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Stable Homotopy Invariance of Teichner's sect Invariant

by

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Abstract

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Given a notion of equivalence of 4-manifolds, there is a corresponding notion of stable equivalence: \( M \) is stably equivalent to \( N \) if \( M \# rS^2 \times S^2 \) is equivalent to \( N \# sS^2 \times S^2 \) for some non-negative integers \( r, s \). Any equivalence relation which extends over an \( S^2 \times S^2 \) summand gives a well-defined equivalence relation, and homotopy equivalence is such a relation. In this paper, we examine how the invariant \( \text{sec} \) of a 4-manifold \( M \) with finite fundamental group and spin universal cover relates to the stable homotopy type of \( M \). The \( \text{sec} \) invariant of a manifold \( M \) may be defined in terms of a characteristic 3-dimensional homology class \( w_2 + w \) on a null-cobordism of \( M \). In the case where \( \text{sec} = 0 \), we are able to conclude some geometric information about \( w_2 + w \): namely, that \( w_2 + w \) is represented by \( S^3 \). This allows us to prove that \( \text{sec}(M) \) determines the stable homotopy type of \( M \), or more generally, that manifolds \( M \) and \( N \) for which \( \text{sec}(M - N) \) is defined and equal to 0, are stably homotopy equivalent. We also prove a partial converse to this theorem. If \( M \) and \( N \) are homotopy equivalent, and there exists a homeomorphism \( M \# \mathbb{CP}^2 \rightarrow N \# \mathbb{CP}^2 \) which preserves the homotopy classes of the core 2-spheres of the \( \mathbb{CP}^2 \), then \( \text{sec}(M - N) = 0 \).
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Chapter 1

Introduction

The homeomorphism classification of 4-manifolds is known to be a difficult problem. There are several ways to simplify it; for example, restrict attention to a single fundamental group, or a certain type of intersection form. In this paper, we will simplify the problem by looking at a different equivalence: stable homeomorphism. Two 4-manifolds are stably homeomorphic if, after taking a connected sum of each with some number of \( S^2 \times S^2 \)'s, they are homeomorphic. More generally, given a notion of equivalence of 4-manifolds, we can make a notion of stable equivalence from it as long as the equivalence extends over \( S^2 \times S^2 \) summands. So in addition to stable homeomorphism, we may talk about stable diffeomorphism, stable homotopy equivalence, and stable cobordism, as well as many other stable relations.

In general, classifying manifolds up to a stable equivalence is easier than classifying them by the equivalence. For example, we know by [1] that a pair of simply connected closed 4-manifolds are homeomorphic if and only if they have the same Kirby-Siebenmann
invariant and isomorphic intersection forms. This reduces the topological classification problem into an algebraic classification problem (namely, classifying forms over \( \mathbb{Z} \)). This algebraic problem still turns out to be quite difficult; for example, the classification of definite forms is still not known. The stable case turns out to be much easier, though. Since a connected sum with \( S^2 \times S^2 \) forces a manifold to have indefinite intersection form, simply connected closed 4-manifolds are stably homeomorphic if and only if they have the same Kirby-Siebenmann invariant, signature, and parity. Each of these invariants is relatively easy to compute, and we may even think of a stable homeomorphism class of a simply connected manifold as being an element of the group \( \mathbb{Z} \times \mathbb{Z}/2 \times \mathbb{Z}/2 \).

Seeing that the simply connected case becomes easier, we might hope the same is true for other fundamental groups. Since we don’t have a homeomorphism classification in general, it’s hard to compare, but there are plenty of groups for which the stable problem can be solved.

In [2], the author develops a theory which mimics the high-dimensional classification of manifolds: he defines a classifying space \( B \) and a fibration \( B \rightarrow BO \), such that a map of a manifold into the fibration describes the homotopy type of the manifold up to a certain point. Although there may be non-homotopy equivalent manifolds which have maps into the same fibration, we still have Kreck’s stable homeomorphism theorem, which we state and prove as Theorem 8. This is analogous to the \( s \)-cobordism theorem: if a pair of manifolds are cobordant, and the cobordism satisfies the appropriate structure conditions, we get an equivalence between the manifolds. As a result, the question of stable homeomorphism becomes a question in algebraic topology: can we compute the cobordism
group $\Omega^B_4 \rightarrow BO$ over each fibration $B \rightarrow BO$. Can we determine when two manifolds represent the same class in $\Omega^B_4 \rightarrow BO$?

In the simply connected case, this becomes a very easy problem, as we saw above. For many other groups, particularly non-solvable infinite groups, this remains a difficult problem. I'm not aware of any method for computing these cobordism groups in general, but for finite fundamental groups. [8] has a spectral sequence (called the James spectral sequence) which allows us to compute the cobordism in most cases. Computing the James spectral sequence requires a group $\Pi$, which is the fundamental group of $B$, the homology $H_\ast(\Pi)$, and in some cases, an element $w \in H^2(\Pi; \mathbb{Z}/2)$. Once we have these data, we have the $E^2$-term of the spectral sequence, and we may compute the $E^3$-term from it. There is no known algorithm for computing the higher-order terms. When we're dealing with 4-manifolds, however, for a given group there are often tricks to compute $E^4 \cong E^\infty$.

Once we know $E^\infty$, we know a graded version of $\Omega^B_4 \rightarrow BO$, and we know the stable homeomorphism classes for the group $\Pi$ and the element $w$. The James spectral sequence divides the cobordism group into 4 subquotients, which give rise to 4 invariants. Two of these are well-known: they are determined by signature and $\pi_1$-fundamental class. The other two, sec and tet, are not as well understood. [8] conjectures that for a manifold $M$, $\text{sec}(M)$ is determined by the stable equivariant intersection form of $M$; in other words, sec is a stable homotopy invariant. We prove a weak form of the conjecture, in which the connected sum $M \# \mathbb{C}P^2$ must have a unique characteristic class. We also prove the converse of the conjecture.

Section 2 defines the components of Kreck's theory of stable homeomorphism, and
discusses some of the motivation from high-dimensional theory. We define the classifying spaces $B$ and their associated fibrations in any dimension, and sketch the proof of the cobordism theorem. We then look at dimension 4, and try to describe the associated 1-universal fibrations. In particular, we find that an oriented 1-universal fibration is determined by its fundamental group $\Pi$, and in certain cases, the element $w \in H^2(\Pi; \mathbb{Z}/2)$.

In Section 3, we define the James spectral sequence, and prove that it exists. Once we know it exists, we develop some computational tools, including the map $d_2 : E^2_{p,q} \rightarrow E^2_{p-2,q+1}$ in low dimensions. This comes from a relationship between the James and Atiyah-Hirzebruch spectral sequences, which we also investigate. In turn, this leads to some facts about cobordism and how it relates to the two spectral sequences. Then we restrict to the case of 1-universal fibrations for manifolds with spin universal cover: these are the only ones where $\text{sec}$ is defined. Using the Atiyah-Hirzebruch spectral sequence, we identify signature and $\pi_1$-fundamental class. Then we prove a geometric characterization of $\text{sec}$ developed in [8], which we will use in later proofs. We finish up by stating Teichner’s conjecture.

Section 4 deals with $\text{sec}$ and its relation to stable homotopy. We prove two major theorems. First, if the element $\text{sec}(M - N)$ is well-defined for the manifolds $M$ and $N$ and equals 0, then the manifolds are stably homotopy equivalent over their 1-universal fibration. This is the converse of the conjecture: if $M$ and $N$ have the same value of $\text{sec}$, then they are stably homotopy equivalent. Second, we show that if $M$ and $N$ have spin universal covers, and the homeomorphism $M \# \mathbb{C}P^2 \rightarrow N \# \mathbb{C}P^2$ preserves the homotopy classes of the cores of the $\mathbb{C}P^2$s, then $\text{sec}(M - N) = 0$.

In general, what stable homeomorphism invariants over the fundamental group are
also stable homotopy invariants over the fundamental group? In the simply connected case, signature and parity are stable homotopy invariants; the Kirby-Siebenmann invariant is not. But Kirby-Siebenmann is the only invariant which distinguishes stable homeomorphism classes of simply connected stably homotopy equivalent manifolds. It does so for any fundamental group, but there are possibly two other such invariants: \textit{sec} and \textit{ter}. Theorem 6.3.2 of \cite{8} shows that \textit{sec}(M) and \textit{ter}(M) may be represented in part by the signature \(\sigma(M)\). Corollary 21 shows \textit{sec} distinguishes between stable homotopy types of stably \(\pi_1\)-cobordant manifolds, which (except for signature) means \textit{ter} is \textit{never} a stable homotopy invariant. \textit{sec} is not a stable homotopy invariant unless, given any two 4-manifolds \(M\) and \(N\) with spin universal cover which are stably homotopy equivalent over their fundamental group, there is a homeomorphism \(h : M \# \mathbb{C}P^2 \rightarrow N \# \mathbb{C}P^2\) which preserves the homotopy class of the cores of the \(\mathbb{C}P^2\)'s. This seems like a weak condition, but is nontrivial. So far, we have been unable to prove it either way.

So the conjecture is solved, except for a (hopefully) small condition. I hope to discover which groups satisfy the condition, and which don't: ideally all groups do, but we have no evidence either way. As for the other invariant, \textit{ter} lives in a subquotient of \(\Omega_4^B - BO\), and there is a nontrivial map \(\sigma : \Omega_4^B - BO \rightarrow \mathbb{Z}\). We know this is nontrivial since for any \([M] \in \Omega_4^B - BO\), \([M \# K3] \in \Omega_4^B - BO\) as well, and \(M \# K3\) isn't even cobordant to \(M\). Thus for any pair \((\Pi, w)\), the image of \(\sigma\) contains \(16\mathbb{Z}\). This is represented by \(E_{0,4}^\infty\); that is, \(\sigma : E_{0,4}^\infty \rightarrow 16\mathbb{Z}\) is an isomorphism. If \(w \neq 0\), then the image of \(\sigma\) is some larger subgroup of \(\mathbb{Z}\). Then taking the quotient of \(\Omega_4^B - BO\) by \(E_{0,4}^\infty\) gives us a map \(\sigma : E_{2,2}^\infty \rightarrow \mathbb{Z}/16\mathbb{Z}\) and \textit{ter} lives in \(E_{2,2}^\infty\). If \(E_{2,2}^\infty\) is not cyclic, then \(\sigma\) must have a nontrivial
kernel, and there must be stably homotopy equivalent 4-manifolds in that case which are not stably homeomorphic. We'd like to know when $E^\infty_{2,2}$ is not cyclic, and more generally, what the image and kernel of $\sigma : E^\infty_{2,2} \to \mathbb{Z}/16\mathbb{Z}$ are. The elements of the kernel of $\sigma$ distinguish between stably homotopy equivalent, non-stably homeomorphic manifolds. We don't know of any finite group $\Pi$ and element $w$ so that $E^\infty_{2,2}$ is not cyclic; but $E^\infty_{2,2}(D_\infty) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ when $w \neq 0$, for example, so stable homotopy type and the Kirby-Siebenmann invariant do not always determine stable homeomorphism type.

Ultimately, the stable classification program should provide some insight into the unstable program. [8] and [3] both prove that under the right conditions, a stable homeomorphism gives rise to a homeomorphism. If we can solve the more algebraic problem of stable homeomorphism classification, and understand how it relates to the unstable problem, then we should be able to solve the unstable problem.
Chapter 2

Classifying Spaces and Stable Homeomorphism

2.1 Fiber Bundles, Structure Groups, and Classifying Spaces

Recall the definition of a vector bundle: an $n$-dimensional (real) vector bundle $T \xrightarrow{\delta} M$ over a space $M$ is a pair $(T, \delta)$ with every point $x \in M$ having a neighborhood $U$ with $\delta^{-1}(U) \cong U \times \mathbb{R}^n$, and if two such neighborhoods intersect, the induced map on the fiber is an element of $GL(\mathbb{R}^n) \sim O(\mathbb{R}^n)$. [4] and [5] are good references for the development of vector bundles. Vector bundles arise naturally as tangent bundles of smooth manifolds. Another type of bundle, useful in manifold theory, is the topological vector bundle. An $n$-dimensional topological vector bundle also has fiber $\mathbb{R}^n$, but the transition functions on the fiber are homeomorphisms of $\mathbb{R}^n$ preserving the origin. These bundles may be considered as tangent bundles to topological manifolds: the fiber over each point is the domain of a chart.
containing the point. Vector bundles, or topological vector bundles, are often described in terms of the structure groups \( O(\mathbb{R}^n) \), or the group of homeomorphisms of \( \mathbb{R}^n \) that preserve the origin (denoted \( \text{Top}(\mathbb{R}^n) \)). To each vector bundle, there is an associated principal \( O(\mathbb{R}^n) \)-bundle, and likewise topological vector bundles have associated \( \text{Top}(\mathbb{R}^n) \)-bundles.

Given a (topological) group \( G \), a principal \( G \)-bundle over a space \( M \) is a pair \((T, \varepsilon)\), where \( \varepsilon : T \rightarrow M \) is a map such that every \( x \in M \) has a neighborhood \( U \) such that there is an isomorphism \( \phi_U : \varepsilon^{-1}(U) \cong U \times G \), and for any two such coordinate neighborhoods \( U, V \), the map \( \phi_V^{-1} \circ \phi_V : (U \cap V) \times G \rightarrow (U \cap V) \times G \) is given by left multiplication by an element of \( G \) on the factor \( G \). The set \( \varepsilon^{-1}(x) \cong G \) is called the fiber, and the maps \( \phi_U^{-1} \circ \phi_V \) are called the transition maps. The structure group of a principal \( G \)-bundle is \( G \) itself, by definition. There is a one-to-one correspondence between isomorphism classes of \( n \)-dimensional vector bundles and isomorphism classes of principal \( O(\mathbb{R}^n) \)-bundles, and between isomorphism classes of \( n \)-dimensional topological vector bundles and isomorphism classes of principal \( \text{Top}(\mathbb{R}^n) \)-bundles. Since \( O(\mathbb{R}^n) \) acts on \( \mathbb{R}^n \), the fiber of a principal \( O(\mathbb{R}^n) \)-bundle acts on \( \mathbb{R}^n \). Given a principal \( O(\mathbb{R}^n) \) bundle \((T, \varepsilon)\) over \( M \), we may define an associated vector bundle over \( M \): for each \( x \in M \), there is a neighborhood \( U \) of \( x \) and a homeomorphism \( \varphi : \varepsilon^{-1}(U) \rightarrow U \times G \). Then points in \( T \) are described by pairs \((g, x)\), with \( g \in G \). This description depends on the local trivialization \( \varphi \), though. The vector bundle associated to \((T, \varepsilon)\) has the total space which is a quotient of \( \mathbb{R}^n \times T \) by the relation \((v,(g,x)) = (g^{-1}(v),x)\), and the map \( \delta(v,(g,x)) = x \). Even though the coordinates \((g,x)\) may not be well-defined, we still obtain a vector bundle. We may also obtain principal bundles from fiber bundles: in fact, there's a one-to-one correspondence between
isomorphism classes of principal $O(\mathbb{R}^n)$-bundles and isomorphism classes of $n$-dimensional vector bundles.

There is a nice relationship between principal bundle theory and homotopy theory. If $\delta : T \to M$ is a principal $G$-bundle and $f : N \to M$ is a continuous map, then there is an induced bundle (the pullback bundle) $f^*(\delta)$ over $N$, whose fibers are the same as the fibers of images in $M$. Every principal $G$-bundle over a CW-complex (and as a result, every fiber bundle with that structure group) is the pullback of a certain universal bundle. The universal bundle for a group $G$ is a principal bundle $EG \to BG$ with fiber $G$, such that the total space $EG$ is contractible. Every principal $G$-bundle over a CW-complex $M$ is the pullback of the universal $G$-bundle for some map $h : M \to BG$. Two such maps are homotopic if and only if they induce isomorphic principal bundles. Thus the set of principal $G$-bundles over $M$ is in one-to-one correspondence with $[M, BG]$, the set of homotopy classes of maps from $M$ to $BG$.

$BG$ is called the classifying space of $G$. It is only well-defined up to homotopy: from its bundle definition, we conclude that $\pi_i(BG) \cong \pi_{i-1}(G)$. In particular, the set of principal $G$-bundles over $S^n$ is in one-to-one correspondence with the group $\pi_{n-1}(G)$. The classifying space $BO(\mathbb{R}^n)$ is the infinite Grassmann manifold of $n$-planes in $\mathbb{R}^\infty$, and $BSO(\mathbb{R}^n)$ is the infinite Grassmannian of oriented $n$-planes in $\mathbb{R}^\infty$. However, classifying spaces are generally difficult to construct: I'm not familiar with an explicit description of $BTop(\mathbb{R}^n)$.
2.2 Vector Bundles and Manifold Theory

Now we shift focus back to vector bundles. A stabilization of a vector bundle \( T \xrightarrow{\delta} M \) is the Whitney sum of \( \delta \) with a 1-dimensional trivial bundle. Two bundles which are isomorphic after some number of stabilizations are called stably equivalent: a stable equivalence class of vector bundles is known as a stable vector bundle. Strictly speaking, stable vector bundles are not truly vector bundles, since there is no well-defined finite-dimensional fiber. We may consider them to be bundles with fiber \( \mathbb{R}^\infty \) by taking a Whitney sum with a trivial \( \mathbb{R}^\infty \)-bundle; clearly, stably equivalent bundles will be isomorphic after adding \( \mathbb{R}^\infty \). Not all \( \mathbb{R}^\infty \)-bundles occur as stable vector bundles, though. These stable vector bundles have several nice properties that finite-dimensional bundles don't have. For example, stable vector bundles over a space \( M \) form a group \( K_0(M) \). There is also a well-defined structure group for stable vector bundles. \( O(\mathbb{R}^n) \) includes into \( O(\mathbb{R}^{n+1}) \) by fixing a basis of \( \mathbb{R}^{n+1} \) and letting elements of \( O(\mathbb{R}^n) \) act as the identity on the last basis element. This inclusion preserves the stable structure of a stabilization, and in fact the vector bundle induced from the inclusion of fibers of a principal bundle is a stabilization. Taking a direct limit, we obtain the structure group \( O \) of stable vector bundles, with classifying space \( BO \). \( O \) is a subgroup of \( O(\mathbb{R}^\infty) \), which consists of all the images of elements of \( O(\mathbb{R}^k) \) for all \( k \).

The space \( BO \) is, in some sense, a classifying space for smooth manifolds as well. A smooth \( n \)-manifold \( M \) has tangent bundle \( TM \), which may be stabilized. Then the stable tangent bundle has structure group \( O \), giving a map \( t : M \to BO \) characterizing the stable tangent bundle of \( M \). If there is another manifold \( N \), homotopy equivalent to \( M \), then there is a map \( s : N \to BO \) as well. But classifying maps into \( BO \) are only interesting
up to homotopy, so we may consider $s : M \to BO$ to be a different smooth structure on the topological space $M$. In fact, a smooth structure on $M$ is given by the transition functions of charts of $M$, which are elements of $C^\infty(\mathbb{R}^n)$. $C^\infty(\mathbb{R}^n)$ is homotopy equivalent to $O(\mathbb{R}^n)$, so that $BC^\infty(\mathbb{R}^n) \sim BO(\mathbb{R}^n)$. Thus a smooth structure on a manifold $M$ is partly determined by the classifying map of the stable tangent bundle $M \to BO$.

We will be less interested in $BO$, since it's not simply connected. $\pi_1(BO) \cong \pi_0(BO) \cong \mathbb{Z}/2$, which is in one-to-one correspondence with stable vector bundles over $S^1$. The trivial bundle is one, and the nonorientable stable vector bundle is not equivalent to the trivial bundle. More generally, if $H_1(M) \to H_1(BO)$ is onto, then $M$ is nonorientable. We will be working only with orientable or oriented spaces. A stable oriented vector bundle has structure group $SO$ (the direct limit of $SO(\mathbb{R}^n)$). with classifying space $BSO$. Stable spin bundles have structure group $Spin$, the universal cover of $SO$, with classifying space $BSpin$. The covering map $Spin \to SO$ induces a fibration $BSpin \to BSO$, but the fiber is no longer discrete: up to homotopy, it's $\mathbb{RP}^\infty = K(\mathbb{Z}/2,1)$.

$\pi_2(BSO) \cong \pi_1(SO) \cong \mathbb{Z}/2$, and if $\pi_2(M) \to \pi_2(BSO)$ is onto, then the induced bundle over $M$ is not spin. If $f : M \to BO$ is the map classifying the smooth structure of a manifold $M$, then $M$ is nonorientable if and only if $f_* : H_1(M) \to H_1(BO)$ is onto. Similarly, if $M$ is oriented, then $M$ is spin if and only if $f_* : H_2(M) \to H_2(BSO)$ is onto. These statements are probably more familiar in terms of the first and second Stiefel-Whitney classes: $M$ is orientable if and only if $w_1(M) = 0$, and is spin if and only if $w_1(M) = w_2(M) = 0$. There are universal classes $w_1 \in H^1(BO; \mathbb{Z}/2) \cong \mathbb{Z}/2$ and $w_2 \in H^2(BO; \mathbb{Z}/2) \cong \mathbb{Z}/2$, with $w_1(M) = f^*(w_1)$ and $w_2(M) = f^*(w_2)$. 
If \( w_1(M) = 0 \), there is a lift of the classifying map \( f \) from \( BO \) to \( BSO \). If in turn \( w_2(M) = 0 \), there is a lift to \( BSpin \). These lifts are generally not unique, and correspond to assigning an orientation or spin structure to \( M \). This description is a bit more cumbersome than using Stiefel-Whitney classes, but works much better for the topological case.

Let \( M \) be a topological manifold; then \( M \) has a map \( M \to BT_{\text{op}} \) classifying its stable topological tangent bundle, which in turn helps classify the topological structure of \( M \). \( \pi_0(Top) \cong \mathbb{Z}/2 \), and the component of \( Top \) containing the identity is called \( ST_{\text{op}} \). \( ST_{\text{op}}(\mathbb{R}^n) \) consist of all homeomorphisms of \( \mathbb{R}^n \) which preserve the origin and orientation. The universal cover of \( ST_{\text{op}} \) is called \( Top_{\text{Spin}} \). A topological manifold is \textit{topologically orientable} if there is a lift of its classifying map to \( BST_{\text{op}} \). This is no different from the usual notion of orientability. An orientable topological manifold is \textit{topologically spin} if there is a lift of its classifying map from \( BST_{\text{op}} \) to \( BT_{\text{op}}_{\text{Spin}} \).

### 2.3 Stable Homeomorphism via Classifying Spaces

In order to use classifying maps to get a full homeomorphism classification of manifolds, we need to know when two manifolds' classifying maps are homotopic. If two manifolds are not homotopy equivalent, then their classifying maps can't possibly be homotopic, and we know they're not homeomorphic. In fact, it doesn't make sense to ask if the classifying maps of two manifolds are homotopic unless the manifolds are homotopy equivalent to begin with. So in order to start this classification program through classifying maps, we need to know the homotopy classes of manifolds. Ideally, we could phrase this in terms of classifying spaces: but the homotopy type of a space isn't determined by a bundle
over the space. Homotopy type is determined by algebraic information, though; we can tell if two spaces are homotopy equivalent by checking whether there is a map between them inducing an isomorphism on their homotopy groups. It's a bit more difficult to codify both homotopy type and classifying maps into the same space, since manifolds don't necessarily have the same homotopy groups as $BO$. We will try to compromise by defining a space which determines half the homotopy type of a manifold.

[2] develops a theory of stable classification of manifolds by mixing the homotopy classification of manifolds with the bundle theory. Before we describe this, we must define a \textit{structure in a fibration}. Let $\xi : B \to C$ be a fibration, whose base $C$ is some sort of classifying space; i.e., a manifold $M$ has a given map $\mu : M \to C$. A structure for $M$ in the fibration $\xi$ is a map $\bar{\mu} : M \to B$ such that $\xi \circ \bar{\mu} = \mu$. A structure of $M$ in $\xi$ encodes structures in both $C$ and $B$, as well as a relationship between them. Thus

\textbf{Definition 1}  
1. [2] Given a fibration $\xi : B \to BO$, a normal $B$-structure on a smooth $n$-manifold $M$ is a lift of the classifying map $\mu : M \to BO$ to a map $\bar{\mu} : M \to B$.

2. A normal $B$-structure $\bar{\mu} : M \to B$ is a normal $k$-smoothing if it is a $(k+1)$-equivalence.

3. $\xi : B \to BO$ is $k$-universal if the fiber is connected and its homotopy groups vanish in dimensions $\geq k + 1$.

Normal $k$-smoothings classify a portion of the homotopy structure of $M$. $k$ need not be the dimension of $M$; in general, if $n$ is the dimension of the manifold, then we will be interested in normal $\left[\frac{n}{2}\right] - 1$-smoothings. Normal $n$-smoothings give a complete homotopy description of $M$ as well as the smooth structure, but are usually too complicated to be helpful.
Even if we can determine when two manifolds are homotopy equivalent, we would still need a way to figure out when their classifying maps are homotopic. Suppose $\bar{\mu}, \bar{v} : M \rightarrow B$ are $n$-smoothings: i.e., two homotopy equivalent manifolds. From the homotopy theoretic standpoint, we want to know when $\bar{\mu} \amalg \bar{v} : M \amalg M \rightarrow B$ can be extended over $M \times I$. From the manifold theoretic standpoint, it's enough to have some $n+1$- manifold $V$ whose boundary is $M \amalg M$, so that $\bar{\mu} \amalg \bar{v}$ can be extended over $V$. When $M$ has dimension at least 5, the s-cobordism theorem tells us that $V = M \times I$. In dimensions 4 or lower, we don't necessarily know that $V = M \times I$, but we may still gain some information.

In general, given $k$-smoothings of $n$-manifolds, we'll be interested in finding cobordisms so that the $k$-smoothings extend over the cobordism. For $n$-smoothings, this is the same as an $s$-cobordism. For smoothings of lower dimension, such as $\left[\frac{n}{2}\right] - 1$, a cobordism over the $k$-smoothing has isomorphisms between the lower homotopy groups of it and its ends, but is rarely an $s$-cobordism. With $M$ a 4-manifold, for example, a 1-smoothing induces an isomorphism $\bar{\mu}_* : \pi_1(M) \rightarrow \pi_1(B)$, and any cobordism $V$ of $M$ which extends the 1-smoothing must have an isomorphism $i_* : \pi_1(M) \rightarrow \pi_1(V)$. For more information on cobordisms, look in [7].

**Definition 2** If $\Pi$ is any group, and $M, N$ are (oriented) manifolds with homomorphisms $\pi_1(M) \rightarrow \Pi$ and $\pi_1(N) \rightarrow \Pi$. A $\Pi$-cobordism from $M$ to $N$ is a connected manifold $V$ with $\partial V = M \amalg N$ making the following diagram commute:

$$
\begin{align*}
\pi_1(M) & \rightarrow \pi_1(V) & \rightarrow \pi_1(N) \\
\downarrow & \downarrow & \downarrow \\
\Pi & \end{align*}
$$
**Definition 3** A $\Pi$-null or $\Pi$-zero cobordism of a manifold $M$ is a $\Pi$-cobordism from $M$ to the empty manifold $\emptyset$.

**Definition 4** If $\xi : B \to BO$ is $k$-universal (or more generally, any fibration over a classifying space) and manifolds $M$ and $N$ have normal $B$-structures $\overline{\mu}$ and $\overline{\nu}$ respectively, then a $B$-cobordism from $M$ to $N$ is a manifold $V$ with $\partial V = M \sqcup -N$ and a normal $B$-structure $\Psi : V \to B$ so that $\Psi|_M = \overline{\mu}$ and $\Psi|_N = \overline{\nu}$.

**Definition 5** A $B$-null or $B$-zero bordism of $M$ is a $B$-cobordism from $M$ to the empty manifold $\emptyset$.

A normal universal $[\frac{n}{2}] - 1$-smoothing provides some data about homotopy type and smooth structure. Once we have a manifold and its associated $[\frac{n}{2}] - 1$-universal bundle, the next step is to try to construct $s$-cobordisms of manifolds using their normal universal $[\frac{n}{2}] - 1$-smoothings. However, a true $s$-cobordism requires more homotopy data than the normal universal smoothing provides. We can still obtain a classification result, but the conclusion is slightly weaker.

Before describing this result, we need to define a few terms.

**Definition 6** Given a $2q$-dimensional manifold $M$, a stabilization of $M$ is $M \# r(S^q \times S^q)$ for some non-negative integer $r$.

Two manifolds $M$ and $N$ are stably diffeomorphic if, after some stabilization, they become diffeomorphic. More precisely, $M$ and $N$ are stably diffeomorphic if $M \# r(S^q \times S^q)$ is diffeomorphic to $N \# s(S^q \times S^q)$ for some non-negative integers $r, s$. Stable diffeomorphism is an equivalence relation, since a diffeomorphism $d : M \to N$ may be extended over
$S^q \times S^q$ to give a diffeomorphism $d\# id : M\# S^q \times S^q \rightarrow N\# S^q \times S^q$. More generally, any equivalence relation which can be extended over the stabilizing factor can be made into a stable equivalence relation. Stable homeomorphism is an equivalence relation, as are stable homotopy equivalence and stable cobordism. We will prove that the other stable relations are actually equivalence relations later.

In order to stabilize the normal structures we'll be working with, we need to have a canonical normal structure on $S^q \times S^q$. $D^{q+1}$ has a unique normal structure $\eta$ in a given $B$. up to homotopy. Then crossing $\eta$ with a constant map gives a normal structure for $S^q \times D^{q+1}$. The restriction of that structure to the boundary $S^q \times S^q$ is called $\nu_c$.

**Definition 7** [2] If $V$ is a $B$-zero bordism of $M^{2q}$ and there is an imbedded torus $S^q \times D^{q+1} \rightarrow V$, join $\partial(S^q \times D^{q+1}) \setminus \partial V$ by an imbedded thickened arc $I \times D^q$ which meets $\partial(S^q \times D^{q+1})$ and $\partial V$ transversely in $\{0\} \times D^{2q}$ and $\{1\} \times D^{2q}$. The result $V'$ of removing the thickened arc and $S^q \times D^{q+1}$ is said to be obtained by subtracting a torus from $V$. Note $\partial V' = \partial V \# S^q \times S^q$.

**Theorem 8** [2] Let $M_0$ and $M_1$ be compact connected $2q$-dimensional manifolds with normal $(q-1)$-smoothings in a fibration $B$. Let $f : \partial M_0 \rightarrow \partial M_1$ be a diffeomorphism compatible with the normal $q-1$-smoothings. By a finite sequence of surgeries and compatible subtractions of tori, a normal $B$-zero bordism $W$ of $M_0 \cup_f M_1$ can be replaced by a relative $s$-cobordism between $M_0 \# r(S^q \times S^q)$ and $M_1 \# s(S^q \times S^q)$.

**Corollary 9** [2] Under the same conditions $f : \partial M_0 \rightarrow \partial M_1$ may be extended to a diffeomorphism $F : M_0 \# r(S^q \times S^q) \rightarrow M_1 \# s(S^q \times S^q)$. This diffeomorphism commutes up to
homotopy with the normal \((q - 1)\)-smoothings in \(B\) given by the normal \((q - 1)\)-smoothing on \(M_i\) and \(\nu_c\) on \(S^q \times S^q\).

We sketch the proof, since we will need to use a slightly modified version of it later.

**Proof.** Denote the group ring \(\mathbb{Z}[\pi_1(W)]\) by \(\Lambda\). \(W\) is a relative \(s\)-cobordism if and only if

i) \(\pi_1(M_i) \rightarrow \pi_1(W)\) is an isomorphism for \(i = 0, 1\)

ii) \(H_k(W, M_i; \Lambda) = 0\) for \(i = 0, 1\) and \(k \leq q\)

iii) The Whitehead torsion \(\tau(W, M_i)\) vanishes for \(i = 0, 1\).

A normal \((q - 1)\)-smoothing of \(M_0\) is a \(q\)-equivalence, so i) and ii) hold for \(k < q\).

Consider the diagram of exact sequences:

\[
\begin{array}{cccccccc}
& & H_{q+1}(B, W; \Lambda) & & \\
& \downarrow & & & \uparrow \\
& H_q(M_i; \Lambda) & \longrightarrow & H_q(W; \Lambda) & \longrightarrow & H_q(W, M_i; \Lambda) & \longrightarrow & 0 \\
& \downarrow & & & \downarrow \\
& H_q(B; \Lambda) & & & & & & 0 \\
\end{array}
\]

Since \(H_q(M_i; \Lambda) \longrightarrow H_q(B; \Lambda)\) is surjective, the same follows for \(H_{q+1}(B, W; \Lambda) \longrightarrow H_q(W, M_i; \Lambda)\). By the relative Hurewicz theorem, \(\pi_{q+1}(B, W) \longrightarrow H_{q+1}(B, W; \Lambda)\) is an isomorphism of groups. Since \(W\) and \(M_i\) are compact, there is a finite set of generators of \(\pi_{q+1}(B, W)\) mapping to generators of \(H_q(W, M_0; \Lambda)\). In particular, there is a finite set of generators of \(H_q(W, M_0; \Lambda)\) which are represented by disjointly embedded spheres. Subtract the corresponding tori to obtain a new cobordism \(W'\) with \(M_0\) replaced by \(M_0 \# r(S^q \times S^q)\).
By examining the long exact homology sequence of $(W, M_0 \cup [S^q \times D^{q+1} \cup I \times D^{2q}], M_0)$, we see that $H_q(W, M_0 \# r(S^q \times S^q); \Lambda) = 0$.

Now we must look at $H_q(W', M_1; \Lambda)$. It's a stably free module, so choose trivial disjointly embedded $q$-spheres in $W'$, join them to $M_0$, and subtract the corresponding tori. This has the effect of adding a free summand of rank the number of spheres to $H_q(W', M_1; \Lambda)$. Thus we may assume $H_q(W', M_1; \Lambda)$ is free. Choose a basis of $H_q(W', M_1; \Lambda)$ so that the torsion vanishes: we know this basis is represented by a set of disjointly embedded spheres. Join them to $M_1$ and subtract the corresponding tori to obtain $W''$: then $W''$ is a relative s-cobordism. ■

So the cobordism class of a normal $B$-structure $(M, \nu)$ determines the homeomorphism class of $M$ over the fibration $\xi : B \to BO$, up to connected sum with $S^q \times S^q$. In other words, the cobordism class of $(M, \nu)$ over $\xi$ determines the stable homeomorphism class of $(M, \nu)$ over $\xi$. The cobordism group is denoted $\Omega^B_{2q}$ or $\Omega^\xi_{2q}$, where $\xi : B \to BO$. This is particularly useful when $\xi$ is $q$-universal. Every $2q$-manifold has a smoothing in a $(q - 1)$-universal fibration, and two manifolds have smoothings in the same $(q - 1)$-universal fibration if and only if there is a $(q - 1)$-equivalence between them. See Proposition 10 for the 4-manifold case.

Unfortunately, the class of $(M, \nu)$ in $\Omega^\xi_{2q}$ does not necessarily determine the stable homeomorphism type of $M$. even for a $(q - 1)$-universal fibration $\xi$. The cobordism class depends on the structure map $\nu$, and a given manifold $M$ may have many possible structures in $\xi$. But if $M$ has two different structures in $\xi$, then the difference between the structures is represented by a fiber self-homotopy equivalence of $B$. Conversely, the set of fiber self-
homotopy equivalences $\text{Aut}(B)$ of $B$ acts on the set of structures of $M$, and in turn acts on $\Omega_{2q}^\xi$. Then the set of stable homeomorphism classes of manifolds with structures in $\xi$ is in one-to-one correspondence with $\Omega_{2q}^\xi/\text{Aut}(B)$. Neither of these groups is terribly easy to compute, although $\Omega_{2q}^\xi$ is generally much easier to compute than the set of smooth 2q-manifolds with structures in $BO$, for example. There's no reason $\Omega_{2q}^\xi/\text{Aut}(B)$ should be a group, or have any natural algebraic structure, unless a nice algebraic relationship between $B$-null cobordisms and $\text{Aut}(B)$ can be found.

2.4 Stable Homeomorphism in dimension 4

Now we specialize to the case $q = 2$. We'll be dealing with 4-manifolds $M$, with smoothings $\nu : M \to B$ which are 2-equivalences. These manifolds are stabilized by taking a connected sum with $S^2 \times S^2$. Every topological 4-manifold has a smoothing in a topological 1-universal fibration, which is a map $\xi : B \to B\text{Top}$ whose fiber's homotopy groups vanish in dimensions $\geq 2$. Since a 1-smoothing induces an isomorphism on fundamental groups. $M$ has a 1-smoothing in a 1-universal fibration whose fiber is $K(\pi_1(M), 1)$. Since we'll be working primarily with oriented, topological manifolds, we'll be interested in two possible types of 1-universal fibrations. Since the 1-smoothing is a 2-equivalence, $\pi_2(M) \to \pi_2(B)$ is onto. If $\widetilde{M}$ is spin, $\pi_2(M) \to \pi_2(B\text{Top})$ is the zero map, so $\pi_2(B) = 0$. Otherwise, $\pi_2(B) \cong \mathbb{Z}/2$. By [8], Section 8, a pair of 4-manifolds with structures in a given 1-universal fibration are stably homeomorphic if and only if they represent the same class in $\Omega_{2}^\xi$ and have the same Kirby-Siebenmann invariant. Thus we will be able to use a smooth calculation for the topological case.
For the spin case, there is an important invariant \( w \in H^2(\Pi; \mathbb{Z}/2) \), called the \( w_2 \)-class of \( M \). If \( \widetilde{M} \) is spin, \( M \) need not be: but the failure to be spin must somehow be induced from the action of \( \Pi \) on \( \widetilde{M} \). The \( w_2 \)-class of \( M \) is a measure of how the \( \Pi \)-action causes \( M \) to fail to be spin. There is a homotopy fibration \( \widetilde{M} \overset{p}{\longrightarrow} M \overset{\mu}{\longrightarrow} K(\Pi, 1) \), whose cohomology spectral sequence gives an exact sequence

\[
H^3(\widetilde{M}; \mathbb{Z}/2) \xrightarrow{d_2} H^2(\Pi; \mathbb{Z}/2) \xrightarrow{\mu^*} H^2(M; \mathbb{Z}/2) \xrightarrow{p^*} H^2(\widetilde{M}; \mathbb{Z}/2) \longrightarrow 0
\]

\( \widetilde{M} \) is a compact, simply connected 4-manifold, so \( H^3(\widetilde{M}; \mathbb{Z}/2) = 0 \). \( p^*(w_2(M)) = w_2(\widetilde{M}) = 0 \) by hypothesis, so there is a unique, well-defined element \( w \in H^2(\Pi; \mathbb{Z}/2) \) such that \( \mu^*(w) = w_2(M) \). \( w \) is an invariant of homeomorphisms which map \( \Pi \) identically.

Perhaps the simplest way to approach the problem of calculating \( \Omega_4^\xi \) is to determine the total space \( B \). We know \( \Omega_4^{BSTop} \cong \mathbb{Z} \), and \( \Omega_4^{BTop_{Spin}} \) has index 16 in \( \Omega_4^{BSTop} \), so we always know the cobordism of the base of \( \xi \). As we'll see later, \( \Omega_4^\Pi \) is also not too difficult to compute.

**Theorem 10** [8] *The 1-universal fibration corresponding to a (possibly nonorientable) 4-manifold \( M \) is determined by \( \pi_1(M) \), \( w_1(M) \in H^1(\Pi; \mathbb{Z}/2) \) and (if \( \widetilde{M} \) is spin) \( w \in H^2(\Pi; \mathbb{Z}/2) \). Recall that elements of \( H^1(X; \mathbb{Z}/2) \) are in one-to-one correspondence with the homotopy classes of maps \([X, K(\mathbb{Z}/2, 1)]\). When \( \widetilde{M} \) is not spin, \( B \) is defined by the following pullback*

\[
\begin{array}{ccc}
B & \overset{\mu}{\longrightarrow} & K(\Pi, 1) \\
\downarrow{\xi} & & \downarrow{w_1(M)} \\
B_{Top} & \overset{w_1}{\longrightarrow} & K(\mathbb{Z}/2, 1)
\end{array}
\]
When $\widetilde{M}$ is spin, $B$ is defined by

$$
\begin{array}{ccc}
B & \xrightarrow{\mu} & K(\Pi, 1) \\
\xi & & \downarrow w_1(M) \times w \\
BTop & \xrightarrow{w_1 \times w_2} & K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)
\end{array}
$$

Proof. If $\widetilde{M}$ is not spin, then $\pi_2(M) \to \pi_2(BTop)$ is onto. In order for $\xi$ to be a 1-smoothing, $\mu_* : \pi_1(B) \to \Pi$ must be an isomorphism. Since $\pi_1(M) \to \Pi$ is an isomorphism. Then if $\nu : M \to B$ is a classifying map, $(\xi \circ \nu)_* : \pi_1(M) \to \pi_1(BTop)$ must be onto. But $\pi_1(BTop) \cong \mathbb{Z}/2$, and $w_1$ classifies $\pi_1(BTop)$. Then we must also have $(\nu \circ w_1(M) \circ \mu)_* = (\nu \circ w_1 \circ \xi)_*$: but $\nu_*$ is an isomorphism, and homotopy classes of maps of a space $X$ into $K(\mathbb{Z}/2, 1)$ are in one-to-one correspondence with $H^1(X; \mathbb{Z}/2)$, and so are classified by the induced map on $\pi_1(X)$. Thus $\xi : B \to BTop$ is 1-universal.

When $\widetilde{M}$ is spin, the argument is identical, except that $\pi_2(M) \to \pi_2(BTop)$ is not onto. Since $h : \pi_2(M) \to H_2(M; \mathbb{Z}/2)$ is not onto, the homotopy class of the map $w_2 : M \to K(\mathbb{Z}/2, 2)$ is not determined by its action on $\pi_2(M)$. In fact, we have an exact sequence $\pi_2(M) \otimes \mathbb{Z}/2 \to H_2(M; \mathbb{Z}/2) \to H_2(\Pi; \mathbb{Z}/2) \to 0$ from the spectral sequence for the homotopy fibration $\widetilde{M} \to M \to K(\Pi, 1)$. Then $w_2$ is determined by $w \in H^2(\Pi; \mathbb{Z}/2)$, which corresponds to $w : K(\Pi, 1) \to K(\mathbb{Z}/2, 2)$. Then in the spin case, $\xi : B \to BTop$ is 1-universal also. ■

When $M$ is orientable, $w_1(M) = 0$, and the map $w_1(M) : K(\Pi, 1) \to K(\mathbb{Z}/2, 1)$ is homotopically trivial. For the case $\widetilde{M}$ not spin, this means the fibration $\xi : B \to BTop$ is the pullback of a trivial fibration. so $B$ is expressed as a product $B = BSTop \times K(\Pi, 1)$. $\xi$ is projection onto the first factor, followed by the double cover of $BSTop$ onto $BTop$. For the other case, $B = BTopSpin \times K(\Pi, 1)$ if $w_1(M) = 0$ and $w = 0$, but the fibration will
generally not be trivial otherwise.
Chapter 3

The James Spectral Sequence

3.1 Definition of the Spectral Sequence

If $\xi : B \to BSTop$ is a 1-universal fibration, we would like to compute $\Omega_4(\xi)$. If the structure of $B$ is, for example, $B = BSTop \times K(\Pi, 1)$, then we can compute $\Omega_4(\xi)$ using homology theory. Cobordism is an extraordinary homology theory and so satisfies the Künneth formula. Oriented topological cobordism is well-known for dimensions 4 and below, and $\Omega_4(\Pi)$ may be computed from the Atiyah-Hirzebruch spectral sequence. This gives $\Omega_4(\xi) = \Omega_4^{STop} \oplus H_4(\Pi; \mathbb{Z})$. In general, we will not even know the structure of $B$. We are fixing a fundamental group $\Pi$ and the $w_2$-class $w \in H^2(\Pi; \mathbb{Z}/2)$, but for any given $\Pi$, there may be many possible 1-universal fibrations. Even if we knew the structure, it's not clear how to compute $\Omega_4(\xi)$ unless $B$ is a product. The James spectral sequence, developed in [8], Section 6, gives a method for computing $\Omega_4(\xi)$ from the homology of the fiber and certain cobordism groups, which depend on the fibration.
Theorem 11 \cite{8} Let $h$ be a generalized homology theory which is connected, i.e., $\pi_i(h) = 0$ $\forall i < 0$. Furthermore, let $F \to B \xrightarrow{f} K$ be an $h$-orientable fibration, $\xi : B \to BSTop$ a stable topological vector bundle, and $M\xi$ the Thom spectrum corresponding to the fibration $\xi$. Then there exists a spectral sequence (called the James spectral sequence) whose $E^2$-term $E^2_{p,q} \cong H_p(K; h_q(M(\xi|F)))$, and which converges to the generalized homology $h_{p+q}(M\xi)$ of the spectrum $M\xi$.

Proof. There is a contractible space $ESTop$, which is the total space of the principal universal fibration of $STop$, so that $ESTop/STop$ is a model for $BSTop$. Each $STop(\mathbb{R}^n)$ is a subgroup of $STop$, and so acts freely on $ESTop$, so we have models $BSTop(\mathbb{R}^n) = ESTop/BSTop(\mathbb{R}^n)$. Then the maps $i_n : BSTop(\mathbb{R}^n) \to BSTop$ are fiber bundles with fibers $STop/STop(\mathbb{R}^n)$. Similarly, $i_n^{n+1} : BSTop(\mathbb{R}^{n+1}) \to BSTop(\mathbb{R}^{n+1})$ are fiber bundles with fibers $STop(\mathbb{R}^{n+1})/STop(\mathbb{R}^n)$. We may now construct a sequence of fiber bundles over $B$ by pulling back the original stable topological vector bundle:

$$
\begin{array}{cccc}
B_n & \xrightarrow{b_n} & B \\
\downarrow{\xi_n} & & \downarrow{\xi} \\
BSTop(\mathbb{R}^n) & \xrightarrow{i_n} & BSTop
\end{array}
$$

Composing the maps $b_n$ with the original fibration $f : B \to K$ we get a sequence of fibrations $f_n : B_n \to K$ with fibers $F_n$ together with topological vector bundles $\xi_n$ over each $B_n$ so that the following diagrams commute:

$$
\begin{array}{cccc}
F_n & \xrightarrow{F_n} & B_n & \xrightarrow{f_n} & K \\
\downarrow & & \downarrow{\text{id}} & & \downarrow{\iota_n} \\
F_{n+1} & \xrightarrow{F_{n+1}} & B_{n+1} & \xrightarrow{f_{n+1}} & K \\
\downarrow & & \downarrow{\iota_{n+1}} & & \downarrow{\text{id}} \\
BSTop(\mathbb{R}^n) & \xrightarrow{i_n} & BSTop(\mathbb{R}^{n+1})
\end{array}
$$

The Thom spectrum $M\xi$ consists of the family of Thom spaces $\{T(\xi_n), s_n : S^1 \wedge T(\xi_n) \to T(\xi_{n+1})\}$, and similarly $M\xi|F = \{T(\xi_n|F_n), s_n|S^1 \wedge T(\xi_n|F_n)\}$. For any $n > 0$, the disc-
sphere bundle pair \((D(\xi_n), S(\xi_n))\) is a relative fibration over \(B_n\) with fiber \((D^n, S^{n-1})\). Then composing with \(f_n : B_n \rightarrow K\) gives a fibration with fiber \((D(\xi_n|F), S(\xi_n|F))\). Then we have a relative Serre spectral sequence \(^nE\) for each \(n\): 
\[ H_p(K; h_q(D(\xi_n|F_n), S(\xi_n|F_n))) \Rightarrow h_{p+q}(D(\xi_n), S(\xi_n)) \]
If we stabilize the bundle \(\xi_n\) by adding a trivial bundle, then \(T(\xi_n \oplus \varepsilon) = S^1 \wedge T(\xi_n)\), and the resulting spectral sequence is isomorphic using the suspension isomorphism for \(h\). Then by composing with \(\xi_n \oplus \varepsilon \rightarrow \xi_{n+1}\), we obtain a map of spectral sequences \(^nE \rightarrow ^{n+1}E\). Taking the direct limit, we get a spectral sequence \((E^\infty_{p,q}, d^\infty_p) = (\lim^n E^1_{p,q}, \lim^n d^1_p)\). Since \(\lim\) is an exact functor, this graded algebra with differentials is actually a spectral sequence.

\[
E^2_{p,q} \cong \lim^n E^2_{p,q} \cong \lim H_p(K; h_q(D(\xi_n|F_n), S(\xi_n|F_n))) \cong H_p(K; \lim h_q(T(\xi_n|F_n)))
\]
\[
\cong H_p(K; h_q(M\xi|F)).
\]
Furthermore, \(E^\infty_{p,q} \cong \lim^n E^\infty_{p,q} \cong \lim h_{p+q}(D(\xi_n|F_n), S(\xi_n|F_n)) \cong \lim h_{p+q}(T(\xi_n)) \cong h_{p+q}(M\xi)\). ■

If \(\xi : B \rightarrow BTop\) is an orientable topological 1-universal fibration, then there is a map \(\mu : B \rightarrow K(\Pi, 1)\) inducing an isomorphism on \(\pi_1\). In fact, when \(B\) is associated to a manifold without spin universal cover, \(B = BTop \times K(\Pi, 1)\). If \(B\) is associated to something that has spin universal cover, then \(\mu\) defines \(B\) as a pullback of \(\omega_2 : BTop \rightarrow K(\mathbb{Z}/2, 2)\). In the former case, clearly the fiber \(F = BTop\). In the latter case, recall that \(TopSpin\) is the double cover of \(STop\): then there is a fibration \(TopSpin \rightarrow STop \rightarrow K(\mathbb{Z}/2, 1)\). This has the property \(\omega_2 = B\nu\), so that the fiber of \(\mu\) is equal to the fiber of \(\omega_2\), which is \(BTopSpin\).
3.2 Application to the Stable Homeomorphism Problem

Letting the extraordinary homology theory $h$ be stable homotopy $\pi^*$, we apply the James spectral sequence to the fibration $\mu: B \to K(\Pi, 1)$. $E^2_{p,q} \cong H_p(K(\Pi, 1); \pi^*_q(M(\xi|F))) \Rightarrow \pi^*_{p+q}(M\xi)$. By Thom's theorem, $\pi^*_q(M\xi) \cong \Omega_q(\xi)$, so $E^2_{p,q} \cong H_p(K; \Omega_q(\xi|F)) \Rightarrow \Omega_{p+q}(\xi)$. If $F$ is a point, then $M(\xi|F)$ is the spectrum of a point and $\Omega_q(\xi|F)$ is just $\Omega_q$. Also, $\xi = id$, so $\Omega_{p+q}(\xi) = \Omega_{p+q}(K(\Pi, 1)) = \Omega_{p+q}(\Pi)$. Then for $F$ a point, the James spectral sequence for stable homotopy collapses to the Atiyah-Hirzebruch spectral sequence for cobordism.

**Proposition 12** [8] Two 4-manifolds $M$ and $N$ with an isomorphism $\pi_1(M) \to \pi_1(N)$ and non-spin universal covers are stably homeomorphic over $\Pi$ if and only if they are $\Pi$-cobordant and have the same Kirby-Siebenmann invariant.

This is not difficult to prove using the fact that $B = BSTop \times K(\Pi, 1)$ in this case, but the proof is more elegant using the James spectral sequence.

**Proof.** If $M$ and $N$ are 4-manifolds with an isomorphism $\pi_1(M) \to \pi_1(N)$ and non-spin universal covers, then they share a 1-universal fibration $\xi: BSTop \times K(\Pi, 1) \to BTop$. They are stably homeomorphic over $\Pi$ if and only if they represent the same cobordism class in $\Omega_4(\xi)$, so we use the James spectral sequence to compute $\Omega_4(\xi)$. $E^2_{p,q} \cong H_p(\Pi; \Omega^S_{q}(\Pi))$, where $\Omega^{STop}$ is just oriented topological cobordism. These groups are well-known in low dimensions: $\Omega^S_0 = \Omega^S_4 = \mathbb{Z}$, and $\Omega^S_2 = \Omega^S_3 = \mathbb{Z}$. Then $E^2_{1,3} = E^2_{2,2} = E^2_{3,1} = E^2_{4,1} = E^2_{3,2} = E^2_{4,1} = E^2_{0,3} = 0$. and $E^2_{4,0} \cong H_4(\Pi; \mathbb{Z})$. $E^2_{5,0} \cong H_5(\Pi; \mathbb{Z})$. and $E^2_{0,4} \cong H_0(\Pi; \mathbb{Z})$. Then all the $d_2$ differentials in and out of $E^2_{p,q-p}$ vanish for all
$p$: in fact, the only differential which could affect dimension 4 is $d_4 : E^4_{3,0} \rightarrow E^4_{0,4}$. But $E^4_{3,0} \cong E^2_{8,0} \cong H_5(\Pi; \mathbb{Z})$ is finite, since $\Pi$ is finite, and $E^4_{0,4} \cong E^2_{0,4} \cong \mathbb{Z}$, so $d_4 = 0$ as well.

The map $f_*$ of the James spectral sequence to the Atiyah-Hirzebruch spectral sequence induces an isomorphism on $E^2_{p,4-p}$ and $E^2_{8,0}$ for all $p$, since $f_* : \pi_1(B) \rightarrow \pi_1(K(\Pi, 1))$ is an isomorphism. As in the James spectral sequence, all differentials vanish in the dimensions we care about, and so in fact $f_*$ is an isomorphism of spectral sequences. Thus $M$ and $N$ are cobordant over $\xi$ if and only if they represent the same class in the $E^\infty$-term of the Atiyah-Hirzebruch spectral sequence; i.e., if they are $\Pi$-cobordant. ■

As a result, the non-spin classification is reasonably easy, and we will be most concerned with the spin case.

### 3.3 Calculational Tools

**Theorem 13** [8] Let $Sq_w^2 : H^{p-2}(\Pi; \mathbb{Z}/2) \rightarrow H^p(\Pi; \mathbb{Z}/2)$ be the map given by $Sq_w^2(x) = Sq^2(x) + x \cup w$. Then

1. For $p \leq 4$, the differential $d_2 : H_p(\Pi; \Omega^{TopSpin}_1) \rightarrow H_{p-2}(\Pi; \Omega^{TopSpin}_1)$ is the dual of $Sq_w^2$.

2. For $p \leq 5$, the differential $d_2 : H_p(\Pi; \Omega^{TopSpin}_2) \rightarrow H_{p-2}(\Pi; \Omega^{TopSpin}_1)$ is the composition $(Sq_w^2)^* \circ r_2$ of reduction mod 2 with the dual of $Sq_w^2$.

**Proof.** There is a commutative diagram of fibrations

\[
\begin{array}{ccc}
\{pt\} & \longrightarrow & B \\
\downarrow & & \downarrow id \\
BT_{TopSpin} & \longrightarrow & B \\
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow id & & \downarrow f \\
B & \longrightarrow & K(\Pi, 1)
\end{array}
\]
If we apply the James spectral sequence in stable homotopy, then we have a map $f_\ast$ of spectral sequences:

$$H_p(B; \Omega_q) \xrightarrow{f_\ast} H_p(\Pi; \Omega_q(\xi|F))$$

$$d_2 \downarrow \quad \quad \quad \quad \quad d_2 \downarrow$$

$$H_{p-2}(B; \Omega_{q+1}) \xrightarrow{f_\ast} H_{p-2}(\Pi; \Omega_{q+1}(\xi|F))$$

For the Atiyah-Hirzebruch spectral sequence, $d_2$ is the dual of $Sq^2_\ast$, so that for $x \in E_{p,q}^2$ and $x = f_\ast(y)$, $d_2(x) = d_2(f_\ast(y)) = f_\ast((Sq^2)^\ast(y)) = (Sq^2_\ast)^\ast(f_\ast(y)) = (Sq^2_\ast)^\ast(x)$. The identity $f_\ast \circ Sq^2_\ast = Sq^2_\ast \circ f_\ast$ is really just a restatement of the fact that $w = w_2(\xi)$. $f$ is a 3-equivalence, so when $p \leq 4$, the Hurewicz theorem implies $f_\ast$ is onto. $f_\ast : H_5(B; \mathbb{Z}) \longrightarrow H_5(\Pi; \mathbb{Z})$ is onto if and only if $d_4 : H_5(\Pi; \mathbb{Z}) \longrightarrow H_4(BTopSpin; \mathbb{Z}) \cong \mathbb{Z}$ vanishes. But $\Pi$ is finite, so $H_5(\Pi; \mathbb{Z})$ is finite as well, and $d_4$ vanishes. Then the theorem is proved for the $p = 5$ case as well. ■

$$\Omega^\text{TopSpin}_3 \cong \Omega^\text{TopSpin}_4 \cong \mathbb{Z} \quad \text{and} \quad \Omega^\text{TopSpin}_3 = \Omega^\text{TopSpin}_4 = 0,$$

as in the non-spin case, but $\Omega^\text{TopSpin}_1 \cong \Omega^\text{TopSpin}_2 \cong \mathbb{Z}/2$. As a result, the differentials need not vanish in most cases. The theorem gives us a way to compute the $E^1$-term of the James spectral sequence, and most differentials $d_3$ vanish. The only one that may possibly be nontrivial is $d_3 : E^3_{5,0} \longrightarrow E^3_{2,2}$. Assuming it’s computable, then $d_4$ vanishes using the argument above, and $E^\infty_{p,4-p}$ is computable. Unfortunately, I know of no general formula describing $d_3$, and it’s computable only in special cases. [8] does the computations for cyclic and quaternionic fundamental groups, and uses facts about the cohomology rings and signatures of manifolds to determine $d_3$. 
3.4 Invariants in the James Spectral Sequence

There are 4 invariants coming from the James spectral sequence in dimension 4 (since $E^2_{1,3} = 0$). If $[M] \in \Omega_4(\xi)$, then $[M] \rightarrow E^\infty_{4,0} \rightarrow H_4(\Pi; \mathbb{Z})$, the (4,0) term of the Atiyah-Hirzebruch spectral sequence. The image of $[M]$ is $\mu_*(M) \in H_4(\Pi; \mathbb{Z})$, where $\mu : M \rightarrow K(\Pi, 1)$ classifies $\pi_1(M)$, and $\mu_*(M)$ is called the $\pi_1$-fundamental class of $M$. As long as $\Pi$ is finite, $E^\infty_{0,1} \cong \Omega^\text{TopSpin}_4$. The inclusion $\Omega^\text{TopSpin}_4 \rightarrow \Omega_4(\xi)$ can be defined by mapping a representative $M$ of $[M] \in \Omega^\text{TopSpin}_4$ homotopically trivial into $\xi$. The signature homomorphism $\sigma : \Omega_4(\xi) \rightarrow \mathbb{Z}$ maps $\Omega^\text{TopSpin}_4 \cong E^\infty_{0,1}$ to $16\mathbb{Z}$. The other two invariants are less familiar: if the $\pi_1$-fundamental class of $M$ vanishes, then the image of $[M]$ in $E^\infty_{3,1}$ is well-defined, and is called $\text{sec}$. When both $\text{sec}$ and the $\pi_1$-fundamental class vanish, the image in $E^\infty_{2,2}$ is called $\text{ter}$. As it has been noted, both signature and $\pi_1$-fundamental class are invariants of $\Pi$-cobordism. Thus stably homeomorphic manifolds which are $\Pi$-cobordant are distinguished by $\text{sec}$ and $\text{ter}$: as long as at least one of $E^\infty_{3,1}$ and $E^\infty_{2,2}$ is nonzero, and $\sigma|_{E^\infty_{3,1}}$ or $\sigma|_{E^\infty_{2,2}}$ is not an isomorphism, there are $\Pi$-cobordant manifolds which are not stably homeomorphic.

If $M$ is $\Pi$-null cobordant, there is a nice description of $\text{sec}$ in terms of the null cobordism. Suppose $W$ is a $\Pi$-null cobordism of $M$; then we have a given map $\nu : W \rightarrow K(\Pi, 1)$ inducing an isomorphism on the fundamental group. The class $w_2(W) + \nu^*(w)$, which we call $w_2 + w$ for short, vanishes on $M$: $w_2(M) = \mu^*(w) = i^*\nu^*(w)$. From the relative cohomology exact sequence for the pair $(W, M)$, we see there is a unique class $a$ in $H^2(W; \mathbb{Z}/2)$ such that $\partial(w_2 + w) = a$. The the Poincaré dual $PD(a)$ is an element of $H_3(W; \mathbb{Z}/2)$.
Proposition 14 [8] Let $W$, $M$, and $a$ be as above. Then $\sec(M) = \nu_*(PD(a)) \in H_3(\Pi; \mathbb{Z}/2)$.

Proof. In the James spectral sequence for a given extraordinary homology theory $h$, we denote the map $h_{p+q}(M\xi) \to E^\infty_{p+q,0}$ by $\varepsilon \theta$.

Since $W$ and $M$ are orientable, we may take the double cover of the base of $\xi : B \to BTop$ to obtain a new fibration $\xi' : B \to BSTop$. This fibration fits into the pullback diagram

$$
\begin{array}{c}
B \xrightarrow{p} K(\Pi, 1) \\
\varepsilon' \downarrow \quad w \downarrow \\
BSTop \xrightarrow{\omega_2} K(\mathbb{Z}/2, 2)
\end{array}
$$

As a result, we have the commutative diagram

$$
\begin{array}{c}
B \xrightarrow{p} PK(\mathbb{Z}/2, 2) \\
\varepsilon' \times_p \downarrow \\
BSTop \times K(\Pi, 1) \xrightarrow{\omega_2 + w = u} K(\mathbb{Z}/2, 2)
\end{array}
$$

Then we have the commutative diagrams of fibrations

$$
\begin{array}{c}
F = BTopSpin \xrightarrow{\pi} B \xrightarrow{p} K(\Pi, 1) \\
\varepsilon' \times_p \downarrow \quad \text{id} \downarrow \\
F' = BSTop \xrightarrow{p_2} BSTop \times K(\Pi, 1) \xrightarrow{\pi} K(\Pi, 1)
\end{array}
$$

Thus we may form the relative James spectral sequence for the relative fibration $(BSTop \times K(\Pi, 1), B)$. The pair $(W, M)$ represents an element of the cobordism of this relative fibration. and if $\partial : H_p(\Pi; \Omega_{q+1}(\xi|F', \xi|F)) \to H_p(\Pi; \Omega_q(\xi|F))$ is the induced map of spectral sequences, then $\partial([W, M]) = [M]$. If $E$ is the spectrum $\Sigma^2 H(\mathbb{Z}/2)$, so that $\pi_k^*(E \wedge X_+) = H_{k-2}(X; \mathbb{Z}/2)$ (where $X_+$ is the spectrum of $X$), then there is a map of spectra $u : M_{p_1} \to E$ induced from $u : BSTop \times K(\Pi, 1) \to K(\mathbb{Z}/2, 2)$. Since $u_\pi(\xi' \times p)$ is
homotopically trivial, we get a map of pairs of spectra $U : (Mp_2, M\xi) \to (E \land K(\Pi, 1)_+, \bullet)$. If we apply the James spectral sequence to these spectra, we get a commutative diagram

\[
\begin{array}{cc}
H_p(\Pi; \Omega_q(p_1|F', \xi|F)) & \Longrightarrow & \pi_{p+q}(Mp_1, M\xi) \cong \Omega_{p+q}^{(p_1, \xi)} \\
\downarrow & & \downarrow \\
H_p(\Pi; \pi_q(E)) & \Longrightarrow & \pi_{p+q}(E \land K(\Pi, 1)_+) \cong H_{p-q-2}(\Pi; \mathbb{Z}/2)
\end{array}
\]

For $p + q = 5$, we have a map $U_* : \Omega_5^{(p_1, \xi)} \to H_5(\Pi; \mathbb{Z}/2)$, and by naturality, $\text{sec}(M)$ is equal to the image of $[W, M]$ under this map. Thus we are done if $[W, M] \to \nu_*(\circ)$. If $<W, M> \in H_5(W, M; \mathbb{Z})$ is the fundamental class of the pair, then $\nu((W, M)) = \nu(<W, M>) \in H_5(BSt_{op} \times K(\Pi, 1), B; \mathbb{Z}/2)$. Naturality also implies that $\nu((U_*([W, M]))) = \nu((W, M)) \cap \circ$. The class $\nu_*(\circ) = \circ \in H^2(W, M; \mathbb{Z}/2)$, so $\text{sec}(M) = \nu_*(<W, M>) \cap \circ = \nu_*(<W, M> \cap \circ) = \nu_*(PD(\circ))$. ■

If $s \in \pi_2(W)$ is in the image of $i_* : \pi_2(M) \to \pi_2(W)$, then $(w_2 - w)(s) = 0$. Given any $s \in \pi_2(W)$ or $H_2(W; \mathbb{Z}/2)$, $(w_2 - w)(s) = 0$ if and only if either $s$ has spin normal bundle, or $s$ has a non-spin normal bundle and $\nu_*(s)$ is a nontrivial element of $H_2(\Pi; \mathbb{Z}/2)$ on which $w \in H^2(\Pi; \mathbb{Z}/2)$ evaluates to 1. In the latter case, $s$ is clearly not represented by a sphere, since $\pi_2(K(\Pi, 1)) = 0$. Thus $w_2 - w$ somehow measures the failure of the $\Pi$ action on $\tilde{M}$ to extend over a spin, simply connected manifold $\tilde{W}$. Thus we expect $\text{sec}$ should measure the same or something similar, but information is lost in the map $\nu_*$.

If $M$ and $N$ are $\Pi$-cobordant 4-manifolds, then a $\Pi$-cobordism $V$ between them also has a well-defined class $w_2 + w \in H^2(V, \partial V)$. This can be used to define an invariant $\text{sec}(M - N)$, which we will see later in more detail.

**Proposition 15** If $M$ and $N$ have a $\Pi$-cobordism $V$ with $w_2 + w = 0$, then $M$ and $N$ are
cobordant over the 1-universal fibration $\xi$.

Proof. Let $s \in H_2(V, M)$. The pair $(V, M)$ is simply connected, so $s$ is represented by $S^2 \to V$, and since $w(s) = 0$, $w_2(s) = 0$. Then we may find a basis for $H_2(V, M)$ and/or $H_2(V, N)$ represented by disjointly embedded 2-spheres and the corresponding tori may be deleted to form a stable $s$-cobordism between $M$ and $N$. Thus $M$ and $N$ are stably homeomorphic, so are cobordant over $\xi$. \[\square\]

If the $E_2^{22}$ of $\xi$'s James spectral sequence is nonzero, then there are manifolds $M$, $N$ which are not stably homeomorphic over $\xi$, but which have $sec(M - N) = 0$. Thus $ter$ is somehow determined by $w_2 + w$, but we don't yet know how.

The natural question at this point is, why are $sec$ and $ter$ separate invariants? It's useful to separate them when using the James spectral sequence, but there is evidence that they measure different things. In particular.

Conjecture 16. \cite{8} $sec(M)$ is determined by the stable equivariant intersection form of $M$.

It may be easier to think about this as a statement about the homotopy type of $M$: $sec(M)$ is determined by the stable homotopy type of $M$. It seems unlikely that, in general, the stable homeomorphism type of a manifold would be determined by its homotopy type and Kirby-Siebenmann invariant, so $ter$ should not be determined by stable homotopy. Ideally, $ter$ would only be well-defined between homotopy equivalent manifolds, but there is often signature information contained in $ter$. For the case $\Pi \cong \mathbb{Z}/2 \ast \mathbb{Z}/2$, the infinite dihedral group, \cite{6} describes a pair of manifolds with boundary $Y$ and $Y'$ which are homotopy equivalent rel boundary but not stably homeomorphic. $Y \# CP^2$ and $Y' \# CP^2$ are homeomorphic. Taking $N = Y \cup_{\partial} Y$ and $N' = Y \cup_{\partial} Y'$, we see that $N$ and $N'$ have the same
Kirby-Siebenmann invariant and are homotopy equivalent, but not stably homeomorphic.

We will examine $Y$ and $Y'$ in more detail later, where we will also see $\text{sec}(N - N') = 0$.

Thus (possibly outside the class of finite groups) $\text{tr} r$ is not determined by homotopy type.

The evidence for the conjecture comes from calculations in [8] for two classes of finite 2-groups: cyclic groups and quaternionic groups.
Chapter 4

Stable Homotopy

4.1 Basic Definitions

If $M$ and $N$ are manifolds which are stably homotopy equivalent over their (common) 1-universal fibration, then they are homotopy equivalent over $K(\Pi,1)$. Then $\mu_*([M]) = \mu_*([N])$. $M$ and $N$ also have the same signature, since homotopy equivalence preserves signature. Then $M$ and $N$ are $\Pi$-cobordant, and in fact, if they have non-spin universal covers and the same Kirby-Siebenmann invariant, are stably homeomorphic. In general, we would like to know when two $\Pi$-cobordant manifolds are stably homotopy equivalent over their 1-universal fibration. If $M$ and $N$ do have spin universal covers, then a stable homotopy equivalence between them need not be guaranteed if they only have equal signatures and $\pi_1$-fundamental classes. There is also a well-defined element $\sec(M - N)$, which is 0 if and only if there is a certain type of stable homotopy equivalence between $M$ and $N$.

Definition 17 If $M$ and $N$ are manifolds such that the fundamental group of $N$ has a finite
generating set $S_N$, and a map $\phi : S_N \rightarrow \pi_1(M)$ inducing a homomorphism from $\pi_1(N)$ to $\pi_1(M)$. then the connected sum $M \# \pi_1(M)N$ of $M$ and $N$ over $\pi_1(M)$ is the standard connected sum $M \# N$, followed by surgery on a loop of the form $g^{-1}\phi(g)$ for each nontrivial $g \in S_N$.

Most of the time, we will deal with manifolds $M$ and $N$ with given isomorphisms $\gamma_N : \pi_1(N) \rightarrow \Pi$ and $\gamma_M : \pi_1(M) \rightarrow \Pi$. In this case, we choose a generating set $S_N$ arbitrarily. This doesn’t give a manifold $M \#_{\Pi} N$ which is well-defined in terms of $M$ and $N$, but for our purposes, any choice of finite generating set will do.

Given a $\Pi$-cobordism $V$ from $M$ to $N$ (which, by definition, gives an isomorphism $\pi_1(M) \rightarrow \pi_1(N)$), one may obtain a 5-manifold $W$ from $V$ such that $\partial W = M \#_{\Pi} N$ and $i_* : \pi_1(M \#_{\Pi} N) \rightarrow \pi_1(W)$ is the canonical isomorphism. Since $V$ is connected, there is an embedded arc from $M$ to $N$ having a trivial regular neighborhood. Remove an open neighborhood of this arc. Similarly, for each $g \in S_N$, the loop $g \phi(g^{-1})$ bounds an embedded 2-disk in $V$ with trivial regular neighborhood. Remove an open neighborhood of each such disk to obtain $W$.

**Proposition 18** For a given 1-universal fibration $B \xrightarrow{\xi} BSTop$, if $[M]$ denotes the class of $M$ in $\Omega_4(\xi)$, then $[M \#_{\Pi} N] = [M] + [N]$.

**Proof.** $M$ and $N$ are (by definition) equipped with maps $\mu : M \rightarrow B$ and $\nu : N \rightarrow B$. Then we may define the map $\mu \Pi \nu : M \Pi N \rightarrow B$. Since $M \#_{\Pi} N$ is obtained by surgery from $M \Pi N$, there is a cobordism (not a $\Pi$-cobordism) from $M \Pi N$ to $M \#_{\Pi} N$, formed by attaching a 1-handle and $|S_N|$ 2-handles to $(M \Pi N) \times I$. Since these handles are attached along classes which are homotopically trivial under the maps $\mu$,
\[ \nu, \mu \# \nu, \text{ and } \mu \# \Pi \nu. \] the maps on the boundary may be extended over the cobordism. Then

\[ [M \# \Pi N] = [M] + [N]. \]

\[ \square \]

We define \( se(M - N) = se(M \# \Pi - N). \) We could also define \( se(M - N) \) in terms of a \( \Pi \)-cobordism from \( M \) to \( N \): if \( V \) is such a cobordism, and \( \mu : V \to K(\Pi,1) \), then \( se(M - N) = \mu_* (PD(w_2 + w)). \) These two definitions are equivalent, though, since the subsets of \( V \) which are removed to make a \( \Pi \)-null cobordism of \( M \# \Pi - N \) intersect \( PD(w_2 + w) \) an even number of times. When the invariants are defined, note that the Proposition implies \( \sigma(M \# \Pi N) = \sigma(M) + \sigma(N), se(M \# \Pi N) = se(M) + se(N), \)

\[ \text{ter}(M \# \Pi N) = \text{ter}(M) + \text{ter}(N), \] and the \( \pi_1 \)-fundamental class of \( M \# \Pi N \) is the sum of the \( \pi_1 \)-fundamental classes of \( M \) and \( N \). In particular, if \( M \) and \( N \) are \( \Pi \)-cobordant, then \( \sigma(M \# \Pi - N) = 0 \) and the \( \pi_1 \)-fundamental class of \( M \# \Pi - N \) is 0.

4.2 Converse of the Conjecture

**Lemma 19** Let \( M \) be a 4-manifold with finite fundamental group \( \Pi \) and universal cover \( \tilde{M} \) topologically spin. If \( V \) is a stable \( \Pi \)-null cobordism of \( M \) with \( se(M) = 0 \), then (possibly after changing the cobordism \( V \)) the Poincaré dual to the mod 2 homology class \( w_2 - w \) is the reduction mod 2 of an integral class.

**Proof.** There are 3-manifolds representing \( w \) and \( w_2 \), but they need not be orientable. If they are not orientable, then define \( J \) and \( K \) to be those representatives, with a regular neighborhood of the Poincaré dual to \( w_1 \) removed. Then we define a map into \( V \) by the quotient of \( J \) or \( K \) onto the non-orientable 3-manifold, followed by the map representing \( w \) or \( w_2 \). Otherwise, \( J \) and \( K \) are the manifolds representing \( w \) and \( w_2 \), with
the appropriate maps. Thus we may assume the Poincaré duals to \( w, w_2 \in H^2(V; \mathbb{Z}/2) \) are represented by maps of oriented 3-manifolds with boundary \( J \to V \) and \( K \to V \) respectively. In particular, if \( M \) is not spin, then some boundary component of \( J \) and \( K \) represent the Poincaré duals of \( w, w_2 \in H^2(M; \mathbb{Z}/2) \), but there may be other boundary components. The maps need not be proper; in particular, \( \partial J \) and \( \partial K \) may map into \( V \) so that they are not homotopic to maps \( \partial J, \partial K \to M \). We will abuse notation by referring to the manifold, map, and image as \( J \) or \( K \); it should be clear from context which we’re working with. From the relative exact sequence in homology, \( \partial J \) and \( \partial K \) are representatives mod 2 of the Poincaré duals to \( w(M) \) and \( w_2(M) \). If \( \mu : M \to K(\Pi, 1) \) is the map classifying the fundamental group of \( M \), then \( \mu_*(PD(w)) = \mu_*(PD(w_2(M))) = 0 \). Then

\[
\begin{array}{c}
H_2(M; \mathbb{Z}) \\
\mu_* \\
\mu_* \\
H_2(\Pi; \mathbb{Z}/2)
\end{array}
\]

Since \( \mu_* : H_1(M; \mathbb{Z}) \to H_1(\Pi; \mathbb{Z}) \) is an isomorphism, the image of \( PD(w) = PD(w_2) \) under the homology Bockstein is 0. Thus \( PD(w) \) is represented by a class in \( H_2(M; \mathbb{Z}) \). The coefficient exact sequence in homology gives torsion elements \( \beta \) and \( \gamma \) (called the half-boundaries of \( J \) and \( K \), respectively) in \( H_2(V, M; \mathbb{Z}) \). Since \( H_2(M; \mathbb{Z}) \to H_2(V; \mathbb{Z}) \to H_2(V, M; \mathbb{Z}) \) is exact, we may find well-defined elements in \( H_2(V; \mathbb{Z}) \) representing \( \partial J \) and \( \partial K \).

\[
\begin{array}{ccc}
\pi_2(V) & \longrightarrow & \pi_2(V, M) \\
\downarrow h & & \downarrow h \\
H_2(V; \mathbb{Z}) & \overset{J_*}{\longrightarrow} & H_2(V, M; \mathbb{Z}) \\
& & \downarrow \\
& & H_2(V; \mathbb{Z}) \to H_2(V, M; \mathbb{Z}) \to 0
\end{array}
\]

Since the pair \((V, M)\) is simply connected, \( \beta \) and \( \gamma \) are represented by maps \((B^2, S^1) \to (V, M)\). But from the relative homotopy exact sequence, \( \pi_2(V) \to \pi_2(V, M) \) is
onto, and thus $\beta$ and $\gamma$ are represented by $(S^2, \varphi) \to (V, M)$. Therefore we may express the boundaries of $J$ and $K$ as the sums of spheres representing twice $\beta$ and $\gamma$ with characteristic classes for $M$. The spheres representing twice $\beta$ and $\gamma$ have trivial normal bundle, so we may surger them. Since we are surgering $S^2$s in a 5-manifold, we will not change the normal bundle of any existing elements of $H_2(V; \mathbb{Z}/2)$. By choosing the surgery framing correctly, we insure that we are not adding any new elements with nontrivial normal bundle. Thus, possibly after surgering $V$. $PD(w)$ and $PD(w_2)$ are represented by proper maps of 3-manifolds $J$ and $K$ into $V$.

The spectral sequence for the homotopy fibration $\tilde{M} \to M \to K(\Pi, 1)$ gives an exact sequence $\pi_2(M) \to H_2(M; \mathbb{Z}) \xrightarrow{\mu_*} H_2(\Pi; \mathbb{Z}) \to 0$. If $\alpha \in H_2(M; \mathbb{Z})$ and $\mu_*(\alpha) = 0$, $\alpha$ is spherical. But $w$ has a representative in $H^2(M; \mathbb{Z})$, so $PD(w)$ is in $H_2(M; \mathbb{Z})$, and as we've seen, $\mu_*(PD(w)) = 0$. Thus $\partial J$ is spherical. Although $w = w_2(M)$, their integral representatives need not be equal: in particular, we do not know that $\partial J = \partial K$.

$$
\begin{array}{ccc}
H_3(V, M; \mathbb{Z}) & \xrightarrow{\partial} & H_2(M; \mathbb{Z}) \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\mu_*} & H_2(\Pi; \mathbb{Z}) \\
& & \downarrow \\
& & H_2(\Pi; \mathbb{Z})
\end{array}
$$

$K$ represents a class in $H_3(V, M; \mathbb{Z})$, so $\mu_*(i_*(\partial K)) = 0$. $\partial K$ is an integral characteristic class for $w_2$ in $M$, and since the square above commutes, we have $\mu_*(\partial K) = 0$. Thus the characteristic class for $w_2$ is spherical as well. Then $PD(w_2 + w)$ is represented by the sum of $J$ and $K$, and its boundary is a spherical class in $M$ which goes to 0 when reduced mod 2. Pushing this spherical class into $V$, we may surger it, and we get a map of a 3-manifold $(L, \phi) \to (V, M)$ representing $PD(w_2 + w)$.

**Theorem 20** If $M, V$ are as above, then (possibly after surgering $V$) there is a map $S^3 \to V$. 

representing \( PD(w_2 + w) \).

**Proof.** By the Lemma, there is a map of a closed orientable 3-manifold \( L \to V \) representing \( PD(w_2 + w) \). \( \mu(L) \) represents \( \text{sec}(M) = 0 \), so the class that \( L \) represents in \( H_3(\Pi; \mathbb{Z}) \) is even: namely, there is some \( \delta \in H_3(\Pi; \mathbb{Z}) \) such that \([L] = 2\delta\). If \( \delta \) itself is the image of some \( d \in H_3(V) \) under \( \mu_* \), then \( L - 2d \) also represents \( PD(w_2 + w) \), and \( \mu_*(L - 2d) = 0 \).

Again, we have the exact sequence

\[
H_3(V; \mathbb{Z}) \xrightarrow{\mu_*} H_3(\Pi; \mathbb{Z}) \xrightarrow{d_3} H_2(\tilde{V}; H_0(\Pi)) \xrightarrow{h} H_2(V; \mathbb{Z}) \xrightarrow{\mu_*} H_2(\Pi; \mathbb{Z}) \to 0
\]

\( H_0(\Pi) \cong \mathbb{Z} \), and there is an action of \( \Pi \) or \( H_0(\Pi) \), given by a map \( \Pi \to Aut(\mathbb{Z}) \cong \mathbb{Z}/2 \), or equivalently, an element of \( H^1(\Pi; \mathbb{Z}/2) \). If this element is nonzero, there is some \( g \in \Pi \) such that \( g \) acts on \( \mathbb{Z} \) nontrivially. In other words, if we start at the basepoint of \( K(\Pi, 1) \) and travel along the loop \( g \), we will end up at the basepoint with reversed orientation. But \( V \) is orientable, so \( \Pi \) acts on \( H_0(\Pi) \) trivially. Then \( H_2(\tilde{V}; H_0(\Pi)) \) is the image of \( H_2(V; \mathbb{Z}\Pi) \) under the coefficient homomorphism \( \mathbb{Z}\Pi \to \mathbb{Z} \), the augmentation homomorphism. If \( d_3(\delta) = 0 \), then \( \delta \) is in the image of \( \mu_* \). This need not be the case, so we will try to surger \( V \) so that \( d_3(\delta) = 0 \). Note that \( d_3(\delta) \) is represented by a map \( j : S^2 \to V \), and it represents the 0 homology class in \( V \). Then \([j] \cdot [L] = 0\). If we pair the intersection points with opposite sign, draw an arc on the image of \( j \) connecting each pair; then surger \( L \) along these arcs. \( L \) represents the same homology class, and is disjoint from the image of \( j \). Since \([j]\) is the trivial homology class, its normal bundle is trivial, and we may surger the image of \( j \). We have not changed the fundamental group of \( V \), so \( \mu_*(L) \) is the same: then \( d_3(\delta) = 0 \), so there is some \( d \in H_3(V) \) such that \( \mu_*(d) = \delta \). and
\[ \mu_4(L - 2d) = 0. \]

Thus we have maps \( L \to V \xrightarrow{\mu_*} K(\Pi, 1) \), and some 4-manifold \( P \) with \( \partial P = L \) and an extension to \( P \to K(\Pi, 1) \). Then \( P \) may be built from \( L \times I \) by attaching a sequence of 4-dimensional handles. We may assume \( L \) is connected before building \( P \), so any 1-handles attached to \( L \) would create extra elements in \( \pi_1(L) \) instead of killing off \( \pi_0(L) \). We may also assume there is only one 4-handle, whose attaching region is \( S^3 \). If we avoid attaching the 4-handle altogether, we have a cobordism \( P' \) from \( L \) to \( S^3 \). Each 2-handle is attached along a loop, and thus that loop is homotopically trivial in \( P \). Since \( P \to K(\Pi, 1) \), any loop along which a 2-handle is attached is homotopically trivial in \( K(\Pi, 1) \). But any loop in \( V \) which maps to a homotopically trivial loop in \( K(\Pi, 1) \) is itself homotopically trivial. When we attach a 2-handle to \( L \times I \) along a loop in \( L \), the boundary gets a Dehn surgery along that loop. Since the loop is homotopically trivial in \( V \), the Dehn surgery can be realized in \( V \). After performing those prescribed Dehn surgeries, we are left with a new 3-manifold \( L' \) which is cobordant to \( S^3 \) through a cobordism that only uses 3-handles. Each 3-handle is attached along a 2-sphere in \( L' \), so \( \pi_2(L') \) is nontrivial. Dually, there is a 4-manifold built from \( S^3 \) using only 1-handles whose boundary is the disjoint union of \( S^4 \) and \( L' \). Since \( L' \) represents a nontrivial integral homology class in \( V \), it is an orientable manifold. Therefore, \( L' \) is some connected sum of \( S^2 \times S^1 \)'s. We may assume each \( S^1 \) factor represents a nontrivial loop in \( V \) and \( K(\Pi, 1) \), since otherwise there would be a Dehn surgery available to kill that factor. Then in order to realize a cobordism (or more precisely, a homology) between \( L' \) and \( S^3 \) in \( V \), we must compress the \( S^2 \) factors.
Given an $S^2$ factor of $L'$, the compression along the $S^2$ can be performed in $V$ only if the $S^2$ bounds a 3-ball: in other words, if it represents a trivial class in $\pi_2(V)$. If the $S^2$ does not represent a trivial homotopy class, but has trivial normal bundle, then we may surger the $S^2$. Otherwise, we have a homotopically nontrivial $S^2$ with nontrivial normal bundle in $V$. The "dual" $S^1$ in $L'$ is a homotopically nontrivial loop, called $\gamma$. We modify $V$ as follows: Remove a regular neighborhood of $\gamma$, which has boundary $S^1 \times S^3$, since $V$ is orientable. Similarly, take $\mathbb{C}P^2 \times S^1$ and remove a regular neighborhood of a generator of $\pi_1$. Attach these two manifolds along the common $S^1 \times S^3$ in their boundary, and call the result $V'$.

Note that $H_2(S^1 \times S^3) = 0$ and $H_1(S^1 \times S^3) \rightarrow H_1(\mathbb{C}P^2 \times S^1)$ is an isomorphism. The Mayer-Vietoris sequence gives us the isomorphisms $H_2(V') \cong H_2(V) \oplus H_2(\mathbb{C}P^2 \times S^1)$. $H_2(V'; \mathbb{Z}/2) \cong H_2(V; \mathbb{Z}/2) \oplus H_2(\mathbb{C}P^2 \times S^1; \mathbb{Z}/2)$. $H^2(V') \cong H^2(V) \oplus H^2(\mathbb{C}P^2 \times S^1)$. and $H^2(V'; \mathbb{Z}/2) \cong H^2(V; \mathbb{Z}/2) \oplus H^2(\mathbb{C}P^2 \times S^1; \mathbb{Z}/2)$. Therefore, in order to determine $w_2(V')$ and the $w_2$-class of $V'$, it suffices to check how each acts on the subgroups $H_2(V; \mathbb{Z}/2)$ and $H_2(\mathbb{C}P^2 \times S^1; \mathbb{Z}/2)$.

The projection onto the second factor $p_2 : \mathbb{C}P^2 \times S^1 \rightarrow S^1$ is a classifying map for $\pi_1(\mathbb{C}P^2 \times S^1)$. Compose this with the map of $S^1$ to the homotopy class $\gamma \in \pi_1(K(P,1))$. This gives $\mathbb{C}P^2 \times S^1$ a structure over $P$, which allows the map $\mu : V \rightarrow K(P,1)$ to extend over $V'$. Since $H_2(\mathbb{Z}) = 0$, $\mu_* : H_2(\mathbb{C}P^2 \times S^1) \rightarrow H_2(\mathbb{Z})$ is trivial, as is $\mu_* : H_2(\mathbb{C}P^2 \times S^1; \mathbb{Z}/2) \rightarrow H_2(\mathbb{Z}/2)$. Thus $\mu^* : H^2(\mathbb{Z}/2) \rightarrow H^2(\mathbb{C}P^2 \times S^1; \mathbb{Z}/2)$ is also trivial. Since $\mu$ extended from $V$ over $V'$, the map $\mu^* : H^2(\mathbb{Z}/2) \rightarrow H^2(V'; \mathbb{Z}/2)$ factors through $\mu^* : H^2(\mathbb{Z}/2) \rightarrow H^2(V; \mathbb{Z}/2)$. In other words, the $w_2$-class $w$ evaluates to 0 on
any element of the subgroup $H_2(\mathbb{C}P^2 \times S^1; \mathbb{Z}/2)$.

We know that $w_2(V) \in H^2(V; \mathbb{Z}/2)$ determines the action of $w_2(V')$ on the subgroup $H_2(V; \mathbb{Z}/2)$. $H_2(\mathbb{C}P^2 \times S^1; \mathbb{Z}/2)$ is cyclic, generated by the core $\mathbb{C}P^1$. This sphere has a nontrivial normal bundle, so $w_2(\mathbb{C}P^2 \times S^1)$ is the nontrivial class in $H^2(\mathbb{C}P^2 \times S^1; \mathbb{Z}/2)$. Then the characteristic class in $H_3(\mathbb{C}P^2 \times S^1; \mathbb{Z}/2)$ is represented by $\mathbb{C}P^1 \times S^1$.

$w_2 + w$ is also characterized by its action on each subgroup. Since we know $w_2$ and $w$, we know $w_2 + w$, and its Poincaré dual is represented by the sum of the homology classes $[L']$ and $[\mathbb{C}P^1 \times S^1]$. Since the loop $\gamma$ is nontrivial in each, we may compress $L' \cup \mathbb{C}P^1 \times S^1$ along $\gamma$. The homology class $PD(w_2 + w) \in H_3(V'; \mathbb{Z}/2)$ is still represented by a connected sum of $S^1 \times S^2$s, but the $S^2$ factor which had a nontrivial normal bundle, now has a trivial normal bundle.

So possibly after modifying $V'$, we have a $\Pi$-null cobordism of $M \# \Pi - \mathcal{V}$ with the class $PD(w_2 + w)$ represented by a connected sum of $S^2 \times S^1$, with each $S^2$ factor having a trivial normal bundle in $V'$. If any $S^2$ factor is homotopically nontrivial, then we surger each such factor. In the case where the $S^2$ factor represented a nontrivial homology class, the surgery kills off that homology class, and doesn’t affect $PD(w_2 + w)$. But the $S^2$ factor now bounds a 3-ball, and so we may compress along the $S^2$. If the $S^2$ represents a trivial homology class, then the surgery may create a new homology class $\zeta$. $\zeta$ is represented by a sphere which was originally the boundary of the dual 3-ball to the $S^2$ factor. After the surgery, the $S^2$ factor bounds a 3-ball which is an algebraic dual to $\zeta$. Since the spherical representative of $\zeta$ bounded a ball before the surgery, it intersects $L'$ an even number of times. When we do the compression, we have replaced an imbedded $S^2 \times I$ in $L'$ (which $\zeta$
intersected an even number of times) with $D^3 \times S^0$. $\zeta$ intersects that $D^3$ exactly once, so it intersects $D^3 \times S^0$ twice. In fact, the algebraic intersection of $\zeta$ with $D^3 \times S^0$ is 0. In any case, the compression has not affected the mod 2 intersection number of $\zeta$ with $L'$, and so the compression results in a 3-manifold which still represents $PD(w_2 + w)$.

Since we can compress each 2-sphere factor in $L'$, we are left with a simply connected, closed 3-manifold. This is a homotopy $S^3$, and it represents $PD(w_2 + w)$, and the conclusion follows. ■

Given $M \# CP^2$, there is an "un-connected sum" operation giving a homotopy $M$. $CP^2$ has a core $CP^1$, which generates $\pi_2(CP^2)$. Attaching a 3-cell to this $CP^1$ kills the generator of $\pi_2$ and adds no new homotopy in dimensions 2 or 3; call the result $D$. Then $\pi_4(D) \cong H_4(D) \cong \mathbb{Z}$, and there is a map $S^4 \to D$ representing a generator of $\pi_4(D)$. This map is a homology equivalence between simply connected CW-complexes, so is a homotopy equivalence. $D$ is not a manifold, but has an open subset, obtained by deleting the core $CP^1$ from the $CP^2$, which is a 4-manifold. The connected sum $M \# D$ makes sense, and if we attach a 3-cell to the core $CP^1$ in $M \# CP^2$, we get $M \# D$, which is homotopy equivalent to $M \# S^4 = M$.

**Corollary 21** If $M, N$ are 4-manifolds with the same signature and isomorphic finite fundamental group $\Pi$, and $\text{sec}(M - N) = 0$, then $M$ is stably homotopy equivalent to $N$ over their common 1-universal fibration $\xi$.

**Proof.** By Theorem 20, there is a $\Pi$-null cobordism $V$ of $\partial M \# \Pi - N$ such that $PD(w_2 + w)$ is represented by $S^3 \to V$. Adding 3-handles along the disks representing the relations $g^{-1} \phi(g)$ in $M \# \Pi N$, we get a null-cobordism (not over $\Pi$) of $M \# - N$. Then adding
a 4-handle along the $S^3 \times I$, we get a $\Pi$-cobordism $W$ from $M$ to $N$. In $W$, $PD(w_2 + w)$ is still spherical, since the only homology possibly created by attaching the handles is in dimensions 3 and 4. Now we define a $\Pi$-cobordism $U$ from $\widetilde{M \# CP^2}$ to $\widetilde{N \# CP^2}$: Choose an imbedded arc in $W$ from $M$ to $N$, and remove a regular neighborhood of it. Similarly, remove a regular neighborhood of an imbedded arc in $CP^2 \times I$, and attach the two manifolds along their common $S^3 \times I$. The result is the cobordism $U$. Since $w_2(\widetilde{M \# CP^2}) \neq 0$ and $w_2(\widetilde{N \# CP^2}) \neq 0$, $U$ is a cobordism over its 1-universal fibration. By Theorem 8, there is a sequence of spheres, whose corresponding tori may be subtracted to make $U$ into an s-cobordism.

Since $(CP^2 \setminus B^4) \times I$ deformation retracts onto either boundary component, the spheres which need to be deleted to make $U$ an s-cobordism are supported entirely in $W$. However, if $w_2 + w$ is nontrivial, then some of these spheres in $W$ are not boundary-parallel and have nontrivial normal bundle. They also intersect $PD(w_2 + w)$ an odd number of times. Take an arc from $CP^1 \times I$ to $L'$, and tube $(CP^1 \times I)$ to $L'$ along this arc to get $(CP^1 \times I) \# S^3 = CP^1 \times I$. This has a geometric dual, which is the core $CP^1$. Given a sphere $T$ which has odd intersection with $L'$, for each intersection of $T$ with $L'$, tube $T$ to the dual $CP^1$ along an arc in $CP^1 \times I$, so that the resulting sphere $S$ is disjoint from $CP^1 \times I$. Now $S$ has trivial normal bundle, and $[S] = [T]$ in $H_2(U, \partial U)$. Therefore, we may assume that there is a basis for $H_2(U, \partial U)$ represented by 2-spheres with trivial normal bundle, and which are disjoint from $CP^1 \times I$. Subtracting the corresponding tori results in an s-cobordism between $M \# CP^2 \# r(S^2 \times S^2)$ and $N \# CP^2 \# s(S^2 \times S^2)$ for some non-negative integers $r,s$. This s-cobordism contains a (not necessarily embedded) $CP^1 \times I$ whose boundary is the core of the
\( \mathbb{C}P^2 \)'s, so the homeomorphism induced from the \( s \)-cobordism preserves the homotopy classes of the cores of the \( \mathbb{C}P^2 \)'s. Then \( M \# \mathbb{C}P^2 \# r(S^2 \times S^2) \cup_f B^3 = N \# \mathbb{C}P^2 \# s(S^2 \times S^2) \cup_g B^3 \), where \( f \) and \( g \) map \( S^2 \) to each core \( \mathbb{C}P^1 \). Since the core \( \mathbb{C}P^1 \)'s are homotopic in the \( s \)-cobordism, the homeomorphism may be extended to a homotopy equivalence of the two manifolds with \( B^3 \) attached. But \( M \# \mathbb{C}P^2 \# r(S^2 \times S^2) \cup_f B^3 \) is homotopy equivalent to \( M \# r(S^2 \times S^2) \) and \( N \# \mathbb{C}P^2 \# s(S^2 \times S^2) \cup_g B^3 \) is homotopy equivalent to \( N \# s(S^2 \times S^2) \), so that \( M \# r(S^2 \times S^2) \) and \( N \# s(S^2 \times S^2) \) are homotopy equivalent. ■

4.3 Main Theorem

The main theorem states that \( \text{sec}(M - M') \) is a stable homotopy invariant, as long as \( M \) and \( M' \) satisfy an additional hypothesis. The hypothesis is necessary to apply the machinery of [6], which gives a geometric description of the difference between manifolds which become stably homeomorphic after adding a \( \mathbb{C}P^2 \) summand. The difference is in the manifolds \( Y \) and \( Y' \), which we will prove are homotopy equivalent and cobordant over their fundamental group.

**Definition 22** The 4-manifold \( Y \) with boundary is given by the Kirby diagram below. The union of the 0-handle of \( Y \) with each 1-handle contains an annulus whose boundary is the disjoint union of two copies of the core of the linking 2-handle. In \( Y \# \mathbb{C}P^2 \), identify one boundary component of one of the annuli to one boundary component of the other annulus and tube the result to the core of the \( \mathbb{C}P^2 \). The result is an annulus, which we cap off with copies of the core of the \( \mathbb{C}P^2 \). This gives us an embedded 2-sphere in \( Y \# \mathbb{C}P^2 \) with framing 1. That 2-sphere represents the core of the \( \mathbb{C}P^2 \) in \( Y' \# \mathbb{C}P^2 \); or equivalently, removing a
neighborhood of the 2-sphere and glue in a 4-ball gives $Y'$. Note that the cores of the $\mathbb{C}P^2$s in $Y \# \mathbb{C}P^2$ and $Y' \# \mathbb{C}P^2$ are homotopic.

**Definition 23** $N = Y \cup Y'$.

**Lemma 24** $Y$ is homotopy equivalent rel boundary to $Y'$. $\pi_1(Y) \cong \mathbb{Z}/2 \ast \mathbb{Z}/2$. and $N$ has spin universal cover.

**Proof.** From the Kirby diagram for $Y$, it is clear that $\pi_1(Y) \cong \mathbb{Z}/2 \ast \mathbb{Z}/2$. We may get the boundary of $Y$ by Dehn surgery on the same link in $S^3$ as the Kirby diagram, with the dotted circles replaced by circles with framing zero. The elements of $\pi_1(Y)$ generating the two $\mathbb{Z}/2$ factors are represented by meridians to the dotted circles in the Kirby diagram. These elements generate $\pi_1(Y)$, so if they are in the image of the inclusion of $\pi_1(\partial Y)$, then the inclusion is onto $\pi_1(Y)$. In fact, the meridians to the zero-framed circles in the Dehn surgery diagram are parallel to the meridians of the dotted circles. Since their image in $\pi_1(Y)$ is nontrivial, they must be nontrivial elements of $\pi_1(\partial Y)$, and thus if $i : \partial Y \rightarrow Y$ is the inclusion, then $i_* : \pi_1(\partial Y) \rightarrow \pi_1(Y)$ is onto.
The homeomorphism $Y \# \mathbb{C}P^2 \to Y' \# \mathbb{C}P^2$ preserves the homotopy classes of the core $\mathbb{C}P^1$s. so we may perform the un-connected sum operation on them. This gives us a homotopy equivalence $(Y \# \mathbb{C}P^2) \cup B^3 \to (Y' \# \mathbb{C}P^2) \cup B^3$. Then the homotopy equivalence of $(Y \# \mathbb{C}P^2) \cup B^3$ with $Y$ is the identity away from a neighborhood of the $\mathbb{C}P^2 \cup B^3$: in particular, the homotopy equivalence is the identity on the boundary. So $Y$ is homotopy equivalent to $Y'$ rel boundary.

Recall $i_* : \pi_1(\partial Y) \to \pi_1(Y)$ is onto. Then the Seifert-VanKampen theorem gives the isomorphism $\pi_1(\mathcal{N}) \cong \pi_1(Y) *_{i_*\pi_1(\partial Y)} \pi_1(Y') \cong \pi_1(Y) \cong \mathbb{Z}/2 * \mathbb{Z}/2$. In fact, since $Y$ is homotopy equivalent to $Y'$ rel boundary, the generators of $\pi_1(\mathcal{N})$ are represented on either $Y$ or $Y'$.

Having identified $\pi_1(\mathcal{N})$, we wish to find out if $\mathcal{N}$ actually has spin universal cover. It's sufficient to find a finite cover which is spin. so consider the exact sequence $1 \to \mathbb{Z} \to \mathbb{Z}/2 * \mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 1$. Then $\mathcal{N}$ has a cover $\tilde{\mathcal{N}}$ with fundamental group $\mathbb{Z}$, and deck translation group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Since $\chi(\mathcal{N}) = 2$, $\chi(\tilde{\mathcal{N}}) = 8$, and $H_2(\tilde{\mathcal{N}}) \cong H^2(\tilde{\mathcal{N}}) \cong \mathbb{Z}^8$. The homology of $\tilde{\mathcal{N}}$ is torsion-free, so reduction mod 2 is surjective in each dimension. Thus $H^2(\tilde{\mathcal{N}}; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^8$. $H_2(\mathcal{N}; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^4$. with generators represented by the 2-cells $X_1$, $X_2$ of $Y$ and their corresponding dual spheres $S_1$ and $S_2$. Each dual sphere has a 4-fold cover in $\tilde{\mathcal{N}}$, which must consist of 4 spheres. But each dual 2-sphere is 2-torsion in $H_2(\mathcal{N})$, so the cover must not consist of 4 independent homology classes of sphere. More precisely, since $S_1$ intersects $X_1$ once and is disjoint from $X_2$, and vice versa, the cover of $S_1$ or $S_2$ must consist of 2 homology classes of spheres. Thus there is a factor $\mathbb{Z}^4 \subset H_2(\tilde{\mathcal{N}})$, generated by the covers of $S_1$ and $S_2$, that maps surjectively to $H_2(\mathcal{N})$. The dual 2-spheres
in $N$ have spin normal bundles, so their covers must have spin normal bundles as well. Thus $w_2(\tilde{N})$ evaluates to 0 on any lift of a dual 2-sphere. The 2-cells of $Y$ represent relations on $\pi_1(Y) \cong \pi_1(N)$, and in fact $\pi_1(N)$ is represented on a pair of $\mathbb{R}P^2$s formed from the 1- and 2-cells of $Y$. These $\mathbb{R}P^2$s do not represent integral homology classes of $\tilde{N}$, but they do represent elements of $H_2(N; \mathbb{Z}/2)$; in fact, they represent the support of $w_2(N)$. Each $\mathbb{R}P^2$ has a loop representing one of the $\mathbb{Z}/2$ factors of $\pi_1(N)$, but not the other factor. Thus each $\mathbb{R}P^2$ must be covered by a pair of spheres in $\tilde{N}$. Let $\mathbb{Z}/2 \oplus \mathbb{Z}/2 = \langle a, b \mid a^2, b^2, aba^{-1}b^{-1} \rangle$ so that $a$ is represented on $X_1$ and $b$ is represented on $X_2$. Then the deck translation of $\tilde{N}$ corresponding to $a$ must act on any cover of $X_1$ by reversing orientation, but must permute the covers of $X_2$ nontrivially. Similarly, the deck translation corresponding to $b$ must reverse the orientation of a cover of $X_2$, but not $X_1$. In other words, there is an explicit representation of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ on the free abelian group generated by the covers of $X_1$ and $X_2$, given by $a \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ and $b \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. This representation shows that the four spheres covering the $\mathbb{R}P^2$s in $\tilde{N}$ represent distinct homology classes. Thus every element of $H_2(\tilde{N}; \mathbb{Z}/2)$ is represented by a sum of spheres, each covering either one of the $\mathbb{R}P^2$s in $N$, or one of their dual spheres. The elements covering the dual spheres must have spin (in fact, trivial) normal bundles, since they cover the spin bundles of the dual spheres. The 2-handles of $Y$ have framing 1, so a push-off of a 2-handle intersects the 2-handle once. If $T_1$, $T_2$ are spheres covering one of the $\mathbb{R}P^2$s, then the self-intersection of $T_1 + T_2$ is a 4-fold cover of the self-intersection of the $\mathbb{R}P^2$. In other words, $T_1 + T_2$ intersects
itself in 4 points. so \(([T_1] + [T_2]) \cdot ([T_1] + [T_2]) = 0, 2, \text{ or } 4\). Since \(N\) is orientable, the deck translations of \(\tilde{N}\) are orientation-preserving homeomorphisms, and they act transitively on the set of points of the self-intersection. Thus all 4 points must have the same sign, and
\[4 = ([T_1] + [T_2]) \cdot ([T_1] + [T_2]) = |T_1|^2 + 2[T_1] \cdot [T_2] + [T_2]^2.\]
\(T_1\) and \(T_2\) cannot intersect and must have the same self-intersection, so \(|T_1|^2 = |T_2|^2 = 2\) and thus these two spheres have spin normal bundles. Therefore, \(\tilde{N}\) is spin, which implies that \(\tilde{N}\) is spin. 

Lemma 25 \(H^*(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2[a, b]/ < ab > \) with both generators in dimension 1, and \(H^*(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}[x, y]/ < 2x, 2y, xy > \) with both generators in dimension 2.

Proof. Since \(\mathbb{R}P^\infty\) is an example of \(K(\mathbb{Z}/2, 1)\), \(K(\mathbb{Z}/2 \ast \mathbb{Z}/2, 1) = \mathbb{R}P^\infty \vee \mathbb{R}P^\infty\). Thus there are inclusions \(K(\mathbb{Z}/2, 1) \rightarrow K(\mathbb{Z}/2 \ast \mathbb{Z}/2, 1)\) and projections \(K(\mathbb{Z}/2 \ast \mathbb{Z}/2, 1) \rightarrow K(\mathbb{Z}/2, 1)\), which imply that there are two polynomial subalgebras \(\mathbb{Z}/2[a]\) and \(\mathbb{Z}/2[b]\) of \(H^*(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2)\) with 1-dimensional generators. However, define \(Q^{2k+2}\) to be the \(2k+2\)-dimensional manifold which is an \(S^1\)-fiber bundle over \(\mathbb{R}P^{2k+1}\), such that \(\pi_1(Q) \cong \mathbb{Z}/2 \ast \mathbb{Z}/2\). \(Q^4\) is double covered by \(S^1 \times S^3\), and thus \(Q^4\) is a 3-dimensional approximation to \(K(\mathbb{Z}/2 \ast \mathbb{Z}/2, 1)\). In particular, \(H^2(Q^4; \mathbb{Z}/2) \cong H^2(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2)\). \(Q^4\) is closed and has \(\pi_1(Q^4) \cong \mathbb{Z}/2 \ast \mathbb{Z}/2\), so \(H^1(Q^4; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2\). Since \(\chi(S^1 \times S^3) = 0\), then \(\chi(Q^4) = 0\) and Poincaré duality implies \(H^2(Q^4; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2\). Thus there is some relation between \(a\) and \(b\): since each generator is free, and \(\mathbb{Z}/2 \ast \mathbb{Z}/2\) is symmetric with respect to its generators of order 2, the relation must be either \(ab = 0\) or \(a^2 + ab + b^2 = 0\).

In fact, the existence of such a relation immediately implies that \(H^*(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2\) whenever \(* > 0\). Furthermore, we may use Poincaré duality and Euler characteristic to compute the cohomology of \(Q^{2k+2}\) from that of \(Q^{2k}\). We get \(H^0(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2\).
\[ \mathbb{Z}/2; \mathbb{Z} \cong \mathbb{Z}, H^*(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \text{ whenever } \ast \text{ is even and positive, and } H^*(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}) = 0 \text{ otherwise.} \] In particular, \( S^1 : H^{2k-1}(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2) \to H^{2k}(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2) \) is an isomorphism. And \( S^1 : H^{2k}(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2) \to H^{2k+1}(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2) \) is the zero map. Thus \( a(a^2 + ab + b^2) = 0 \) if and only if \( S^1(a^3 + a^2b + ab^2) = 0 \). Using the Cartan product formula, \( S^1(a^3 + a^2b + ab^2) = a^3 \), but we know \( a^3 \) is a nonzero class in \( H^4(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2) \). Thus the relation \( ab = 0 \) must hold. Reduction mod 2 is an isomorphism on \( H^{2k}(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}) \), so we may find elements \( x, y \in H^2(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}) \) so that \( r_2(x) = a^2 \), \( r_2(y) = b^2 \). Then \( r_2(x^2 + xy + y^2) = a^4 + b^4 \neq 0 \), but \( r_2(xy) = 0 \). Thus \( H^*(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}) \cong \mathbb{Z}[x_2, y_2]/\langle 2x_2y_2 \rangle \) and \( H^*(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2[a_1, b_1]/\langle ab \rangle \) (where the subscripts denote the dimension of the generators).

**Theorem 26** Let \( M \) and \( M' \) be 4-manifolds with finite fundamental group \( \Pi \), spin universal covers, and common 1-universal fibration \( B \to BTop \). If there exists a homeomorphism \( h : M \# \mathbb{CP}^2 \to M' \# \mathbb{CP}^2 \) such that the automorphism preserves the homotopy class of the core \( \mathbb{CP}^2 \)-s, and such that the un-connected sum operation on \( h \) produces a homotopy equivalence \( M \to M' \), then \( \text{sec}(M - M') = 0 \).

**Proof.** Let \( M \) and \( M' \) be topological 4-manifolds with isomorphic finite fundamental groups and spin universal covers, that are homotopy equivalent over their fundamental group and have the same Kirby-Siebenmann invariant. As usual, if \( \xi : B \to BTopSpin \) is a 1-universal fibration for \( M \) and \( M' \), then we require that the two manifolds are equipped with classifying maps \( \overline{\nu} : M \to B \) and \( \overline{\nu}' : M' \to B \). Since \( M \), \( M' \) are homotopy equivalent over \( \Pi \), Proposition 12 implies that \( M \# \mathbb{CP}^2 \) and \( M' \# \mathbb{CP}^2 \) are stably homeomorphic. Replace \( M \) and \( M' \) by stabilized versions, so that \( M \# \mathbb{CP}^2 \) and \( M' \# \mathbb{CP}^2 \) are homeomorphic.
If $M$ and $M'$ need to be stabilized further at any point during the proof, we will do so without stating it explicitly.

By hypothesis, the homeomorphism takes the homotopy class of the core of one $\mathbb{C}P^2$ to the homotopy class of the core of the other. So by [6] Section 3, we may find some finite list of copies of $Y$ imbedded in $M$, which give $M'$ when they're replaced by $Y'$. It's sufficient to show that replacing an imbedded copy of $Y$ with $Y'$ doesn't change the invariant sec: more precisely, assume $M'$ is obtained from $M$ by replacing one copy of $Y$ with $Y'$. Then we want to show $\sec(M - M') = 0$. It's sufficient to find a $\Pi$-cobordism $W$ from $M$ to $M'$ such that $\mu_*(PD(w_2 + w)) = 0$.

We will construct $W$ by constructing a cobordism $P$ rel boundary between $Y$ and $Y'$ (or equivalently, a $\mathbb{Z}/2 \times \mathbb{Z}/2$-null cobordism of $N$), and replacing an imbedded copy of $Y \times I$ in $M \times I$ with $P$. $Y$ is homotopy equivalent to $\mathbb{R}P^2 \vee \mathbb{R}P^2$, so $H_1(Y) = (\mathbb{Z}/2)^2$ and $H_2(Y) = 0$. Since $Y$ is homotopy equivalent to $Y'$, the homology groups of $Y'$ are isomorphic to those of $Y$. $Y \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is homeomorphic to $Y \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ bounds the 5-manifold $((\mathbb{C}P^2 \setminus B^4) \times I)$, so $(Y \times I) \# ((\mathbb{C}P^2 \setminus B^4) \times I)$ has boundary $(Y \cup_0 Y') \# (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}) = Y \cup_0 (Y \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$. Similarly, $(Y' \times I) \# ((\mathbb{C}P^2 \setminus B^4) \times I)$ has boundary $Y' \cup_0 (Y' \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$. Since $Y \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is homeomorphic to $Y' \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, we define $P = (Y \times I) \# ((\mathbb{C}P^2 \setminus B^4) \times I) \cup (Y' \times I) \# ((\mathbb{C}P^2 \setminus B^4) \times I)$, where the union is taken along the $Y \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in each boundary.

Note $\partial P = N$. We wish to compute $\sec(N)$ using the machinery of the James spectral sequence $[8]$. We will need to find $w(N)$. The Mayer-Vietoris sequence for $N$ in
dimensions 1 through 4 is:

\[ H_4(N) \rightarrow H_3(\partial Y) \rightarrow H_3(Y) \oplus H_3(Y') \rightarrow H_3(N) \rightarrow \]

\[ H_2(\partial Y) \rightarrow H_2(Y) \oplus H_2(Y') \rightarrow H_2(N) \rightarrow \]

\[ H_1(\partial Y) \rightarrow H_1(Y) \oplus H_1(Y') \rightarrow H_1(N) \]

\( N \) is a closed manifold, and its fundamental group has abelianization \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \), so \( H_4(\mathcal{N}) \cong \mathbb{Z} \). \( H_1(\mathcal{N}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \), and \( H_3(\mathcal{N}) = 0 \). \( H_1(Y) \rightarrow H_1(\mathcal{N}) \) and \( H_1(Y') \rightarrow H_1(\mathcal{N}) \) are both isomorphisms, so the image of \( H_1(\partial Y) \rightarrow H_1(Y) \oplus H_1(Y') \) projects isomorphically onto each of the factors \( H_1(Y) \) and \( H_1(Y') \). In fact, we have a short exact sequence

\[ 0 \rightarrow H_2(\mathcal{N}) \rightarrow H_1(\partial Y) \overset{\iota}{\rightarrow} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow 0 \]

The relative exact sequence for \( Y \) and \( \partial Y \) in dimensions 1 and 2 is:

\[ H_2(Y) \rightarrow H_2(Y, \partial Y) \rightarrow H_1(\partial Y) \overset{\iota}{\rightarrow} H_1(Y) \rightarrow H_1(Y, \partial Y) \]

The first and last groups in this diagram are both trivial, so we get an isomorphism \( H_2(\mathcal{N}) \cong H_2(Y, \partial Y) \). By Poincaré-Lefschetz duality applied to the pair \((Y, \partial Y)\). \( H^2(Y, \partial Y) = 0 \) and \( H^3(Y, \partial Y) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \), which implies that \( H_2(\mathcal{N}) \cong H_2(Y, \partial Y) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \).

This isomorphism has a nice geometric realization. Each of the 2-handles used to build \( Y \) has a dual 2-disk, which intersects the core of the 2-handle once and has boundary on the boundary of the 2-handle. In particular, this disk represents a homology class in \( H_2(Y, \partial Y) \) which maps to a nontrivial element of \( H_1(\partial Y) \). Since \( Y \) and \( Y' \) are homotopy equivalent, there's a corresponding disk in \( Y' \) whose boundary is the same as the one in \( Y \). In particular, the union of these disks is a sphere in \( \mathcal{N} \). This sphere intersects one of
the $\mathbb{R}P^2$s in $Y$ in exactly one point, and the $\mathbb{R}P^2$ represents a $\mathbb{Z}/2$ homology class, so the sphere represents a nontrivial homology class.

Thus $H_2(N) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, and $H_2(N; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^4$ generated by the 2-handles of $Y$ and the spheres described above. Each dual disk of the 2-handles of $Y$ has a trivial normal bundle, and since the homotopy equivalence of $Y$ to $Y'$ may be regarded as the identity on the boundary, the disks are attached so that the framings agree. In other words, the spheres have spin normal bundles. The 2-handles both have framing 1, so the cores of the 2-handles do not have spin normal bundles. Let $\mu : N \to K(\mathbb{Z}/2 \ast \mathbb{Z}/2, 1)$ define the isomorphism $\pi_1(N) \cong \mathbb{Z}/2 \ast \mathbb{Z}/2$: then $K(\mathbb{Z}/2 \ast \mathbb{Z}/2, 1)$ may be built from $N$ by attaching cells of dimension 3 or higher. In particular, $\mu_* : H_2(N; \mathbb{Z}/2) \to H_2(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2)$ is onto, and thus $\mu^* : H^2(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2) \to H^2(N; \mathbb{Z}/2)$ is injective. As long as $w_2(N)$ is in the image of $\mu^*$, we may use this to compute the $w_2$-class of the 1-universal fibration.

If there is a class $w \in H^2(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2)$ so that $\mu^*(w) = w_2(N)$, then $w$ is the $w_2$-class of $N$. Since the 2-cells of $Y$ form the relations on $\pi_1(N)$, the $\mathbb{Z}/2$ homology classes they represent map nontrivially under $\mu_*$. Note $\mathbb{R}P^\infty \vee \mathbb{R}P^\infty$ may be built with only those 2-cells, so any other $\mathbb{Z}/2$ homology classes of $N$ are trivial under $\mu_*$. Thus $\mu^*(a^2 + b^2)$ evaluates to 1 on each of the 2-cells of $Y$, and 0 on the unions of the dual disks, so $\mu^*(a^2 + b^2) = w_2(N)$, and $w(N) = a^2 + b^2$.

Now we have the pieces necessary to compute the James spectral sequence for $N$. Since we’re investigating sec. the most important position will be $E_{3,1}$. As we’ve seen, in the James spectral sequence, $E_{3,1}^3 = E_{3,1}^\infty$. $E_{3,1}^2 \cong H_3(\mathbb{Z}/2 \ast \mathbb{Z}/2; \Omega_1^{Span}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. We have maps $d_2 : E_{3,1}^2 \to E_{1,2}^1$ and $d_2 : E_{3,0}^2 \to E_{3,1}^2$. The first is the dual of $Sq_{a^2}$.
$H^1(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2) \rightarrow H^3(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2)$ (Proposition 13). The second is the dual of $Sq^2_w$ composed with reduction mod 2. But reduction mod 2 is an isomorphism on odd-dimensional homology of $\mathbb{Z}/2 \ast \mathbb{Z}/2$, so we may view the two maps as $(Sq^2_w)^*: H_3(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2) \rightarrow H_1(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2)$ and $(Sq^2_w)^*: H_3(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2) \rightarrow H_3(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2)$.

If $x \in H^1(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2)$. then $Sq^2(x) = 0$. so $Sq^2_w(a) = a^3$ and $Sq^2_w(b) = b^3$. Thus $Sq^2_w: H^1(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2) \rightarrow H^3(\mathbb{Z}/2 \ast \mathbb{Z}/2; \mathbb{Z}/2)$ is an isomorphism and so is its dual $d_2$.

Thus $Ker(d_2) = 0$, so $E^3_{3,1} = 0$ and $sec(N) = 0$.

Now $sec$ is only well-defined in terms of $P$ if the inclusion $i_*: \pi_1(N) \rightarrow \pi_1(P)$ is an isomorphism. $\pi_1((Y \times I) \# ((\mathbb{C}P^2 \setminus B^4) \times I)) \cong \mathbb{Z}/2 \ast \mathbb{Z}/2$, as does $\pi_1((Y' \times I) \# ((\mathbb{C}P^2 \setminus B^4) \times I))$.

and $P$ is the union of the two along the common $Y \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in their boundary. so $i_*: \pi_1(N) \rightarrow \pi_1(P)$ is an isomorphism. Thus $P$ is a $\mathbb{Z}/2 \ast \mathbb{Z}/2$-cobordism of $Y$ and $Y'$. By Theorem 14, there is a class $s \in H_3(P; \mathbb{Z}/2)$ such that $\mu_*(s) = sec(N)$. The class $s$ is the Poincaré dual to the first obstruction to the universal cover of $P$ being spin. $(w_2 + w)(P)$ is supported on the core $\mathbb{C}P^1$ of $((\mathbb{C}P^2 \setminus B^4) \times I)$. Thus $s$ is represented by a 3-manifold $K$ embedded in $P$ which intersects the core $\mathbb{C}P^1$ once, but has intersection 0 with any element in the image $H_2(N; \mathbb{Z}/2) \rightarrow H_2(P; \mathbb{Z}/2)$. For example, let $K$ be the union of the $\mathbb{C}P^1 \times I$'s. Then $\mu(K)$ bounds in $K(\mathbb{Z}/2 \ast \mathbb{Z}/2, 1)$.

We define the cobordism $W$ between $M$ and $M'$ as follows. Take $M \times I$: this 5-manifold with boundary contains an embedded copy of $Y \times I$, which we may remove and replace with $P$. The result of this replacement is $W$: since $\partial P = N = Y \cup_{\partial Y} Y'$. we get $\partial W$ is the disjoint union of a copy of $M$, and a copy of $M$ with its embedded $Y$ replaced by $Y'$. Thus $\partial W = M \amalg M'$, and $W$ is the desired cobordism. The universal cover $\tilde{W}$
of $W$ is the union of some number of copies of $(M \times I) \setminus (Y \times I)$ (which is spin) with some number of copies of covers of $P$, so the obstruction $(w_2 + w)(W)$ is supported entirely on $P$. The Mayer-Vietoris sequence for $W$ in dimensions 1 and 2 is:

$$
H_2(\partial Y) \longrightarrow H_2(P) \oplus H_2((M \times I) \setminus (Y \times I)) \longrightarrow H_2(W) \longrightarrow
$$

$$
H_1(\partial Y \times I) \longrightarrow H_1(P) \oplus H_1((M \times I) \setminus (Y \times I)) \longrightarrow H_1(W)
$$

The image of the map $H_2(W) \to H_1(\partial Y \times I)$ is at most $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. If $[c]$ is in the image, then the curve $c$ bounds disks in both $(M \times I) \setminus (Y \times I)$ and $P$. The spin structures on the two disks must match, since we may view the curve as lying entirely within a copy of $M$; then we may choose the disks so that they lie within a copy of $M \times I$, which means the resulting sphere has a spin normal bundle. Therefore $PD((w_2 + w)(W)) = i_*PD((w_2 + w)(P))$, and we may choose the Poincaré dual $s$ of $(w_2 + w)(W)$ to be represented by the 3-manifold $K$ described above. Now $K$ bounds in $K(\mathbb{Z}/2 \ast \mathbb{Z}/2, 1)$, and by construction of $Y$, there is a map $i_* : K(\mathbb{Z}/2 \ast \mathbb{Z}/2, 1) \to K(\Pi, 1)$ induced from $i : P \hookrightarrow W$. Then since $s(W) = i_*(s(P))$, we must have $\text{sec}(M - M') = i_*(\text{sec}(Y)) = 0$. ■
Bibliography


