cutting-plane proofs to trivially obtain a lower bound of \( n/(2 \log n) \)). Our second example, with only one inequality and Chvátal rank 1, has \( N_+ \)-rank \( n \) (Theorem 4.2). Also, \( N, N_0 \), and \( N_+ \) are identical for this example (generally \( N_+ \) is stronger than \( N_0 \) and \( N \)).

A tool we use in these results is an analogue of an important property of the Gomory-Chvátal operator: the behaviour of \( N_+ \) on a face of a polytope is completely determined by the face (Lemma 3.6).

**Cutting-plane proofs**

If an inequality is satisfied by 0-1 vectors in \( P \), this fact can be established by a cutting-plane proof using matrix-cuts (this follows easily from the results of Lovász and Schrijver). The depth of a proof is related to the rank of \( P \), and the length of proofs to the question of whether \( \text{NP}=\text{co-NP} \).

Lovász suggested the study of matrix-cut based cutting-plane proofs. These can polynomially simulate resolution proofs, while the converse is not true (see [102],[12]). An important question is to determine the complexity of such proofs; their relationship with Gomory-Chvátal proofs is also an interesting issue. In an attempt to prove exponential lower bounds on the lengths of matrix-cut proofs, Pudlák (1999) derives an effective interpolation result for proofs with \( N \)-cuts and asks whether a similar property holds for \( N_+ \)-cuts.

Our main result on this topic is an exponential lower bound on the length of \( N_0 \)-cutting-plane proofs (Theorem 7.11). We first show that \( N_+ \)-cuts have an effective interpolation property (Proposition 7.6). This answers Pudlák's question above. Our approach is more geometric than Pudlák's and yields his interpolation result on \( N \)-cuts as a consequence (and also a result for \( N_0 \)-cuts). The geometric characterization of the \( N_0 \) operator in [92] is then used to obtain a monotone interpolation
property for $N_0$-cuts, which yields the exponential lower bound. This is an appli-
cation of Pudlák's (1997) approach to proving exponential lower bounds for the length
of Gomory-Chvátal cutting-plane proofs.

As a consequence, any algorithm, which sequentially generates $N_0$-cuts to solve 0-
1 integer programs, must take exponential time in the worst case. The lift-and-project
cuts of Balas, Ceria and Cornuéjols (1993), and the simple disjunctive cuts of Balas
(1975) are also $N_0$-cuts; cutting-plane algorithms based on these, such as the
specialized cutting-plane algorithm in [10], have exponential time complexity.

Computation

Semidefinite programming is now an important tool in combinatorial optimization.
Various problem-specific semidefinite relaxations, have been used in computation and
also in approximation algorithms. Low-rank approximations of such relaxations have
recently been used by Burer, Monteiro and Zhang (see [23], [24]) to get fast algorithms
for the stable set problem and the maximum cut problem.

The $N_+$ operator is a general tool to obtain semidefinite relaxations for 0-1 integer
programs; however, the resulting relaxations have too many inequalities. State-of-the-
art solvers for semidefinite programs are unable to solve such large problems quickly.
In this context we examine whether we can optimize over $N_+$ in reasonable time, and
if doing so yields high quality bounds for the associated integer program.

We develop a code, which solves a sequence of semidefinite programs, each with
fewer inequalities than $N_+$, to optimize over $N_+$. The SDPA solver of Fujisawa and
Kojima [50] is used to solve individual semidefinite programs. We consider instances
of the stable set problem, an integer programming formulation of a problem of Erdős
and Turán, and some MIPLIB problems. We optimize over $N_+$ for these problem
instances. Our results are not conclusive; for the first two cases, we do get improved
bounds, but for the MIPLIB problems, the bounds we get are disappointing.
We then give an analogue of a result of Burer, Monteiro and Zhang [24] in the case of the $N_+$ operator. This allows us to extend the applicability of the low-rank approximation ideas in [24] to the $N_+$-relaxations of packing problems (Proposition 6.1).

1.2 Organization

In Chapter 2, we collect standard results from linear algebra, linear programming and semidefinite programming, which are necessary later on. We introduce the matrix cuts of Lovász and Schrijver in Chapter 3, and prove some basic properties, including some analogues of the Chvátal operator.

Chapter 4 contains the examples of polytopes with high rank; we also establish the rank of the standard relaxation of the traveling salesman problem.

In Chapter 5, we consider the division operator $\bar{N}(P)$. This is intriguing in that some types of Gomory-Chvátal cuts are valid for the convex set $\bar{N}(P)$. We prove that the standard relaxation of the matching polytope has division-rank greater than 1. Therefore, all Gomory-Chvátal cuts are not generated by the division operator; this answers a question of Lovász.

In Chapter 6, we optimize linear functions over semidefinite relaxations obtained by applying the $N_+$ operator to the stable set problem, a problem of Erdős and Turán, and some MIPLIB problem instances. We also discuss some recent work on low-rank approximations of semidefinite relaxations in the context of the $N_+$ operator, and extend a result in [24].

We define cutting-plane proofs based on matrix-cuts in Chapter 7. The main result of this chapter is an exponential lower bound on the length of $N_0$-cutting-plane proofs. We first prove some simple connections between the depth and length of matrix-cut based proofs. We then discuss the idea of interpolation applied to cutting-plane proofs, and derive an effective interpolation result for $N_+$-cuts. Finally, we use Pudlák's monotone interpolation idea to prove the exponential lower bound result mentioned above, and also collect some open problems related to matrix cuts.
Chapter 2

Preliminaries

2.1 Linear Algebra

An excellent reference for linear algebra is the book by Horn and Johnson (1985). We denote the $n$-dimensional Euclidean space by $\mathbb{R}^n$ and the vector space of $m \times n$ real matrices by $\mathbb{R}^{m \times n}$. The length of vectors in $\mathbb{R}^n$ is expressed by $\|x\|$, where $\|x\|^2 = x^T x$.

Two vectors $x$ and $y$ in $\mathbb{R}^n$ are orthogonal, denoted by $x \perp y$, if $x^T y = 0$. For a subset $S$ of $\mathbb{R}^n$, its orthogonal complement $S^\perp$, is defined as

$$S^\perp = \{y \mid y \perp x, \text{ for all } x \in S\},$$

and is a linear subspace of $\mathbb{R}^n$.

The set of all finite linear combinations of vectors in a set $S \subseteq \mathbb{R}^n$ is called $\text{span}(S)$. If $b_1, \ldots, b_m$ are vectors in $\mathbb{R}^n$, then

$$\text{span}\{b_1, \ldots, b_m\} = \{\lambda_1 b_1 + \ldots + \lambda_m b_m \mid \lambda_i \in \mathbb{R}, i = 1, \ldots, m\}.$$

A set of vectors $\{b_0, b_1, \ldots, b_m\}$ is said to be affinely independent if the vectors $b_1 - b_0, b_2 - b_0, \ldots, b_m - b_0$ are linearly independent. The dimension of a set in $\mathbb{R}^n$ is one less than the size of the largest affinely independent set contained in it. For linear subspaces, this definition of dimension is consistent with the standard definition in terms of bases.

A system of linear equations $a_i^T x = b_i$ ($i = 1, \ldots, m$) in $\mathbb{R}^n$ is often denoted by $Ax = b$ where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The set of solutions of a linear system $Ax = b$ is known as an affine subspace of $\mathbb{R}^n$; if $b = 0$, the affine subspace is a linear subspace. Any linear subspace $L$ of $\mathbb{R}^n$ can be expressed as

$$L = \{x \mid a_i^T x = 0, \ldots, a_k^T x = 0\} = \text{span}\{b_1, b_2, \ldots, b_l\}$$
for vectors $a_1, \ldots, a_k$ and $b_1, \ldots, b_l$, where $k + l = n$. $L^\perp$ is spanned by the vectors $a_1, \ldots, a_k$, and is also the set of solutions of the equations $b_i^T x = 0, \ldots, b_l^T x = 0$ (the vectors in $L^\perp$ define the linear equations satisfied by $L$). Also, $L^{\perp \perp} = L$.

We usually denote matrices by upper-case letters $A, B, \ldots$ and their coefficients by the corresponding lower-case letters; thus $A$ represents the matrix $(a_{ij})$. Orthogonal complements and linear functions in $R^{m \times n}$ will be represented by using the inner product defined on this space. The inner product of two $m \times n$ matrices $A$ and $B$, is given by $A \cdot B = \sum_{i,j} a_{ij} b_{ij}$. The trace of a matrix $A$, denoted by $tr(A)$, is defined by $tr(A) = \sum_i a_{ii}$ and the trace operator satisfies $tr(AB^T) = tr(B^T A)$ for $A, B \in R^{m \times n}$. The inner product of $A$ and $B$ can be written as $tr(A^TB)$.

We consider various subspaces of the set of $n \times n$ matrices $R^{n \times n}$. The set of symmetric matrices forms a subspace of $R^{n \times n}$—called $S_n$—of dimension $C_n^{n+1}$. Similarly, the set of skew-symmetric matrices forms a $C_n^n$-dimensional subspace of $R^{n \times n}$. The orthogonal complement of $S_n$ is precisely the subspace of skew-symmetric matrices. Therefore, $S_n$ is spanned by the matrices $e_i e_j^T + e_j e_i^T$ and $e_i e_i^T$ for all $i, j$ with $i \neq j$, and also

$$S_n = \{X \mid A \cdot X = 0, \text{ for all skew-symmetric matrices } A \}.$$

A set of vectors $\{v_1, \ldots, v_k\}$ is said to be orthonormal if each vector in the set has unit length, and the vectors are mutually orthogonal. A square matrix $A$, satisfying $A^T A = I$, is called an orthogonal matrix and is non-singular.

An eigenvalue of a square matrix $A$ is a number $\lambda$, real or complex, such that $Ax = \lambda x$ for some non-zero vector $x$. The vector $x$ is said to be an eigenvector corresponding to $\lambda$. Every $n \times n$ matrix has $n$ eigenvalues, some of which may be repeated, and the sum of the eigenvalues equals the trace of the matrix.

Every $n \times n$ symmetric matrix has real eigenvalues and a set of $n$ orthonormal eigenvectors. This yields the spectral decomposition of a symmetric matrix: if $A \in S_n$, then $A = UDU^T$, where $U$ is an orthogonal matrix whose columns are eigenvectors of $A$, and $D$ is a diagonal matrix with the eigenvalues of $A$ as its diagonal entries.
We can also write $A$ as
\[ A = \sum_{i=1}^{n} \lambda_i u_i u_i^T, \]
where $\lambda_i$ is the $i$th eigenvalue, and $u_i$ is a unit-length eigenvector corresponding to $\lambda_i$.

### 2.2 Linear programming

A *linear program* is an optimization problem of the form
\[
\max \{ c^T x \mid Ax \leq b \}, \tag{2.1}
\]
and linear programming is concerned with solving linear programs. We often denote the system of linear inequalities $a_i^T x \leq b_i$ ($i = 1, \ldots, m$) in $\mathbb{R}^n$, by $Ax \leq b$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Linear programs can be written in a variety of equivalent forms, for example, as
\[
\min \{ c^T x \mid Ax = b, x \geq 0 \}.
\]

In this section we collect some basic results on linear programming. An excellent general reference is the book of Schrijver (1986).

A complete characterization of when linear inequality systems have solutions is provided by Farkas’ Lemma.

**Lemma 2.1 (Farkas’ Lemma)** Let $Ax \leq b$ be a system of inequalities. Either $Ax \leq b$ has a solution or there is a vector $y$ with $y \geq 0$, $y^T A = 0$ and $y^T b < 0$.

If $Ax \leq b$ has a solution $x$, the vector $x$ is a “certificate” of this fact; if there is no solution, then the vector $y$ in the Farkas’ Lemma provides a certificate for the non-existence of a solution.

A useful variant of the above lemma is the following:

**Lemma 2.2 (Farkas’ Lemma)** Let $Ax = b, x \geq 0$ be a system of inequalities. Either this system has a solution or there exists a vector $y$ with $y^T A \geq 0$ and $y^T b < 0$. 
Any solution of \( Ax \leq b \) is said to be a feasible solution of the linear program (2.1). One that maximizes the linear function \( c^T x \), is said to be an optimal solution. The dual of the linear program (2.1) is

\[
\min \{ y^T b \mid y^T A = c^T, y \geq 0 \}. \tag{2.2}
\]

The dual of the linear program in (2.2) is precisely (2.1); each is a dual of the other. The important Duality theorem of linear programming (proved by Von Neumann in 1947) characterizes optimal solutions of linear programs in terms of dual linear programs.

**Theorem 2.3** (Duality Theorem) If the linear program \( \max \{ c^T x \mid Ax \leq b \} \) and its dual have feasible solutions, then \( \max \{ c^T x \mid Ax \leq b \} = \min \{ y^T b \mid y^T A = c^T, y \geq 0 \} \).

If a linear inequality is valid for all solutions of a linear system \( Ax \leq b \), there is a "proof" of the validity of this inequality. This follows from the affine form of Farkas' Lemma:

**Lemma 2.4** Let \( Ax \leq b \) be a system of inequalities which has at least one solution, and assume that all solutions of \( Ax \leq b \) satisfy \( c^T x \leq d \). Then there is a vector \( y \geq 0 \) such that \( y^T A = c^T \), and \( y^T b = d' \leq d \).

A basic solution of (2.2) is one in which the non-zero components of \( y \) correspond to linearly independent rows of \( A \). This means that if \( Ax \leq b \) is a linear system in \( \mathbb{R}^n \), then a basic solution of (2.2) has at most \( n \) non-zero components.

**Lemma 2.5** (Carathéodory's Theorem). If (2.2) has a feasible solution, then it has a basic feasible solution.

It is well known that if (2.2) has an optimal solution, then it has a basic optimal solution.

Optimal solutions of a primal and dual pair of linear programs satisfy complementary slackness.
Lemma 2.6 Let $x$ and $y$ be solutions of (2.1) and (2.2) respectively. Then $x$ and $y$ are optimal solutions (that is, $c^T x = y^T b$) if and only if $y^T (b - A x) = 0$, i.e., whenever $y_i > 0$, then the corresponding inequality $a_i^T x \leq b_i$ holds with equality, for $i = 1, \ldots, m$.

There are a number of algorithms to solve linear programs. The simplex method of Dantzig, though non-polynomial, is very fast, on the average. Interior-point methods, inspired by Karmarkar's (1984) polynomial-time algorithm, are efficient in both theory and practice; see the book by Wright (1997) for a nice survey.

2.3 Cones and Polyhedra

A polyhedron in $\mathbb{R}^n$ is the set of solutions of a linear inequality system; a bounded polyhedron is called a polytope. If all coefficients of $A x \leq b$ are rational, then $P = \{x \mid A x \leq b\}$ is called a rational polyhedron. We will deal mainly with rational polytopes contained in $Q_n = [0, 1]^n$, the 0-1 cube in $\mathbb{R}^n$. If the dimension is obvious from the context, we denote the 0-1 cube by $Q$. If $S$ is a set contained in $\mathbb{R}^n$, let $S_I$ be the convex hull of integral vectors in $S$ (also called the integer hull). For sets in $Q$, the integral hull is a polytope.

A convex set in $\mathbb{R}^n$ is full-dimensional if it has dimension $n$. Full-dimensional polyhedra have unique descriptions, up to scalar multiplication, in terms of linear inequalities. Let $P = \{x \mid A x \leq b\}$. Every face $F$ of $P$ can be written as $F = \{x \mid A' x = b', A'' x \leq b''\}$, where the linear systems $A' x \leq b'$ and $A'' x \leq b''$ constitute a partition of the inequalities in $A x \leq b$. A facet is a maximal face and has dimension one less than the dimension of $P$, whereas a vertex is a minimal face and has dimension one.

A cone is a set of vectors closed under addition and multiplication by non-negative scalars. A polyhedral cone is the set of solutions of a homogenous system of inequalities
$Ax \geq 0$. Let

$$cone(\{b_1, b_2, \ldots, b_m\}) = \{ \lambda_1 b_1 + \ldots + \lambda_m b_m \mid \lambda_i \geq 0 \text{ and } \lambda_i \in R, i = 1, \ldots, m \}$$

be the cone generated by the vectors $b_1, \ldots, b_m$. A finitely generated cone is one of this form, i.e. generated by a finite set of vectors. Let $cone(S)$ stand for the set of all finite non-negative linear combinations of vectors in $S$. Polyhedral cones are finitely generated and vice versa (Farkas-Minkowski-Weyl Theorem).

Cones need not be closed whereas polyhedra are (and hence also polyhedral cones). If $C$ is a cone, its polar cone (also called dual cone by some authors) is $C^*$, defined by

$$C^* = \{ y \mid y^T x \geq 0 \text{ for all } x \in C \}.$$ 

The polar cone is simply the cone spanned by the vectors defining valid inequalities for $C$. $C^*$ is a closed set. If $C$ is closed, then $C^{**} = C$, otherwise $C^{**}$ contains $C$ and equals the closure of $C$. The polar cone of a linear subspace (which is also a polyhedral cone) is simply its orthogonal complement. For cones of matrices, the polar cone is defined via the associated inner product. The polar cone of the cone of symmetric matrices is the cone of skew-symmetric matrices.

For sets of vectors $A$ and $B$, let $A + B = \{ x + y \mid x \in A, y \in B \}$. If $C_1$ and $C_2$ are cones, then both $C_1 \cap C_2$ and $C_1 + C_2$ are also cones. If $C_1$ and $C_2$ are closed, then $(C_1 \cap C_2)^* = C_1^* + C_2^*$. What this means is that if a cone is defined by two sets of inequalities (e.g. $C_1 \cap C_2$), then the polar cone must be generated by both sets of inequalities. Also $(C_1 + C_2)^* = C_1^* \cap C_2^*$. See Schrijver (1986) for results on polyhedra and Knuth (1990) for simple proofs of facts about polar cones.

### 2.4 Semidefinite matrices

A symmetric matrix $A$ is said to be positive semidefinite if $x^T Ax \geq 0$ for all vectors $x$. If for all $x \neq 0$, $x^T Ax > 0$, then $A$ is said to be positive definite. If a matrix $A$ is positive semidefinite, we denote this by $A \succeq 0$, and $A > 0$ means that $A$ is
positive definite. $A \succeq B$ means that $A - B \succeq 0$. We will refer to the set of positive semidefinite matrices as $S^n_+$. 

There are several equivalent definitions of positive semidefinite matrices; see Horn and Johnson (1985). In what follows, we will assume all matrices are square and symmetric.

**Proposition 2.7** A matrix $A$ is positive semidefinite if and only if any of the following conditions hold:

1. $x^TAx \geq 0$ for all $x$.

2. All eigenvalues of $A$ are non-negative.

3. $A = U^TU$ for some matrix $U$.

The above conditions can be derived using the spectral decomposition of symmetric matrices.

The third condition in Proposition 2.7 means a positive semidefinite matrix can be written as a *Gram matrix*: if $A \succeq 0$ and $A \in R^{n \times n}$, then for some $u_1, \ldots, u_n \in R^n$, $a_{ij} = u_i^Tu_j$ for $i = 1, \ldots, n, j = 1, \ldots, n$. Further, if $A = U^TU$, then $A$ has the same rank as $U$. Also, $A$ can be written as

$$A = \sum_{i=1}^{n} v_i v_i^T, \text{ for some } v_1, \ldots, v_n \in R^n. \tag{2.3}$$

Observe that (2.3) implies the following: if $A \succeq 0$, then the quadratic function $x^TAx$ can be expressed as the sum of $n$ squares of linear functions $v_i^Tx$. Also, if we take the sum of an arbitrary number of rank-one matrices of the form $uu^T$, we can replace this by the sum of $n$ matrices of the form $uu^T$.

We will use various properties of positive semidefinite matrices. A *principal submatrix* of a matrix is a square submatrix obtained by deleting some rows and the corresponding columns from the matrix. If $A \succeq 0$, then
1. Every principal submatrix of $A$ is positive semidefinite and has non-negative determinant.

2. For all $i$, $a_{ii} \geq 0$. If $a_{ii} = 0$ for some $i$, then $a_{ij} = a_{ji} = 0$ for all $j$.

3. If $B$ is non-singular, then $B^T A B \succeq 0$.

Conditions 1 and 3 are also sufficient for a matrix to be positive semidefinite.

If a matrix $A$ has a block-diagonal decomposition, then $A \succeq 0 \iff$ every block is positive semidefinite. For example,

$$\text{if } A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \text{ then } A \succeq 0 \iff A_1 \succeq 0, A_2 \succeq 0. \quad (2.4)$$

**Proposition 2.8** *(Schur complement)* Let $A$ be a non-singular matrix, and let $B$ and $C$ be matrices.

If $D = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$, then $D \succeq 0 \iff C - B^T A^{-1} B \succeq 0$.

It is easy to see that $S_n^+$ is a convex cone; it is defined by the inequalities $A \bullet xx^T \succeq 0$, for all $x$. It is well known that

**Proposition 2.9** The cone $S_n^+$ of positive semidefinite matrices is pointed and full-dimensional, and it is self dual, that is $(S_n^+)^* = S_n^+$.

The fact that $S_n^+$ is self-dual follows from the fact (Fejer’s Theorem) that $A$ is positive semidefinite iff $A \bullet B \succeq 0$ for all positive semidefinite matrices $B$.

A positive semidefinite matrix has a *Cholesky factorization*; if $A \succeq 0$, then $A = LL^T$, for some lower triangular matrix $L$. This can be computed in $O(n^3)$ arithmetic operations; if $A$ is rational, the Cholesky factorization can be obtained in polynomial time and yields a polynomial-time test for positive semidefiniteness.

Positive definite matrices can be characterized in a manner similar to Proposition 2.7.
Proposition 2.10 A matrix $A$ is positive definite if and only if any of the following conditions hold:

1. $x^T A x > 0$ for all $x \neq 0$.

2. All eigenvalues of $A$ are positive.

3. $A = U^T U$ for some matrix non-singular matrix $U$.

2.5 Semidefinite programming

A semidefinite program is a problem of the form

$$\begin{align*}
\min & \quad C \cdot X \\
\text{s.t.} & \quad A_i \cdot X = b_i, \text{ for } i = 1, \ldots, m, \\
& \quad X \succeq 0.
\end{align*} \tag{2.5}$$

Here $C, A_i$ and $X$ are $n \times n$ matrices (and s.t. stands for "subject to"). Thus, the problem of minimizing a linear function over the intersection of an affine subspace with the cone of positive semidefinite matrices is called a semidefinite program. A linear program is a special type of semidefinite program: the off-diagonal variables in $X$ in (2.5) are set to zero and then $X \succeq 0 \iff x_{ii} \geq 0$, for all $i$.

As in linear programming, (2.5) has a dual semidefinite program

$$\begin{align*}
\max & \quad \sum_{i=1}^{m} y_i b_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} y_i A_i \preceq C, \tag{2.6}
\end{align*}$$

and the dual of (2.6) is precisely (2.5). However duality in semidefinite programs is more complicated than in linear programming. If $X$ and $y$ are feasible solutions of (2.5) and (2.6), then

$$C \cdot X \geq \sum_{i=1}^{m} y_i b_i.$$
We can infer this from the fact that \(0 \leq (C - \sum_i y_i A_i) \cdot X = C \cdot X - \sum_i y_i b_i\). If \(v_p\) and \(v_d\) are the optimal values of the primal and dual semidefinite programs, then

\[v_p \geq v_d,\]

and this phenomenon is called \textit{weak duality}.

We need to be careful here. Even if a semidefinite program is feasible and the objective function is bounded, there may not be a solution which attains the minimum. For example consider

\[
\begin{align*}
\min & \quad x_1 \\
\text{s.t.} & \quad \begin{pmatrix} x_1 & 1 \\ 1 & x_2 \end{pmatrix} \succeq 0.
\end{align*}
\]

In this example, the semidefiniteness constraint is equivalent to \(x_1 \geq 0, x_2 \geq 0,\) and \(x_1 x_2 \geq 1\). This means that the infimum of the objective function is 0, but this value is not attained, as \(x_1 > 0,\) for all feasible solutions. We will therefore assume that \(v_p\) and \(v_d\) stand for the infimum and supremum of the associated semidefinite programs, in place of the minimum and maximum. In the example (see Helmberg 2000)

\[
\begin{align*}
\min & \quad x_1 \\
\text{s.t.} & \quad \begin{pmatrix} 0 & x_1 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & x_1 + 1 \end{pmatrix} \succeq 0,
\end{align*}
\]

both \(v_p\) and \(v_d\) are attained, but \(v_p = 0,\) while \(v_d = -1\).

To guarantee the existence of solutions attaining \(v_p\) and \(v_d\) and satisfying \(v_p = v_d,\) (this is called \textit{strong duality}) additional assumptions are required. A \textit{strictly feasible} solution of (2.5) is a feasible solution \(X\) which is positive definite.

\textbf{Theorem 2.11} Assume both (2.5) and (2.6) have feasible solutions and assume that the primal program has a strictly feasible solution. Then \(v_p = v_d\) and the dual optimum is attained. If both the primal and dual programs have strictly feasible solutions, then the primal and dual optima are both attained and are equal.
A short proof can be found in the lecture notes by Lovász (2000). As in linear programming, semidefinite programs which have primal and dual optimal solutions have complementary slackness properties.

Unlike linear programs, semidefinite programs typically cannot be solved exactly, in polynomial time. This can happen for a number of reasons. Firstly, even if the input data is rational, the solution can be irrational. For example, the optimal solution of

\[
\begin{align*}
\text{max } & \quad x \\
\text{s.t. } & \quad \begin{pmatrix} 1 & x \\ x & 2 \end{pmatrix} \succeq 0
\end{align*}
\]

is \(x = \sqrt{2}\) as the positive semidefiniteness condition is equivalent to \(x^2 \leq 2\). Secondly, the optimal solution might require an exponential number of bits to write down. Consider the following example of Khachiyan (given in Ramana 1997):

\[
\begin{align*}
\text{min } & \quad x_n \\
\text{s.t. } & \quad x_1 \geq 2 \\
& \quad \begin{pmatrix} 1 & x_i \\ x_i & x_{i+1} \end{pmatrix} \succeq 0, \text{ for } i = 1, \ldots, n - 1.
\end{align*}
\]

In this example, the constraints (besides the first one) are equivalent to \(x_{i+1} \geq x_i^2\), which means that the optimum value is \(2^n\). Hence \(2^n\) bits are required to write down the solution but about \(O(n)\) bits are required to represent the problem. See Ramana (1997) and Porkoláb and Khachiyan (1997) for more information on complexity issues.

Semidefinite programs can however be solved to within a prescribed error tolerance, as long as some additional information, such as bounds on the sizes of optimal solutions, is provided. The first polynomial-time algorithm was given by Grötschel, Lovász and Schrijver (1988) and was based on the use of the ellipsoid method, and the equivalence of weak separation and weak optimization. See the book cited above for more details. This is not a practical method. More efficient interior-point methods were given by Nesterov and Nemirovskii (1994), Alizadeh (1995) and others. See Section 6.1.1 for additional information on semidefinite programming algorithms.
Chapter 3

The matrix-cut operators

3.1 Introduction

Various methods to derive cutting planes for mixed integer programs and pure integer programs have been described in the literature (see the paper by Cornuéjols and Li (2001) for a comparison of many of these methods). Most of these cutting planes, including Gomory-Chvátal cuts, can be specialized to 0-1 systems. Lovász and Schrijver (1991) introduced a set of techniques to derive cutting planes, called matrix cuts, for 0-1 vectors in polyhedra. A common theme behind the different types of matrix cuts is the use of quadratic inequalities. We introduce the derivation of matrix cuts through an example.

![Figure 3.1: P](image)

Example 1

Consider the polytope $P$ (see Figure 3.1) in the two-dimensional 0-1 cube defined by the inequalities

\[
1.5 - x_1 - x_2 \geq 0,
\]

\[
0.5 + x_1 - x_2 \geq 0.
\]
\[ x_1 \geq 0, 1 - x_1 \geq 0. \]
\[ x_2 \geq 0, 1 - x_2 \geq 0. \]

**Step 1. Multiply** (by \( x_i \geq 0 \) and \( 1 - x_i \geq 0 \)): As \( 1.5 - x_1 - x_2 \) and \( x_1 \) are non-negative for every point in \( P \), the quadratic inequality \((1.5 - x_1 - x_2)x_1 \geq 0\) is valid for \( P \). Similarly, the following inequalities are valid for \( P \):

\[(1.5 - x_1 - x_2)x_1 \geq 0.\]
\[(0.5 + x_1 - x_2)x_2 \geq 0, \quad (3.2)\]
\[(1.5 - x_1 - x_2)x_2 \geq 0.\]
\[(0.5 + x_1 - x_2)(1 - x_1) \geq 0.\]

**Step 2. Replace** \( (x_i^2 \text{ by } x_i) \): All 0-1 points in \( Q \) satisfy \( x_1^2 = x_1 \) and \( x_2^2 = x_2 \). Replacing \( x_1^2 \) by \( x_1 \) in the first inequality in (3.2), \( 1.5x_1 - x_1^2 - x_1x_2 \geq 0 \), we infer that \( 0.5x_1 - x_1x_2 \geq 0 \) is satisfied by all 0-1 points in \( P \). Observe that replacing \( x_1^2 \) by \( x_1 \) is equivalent to adding appropriate multiples (positive or negative) of the function \( x_1^2 - x_1 \). In a similar manner, from (3.2) we get, respectively,

\[ 0.5x_1 - x_1x_2 \geq 0, \]
\[ -0.5x_2 + x_1x_2 \geq 0, \quad (3.3)\]
\[ 0.5x_2 - x_1x_2 \geq 0, \]
\[ 0.5 - 0.5x_1 - x_2 + x_1x_2 \geq 0. \]

which are valid for 0-1 points in \( P \) (but not necessarily for \( P_i \), as they are non-linear).

**Step 3. Add** (valid inequalities): Adding the first two inequalities in (3.3) and the third and the fourth, the quadratic terms cancel out, and we get the cutting planes

\[ x_1 - x_2 \geq 0, \quad (3.4) \]
1 - x_1 - x_2 \geq 0

which are valid for \( P_I \) but not for \( P \).

Such cutting planes are called \( N \)-cuts. The set of points satisfying all \( N \)-cuts for \( P \) is called \( N(P) \). We will see later that \( N(P) \) is always a polytope; thus a finite number of \( N \)-cuts imply the rest. In the example above, the inequalities in (3.4) define \( N(P) \); see Figure 3.2.

We can iterate this process and derive new cuts from (3.4). Multiplying these two inequalities by \( x_2 \geq 0 \) as in Step 1, replacing \( x_2^2 \) by \( x_2 \) as in Step 2, and applying Step 3 to add the resulting inequalities, we obtain \(-x_2 \geq 0 \) or \( x_2 \leq 0 \). Therefore \( x_2 \leq 0 \) is an \( N \)-cut for \( N(P) \) and is valid for \( P_I \) (along with the bounds on the variables, it defines \( P_I \)). The inequalities in (3.1), (3.4) and the inequality \( x_2 \leq 0 \), arranged in a sequence, along with the derivation of each from the previous ones, constitute a type of \textit{cutting-plane proof} of \( x_2 \leq 0 \). We will formally define such proofs in Section 7.2.

An alternative way of iterating the process of obtaining \( N \)-cuts, is to derive all \( N \)-cuts valid for 0-1 points in \( N(P) \) and obtain \( N(N(P)) \) and so on. For \( P \) in Example 1, \( x_2 \leq 0 \) is an \( N \)-cut for the polytope \( N(P) \), and \( N(N(P)) = P_I \). Such a result (that two iterations yield the integral hull) holds for all polytopes in two dimensions.

Figure 3.2 : (a) \( P \), (b) \( N(P) \), (c) \( N(N(P)) = P_I \)

Another class of cutting planes, called \( N_+ \)-cuts, was also defined by Lovász and Schrijver (1991). The set of points in a polytope \( P \) which satisfy all \( N_+ \)-cuts is called \( N_+(P) \). The class of \( N_+ \)-cuts includes all \( N \)-cuts and, in some cases, many other cuts besides. Consider Example 1, and the inequalities obtained in (3.2). Squaring the
linear function $0.5 + x_1 - x_2$, we get an identically non-negative quadratic function. Thus
\[(\frac{1}{2} + x_1 - x_2)^2 = \frac{1}{4} + x_1^2 + x_2^2 + x_1 - x_2 - 2x_1x_2 \geq 0\]
holds for all $x \in R^2$. Replacing $x_1^2$ and $x_2^2$ as in Step 2 above, we see that
\[\frac{1}{4} + 2x_1 - 2x_1x_2 \geq 0\]
is valid for 0-1 vectors in $R^2$. Adding this to two times the second inequality in (3.3), we get a linear inequality valid for $P_i$, namely
\[\frac{1}{4} + 2x_1 - x_2 \geq 0.\] (3.5)

Notice that some points in $P$, e.g. $(0,1/2)$, do not satisfy this inequality (see Figure 3.3). In this example it turns out that all $N_+\text{-}cuts$ are implied by the $N\text{-}cuts$ in (3.4); hence $N_+(P) = N(P)$.

![Figure 3.3: The $N_+\text{-}cut \ \frac{1}{4} + 2x_1 - x_2 \geq 0$](image)

### 3.2 Matrix cuts and associated operators

In this section we formally define the different types of matrix-cuts of Lovász and Schrijver and also the elementary closures associated with these cuts. Let $Q_n$ be the 0-1 cube in $R^n$, that is $Q_n = [0,1]^n$. If the dimension is obvious from the context, we denote the 0-1 cube by $Q$. Let $P$ be a polytope defined by

\[P = \{x \in Q \mid a_i^T x \leq b_i, \ i = 1, \ldots, m\},\] (3.6)
where $a_i^T x \leq b_i \ (i = 1, \ldots, m)$ is a system of linear inequalities in $R^n$. We denote this system by $Ax \leq b$. We assume, for the sake of convenience, that the inequalities $0 \leq x_j \leq 1 \ (j = 1, \ldots, n)$ are included in the system $Ax \leq b$. As in the previous section, we conclude (by multiplication) that the inequalities

\begin{equation}
(b_i - a_i^T x) x_j \geq 0,\\
(b_i - a_i^T x)(1 - x_j) \geq 0, \text{ for } i = 1, \ldots, m, j = 1, \ldots, n,
\end{equation}

are valid for $P$; also, 0-1 vectors (and no others) in $P$ satisfy the equations

\begin{equation}
x_j^2 - x_j = 0, \text{ for } j = 1, \ldots, n.
\end{equation}

An inequality $c^T x \leq d$ or $d - c^T x \geq 0$ is called an $N(P, Q)$ cutting plane if

\begin{equation}
d - c^T x = \sum_{i,j} \alpha_{ij} (b_i - a_i^T x)x_j + \sum_{i,j} \beta_{ij} (b_i - a_i^T x)(1 - x_j) + \sum_j \lambda_j (x_j^2 - x_j),
\end{equation}

where $\alpha_{ij}, \beta_{ij} \geq 0$ for $i = 1, \ldots, m, j = 1, \ldots, n$ and $\lambda_j$ is a real number, for $j = 1, \ldots, n$. For shortness, we refer to an $N(P, Q)$ cutting plane as an $N$-cut. Adding the terms in (3.8) is a crucial ingredient in the derivation of $N$-cuts (this achieves the same purpose as replacing $x_j^2$ by $x_j$). The first two sets of terms in (3.9) can only yield linear inequalities satisfied by all vectors in $P$.

In (3.9), we take products of inequalities defining $P$ and $Q$; hence the name $N(P, Q)$ cutting plane. We could instead, in (3.9), consider products of inequalities defining $P$ and any polytope containing $P$, to get valid cutting planes for $P$; if we let the second polytope be $P$ instead of $Q$, we get $N(P, P)$ cutting planes. A weakening of $N(P, Q)$-cuts, called $N_0(P, Q)$-cuts, can be obtained if in (3.9) we insist that $x_i x_j$ and $x_j x_i$ are distinct terms, for all $i, j$ with $i \neq j$. That is, we are not allowed to combine $x_i x_j$ and $x_j x_i$ terms. Though this definition seems less intuitive, we will see later on that there is a nice geometric interpretation of such cuts.
An inequality $c^T x \leq d$ or $d - c^T x \geq 0$ is called an $N_+(P, Q)$ cutting plane (also called an $N_+$-cut) if

$$d - c^T x = \sum_{i,j} \alpha_{ij} (b_i - a_i^T x)x_j + \sum_{i,j} \beta_{ij} (b_i - a_i^T x)(1 - x_j) + \sum_j \lambda_j (x_j^2 - x_j) + \sum_k (g_k + h_k^T x)^2,$$

(3.10)

where $\alpha_{ij}, \beta_{ij} \geq 0$, $\lambda_j$ is a real number, for $i = 1, \ldots, m$, $j = 1, \ldots, n$, and $g_k + h_k^T x$ a linear function, for $k = 1, \ldots, n + 1$. One can also define $N_+(P, P)$ cutting planes. Observe that in (3.10) we have only $n + 1$ terms $(g_k + h_k^T x)^2$. This is because the sum of squares of an arbitrary number of linear functions in $\mathbb{R}^n$ is identically equal to the sum of squares of $n$ linear functions (see (2.3) and the discussion that follows).

Note that, in both (3.9) and (3.10), we take combinations of the $2mn$ inequalities in (3.7). We can use a smaller number, bounded above by $n^2$, of inequalities, to get the same cuts (see Lemma 3.3). Indeed, to define $N$-cuts and $N_+$-cuts for closed convex sets, a bound on the number of inequalities in (say) (3.9), which is independent of $m$, is required. This is because a convex set may require an infinite number of non-redundant linear inequalities to define it.

The sets $N(P, Q)$ and $N_+(P, Q)$ (abbreviated as $N(P)$ and $N_+(P)$) are defined as follows:

$$N(P, Q) \text{ is the set of points satisfying all } N(P, Q) \text{ cutting planes},$$

(3.11)

$$N_+(P, Q) \text{ is the set of points satisfying all } N_+(P, Q) \text{ cutting planes},$$

(3.12)

$N_0(P, Q)$ (or $N_0(P)$), $N(P, P)$ and $N_+(P, P)$ are defined in a similar manner in terms of the corresponding cutting planes. We refer to $N, N_0$ and $N_+$ as, respectively, the commutative, non-commutative and semidefinite operators and collectively as the matrix-cut operators. $N(P)$ is also defined in Sherali and Adams (1990), but used
in a different setting. $N_0(P)$ is actually defined in Lovász and Schrijver (1991) via a geometric characterization; see Lemma 3.9.

As discussed above, all 0-1 vectors in $P$ satisfy all $N$-cuts and $N_+$-cuts; it is clear from definitions (3.9) and (3.10) that every $N$-cut is also an $N_+$-cut. Further, since

$$b_i - a_i^T x = (b_i - a_i^T x)x_j + (b_i - a_i^T x)(1 - x_j)$$

for any $x_j$, the inequalities defining $P$ are also $N$-cuts (and $N_0$-cuts) for $P$. Hence, the relationship (Lemma 1.1 in Lovász and Schrijver 1991)

$$P_I \subseteq N_+(P) \subseteq N(P) \subseteq N_0(P) \subseteq P$$ (3.13)

holds for any polytope $P \subseteq Q$.

The matrix-cut operators can be iterated. Let $N^0(P) = P$ and $N^{t+1}(P) = N(N^t(P))$ if $t$ is a non-negative integer. Lovász and Schrijver (1991) proved the following important result.

**Theorem 3.1** Let $P \subseteq Q_n$ be a polytope. Then $N^n(P) = P_I$. □

(This also follows from a result of Balas (1975); see Section 3.4). Moreover, Lovász and Schrijver showed that for any fixed value of $t$, it is possible to optimize linear functions over $N^t(P)$ in polynomial time (see [92] for a precise statement). Identical results hold for the $N_0$ and $N_+$ operators; we can also replace polytopes by closed convex sets in $Q_n$. For a proof of Theorem 3.1 see Theorem 3.10; there we prove a slightly stronger statement.

Theorem 3.1, along with the algorithmic tractability of $N(P)$ and $N_+(P)$, provides the major motivation for studying the $N$ and $N_+$ operators. In contrast to the matrix-cut operators, the optimization problem for $P'$ is NP-hard in general (Eisenbrand 1999). However, the Chvátal operator $P'$ can also be iterated to obtain the integral hull of polytopes in $Q$. Let $P^{(t)}$ stand for $P'$ iterated $t$ times. The smallest number $t$ for which $P^{(t)} = P_I$ is called the Chvátal rank of $P$. Bockmayr and Eisenbrand
(1997) proved that polytopes in $Q$ have Chvátal rank at most $6n^3 \log n$; this has been improved to $3n^2 \log n$ by Eisenbrand and Schulz (1999).

The main reason we are interested in the $N_0$ operator is it is easier to visualize. In certain cases the $N_0$ operator is identical to the other two; in such cases we can visualize the effect of the other operators.

We follow Lipták (1999) and define the non-commutative rank of a polytope $P$ to be the least integer $t \geq 0$ such that $N_0^t(P) = P_t$. The commutative rank (also in [87]) and semidefinite rank are defined analogously for the $N$ and $N_+$ operators respectively. Given an inequality $c^T x \leq d$ valid for $P_t$, it’s depth with respect to the $N_0$ operator is the least non-negative integer $t$ such that $c^T x \leq d$ is valid for $N_0^t(P)$; we will sometimes abbreviate this and refer to the $N_0$-depth of $c^T x \leq d$. The terms $N$-depth and $N_+$-depth are similarly defined.

3.3 Dual characterizations

The operators $N(P)$ and $N_+(P)$ have alternative “dual” characterizations which are often easier to handle. They involve expressing the sets $N(P)$ and $N_+(P)$ as projections of certain higher-dimensional sets, which are derived from $P$.

First observe that in (3.9) and (3.10), the quadratic terms cancel out in obtaining the final linear expression. Thus the coefficients of $x_ix_j$ and $x_i^2$ (after being scaled by the appropriate multipliers) add up to zero. If $d - c^T x \geq 0$ is an $N$-cut, replacing every $x_ix_j$ by $y_{ij}$ and $x_i^2$ by $y_{ii}$ in (3.9), the $y_{ij}$ and $y_{ii}$ terms cancel out to yield the same $N$-cut.

Let $F$ be the set of expressions, linear in $x_j$ and $y_{ij}$, obtained by performing the above replacements in (3.7) and (3.8). Let us rewrite the inequalities in $F$ as $Ex + Fy \leq g$, (here $y$ is a vector with components $y_{ij}$, $i = 1, \ldots, n$, $j = i + 1, \ldots, n$). Then

$$c^T x \leq d \text{ is an } N \text{-cut } \iff \exists z \geq 0 \text{ such that } z^T E = c, z^T F = 0, z^T g = d.$$
Finally we can express $N(P)$ as follows:

$$N(P) = \{ x \mid \exists y \text{ such that } (x, y) \text{ satisfies } Ex + Fy \leq g \}. $$

This definition is equivalent to (3.11). Let $(x', y')$ satisfy $Ex' + Fy' \leq g$. It follows that

$$\text{if } z^TF = 0 \text{ with } z \geq 0, \text{ then } z^TEx' = z^T(Ex' + Fy') \leq z^Tg.$$  

Therefore $x'$ satisfies all $N$-cuts for $P$. On the other hand, let $x'$ belong to $N(P)$, and assume there is no $y$ which solves $Ex' + Fy \leq g$ or $Fy \leq g - Ex'$. Then, by Farkas' Lemma,

$$\exists z \geq 0 \text{ such that } z^TF = 0, \text{ and } z^T(g - Ex') < 0.$$  

This implies that $z^TEx' > z^Tg$ and the $N$-cut $z^TEx \leq z^Tg$ is violated by $x'$. This contradicts the assumption that $x' \in N(P)$.

Therefore $N(P)$ can be expressed as the projection of the set of vectors $(x, y)$ satisfying $Ex + Fy \leq g$ and is a polytope. To formalize this notion, we need a few definitions. For $x \in \mathbb{R}^n$, let

$$\bar{x} = \begin{pmatrix} 1 \\ x \end{pmatrix} \in \mathbb{R}^{n+1}.$$ We refer to the additional coordinate as the 0th coordinate; thus $\bar{x}_0 = 1$. The polytope $P$ in (3.6) can be expressed as

$$P = \{ x \mid u_i^T\bar{x} \geq 0, \text{ for } i = 1, \ldots, n \} \text{ where } u_i = \begin{pmatrix} b_i \\ -a_i \end{pmatrix}. \tag{3.14}$$

(We call $u_i^T\bar{x} \geq 0$ the homogenized form of $a_i^Tx \leq b_i$).

A convenient way of representing a quadratic inequality is to arrange its coefficients in a matrix. Given $q(x) \geq 0$, where $x \in \mathbb{R}^n$, we express it as

$$\bar{x}^TQ\bar{x} \geq 0 \text{ or } Q \cdot \bar{x}^T \geq 0$$

for some $Q \in \mathbb{R}^{(n+1)\times(n+1)}$. The coefficient of $x_ix_j$ in $q(x)$, for $i, j > 0$, is $Q_{ij} + Q_{ji}$ and the coefficient of $x_i$ is $Q_{ii} + Q_{1i}$; the constant term is obviously $Q_{00}$. We say that
$Q$ represents the quadratic function $q(x)$. $Q$ is not uniquely defined. If $Q' = Q + A$, where $A$ is any skew-symmetric matrix, then $Q'_{ij} + Q'_{ji} = Q_{ij} + Q_{ji}$. Let $Q^g$ be the unique symmetric matrix representing $q(x)$. Any matrix in the affine subspace

$$\{ Q^g + A \mid A \text{ is a skew-symmetric matrix} \}$$

also represents $q(x)$.

Now let the $i$th unit vector in $R^{(n+1)}$ be $e_i$ and let $f_i \in R^{(n+1)}$ stand for $e_0 - e_i$. The quadratic inequalities $(b_i - a_i^T x)x_j \geq 0$ in (3.7) can be written, using the homogenized form in (3.14), as

$$(u_i^T \bar{x})(\bar{x}^T e_j) \geq 0 \text{ or } u_i e_j^T \cdot \bar{x}^T \geq 0.$$ 

As in the definition of $F$ above, we replace the $x_i x_j$ terms by $y_{ij}$ terms. In fact, we substitute $\bar{x}^T \bar{x}$ with the matrix $Y = (y_{ij})$; then $y_{ij} = x_i x_j$ and $y_0 = y_0 = x_i$ for $i = 1, \ldots, n$. The equations in (3.7) and (3.8) become

\begin{align*}
    u_i e_j^T \cdot Y & \geq 0. \\
    u_i (e_0 - e_j)^T \cdot Y & \geq 0, \\
    y_0 = y_0 = y_i & \text{ for } i = 1, \ldots, n.
\end{align*}

for $i = 1, \ldots, m, j = 1, \ldots, n$. As $x_i x_j$ is the same as $x_j x_i$, we add the condition

$$Y \text{ is symmetric.}$$

Note that (3.17) is the same as

$$e_j (e_0 - e_j)^T \cdot Y = (e_0 - e_j) e_j^T \cdot Y = 0,$$

and (3.18) is equivalent to the condition

$$A \cdot Y = 0 \text{ for every skew-symmetric matrix } A.$$ 

A linear inequality $w^T \bar{x} \geq 0$ can be written as $we_0^T \cdot \bar{x}^T \geq 0$. From the discussion above, it follows that $w^T \bar{x} \geq 0$ is an $N$-cut if and only if

$$we_0^T = \sum_{i,j} \alpha_{ij} u_i e_j^T + \sum_{i,j} \beta_{ij} u_i (e_0 - e_j)^T + \sum_{j} \lambda_j e_j (e_0 - e_j)^T + A,$$

(3.19)
where \(\alpha_{ij}, \beta_{ij},\) and \(\lambda_j\) are as in (3.9) and the matrix \(A\) is skew-symmetric.

To derive a similar expression for \(N_+\)-cuts, we need a concise way of expressing sums of squares of linear functions. Now, the terms \((g_j + h_j^T x)^2\) in (3.10) can be expressed as \((v_j^T x)^2\) or \(v_j v_j^T \cdot x x^T\), where \(v_j^T = (g_j, h_j^T)\). After replacing \(x x^T\) by \(Y\), the inequalities \((g_j + h_j^T x)^2 \geq 0\) are translated to

\[
v_j v_j^T \cdot Y \geq 0 \quad \text{for all } v \in \mathbb{R}^{n+1}. \tag{3.20}
\]

Condition (3.20) is equivalent to

\[
Y \text{ is positive semidefinite.} \tag{3.21}
\]

Finally, \(w^T x \geq 0\) is an \(N_+\)-cut if and only if

\[
w e_0^T = \sum_{i,j} \alpha_{ij} u_i e_j^T + \sum_{i,j} \beta_{ij} u_i (e_0 - e_j)^T + \sum_j \lambda_j e_j (e_0 - e_j)^T + A + B, \tag{3.22}
\]

where the matrix \(B\) is symmetric positive semidefinite and the rest is as in (3.19). A direct translation of (3.10) would result in having \(\sum v_j v_j^T\), for some \(n + 1\) vectors \(v_j\), in place of \(B\). But as a matrix \(B\) is positive semidefinite if and only if \(B = \sum v_j v_j^T\) for some vectors \(v_j\), the replacement is justified.

Consider the sets of matrices derived from the polytope \(P\):

\[
M(P) = \{ Y \in \mathbb{R}^{(n+1) \times (n+1)} | Y \text{ satisfies conditions (3.15) -- (3.18)} \}. \tag{3.23}
\]

\[
M_0(P) = \{ Y \in \mathbb{R}^{(n+1) \times (n+1)} | Y \text{ satisfies conditions (3.15) -- (3.17)} \}, \tag{3.24}
\]

\[
M_+(P) = \{ Y \in \mathbb{R}^{(n+1) \times (n+1)} | Y \text{ satisfies conditions (3.15) -- (3.18), (3.20)} \}. \tag{3.25}
\]

Each of the above sets is a cone. If \(P\) is a polytope, \(M(P)\) and \(M_0(P)\) are polyhedral cones; the conditions on the matrices \(Y\) in (3.23) and (3.24) can be expressed as linear inequalities over the elements of \(Y\). Since the cone of positive semidefinite matrices is not polyhedral, \(M_+(P)\) is generally not polyhedral.

\(M(P)^*\), the polar cone of \(M(P)\), is precisely the set of matrices having the same form as the right hand side of (3.19). Similarly \(M_+(P)^*\) is defined by the right hand
side of (3.22). Therefore \( w^T \bar{x} \geq 0 \) is an \( N \)-cut if and only if \( we_0^T \) belongs to \( M(P)^* \). A similar statement holds for \( N_+ \)-cuts.

Finally, we can write down the "dual" characterizations of the matrix-cut operators (given in Lovász (1992) or Lovász and Schrijver (1991)).

**Lemma 3.2** Let \( P \subseteq Q \) be a polytope. Then

\[
N(P) = \{ x \mid \bar{x} = Y e_0, \ \text{where} \ Y \in M(P) \}, \tag{3.26}
\]

\[
N_0(P) = \{ x \mid \bar{x} = Y e_0, \ \text{where} \ Y \in M_0(P) \}, \tag{3.27}
\]

\[
N_+(P) = \{ x \mid \bar{x} = Y e_0, \ \text{where} \ Y \in M_+(P) \}. \tag{3.28}
\]

**Proof:** We prove the result for \( N_+(P) \); the other cases can be handled similarly. For some \( x \in Q \), assume there is a matrix \( Y \in M_+(P) \) such that \( \bar{x} = Y e_0 \). Given an \( N_+ \)-cut \( w^T \bar{x} \geq 0 \), \( we_0^T \in M_+(P)^* \). Hence \( we_0^T \bullet Y \geq 0 \) or \( w^T Y e_0 = w^T \bar{x} \geq 0 \). On the other hand if \( x \in N_+(P) \), but \( \bar{x} \) does not belong to the cone \( M_+(P)e_0 \), then there is a vector \( w \in \mathbb{R}^{n+1} \) separating \( \bar{x} \) from \( M_+(P)e_0 \). Hence \( w^T Y e_0 \geq 0 \) for all \( Y \in M_+(P) \) and \( w^T \bar{x} < 0 \). But this means that \( x \) violates the \( N_+ \)-cut \( w^T \bar{x} \geq 0 \). \( \square \)

We note that in the derivation of the matrix-cut operators, \( P \) is first mapped to a higher-dimensional body by mapping each vector \( x \) to a matrix \( \bar{x}x^T \) and then replacing \( \bar{x}x^T \) by \( Y \). Then certain "cuts" are applied to this set of matrices to obtain \( M(P), M_0(P) \) etc. and the resulting system is projected down to the original space to yield a convex set which is a better approximation to the integer hull than the original polytope; hence the name "matrix-cut operators". Observe that if \( P \) is a polytope then \( N_0(P) \) and \( N(P) \) are also polytopes.

We take another look at the polytope \( P \) in Example 1 (see Figure 3.1). Let the polytope defined by (3.4) be \( P_N \); because the inequalities defining \( P_N \) are \( N \)-cuts,
\( N_+(P) \subseteq N(P) \subseteq P_N \). One can verify that the matrix \( Y \) defined by

\[
Y = \begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{4} \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{2}
\end{pmatrix}
\]

belongs to \( M(P) \). Further \( Y \) is positive semidefinite. Hence \( Y \in M_+(P) \). This implies that \( \left( \frac{1}{2}, \frac{1}{2} \right) \in N_+(P) \). Also, the points \( (0, 1) \) and \( (1, 0) \) belong to \( P_l \subseteq N_+(P) \). It follows that \( P_N \subseteq N_+(P) \) (as \( N_+(P) \) is convex) and therefore \( P_N = N_+(P) = N(P) \).

The following lemma is necessary to define \( N \)-cuts and \( N_+ \)-cuts for closed convex sets. We will also use it in cutting-plane proofs. If \( u_i \) and \( v_j \) are defined as in (3.19) and (3.22), let \( q_{ij}(x) \) be the quadratic function obtained from \( (u_i^T \mathbf{x})(v_j^T \mathbf{x}) \) by replacing terms of the form \( x_k^2 \) by \( x_k \).

**Lemma 3.3** If \( c^T x \leq d \) is an \( N \)-cut derived as in (3.9), then \( c^T x \leq d \) can be obtained with at most \( \frac{1}{2}n(n + 1) + 1 \) of the numbers \( \alpha_{ij} \) and \( \beta_{ij} \) nonzero. If \( N(P) \neq \emptyset \), then at most \( \frac{1}{2}n(n + 1) \) of these numbers can be assumed to be non-zero.

**Proof:** Let \( p_{ij} \) and \( q_{ij} \) be symmetric matrices representing the quadratic functions obtained after replacing \( x_j^2 \) by \( x_j \) in the functions represented by \( u_i e_j^T \) and \( u_i (e_0 - e_j)^T \) in equation (3.19). Any \( N \)-cut obtained in (3.19) can be derived with non-negative combinations of the matrices \( p_{ij} \) and \( q_{ij} \). As these are contained in an \( \frac{1}{2}n(n + 1) + 1 \) dimensional space, Carathéodory’s Theorem implies the first part of the lemma. If \( N(P) \) is non-empty, then there is a non-zero matrix \( Y \in M(P) \) such that \( Y \cdot p_{ij} = 0 \) and \( Y \cdot q_{ij} = 0 \). The second part of the lemma follows. \( \Box \)

A similar result holds for \( N_+ \)-cuts. Note that even if \( N(P) = \emptyset \), any inequality valid for \( N(P) \) (that is, every inequality) can be derived from \( P \), not just the inequality \( 0^T x \leq -1 \).
3.4 Related work and variations

Sherali and Adams (1990) described a sequence of operators \( \overline{N}(P) \supseteq \overline{N}^2(P) \supseteq \ldots \), such that \( \overline{N}^n(P) = P_I \), for polytopes in \( Q_n \). Their idea is as follows. Fix some \( r \) between 1 and \( n \). Consider all the terms of the form

\[
P_{S_1,S_2}(x) = x_{i_1} x_{i_2} \cdots x_{i_k} (1 - x_{j_1}) (1 - x_{j_2}) \cdots (1 - x_{j_l}),
\]

where \( i_1, \ldots, i_k \in S_1 \) and \( j_1, \ldots, j_l \in S_2 \) and \( S_1, S_2 \) are disjoint with \( |S_1| + |S_2| = r \). The polynomials \( P_{S_1,S_2}(x) \) are non-negative for all \( x \) in \( Q \). As in (3.9), multiply inequalities defining \( P \) with \( P_{S_1,S_2}(x) \), replace \( x_j^2 \) by \( x_j \), and cancel non-linear terms by non-negative linear combinations to get valid inequalities for \( P_I \). \( \overline{N}^r(P) \) is the polytope defined by these inequalities. The commutative operator is the same as \( \overline{N}^1(P) \).

The disjunctive cuts of Balas (1979) can be specialized to 0-1 integer programming problems and to disjunctions of the form \( P \cap \{x_k \leq 0\} \) and \( P \cap \{x_k \geq 1\} \). In this case we shall call them simple disjunctive cuts (see [39]). Balas, Ceria and Cornuéjols (1993) define a lift-and-project operator related to simple disjunctive cuts. The lift-and-project operator \( P_k \) is obtained by choosing \( P_{S_1,S_2}(x) \) with \( S_1 \) and \( S_2 \) restricted to containing a fixed index \( k \), and \( |S_1| + |S_2| = 1 \). For any \( k \),

\[
P_k = \text{conv}((P \cap \{x_k = 0\}) \cup (P \cap \{x_k = 1\})).
\]

Linear inequalities valid for any \( P_k \) are simple disjunctive cuts. These ideas are related to \( N_0(P) \); see Lemma 3.9. Balas also proved that, if \( i_1, i_2, \ldots, i_n \) is any permutation of \( \{1, \ldots, n\} \), then

\[
(\cdots((P_{i_1}i_2)\cdots)i_n) = P_I,
\]

where \( P \) is a polytope in \( Q_n \). Theorem 3.1 is a consequence of this result.

The semidefinite operator is related to the theta function \( \vartheta(G) \) of Lovász (1979). This is defined in terms of orthogonal labellings of graphs. Given a graph \( G \) on \( n \) nodes, with edge set \( E \), a set of vectors \( v_1, v_2, \ldots, v_n \in R^k \) (\( k \leq n \)) defines an
orthogonal labelling of $G$ if

$$ij \in E \Rightarrow v_i^Tv_j = 0.$$  

Note that if a 0-1 vector $x$ is the incidence vector of a stable set, then

$$ij \in E \Rightarrow x_i x_j = 0.$$

A node variable is mapped to a vector in some $k$-dimensional space and the quadratic equation $x_i x_j = 0$ is transformed into a vector equation.

The theta body $TH(G)$ associated with a graph $G$ (see [62]) is a convex set containing the stable set polytope and contained in the relaxation of the stable set polytope defined by the clique inequalities. The importance of $TH(G)$ is that one can optimize any linear function over $TH(G)$ in polynomial time by solving a semidefinite program. Lovász and Schrijver showed that

$$N_+(P(G)) \subseteq TH(G),$$

where $P(G)$ is the canonical linear relaxation of the stable set polytope (see Section 4.1).

Many ideas, which are relevant to the theta function, can be extended to the semidefinite operator (Lovász and Schrijver mention that their work on $N_+$ was motivated by the theta function). One, which seems fairly interesting, is the following. Just as the theta function maps variables to vectors, $N_+$ can be thought of as mapping variables to vectors in $R^{n+1}$. Let $P$ be defined by (3.6). Set $x_0 = 1$, and replace (3.7) and (3.8) by

$$
(b_i x_0 - a_i^T x) x_j \geq 0,
$$

$$
(b_i x_0 - a_i^T x)(x_0 - x_j) \geq 0,
$$

$$
x_j^2 - x_0 x_j = 0,
$$

(3.29)

for $i = 1, \ldots, m, j = 1, \ldots, n$. To obtain $N_+(P)$, $x_i x_j$ terms are replaced by $y_{ij}$, and the condition, $Y = (y_{ij}) \succeq 0$, is added, yielding a semidefinite program. A
semidefinite matrix is a Gram matrix. This means there are vectors $\tilde{x}_i$, such that $x_i x_j$ is replaced by $\tilde{x}_i^T \tilde{x}_j$ and we have a vector optimization problem, with vectors from $\mathbb{R}^{n+1}$.

If we insist that $\tilde{x}_i$ lie in $\mathbb{R}$, we get back the original integer program. On the other extreme, $\tilde{x}_i \in \mathbb{R}^{n+1}$ implies that we have a polynomially solvable semidefinite program. With $\tilde{x}_i$ constrained to lie in $\mathbb{R}^k$, where $k$ lies between 1 and $n + 1$, we get nonlinear programming problems of varying difficulty which yield relaxations of different strengths.

Recently, Burer, Monteiro and Zhang (2000) show that, choosing $\tilde{x}_i$ from $\mathbb{R}^2$, in the semidefinite programs arising from the theta function and the max-cut relaxation of Goemans and Williamson, allows the design of fast and effective algorithms. This is because the rank-two relaxations, though non-linear and non-convex, are reasonably easy to solve, yet yield strong relaxations. The semidefinite operator can be used to extend their ideas to different problems, by providing a ready-made semidefinite relaxation. In Chapter 6, we remark upon their work, and extend a result of theirs on rank-two relaxations for the stable set problem.

### 3.5 Basic properties

In this section we collect some properties of the matrix-cut operators applied to polytopes. Some of these properties are analogous to those of the Chvátal operator. We note that the matrix-cut operators can be applied to closed convex sets contained in $Q$.

Given a convex set $S \subseteq \mathbb{R}^n$, we define an associated convex cone $\overline{S}$ by

$$\overline{S} = cone(\{\overline{x} \in \mathbb{R}^{n+1} | x \in S\}).$$

If $P \subseteq Q$ is defined by (3.6), it follows that

$$\overline{P} = \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^{n+1} | bx_0 - Ax \geq 0 \right\}$$
For the empty set $\emptyset$, we adopt the convention that $\overline{0} = \{0\}$ (here $0$ ($1$) refers to the vector of all zeros (all ones) in the appropriate dimension). Of special interest will be the cone

$$
\overline{Q} = \{x \in \mathbb{R}^{n+1} \mid x_0 - x_i \geq 0, x_i \geq 0 \ \forall i \geq 1\}.
$$

If $y \in \overline{Q}$ with $y_0 > 0$, we define the image of $y$ in $\mathbb{R}^n$ by $\check{y}$ where $\lambda y = \begin{pmatrix} 1 \\ \check{y} \end{pmatrix}$ for some $\lambda > 0$. For a subcone $K$ of $\overline{Q}$, let

$$
\check{K} = \{\check{y} \mid y \in K\}.
$$

While establishing properties of the matrix-cut operators, we will often need to check that a matrix $Y$ belongs to $M_+(P)$ for some $P$. For this purpose, it is useful to set down the following condition, which is equivalent to (3.15) and (3.16):

$$
Ye_i \in \overline{P} \text{ and } Y(e_0 - e_i) \in \overline{P} \text{ for } i = 1, \ldots, n. \tag{3.30}
$$

Also, if $Y = (y_{ij})$ is a matrix satisfying (3.30), then (since $Y$ is contained in $\overline{Q}$)

$$
y_{ij} \geq 0, \quad y_{0j} \geq y_{ij}, \quad y_{ai} \geq y_{ij},
$$

$$
y_{ij} \geq y_{00} + y_{0j} - y_{00} \quad \text{whenever } \quad i \geq 0, j \geq 0. \tag{3.31}
$$

The matrix-cut operators commute with some operations on polytopes. We look at a few here. The first two are mentioned in [92]. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ corresponds to a flipping operation if it 'flips' some coordinates. That is, if $J \subseteq \{1, \ldots, n\}$ and $f$ flips the coordinates in $J$, then

$$
y = f(x) \Rightarrow y_i = \begin{cases} x_i & \text{if } i \notin J, \\ 1 - x_i & \text{if } i \in J. \end{cases} \tag{3.32}
$$

The function $f$ corresponds to an embedding operation if $f : \mathbb{R}^n \to \mathbb{R}^{n+k}$ and

$$
y = f(x) \Rightarrow y_i = \begin{cases} x_i & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } n < i \leq n + k_1, \\ 1 & \text{if } n + k_1 < i \leq n + k. \end{cases} \tag{3.33}
$$
where $0 \leq k_1 \leq k$. Note that we can always renumber the coordinates so that the additional coordinates with values 0 or 1 are interspersed with the original ones and not grouped at the end.

Given a face $F$ of $Q$, $f_F$ will denote the embedding function defined by

$$f_F \text{ embeds } Q_{dim(F)} \text{ in } F.$$  \hfill (3.34)

Figure 3.4: flipping operation

Figure 3.5: embedding operation

Figure 3.6: duplication operation
Consider a $k$-tuple of coordinates $\{j_1, \ldots, j_k\}$, which are not necessarily distinct, such that $j_i \in \{1, \ldots, n\}$ for $i = 1, \ldots, k$. If $f : \mathbb{R}^n \to \mathbb{R}^{n+k}$ and

$$y = f(x) \Rightarrow y_i = \begin{cases} x_i & \text{if } 1 \leq i \leq n, \\ x_{j_i} & \text{if } n < i \leq n + k, \end{cases}$$

(3.35)

then $f$ corresponds to a duplication operation.

Given a set $S \subseteq \mathbb{R}^n$, we define the set $f(S)$ by $f(S) = \{f(x) | x \in S\}$. The following easy lemma will be used in a number of places (see the discussion on flipping and embedding in Lovász and Schrijver 1991 and the discussion on embedding in Stephen and Tunçel 1999).

**Lemma 3.4** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ correspond to a flipping operation, an embedding operation, or a duplication operation and let $P \subseteq Q$ be a polytope. Then $N_+(f(P)) = f(N_+(P))$. This equation is also valid for the $N_0$ and $N$ operators.

**Proof:** We give the proof only for the $N_+$ operator. Let $f$ be a flipping operation; we can assume that $f$ flips only $x_n$. Then $y = f(x) \Rightarrow y_n = 1 - x_n$ and for $i = 1, \ldots, n - 1$, $y_i = x_i$. It suffices to prove that $f(N_+(P)) \subseteq N_+(f(P))$: the desired conclusion will follow from the fact that $f(f(S)) = S$ for any set $S$ in $Q$. Let $A$ be an $(n + 1) \times (n + 1)$ matrix defined by $A = (e_0, \ldots, e_{n-1}, f_n)^T$. If $x \in N_+(P)$, then $\bar{x} = Ye_0$ for some $Y \in M_+(P)$. It is easily checked that $AYA^T$ belongs to $M_+(f(P))$ ($AYA^T$ is positive semidefinite as $A$ is non-singular; see Section 2.4). As $AYA^T e_0 = AY e_0 = \bar{f}(x)$, this means that $f(x) \in N_+(f(P))$.

Let $f$ correspond to an embedding operation and assume that $f(x) = \left( \begin{array}{c} x \\ 0 \end{array} \right)$. For a matrix $Y$, let

$$Y' = \left( \begin{array}{cc} Y & 0 \\ 0^T & 0 \end{array} \right).$$

Then $Y \in M_+(P) \iff Y' \in M_+(f(P))$ (note that, by (2.4), $Y$ is positive semidefinite if and only if $Y'$ is). Therefore $N_+(f(P)) = \{ \left( \begin{array}{c} x \\ 0 \end{array} \right) | x \in N_+(P) \} = f(N_+(P))$. 


Adding a new coordinate with value 1 is equivalent to first adding the new coordinate with value 0 and then flipping the coordinate.

For the last case, assume that \( f \) duplicates only \( x_n \), that is \( f : \mathbb{R}^n \to \mathbb{R}^{n+1} \) and \( y = f(x) \Rightarrow y_{n+1} = x_n \) and \( y_i = x_i \) for \( 1 \leq i \leq n \). For a matrix \( Y = (y_{ij}) \), let

\[
Y' = \begin{pmatrix}
Y & Ye_n \\
e_n^TY & y_{nn}
\end{pmatrix}
\]

(we repeat the last row and column and also the last diagonal element). It is easy to see that \( Y \in M(P) \iff Y' \in M(f(P)) \). If \( Y \) is positive semidefinite, then \( Y = U^TU \), for some matrix \( U \). Let

\[
U' = \begin{pmatrix}
U & Ue_n \\
0^T & 0
\end{pmatrix}
\]

then \( Y' = U'^TU' \). As \( Y \) is a principal submatrix of \( Y' \), \( Y \) is positive semidefinite if \( Y' \) is. This implies that \( Y \in M_+(P) \iff Y' \in M_+(f(P)) \). Hence \( N_+(f(P)) = \{ \begin{pmatrix} x \\ x_n \end{pmatrix} \mid x \in N_+(P) \} = f(N_+(P)) \). \( \Box \)

**Remark 3.5** If we have a polytope \( P \) contained in some face \( F \) of \( Q \), we can project out the coordinates which are fixed to 0 or 1 to obtain a polytope \( P_1 \subseteq Q_k \) for some \( k < n \), apply (for example) the \( N_+ \) operator to \( P_1 \) in the lower-dimensional space, and then embed \( N_+(P_1) \) in the face \( F \) by adding back the coordinates fixed to 0 or 1. This, in conjunction with Lemma 3.6 below, is useful in inductive proofs for the rank of a polytope.

A useful property of the Chvátal operator is that \( P' \cap F = F' \) where \( F \) is a face of a rational polyhedron \( P \). It is interesting that a similar property holds for the matrix-cut operators applied to all polyhedra.

**Lemma 3.6** If \( F \) is a face of a polytope \( P \subseteq Q \), then \( N_+(F) = N_+(P) \cap F \). This equation is also valid for the \( N \) and \( N_0 \) operators.

**Proof:** Assume \( F \) is a face of \( P \). Then there is a supporting hyperplane \( H = \{ x \mid c^Tx = d \} \) of \( P \), where \( c^Tx \leq d \) is valid for \( P \), such that \( F = P \cap H \). Let
\[ u^T \bar{x} \geq 0 \] be the homogenized form of \[ c^T x \leq d \] (see (3.14)). By definition \( N_+(F) = N_+(P \cap H) \subseteq N_+(P) \cap H \). Let \( x \in N_+(P) \cap H \). Then \( \bar{x} = Ye_0 \) for some \( Y \in M_+(P) \).

As \( H \) is a supporting hyperplane of \( P \), we have \( u^T Ye_i \geq 0 \) and \( u^T Y(e_0 - e_i) \geq 0 \) for \( i = 1, \ldots, n \). As \( x \in H \),

\[ 0 = u^T \bar{x} = u^T Ye_i + u^T Y(e_0 - e_i). \]

Hence, \( Ye_i \in \overline{H} \) and \( Y(e_0 - e_i) \in \overline{H} \) for \( i = 1, \ldots, n \). This implies that \( x \in N_+(P \cap H) \) and the lemma follows. It is clear that the proof applies to the \( N \) and \( N_0 \) operators.

□

Lemma 3.6 is useful in many contexts. Let \( P \) be a polytope with a non-integral vertex. Denoting the vertex as the face \( F \), we have \( N_+(F) = \emptyset = N_+(P) \cap F \) (see Theorem 3.10). Hence every non-integral vertex of \( P \) is cut off by an \( N_- \)-cut for \( P \), and also by an \( N \)-cut. So is every non-integral extreme point, if \( P \) is a closed convex set. Another application is the following. Let \( F \) be a face of a polytope \( P \) and let \( c^T x \leq d \) be an \( N_+ \)-cut for \( F \). From Lemma 3.6, \( N_+(F) \) is a face of \( N_+(P) \). Therefore, we can "rotate" \( c^T x \leq d \) to get an inequality \( (c')^T x \leq d' \) valid for \( N_+(P) \), and hence an \( N_+ \)-cut for \( P \), such that

\[ F \cap \{ x \mid c^T x \leq d \} = F \cap \{ x \mid (c')^T x \leq d' \}. \]

Compare this with Lemma 6.33 in Cook, Cunningham, Pullyblank, and Schrijver (the same result for \( P' \)). Lastly, we will often use the following corollary of Lemma 3.6:

\[ F \text{ is a face of } Q \Rightarrow N_+(P \cap F) = N_+(P) \cap F. \]

Let the facets of \( Q \) be denoted by \( F^0_i = \{ x \in Q \mid x_i = 0 \} \) and \( F^1_i = \{ x \in Q \mid x_i = 1 \} \), for \( i = 1, \ldots, n \). Goemans and Tunçel (2000) observe that \( N_+(P) \) is completely determined by the intersection of \( P \) with the facets of \( Q \). This follows from (3.30): if \( Y \) satisfies this condition, then \( Ye_i \in F^1_i \) and \( Y(e_0 - e_i) \in F^0_i \).

If an inequality is valid for the intersection of \( P \) with two opposite facets, then it is valid for \( N_0(P) \). Observe that this is not necessarily true for the Chvátal operator.
Lemma 3.7 ([92]) Assume that \( a^T x \leq b \) is valid for \( P \cap \{ x \mid x_i = 0 \} \) and \( P \cap \{ x \mid x_i = 1 \} \), where \( P \) is a polytope, and \( 1 \leq i \leq n \). Then \( a^T x \leq b \) is valid for \( N(P) \) and also for \( N_0(P) \).

**Proof:** It is easy to see that there are numbers \( \alpha \) and \( \beta \), such that

\[
a^T x - \alpha x_i \leq b \text{ is valid for } P,
\]

\[
a^T x - \beta (1 - x_i) \leq b \text{ is valid for } P.
\]

Multiplying the first inequality by \( 1 - x_i \) and the second by \( x_i \), replacing \( x_i^2 \) by \( x_i \) and adding, we see that \( a^T x \leq b \) is an \( N \)-cut. □

Lemma 3.7 implies that, for a polytope \( P \),

\[
N_0(P) \subseteq \text{conv}((P \cap F_i^0) \cup (P \cap F_i^1)), \text{ for } i = 1, \ldots, n
\]  \hspace{1cm} (3.36)

(recall from Section 3.4 that the right-hand side above is the polytope \( P_i \)). Hence, if \( P \) does not intersect some facet of \( Q \) (say \( F_i^0 \)), then \( N_0(P) \) is contained in the opposite facet \( (F_i^1) \). This fact, together with Lemma 3.6, has a useful corollary.

**Corollary 3.8** If \( P \cap F_i^0 = \emptyset \), then \( N_0(P) = N_0(P) \cap F_i^1 = N_0(P \cap F_i^1) \). □

If \( P \) does not intersect some pair of opposing facets of \( Q \), then \( N_0(P) = \emptyset \). As \( N_+(P) \subseteq N(P) \subseteq N_0(P) \), the same (and Corollary 3.8) is true for the \( N \) and \( N_+ \) operators.

In fact, Lovász and Schrijver gave the following characterization of \( N_0 \).

**Lemma 3.9** ([92]) \( N_0(P) = \cap_i \text{conv}((P \cap F_i^0) \cup (P \cap F_i^1)) \).

**Proof:** That \( N_0(P) \) is contained in the right-hand side, follows from (3.36). To prove the reverse inclusion, assume \( x \) is contained in \( \text{conv}((P \cap F_i^1) \cup (P \cap F_i^0)) \), for \( i = 1, \ldots, n \). Then \( x = \lambda_i h_i + (1 - \lambda_i) g_i \), where \( h_i \in P \cap F_i^1 \) and \( g_i \in P \cap F_i^0 \), and \( 0 \leq \lambda_i \leq 1 \), for \( i = 1, \ldots, n \). Define a matrix \( Y = (y_{ij}) \) by setting

\[
y_{ij} = \lambda_i h_i.
\]

It is easy to check that \( Y \in M_0(P) \). Hence \( x \) belongs to \( N_0(P) \). □
It is shown in Bockmayr and Eisenbrand (1997) that if $P \subseteq Q$ is a polytope with empty integer hull, then $P^{(d)} = P_I = \emptyset$, where $d$ is the dimension of $P$ (the case $d = 0$ is treated differently). The next theorem states that a similar result holds for the matrix-cut operators; however, $P_I$ need not be empty. This is a slight strengthening of Theorem 3.1. We follow the proof of Lemma 5 in [16] closely. Let $\text{dim}(P)$ stand for the dimension of $P$.

**Theorem 3.10** Let $P \subseteq Q$ be a polytope and let $\text{dim}(P) = d$. If $d = 0$, then $N_0(P) = P_I$. If $d > 0$, then $N_0^d(P) = P_I$.

**Proof:** If $P = P_I$, then $P$ has non-commutative rank 0. Assume $P \neq P_I$. If $d = 0$, then $P = \{a\}$ for some $a \notin \mathbb{Z}^n$. Hence $P$ does not intersect some pair of opposite facets of $Q$ and $N_0(P) = \emptyset$. If $d = 1$, then $P = \text{conv}\{a, b\}$ for some $a, b \in [0, 1]^n$. Since $P \neq P_I$, we can assume that $a \notin \mathbb{Z}^n$ and $0 < a_i < 1$ for some $i$ ($1 \leq i \leq n$). If $0 < b_i < 1$, then $N_0(P) = \emptyset$. So assume $b_i = 1$ (the same argument applies if $b_i = 0$). Since $P \cap F_i^Q = \emptyset$, $N(P) = N(P \cap F_i^1) = \{b\}_I$ (as $\{b\}$ is a polytope of dimension 0). But $\{b\}_I = P_I$ and hence $N_0(P) = P_I$.

We now prove the theorem, by induction on $n$, for the case $1 \leq d \leq n$. The theorem is obviously true for $n = 1$. Assume the theorem is true for polytopes in $Q_{n-1}$ and let $P \subseteq Q = Q_n$ with $\text{dim}(P) = d > 1$. If $P$ is contained in some facet $F$ of $Q$, then $N_0^d(P) = P_I$ by the induction hypothesis and Lemma 3.4. Therefore assume
that $P$ is not contained in any facet of $Q$. Consider some index $i$ with $1 \leq i \leq n$. It is obvious that

$$P_i = \text{conv}((P \cap F_i^0) \cup (P \cap F_i^1)_i).$$

But both $P \cap F_i^0$ and $P \cap F_i^1$ are polytopes with dimension $\leq d - 1$ contained (essentially) in $Q_{n-1}$. Since $d - 1 > 0$, both have rank at most $d - 1$. Denote $N_0^{d-1}(P)$ by $P_i$. Then

$$P_i \cap F_i^0 = N_0^{d-1}(P \cap F_i^0) = (P \cap F_i^0)_i.$$

Similarly $P_i \cap F_i^1 = (P \cap F_i^1)_i$. But $N_0^d(P) = N_0(P_i)$, and by Lemma 3.9, is contained in

$$\text{conv}((P_i \cap F_i^0) \cup (P_i \cap F_i^1)) = P.$$

This implies that the non-commutative rank of $P$ is at most $d$. \quad \Box

A non-empty convex set $S$ is said to be of anti-blocking type if $S \subseteq \mathbb{R}^n_+$ and $x \in S, 0 \leq y \leq x \Rightarrow y \in S$. A convex set $S$ is of blocking type if $S \subseteq \mathbb{R}^n_+$ and $x \in S, y \geq x \Rightarrow y \in S$. The integer hull of an anti-blocking set is an anti-blocking polyhedron. Hammer, Johnson, and Peled (1975) observe that a polyhedron is anti-blocking if and only if it can be defined by $Ax \leq b, x \geq 0$, for some $A$ and $b$ with non-negative coefficients. It is shown in Hartmann (1988) that if $P$ is a polytope of anti-blocking type, then so is $P'$; a similar result turns out to be true for the matrix-cut operators. See Schrijver (1986) for a discussion of anti-blocking and blocking polyhedra.

As polytopes in $Q$ cannot be of blocking type, we use a modified definition. We say a non-empty convex set $S \subseteq Q$ is of blocking type if $y \in Q$ and $y \geq x \in S \Rightarrow y \in S$. The next lemma can also be found in Goemans and Tunçel (2000).

**Lemma 3.11** Let $P \subseteq Q$ be a non-empty anti-blocking (blocking) polytope. Then $N_+(P)$ is a convex set with the anti-blocking (blocking) property. $N_0(P)$ and $N(P)$ are anti-blocking (blocking) polytopes.
Proof: Let \( x \in N_+(P) \). Then \( z = Ye_0 \) with \( Y \in M_+(P) \). Consider \( K \subseteq \{1, \ldots, n\} \) and define \( Y' \) by
\[
Y'_{ij} = \begin{cases} 
0 & \text{if } i \in K \text{ or } j \in K, \\
Y_{ij} & \text{otherwise.}
\end{cases}
\]
Let the \( i \)th column of \( Y \) be \( y_i \) and let \( z_i = y_0 - y_i \) (we define \( y'_i \) and \( z'_i \) analogously). Then \( y'_i \leq y_i \Rightarrow y'_i \in \overline{P} \) (as \( P \) is an anti-blocking polytope). Similarly \( z'_i \in \overline{P} \).

The matrix \( Y' \) is positive semidefinite as the non-zero elements in \( Y' \) form a principal submatrix of \( Y \) and \( Y \) is positive semidefinite. Hence \( Y' \in M_+(P) \). Defining \( x^K = y'_0 \), it follows that \( x^K \in N_+(P) \) (\( x^K \) is the same as \( x \) with the components in \( K \) being set to zero). Since
\[
0 \leq y \leq x \Rightarrow y \in \text{conv}(\{x^K | K \subseteq \{1, \ldots, n\}\}) \subseteq N_+(P).
\]
the result for anti-blocking polytopes follows. A non-empty blocking polytope can be transformed into an anti-blocking one, by flipping all coordinates; one can then apply the above result and Lemma 3.4. \( \square \)

Let \( P \) be a polytope in \( Q \) and let \( c^T x \leq d \) be an inequality, with \( c \geq 0, d \geq 0 \), which is valid for \( P \cap F_i^1 \) whenever \( c_i > 0 \). It is shown in [92, Lemma 1.5] that the above assumptions imply \( c^T x \leq d \) is valid for \( N_+(P) \). We use this result to obtain an upper bound on the semidefinite rank of an anti-blocking polytope (we generalize Corollary 2.19 in [92] which provides a similar bound for stable set polytopes). This can also be derived from a result of Goemans (1998, Theorem 2). See Hartmann (1988) and Chvátal, Cook and Hartmann (1989) for results on the depth of inequalities for the Chvátal operator, with respect to anti-blocking polyhedra.

**Lemma 3.12** Let \( P \subseteq Q \) be a non-empty anti-blocking polytope with \( \max\{1^T x | x \in P_I\} = k \). Then the semidefinite rank of \( P \) is at most \( k + 1 \).

Proof: We prove the theorem by induction on \( \max\{1^T x | x \in P_I\} \) (which we denote by \( k \)). Let \( k = 0 \). Then \( P_I = \{0\} \). Now, \( P \cap F_i^1 \neq \emptyset \) would imply that \( e_i \in P_I \) by
the anti-blocking property of $P$. Hence, for $i = 1, \ldots, n$ we have $P \cap F_i^1 = \emptyset$. By Corollary 3.8 we have

$$N_+(P) \subseteq \cap_i F_i^0 = \{0\} = P_1.$$ 

Now consider some $k > 0$ and assume that the theorem is true whenever $\max\{1^T x \mid x \in P_i\} < k$. Let $P$ satisfy the conditions of the theorem (with this value of $k$). As $P$ is an anti-blocking polytope, so is $P_i$, and $P_i = \{x \in Q \mid Ax \leq b\}$ for some matrix $A \geq 0$ and vector $b \geq 0$. If $f \equiv f_{F_i^1}$, then $P \cap F_i^1 = f(P_i)$, where $P_i$ is a lower-dimensional anti-blocking polytope satisfying

$$\max\{1^T x \mid x \in (P_i)_I\} \leq k - 1.$$ 

Therefore $P_i$ has semidefinite rank at most $k$, and

$$N^k_+(P) \cap F_i^1 = N^k_+(P \cap F_i^1) = P_i \cap F_i^1.$$ 

Let $c^T x \leq d$ be an inequality in the system $Ax \leq b$; $c^T x \leq d$ is valid for $P_i$ and also for $N^k_+(P) \cap F_i^1$. We can conclude from [92, Lemma 1.5] that $c^T x \leq d$ is valid for $N^k_{k+1}(P)$; thus $N^k_{k+1}(P) \subseteq P_i$. □
Chapter 4

Rank of polytopes

4.1 The stable set polytope

Lovász and Schrijver (1991) applied the $N$ and $N_+$ operators to the stable set problem and proved a number of important properties. We summarize some of their results and mention an interesting question. Let $G = (V, E)$ denote a graph with vertex-set $V$ and edge-set $E$. If $x \in R^V$ and $S \subseteq V$, we define $x(S)$ to be the sum $\sum_{v \in V} x_v$. For a subset $S$ of $V$, let $\delta(S) = \{(v, w) \in E : v \in S, w \in V \setminus S\}$ and let $\gamma(S) = \{(v, w) \in E : v, w \in S\}$. Let $\chi^S \in \{0, 1\}^V$ stand for the incidence vector of the set $S$.

A stable set is a collection of mutually non-adjacent vertices. The stable set polytope of $G$ is defined by

$$STAB(G) = \text{conv}(\{\chi^S : S \text{ is a stable set in } G\}),$$

and the fractional stable set polytope is defined by

$$FS(G) = \{x \in Q : x_u + x_v \leq 1 \text{ for all } uv \in E\}. \quad (4.1)$$

The inequalities $x_u + x_v \leq 1$ are called edge inequalities. The 0-1 solutions of $FS(G)$ correspond to stable sets of $G$ and $FS(G)_1 = STAB(G)$. The problem of maximizing a linear function over $STAB(G)$ is called the weighted stable set problem.

We discuss two well-known classes of inequalities valid for $STAB(G)$ and consider their $N$-depth and $N_+$-depth (starting from $FS(G)$). We assume, in this section, that $G$ is a connected graph. If $C \subseteq V$ is a clique, then the clique inequality $x(C) \leq 1$ is valid for $STAB(G)$ (though not necessarily for $FS(G)$). If $C$ is an odd hole ($C \subseteq V$ and $C$ induces a chordless odd cycle) then $x(C) \leq \frac{1}{3}(|C| - 1)$ is called an odd hole inequality and is valid for the stable set polytope.
Clique inequalities have $N_+\text{-depth } 1$ and $N\text{-depth } |C| - 2$ for a clique $C$ (see [92]). Let $C$ be the family of all cliques in $G$. It is known that if $G$ is a perfect graph, then $STAB(G) = \{x \in Q : x(C) \leq 1 \text{ for } C \in C\}$; thus, the semidefinite rank of $FS(G)$ is exactly 1. As the size of the inequality system in (4.1) is bounded by a polynomial function of the size of $G$, it is possible to optimize linear functions over $N_+(FS(G))$ in time bounded by some polynomial function of the size of $G$. Thus, the weighted stable set problem for perfect graphs can be solved in polynomial time. For perfect graphs, the commutative rank (or even the Chvátal rank) can be significantly higher than the semidefinite rank. It is shown in [92] that the commutative rank of $FS(G)$ is exactly $\omega(G) - 2$, where $\omega(G)$ is the size of the largest clique in $G$. Complete graphs are perfect; the Chvátal rank of $FS(G)$ for an $n$-vertex complete graph is exactly $\lceil \log_2(n - 1) \rceil$ (see Hartmann 1988).

Graphs for which the stable set polytope is defined by the edge inequalities and odd hole inequalities are said to be $t$-perfect. If $C$ is an odd hole and $v \in C$, then the odd hole inequality for $C$ is valid for $FS(G) \cap \{x: x_v = 1\}$ and for $FS(G) \cap \{x: x_v = 0\}$. Lemma 3.9 implies that odd hole inequalities are valid for $N(FS(G))$. Therefore, for $t$-perfect graphs, the commutative (and non-commutative) rank of $FS(G)$ is 1 and the weighted stable set problem can be solved in polynomial time.

In [92] the concept of the defect of an inequality valid for $STAB(G)$ is introduced. Let $a^Tx \leq b$ be valid for $STAB(G)$ with $a \succeq 0$. The defect of $a^Tx \leq b$ is the number $\sum_i a_i - 2b$ (actually the defect is defined differently in this paper, but [92, Lemma 2.10] states that the two definitions are equivalent). The defect is always non-negative. The edge inequalities are the only ones with defect 0; odd hole inequalities have defect 1. The authors show that the $N$-depth of $a^Tx \leq b$ is at most its defect and at least the defect divided by $b$. Let $G_r$ be the collection of graphs $G$ for which every facet of $STAB(G)$ has defect bounded above by $r$. For a fixed value of $r$, the stable set problem can be solved in polynomial time for graphs in $G_r$. $G_0$ is the class of bipartite graphs and $G_1$ the class of $t$-perfect graphs. Unfortunately, not much is known about
for $r \geq 2$. Some aspects of the defect of inequalities have been studied by Sewell (1990) and Lipták (1999).

It turns out that $N(\mathcal{F}(G))$ is defined by the edge inequalities and odd hole inequalities. So is $N_0(\mathcal{F}(G))$. No such characterization is known for $N^k(\mathcal{F}(G))$ when $k \geq 2$. If $G$ is a perfect graph, then it has no chordless odd cycles with more than 3 vertices. Hence $N_0(\mathcal{F}(G)) = N(\mathcal{F}(G)) = \{x \in Q : x(C) \leq 1 \text{ for } C \in \mathcal{C} \text{ with } |C| \leq 3\}$. For $k = \omega(G) - 2$, $N^k(\mathcal{F}(G))$ is defined by all clique inequalities (with size $\leq k + 2$). This motivates the following question [89, Problem 5].

**Question 4.1** Is it true that $N^k(\mathcal{F}(G)) = \{x \in Q : x(C) \leq 1 \text{ for } C \in \mathcal{C} \text{ with } |C| \leq k + 2\}$ when $G$ is perfect?

Lipták (1999) has recently established some properties of $N^2_0(\mathcal{F}(G))$; he also presents some related conjectures. One can replace $N$ by $N_0$ in Question 4.1. If Question 4.1 can be answered in the affirmative, then a possible approach to proving the perfection of some class of graphs would be to show that $N^k(\mathcal{F}(G))$ or $N^h_0(\mathcal{F}(G))$ behave as in Question 4.1.

A matching in a graph $G$ corresponds to stable sets in the line graph of $G$. Let the matching polytope $P(G)$ be defined by $P(G) = \text{conv} \{x^M : M$ is a matching in $G\}$. Here $x^M \in \{0, 1\}^E$ stands for the incidence vector of the matching $M \subseteq E$. Let the fractional matching polytope be defined by

$$FM(G) = \{x \in Q_{[\mathcal{E}]} : x(\delta(\{v\})) \leq 1 \text{ for all } v \in V\}. \quad (4.2)$$

Edmonds (1965) proved that $P(G)$ is the set of all $x \in R^E$ satisfying

$$x_e \geq 0 \text{ for all } e \in E,$$

$$x(\delta(\{v\})) \leq 1 \text{ for all } v \in V, \quad (4.3)$$

$$x(\gamma(W)) \leq \frac{1}{2}(|W| - 1) \text{ for all } W \subseteq V \text{ with } W \text{ odd} .$$

It follows from (4.3) that $FM(G)' = FM(G)_r = P(G)$. Stephen and Tunçel (1999) proved that if $G$ is a complete graph with $2n + 1$ vertices, then the semidefinite rank
of $FM(G)$ is exactly $n$. In this case $FM(G) \subseteq Q_d$ where $d = n(2n + 1)$ and the semidefinite rank is approximately $\sqrt{d/2}$. This implies a similar lower bound for the rank of $FS(G)$ when $G$ is the line graph of a complete graph. In fact, if we start out with the stronger system

$$FC(G) = \{x \in Q : x(C) \leq 1 \text{ for all } C \in C\},$$

where $G$ is the line graph of a complete graph (say $G'$), then the semidefinite rank of $FC(G)$ differs from the rank of $FS(G)$ by at most 1. This follows from the fact that cliques in $G$ are either odd cycles of length 3, or correspond to the edges incident to a vertex in $G'$ (for such cliques, the inequality $x(C) \leq 1$ is the same as $x(\delta(\{v\})) \leq 1$ for some $v$ in $V(G')$).

## 4.2 Polytopes with high rank

For any polytope in $Q_n$, $n$ iterations of the $N$ and $N_+$ operators are sufficient to obtain the integer hull (Theorem 3.1). This is essentially the best possible upper bound for the commutative operator as there are examples with commutative rank $n - 2$ (e.g., the fractional stable set polytope of a complete graph). But for most of the examples in Section 4.1 the semidefinite rank is much less than $n$. In the case of perfect graphs, $FS(G)$ has semidefinite rank 1, and for complete graphs, $FM(G)$ requires roughly $\sqrt{n/2}$ iterations of the $N_+$ operator (here $n$ is the number of edges).

For general polyhedra, Goemans (1997) raised the question of determining the maximum possible value of the semidefinite rank. Recently Goemans and Tunçel (2000) presented an example for which $n/2$ iterations of the $N_+$ operator are required. Let (assume $n$ is even)

$$P = \{x \in Q_n : \sum_{i \in S} x_i \leq \frac{n}{2} \text{ for all } S \subseteq \{1, \ldots, n\} \text{ with } |S| = \frac{n}{2} + 1\}. \quad (4.4)$$

Here $P_t = \{x \in Q_n : \sum_{i=1}^n x_i \leq n/2\}$. The semidefinite rank of $P$ is $n/2$ while the commutative rank of $P$ is $n - 2$. Goemans and Tunçel also showed that

$$\max\{\sum x_i : x \in N^k(P)\} = \max\{\sum x_i : x \in N^k_+(P)\} \text{ for } k \leq n/2 - \sqrt{n} + 3/2.$$
Adding the semidefiniteness condition to $N$ does not seem to make much of a difference for \( n - o(n) \) iterations.

We do not include a proof of this result. However, we can easily show that the semidefinite rank of the polytopes in (4.4) is unbounded with increasing $n$. In fact, we will use results from Section 7.2 and prove a lower bound on the rank of $P$, which is greater than the $\sqrt{n/2}$ bound for matching polytopes. For $P$ defined by (4.4) and the inequality $\sum_{i}^n x_i \leq n/2$, the conditions of Lemma 7.2 are satisfied with $t = C_{n/2+1}^n$ and $q = \frac{1}{2}n(n+1)$. Any $N_\perp$-cutting-plane proof of $\sum_{i}^n x_i \leq n/2$ from (4.4) must have length at least $(t - 1)/(q - 1)$. If the semidefinite rank of $P$ is $r$, from Lemma 7.1 we have

\[
(q' - 1)(q - 1) \geq (t - 1)/(q - 1) \quad \text{and} \quad r \geq \log t / \log q;
\]

the first inequality implies that $q' \geq t$, and the second inequality follows. Here all logarithms are assumed to have base two. Then $C_{n/2+1}^n = (1 + 2/n)^{-1}C_{n/2}^n$. Using Stirling’s approximation to replace $C_{n/2}^n$ by $(2/(\pi n))^{1/2}2^n$ when $n$ is large, we get a lower bound of (roughly) $n/(2 \log n)$ for the semidefinite rank of $P$ for large $n$. However the gap between this bound and the actual bound of $n/2$ is quite large.

![Figure 4.1: (a) $P_2$; (b) $(P_2)_t$](image)

We now show that $n$ iterations of the $N_\perp$ operator are sometimes needed to obtain the integer hull of polytopes contained in $Q_n$. We will also show that the $N_\perp$ operator is identical to the $N_0$ operator on occasion. Consider the polytope defined by

\[
P_n = \{x \in Q_n : x_1 + \cdots + x_n \geq \frac{1}{2}\}.
\]

(4.5)
It is obvious that \((P_n)_I = \{x \in Q_n : x_1 + \cdots + x_n \geq 1\}\) and the Chvátal rank of \(P_n\) is 1. See Figure 4.1

**Theorem 4.2** Let \(P_n\) be defined as in (4.5). Then the semidefinite rank of \(P_n\) is \(n\). Further, \(N^k_0(P_n) = N^k(P_n) = N^k_+(P_n)\) for all integers \(k \geq 0\).

**Proof:** We first show that

\[
\frac{1}{2n - k} 1 \in N^k(P_n) \text{ if } k \leq n
\]  

by induction on \(n\). As the result is trivial for \(k = 0\), we will prove the result for positive values of \(k\). Certainly (4.6) is true for \(n = 1\) and \(k = 0\) or 1. Let \(n \geq 2\) and assume (4.6) holds for \(P_{n-1}\). Consider some non-zero \(k\) which satisfies \(k \leq n\).

Consider the matrix \(Y = (y_{ij}) \in R^{(n+1) \times (n+1)}\) defined by

\[
y_{ij} = \begin{cases} 
1 & \text{if } i = j = 0, \\
\frac{1}{2n - k} & \text{if } i = 0, j \geq 1 \text{ or } i \geq 1, j = 0 \text{ or } i = j \geq 1, \\
0 & \text{otherwise}.
\end{cases}
\]  

Let the \(i\)th column of \(Y\) be \(y_i\). Then \(\tilde{y}_i \in N^{k-1}(P_n)\) as \(\tilde{y}_i = e_i\) belongs to \((P_n)_I\). Let \(z_i = y_0 - y_i\). Then

\[
z_i = \frac{2n - k - 1}{2n - k} e_0 + \sum_{j \neq i, 0} \frac{1}{2n - k} e_j \Rightarrow \tilde{z}_i = \sum_{j \neq i, 0} \frac{1}{2n - k - 1} e_j.
\]

Now \(\tilde{z}_i \in F^0_i\). Let \(f\) be the embedding function defined by \(f \equiv f_{F_i^0}\). Then \(P_n \cap F^0_i = f(P_{n-1})\). By the induction hypothesis

\[
\frac{1}{2n - k - 1} 1 = \frac{1}{2(n - 1) - (k - 1)} 1 \in N^{k-1}(P_{n-1})
\]

This implies that

\[
\tilde{z}_i \in f(N^{k-1}(P_{n-1})) = N^{k-1}(P_n \cap F^0_i) \subseteq N^{k-1}(P_n).
\]

Hence \(Y \in M(N^{k-1}(P_n))\) and (4.6) follows (take the vector \(Ye_0\)). Since \(n\) is strictly less than \(2n - k\) whenever \(k < n\), it follows that \(\frac{1}{2n - k} 1 \notin (P_n)_I\) for \(k < n\) and the commutative rank of \(P_n\) is exactly \(n\).
We now show, by induction on \( k \), that

\[
N^k_0(P_n) = N^k_+(P_n) \text{ for } k \leq n
\]  
(4.8)

(we will refer to \( P_n \) as \( P \) as we do not need to consider \( P_n \) for varying \( n \) any more). Let \( k \) be a positive integer with \( k \leq n \). Assume (4.8) is true for \( k - 1 \) and let \( T = N^{k-1}_0(P) \) (observe that \( T_i = P_i \)). Consider some \( x \in N_0(T) \). If \( \sum_{i=1}^n x_i \geq 1 \) then \( x \) is contained in \( P_i \) and hence in \( x \in N_+(T) \).

So assume \( \sum_{i=1}^n x_i < 1 \). Now \( \bar{x} = Ye_0 \) for some \( Y = (y_{ij}) \in M_0(T) \). For \( 1 \leq i \leq n \) both \( y_i \) and \( y_0 - y_i \in \overline{T} \). Define \( Y' = (y'_{ij}) \) by

\[
y'_{ij} = \begin{cases} y_{ij} & \text{if } i = j \text{ or } i = 0 \text{ or } j = 0, \\ 0 & \text{otherwise.} \end{cases}
\]  
(4.9)

It is easy to check that the \( i \)th column of \( Y' \) lies in \( \overline{T} \). Now \( y'_0 - y'_i \) equals \( y_0 - y_i \) in the 0th co-ordinate and \( y'_0 - y'_i \geq y_0 - y_i \). Further, \( \bar{x} = y'_0 \) and \( \sum_{i=1}^n x_i < 1 \) together imply that \( y'_0 - y'_i \in \overline{Q} \). Since \( P \) is a non-empty blocking polytope, it follows from Lemma 3.11 that \( T \) is also a non-empty blocking polytope. Therefore \( y'_0 - y'_i \in \overline{T} \). \( Y' \) is obviously symmetric. Now consider any \( u \in R^{n+1} \). We have from (4.9) that

\[
u^T Y'u = \sum_{i=0}^n u_i^2 y_{ii} + 2 \sum_{i=1}^n u_0 u_i y_{i0}.
\]

Using \( y_{00} = 1 \) and \( y_{i0} = x_i \),

\[
u^T Y'u = u_0^2 + \sum_{i=1}^n x_i (u_i^2 + 2u_i u_0) \geq \sum_{i=1}^n x_i (u_0^2 + u_i^2 + 2u_i u_0) = \sum_{i=1}^n x_i (u_0 + u_i)^2 \geq 0.
\]

The first inequality follows from that fact that \( \sum_{i=1}^n x_i < 1 \) and the second from \( x_i \geq 0 \). Hence \( Y' \) is positive semidefinite and \( Y' \in M_+(T) \). But \( \bar{x} = y_0 = y'_0 \Rightarrow x \in N_+(T) \).

This implies that \( N_0(T) \subseteq N_+(T) \). Hence \( N^k_0(P_n) = N^k(P_n) = N^k_+(P_n) \) and the theorem follows. \( \square \)

The polytopes of the previous example have high semidefinite rank, but low Chvátal rank. In Section 4.1 we discussed examples where the reverse is true. Some
polytopes however have high semidefinite rank as well as high Chvátal rank, as we now discuss.

Chvátal, Cook and Hartmann (1989, Lemma 7.2) give an example with empty integer hull and high Chvátal rank. In this example, polytopes in $Q_n$ are defined by $2^n$ inequalities, in addition to the bounds $0 \leq x \leq 1$. From the discussion in the beginning of this section, we know that such polytopes are likely to have high rank, if removing any inequality causes the inclusion of integer points. On the other hand, if a polytope $P$ has empty integer hull and Chvátal rank $n$, Eisenbrand and Schulz (1999) showed that a defining linear system for $P$ must have at least $2^n$ inequalities. An identical result holds for the matrix-cut operators; polytopes with empty integer hull and high rank must have many inequalities. We adapt the proof of Proposition 5 in Eisenbrand and Schulz (1999) and obtain the following fact.

**Proposition 4.3** Let $P \subseteq Q_n$ be a polytope with $P_I = \emptyset$ and non-commutative rank $n$. Then any system of linear inequalities defining $P$ must contain at least $2^n$ inequalities different from the bounds $0 \leq x \leq 1$.

**Proof:** We observe that if the rank of $P$ is $n$, then both $P \cap F_i^0$ and $P \cap F_i^1$ have non-commutative rank $n - 1$. For if $P \cap F_i^0$ (and similarly $P \cap F_i^1$) has rank $\leq n - 2$, then $N_0^{n-2}(P) \cap F_i^0 = N_0^{n-2}(P \cap F_i^0) = \emptyset$ and hence $N_0^{n-1}(P) = N_0^{n-1}(P \cap F_i^1) = \emptyset$. We can argue as above for faces of $P \cap F_i^0$ and $P \cap F_i^1$ and obtain by induction that for any 1-dimensional face $F$ of $Q$, $P \cap F$ has non-commutative rank 1 and hence $P \cap F \neq \emptyset$. As $P_I = \emptyset$, for every vertex of $Q$ there must be some inequality in any linear system defining $P$ which separates that vertex from $P$. If some inequality separates two 0-1 vectors from $P$, then it separates some 1-dimensional face of $Q$ from $P$. But this is a contradiction and hence the proposition follows. \qed

Clearly the bound of $2^n$ in Proposition 4.3 cannot be raised; any polytope $P \subseteq Q_n$ with $P_I = \emptyset$ is contained in a polytope $T$ with $T_I = \emptyset$ which has a defining system of $2^n$ inequalities (besides the bounds on the variables). In addition, if $P$ has rank $n$, then so does $T$. 
Consider the polytope $P_n$, with empty integer hull, defined by

$$P_n = \{ x \in Q_n : \sum_{i \in J} x_i + \sum_{i \notin J} (1 - x_i) \geq \frac{1}{2}, \text{ for all } J \subseteq \{1, \ldots, n\}\}.$$  \hspace{1cm} (4.10)

See Figure 4.2 We can again apply Lemma 7.2 and Lemma 7.1 to obtain a lower bound of $n/(2 \log n)$ for the semidefinite rank. However, this bound is quite far from the actual value which we establish in Theorem 4.4. This example was also considered by Goemans and Tunçel (2000) and they also prove Theorem 4.4.

![Diagram](image)

Figure 4.2: (a) $P_2$; (b) $(P_2)_{1}$

We will need an inductive characterization of some of the faces of $P_n$: if $F$ is a face of $Q$ with $\dim(F) = q$, and $f_F$ is defined as in (3.34), then

$$P_n \cap F = f_F(P_q).$$  \hspace{1cm} (4.11)

**Theorem 4.4** Let $P_n$ be defined as in (4.10). Then the semidefinite rank of $P_n$ is $n$.

**Proof:** We will prove by induction on $n$, that

$$\frac{1}{2} 1 \in N^{n-1}_{+}(P_n).$$  \hspace{1cm} (4.12)

The case $n = 1$ is trivial; assume $\frac{1}{2} 1 \in N^{n-2}_{+}(P_{n-1})$. Consider the matrix $Y = (y_{ij})$ defined by

$$y_{ij} = \begin{cases} 1 & \text{if } i = j = 0, \\ \frac{1}{2} & \text{if } i = 0, j \geq 1 \text{ or } i \geq 1, j = 0 \text{ or } i = j \geq 1, \\ \frac{1}{4} & \text{otherwise}. \end{cases}$$  \hspace{1cm} (4.13)
Let the ith row of $Y$ be $y_i$ and let $z_i = y_0 - y_i$. Then, if $i \geq 1$, $\bar{y}_i \in P_n \cap F^1_i$; the ith co-ordinate of $\bar{y}_i$ has value 1 while the rest have value $\frac{1}{2}$. Let $f \equiv f_{F^1_i}$. By the induction hypothesis and (4.11), $\bar{y}_i = f(\frac{1}{2}1) \in f(N^\tau^-2(P_{n-1})) = N^\tau^-2(P_n \cap F^1_i) \subseteq N^\tau^-2(P_n)$. Similarly one shows that $\bar{z}_i \in N^\tau^-2(P_n \cap F^0_i) \subseteq N^\tau^-2(P_n)$. To show that $Y$ is semidefinite, consider $u \in \mathbb{R}^{n+1}$. Then
\[
u^T Y u = u_0^2 + \frac{1}{2} \sum_{i=1}^n u_i^2 + \sum_{i=1}^n u_i u_0 + \frac{1}{2} \sum_{i=1}^n \sum_{j>i} u_i u_j
= (u_0 + \frac{1}{2} \sum_{i=1}^n u_i)^2 + \frac{1}{4} \sum_{i=1}^n u_i^2 \geq 0.
\]
Hence $Y \in M_+(N^\tau^-2(P_n))$ and (4.12) follows. This implies that the semidefinite rank of $P_n$ is $n$ (since $(P_n)_I = \emptyset$). \[\square\]

Goemans and Tunçel also show that $N^k_0(P_n) = N^k_+(P_n)$, for all $k \leq n$, for the example in Theorem 4.4.

The Chvátal rank of $P_n$ is shown in [29] to be at least $n$; that the rank is exactly $n$ follows from the fact that $(P_n)_I = \emptyset$ (see the result of Bocmayer and Eisenbrand (1997) cited in Section 3.5). Hence we have a family of polytopes that have high Chvátal rank as well as high semidefinite rank. Let us combine both the operators to obtain a stronger operator $N_+$ defined by
\[
N_+(P) = N_+(P) \cap P'.
\]
(4.14)

The rank of a polytope with respect to $N_+$ will be defined as in the case of the other operators. We will show that even with this strengthened operator, $P_n$ has rank $n$.

We define $S_j$ to be the set of all vectors which have $j$ components equal to 1/2 and the remaining components equal to 0 or 1.

Chvátal, Cook and Hartmann [29, Lemma 7.2] show that the rank of $P_n$ is at least $n$ by establishing the auxiliary result
\[
S_j \subseteq P \Rightarrow S_{j+1} \subseteq P' \text{ for } j \geq 1.
\]
(4.15)
To obtain a similar result for the $N_+$ operator observe that the proof of (4.12) yields

$$S_{n-1} \subseteq P \Rightarrow S_n = \left\{ \frac{1}{2} 1 \right\} \subseteq N_+(P) \quad (4.16)$$

(as the vectors $\tilde{y}_i$ and $\tilde{z}_i$ belong to $S_{n-1}$ which is contained in $P = N_+^{n-2}(P_n)$). We use (4.16) to prove the following lemma.

**Lemma 4.5** Let $P \subseteq Q$ be a polytope and let $S_j \subseteq P$, where $1 \leq j < n$. Then $S_{j+1} \subseteq N_+(P)$.

**Proof:** Assume $S_j \subseteq P$ for some $j \geq 1$. Let $x \in S_{j+1}$ and consider the face $F$ of $Q$ defined by

$$F = \{ y \in Q : y_i = 1 \text{ if } x_i = 1, \ y_i = 0 \text{ if } x_i = 0 \}.$$

Then $\dim(F) = j + 1$. Let $S'_j$ denote the collection of vectors in $R^{j+1}$ with $j$ components equal to $1/2$ and the remaining component equal to $0$ or $1$. The polytope $P \cap F$ can be written as $f_F(P_1)$ for some polytope $P_1 \subseteq Q_{j+1}$ where $f_F$ is defined as in (3.34). ($P_1$ is obtained by dropping the fixed components of $P \cap F$). Then $S_j \cap F = f_F(S'_j)$ and $S'_j \subseteq P_1$. From (4.16) we obtain that $x = f_F(\frac{1}{2} 1) \in f_F(N_-(P_1)) \Rightarrow x \in N_+(P)$. Hence $S_{j+1} \subseteq N_+(P)$. \qed

Since $S_1$ belongs to $P_n$, we can combine Lemma 4.5 with (4.15) and conclude that $S_j \subseteq N_+^{j-1}(P_n)$.

**Corollary 4.6** Let $P_n$ be defined as in (4.10). Then $\frac{1}{2} 1 \in N_+^{n-1}(P_n)$ and the rank of $P_n$ is $n$ with respect to the $N_+$ operator. \qed

The following easy result will be useful in applying Corollary 4.6 to the traveling salesman problem.

**Lemma 4.7** Let $f : R^n \to R^m$ be a function defined as a composition of the embedding, flipping and duplication operations. Let $S \subseteq Q_n$ and $T \subseteq Q_m$ be polytopes such that $f(S) \subseteq T$. Then for any positive integer $t$, $f(N^t_+(S)) \subseteq N^t_+(T)$.
Proof: Lemma 3.4 implies that $f(N^*_+(S)) = N^*_+(f(S)) \subseteq N^*_+(T)$. It is obvious that $f$ can be represented by $f(x) = Ax + b$ for some integral $A$ and $b$. It is known that (see [29, Lemma 2.2]) for such $f$, $f(S) \subseteq T$ implies $f(S^{(t)}) \subseteq T^{(t)}$. Hence $f(S^{(t)} \cap N^*_+(S)) \subseteq T^{(t)} \cap N^*_+(T)$ and the result follows. \qed

4.3 The traveling salesman problem

Let $G = (V, E)$ denote a complete graph with vertex-set $V$ and edge-set $E$. If $x \in R^E$ and $D \subseteq E$, we define $x(D)$ to be the sum $\sum_{e \in D} x_e$. For a subset $S$ of $V$, let $\delta(S) = \{(v, w) \in E : v \in S, w \in V \setminus S\}$ and let $\gamma(S) = \{(v, w) \in E : v, w \in S\}$. Consider the polytope $H(G)$ (or $H$) defined as the set of all $x \in R^E$ satisfying

\[
x(\delta(\{v\})) = 2 \quad \text{for all } v \in V,
\]

\[
x(\delta(W)) \geq 2 \quad \text{for all } W \subseteq V \text{ with } \emptyset \neq W \neq V,
\]

\[
0 \leq x_e \leq 1 \quad \text{for all } e \in E.
\]  

The integral vectors in $H$ are the incidence vectors of Hamiltonian circuits in $G$; the problem of maximizing a linear function over this set of integral vectors is the traveling salesman problem (TSP). Dantzig, Fulkerson, and Johnson (1954) introduced $H$ as a relaxation of the TSP and developed the cutting-plane method for optimizing over $H_I$. The most successful algorithms for solving large TSP instances all adopt the Dantzig, Fulkerson, and Johnson approach (see Jünger, Reinelt, and Rinaldi (1995) for a survey of this work).

Chvátal [26] conjectured that the Chvátal rank of $H(G)$ tends to infinity with the number of vertices $n$; Chvátal, Cook, and Hartmann (1989) proved this by establishing that the Chvátal rank of $H(G)$ is at least $[n/8]$. We will adapt the proof in the above paper to show that the $N_*$ rank of $H(G)$ is also at least $[n/8]$. This bound cannot be improved by more than a constant factor; we establish an upper bound of $n + 1$ on the $N_*$ rank of $H(G)$ (as pointed out by a referee, this can be improved slightly to $n - 2$ using a result in Goemans 1998). The number of variables in (4.17) is $\frac{1}{2}n(n - 1)$,
so these results establish that the $N_*$ rank of $H(G)$ is within a constant factor of the square root of the number of variables in the problem. This is similar to the Stephen and Tunçel (1999) result for the semidefinite rank of the standard relaxation of the matching polytope; note however that the Chvátal rank (and hence the $N_*$ rank) of the matching relaxation is 1.

We begin by identifying two subsets of edges used in Chvátal, Cook, and Hartmann (1989). Let $k = \lfloor n/8 \rfloor$ and $r = n - 8k$. Label the $n$ vertices in $V$ as $a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i$ for $i = 1, \ldots, k$, and $w_j$ for $j = 1, \ldots, r$; for convenience we set $w_0 = e_k$ and $w_{r+1} = a_1$. Let $E_{\frac{k}{2}}$ denote the edge-set

$$(a_i, b_i), (b_i, c_i), (c_i, d_i), (d_i, e_i), (e_i, f_i), (f_i, g_i), (g_i, h_i), (h_i, a_i), \quad i = 1, \ldots, k$$

and let $E_1$ denote the edge-set

$$(h_i, d_i), (b_i, f_i), \quad i = 1, \ldots, k,$$

$$(c_i, g_{i+1}), (e_i, a_{i+1}), \quad i = 1, \ldots, k - 1,$$

$$(c_k, g_1),$$

$$(w_j, w_{j+1}), \quad j = 0, \ldots, r.$$ The two sets are illustrated in Figure 4.3.

It is easy to verify that no Hamiltonian circuit contained entirely in $E_{\frac{k}{2}} \cup E_1$ can use every edge in $E_1$. In other words, each 0-1 vector in $H$ satisfies the inequality

$$x(E_{\frac{k}{2}}) + 2x(E_1) \leq (n - 1) + |E_1|. \quad (4.18)$$

Notice that this inequality is violated by the vector $x^*$ obtained by setting $x^*_e = 1$ if $e \in E_1$, $x^*_e = \frac{1}{2}$ if $e \in E_{\frac{k}{2}}$, and $x^*_e = 0$ otherwise; therefore, $x^*$ is not a member of $H_f$. We will obtain a lower bound on the $N_*$ rank of $H$ by showing that $N^{k-1}_e(H)$ contains $x^*$.

**Theorem 4.8** Let $G$ be a complete graph with $n$ vertices. The rank of $H(G)$ (as defined in (4.17)) with respect to the $N_*$ operator is at least $\lfloor n/8 \rfloor$ and at most $n + 1$. 
Figure 4.3: Dark edges are in $E_1$ and light edges are in $E_{1/2}$; $n = 26$.

**Proof:** Consider the polytope $T(G)$ (or $T$) defined as the set of all $x \in \mathbb{R}^E$ satisfying

$$x(\delta(\{v\})) \leq 2 \quad \text{for all } v \in V,$$

$$x(\gamma(W)) \leq |W| - 1 \quad \text{for all } W \subseteq V \text{ with } \emptyset \neq W \neq V,$$

$$0 \leq x_e \leq 1 \quad \text{for all } e \in E.$$

Let $F = \{x \in Q : x(\delta(\{v\})) = 2 \text{ for all } v \in V\}$. It can be shown that $H(G)$ is a face of $T(G)$ (see Grötschel and Padberg (1985) for a discussion); in particular we have $H = T \cap F$ and $H_I = T_I \cap F$. Any 0-1 vector in $T$ can have at most $n$ ones; thus $\max\{1^T x : x \in T_I\} = n$. Since $T$ is an anti-blocking polytope, it follows from Lemma 3.12 that the semidefinite rank of $T$ is at most $n + 1$. By Lemma 3.6 we have $N_{n+1}^+(H) = N_{n+1}^+(T \cap F) = N_{n+1}^+(T) \cap F = T_I \cap F = H_I$. Hence the $N_*$ (or $N_+$) rank of $H$ is at most $n + 1$.

To obtain the lower bound, we show the existence of a function $f : Q_{[n/8]} \rightarrow \mathbb{R}^E$ satisfying the following properties (let $k = \lfloor n/8 \rfloor$):

(i) $f$ is a composition of the embedding, flipping and duplication operations;

(ii) $f(P_k) \subseteq H$, where $P_k$ is defined as in (4.10);
(iii) $x^* = f(\frac{1}{2}1) \notin H_r$.

Given such an $f$, Corollary 4.6 and Lemma 4.7 together imply that $x^* = f(\frac{1}{2}1) \in f(N_{k-1}^k(P_k)) \subseteq N_{k-1}^k(H)$. Hence (iii) implies that the $N_*$ rank of $H$ is at least $k$.

We will construct $f$ as in Chvátal, Cook, and Hartmann (1989). If $y \in Q_k$, let $f(y)$ be the vector $x \in R^E$ defined by

$$x_e = \begin{cases} 
1 & \text{if } e \in E_1, \\
y_i & \text{if } e \in \{(a_i, b_i), (c_i, d_i), (e_i, f_i), (g_i, h_i)\}, \\
1 - y_i & \text{if } e \in \{(b_i, c_i), (d_i, e_i), (f_i, g_i), (h_i, a_i)\}, \\
0 & \text{otherwise}.
\end{cases}$$

It is clear that (i) and (iii) hold for $f$ defined in this way. A short proof that (ii) holds can be found in Lemma 8.2 of Chvátal, Cook, and Hartmann (1989); for completeness we repeat the argument below.

Consider an arbitrary vector $y \in P_k$ and let $x'$ denote $f(y)$. We must show that $x'$ satisfies each inequality in (4.17). Clearly $0 \leq x' \leq 1$ and $x'((\delta\{u\})) = 2$ for all $v \in V$. It remains to show that $x'$ satisfies $x(\delta(W)) \geq 2$ for all proper subsets $W \subseteq V$.

For $J \subseteq \{1, \ldots, k\}$, let $y_J \in Q_k$ denote the incidence vector of $J$ and observe that $f(y_J)$ is the incidence vector of two circuits in $G$, one spanning the set

$$W_J = (\bigcup_{i \in J} \{g_i, h_i, d_i, c_i\}) \cup (\bigcup_{i \notin J} \{g_i, f_i, b_i, c_i\})$$

and the other spanning $V \setminus W_J$. Therefore, $f(y_J)$ satisfies $x(\delta(W)) \geq 2$ for each proper subset $W \subseteq V$ other than $W_J$. Let $W$ be any proper subset of $V$. Case 1: $W \neq W_J$ for all $J$. Since $y$ is a convex combination of vectors $y_J$ (as $y$ is in $Q_k$), we can conclude that $x'$ satisfies $x(\delta(W)) \geq 2$. Case 2: $W = W_J$ for some $J$. We have

$$x'(\delta(W_J)) = 4\sum_{i \in J}(1 - y_i) + 4\sum_{i \notin J} y_i.$$

Therefore, since $y$ satisfies the inequalities (4.10), we know that $x'(\delta(W_J)) \geq 2$.  \qed
A similar result can be proven for the standard relaxation of the asymmetric traveling salesman problem; the proof is again an easy application of Corollary 4.6 and the proof method used in Chvátal, Cook, and Hartmann (1989).

We can also extend some of the results in [29] on the lengths of Chvátal cutting-plane proofs to \( N_* \)-cutting-plane proofs. Using the notation of the proof of Theorem 4.8, let \( J \subseteq \{1, \ldots, k\} \) and let \( G \) be a complete graph with \( n \) vertices. If we remove \( x(\delta(W_J)) \geq 2 \) from the system (4.17), then \( f(y_J) \) satisfies the remaining inequalities in (4.17) but violates (4.18). Hence the conditions of Lemma 7.2 are satisfied with \( t = 2^k \) and \( q = n^4 + 1 \) (there are \( n(n - 1)/2 \) variables in (4.17)). Using the analogue of Lemma 7.2 for \( N_* \)-cutting-plane proofs, we can (partially) restate Theorem 8.8 in [29].

**Theorem 4.9** Every \( N_* \)-cutting-plane proof of (4.18) from (4.17) must have length at least \( 2^{n/8}/n^4 \). \( \square \)

Cutting planes are often embedded in branch-and-bound methods to solve traveling salesman problems. Cook and Hartmann [37, Theorem 1] gave exponential lower bounds on the number of operations required to solve traveling salesman problems, where the operations include branching, separating over the polytope (4.17) and the derivation of Chvátal cuts. The proof technique used in [37] can easily be modified to handle the case where \( N_* \)-cuts are used instead of Chvátal cuts; similar exponential lower bounds can be derived. As this is fairly straightforward, we do not discuss this extension. One point to note is that the system (4.17) has exponentially many inequalities, and so the lower bounds mentioned above are not exponential in the size of the input.
Chapter 5

The division operator

5.1 Introduction

In addition to the commutative and semidefinite operators, Lovász and Schrijver (1991) defined another matrix-cut operator, called the division operator, denoted by $\tilde{N}(P)$ ($\tilde{N}$ operates on polytopes in $Q$). We introduce this operator by first describing how to generate inequalities valid for $\tilde{N}(P)$.

Consider the polytope $P$ in Example 1 in Section 3.1. From (3.3) we know that the inequalities

$$0.5x_1 - x_1x_2 \geq 0,$$

$$0.5 - 0.5x_1 - x_2 + x_1x_2 \geq 0,$$

are valid for 0-1 vectors in $P$; so are $x_2^2 - x_2 = 0$ and $x_1x_2 \geq 0$. Now

$$(1 - 3x_2)(2 - 0.5x_1 - x_2) = 2 - 0.5x_1 - 7x_2 + 3x_2^2 + 1.5x_1x_2$$

$$= 4(0.5 - 0.5x_1 - x_2 + x_1x_2) + 0.5x_1x_2 + 3(0.5x_1 - x_1x_2) + 3(x_2^2 - x_2).$$

Hence $(1 - 3x_2)(2 - 0.5x_1 - x_2) \geq 0$ is valid for the 0-1 vectors in $P$. As the term $2 - 0.5x_1 - x_2$ is positive for all points in $Q$, we can divide it out from the previous inequality to conclude that

$$1 - 3x_2 \geq 0$$

is valid for $P_I$; $1 - 3x_2 \geq 0$ is called an $\tilde{N}$-cut. Observe that this $\tilde{N}$-cut is stronger than any $N$-cut or $N_+$-cut for this example. It turns out that for any $\alpha > 0$, $x_2 \leq \alpha$.
is also an $\tilde{N}$-cut for $P$. Therefore, the set of points satisfying all $\tilde{N}$-cuts is precisely $P_I$; see Figure 5.1.

In what follows we will assume (by scaling) that a linear function which is strictly positive for all $x$ in $Q$, is of the form $1 + (g + h^T x)$ where $g + h^T x \geq 0$ is valid for $Q$.

Let $P$ be a polytope in the 0-1 cube defined by (3.6). An inequality $c^T x \leq d$ is said to be an $\tilde{N}(P, Q)$ cutting plane (or $\tilde{N}$-cut for $P$) if $g + h^T x \geq 0$ is some inequality valid for $Q$ such that

\[
(d - c^T x)(1 + g + h^T x) = \sum_{i} \alpha_{ij}(b_i - a_i^T x)x_j + \\
\sum_{i} \beta_{ij}(b_i - a_i^T x)(1 - x_j) + \\
\sum_{j} \lambda_j(x_j^2 - x_j),
\]

(5.1)

where $\alpha_{ij}, \beta_{ij} \geq 0$ for $i = 1, \ldots, m, j = 1, \ldots, n$ and $\lambda_j$ is a real number, for $j = 1, \ldots, n$.

Observe that (5.1) implies that $w^T \bar{x} \geq 0$ is an $\tilde{N}$-cut for $P$ if and only if there is some $u^T \bar{x} \geq 0$, satisfied by all $x$ in $Q$, such that $w(e_0 + u)^T \in M(P)^*$ (here $w, u$ and $e_0 \in \mathbb{R}^{n+1}$). To see this, compare (5.1) with equations (3.9) and (3.19). The matrix $w(e_0 + u)^T$ corresponds to the quadratic function $(w^T \bar{x})(1 + u^T \bar{x})$. 
5.2 The division operator

The division operator \( \hat{N}(P) \) is defined as follows:

\[
\hat{N}(P) = \{ x \mid x \text{ satisfies all } \hat{N} \text{-cuts for } P \}.
\]

Every \( \hat{N} \)-cut is also an \( N \)-cut; this is seen by taking \( g = 0 \) and \( h = 0 \) in (5.1). Hence, \( \hat{N}(P) \) is a closed, convex set and satisfies

\[
P_t \subseteq \hat{N}(P) \subseteq N(P).
\]

The division rank of a polytope is the least integer \( t \) such that \( \hat{N}^t(P) = P_t \); the \( \hat{N} \)-depth of an inequality is defined similarly to its \( N \)-depth. We can similarly define \( \hat{N}_+ \)-cuts and the operator \( \hat{N}_+(P) \).

Lovász and Schrijver showed that \( \hat{N}(P) \) can also be expressed as a projection operator. Lemma 3.2 implies that \( N(P) \) is the set of all \( x \) such that \( x \in M(P)e_0 \), where \( M(P)u = \{ Yu \mid Y \in M(P) \} \) when \( u \in R^{n+1} \). Linear inequalities which are positive for all points in \( Q \) correspond to vectors contained in the interior of \( \overline{Q}^* \). Further, if a vector \( v \) is contained in the interior of \( \overline{Q}^* \), then \( v = \alpha(e_0 + u) \) for some \( \alpha > 0 \) and \( u \in \overline{Q}^* \). The following was shown in [92, Lemma 1.7].

**Lemma 5.1** Let \( P \subseteq Q \) be a polytope. Then \( \hat{N}(P) = \{ x \mid x \in \bigcap_{u \in \overline{Q}^*} M(P)(e_0 + u) \} \).

\( \square \)

Lovász and Schrijver (1991) mention that the following complexity issue is unresolved:

Is it possible to optimize a linear function over \( \hat{N}(P) \) in polynomial time?

An interesting result on \( \hat{N} \)-cuts is proved in [92].

**Lemma 5.2** Let \( P \subseteq Q \) be a polytope. If \( N(P \cap \{ x \mid a^T x \geq b \}) = \emptyset \), then \( a^T x \leq b \) is an \( \hat{N} \)-cut for \( P \). \( \square \)

This lemma can often be used to show the existence of \( \hat{N} \)-cuts which are not \( N \)-cuts. For the polytope in Example 1, given a number \( \alpha \in (0, 1) \), \( N(P \cap \{ x \mid x_2 \geq \alpha \}) = \emptyset \).
Therefore, for every $\alpha > 0$, $x_2 \leq \alpha$ is an $\tilde{N}$-cut, and the division rank of the polytope is $1$. The gap between the division rank and the commutative rank or semidefinite rank can be as high as $n - 1$ for polytopes in $Q_n$. For example, let $n$ be greater than 1, and consider $P_n$ defined in (4.5). Using Lemma 5.2, we can infer that for every $\alpha < 1$, $\sum_{i=1}^{n} x_i \geq \alpha$ is valid for $\tilde{N}(P_n)$. As $\tilde{N}(P_n)$ is closed, this implies that

$$x_1 + \cdots + x_n \geq 1$$

(5.2)

is valid for $\tilde{N}(P_n)$. Hence the division rank of $P_n$ is 1. But the semidefinite rank of $P_n$ is exactly $n$ by Theorem 4.2. We do not know if the division rank is always less than the commutative rank.

Let the cones of inequalities valid for $N(P)$ and $\tilde{N}(P)$ be denoted by $N(P)^*$ and $\tilde{N}(P)^*$ respectively (to be precise we should use $\overline{N(P)}^*$ etc.). In Section 3.2, it is show that $N(P)^*$ is identical to the cone of $N$-cuts. It is stated in [92, Lemma 1.7]) that $\tilde{N}(P)^*$ is spanned by the set of $\tilde{N}$-cuts. However, it turns that the cone of $\tilde{N}$-cuts is not always closed and is not always equal to $\tilde{N}(P)^*$. That is, not all inequalities valid for $\tilde{N}(P)$ are $\tilde{N}$-cuts. We illustrate this fact in the following example.

**Example 3**

Let $P_n$ be defined as in (4.5), where $n$ is greater than 1. We will show that (5.2) is not an $\tilde{N}$-cut, even though it is valid for $\tilde{N}(P_n)$. To this end, consider the matrix $Y = (y_{ij})$, defined in (4.7), with $k = 1$. Then

$$y_{ij} = \begin{cases} 
1 & \text{if } i = j = 0, \\
\frac{1}{2n-1} & \text{if } i = 0, j \geq 1 \text{ or } i \geq 1, j = 0 \text{ or } i = j \geq 1, \\
0 & \text{otherwise},
\end{cases}$$

(5.3)

and $Y \in M(P_n)$. Let $w \in R^{n+1}$ be defined by $w_0 = -1$ and $w_i = 1$ for $i = 1, \ldots, n$ ($w^T \bar{x} \geq 0$ is the same as (5.2)). Now, for any $u \in Q^+$, $Y \bullet w(e_0 + u)^T = w^T Y (e_0 + u)$
and

$$w^T Y = (\delta, 0, \ldots, 0), \text{ where } \delta = \frac{n}{2n - 1} - 1 < 0.$$  

This means that $w^T Y(e_0 + u) < 0$ for every $u \in \overline{Q}^*$. Hence, $w(e_0 + u)^T \notin M(P_n)^*$ for any $u \in \overline{Q}^*$ and $w^T x \geq 0$ is not an $\hat{N}$-cut for $P_n$.

We see, from the above discussion, that the Chvátal cut $\sum_{i=1}^n x_i \geq 1$ for $P_n$ is valid for $\hat{N}(P)$. Lovász posed the following question: does $\hat{N}(P)$ satisfy all the Chvátal cuts for $P$? Can the matching polytope $P(G)$ be obtained from the fractional matching polytope $FM(G)$ in one iteration of the $\hat{N}$ operator? We will show that the division rank of $FM(G)$ is greater than 1 (for appropriate $G$). Let

$$\hat{N}_+(P) = \{x | \exists \in \cap_{u \in \overline{Q}^*} M_+(P)(e_0 + u)\}.$$

**Proposition 5.3** Let $G = (V, E)$ be the complete graph with $2n + 1$ vertices, where $n > 1$. Let $P$ be defined by (4.2). Then the rank of $P$, with respect to the operator $\hat{N}_+$, is greater than 1.

**Proof:** As $G$ has $n(2n + 1)$ edges, $P$ is contained in $Q_m$, where $m = n(2n + 1)$. We will show that the inequality

$$x(E) \leq n,$$  

valid for $P_1$, is not valid for $\hat{N}_+(P)$. We will prove this by demonstrating that (5.4) is not contained in the closure of the set $N$ of $\hat{N}_+$-cuts (by Example 3, it is not enough to show that (5.4) is not contained in $N$). We will use a class of matrices discussed in Stephen and Tunçel (1999). Let $Y = (y_{ij}) \in R^{(m+1) \times (m+1)}$ be defined by

$$y_{ij} = \begin{cases} 
1 & \text{if } i = j = 0, \\
\frac{1}{2n} & \text{if } i = 0, j \geq 1 \text{ or } i \geq 1, j = 0 \text{ or } i = j \geq 1, \\
\frac{1}{2n(2n-2)} & \text{if edge } i \text{ and edge } j \text{ are non-adjacent,} \\
0 & \text{otherwise.} 
\end{cases}$$  

Stephen and Tunçel show that $Y \in M_+(N_+^{n-2}(P))$. For our purpose it is enough that $Y \in M_+(P)$ (as $n \geq 2$). Let $w \in R^{m+1}$ and assume $w_0 = n$ and $w_i = -1$ for
1 \leq i \leq m; w$ corresponds to the inequality (5.4). We will show that there is no sequence of $\tilde{N}_+$-cuts (or their corresponding vectors) converging to $w$. Suppose, on the contrary, that there is a sequence $v_1, v_2, \ldots,$ converging to $w$, where each $v_k$ is an $\tilde{N}_+$-cut. Now,

$$w^T Y = (-\frac{1}{2}, -\frac{1}{4n}, \ldots, -\frac{1}{4n}). \quad (5.6)$$

Observe that $w^T Y e_i < 0$ and $w^T Y f_i < 0$ for $1 \leq i \leq m$. As the sequence $\{v_k\}$ converges to $w$, we can scale the vectors $v_k$ and assume that the zeroth component of each $v_k$ is $n$. Also $\|w - v_k\|_{\infty} \leq \epsilon_k$, where $\{\epsilon_k\}$ is a sequence of non-negative numbers converging to zero. The vectors $v_k^T Y$ converge to $w^T Y$; if we choose $\epsilon_k$ small enough, then $v_k^T Y e_i < 0$ and $v_k^T Y f_i < 0$ for $1 \leq i \leq m$. This implies that, for large enough $k$, $v_k^T Y (e_0 + u) < 0$ for every $u$ in $Q^*$ and $v_k$ is not an $\tilde{N}_+$-cut; this is a contradiction. Hence $w \notin \mathcal{N}$ and (5.4) is not valid for $\tilde{N}_+(P). \quad \Box$

Another approach might be to prove that there is a point $x$ in $\tilde{N}_+(P)$ which does not belong to $P_I$. However, showing that $x \in \tilde{N}(P)$ (or $\tilde{N}_+(P)$) seems to be a complicated task; this also makes it difficult to compute the exact rank of $P$. Note that Proposition 5.3 implies that $\tilde{N}_+(P)$ does not satisfy all Gomory-Chvátal cuts. In fact, this does not seem to be a common phenomenon. It would be interesting to find interesting classes of cuts which can be derived using the division operator.
Chapter 6

Computation

6.1 The semidefinite operator and computation

In this chapter, we examine the use of the matrix-cut operators in solving integer programs. We concentrate on the semidefinite operator $N_+$, and apply it to different types of problems. We investigate the quality of bounds we get by solving semidefinite programming relaxations of 0-1 integer programs, obtained by applying $N_+$. To the best of our knowledge, the operator $N_+$ has not previously been studied in a computational setting.

An important consideration in introducing a relaxation of an integer program is that the relaxation should be relatively "easy" to solve. Ease of solvability is often associated with the existence of polynomial time algorithms. However, what matters in practice is that the time spent in solving the relaxation should be low for most problems of interest, even if a guarantee of polynomial-time solvability cannot be provided. It is possible to optimize linear functions over $N_+(P)$ in polynomial time, if $P$ is a polytope over which linear functions can be optimized in polynomial time. We consider various issues in optimizing over $N_+(P)$ effectively.

Linear programming based branch-and-bound has been the preferred option for solving general integer programs for quite some time. Linear programming relaxations are easy to solve, and can be incorporated easily in a branch-and-bound framework to solve integer programs. Recently, many authors have examined semidefinite relaxations of integer programs, and have shown that these can yield strong bounds. Most of these relaxations are problem-specific.

Lovász (1979) introduced a number of semidefinite relaxations for the stable set
problem. Semidefinite relaxations have also been considered for the stable set problem, by Grötschel, Lovász and Schrijver (1986); for the maximum cut problem, by Delorme and Poljak (1993) and Goemans and Williamson (1995); for max sat and max 2-sat, by Goemans and Williamson (1995); for graph coloring, by Karger, Motwani and Sudan (1994); for max k-cut, by Frieze and Jerrum (1995), for quadratic knapsack problems, by Helmberg, Rendl and Weismantel (2000); for machine scheduling problems, by Skutella (1998); for vertex cover, by Goemans and Kleinberg (1995); and for discrepancy problems in number theory, by Lovász (2000). Approximation algorithms with good approximation guarantees have been obtained, using semidefinite relaxations, by Goemans and Williamson (1995), Karger, Motwani and Sudan (1994), Karloff and Zwick (1997), Ye (1999) and others. Computational studies, with semidefinite relaxations used only as a mechanism to obtain bounds, have been performed by Benson, Ye and Zhang (1998), Burer, Monteiro and Zhang (1999), and others. Helmberg and Rendl (1998), Helmberg, Rendl and Weismantel (2000) and Gruber and Rendl (1999) investigate the combined use of semidefinite relaxations and cutting planes, while Karisch, Rendl and Klausen (2000) solve graph bisection problems by using semidefinite programming in a branch and bound framework.

The main motivation for using the $N_+$ operator is that we automatically get semidefinite relaxations for a 0-1 integer program. Further, we can often optimize over these relaxations in polynomial time. In some cases, as in the stable set problem, $N_+(P)$, where $P$ is defined by (6.4), is stronger than the semidefinite relaxation for the $\phi$-function, which has been extensively studied. However, to get practical implementations, it seems necessary to add problem specific manipulations while applying the semidefinite operator to different problems. See the next section for more on this issue.

The matrix-cut operators can also be used in separation routines by generating $N$-cuts or $N_+$-cuts, as it is possible to separate over $N(P)$ and $N_+(P)$ in polynomial time. This is in contrast to the situation for Gomory–Chvátal cutting planes, where
the problem of separating over the elementary closure of \( P \) is NP-hard. We will not
look at cutting-plane algorithms based on \( N \) and \( N_+ \); such algorithms which solve
0-1 integer programs (and also mixed 0-1 programs) in finite time, can be devised.
This follows, for example, from the work of Jeroslow (1980), or from Theorem 3.1 in
Balas, Ceria and Cornuéjols (1993). See also, the chapter on cutting planes in Wolsey

The main goal of this chapter is to demonstrate that we can often compute
\( \max \{ c^T x \mid x \in N_+(P) \} \), where \( P \) is a linear programming relaxation of an integer
program, in reasonable time. Secondly, the time spent in computing this bound, is
justified in some cases, in the sense that we obtain good upper bounds on the op-
timal solution of the integer program. We show that we can obtain these bounds
by using general purpose semidefinite programming codes. In Section 6.4 we men-
tion some extensions of the work of Burer, Monteiro, and Zhang [23, 24] on low-rank
approximations of semidefinite relaxations.

6.1.1 Algorithms and codes

Good algorithms and software for semidefinite programs are a crucial ingredient in
the effective use of \( N_+(P) \). Grötschel, Lovász and Schrijver (1981, 1988) provided
a polynomial-time algorithm for semidefinite programming. Interior-point methods
were first derived by Nesterov and Nemirovskii (1994) and Alizadeh (1995). In the
last decade, other algorithms for semidefinite programs have been proposed: primal-
dual interior point methods by Helmberg, Rendl, Vanderbei, and Wolkowicz (1996),
Kojima, Shindoh, and Hara (1997), Monteiro (1997); spectral bundle methods by
Helmberg and Rendl (2000); and methods based on nonlinear programming, by Burer,
Monteiro, and Zhang (1999).

Some publicly available codes for semidefinite programming are \textit{SBmethod},
by Helmberg [72], \textit{CSDP}, by Borchers [20], \textit{SDPpack}, by Alizadeh, Haebeler,
Nayakkankuppam, Overton and Schmieta [4], \textit{DSDP}, by Benson, Ye and Zhang [14],
and SDPA, by Fujisawa and Kojima [50]. More details about these codes, including benchmarks, can be found at Hans Mittelmann's website [94]. Many specialized codes have been designed for specific semidefinite relaxations; we do not mention these here.

6.2 Implementation details

6.2.1 Choice of software

For our numerical experiments, we use the SDPA (version 5.0) code of Fujisawa and Kojima [50]. We chose this code based on the benchmark tests of Hans Mittelmann [94]. This is a general purpose solver for semidefinite programs, based on a predictor-corrector primal-dual interior-point method (see [84], [74], [95]). It is written in C++, handles sparse semidefinite programs, and has a callable library, which we access. A useful feature is that it provides a convenient method for specifying sparse equations.

SDPA reverses the definition of primal and dual semidefinite programs used in (2.5) and (2.6) (and also has min and max interchanged). It defines the standard form semidefinite program as

\[
\begin{align*}
\min & \quad \sum_{i=1}^{m} x_i b_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} x_i A_i \preceq C,
\end{align*}
\]

(6.1)

and its dual as the semidefinite program

\[
\begin{align*}
\max & \quad C \bullet Y \\
\text{s.t.} & \quad A_i \bullet Y = b_i, \text{ for } i = 1, \ldots, m, \\
& \quad Y \succeq 0
\end{align*}
\]

(6.2)

where \(Y, C,\) and \(A_i\) are \(n \times n\) matrices and \(y_i, b_i\) are numbers. The matrix \(Y\) can have a block-diagonal approach. One of the limitations of this approach is that dealing with additional variables, which are not part of \(Y\), takes some work. For example, if we have a mixed 0-1 program, then the real variables have to be shifted to be
nonnegative and then placed on the diagonal of $Y$. Though, this can be done, writing code becomes more complicated.

### 6.2.2 Basic Approach

Given a polytope $P$ defined in (3.6), we wish to compute $\max\{c^T x \mid x \in N_+ (P)\}$. It follows, from definitions (3.28) and (3.25), that this is equivalent to solving

$$\begin{align*}
\max \quad & C \cdot Y \\
\text{s.t.} \quad & Y \text{ satisfies (3.15) – (3.18)}, \\
& Y \succeq 0,
\end{align*}$$

where $C = e_0(0 \ c^T)$ and $Y$ and $C$ are $(n + 1) \times (n + 1)$ matrices. Note that this is in the same form as (6.2). The conditions (3.15) and (3.16) are inequalities. To model these, we need the fact that SDPA allows the matrix variable in (6.2) to have a block diagonal structure. An inequality $A_i \cdot Y \geq b_i$, is written as $A_i \cdot Y - z_i = b_i$, with the introduction of a slack variable $z_i \geq 0$; the slack variables are arranged as a diagonal matrix forming a diagonal block of the matrix in (6.2).

If the system $Ax \leq b$ defining $P$ has $m$ inequalities and $n$ variables, then (6.3) has $2mn$ inequalities involving the $n \times n$ matrix $Y$. For a reasonably sized integer program, with a few hundred variables, and a thousand or more inequalities, $2mn$ is too large and SDPA (or other solvers) cannot handle such large semidefinite programs. Hence, instead of solving a single semidefinite program with $2mn$ inequalities, we solve a sequence of semidefinite programs, each having much fewer inequalities. This is essentially an active set approach, and is related to methods used in many nonlinear programming solvers, the simplex method, and routines to solve integer programs with exponentially many constraints. The justification for this is that at most $n(n + 1)/2$ constraints will be active at the optimal solution; see Lemma 3.3.

To obtain such a sequence $S_1, S_2, \ldots S_k$ of semidefinite programs, with the final one $S_k$ yielding the optimal solution, we need to tackle three issues:
1. How do we select an initial set of active inequalities?

2. What criteria do we use to delete inactive inequalities?

3. How do we select a subset of violated inequalities to make active?

In our code, we attempt to handle these issues uniformly for any input $P$.

Now let $S_i$ be the current problem in the above sequence, and let $Y$ be the solution in (6.2). Let active inequalities be those inequalities in $S_i$ which are tight for $Y$. We place an upper bound of $n^2$ on the number of inequalities in $S_i$, for every $i$; though roughly $n^2/2$ will suffice, this allows (hopefully) the final active set to be located in fewer iterations.

To determine the inequalities to be eliminated, we draw upon some ideas used in the CONCORDE library of Applegate et al (1995, 1998). We determine whether an inequality $A_i \cdot Y \geq b_i$ is active or not by looking at the associated (scaled) dual variable $x_{i,i}$; if $|x_{i,i}|$ is larger than some epsilon, the inequality is active. We maintain a simple form of aging for every inequality in $S_i$. An inequality which is present, but inactive, in $S_i$, has its age incremented by one, but does not immediately die. We delete inequalities when their age crosses an aging parameter. If an inequality becomes active, after growing old for a few iterations, we give it a new lease of life (one unit's worth) and decrease its age by one. Note that the use of the dual variables to determine if an inequality is active or not is justified only if strong duality and complementary slackness hold. It can be argued that this is the case for $N_+(P)$.

Having done this, we determine all inequalities in (6.3) violated by $Y$ (we do not use any clever separation heuristics, as our code is designed for arbitrary $P$). We then sort in decreasing order of amount of violation, and chose $\text{maxv}$ of the most violated ones, keeping in mind that the total should not cross $n^2$. The parameter $\text{maxv}$ is initially small, and then increases after a few iterations (currently 25). This is so that simple problems, where the optimal active set is small and quickly found, are not unduly affected; adding lots of inequalities slows down the semidefinite solver.
a lot.

For the three classes of integer programs we tackle, to obtain improved efficiency, we allow a problem specific selection of the initial active set of inequalities. We do incorporate a general heuristic – this consists of solving the linear program for $P$, and then choosing the active inequalities in the optimal solution and applying these to the first column of $Y$ – which can be used for all problems. Some of the ideas above can also be found in Helmberg and Rendl (1998).

6.2.3 Problem classes

As optimizing a linear function over $N_+(P)$ takes a lot of time in practice, it is useful to do so only in cases where the integer program is small, and yet hard to solve by standard methods. We looked at some instances of the stable set problem, at a problem posed by Erdős and Turán (1935) in Number Theory, and at some general 0-1 integer problems from the MIPLIB 3.0 [15] test set.

Stable sets

Let $G = (V, E)$ be a graph with $n$ nodes, and let the vertices have associated nonnegative weights. The problem of computing a maximum weight stable set in $G$ is NP-hard. Recent work on this problem includes Sewell (1998), Mannino and Sassano (1994), Pardalos and Xue (1994), and Burer. Monteiro, and Zhang (2000). Sewell (1998) investigates the maximum stable set problem for various classes of graphs. In particular, he reports that the stable set problem for sparse graphs, with about 10% edge density, is difficult to solve exactly (his experiment is on uniform random graphs; see the next section). He also reports that the boundary of solvability for such graphs is about 200 nodes. We perform experiments on such graphs.

Every stable set corresponds to a 0-1 solution $x$ (here $x \in \mathbb{R}^n$) of the inequalities

$$x_i + x_j \leq 1 (ij \in E)$$

$$x_i \geq 0 (i \in V).$$

(6.4)
These inequalities define a linear relaxation of the problem. Let $P$ be the polytope defined by these inequalities. Then, computing $\max\{1^T x \mid x \in N_+(P)\}$ yields an upper bound on the value of the maximum stable set.

**A problem of Erdős and Turán**

Consider the problem of determining the size of the maximum subset of numbers in $1, \ldots, n$, such that no three numbers in the subset are in arithmetic progression. Let $f(n)$ be the size of the maximum subset. Erdős and Turán (1935) conjectured that

$$\frac{f(n)}{n} \to 0, \text{ as } n \to \infty.$$ 

This was proved for Roth in 1952; Roth (1953) also showed that $f(n)/n$ is bounded above by $c/\log \log n$, for some constant $c$, as $n$ tends to infinity. Improved upper bounds were given by Heath-Brown (1987) and Szemeredi (1990). Szemeredi (1975) proved a similar result in the case where no $k$ numbers are allowed to be in arithmetic progression, for fixed $k \geq 3$.

We look at an integer programming formulation and compute bounds on $f(n)$. Consider the inequalities

$$x_{i_1} + x_{i_2} + x_{i_3} \leq 2, \text{ if } i_1 + i_2 = 2i_3, \text{ and } 1 \leq i_1, i_2, i_3 \leq n,$$

$$x_i \in \{0, 1\}, \text{ for } i = 1, \ldots, n. \quad (6.5)$$

Maximizing $x_1 + x_2 + \cdots + x_n$ subject to the inequalities above, yields $f(n)$. We let $P$ stand for the linear programming relaxation for this problem, and compute $\max\{1^T x \mid x \in N_+(P)\}$ to obtain an upper bound on $f(n)$.

Solving the integer program above seems to be quite difficult, for small values of $n$ (for example, for $n = 75$), and hence we chose this class to apply $N_+$ to. We used CPLEX 6.5.2 [40] to solve these problems for different values of $n$, and it took a lot of time to do so; see the next section. Queyranne has recently studied this problem, and has provided an improved integer programming formulation.
MIPLIB problems

We wished to consider some integer programs which arise in practice and are not necessarily of a combinatorial nature. A standard test set of pure and mixed integer programs is MIPLIB 3.0 [15] developed by Bixby, Ceria, McZeal, and Savelsbergh. Most of the problems in MIPLIB are mixed integer problems, as our code does not handle such problems, we looked mainly at some 0-1 integer programs of small size.

6.3 Numerical Results

All experiments were performed on a 500 Mhz Alpha EV6, with the alpha binaries of SDPA and CPLEX 6.5.2. The compiler used was g++, and the default parameters of SDPA and CPLEX were used. Testing for violated inequalities and active inequalities involved the use of tolerances; an inequality was assumed to be violated if the actual violation was greater than $10^{-4}$. Similarly, a scaled dual variable was assumed to be non-zero if it exceeded $10^{-4}$ in magnitude. Problems with an asterisk in the right-most column are those which were solved completely, up to the error tolerances of the solver and our code.

For the stable set problem, the initial inequalities were those defining the theta function. Once the theta function was computed (one iteration of SDPA), we went through a cycle of adding and deleting inequalities, as described before. We test our code on uniform random graphs with 100, 150 and 200 nodes, and edge density 5%, 10% and 15%. As defined in Sewell (1998), a random graph $G_{n,p}$ on nodes $\{1, \ldots, n\}$, defined by selecting each edge with probability $p$ (here $0 < p < 1$), independently of other edges, is called a uniform random graph. The results below are for a specific set of random graphs. For graphs with 150 and 200 nodes, the code could not compute the $N_\star(P)$ upper bound in the allotted time, which was 4000 seconds. As the graphs grew bigger, the time per iteration went up, and the number of iterations we could perform within the maximum time went down.

The MIPLIB problems we considered are trivial for most modern IP solvers, and
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<th>θ</th>
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<td>43</td>
<td>44.2</td>
<td>43.1</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>489</td>
<td>30</td>
<td>33.2</td>
<td>32.0</td>
<td>8*</td>
</tr>
<tr>
<td></td>
<td>15%</td>
<td>771</td>
<td>25</td>
<td>27.7</td>
<td>27.0</td>
<td>6*</td>
</tr>
<tr>
<td>150</td>
<td>5%</td>
<td>555</td>
<td>54</td>
<td>57.6</td>
<td>55.4</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>1033</td>
<td>36</td>
<td>42.0</td>
<td>41.2</td>
<td>4</td>
</tr>
<tr>
<td>200</td>
<td>5%</td>
<td>974</td>
<td>63</td>
<td>70.3</td>
<td>68.1</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>2014</td>
<td>-</td>
<td>49.9</td>
<td>49.3</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 6.1: Comparison of θ and $N_+(P)$ for stable sets.

the $N_+(P)$ approach looks less attractive in comparison. In particular, observe that
the value for stein27 hardly changed from the optimal LP relaxation value. This,
along with Theorem 4.2, seems to suggest that covering problems are not amenable
to the $N_+$ approach.

<table>
<thead>
<tr>
<th>Name</th>
<th>n</th>
<th>m</th>
<th>$LP$</th>
<th>$N_+(P)$</th>
<th>Opt</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>lseu</td>
<td>89</td>
<td>28</td>
<td>834.7</td>
<td>871.5</td>
<td>1120</td>
<td>10</td>
</tr>
<tr>
<td>p0033</td>
<td>33</td>
<td>16</td>
<td>2520.5</td>
<td>2552.8</td>
<td>3089</td>
<td>10</td>
</tr>
<tr>
<td>stein27</td>
<td>27</td>
<td>118</td>
<td>13.0</td>
<td>13.0</td>
<td>18</td>
<td>40*</td>
</tr>
</tbody>
</table>

Table 6.2: Some $N_+(P)$ numbers for MIPLIB problems.

In the case of the arithmetic progression problem, we hoped to establish a stronger
upper bound than the trivial linear programming bound of $2n/3$. For the first set of
problems (with $n$ up to 75), we set a time limit of 2000 seconds and an iteration limit
of 15 iterations, except for the case $n = 30$, where we allowed the algorithm to run till
completion. For the second set of problems ($n = 90$ to $n = 120$), we set a time limit
of 4000 seconds and the iteration limit to 15. Optimizing over $N_+$ for these problems
is challenging; the number of inequalities in (6.5) are $O(n^2)$, and we end up with a semidefinite program with $O(n^3)$ constraints.

These integer programs also seem to be quite difficult. We do not give all the optimal integral values below, as we were not able to compute them for $n \geq 75$. For $n = 75$, CPLEX had not solved the problem after enumerating about 1.3 million nodes.

Examining the table below, we see that we approach a bound of about $n/2$, for smaller values of $n$. This is still linear in $n$. For larger values of $n$, we have many violated inequalities upon termination, and we only have a (possibly weak) upper bound on the $N_+$ value. However, it turns out that we cannot obtain a bound on $f(n)$, which is less than $n/2$, by using $N_+$ once. The matrix $Y$, defined in (4.13), belongs to $M_+(P)$, where $P$ is the linear programming relaxation of (6.5) (this condition is not difficult to check). On the other hand, it can be argued that maximizing $x_1 + \cdots + x_n$ over $N_+(P)$ yields a bound close to $n/2$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f(n)$</th>
<th>$LP$</th>
<th>$N_+(P)$</th>
<th>Iterations</th>
<th>Violated eqns.</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>12</td>
<td>20</td>
<td>16.0</td>
<td>23</td>
<td>solved</td>
</tr>
<tr>
<td>45</td>
<td>16</td>
<td>30</td>
<td>23.5</td>
<td>15</td>
<td>100</td>
</tr>
<tr>
<td>60</td>
<td>19</td>
<td>40</td>
<td>31.2</td>
<td>15</td>
<td>500</td>
</tr>
<tr>
<td>75</td>
<td>-</td>
<td>50</td>
<td>39.4</td>
<td>13</td>
<td>1500</td>
</tr>
<tr>
<td>90</td>
<td>-</td>
<td>60</td>
<td>50.0</td>
<td>13</td>
<td>10000</td>
</tr>
<tr>
<td>120</td>
<td>-</td>
<td>80</td>
<td>66.8</td>
<td>15</td>
<td>19000</td>
</tr>
<tr>
<td>150</td>
<td>-</td>
<td>100</td>
<td>87.6</td>
<td>13</td>
<td>152000</td>
</tr>
</tbody>
</table>

Table 6.3: Arithmetic progression problem of Erdős and Turán
6.4 Low rank relaxations

Modeling a 0-1 integer program by a semidefinite program typically involves squaring the number of variables. In return, we obtain a polynomially solvable convex programming problem, which yields provable (and often good) bounds. However, in practice, the additional cost of optimizing over \( n^2 \) variables (instead of \( n \)) often outweighs the gains due to improved bounds. The cost of solving semidefinite programs is a significant bottleneck in applying semidefinite programming to discrete optimization. For the maximum cut problem and the stable set problem, special purpose codes, based on semidefinite programming, have been developed. These codes exploit problem specific information to improve efficiency, yet are not fast enough.

As mentioned in Section 3.4, Burer, Monteiro and Zhang [23, 24] examine low-rank relaxations for the maximum cut problem and the stable set problem. They derive these approximations from fairly standard semidefinite relaxations for these problems. Yet, they obtain remarkable speed-ups over the time taken by the best codes for the corresponding semidefinite relaxations. The number of variables increases from \( n \) to \( 2n \) (instead of \( n^2 \)). This allows them to solve the problems very quickly. Further, their method scales better. As \( n \) grows, the speed-up over semidefinite programming based methods increases significantly.

We describe their method in the context of the maximum cut problem. Let \( G = (V, E) \) be a weighted graph with \( n \) nodes, and weights \( w_{ij} \geq 0 \) associated with each edge \( ij \in E \). The maximum cut problem is the problem of finding the maximum weight cut in the graph. It is well-known that this problem can be formulated as the quadratic optimization problem in \( n \) variables:

\[
\max \sum_{i,j} \frac{1}{2} w_{ij}(1 - x_i x_j) \\
\text{s.t.} \quad x_i^2 = 1.
\]  

(6.6)

This problem is NP-hard. Goemans and Williamson (1995) proposed the following successful approach. Replace the \( \{-1, +1\} \) variables by vectors in \( \mathbb{R}^n \) (for conve-
nience, we use the same notation for vector inner products and scalar products) to obtain the vector optimization problem:

$$\max \sum_{i,j} w_{ij} \frac{1}{2} (1 - x_i x_j)$$

$$s.t. \quad x_i^2 = 1, \quad x_i \in \mathbb{R}^n.$$  \quad (6.7)

This is now a semidefinite program (see Section 3.4). Solving (6.7), yields an upper bound on the optimal value in (6.6), which is within 1/.878 of the maximum weight cut (Goemans and Williamson 1995). Burer, Monteiro, and Zhang [23] suggested choosing vectors in $\mathbb{R}^2$, instead of $\mathbb{R}^n$, in (6.7). These vectors lie on a circle of radius 1 centered at the origin; in polar coordinates, they can be specified by the angle $\theta$ with the $x$-axis. Thus they obtain a problem with $n$ variables instead of $n^2$ variables. They report significant speedups over the best specialized codes for (6.7). See also Lovász (2000) for a discussion of low-rank relaxations, and for connections to some different low-rank ideas.

Let $\mathcal{P}$ be defined by (3.6). The rank-$k$ relaxation, denoted by

$$\max \{w^T x \mid x \in N_{+,k}(\mathcal{P})\},$$  \quad (6.8)

of the problem $\max \{w^T x \mid x \in N_{+}(\mathcal{P})\}$ is

$$\max \sum_i w_i x_i x_0$$

$$s.t. \quad (b_i x_0 - \sum_l a_{il} x_i) x_j \geq 0 \quad (\forall i, j)$$

$$\quad (b_i x_0 - \sum_l a_{il} x_i)(x_0 - x_j) \geq 0 \quad (\forall i, j)$$  \quad (6.9)

$$\quad x_i^2 - x_i x_0 = 0 \quad (\forall i)$$

$$\quad x_0^2 = 1$$

$$\quad x_i \in \mathbb{R}^k \quad (\forall i).$$

Choosing $k = 1$ and $k = n$ in (6.9), we get, respectively, the 0-1 integer program and $N_{+}(\mathcal{P})$. 
The condition \( x_i^2 - x_i x_0 = 0 \) has a nice geometric interpretation. The vectors \( x_i \) and \( x_i - x_0 \) are orthogonal. This means that the vectors \( x_i \) lie on a circle with center \( \frac{1}{2} x_0 \) and radius \( \| x_0 \| = 1 \). Without loss of generality, we can assume that \( x_0 = e_k \). See Figure 6.1.

Figure 6.1 : Circle centered around \((0,0.5)\) and vectors \( x_0 \), and \( x_1 \) in \( R^2 \).

In [23] it is proved that for the rank-two approximation of a certain semidefinite formulation of the \( \vartheta \)-function, the optimal value equals the largest stable set in \( G \). This implies that solving rank-two relaxations is NP-complete in general. Let \( P \) be the fractional stable set polytope. It is natural to expect that the rank-two relaxation \( \max \{ w^T x \mid x \in N_{+2}(P) \} \) yields the maximum-weight stable set, as \( N_+(P) \subseteq T \text{H}(G) \).

We give a simple proof of this fact (the proof becomes easier than the proof in for \( \vartheta \) in [23]). We can also extend the result to packing problems, that is problems defined by inequalities of the type:

\[
x_1 + x_2 + \cdots + x_k \leq 1.
\]

(6.10)

where the variables \( x_i \) are binary variables. Observe that, in (6.9), if we multiply the above inequalities by \( x_1, x_2 \) etc., we obtain the fact that every solution must satisfy

\[ x_i x_j = 0, \text{ for } 1 \leq i, j \leq k. \]

**Proposition 6.1** Let \( G = (V, E) \) be a graph and let \( P \) be defined by (6.4). Then for every \( w \in R^n \), \( \max \{ w^T x \mid x \in N_{+2}(P) \} = \max \{ w^T x \mid x \in P_I \} \).
Proof: We can assume that \( w \geq 0 \), in the statement of the proposition. Let \( w_+ \) be the maximum of \( w^T x \) over \( N_{+2}(P) \), and let \( w_\star \) be the weight of the maximum weight stable set. Obviously \( w_+ \geq w_\star \). Now assume that \( w_+ \) is attained at the solution \( x_1, \ldots, x_n \) in \( \mathbb{R}^2 \). The vectors \( x_i \) form an orthogonal labelling of \( G \): for every edge \( ij \in E, x_i^T x_j = 0 \). Let \( V' \subseteq V \) correspond to the nonzero vectors from \( x_1, \ldots, x_n \) and let \( G' = (V', E') \) be the subgraph of \( G \) spanned by \( V' \). Assume \( G' \) is connected.

Now if \( ij \) and \( jk \) are edges in \( G' \), then \( x_i x_j = 0 \), and \( x_j x_k = 0 \). But as all vectors satisfy \( x_i^2 - x_0 x_i \), this means that \( x_i = x_k \) (see Figure 6.1). From this we conclude that there can be no odd cycle in \( G' \) and \( G' \) is bipartite with bipartition \( V'_1 \) and \( V'_2 \).

Hence, \( x_i = x_1 \), for all \( i \in V'_1 \), and \( x_i = x_2 \), for all \( i \in V'_2 \), and

\[
w_+ = (\sum_{i \in V'_1} w_i) x_1^2 + (\sum_{i \in V'_2} w_i) x_2^2.
\]

Further \( x_1 x_2 = 0 \); this implies that \( x_1^2 + x_2^2 = 1 \). If \( \sum_{i \in V'_1} w_i \geq \sum_{i \in V'_2} w_i \), then the stable set \( V'_1 \) has weight

\[
w(V'_1) = \sum_{i \in V'_1} w_i = (\sum_{i \in V'_1} w_i)(x_1^2 + x_2^2) \geq w_+.
\]

This implies that \( w_\star \geq w_+ \). If \( G' \) has more than one component, we can similarly construct a stable set (by taking the union of stable sets in each component) with weight at least \( w_+ \). The result follows. \( \Box \)
Chapter 7

Cutting-plane proofs

7.1 Introduction

Chvátal (1973) introduced the idea of using cutting planes to prove that all integral solutions of a linear system of inequalities satisfy another inequality; this results in the notion of a cutting-plane proof. Chvátal developed this notion further in [26], [27], and [28] and showed that various results in combinatorics can be proved using cutting planes. Schrijver (1980) extended the applicability of cutting-plane proofs to rational systems of inequalities (Chvátal's results applied to bounded polyhedra). Cutting-plane proofs of the validity of many classes of inequalities can now be found in the literature. Cutting-plane proofs can be found, for example, in Grötschel and Pulleyblank (1986, for clique tree inequalities), Gerards and Schrijver (1986), and Müller and Schulz (1996). See also [63] and [29].

Cutting-plane algorithms, such as Gomory's algorithm [57], generate cutting-plane proofs in the process of solving an integer program. The finiteness of Gomory's algorithm implies the existence of cutting-plane proofs for inequalities valid for integer points in a rational or bounded polyhedron (assuming there is at least one such integer point). (See [110] and [48] for connections between Gomory's algorithm and cutting-plane proofs). On the other hand, lower bounds on the length of such proofs also result in lower bounds on the minimum number of iterations required to solve an integer program via Gomory's algorithm, or any other cutting-plane algorithm. Pučl (1997) established exponential lower bounds for the length of cutting-plane proofs; see also Cook and Haken (1999). This non-trivial result implies that Gomory's algorithm is non-polynomial for 0-1 integer programs. Gomory's algorithm is known
to be non-polynomial in general; a simple example given by Bondy (see Chvátal 1973) illustrates this fact.

The connection between cutting-plane proofs and the complexity class co-NP is immediate. If an integer program has no solutions, this fact can be established by a cutting-plane proof; if there always exists a short (polynomial length) cutting-plane proof, then the problem of determining if an integer program is feasible is in co-NP. As a consequence, we would have NP=co-NP, as this problem (feasibility of integer programs) is an NP-complete problem. A discussion of cutting-plane proofs in the context of complexity theory can be found in Cook, Couillard, and Turán (1987), Cook (1990), and Pudlák (1999). Complexity issues related to such proofs have also been studied by Goerdt (1991), Buss and Clote (1996), Bonet, Pitassi and Raz (1997), and Bonet, Esteban, Galesi, and Johansen (1998). Short cutting-plane proofs of infeasibility of integer programs do not always exist; this follows from Pudlák's work (1997).

All cutting-plane proofs described in the above papers involve the use of Gomory-Chvátal cuts. Proofs using other types of cutting planes, e.g. disjunctive cuts (Balas 1979), mixed integer cuts (Nemhauser and Wolsey 1990), split cuts (Cook, Kannan and Schrijver 1990), and matrix cuts, can be defined. For each of these classes of cuts, some cutting-plane proof results, similar to ones for Gomory-Chvátal cuts, hold; see the papers listed above. Lovász suggested the study of proofs using matrix-cuts, and Pudlák (1999) presents some results on such proofs. Among other things, Pudlák observes that matrix-cuts can be used to polynomially simulate resolution proofs, but not vice-versa.

We examine cutting-plane proofs based on matrix cuts, and compare such proofs with Gomory-Chvátal cutting-plane proofs. Some relationships between rank and proof length, which easily follow from [29], are mentioned in the next section. We then discuss the concept of interpolation and its usefulness in establishing lower bounds on lengths of cutting-plane proofs. We give an alternative proof of Pudlák's result
on interpolation of matrix-cut based proofs, and extend his result to $N_\epsilon$-cut based proofs.

In Section 7.4, the concept of monotone interpolation is introduced, and the main result of this chapter, an exponential lower bound on the length of $N_\epsilon$-cutting-plane proofs, is proved. Finally, we state some known relationships between matrix-cut proofs and other proof systems, and pose some questions.

### 7.2 Definition

In this section, we formalize the notion of an $N$-cutting-plane proof, and mention a few results on Gomory-Chvátal cutting-plane proofs which trivially carry over. Let

$$a_i^T x \leq b_i \ (i = 1, \ldots, m) \quad (7.1)$$

be a system of linear inequalities in $\mathbb{R}^n$, and let $Ax \leq b$ denote this system. Assume $c^T x \leq d$ is valid for all 0-1 solutions of $Ax \leq b$. An $N$-cutting-plane proof of $c^T x \leq d$ from $Ax \leq b$ is a sequence of inequalities,

$$a_{m+k}^T x \leq b_{m+k} \ (k = 1, \ldots, M), \quad (7.2)$$

with $c^T x \leq d$ the last inequality in the sequence, and a collection of numbers

$$\alpha_{jl}^k, \beta_{jl}^k \geq 0 \ (j = 1, \ldots, m + k - 1, l = 1, \ldots, n) \quad (7.3)$$

such that, for $k = 1, \ldots, M$, $a_{m+k}^T x \leq b_{m+k}$ is derived, as in (3.9), using $\alpha_{jl}^k, \beta_{jl}^k$, from

$$a_j^T x \leq b_j \ (j = 1, \ldots, m + k - 1).$$

Informally, each inequality in the sequence is an $N$-cut for the previous ones. The length of the cutting-plane proof is $M$ and its size is the sum of the sizes of the inequalities and numbers $\alpha_{jl}^k, \beta_{jl}^k$ in the proof. (Here, by the size of a proof, we mean the number of binary digits required to write down the proof). If an inequality belongs to $Ax \leq b$, then we say that it has an $N$-cutting-plane proof of length 0 from
\[ Ax \leq b. \] We will assume that inequalities in (7.1) and (7.2) are rational; we can then scale them to be integral.

To define an \( N^+ \)-cutting-plane proof, we need the additional squares of linear functions used in (3.10) We can instead replace these by an identically non-negative quadratic functions (represented by a positive semidefinite matrix; see (3.22)). Then, the inequalities \( a^T_{m+k} x \leq b_m \) are derived from \( a_j^T x \leq b_j \) \( (j = 1, \ldots, m + k - 1) \) and \( h_{m+k}(x) \), where \( h_{m+k}(x) \) is an identically non-negative quadratic function.

A convenient way of representing a cutting-plane proof, suggested by Chvátal (1984), is by a directed acyclic graph whose nodes correspond to the original inequalities and those in the proof. An arc from \( i \) to \( j \) means that the \( i \)th inequality is used in deriving the \( j \)th inequality (i.e., one of \( \alpha^i d \) and \( \beta^i d \) is nonzero) as an \( N \)-cut. The depth of an inequality corresponds to the length of the longest directed path ending at the final inequality. A tree-like cutting-plane proof is one in which every node in the graph, other than the ones corresponding to the initial inequalities, have out-degree 1. Tree-like Chvátal proofs have been studied in [77], [17] and [18] (we abbreviate Gomory-Chvátal cutting-plane proofs as Chvátal proofs).

We now restate a few properties of Chvátal proofs in the context of \( N \)-cutting-plane proofs and \( N^- \)-cutting-plane proofs; see [29, Theorem 6.1, Lemma 7.1]. We note a few trivial facts first. The length of an \( N \)-cutting-plane proof for an inequality is obviously greater than its \( N \)-depth. Secondly, any inequality valid for \( N(P) \) is an \( N \)-cut (unlike \( P' \)). Also, if \( N(P) \) is empty, then every inequality, not just \( 0^T x \leq -1 \), is an \( N \)-cut for \( P \).

**Lemma 7.1** Let \( P \subseteq Q \) be a polytope and let \( c^T x \leq d \) be an inequality valid for \( N^t(P) \). Let \( q = \frac{1}{2}n(n + 1) \).

(i) If \( N^t(P) \neq \emptyset \), then there is an \( N \)-cutting-plane proof of \( c^T x \leq d \) from \( P \) of length at most \((q^t - 1)/(q - 1)\).

(ii) If \( N^t(P) = \emptyset \), but \( N^{t-1}(P) \neq \emptyset \), then there is an \( N \)-cutting-plane proof of
\[ c^T x \leq d \text{ from } P \text{ of length at most } (q^t - 1)/(q - 1) + (q^{t-1} - 1)/(q - 1). \]

**Proof:** First assume that \( N^t(P) \) is non-empty. If \( t = 1 \), then \( c^T x \leq d \) is an \( N \)-cut. Consider some \( t > 1 \) and assume that (i) holds whenever \( c^T x \leq d \) is valid for \( N^k(P) \) with \( k < t \). From Lemma 3.3, \( c^T x \leq d \) has an \( N \)-cutting-plane proof of length one from at most \( q \) inequalities valid for \( N^{t-1}(P) \). By the induction hypothesis, each of these \( q \) inequalities has an \( N \)-cutting-plane proof of length at most \( (q^{t-1} - 1)/(q - 1) \). These cutting-plane proofs can be combined to get one for \( c^T x \leq d \) with length at most

\[ q(q^{t-1} - 1)/(q - 1) + 1, \]

and the result follows. To prove (ii), use the fact that \( c^T x \leq d \) is an \( N \)-cut for some \( q + 1 \) inequalities valid for \( N^{t-1}(P) \), and apply (i). \( \square \)

As the integral hull of \( P \) can be obtained in a finite number of iterations of the operator \( N(P) \), inequalities valid for \( P_t \) have \( N \)-cutting-plane proofs from \( P \). If \( P_t \) is empty, we will refer to a cutting-plane proof of \( 0^T x \leq -1 \) as a cutting-plane proof of infeasibility.

An inequality valid for \( P_t \) might not remain valid if we remove some constraint from the defining system for \( P \) (and change \( P_t \)). Any cutting-plane proof for the inequality must use every such constraint. This results in a bound on the length of cutting-plane proofs.

**Lemma 7.2** Let \( P \) be defined by (7.1). Let \( c^T x \leq d \) be valid for \( P_t \) and assume that the system

\[ a_i^T x \leq b_i \quad (i = 1, \ldots, m; \ i \neq k), \ c^T x > d \]

has a 0-1 solution for at least \( t \) values of \( k \). Let \( q = \frac{1}{2}n(n + 1) + 1 \). Then any \( N \)-cutting-plane proof of \( c^T x \leq d \) from (7.1) has length at least \( (t - 1)/(q - 1) \).

**Proof:** Let \( M \) be the minimum length of an \( N \)-cutting-plane proof of \( c^T x \leq d \), and assume that the inequalities in (7.2) constitute this proof. Let \( l_k \) be the number of
inequalities used in the derivation of $a_{m+k}^T x \leq b_{m+k}$. Then

$$L = \sum_{k=1}^{M} l_k \leq \sum_{k=1}^{M} q = qM;$$

from Lemma 3.3, we can assume that $l_k \leq q$. Every inequality $a_{m+k}^T x \leq b_{m+k}$, where $1 \leq k < M$, must be used in the derivation of some subsequent inequality in the sequence, otherwise the inequality can be omitted and a shorter cutting-plane proof obtained. Also, at least $t$ inequalities of the system (7.1) must be used. Hence $L \geq M - 1 + t$. Since $qM \geq L$, this implies $M \geq (t - 1)/(q - 1)$. \(\square\)

We can use both Gomory-Chvátal cuts and $N_+$-cuts in a cutting-plane proof; informally, an $N_+$-cutting-plane proof is a sequence of inequalities where each inequality is either a Gomory-Chvátal cut or an $N_+$-cut for the previous inequalities. The extension of Lemma 7.2 to $N_+$-cutting-plane proofs is trivial. We also define $N_-$-cutting-plane proofs; each inequality is either an $N$-cut or a Gomory-Chvátal cut for the previous ones. We abbreviate $N$-cutting-plane proofs by $N$-proofs; we abbreviate the other types of cutting-plane proofs mentioned in this section in a similar manner.

In the context of complexity theory, we are interested in the sizes of proofs. Cook, Coullard, and Turán (1987) showed that a Chvátal proof of $c^T x \leq b$, from $Ax \leq b$, can be mapped to another Chvátal proof of $c^T x \leq d$, of the same length and with size polynomial in the sizes of $Ax \leq b$ and $c^T x \leq d$, and the length of the proof. Thus, bounds on lengths of Chvátal proofs are equivalent, up to a polynomial factor, to bounds on sizes of such proofs. We do not know if a similar result holds for $N$-proofs and $N_+$-proofs.

### 7.3 Interpolation and cutting-plane proofs

It is a non-trivial task to prove strong (e.g., super-polynomial) lower bounds for lengths of proofs in a propositional proof system. See Cook and Rekhow (1979) for a precise definition of such proof systems, and see Beame and Pitassi (1998), and also [102], for recent surveys on this topic. Some of the methods developed recently
are, the bottleneck counting method of Haken (1985), the restriction method of Ajtai (1994), and the interpolation method.

Krajíček (1994) proposed the idea of using effective interpolation to establish lower bounds on the lengths of proofs in different proof systems. See also [86]. Razborov (1995), and independently, Bonet, Pitassi, Raz (1997), were the first to use interpolation in proving exponential lower bounds for some proof systems. Pudlák (1997) derived exponential lower bounds for lengths of Chvátal proofs. In Bonet, Pitassi, Raz (1997), an exponential lower bound for a restricted version of the Chvátal proof system is presented.

In this section we discuss interpolation in the context of cutting-plane proofs and extend a result of Pudlák's on N-cutting-plane proofs (see [102, Theorem 2]).

Suppose we have the infeasible system of inequalities

\[ Ax + Cz \leq e. \]
\[ By + Dz \leq f, \]
\[ x, y, z \text{ are 0-1.} \quad (7.4) \]

Then, the inequality \( 0^T x \leq -1 \) has a Chvátal proof \( P \) from (7.4). Let \( z' \) denote some 0-1 assignment to the variable \( z \). The system of inequalities.

\[ Ax \leq e - Cz', \]
\[ By \leq f - Dz', \]
\[ x, y \text{ are 0-1,} \quad (7.5) \]

obtained from (7.4), is still infeasible. Now, \( P \) can easily be modified to a proof \( P' \) of infeasibility of (7.5), with the same length (here, let \( P_i \) stand for the \( i \)th inequality in \( P \)):

\[ \text{if } P_i \text{ is } a^T x + b^T y + c^T z \leq d \text{ then } P'_i \text{ is } a^T x + b^T y \leq d - c^T z'. \quad (7.6) \]

Observe that, in (7.5), we have two systems of inequalities, with no variables in common, and at least one of the two is infeasible. Information about which of the two
systems is infeasible, can be extracted from $\mathcal{P}$. Pudlák (1997) showed the following: 

*Given any $z'$, it is possible to construct, in polynomial time (in the size of the proof $\mathcal{P}$), two Chvátal proofs, one involving $x$ alone, and the other involving only $y$, such that the last inequality in one of the two, is $0^T x \leq -1$.*

This results in a polynomial-time algorithm $\mathcal{F}_\mathcal{P}(z)$, for each composite Chvátal proof $\mathcal{P}$, which takes as input $z'$, and decides which of the two systems in (7.5) is infeasible (one can simultaneously extract a Chvátal proof of infeasibility). This is called **effective interpolation**.

The reason for obtaining interpolation results is that lower bounds can sometimes be derived for the complexity of the interpolating algorithm $\mathcal{F}_\mathcal{P}$; we immediately get a lower bound for the length of $\mathcal{P}$. We discuss this later.

Similar effective interpolation results can be proved for cutting-plane proofs with $N$-cuts or $N_+$-cuts. First we give Pudlák's effective interpolation result for Chvátal proofs [101, Theorem 3]. This follows from Proposition 7.3.

Assume that the following inequality system, in $n$ variables and $m$ inequalities, is infeasible:

$$Ax \leq e, \quad x \text{ is 0-1 ,} \quad (7.7)$$

$$By \leq f, \quad y \text{ is 0-1 ,} \quad (7.8)$$

and assume that $x$ and $y$ have no variables in common.

**Proposition 7.3** [101] Let $\mathcal{R}$ be a Chvátal proof of $0^T x \leq -1$ from (7.7) and (7.8). In polynomial time (in the size of $\mathcal{R}$), a Chvátal proof of infeasibility of either (7.7), or of (7.8), can be constructed from $\mathcal{R}$.

**Proof:** Let $a_i^T x + b_i^T y \leq d_i$ be the $i$th inequality in $\mathcal{R}$ and call this $\mathcal{R}_i$. Now, $\mathcal{R}_1, \ldots, \mathcal{R}_m$ are just the inequalities in (7.7) and (7.8) and the last inequality $\mathcal{R}_k$ (for some $k$) is precisely $0^T x \leq -1$. We can assume that inequalities in $\mathcal{R}$ have integral coefficients. We say that $\mathcal{R}_i$ is derived from $\mathcal{R}_j$, for $j = 1, \ldots, i - 1$, if

$$a_i^T x + b_i^T y = \sum_j \lambda_{ij} (a_j^T x + b_j^T y) \quad \text{and} \quad \left| \sum_j \lambda_{ij} d_j \right| \leq d_i,$$
where $\lambda_{ij} \geq 0$.

We construct a sequence of inequalities $S_i$, involving only $x$, and another sequence, $T_i$, involving only $y$, such that $S_i$ and $T_i$ together imply $R_i$. For the first $m$ inequalities in $R$, if $R_i$ involves only $x$, then set $S_i$ to $R_i$ and $T_i$ to $0^T x \leq 0$, otherwise reverse this assignment. Define subsequent terms of $S$ and $T$ as follows: for $i = m + 1, \ldots, k$, if $R_i$ is derived from $R_j$ with the numbers $\lambda_{ij} \geq 0$, then let $S_i$ be derived from $S_j$ and let $T_i$ be derived from $T_j$, with the same numbers $\lambda_{ij}$. If the right-hand sides of $S_i$ and $T_i$ are $g_i$ and $h_i$ respectively, we can conclude that

$$S_i \equiv a_i^T x \leq g_i, \quad T_i \equiv b_i^T y \leq h_i,$$

with $g_i + h_i \leq d_i$.

Therefore, the last inequalities in $S$ and $T$ are, respectively $0^T x \leq g_k$ and $0^T x \leq h_k$. Since $d_k = -1$, one of $g_k$ and $h_k$ is at most -1, and we have a Chvátal proof of infeasibility of either (7.7) or (7.8). This is a polynomial-time construction. 

Suppose $P$ is a Chvátal proof of $0^T x \leq -1$ from (7.4). The polynomial-time algorithm $F_P(z)$, mentioned above, broadly does the following. If $z'$ is a 0-1 assignment to $z$, then $F_P$ first computes (7.5) and $P'$ (from (7.6)). Then it computes one of the two proofs in Proposition 7.3, say $S$. If in the last inequality $S_k$, $g_k \leq -1$, then $F_P$ returns a 0 to indicate that $Ax \leq e - Cz'$ is infeasible, else it knows that $h_k \leq -1$, and it returns a 1.

Observe that we do not need to compute $T$. In fact, $F_P$ only needs to compute $e - Cz'$ and the numbers $g_i$ to decide which of the two systems in (7.5) is infeasible.

To apply a similar idea to $N$-proofs and $N_+$-proofs, Padlák proved the following result.

**Theorem 7.4** ([102]) Let $R$ be an $N_\#$-proof of $0^T x \leq -1$ from (7.7) and (7.8). In polynomial time (in the size of $R$), an $N_\#$-proof of infeasibility of either (7.7), or of (7.8), can be constructed from $R$.

We have restated the actual result slightly. Observe that, in the above proposition, the proofs are $N_\#$-proofs and involve rounding.
We present a more geometric approach than Pudlák’s proof to proving Proposition 7.4. This will allow us to extend the idea to $N_+\,$-proofs (more precisely $N_\ast\,$-proofs), which is mentioned as being unsolved by Pudlák (1999). See [102, page 11]. This generalization will follow from the next lemma.

**Lemma 7.5** Let $P_1$ and $P_2$ be polytopes defined by the inequalities in (7.7) and (7.8). Then $N_+(P_1 \cap P_2) = N_+(P_1) \cap N_+(P_2)$. An identical result holds for the $N$ and $N_0$ operators.

**Proof:** We have $P_1 = \{(x, y)|Ax \leq e\}$ and $P_2 = \{(x, y)|By \leq f\}$. For any two polytopes $P_1$ and $P_2$, it is true that $N_+(P_1 \cap P_2) \subseteq N_+(P_1) \cap N_+(P_2)$. To prove the reverse inclusion, assume that

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \in N_+(P_1) \cap N_+(P_2).$$

Then there are symmetric matrices $X \in M_+(P_1)$ and $Y \in M_+(P_2)$, such that $z = Xe_0 = Ye_0$, where

$$X = \begin{pmatrix} 1 & x^T & y^T \\ x & X_{11} & X_{12} \\ y & X_{12} & X_{22} \end{pmatrix}, \quad \text{and} \quad Y = \begin{pmatrix} 1 & x^T & y^T \\ x & Y_{11} & Y_{12} \\ y & Y_{12} & Y_{22} \end{pmatrix}. \quad (7.9)$$

We can conclude that $X_{11} - xx^T$ and $Y_{22} - yy^T$ are both positive semidefinite. This is true because of Proposition 2.8, and the fact that

$$\begin{pmatrix} 1 & x^T \\ x & X_{11} \end{pmatrix} \succeq 0, \quad \text{and} \quad \begin{pmatrix} 1 & y^T \\ y & Y_{22} \end{pmatrix} \succeq 0$$

(set $A = 1$ in Proposition 2.8); the above matrices are principal submatrices of $X$ and $Y$.

Now, let $Z$ be the matrix defined by

$$Z = \begin{pmatrix} 1 & x^T & y^T \\ x & X_{11} & xy^T \\ y & yx^T & Y_{22} \end{pmatrix}.$$
It is not difficult to verify that \( Z \) is contained in \( M(P_1 \cap P_2) \). Also, observe that \( Z - \overline{z} \overline{z}^T \) is a block diagonal matrix with nonzero blocks \( X_{11} - xx^T \) and \( Y_{22} - yy^T \). Therefore \( Z \) is positive semidefinite. Since \( \overline{z} = Ze_0 \), we have shown that \( z \in N_+(P_1 \cap P_2) \), and the result follows for the semidefinite operator.

Observe that, in (7.9), if we start out with \( X \) in \( M(P_1) \) and \( Y \) in \( M(P_2) \), then \( Z \) yields the result for the commutative operator. Let \( X \) and \( Y \) belong to \( M_0(P_1) \) and \( M_0(P_2) \) respectively. Then \( X \) and \( Y \) are as in (7.9), except that they are non-symmetric and \( X_{12}^T \) is replaced by \( X_{21} \), and \( Y_{12}^T \) by \( Y_{21} \). The matrix \( Z \) above, formed from \( X \) and \( Y \), belongs to \( M_0(P_1 \cap P_2) \), and the result for the non-commutative operator follows. \( \square \)

We can now prove the following result.

**Proposition 7.6** Let \( \mathcal{R} \) be an \( N_+ \)-proof of \( 0^T x \leq -1 \) from (7.7) and (7.8). In polynomial time (in the size of \( \mathcal{R} \)), an \( N_+ \)-proof of infeasibility of either (7.7), or of (7.8), can be constructed from \( \mathcal{R} \).

**Proof:** We can assume, by scaling, that all inequalities in \( \mathcal{R} \) are integral. We show the result for the case that all inequalities are \( N_+ \)-cuts for the previous ones in \( \mathcal{R} \). An inequality which is a Gomory-Chvátal cut for the previous ones in the proof can be handled as in Proposition 7.3.

Let \( \mathcal{S} \) and \( \mathcal{T} \), and the first \( m \) inequalities in \( \mathcal{S} \) and \( \mathcal{T} \), be as in Proposition 7.3. Assume that we have obtained \( \mathcal{S}_j \) and \( \mathcal{T}_j \) for all \( j < i \), where \( i \) is some number greater than \( m \). Also assume that

\[
\mathcal{S}_j \equiv a_j^T x \leq g_j, \quad \text{and} \quad \mathcal{T}_j \equiv b_j^T y \leq h_j, \quad \text{with} \quad g_j + h_j \leq d_j, \tag{7.10}
\]

for every \( j < i \), where \( \mathcal{R}_j \) is the inequality \( a_j^T x + b_j^T y \leq d_j \). Note that (7.10) obviously holds for \( i \leq m + 1 \); we prove, by induction, that it holds for all \( i \). Let \( P_1 \) be the polytope defined by the inequalities \( \mathcal{S}_1, \ldots, \mathcal{S}_{i-1} \). Similarly, let \( P_2 \) be defined by \( \mathcal{T}_1, \ldots, \mathcal{T}_{i-1} \). Now, if \( P \) is defined by the inequalities \( \mathcal{R}_1, \ldots, \mathcal{R}_{i-1} \), then \( P_1 \cap P_2 \subseteq P \).
It follows that $a_i^T x + b_i^T y \leq d_i$ is valid for $N_+(P_1 \cap P_2)$ and is an $N_+(P_1 \cap P_2)$-cut. Take the union of the sets of inequalities defining $N_+(P_1)$ and $N_+(P_2)$; this completely defines $N_+(P_1 \cap P_2)$ because of Lemma 7.5. This means, by Carathéodory’s Theorem, that
\[ a_i^T x + b_i^T y = \sum_j \alpha_j p_j^T x + \sum_k \beta_k q_k^T y \quad \text{and} \quad \sum_j \alpha_j r_j + \sum_k \beta_k s_k \leq d_i, \]
where $\alpha_j, \beta_k \geq 0$, and $p_j^T x \leq r_j$ are $N_+(P_1)$-cuts and $q_k^T x \leq s_k$ are $N_+(P_2)$-cuts. Adding separately the $N_+(P_1)$-cuts and the $N_+(P_2)$-cuts, we see that
\[ a_i^T x \leq g'_i \text{ is an } N_+(P_1)-\text{cut and } b_i^T x \leq h'_i \text{ is an } N_+(P_2)-\text{cut}, \]
where $g'_i$ and $h'_i$ are real numbers such that $g'_i + h'_i \leq d_i$. The numbers $g'_i$ and $h'_i$ can be computed as
\[ g'_i = \max \{ a_i^T x | (x, y) \in N_+(P_1) \}, \quad h'_i = \max \{ b_i^T x | (x, y) \in N_+(P_2) \}. \quad (7.11) \]

To get the $i$th terms of $S$ and $T$, we compute $g'_i$ and $h'_i$, as in (7.11), by solving two semidefinite programs. As these are not necessarily integers, we round them down to get $g_i$ and $h_i$; we have completed the construction of the $i$th terms in $S$ and $T$.

Finally, repeating this process, we get, as the last inequalities in $S$ and $T$, $0^T x \leq g_k$, and $0^T x \leq h_k$, where at least one of $g_k$ or $h_k$ is at most $-1$. \( \square \)

In the proof above, we can replace $N_+(P)$ by $N(P)$, and we get Pudlák’s result, Theorem 7.4, by combining with Proposition 7.3. We can also use $N_0(P)$ instead of $N_+(P)$.

Note that, in (7.11), in polynomial time we can only approximate $g'_i$ and $h'_i$. Thus, we can only certify that
\[ g'_i + h'_i \leq d_i + \epsilon, \text{ for fixed } \epsilon > 0. \]

As $d_i$ is an integer in (7.11), and we round down $g'_i$ and $h'_i$ to integers, as long as we choose $\epsilon$ to be smaller than one, we will obtain, after rounding down,
\[ g_i + h_i = |g'_i| + |h'_i| \leq d_i. \]
We can then continue the induction step in Proposition 7.6.

Another potential problem is that computing \( g'_i \) in (7.11) is polynomial in \( a^T_i x \leq b_1, \ldots, a^T_{i-1} x \leq g_{i-1} \), and thus depends on the numbers \( g_k \) generated along the way. We need to ensure that the \( |g_k| \) does not become too large. This is easy to see: if \( g_i \) is much less than zero, then \( a^T_i x \leq g_i \) cannot be satisfied by any 0-1 vector, and we know that (7.7) is unsatisfiable. The same holds for \( h_i \). As neither \( g_i \) nor \( h_i \) can be too small, and their sum is bounded above by \( d_i \), they cannot be too large either.

### 7.4 Monotone interpolation

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a real-valued non-decreasing function, that is, if \( x \leq y \), with \( x, y \) in \( \mathbb{R}^n \), then \( f(x) \leq f(y) \). Such a function is called a *monotone function*. Examples of monotone unary and binary functions (referred to as *monotone operations*) are

\[
tx, \quad r + x, \quad x + y, \quad [x], \quad thr(x, -1)
\]

where \( t \) is a non-negative number, \( x \) and \( y \) are real variables, and \( r \) is a real number; \( thr(x, -1) \) is a *threshold function* which returns 0, if \( x \leq -1 \), and returns 1 otherwise.

Consider a fixed system in (7.4), and the algorithm \( \mathcal{F}_p \), mentioned earlier, in the context of Chvátal proofs. Observe that, once we have computed \( e - Cz' \), all further computations to obtain the numbers \( g_i \), are monotone. In fact, we can perform a sequence of monotone operations, with precisely the ones in (7.12), to get the \( g_i \), and then apply \( thr(x, -1) \) to the last one. Further, if all coefficients in \( C \) are negative, then \( e - Cz' \) is obtained through monotone operations. Therefore, from \( z' \), we get the desired outputs 0 or 1, by monotone operations only. This is an example of *monotone interpolation*. This result is a crucial step in the proof of an exponential lower bound on the length of Chvátal proofs.

Razborov (1985), and independently Andreev (1985), proved super-polynomial lower bounds on the lengths of sequences of monotone boolean operations ('and' and 'or') required to compute monotone boolean functions. Alon and Boppana (1987)
improved this to an exponential lower bound; see Boppana and Sipser (1990) for a nice exposition of this result. Pudlák (1997) and Cook and Haken (1999) extended the above results to arbitrary monotone operations (the individual monotone computations can also have a bounded number of inputs). This, along with Pudlák's monotone interpolation result mentioned above, yields an exponential lower bound on the length of Chvátal proofs.

We state an effective interpolation result for \( N^*_\cdot \)-cuts. Further, all computations in the corresponding interpolating algorithm are monotone. However, this does not result in exponential lower bounds, as some of the monotone computations do not have a bounded number of inputs.

**Theorem 7.7** Consider the infeasible system of inequalities \( (7.4) \) and assume that \( C \leq 0 \). Also, let \( P \) be an \( N^*_\cdot \)-proof of infeasibility. Then, there is a polynomial time algorithm \( F_P(z) \), such that, if \( z' \) is any 0-1 assignment to \( z \), then \( F_P \) computes \( 0 \) if \( Ax \leq e - cz' \) is infeasible, and \( 1 \) otherwise. Further, \( F_P \) performs only monotone arithmetic computations.

**Proof:** If \( P \) is an \( N^*_\cdot \)-proof, then \( P' \), defined by \( (7.6) \), is an \( N^*_\cdot \)-proof of the infeasibility of \( (7.5) \). This is true because of Lemma 3.6, as \( (7.5) \) defines a face of the polytope defined by \( (7.4) \). We then proceed as in the case of Chvátal proofs, and apply Proposition 7.6 to show the existence of a polynomial-time algorithm \( F_P \) obtaining the desired output. For monotonicity, observe that the following computation

\[
\max \{ a_i^T x | x \in N^*_\cdot (P) \},
\]

\[
P = \{ x \in Q | a_1^T x \leq g_1, \ldots, a_i^T x \leq g_{i-1} \},
\]

(7.13)

computed in \( (7.11) \), is monotone in the inputs \( g_1, g_2, \ldots, g_{i-1} \). If the numbers \( g_1, \ldots, g_{i-1} \) are increased, then \( P \) is larger, and so is \( N^*_\cdot (P) \), and the maximum in \( (7.13) \) increases. \( \square \)
Even using Lemma 3.3, we need up to $O(n^2)$ of the inequalities $a_1^T x \leq g_1, \ldots, a_{l-1}^T x \leq g_{l-1}$ in (7.13). If $P$ in (7.13) is defined by a fixed number of inequalities, then we can get exponential lower bounds. It turns out that if we replace $N_+$-proofs by $N_0$-proofs in Theorem 7.7, we can choose $P$ in (7.13) defined by at most two inequalities (besides the bounds $0 \leq x \leq 1$). To see this we will need the following lemma.

**Lemma 7.8** Let $P$ be an $N_0$-cutting-plane proof of $c^T x \leq d$, from some polytope in $Q_n$, of length $L$. There exists a sequence of inequalities, of length at most $2n^2 L$, with $c^T x \leq d$ as the last inequality, such that every inequality in the sequence is either an $N_0$-cut for at most two previous inequalities, or the nonnegative linear combination of previous inequalities.

**Proof:** Let $a^T x \leq b$ be an inequality in the proof (not in $Ax \leq b$). Let $P = \{x | Ax \leq b\}$ be the polytope defined by the inequalities used in deriving $a^T x \leq b$ as an $N_0$-cut. By Lemma 3.9, $N_0(P) = \cap_i P_i$, where $P_i = \text{conv}(P \cap F_i^0) \cup (P \cap F_i^1)$. (Here $P_i$ is simply the lift-and-project operator (for i) from Section 3.4). Therefore $N_0(P)$ is completely defined by the inequalities defining the polytopes $P_i$. Hence, by Carathéodory's Theorem, $a^T x \leq b$ is a nonnegative linear combination of $n$ inequalities valid for the polytopes $P_i$ ($1 \leq i \leq n$). We can conclude, from Lemma 3.7 and its proof, that any inequality valid for $P_i$, say $g_i^T x \leq h_i$, is an $N_0$-cut for two inequalities valid for $P$, say $g_1^T x \leq h_1$ and $g_2^T x \leq h_2$. Further, both $g_1^T x \leq h_1$ and $g_2^T x \leq h_2$ are obtained by taking nonnegative linear combinations of at most $n$ inequalities in $Ax \leq b$. Thus, we add $2n^2$ inequalities for $a^T x \leq b$, and the result follows. □

The sequence generated in Lemma 7.8 proves the validity of $c^T x \leq d$. We refer to such a sequence as a simplified $N_0$-proof. If we allow rounding (or Gomory-Chvátal cuts), we call such a cutting-plane proof an $N_1$-proof.

Finally, replacing $N_+$-proofs by $N_1$-proofs in Theorem 7.7, we get the following result.
Theorem 7.9 $C \leq 0$. Let $\mathcal{P}$ be an $N_1$-proof of infeasibility of (7.4) and assume that $C \leq 0$. Then, there is a polynomial time algorithm $\mathcal{F}_P(z)$, which uses monotone arithmetic operations only, such that, if $z'$ is any 0-1 assignment to $z$, then $\mathcal{F}_P$ computes 0 if $Ax = e - cz'$ is infeasible, and 1 otherwise. Further, $\mathcal{F}_P$ performs only monotone arithmetic computations.

Proof: Replace $N_\star$ by $N_1$ in Theorem 7.7, and also in Proposition 7.6, to get corresponding results. Also observe that after such a replacement, (7.13) is replaced by

$$\max \{ a^T \mathcal{x} | \mathcal{x} \in N_1(P) \},$$

$$P = \{ \mathcal{x} \in Q | (a')^T \mathcal{x} \leq g', (a'')^T \mathcal{x} \leq g'' \},$$

(7.14)

where $(a')^T \mathcal{x} \leq g'$ and $(a'')^T \mathcal{x} \leq g''$ are two inequalities among the first $i - 1$ inequalities. This computation is a monotone operation in the inputs $g'$ and $g''$. $\square$

We now state the exponential lower bound of Cook and Haken (1999); we could also use Pudlák's (1997) lower bound result. We need two families of graphs, $\mathcal{G}$ and $\mathcal{B}$, defined on $n$ nodes, where $n = m^2 - 2$, for some $m$. Every graph in $\mathcal{G}$ is the union of $m - 1$ cliques of size $m$, and one clique of size $m - 2$; also $\mathcal{G}$ contains every graph of this type. Similarly, let $\mathcal{B}$ be the set of complements of graphs in $\mathcal{G}$. $\mathcal{G}$ and $\mathcal{B}$ have the same cardinality, and are disjoint. Cook and Haken (1999) call the graphs in $\mathcal{G}$ "good" graphs, and the ones in $\mathcal{B}$, "bad" graphs.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7_1.png}
\caption{A good graph; here $m = 3$ and $n = 7$.}
\end{figure}
Figure 7.2: A bad graph; here $m = 3$ and $n = 7$. Dotted lines are connected.

Now we define a system of inequalities such that every 0-1 solution corresponds to a graph containing a good graph and also a bad graph; as this is not possible, the system will be unsatisfiable. To this end, assume we have nodes $1, \ldots, n$, and two sets of labels $A = \{1, \ldots, n\}$ and $B = \{1, \ldots, n\}$. We define variables $x_{ij}, y_{ij}$ and $z_{ij}$ such that

$$z_{ij} = 1, \text{ if there is an edge } ij, \text{ 0 otherwise},$$

$$x_{ij} = 1, \text{ if } A_i \text{ is mapped to node } j, \text{ 0 otherwise},$$

$$y_{ij} = 1, \text{ if } B_i \text{ is mapped to node } j, \text{ 0 otherwise}.$$

We want $x_{ij}$ and $y_{ij}$ to define one-to-one mappings. We add the equations

$$\sum_i x_{ij} = 1 \forall j, \sum_j x_{ij} = 1 \forall i, \quad (7.15)$$

$$\sum_i y_{ij} = 1 \forall j, \sum_j y_{ij} = 1 \forall i. \quad (7.16)$$

We want the first $m$ labels in $A$ to represent a clique, the next $m$ labels to represent another clique, and so on until the last $m - 2$ labels, which represent a clique of size $m - 2$. In a similar manner, labels in $B$ represent stable sets. We add the inequalities

$$x_{si} + x_{tj} \leq 1 + z_{ij} \forall ij \text{ and } \forall s, t \text{ in the same clique}, \quad (7.17)$$

$$y_{si} + y_{tj} \leq 2 - z_{ij} \forall ij \text{ and } \forall s, t \text{ in the same stable set.} \quad (7.18)$$

Finally let $x, y, z$ be vectors of variables, with components $x_{ij}, y_{ij}$ and $z_{ij}$, respectively. Let $z'$ be a 0-1 assignment to $z$; then $z'$ corresponds to a graph. If $z'$ satisfies
the inequalities (7.15) and (7.17), for some 0-1 \( z \), then \( z' \) contains a good graph as a subgraph. If it satisfies (7.16) and (7.18), for some \( y \), then \( z' \) contains a bad graph. If we let (7.4) stand for the inequalities (7.15) - (7.18), then (7.4) will be unsatisfiable. Also, note that \( C \leq 0 \) in (7.4).

Now, consider the subset of 0-1 assignments \( z' \) corresponding only to good graphs or bad graphs. The algorithms \( \mathcal{F}_P \) defined in Theorems 7.9 and in the context of Chvátal proofs, return 1 if \( z' \) is a good graph, and 0 if \( z' \) is a bad graph, by sequences of monotone operations. Cook and Haken (1999) proved the following result.

**Theorem 7.10 [31, Theorem 1]** Any monotone circuit, with input \( z \) on \( n \) nodes, which returns 1 if \( z' \) is a good graph, and returns 0 if \( z' \) is a bad graph, must contain at least \( 2^{Kn^{1/8}} \) gates, for some positive constant \( K \).

A monotone circuit corresponds to a sequence of monotone operations, and a gate in such a circuit corresponds to a single monotone operation. This results in exponential lower bounds on the length of Chvátal proofs or \( N_1 \)-proofs of unsatisfiability of inequalities (7.15) - (7.18). By Lemma 7.8, the length of an \( N_1 \)-proof is polynomially bounded by the length of an \( N_0 \)-proof.

**Theorem 7.11** Any \( N_0 \)-proof of unsatisfiability of the \( O(n^4) \) inequalities (7.15) - (7.18) in \( O(n^3) \) variables has length at least \( 2^{Kn^{1/8}}/(2n^2) \), for some positive constant \( K \).

### 7.5 Relationships with other systems

Let \( Ax \leq b \) define a polyhedron \( P \) contained in the \( n \)-dimensional 0-1 cube. Suppose \( c^T x \leq d \) is valid for \( P \). As the Chvátal rank of \( P \) is bounded by \( 3n^2 \log n \) (see Eisenbrand 2000), there is a Gomory-Chvátal cutting-plane proof of \( c^T x \leq d \), from \( Ax \leq b \), of length bounded above by \( n^{O(n^2 \log n)} \). This follows from the analogue of Lemma 7.2 for Gomory-Chvátal cuts.

An interesting question is the following.
Question 7.12 If $c^T x \leq d$ is an $N$-cut ($N_+$-cut) for $Ax \leq b$, a polytope in $Q_n$, does there always exist a Gomory-Chvátal cutting-plane proof of $c^T x \leq d$ from $Ax \leq b$ with length bounded by a polynomial function of $n$?

If the answer to this question is positive, we will say that the Chvátal proof system (defined by Gomory-Chvátal cuts) polynomially simulates the $N$-cut proof system. Let $f(n)$ be the bound in question 7.12; if we replace $N$-cuts by $N_+$-cuts, let the bound be $f_+(n)$. We do not know the answer to the above question, i.e., we do not know if either of $f(n)$ or $f_+(n)$ is a polynomial.

However, we can say the following.

Proposition 7.13 If $f(n)$ is a polynomial function, then the Chvátal rank of polytopes in $Q_n$, will be bounded above by $nf(n)$, also a polynomial in $n$. Further, there will exist infeasible integer programs, with polynomially many (in $n$) variables, but requiring an exponential length $N$-cutting-plane proof.

This is easy to see. Firstly, for any polytope $P$, $P^{(f(n))}$ is contained in $N(P)$. Further, if $f(n)$ is polynomial, then a polynomial-length $N$-proof can be translated into a polynomial-length Chvátal-proof, and, if the final inequality is $0^T x \leq -1$, into a polynomial-size Chvátal-proof by results in Cook, Coullard, and Turán (1987). Then Pudlák’s result, that for an infinite class of integer programs, every Chvátal-proof of infeasibility must have exponentially length (and size), would imply the second statement.

We can then say that $f(n) > 1$ as there are examples of polytopes with Chvátal rank greater than $n$ (see Eisenbrand and Schulz 1999). Also

$$f_+(n) \geq \lceil \log_2(n - 1) \rceil;$$

this follows from Hartmann [69, Theorem 3.1.1], where the Chvátal rank of the fractional stable set polytope of the complete graph is shown to equal the right hand-side of the above equation (the $N_+$-rank is 1 in this case).

Consider the reverse problem.
**Question 7.14** Is it possible to polynomially simulate Gomory-Chvátal cuts by $N$-cuts or $N_+$-cuts.

Again letting $g(n)$ ($g_+(n)$) stand for the maximum length of an $N$-cut based proof of a Gomory-Chvátal cut, we know from Theorem 4.2 that $g(n)$ or $g_+(n)$ is at least $n$.

Let $CP$ stand for the Chvátal proof system, and let $CP_2$ stand for the restriction of $CP$, where only division by 2 is allowed. This means that while taking nonnegative combinations of inequalities, we are only allowed to multiply inequalities with multiples of $\frac{1}{2}$. Buss and Clote (1996) proved the following interesting result.

**Proposition 7.15** [21] $CP_2$ polynomially simulates $CP$.

This means that Question 7.14 is equivalent to

**Question 7.16** Given an inequality $c^T x \geq d - \frac{1}{2}$. does there exist a polynomial-length $N$-proof (or $N_+$-proof) of the inequality $c^T x \geq d$?

We have already seen that given $x_1 + \cdots + x_n \geq \frac{1}{2}$, any $N_+$-proof of $x_1 + \cdots + x_n \geq 1$ has length at least $n$. (In fact $N$ cuts are enough to derive $x_1 + \cdots + x_n \geq 1$; combining this with the polynomial simulation of resolution by Gomory-Chvátal cutting planes given in [33], we have a polynomial simulation of resolution by $N$-cuts or $N_+$-cuts.)

We believe that any $N_+$-proof of $x_1 + \cdots + x_n \geq \frac{n}{2}$, from

$$x_1 + \cdots + x_n \geq \frac{n}{2} - \frac{1}{2},$$

is exponential in $n$. We have not been able to demonstrate this fact.
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