Deterministic and Random Analysis of Systems with Hysteresis using the Preisach Formalism

by

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A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

Master of Science

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August 2004
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Abstract

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The hysteretic behavior of single-degree-of-freedom systems subjected to external loads is examined using the classical Preisach model. Two cases are considered. First, the response of an oscillator having a non-linear component described by a series of Jenkin’s elements and subjected to monochromatic and white noise excitation is computed. The steady-state amplitude of the response to the deterministic loading is derived analytically using the equivalent linearization method and is compared with the numerical results obtained by direct integration of the equation of motion. For the stochastic excitation, an analytical approximation for the probability density function of the response envelope is obtained by implementing the stochastic averaging method and is compared with the numerical results derived by a pertinent Monte Carlo study. The second case involves a similar oscillator with a shape memory alloy device under harmonic excitation. The steady-state amplitude of the response is computed analytically and numerically using the Preisach formalism for a range of excitation frequencies.
Acknowledgements

Writing these lines forces me to realize that one personal goal is achieved. While the characterization “personal” is undoubtedly correct since it reflects my own efforts in life and research, it fails to show how this result is based on a sort of cumulative energy. This means that while writing this thesis, contributions and inspiration have come from various sources and now it is the time to list the most important.

My advisor, Dr. Pol D. Spanos is my mental “father” in these two years of graduate studies. His knowledge, his supreme understanding of science and its applications, his own work and most of all his personality have so much influence on my work that I consider this thesis a result of his prolonged efforts to show me a new way in research. Thus, I owe to thank you professor for sharing with me your passion for science.

I would also like to thank Dr. M. K. O'Malley and Professor A.J. Meade Jr. for serving as members of the Thesis Committee. The professional assistance to administrative and organization issues related to this thesis received by our secretary Mrs. A.M. Turney is also greatly acknowledged.

The financial support that I have received from Rice University, the Onassis Foundation, the Greek Technical Chamber of Professional Engineers, and the Hellenic Professional Society of Texas in form of fellowships, scholarships, and awards is highly appreciated, since the completion of this work would be impossible without them.
Furthermore, I would like at this point to thank my friend and officemate Mr. Nikolaos Politis for his continuous and "broad-band" help. Nikos has been a constant source of information about life and research in the United States since the day I came to Rice. This thesis would not have been completed at this instant of time without his voluntary help.

Last, I must mention that being here is the successful result of a societal experiment that my parents Nikolaos and Eleni have been conducting for as long as I exist. Naturally, this work is an outcome of this "parallel process" and the least I can do is to acknowledge their contribution. Many thanks to my brother Periklis for supporting me in the hard times and to Marianna, Jeff, Walt, Tony, Marco and Yiannis for being close to me all this time.

Finally, I would like to dedicate this thesis to my aunt Vasso for being by far the first role-model person in my life.
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Chapter 1

Introduction

1.1 General remarks – The phenomenon of hysteresis

Nature is fundamentally non-linear and hysteretic. Thus, hysteresis is the dominant phenomenon in various physical processes. It can be the byproduct of fundamental physical mechanisms (such as phase transitions [6]), or the consequence of degradation and imperfections (for example energy losses in mechanical systems) or it may be deliberately “built” into a system in order to monitor its behavior (for example use of the Shape Memory Effect of the Shape Memory Alloys to control strain [33]).

Since hysteresis may manifest itself in so many and different ways, it has been a subject of interest in various fields of science for over a century. The first use of the notion must be credited to ferromagnetism where Warburg [68] as early as 1881 noticed that when a ferromagnetic specimen is subjected to a cyclic magnetic field, its magnetic state will not follow exactly the magnetization process. Actually, it will lag behind. However, the first use of the term “hysteresis” was made by Ewing [18], also in 1881, who was working in the field of magnetism. Research on hysteresis related phenomena in magnetism remained active in the following years, primarily by the work of Lord Rayleigh [52], Duhem [16], Preisach [51] and others. Nevertheless, the advances in science and technology made hysteresis popular in other fields such as plasticity, where the engineers von Mises [40], Prandtl [50], Ishlinskii [23], Hill [21] and Prager [49] proposed models
of elasto-plasticity and set the stage for the development of various operators, a concept which constitutes the fundamental mathematical tool for the description of hysteresis.

Today, the list of references or applications where hysteresis is present is very extended and includes examples in control theory and engineering in general [63,66], politics and economics. In spite of the extended awareness of the phenomenon, it is quite remarkable that a mathematical development which could lead to a rigorous description of hysteretic systems has been lagging behind the physical way of approach, constituting in this way "...a sort of hysteresis in hysteretic history", as Visintin [67] successfully remarked. Mathematics where present, of course, at the early works of physicists and engineers, but they appeared in a heuristic way that was far from being rigorous. Actually, it was not until 1966 that hysteresis was given a more formal definition due to an engineering student, Bouc [5] who modeled several hysteretic phenomena. It is the author's opinion that the most important contribution towards a complete mathematical description of the phenomenon must be credited to the Russian mathematicians Krasnoselskii and Pokrovskii who gave a unique mathematical formulation to hysteresis and other related phenomena using the fundamental concept of operators [30]. This pioneering work offered Mayergoyz the proper mathematical background to propose the well-known and extensively used Classical Preisach Model (CPM) [38]. The CPM offers a versatile and general mathematical-geometric way to quantify the effect of hysteresis on systems and presents an efficient tool that can be used in order to fit hysteretic loops obtained by real experimental data. Moreover, the CPM constitutes a general mathematical tool that is extensively used in analytical and numerical computations throughout this thesis and is explicitly defined in Chapter 3.
1.2 Definition of hysteresis

Various definitions and meanings to the phenomenon of hysteresis have been given by scientists depending on their point of view, their research area or their background. Thus, it is essential to define hysteresis in a formal way, in order to describe the predominant characteristics of the phenomenon in a broader perspective. Naturally, the only way that this can be accomplished is through a strict mathematical definition, which could be globally comprehended.

To this aim, it should be noted that hysteretic nonlinearities represent effects typical for transducers. Therefore the control theory approach will be used next in order to define the notion of the phenomenon. In this context, consider a system whose state is characterized by two scalar variables \( u \) and \( w \). Let both variables depend continuously on time, denoted here as \( t \). A schematic representation of the system is shown in Fig 1.1.

![Figure 1.1: General representation of a system](image)

Hence, \( u \) plays the role of the input or independent variable, while \( w \) is the dependent variable or output of the system. Consider now the case where the input-output relationship for this system is the one shown in Fig. 1.2. Based on this behavior, it is noted that when \( u \) increases from \( u_1 \) to \( u_2 \), then the couple \((u,w)\) moves along the path
Similarly, if \( u \) decreases from \( u_2 \) to \( u_1 \), then \((u,w)\) moves along the path \( CDA \). Moreover, if \( u \) inverts its movement when it is somewhere in between the extreme values \( u_1 \) and \( u_2 \), then \((u,w)\) moves into the interior of the region \( S \) bounded by the major loop \( ABCDA \). In general, \((u,w)\) can attain any interior point of \( S \), according to the input function \( u(t) \). It must be also noted that, while the evolution of \( w \) in time depends solely on \( u \), the value of the output at any given instant that \( u \) changes cannot be determined by the value of the input at the same instant. In fact, \( w(t) \) depends on the previous evolution of \( u \) and on the initial state of the system. In other words, there is a sort of memory in the system’s behavior that takes place whenever the input starts to change in time. This is a key remark that presents one of the basic features of hysteretic systems.

![Continuous hysteresis loop](image)

**Figure 1.2:** Continuous hysteresis loop

In general, the input versus output relationship of hysteretic systems is a multibranch nonlinearity as presented in Fig. 1.3.
Another significant feature of this type of non-linear behavior is that the path of the pair 
\((u,w)\) has to be invariant to any increasing time homeomorphism. This means that at any 
given instant of time, the output depends only on previous values of the input and there is 
no dependence of any kind on its derivatives. Consequently, the velocity of input 
variations between values of \(u\) has no influence on the observed nonlinearities. After 
*Truesdell* and *Noll* [64], this property is called *rate independence*. According to 
*Mayergoz* [38], there are three distinct time scales implied by the notion of rate 
independence. The first is the time scale of fast internal dynamics of the system 
(considered here as a transducer with input and output as shown in Fig. 1.1). The second 
is the time scale on which observations (measurements) are performed. This time scale is 
much larger than the time scale of internal system dynamics so that every observation can 
be identified with a specific output value of the system. The third is the time scale of 
input variations. This time scale is much larger than the observation time scale so that 
every measurement can be associated with a specific value of input.

It is also useful to note that in the existing literature on hysteresis, the phenomenon is 
usually associated with the formation of loops (looping). This might be misleading and it 
could create the impression that loops are the essence of hysteresis. However, the effect 
of hysteretic nonlinearities is manifested, in general, through branches which do not 
always assume the form of loops. Hence, a loop is a particular form of a branch that does 
occur very often but is not the only possibility.


Having described the general hysteretic type nonlinearities, it is now more appropriate to try to define hysteresis. Therefore, it could be said that *hysteresis is the rate independent memory effect, where the term memory expresses the lagging of an effect behind its cause.* This includes phenomena in which the output anticipates the input (although by definition hysteresis means to lag behind). Hence, hysteresis generally appears in systems, which do not follow directly the forces applied to them, but react slowly, or do not return completely to their original state. That is, systems whose states depend on their immediate history. Thus, *whenever and wherever the response of a system exhibits a lag compared to the system's input then hysteresis is present.*
1.3 Mathematical modeling of hysteresis

All rate-independent hysteresis nonlinearities can be classified as shown in Fig. 1.4.

![Diagram](image)

Figure 1.4: Types of hysteretic nonlinearities and models used to study their behavior

Obviously, there are two main kinds of hysteretic nonlinearities and each one of them is being modeled by using certain tools.

*Hysteresis nonlinearities with local memories* are identified by one fundamental property, which can be summarized as follows. At any given instant of time, say \( t_0 \), the value of the input \( u(t_0) \), referring to the description mentioned in the previous paragraph, is sufficient to determine the value of the output at the same instant \( f(t_0) \). In this manner, *the past does not exert any influence on the present or future* of the hysteretic system. The last statement is used in probability theory in order to characterize one significant class of stochastic processes called *Markov Processes* and since this concept appears later in the thesis, it is mentioned here to denote an interesting comparison.
A representative loop of this type of hysteretic nonlinearity is shown in Fig. 1.5. Inside the major loop consisting of the ascending and descending branches, there are certain curves that are fully reversible and can be traversed in both directions according to the status of the input (ascending/descending). For this type of hysteresis, branches may occur when the input reaches the major loop. The main feature of this type of behavior and in general of all the hysteretic nonlinearities with local memory, is that every point in the region defined by the major loop corresponds to a uniquely defined state and this state predetermines the behavior of the system in exactly one way for the ascending and one way for the descending input. This practically means that at every point in this region there is one or at most two curves that can represent the future of the system.

The standard way of modeling hysteretic nonlinearities with local memories is by using the class of Differential Models of Hysteresis (DMH). The main advantage of these models is that they may be incorporated directly in the differential equations that govern the motion of a particular system and thus a general equation that takes into consideration the effect of the nonlinearity is obtained. Perhaps the most known example of a DMH is the one proposed by Bouc [5] and subsequently generalized by Wen [70]. In this model the applied force (input) denoted here as $z$, required to produce a displacement $q$ (output) is given in Eq. (1.1).

$$\dot{z} = -\gamma |\dot{q}| |z|^{n-1} - v\dot{q}|z|^n + A\dot{q}$$

(1.1)

Parameters $\gamma$, $v$, $A$ and $n$, control the shape and magnitude of the hysteretic loop. Other examples of DMH include the widely used in deterministic and random vibrations analysis, bilinear [7,8,24] and curvilinear model.
Figure 1.5: Representative hysteretic nonlinearity with local memory and inner curves within the major loop

This kind of representation has in general yielded good results with authors providing evidence that a careful selection of such parameters can result in good approximation of loops obtained by using experimental data. However, it has also been reported that crossing and coincident minor loops might occur. Thus, the existence of these two characteristics suggests that the states of the corresponding hysteretic systems cannot be uniquely defined by their inputs and outputs and the assumption of local memory is not appropriate.

In this case, there is another class of models that can be used to study the behavior of hysteretic systems with nonlocal memories. For this kind of non-linear behavior, future values of the output depend not only on the current value of the input but on its history as
well. Using again notions from probability theory, this type of behavior is analogous to non-Markovian processes.

![Diagram of a bilinear model of hysteresis and its input-output relationship](image)

**Figure 1.6:** The bilinear model of hysteresis and its input-output relationship

Furthermore, at any reachable point in the region defined by a major loop that corresponds to nonlocal memory, there is an infinite number of curves, each one of which
depends on a particular input history, that may represent the future behavior of the system.

The mathematical modeling of hysteretic nonlinearities with nonlocal memories is the theoretical basis of the CPM. In particular, the theory of operators, which was initially proposed by the work of Krasnoselskii and Pokrovskii [30], served as the mathematical background that helped Mayergoyz [38] introduce the CPM, which is essentially based on the superposition of simple operators, called hysterons. The resulting model named after Preisach who first conceived it, is today perhaps the most widely used model in hysteretic studies. Note that the relatively simple geometrical interpretation of the Preisach model has led to efficient numerical techniques, attracting in this way the interest of many researchers in the field. In Chapter 3 the CPM that is used to study the behavior of rate-independent-scalar hysteretic systems will be explicitly presented, highlighting its main advantages and demonstrating its basic properties.

1.4 Thesis organization

Hysteresis is the phenomenon that is being examined in this thesis. Chapter 2 provides a perspective on the problem and presents the goals of the proposed analysis. Hence, hysteresis is regarded from an engineering point of view and in particular in the framework of mechanical vibrations. This means that for the developments of this thesis, hysteresis is a property of a mechanical system which has the form of a Single-Degree-of-Freedom (SDOF) oscillator and is subjected to deterministic and random loads. In this context Chapter 2 serves as the mathematical background of the analysis presented in the
following chapters and, therefore, standard results of deterministic and random vibrations theory are given to help the reader connect the computed results with the existing information on the subject found in the literature. In particular, the method of *Equivalent Linearization (EL)* which is used to compute the response of a *SDOF* system subjected to periodic excitation is explicitly presented since the results in *Chapter 4* are derived using this method. Furthermore, in the case of random vibration analysis, the method of *Stochastic Averaging (SA)* that is used when an approximate solution for the response of a similar *SDOF* oscillator under white noise excitation is needed, is again extensively presented since the results in *Chapter 5* are based on the successful implementation of this method in the case of the system of interest.

*Chapter 3* presents the *CPM*, which constitutes the general mathematical tool used to cope with the hysteresis nonlinearities that are present in the systems that this thesis is considering. The genesis of the model, its basic features, the unique geometrical interpretation of the memory effect that it offers, as well as the manner to numerically implement it are presented in details to familiarize the reader with the kernel of the proposed analysis.

*Chapter 4* presents the response of two systems with hysteresis modeled using the Preisach formalism under deterministic loads. These systems are *SDOF* oscillators with a non-linear component, which is modeled by the *Iwan model (IM)* in the first case and by a *SMA* model in the second. These two cases exhibit some important differences in the way that they use the *CPM* and they have been selected to present the ability of the Preisach method to adjust itself according to the hysteretic characteristics of the system.
of interest. In each case the steady-state amplitude of the oscillator when subjected to a harmonic excitation for a wide range of frequencies is determined. Finally, the effect of certain parameters on the shape of the computed hysteresis loops is examined and it is shown that a suitable choice of their values can deal with various types of loops obtained by real experimental data.

In Chapter 5 the response of the system of the first case considered in Chapter 4 when it is subjected to white noise excitation is examined. To this aim, the SA method presented in Chapter 2 is implemented accordingly to derive analytically the stationary probability density function of the response envelope. The computed results are compared with a pertinent Monte Carlo study.

Finally, in Chapter 6 a recapitulation of what has been presented in the thesis is given and the main conclusions that have been reached are underlined.
Chapter 2

Mathematical background

2.1 Introduction – Motivation and fundamental principles

It became clear in Chapter 1 that hysteresis is the main subject of this thesis. However, this observation is insufficient to include the goals of the thesis and the point of view of the author. In fact, it has been already mentioned that the phenomenon is present in many fields of science and numerous efforts have been made over the past years to define, describe, and quantify the effect of hysteresis on systems, materials, applications, measurements etc. Therefore, it is important to explain at this point the general context in which hysteresis is viewed in the present analysis and which goals are to be achieved.

To this aim, it should be noted that this is a text written by a mechanical engineer for engineers and it might appear useful to them because hysteresis is being considered in the framework of mechanical vibrations. Consider the basic model shown in Fig. 2.1 (similar to Fig. 1.1).

![Diagram](image)

**Figure 2.1:** General representation of the model on which the present analysis is based
Input $x(t)$ induces an output $y(t)$. At this state there are no assumptions whether the system is linear or non-linear, time variant or time invariant or if the input (and consequently the output) is deterministic or stochastic. The only a priori knowledge given is that the system represents a Single-Degree-of-Freedom (SDOF) oscillator. The existing theory on mechanical oscillations in the case that the system is linear and non-linear and in each case, the results under deterministic and random excitations will be used in order to compare known results with the ones computed in the present analysis. In all cases that nonlinearities are appearing, it is assumed that they are of a hysteretic type.

Therefore, since the theoretical framework of the thesis comprises the theory of oscillations, this chapter focuses on main results of linear and non-linear deterministic vibrations and their stochastic counterparts.

### 2.2 Theory of Vibrations - Deterministic case

The deterministic theory of structural vibrations has its roots as early as Lord Rayleigh's celebrated book *The Theory of Sound* published in 1877 [53]. In this book, Rayleigh is concerned with how sound is created and perceived, effort that leads to the vibration and resonance of elastic solids and gases and the propagation of acoustic waves in various media. Engineers dealing with structures adopted the results that Rayleigh derived and in the following years, a new rigorous theory emerged, beginning with Timoshenko's book *Vibration Problems in Engineering* [69]. The theory was primarily motivated by dynamic loading resulting form rotating machinery operation and thus the loads
considered were periodic with known frequencies. In all cases the physical parameters of the system are specified. Today, there are many books that extensively cover the theory of deterministic vibrations including cases where the system is linear [12,39] or nonlinear [3,14,20,41,57] and the excitation is periodic or non-periodic. The case of both linear and nonlinear systems will be examined in the following sections.

2.2.1 Linear system

Modeling a mechanical system is often a necessary step in the design process in order to get an exact analytical description of its behavior under prescribed loads. To this aim, lumped-parameter models or -as they are frequently called- discrete models are used extensively in studies of vibrating mechanical systems. These models consist of dampers and springs that are connected to a mass, whose motion is caused by an excitation force. The SDOF model is one example of a discrete model and in its simple form is given in Fig. 2.2 where \( m \) is the mass, \( C \) the coefficient of viscous damping, \( K \) the spring constant and \( F \) the excitation force. The equation that governs the motion of this system can be found by direct application of Newton's 2\textsuperscript{nd} law as presented in Eq. (2.1)

\[
m \ddot{x}(t) + C \dot{x}(t) + K x(t) = F(t)
\]

(2.1)
Dividing Eq. (2.1) by $m$ yields Eq. (2.2), where the natural frequency $\omega_n$, the viscous damping factor $\zeta$ and the excitation force $f(t)$ are given in Eq. (2.3).

\[
\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n x(t) = f(t) \tag{2.2}
\]

\[
\omega_n = \sqrt{\frac{K}{m}}, \quad \zeta = \frac{C}{2m\omega_n} \quad \text{and} \quad f(t) = \frac{F(t)}{m} \tag{2.3}
\]

Thus, the motion of the oscillator is described by a 2nd order linear Ordinary Differential Equation (ODE), which must satisfy the pertinent initial conditions given in Eq. (2.4), where $x_0$ and $\dot{x}_0$ are the initial displacement and velocity respectively.

\[
x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = \dot{x}_0 \tag{2.4}
\]
For the purposes of the present analysis, the response of an oscillator (like the one given in Fig. 2.2) under periodic excitations is considered. To this aim, assume for convenience that the excitation has only one frequency and thus it is a trigonometric or monochromatic (mono-frequency) force of the form shown in Eq. (2.5).

\[ f(t) = F_o \cos(\omega t) \]  

(2.5)

It is further assumed that the response of the oscillator to this excitation is given by Eq. (2.6), where \( A \) is the response's amplitude.

\[ x(t) = A \cos(\omega t) \]  

(2.6)

Thus, by substitution of Eq. (2.6) in Eq. (2.2) and differentiation, it is straightforward to verify that the response is given by Eq. (2.7), where \( \phi \) is a change in the phase between the excitation and the response defined in Eq. (2.8).

\[ x(t) = \frac{F_o}{\sqrt{(-\omega^2 + \omega_n)^2 + (2\zeta\omega_n)^2}} \cos(\omega t + \phi) \]  

(2.7)

\[ \tan \phi = \frac{2\zeta\omega_n}{\omega^2 - \omega_n^2} \]  

(2.8)

Analogous results can be derived for any type of periodic excitation using a Fourier series representation. This means that any periodic excitation can be represented as a
sum of trigonometric functions like the one defined in Eq. (2.5). In this case the response of a linear oscillator can be computed by taking into account one basic property of linear systems: the superposition principle. According to this principle, since the excitation can be viewed as a linear combination of simple trigonometric functions, the total response is the linear combination of the individual responses of the oscillator to each input.

For reasons of completeness it is mentioned that in the case that the excitation is non-periodic, then the calculation of the response of the oscillator is determined via the convolution integral method, which takes advantage of the impulse response of the system. Impulse response is the response of the oscillator when the excitation is the Dirac delta function. Since in the analysis presented in this thesis the deterministic loads considered do not include the case of non-periodic excitations, the interested reader may consult the suggested references for further details.

2.2.2 Non-Linear system

While vibrations of systems which are linear should not be treated as trivial, since as Schmidt and Tondl [57] notice: "[...] many important vibration problems must be and can be treated as linear", in modern engineering with its continuous refinements of instrumentation and its improved computational capabilities, non-linear oscillations are gaining increasing attention. In fact, linearization is some times just a first approximation of the reality that provides good results in some cases, but it fails to furnish acceptable results in other. Therefore, in some applications it is almost self-evident that
nonlinearities should be included in the analysis in pursuit of a more accurate representation of the physical system.

This refinement though, does not come without a prize. Thus, when nonlinearities are included in the modeling of a physical system the resulting equations of motion are nonlinear differential equations, for which apart from very few exceptions, it is generally not possible to derive a closed-form solution. Therefore, numerical solutions are needed. However, a numerical solution does not always offer an insight into the system’s dependence on some parameters and thus approximate analytical solutions that are able to cope with the non-linear terms in an efficient way are also needed. To this aim, perturbations methods [41] were historically the first class of methods used to deal with this challenge. These methods aim on deriving periodic solutions in terms of a power series of a small parameter, say \( \varepsilon \), which appears in the differential equation, such that when it is equal to zero the non-linear equation degenerates into a linear differential equation with constant coefficients. However, as Bogoliubov and Mitropolsky mention [3], these methods are suitable for small intervals of time and sometimes it is quite cumbersome to compute the solution.

Another natural way of attacking non-linear differential equations is to replace the non-linear equation by an equivalent linear where the difference between the two is minimized in some appropriate sense. This concept was initially proposed by Bogoliubov and Mitropolsky [3] and was independently presented as a method to deal even with non-linear stochastic differential equations by Booton [4] and later by Caughey [10], who used the term equivalent or statistical linearization. In terms of deterministic loads, the
method of equivalent linearization is used in this thesis in order to obtain approximate values of the steady-state amplitude of a SDOF oscillator when subjected to monochromatic excitation and thus it is useful to present its basic principles.

2.2.2.1 The method of Equivalent Linearization

In the following analysis, the response of a non-linear SDOF dynamical system to an external deterministic load is considered. The equation of motion of such a system can be written as in Eq. (2.9).

\[ m \ddot{x}(t) + C \dot{x}(t) + K x(t) + f(x,\dot{x},t) = F(t) \]  

(2.9)

where again \( m \) is the mass, \( C \) the coefficient of viscous damping, \( K \) the spring constant, \( F \) is the excitation force and \( f(x,\dot{x},t) \) is a function that depends on the response \( x \) and its derivative. As mentioned earlier, exact solutions for Eq. (2.9) exist only in cases that the non-linear term has some specific forms. In general, though, approximate solutions constitute the only way to approach analytically these non-linear problems.

The principle of the method is simple: replace the non-linear problem by an auxiliary-equivalent, for which an exact analytical solution is known. This substitution is made in a way that the difference between the original and the equivalent system is optimum according to some criterion. In general, the equivalent problem needs not to be linear but in terms of analysis and considering the extension to MDOF systems the linear case has given readily available results. Therefore, an equivalent linear system that is an
approximation of the problem defined in Eq. (2.9) is the one presented in Eq. (2.10),
where $C_{eq}$ and $k_{eq}$ are called equivalent damping and stiffness terms and they are
determined by the requirement that the difference between the original and the equivalent
system is minimized in a certain way.

$$\ddot{x} + (C + C_{eq})\dot{x} + (K + k_{eq})x = F(t)$$  \hspace{1cm} (2.10)

This difference is represented by $\delta$ and based on Eqs. (2.9) and (2.10) assumes the form
of Eq. (2.11).

$$\delta(x, \dot{x}) = f(x, \dot{x}) - C_{eq}\dot{x} - k_{eq}x$$  \hspace{1cm} (2.11)

In Eq. (2.11) the dependence on time is omitted for reasons of convenience. By
examining Eq. (2.11), it becomes clear that the equivalent terms depend on the response
of the system, and thus the response $x(t)$ that will be derived for the equivalent system
must also be a good approximation of the response of the non-linear system in the case
that $\delta$ is minimized. Furthermore, it is clear that the smaller the non-linear term is, the
better the EL approximation will be.

The minimization criterion that is chosen for this difference according to the EL method
is the mean square of $\delta$ leading to a least-squares approximation. In this case, one
should satisfy the requirement given in Eq. (2.12).
\[ \delta^2 = \text{minimum} \] (2.12)

This minimization takes place in a time averaging sense, which means that if one period of excitation is considered then Eq. (2.12) is equivalent to Eq. (2.13).

\[ \min \left( \int_0^T (f(x, \dot{x}) - C_{eq} \dot{x} - k_{eq} x)^2 \, dt \right) = \text{min}(E) \] (2.13)

Furthermore, since this minimization is made with respect to the unknowns \( C_{eq} \) and \( k_{eq} \), differentiation of \( E \) in Eq. (2.13), as presented in Eq. (2.14), leads to Eq. (2.15).

\[ \frac{\partial E}{\partial C_{eq}} = 0 \quad \text{and} \quad \frac{\partial E}{\partial k_{eq}} = 0 \] (2.14)

\[ C_{eq} = \frac{\int_0^T f(x, \dot{x}) \dot{x} \, dt}{\int_0^T \dot{x}^2 \, dt} \quad \text{and} \quad k_{eq} = \frac{\int_0^T f(x, \dot{x}) x \, dt}{\int_0^T x^2 \, dt} \] (2.15)

Thus, Eq. (2.15) provides the unknown terms that appear due to the equivalent linearization and their substitution in Eq. (2.10) results in a linear ODE, which is an optimum approximation of the original non-linear Eq. (2.9).
2.3 Theory of Vibrations - Stochastic case

Many important modern vibration problems do not include a trigonometric, periodic or even a deterministic loading. Areas such as aerospace and ocean engineering deal with excitations which cannot be described in terms of a deterministic function where there is dependence on a set of independent variables. In fact, the excitation is so complex that it can only be described in a statistical sense. These remarks have induced developments in the theory of vibrations towards the direction of random phenomena and resulted in the incorporation of techniques and results of the well-established theory of probability.

In this probabilistic approach of vibration both the excitation and the response of the physical system are modeled as stochastic processes which in general can be partially determined by statistical parameters such as the mean value, the variance, and other moments and fully by distributions and density functions. The theoretical background of random vibration analysis can be traced in the early work of physicists like Einstein [17] and other researchers who beginning with the study of the Brownian motion they developed probabilistic characterizations for the motion of a linear SDOF system excited by what is called a white noise process. In addition, the development of mathematical tools such as the Fourier analysis, the autocorrelation functions and the spectral density functions, enabled engineers to adapt this theory in the case of structural vibrations arising in turbulence problems from jet engines or wind loading. Thus, in 1963 Crandall in his book entitled Random Vibration in Mechanical systems [13] signaled the start of the modern study of random vibration of mechanical systems and structures.
The purpose of this section cannot be a thorough presentation of the theory of linear and non-linear random vibration theory. However, it is deemed appropriate to familiarize the reader with certain basic concepts of stochastic processes and the methods used in random vibration analysis. Moreover, one can consult the extended literature on the subject, a basic subset of which is given in references \([2,11,34,42,54,72,73]\), to gain some additional insight in the concepts and principles that underlie this modern branch of engineering.

Next, some basic properties of stochastic processes and random vibrations are given in a short but hopefully coherent way, in the context of the results appearing in the following chapters.

### 2.3.1 Random Variables and Stochastic Processes

*Stochastic processes* theory is based primarily on results of *probability theory* (PT) and when talking about probabilities, the mathematical concept of *random variables* (rvs) is the basis of the discussion \([47]\). A rv \(\eta\) is a rule in the sense that it assigns a number under certain assumptions to an event such as “the outcome of a die that is rolled”. There are *discrete* rvs, like the one describing the rolling of a die in which case the set of all possible outcomes are finite, and *continuous* rvs, which characterize physical quantities that are measured continuously in time. This thesis deals only with *continuous* rvs.
A mathematical description of a rv is achieved by means of its cumulative distribution function (cdf) defined in Eq. (2.16), where the symbol \( P \) stands for probability, when asking: “what is the probability that given \( x \) the rv \( \eta \) is smaller or equal to \( x \).

\[
F(x) = P(\eta \leq x)
\]  

(2.16)

If the set of all possible events that the rv is expressing includes all real numbers, \( F(x) \) tends to zero as \( x \to -\infty \) and to one when \( x \to +\infty \). The derivative of a cdf is called probability density function (pdf), which in terms of the definition given in Eq. (2.16), gives the same probability (i.e. that \( \eta \) is smaller or equal to \( x \)) in the way shown in Eq. (2.17).

\[
P(\eta \leq x) = \int_{-\infty}^{x} f(s)ds
\]  

(2.17)

An important pdf is the Gaussian distribution, which for the one-dimensional case that is considered in this presentation has the form presented in Eq. (2.18).

\[
f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]  

(2.18)

The pdf describes completely the statistical properties of a random variable. In many cases, though, the pdf of a rv is not known and some knowledge about the behavior of the rv is obtained by certain statistical measures. The mean, mean value or expected
value of a rv is an example of such a measure. The mean value expresses the central tendency or the average value of all possible values that the rv could assume, when an infinite set of experiments is considered. Its mathematical form is given in Eq. (2.19).

\[ \mu = \mathbb{E}[\eta] = \int_{-\infty}^{\infty} x f(x) \, dx \]  \hspace{1cm} (2.19)

Another important measure is the variance of a rv, which expresses the second moment of the pdf, that is the dispersion around the mean and is defined in Eq. (2.20).

\[ \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx \]  \hspace{1cm} (2.20)

The positive square root of the variance is the standard deviation of the rv, \( \sigma \). In general the n-th moment of a rv is defined as in Eq. (2.21).

\[ \mathbb{E}[\eta^n] = \int_{-\infty}^{\infty} x^n f(x) \, dx \]  \hspace{1cm} (2.21)

It should be noted here that a Gaussian distribution is fully defined by its first two moments (mean and variance). Furthermore, Gaussian rvs play an important role in probability theory due to their unique properties. For example, it is true that any linear combination of Gaussian rvs is also a Gaussian rv, and that the sum of N independent rvs with arbitrary pdfs tends to a Gaussian distribution for large N. The last result
constitutes one of the fundamental theorems of PT, called the _Central Limit Theorem_ (CLT).

Based on these fundamental concepts, it is now appropriate to define the _Stochastic Process_ (SP). Recall that a _rv_ is a rule, which assigns a number to the outcome of an experiment. Assume now, that instead of recording a number, a function of time is obtained each time that the experiment is performed. This sample function will be in general different in each experiment and a repetitive collection of such functions leads to an _ensemble or group of realizations_. This ensemble constitutes the notion of a SP and is associated with the concept of _rvs_ by realizing that at each instant of time, the set of values of the SP that correspond to this instant is in fact a _rv_. The use of time functions should not be regarded as a restriction to the generality of the concept, since there are applications where time and space are viewed as the independent variables of each sample function. Therefore, a SP is an infinite set of correlated _rvs_, each one of which corresponds to a specific instant of time.

A SP is completely defined when the _cdf_ at any given instant of time is known. Alternatively, the joint _pdf_ defined for a vector of _rvs_ \( \eta = [\eta_1, \eta_2, \ldots, \eta_n]^T \) is required for all instants of time. In practice, though, what are easier to compute for a given SP are again some statistical properties, like the _mean_, the _covariance_ and the _autocorrelation function_ defined in Eqs. (2.22), (2.23) and (2.24) respectively.

\[
\mu(t) = E[\eta(t)] \quad (2.22)
\]
\[
\text{Cov}(t_1, t_2) = E\{[\eta(t_1) - \mu(t_1)][\eta(t_2) - \mu(t_2)]\} = R(t_1, t_2) - \mu(t_1)\mu(t_2) \tag{2.23}
\]

\[
R(t_1, t_2) = E[\eta(t_1)\eta(t_2)] \tag{2.24}
\]

A special case of a SP is a Gaussian or White-Noise process, which is the process whose joint pdf at time instants \( t_1, t_2, \ldots, t_n \) is an n-dimensional Gaussian distribution.

One important concept related to SPs is stationarity. Moreover, a SP is called strictly stationary if its statistical properties are not affected by a shift in the time origin. In the case that the complete statistical description of the process is not available, then the concept of stationarity can be restricted in the first two moments by requiring that the mean value is constant and that the covariance is only a function of the time difference \( \tau = t_2 - t_1 \). A process with these properties is called stationary in the wide sense or weakly stationary.

Another important and in many cases useful property of stationary SPs is ergodicity. Formally, ergodicity implies a substitution of the ensemble averages with pertinent temporal ones, when computing moments. For example, if a realization \( \eta_i(t) \) of an ergodic process \( \eta(t) \) is considered, then its mean value is given as,

\[
\mu = \lim_{T \to \infty} \frac{1}{T} \int_0^T \eta_i(t) dt \tag{2.25}
\]
Ergodicity is an essential assumption in cases where only one realization of a stationary SP is available. In this case, if the realization is long enough, it can be treated as ergodic and thus statistical moments can be derived.
Finally, it is important to mention the notion of the *power spectrum* or *spectral density function*. The power spectrum of a SP is defined as the Fourier transform of its autocorrelation function as shown in Eq. (2.26).

\[
S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) \exp(i\omega \tau) d\tau
\]  

(2.26)

The inverse of Eq. (2.26) provides the autocorrelation function and thus the spectral density and the autocorrelation form a very important pair of functions. According to its spectral density, a SP can be *wide band* or *narrow band*. A wide (or broad band) process has a density that covers a broad band of frequencies. In the limit case that the frequency band includes all the frequency range \((-\infty, \infty)\), the process is called *white* and its power spectrum is illustrated in Fig. 2.4. The concept of a process with a white spectrum is purely theoretical, since a spectrum like this would have infinite energy and in practice a process is considered white if its bandwidth is extended enough to include all the frequencies of interest. In the other case where the power spectrum of a SP has a frequency range limited in a small interval, then it is narrow banded.

![Figure 2.4](image)

**Figure 2.4:** Power spectrum of a wide band process. The spectrum is white if \(\omega_1 = 0\) and \(\omega_2 = \infty\).
In conjunction with the analysis and results presented in the following chapters, it is believed that it will be useful to describe briefly a basic method of solving the non-linear Stochastic Differential Equations (SDE) that constitute the most general case of equations of motion found in the modeling procedure of vibration problems that involve random excitation. Existing results of linear and non-linear random vibration theory will be extensively used and the suggested literature is proposed as the best source of information for the interested reader. However, the basic properties on SPs, PT and rvs that have already been given are viewed as sufficient to follow the steps of the forthcoming analysis.

2.3.2 The Stochastic Averaging method

When the excitation of the system is random, SPs model both the excitation and the response of the system. In the case that the system of interest is again the SDOF oscillator of Fig. 2.2, where the excitation \( F(t) \) is now random, Eq. (2.1) is no longer an ODE. In fact, it takes the form of a Stochastic Differential Equation (SDE). The case that the excitation is a White Noise Gaussian Process (WNGP) is particularly popular in random vibration studies due to existing solution techniques of the corresponding SDE. In the framework of this thesis, only such random excitations are considered and thus the following results focus on this case.

The determination of the response of a system, which is excited by a WNGP, is a difficult task and exact solutions have been derived only for specific cases. In general, there are two major types of approximate techniques. The first, are directly applicable to
the SDE and the goal is to determine the form of the solution by means of integration (Itô Calculus) [45]. The second, which includes the method presented in this section are based on the fact that the choice of a white noise excitation allows the development of a rather complete alternative theory since it can be shown that the response in this case is a Markov process for which there exists a significant amount of knowledge.

Stochastic Averaging (SA) is a method that belongs to the second class of techniques and its roots can be found in the work of Stratonovich, who studied the response of systems that are excited by broad-band random excitations, in connection with non-linear self-excited oscillations in electrical engineering due to the presence of noise. In the area of vibrations the basic principles stated by Stratonovich in his book Topics in the theory of random noise [61, 62] where adopted by Caughey [9], Iwan [26] and later Roberts and Spanos [54, 55]. The name SA and the adaptation and generalization of the procedure described by Stratonovich in random vibrations should be credited to their work. The goal in the SA method is to find a solution of the Fokker-Planck Equation (FPE) that corresponds to a Markovian approximation of the response energy or amplitude. This solution provides the exact pdf of the response to WNGP. In general, techniques for finding the solution of a FPE include separation of variables and eigenfunction expansions for the resulting eigenvalue problem, iterative and numerical solutions.

The SA method is based on the fact that the rate of change with respect to time of the total energy of the system is equal to the power input due to the random excitation minus the power dissipation due to the damping mechanism. This energy balance approach is significantly facilitated under the following assumptions. First, if the damping is light and
the supremum of the excitation power spectrum is scaled accordingly, then the energy of the system varies slowly with respect to time and can be treated as a constant over an appropriate period. Second, with light damping and a broad-band random excitation the relaxation time of the response is much greater than the correlation time of the excitation. Thus, it is possible to model the power input due to the excitation as a non-zero mean component plus an additional fluctuating term with the character of white noise.

Furthermore, SA is primarily based on a limit theorem due to Stratonovich and Khasminskii [29]. This theorem is applicable to an SDE of the form shown in Eq. (2.27), where $X(t)$ is an n-vector stochastic process, usually related to the response and $Y(t)$ is an m-vector stochastic process, usually relating to excitation.

$$\dot{X} = \varepsilon^2 f(X, t) + \varepsilon g(X, Y, t)$$  \hspace{1cm} (2.27)

If the elements of $Y(t)$ are broad-band processes, with zero means and under certain assumptions [29] it can be shown that $X(t)$ may be uniformly approximated over a time interval of order $O(\varepsilon^{-1})$ by an n-dimensional Markov process, which satisfies the Itô equation given in Eq. (2.28).

$$dX = \varepsilon^2 m(X)dt + \varepsilon \sigma(X)dW$$  \hspace{1cm} (2.28)

The symbol $W(t)$ denotes an n-vector of independent Wiener processes (that is WNGP) with unit variance and $m$ (drift vector) and $\sigma$ (diffusion matrix) are defined by the
expressions shown in Eq. (2.29) and (2.30), where \( E\{\cdot\} \) denotes the operator of expectation and \( T^{av} \) is a time-averaging operator defined in Eq. (2.31)

\[
m = T^{av} E\{f\} + \int_{-\infty}^{0} E\left\{ \frac{\partial g}{\partial X}(g')(t+\tau) \right\} d\tau
\]

\[
\sigma_{\sigma'} = T^{av} \int_{-\infty}^{\infty} E\{(g)(g')(t+\tau)\} d\tau
\]

\[
T^{av}\{\cdot\} = \lim_{T \to \infty} \frac{1}{T} \int_{t_o}^{t_o+T} \{\cdot\} dt
\]

Here \( t_o \) is the initial time and \( T \) is the period. Now, if \( \varepsilon \) is small then the elements of \( X(t) \) must be slowly varying with respect to time. This is a key remark that will be used in the state space formulation in order to obtain independent response variables.

For the purposes of the following discussion it will be assumed that the response of a non-linear SDOF system under random excitation can be described by Eq. (2.32) [55,58,59].

\[
m \ddot{x}(t) + C \dot{x}(t) + K x(t) + \varepsilon^2 f(x,\dot{x},t) = w(t)
\]
where $m$ is the mass, $C$ is the damping coefficient, $K$ is the stiffness, $w(t)$ is a zero mean white noise process with spectral density $S$ and $f(x,\dot{x},t)$ is a general nonlinear restoring force. The autocorrelation function of the excitation $w(t)$ is given in Eq. (2.33).

$$R(\tau) = E\{w(t)w(t+\tau)\} = 2\pi S\delta(\tau)$$  \hspace{1cm} (2.33)

In the last equation, $\delta(\tau)$ is the Dirac delta function. Naturally, since $\epsilon$ is small, the nonlinear term is small compared to the other terms of Eq. (2.32). The conditions presented in Eqs. (2.34) and (2.35) are additionally required.

$$\frac{c}{m} = O(\epsilon)$$  \hspace{1cm} (2.34)

$$S = O(\epsilon)$$  \hspace{1cm} (2.35)

Using these conditions Stratonovich claimed that the response during a single period of excitation will be nearly harmonic (pseudo-harmonic) as shown in Eq. (2.36). The derivative of the response is given in Eq. (2.37).

$$x(t) = a(t) \cos(\omega_n t + \phi(t))$$  \hspace{1cm} (2.36)

$$\dot{x}(t) = -a(t) \omega_n \sin(\omega_n t + \phi(t))$$  \hspace{1cm} (2.37)
In this Van der Pol type transformation, \( a(t) \) denotes the amplitude and \( \phi(t) \) the phase of the response and under the assumptions made above, both are slowly varying functions of time. The effect of the nonlinear term is incorporated in the change of the frequency. Thus, \( \omega_n \) is different from the angular velocity of the system. Now, from Eqs. (2.36) and (2.37) it can be shown that Eqs. (2.38) and (2.39) represent the evolution of the amplitude and phase in time.

\[
\dot{a} = \frac{\epsilon^2}{\omega_n} f(a \cos(\theta), -a\omega_n \sin(\theta)) \sin(\theta) - \frac{\epsilon z(t)}{\omega_n} \sin(\theta) \tag{2.38}
\]

\[
\dot{\phi} = \frac{\epsilon^2}{a\omega_n} f(a \cos(\theta), -a\omega_n \sin(\theta)) \cos(\theta) - \frac{\epsilon z(t)}{\omega_n} \cos(\theta) \tag{2.39}
\]

In the equations above \( \theta = \omega_n t + \phi \), and the excitation has been replaced by \( \epsilon z(t) \) in order to indicate the fact that when \( \epsilon \) is small the level of the excitation will normally be weak compared with the maximum level of the response.

Clearly, Eqs. (2.38) and (2.39) are in the standard form given in Eq. (2.27). Thus, the limit theorem mentioned above can be applied where it is noted that when \( \epsilon \to 0 \), the pair \( (a(t), \phi(t)) \) approaches a joint Markov process. By evaluating the quantities \( m \) and \( \sigma \) and due to the periodic nature of the terms involved, one obtains the \( Itd \) equations for \( a(t) \) and \( \phi(t) \) presented in Eqs. (2.40) and (2.41).

\[
da = \frac{-\epsilon^2}{\omega_n} F(a) dt + \frac{\pi S}{2a\omega_n^2} - \frac{(\pi S)^2}{\omega_n} dW_1(t) \tag{2.40}
\]
\[ d\phi = \frac{-\varepsilon^2}{\omega_n} G(a) dt - \frac{(\pi S)^2}{a \omega_n} dW(t) \]  \hspace{1cm} (2.41)

\[ F(a) = \frac{1}{2\pi} \int_0^{2\pi} h(a \cos(\theta), -a\omega_n \sin(\theta)) \sin(\theta) d\theta \]  \hspace{1cm} (2.42)

\[ G(a) = \frac{1}{2\pi} \int_0^{2\pi} h(a \cos(\theta), -a\omega_n \sin(\theta)) \cos(\theta) d\theta \]  \hspace{1cm} (2.43)

One major advantage of the procedure followed so far is that from Eq. (2.40) it can be deduced that the amplitude is uncoupled from the phase \( \phi(t) \) and thus \( a(t) \) is a one dimensional Markov process. The transition density function \( p(a, f, t | a_1, f_1, t_1) \) that corresponds to the pair \((a(t), \phi(t))\) is governed by the \textbf{FPE} (or forward Kolmogorov equation) given in Eq. (2.44), where \( p(a, t) \) is the pdf of the response envelope.

\[ \frac{\partial p}{\partial t} = \frac{\partial}{\partial a} \left[ \left( \frac{\varepsilon^2 F(a)}{\omega_n} - \frac{\pi S}{a \omega_n^2} \right) p \right] + \frac{\varepsilon^2 G(a)}{a \omega_n} \frac{\partial p}{\partial \phi} \]

\[ + \frac{\pi S}{2a \omega_n^2} \left[ \frac{\partial^2 p}{\partial a^2} + \frac{1}{a^2} \frac{\partial^2 p}{\partial \phi^2} \right] \]  \hspace{1cm} (2.44)

Now, since \( a(t) \) and \( \phi(t) \) are not coupled, the \textbf{FPE} that corresponds to the transition density function \( p(a, t | a_1, t_1) \) for \( a(t) \) can be written as in Eq. (2.45).

\[ \frac{\partial p}{\partial t} = \frac{\partial}{\partial a} \left[ \left( \frac{\varepsilon^2 F(a)}{\omega_n} - \frac{\pi S}{2a \omega_n^2} \right) p \right] + \frac{\pi S}{2a \omega_n^2} \frac{\partial^2 p}{\partial a^2} \]  \hspace{1cm} (2.45)
In the case that the excitation is stationary, the response will also approach stationarity as time progresses as described in Eq. (2.46), where \( w(a) \) is the stationary \( \text{pdf} \) of the amplitude of the response.

\[
\lim_{t \to \infty} p(a, t | a_0, t_0) \to w(a)
\]

(2.46)

Now, this stationary \( \text{pdf} \) can be obtained by setting Eq. (2.45) equal to zero. This solution has been derived in [58] and is based on the separation of variables technique that \textit{Stratonovitch} used in order to solve the same problem. The result as stated in this reference is given in Eq. (2.47), where \( L \) is a normalization constant.

\[
w(a) = L \ a \ \exp \left\{ -2 \epsilon^2 \omega_0 \ \frac{a^2}{\pi S} \int_0^a F(s) ds \right\}
\]

(2.47)

The same authors provide the form of the stationary \( \text{pdf} \) of the response of a linear system excited by white noise excitation by taking into account results from linear random vibration theory. Eq. (2.48) shows this result.

\[
w(a) = \frac{a}{\sigma^2} \ \exp \left\{ -\frac{a^2}{2\sigma^2} \right\}
\]

(2.48)

Here, \( \sigma^2 \) is the variance obtained assuming a linear system induced by the same excitation and its value is given in Eq. (2.49).
\[ \sigma^2 = \frac{\pi S_m}{2\zeta \omega_0^3} \]

Thus, in the case that the system is linear and the excitation is Gaussian the response of the system according to one of the fundamental results of the linear random vibration theory is also Gaussian and the amplitude has the Rayleigh distribution as given in Eq. (2.47). The same result has been given also by Stratonovich [61].

The principal advantage of the approximate solution given by Eq. (2.47) is that it gives explicit results for various types of damping where exact results do not exist even for the case of white noise excitation. Furthermore, it is rather straightforward to obtain useful statistics of the response. In the case that the excitation is not stationary or when we are interested in the transient response to suddenly applied stationary excitations, then it is necessary to solve Eq. (2.45) in order to obtain the transition density function. In this case, reference [55] presents a complete solution for the linear system and also comments on possible ways to solve the general nonlinear one.

2.3.3 The Monte Carlo method

SDEs are in general harder to treat than their deterministic counterparts. Existing analytical schemes of finding the solution include statistical linearization [48,58], the method of moment closure, perturbation and Markov methods. SA is an example of an approximate analytical solution that is based on the Markovian approximation. Due to the difficulties arising when trying to obtain stationary and non-stationary solutions using
these approximate techniques, alternative numerical methods of estimating the exact response statistics of non-linear systems, which are subjected to random excitation, have been proposed. One of the most important is the Monte Carlo (MC) method [19,27,35,56], which in general is based on the principle of performing numerical "experiments" on a computer. The genesis of the method is found during the development of the atomic energy in the post World War II era, where scientists needed to solve problems like the neutron diffusion or transport through an isotropic medium. The name is due to the famous casino located in the Monte Carlo area close to Italy due to the basis of the method, which is the generation of random numbers, a principle highly related with the games of chance and PT. In terms of solving SDE, MC simulation is based on a fundamental observation according to which a SDE can be regarded as an infinite set of deterministic equations. In random vibration theory, MC method is normally used in order to numerically assess the validity of analytical computations [48,58,65].

MC analysis involves digital simulations in which a large number of experiments are made in order to estimate statistical properties of the non-linear system within some confidence intervals. The first step in the procedure involves the generation of a set of random numbers. The term random means that these numbers belong to a specific distribution for example, the Gaussian distribution. This set is used to form a sample function of the excitation, which is then used as an input to the non-linear system. Moreover, the response is computed by integrating numerically the resulting non-linear differential equation, using any of the solvers that are available (e.g. Runge-Kutta schemes). By repeating the same procedure many times, a collection of response
functions is created, on which statistical analysis is performed. In general, MC can be used for both stationary and non-stationary response statistics. The number of iterations is not specified a priori but in principle the larger the set of response functions the better its statistics will be. In terms of estimating the pdf of the response, this number is of the order of thousands while if only certain moments are needed then it is of the order of hundreds.

In the present analysis, MC simulation will be used in Chapter 5 where the stationary pdf of the response envelope of a non-linear oscillator subjected to white noise excitation with specified spectral density is computed analytically via the SA method. The procedure involves the generation of sample excitation records, the numerical integration of the governing differential equation using these records and repetition of these steps. A relatively large number of response sample functions is created and based on this ensemble the stationary pdf is computed and compared with the analytical results.
Chapter 3

The Classical Preisach Model of Hysteresis

3.1 Introduction

The Classical Preisach Model (CPM) of hysteresis is the mathematical tool that is used to perform all the analytical and numerical computations in this thesis. Therefore, it is of primary importance for the understanding of both the general subject of this thesis and the proposed results, to present explicitly the model, whose successful implementation in the case of dynamical systems was the initial goal of the author. To this aim, this chapter provides information on how the model was originally conceived, the way that it transformed from a physical model to a general mathematical idea and finally on the development of what is now called CPM. The material is presented in a way that is deemed the most appropriate for the reader. However, the interested reader should consult additional references, especially the monograph on the theory of operators written by the Russian mathematicians Krasnoselskii and Pokrovskii [30], as well as the celebrated book by Mayergoyz [38] that formally introduced the concept of the CPM.

3.2 Definition of the CPM

3.2.1 From Preisach to Mayergoyz – The evolution of an idea

Ferenc Preisach in a landmark paper published in 1935 [51] introduced a purely intuitive model to describe hysteresis in ferromagnetic materials. This model was based on
plausible hypotheses and personal understanding of the physical mechanisms underlying the magnetization of this type of materials. Specifically, according to Preisach, these materials were made up of tiny magnetic particles which he named hysterons that possess a magnetic moment that depends on the history of the applied magnetic field. For this reason, this model was first regarded as a physical model of hysteresis and primarily remained known in the area of magnetism, although other researchers over the following years used empirical variations of the model in a number of physical systems. However, in the 1970's a group of Russian mathematicians started a detailed study of systems with hysteresis. It was precisely then that it was realized that the Preisach model introduces a new mathematical idea with the capability to be generalized to a broader class of phenomena and it is not just a phenomenological model. Hence, the mathematicians redefined the model in a rigorous way and proposed a new mathematical tool capable of describing any hysteretic nonlinearity. But, what is the kernel of the Preisach model and how it succeeds to fulfill its goals? These and other related questions will be the subject of the following sections where a general description of the model, its primary characteristics and some basic identification methods will be given.

3.2.2 Description of the Preisach model

The kernel of the Preisach model is a parameterized set of delayed relays or hysteresis operators as they will be called for the remainder of this text, $\hat{\gamma}_{\alpha,\beta}$. Each operator of the set (Fig. 3.1) is characterized by its switching values $\alpha$ and $\beta$ and without loss of generality it is assumed that $\alpha \geq \beta$. The output of each operator can take only two
values, +1 or -1 (however there are applications, as it will be shown in this thesis where the operators may assume different values like 0 and 1) and it remains constant until the input increases above $\alpha$ (where it climbs to +1) or decreases below $\beta$ (where it falls to -1). These movements are irreversible since $\alpha \neq \beta$. Based on this description, consider an input function $u(t) \in C^0[0,T]$. The hysteresis operator is defined as in Eq. (3.1).

$$\hat{\gamma}_{a,\beta} = \begin{cases} +1 & \text{if } u > \alpha \text{ and increasing or } u > \beta \text{ and decreasing} \\ -1 & \text{if } u < \beta \text{ and decreasing or } u < \alpha \text{ and increasing} \end{cases}$$ (3.1)

It is clear that the operators $\hat{\gamma}_{a,\beta}$, as defined here, represent hysteresis nonlinearities with local memories, since given the switching values and a value of the input, the output can be determined, regardless of the input history.

![Diagram](image)

**Figure 3.1:** A typical hysteron

Consider next, a superposition of such hysterons that have the same input $u(t)$ and give contribution to a single output $f(t)$, along with an arbitrary weighting function $\mu(a,\beta)$
which is often called *Preisach function*. Then, the CPM is formally defined in Eq. (3.2), where \( \Gamma \) is an integral operator.

\[
f(t) = \int \int_{\alpha \leq \beta} \mu(\alpha, \beta) \gamma_{\alpha,\beta} u(t) d\alpha d\beta = \Gamma u(t)
\]  

(3.2)

Thus, the output of the hysteretic system is written as a weighted sum of the output of each hysteron. Schematically, using again the control theory approach, Eq. (3.2) is equivalent to the block diagram of Fig. 3.2.

It is noted that the basis of the model is a superposition of simple hysterons with local memories that induces a behavior as shown in Eq. (3.2) which has nonlocal memory.

**Figure 3.2:** Block diagram representation of the CPM of hysteresis
3.2.3 Properties of the CPM – The role of the Preisach plane

Mayergoyz [38] proposed a very convenient geometrical method to present the way that the Preisach model actually works. Specifically, to demonstrate the structure of the model he took advantage of the “binary” output of the hysteresis operators. He, furthermore, proposed a plane, a half plane to be more accurate, where the support of the Preisach function is represented by means of a limiting triangle and each point of the enclosed area has coordinates that correspond to the switching values of each operator.

\[ (\alpha, \beta) \]

\[ \gamma_{\alpha, \beta} \]

\[ T \]

\[ \alpha = \beta \]

**Figure 3.3:** The Preisach plane

In Fig. 3.3, the support of the Preisach function is defined inside the area of the limiting triangle and note that since the output of the operators \( \hat{y}_{\alpha, \beta} \) can only assume two values, each pair \((\alpha, \beta)\) is sufficient to describe the state of one operator. Moreover, note that since, in this geometrical interpretation, a limiting triangle represents the Preisach function, it is implicitly assumed that the major hysteretic loops are closed. This
assumption is applied to most hysteretic nonlinearities observed experimentally and will not substantially limit the generality of the model.

The most important feature of the CPM is that it can represent hysteresis nonlinearities with nonlocal memories. Thus, one of the most important characteristics of the model is its memory formation capability, or the way that it takes into consideration the contribution of the entire input history at any given instant of time that the value of output is computed. This memory formation mechanism of the model will be discussed next.

Consider the input history of Fig. 3.4. To better understand the manner that the CPM stores the contributions of the input history to the final output and forms the nonlocal memory effect, let the input at time \( t_1 \) have a value less than \( \beta_0 \). This state is called "negative saturation" and consequently all the operators at this state have the value -1. Assume now that the input increases until, at time \( t_2 \), it reaches some maximum value \( a_1 \). As \( u(t) \) increases all the operators whose switching up value is less than the current value of the input, are "switched on" and assume the +1 value. In the Preisach plane (see Fig. 3.4), this leads to the division of \( T \) into two areas: \( S^+(t) \) consisting of all points \((a,\beta)\) for which the corresponding hysterons have the value +1, and \( S^-(t) \) which similarly consists of all points that correspond to hysterons that assume the value -1. This division is made by the line \( a = u(t) \) that moves up as the input increases, until it reaches the maximum value \( a_1 \). Let now the input decrease monotonically until it reaches the value \( \beta_1 \) at the time instant \( t_4 \). As the input decreases, the line \( \beta = u(t) \) moves from right to left and causes all the operators whose switching off values are greater than the current value of the input to assume the value -1. Similarly, assume that the input starts at time \( t_5 \)
increase again until it reaches the value $a_2$ at time $t_4$. This change causes a new horizontal link to move upwards, turning the operators with $a$ values less than the instantaneous value of $u(t)$ to assume again the value +1. Schematically, this procedure is presented in Fig 3.4. It can be shown that if $L(t)$ is the interface generated by these two subsets, $S^+(t)$ and $S(t)$, then for a general input history with various extreme points it has a staircase pattern and its vertices coincide with the previous extreme values of the input (Fig. 3.5).

Hence, the memory of the CPM is represented geometrically by this staircase pattern of the interface $L(t)$ between the two subsets whose vertices have coordinates in the Preisach plane that correspond to local extreme values of the input function. The final link of $L(t)$ may be horizontal or vertical according to the input function.

The example above represents the general case, although in practice the CPM may have to be modified according to the specifications of each application. An example of an application that requires such a change is the Shape Memory Alloy (SMA) system presented as a separate case of study in this thesis. For completeness, the corresponding memory formation mechanism, the hysteretic operators used as well as some information about the SMA system itself will be given in Chapter 4 when considering the modeling of this hysteretic system.

Based on the memory formation mechanism of the CPM, it is now appropriate to introduce two basic properties of the model. First, it is clear by the simple example presented in Fig. 3.4 that the CPM does not store all the extreme values of the input function. In fact, the model takes into consideration only some dominant values
according to a few basic rules and simply disregards all the others with no contribution to the output of the hysteretic system.

Figure 3.4: A sample input history and the memory formation mechanism of the CPM
Figure 3.5: Evolution of the interface $L(t)$ during the memory formation of the CPM for a general input history

The following property as given in reference [38] explains why this is happening:

**Property 1:** Each local maximum wipes out the vertices of $L(t)$ whose $\alpha$-coordinates are below this maximum and each local minimum wipes out the vertices whose $\beta$-coordinates are above this minimum (WIPING-OUT PROPERTY).

This practically means that the Preisach model stores only the alternating series of dominant input extreme values. All others are simply wiped-out!

The CPM is also characterized by a second basic property. To explain this property better, consider two sample input functions $u_1(t)$ and $u_2(t)$, which are in general different but they have two identical consecutive extreme values say $u_-$ and $u_+$. Next,
assume that both input functions change in the range \([u_-, u_+]\). Then minor hysteresis loops are produced that may have different position in the output axis, but they are congruent as shown in Fig 3.6.

![Diagram showing hysteresis loops](image)

**Figure 3.6:** The congruency property

Therefore, the following property is an important feature of the CPM.

**Property 2:** All minor hysteresis loops corresponding to back-and-forth variations of inputs between the same two consecutive extreme values are congruent in the geometrical sense (CONGRUENCY PROPERTY).

Based on these two fundamental properties of the Preisach model, there is a fundamental theorem, which gives the necessary and sufficient conditions in order to model a hysteresis nonlinearity using the CPM.
**Theorem A:** Properties 1 and 2 constitute the necessary and sufficient conditions for a hysteretic nonlinearity to be represented by the Preisach model on the set of piecewise monotonic inputs.

These are the theoretical requirements that one should satisfy in order to use the model when a hysteretic nonlinearity is given. The next step before implementing the model is to determine the Preisach function. This leads to an identification problem. A first approach on this subject is based on the so-called first order transition curves that are determined by the experimental data and was proposed by Mayergoyz. This procedure involves numerical differentiation of these curves and therefore it is highly probable to observe error accumulation. Alternative approaches do exist and as an example case the method proposed in references [31,32], which is based on fitting a known function on experimental hysteretic loops will be used in this study. It is noted that this step is of dominant importance, since the derivation of the Preisach function plays an important role in the behavior of the model.

### 3.2.4 Derivation of the Preisach function – Identification problem

The identification problem, as defined in the literature, is the procedure of deriving the Preisach function based on the hysteretic loops of a real experiment. To this aim, Mayergoyz in reference [38] proposed the method of first order transition curves that is presented next.
Consider a system, which is in the state of negative saturation. Let the input $u(t)$ increase monotonically until it reaches the value $\alpha'$ (Fig. 3.7). The corresponding value of the output at this point is $f_{\alpha'}$. A first-order transition curve is formed as the above increase is followed by a subsequent monotonic decrease to some value $\beta'$. The output at this point is $f_{\alpha',\beta'}$.

Figure 3.7: A first order transition curve and corresponding changes in the Preisach Plane
Define now the function,

\[ F(\alpha', \beta') = \frac{1}{2} (f_{\alpha'} - f_{\alpha' + \beta'}) \]  \hspace{1cm} (3.3)

Based on Eqs. (3.2) and (3.3) and using Fig 3.7, one derives Eq. (3.4)

\[ f_{\alpha' + \beta'} - f_{\alpha'} = -2 \int_{\tau(\alpha', \beta')} \mu(\alpha, \beta) \, d\alpha d\beta \Rightarrow F(\alpha', \beta') = \int_{\tau(\alpha', \beta')} \mu(\alpha, \beta) \, d\alpha d\beta \]  \hspace{1cm} (3.4)

Further, differentiating the last expression twice with respect to \( \alpha \) and \( \beta \) yields Eq. (3.5).

\[ \mu(\alpha', \beta') = -\frac{\partial^2 F(\alpha', \beta')}{\partial \alpha' \partial \beta'} = \frac{1}{2} \frac{\partial^2 f_{\alpha' + \beta'}}{\partial \alpha' \partial \beta'} \]  \hspace{1cm} (3.5)

Hence, by using the input-output relationship for a particular hysteretic system and the notion of first-order transition curves it is possible to derive the Preisach function.

One major disadvantage of the method presented above, as it was mentioned earlier, is that it involves numerical differentiation of experimental data and thus there is a chance that the output of this procedure is corrupted by errors, which are inherent in this numerical operation. Therefore, the alternative identification method that has been proposed in references [31,32] and consists of a least-squares fitting of a known statistical distribution and a single major loop of the hysteretic system can also serve as a more efficient identification procedure. An alternative approach is proposed in reference [43].
3.2.5 Numerical Implementation of the CPM

Consider an input function that creates the interface $L(t)$ shown in Fig. 3.8 (the symbol $u(t)$ denotes the current value of the input and $N=3$ in this case).

![Diagram](image)

Figure 3.8: An example of an interface (descending status)

The output that corresponds to this input function can be expressed using reference [15] as in Eq. (3.6).

$$f(t) = \int_{\alpha_0}^{\alpha} \mu(\alpha, \beta) \gamma_{\alpha, \beta} u(t) \, d\alpha \, d\beta \Rightarrow f(t) = \int_{s'(t)} \mu(\alpha, \beta) \, d\alpha \, d\beta - \int_{s'(t)} \mu(\alpha, \beta) \, d\alpha \, d\beta \Rightarrow$$

$$\Rightarrow f(t) = - \int_{s'(t)} \mu(\alpha, \beta) \, d\alpha \, d\beta - 2 \int_{s'(t)} \mu(\alpha, \beta) \, d\alpha \, d\beta$$

(3.6)

Next, based on Eq. (3.4) one can verify Eq. (3.7).

$$\int_{s'(t)} \mu(\alpha, \beta) \, d\alpha \, d\beta = F(\alpha, \beta)$$

(3.7)
Eq. (3.7) practically shows that the area of the support of the Preisach function in the limiting triangle $T$, as it was expected by the definition. Furthermore, the area of the subset $S^t(t)$ can be computed by simple subtractions of appropriate triangles as given in Eq. (3.8).

$$\int_{S^t(t)} \mu(\alpha, \beta) \, d\alpha \, d\beta = \int_{S_1} \mu(\alpha, \beta) \, d\alpha \, d\beta + \int_{S_2} \mu(\alpha, \beta) \, d\alpha \, d\beta + \int_{S_n} \mu(\alpha, \beta) \, d\alpha \, d\beta \Rightarrow$$

$$\Rightarrow \int_{S^t(t)} \mu(\alpha, \beta) \, d\alpha \, d\beta = \left[ \sum_{j=1}^{N-1} F(\alpha_j, \beta_{j-1}) - F(\alpha_j, \beta_j) \right] + [F(\alpha_N, \beta_{N-1}) - F(\alpha_N, u(t))]$$

(3.8)

Now, in case that the final link is horizontal (ascending status, Fig. 3.9), by similar arguments, one derives Eq. (3.9).

$$\int_{S^t(t)} \mu(\alpha, \beta) \, d\alpha \, d\beta = \int_{S_1} \mu(\alpha, \beta) \, d\alpha \, d\beta + \int_{S_2} \mu(\alpha, \beta) \, d\alpha \, d\beta + \int_{S_n} \mu(\alpha, \beta) \, d\alpha \, d\beta \Rightarrow$$

$$\Rightarrow \int_{S^t(t)} \mu(\alpha, \beta) \, d\alpha \, d\beta = \left[ \sum_{j=1}^{N-1} F(\alpha_j, \beta_{j-1}) - F(\alpha_j, \beta_j) \right] + [F(u(t), \beta_{N-1})]$$

(3.9)

Hence, using Eqs. (3.6), (3.7), (3.8) and (3.9) the input-output relationship is expressed in terms of simple algebraic computations between areas of triangles. This of course presents a major advantage of the CPM, and explains the popularity of the method.
Figure 3.9: An example of an interface (ascending status)
Chapter 4

Steady-state response of systems with hysteresis using the Preisach formalism

4.1 Introduction

The first application of the CPM considered in this thesis involves the steady-state dynamic response of two hysteretic systems when subjected to monochromatic excitation. The first system is a single-degree-of-freedom (SDOF) oscillator with a nonlinear component described by the model proposed by Iwan, which consists of a series of elasto-plastic elements [25]. The second involves a similar oscillator where the nonlinear component is now a Shape Memory Alloy (SMA) device. These examples where chosen due to their salient differences, mainly on the kind of operators that they rely on. Thus, they can demonstrate the flexibility of the CPM to adjust itself according to the case of study.

4.2 Case 1: The Iwan model

4.2.1 Description of the model

In 1966, Iwan in an effort to overcome the practical difficulties of the widely used, at that time, bilinear hysteretic model proposed an alternative that consists of a series of ideal elasto-plastic elements known as Jenkin’s elements [25]. Timoshenko initially suggested
this approach and Iwan adopted it in order to create a relatively simple hysteretic model, which can be used in studies of the dynamic response of hysteretic systems.

For reasons of convenience, the model proposed by Iwan is being called IM in the remainder of the text.

Figure 4.1: The Iwan Model (IM)

Consider next one Jenkin's element. Its force-displacement curve is shown in Fig. 4.2.

Figure 4.2: The force-displacement plot of a Jenkins element (f_y is the yielding force and k is Young's modulus)
Based on the presentation of the CPM, the displacement $x(t)$ can be viewed as the input and the force $f(t)$ as the output of the model. Let $x(t)$ be bounded in the range $[-a_0, a_0]$. The objective is to derive the Preisach function for the IM, and to this aim the method of first order transition curves will be used [36,38].

Starting from the state of negative saturation (i.e. from $x = -a_0$), let the input increase to some value $a'$. The corresponding value of the output is $f_{a'}$ (for this model $f_{a'} = f_{a_0}$). Now, if the input is decreased to some value, say $b'$, the output follows a reversal transition curve as is it shown schematically in Fig. 4.3 and it assumes the value $f_{a'b'}$. Based on what is known for this elastoplastic element one can show Eq. (4.1).

$$f_{a'b'} = f_y - k(a' - b') , \quad a' - \frac{2f_y}{k} \leq b' \leq a'$$

(4.1)

Define the function given in Eq. (4.2).

$$F(a', b') = \frac{1}{2}(f_{a'} - f_{a'b'})$$

(4.2)

Then, one can verify the result shown in Eq. (4.3).

$$f_{a'b'} - f_{a'} = -\int_{b'}^{a'} \mu(a', b') d\alpha d\beta$$

(4.3)
Thus the Preisach function can be obtained by differentiating Eq. (4.3) twice with respect to $\alpha'$ and $\beta'$, as presented in Eq. (4.4).

$$
\mu(\alpha', \beta') = -\frac{\partial^2 F(\alpha', \beta')}{\partial \alpha' \partial \beta'} = \frac{1}{2} \frac{\partial^2 f_{\alpha}\beta'}{\partial \alpha' \partial \beta'}
$$

(4.4)

\[\text{Figure 4.3: Major hysteresis loop and first order transition curves of the Jenkin's element}\]

Then, from Eqs. (4.1) and (4.4) one can verify that the Preisach function is given by Eq. (4.5), where $\delta$ is the Dirac delta function.

$$
\mu(\alpha', \beta') = \frac{1}{2} k \left[ \delta(\alpha' - \beta') - \delta(\alpha' - \beta' - \frac{2f_{\text{v}}}{k}) \right]
$$

(4.5)

The support of the Preisach function defined in Eq. (4.5) consists of the two lines defined in Eq. (4.6).
\[ \alpha' = \beta' \quad \text{(elastic part)} \]
\[ \alpha' - \beta' = \frac{2f_y}{k} \quad \text{(slip part)} \] (4.6)

Having defined the Preisach function it is now possible to express the input-output relationship. Eq. (4.7) shows how this may be accomplished.

\[ f(t) = \frac{k}{2} \left( \int_{-a_0}^{a_0} \gamma_{\alpha,\alpha} x(t) \, d\alpha - \int_{\frac{2f_y}{k}}^{\frac{2f_y}{k} - a_0} \gamma_{\alpha,\alpha} x(t) \, d\alpha \right) \] (4.7)

The IM consists of several Jenkin's elements, as the one defined above and therefore in the case that a series of such elements in a parallel arrangement is considered, the input-output equation can be written as in Eq. (4.8), where \( f_{IM}(t) \) is the output of the IM and \( f(f_y, t) \) is the output of each individual element expressed as a function of the yielding force \( f_y \), which is assumed to be uniformly distributed in the interval \([f_{y,\text{min}}, f_{y,\text{max}}] \).

\[ f_{IM}(t) = \int_{f_{y,\text{min}}}^{f_{y,\text{max}}} p(f_y) f(f_y, t) \, df_y \] (4.8)

Substituting Eq. (4.8) into Eq. (4.7) yields Eq. (4.9).

\[ f_{IM}(t) = \frac{k}{2} \left( \int_{-a_0}^{a_0} \gamma_{\alpha,\alpha} x(t) \, d\alpha - \int_{f_{y,\text{min}}}^{f_{y,\text{max}} - \frac{2f_y}{k}} \gamma_{\alpha,\alpha} x(t) \, d\alpha \right) \Rightarrow \]

\[ f_{IM}(t) = \frac{k}{2} \left( \int_{-a_0}^{a_0} \gamma_{\alpha,\alpha} x(t) \, d\alpha - \frac{k}{2} \int_{f_{y,\text{min}}}^{f_{y,\text{max}} - \frac{2f_y}{k}} \gamma_{\alpha,\alpha} x(t) \, d\alpha \right) \]

\[ \Rightarrow f_{IM}(t) = \frac{k}{2} \left( \int_{-a_0}^{a_0} \gamma_{\alpha,\alpha} x(t) \, d\alpha - \frac{k}{2} \right) \int_{f_{y,\text{min}}}^{f_{y,\text{max}}} \gamma_{\alpha,\beta} x(t) \, d\alpha \, d\beta \] (4.9)
Based on Eq. (4.9), the Preisach function of the IM takes the form given in Eq. (4.10), where $H$ denotes the Heaviside function.

$$
\mu_{hl}(\alpha', \beta') = \frac{k}{2} \left\{ \delta(\alpha'-\beta') - \frac{1}{2} \frac{k}{f_{y,max} - f_{y,min}} \left[ H(\alpha'-\beta'+ \frac{2f_{y,max}}{k}) - H(\alpha'-\beta'- \frac{2f_{y,min}}{k}) \right] \right\} \tag{4.10}
$$

The support of this function in the Preisach plane is shown in Fig. 4.4.

Figure 4.4: The support of the Preisach function for the IM

Thus, the Preisach function for the IM was determined following the procedure, which is primarily based on the so-called first-order transition curves that are obtained by plotting experimental data.
4.2.2 Numerical implementation of the IM using the CPM

Based on Eq. (4.9) and the definition of the Preisach plane, the input-output relation of the IM can be written as in Eq. (4.11), where according to the formulation $\alpha_o = -\beta_o = f_y,_{\text{max}}/k$.

\[
f_{IM}(t) = \frac{k}{2} \left( \int_{-\alpha_o}^{\alpha_o} \frac{d\alpha}{x(t)} \right) - \frac{k}{2} \frac{1}{f_{y,_{\text{max}}}-f_{y,_{\text{min}}}} \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} \left( \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} \int_{\gamma_{\text{min}}}^{\gamma_{\text{max}}} \gamma_{\alpha,\beta} x(t) d\alpha d\beta \right) \Rightarrow \]

\[
\Rightarrow f_{IM}(t) = k x(t) - \frac{k^2}{4} \frac{1}{f_{y,_{\text{max}}}-f_{y,_{\text{min}}}} \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} \left( \int_{\gamma_{\text{min}}}^{\gamma_{\text{max}}} \gamma_{\alpha,\beta} x(t) d\alpha d\beta \right) \]

(4.11)

Next, define the function $\Phi(\alpha', \beta')$ as in Eq. (4.12)

\[
\Phi(\alpha', \beta') = -\frac{k^2}{4} \frac{1}{f_{y,_{\text{max}}}-f_{y,_{\text{min}}}} \int_{\beta'}^{\beta_{\text{max}}} \int_{\gamma_{\text{min}}}^{\gamma_{\text{max}}} \gamma_{\alpha,\beta} x(t) d\alpha d\beta = -\frac{k^2}{4} \frac{1}{f_{y,_{\text{max}}}-f_{y,_{\text{min}}}} \Psi(\alpha', \beta') \]  

(4.12)

\[
\Psi(\alpha', \beta') = \frac{[2f_{y,_{\text{min}}} + k(\beta' - \alpha')]^2}{2k^2} \]  

(4.13)

Then, based on the procedure presented in Chapter 3, the output of the IM can be expressed numerically as in Eq. (4.14) for the ascending branch and as in Eq. (4.15) for the corresponding descending.

\[
f_{IM}(t) = k x(t) - \Phi(\alpha_o, \beta_o) + 2 \left\{ \Phi\left( \frac{f_{y,_{\text{min}}}}{k}, \beta_o \right) - \Phi\left( \frac{f_{y,_{\text{min}}}}{k}, \beta_i \right) \right\} + \\
+ \frac{2f_{y,_{\text{min}}}}{k} \Phi(x(t), \beta_{N-1}) + \sum_{j=2}^{N-1} \left\{ \Phi(\alpha_j - \beta_{j-1}) - \Phi(\alpha_j - \beta_j) \right\} \]  

(4.14)
\[ f_{im}(t) = kx(t) - \Phi(\alpha_o, \beta_o) + 2 \left[ \Phi\left(\frac{f_{y,\text{min}}}{k}, \beta_o\right) - \Phi\left(\frac{f_{y,\text{min}}}{k}, \beta_o\right) \right] + \Phi(\alpha_N, \beta_{N-1}) \\
- H(\alpha_o - \frac{2f_{y,\text{min}}}{k} - x(t)) \Phi(\alpha_N, x(t)) + \sum_{j=2}^{N-1} \left[ \Phi(\alpha_j - \beta_{j-1}) - \Phi(\alpha_j - \beta_j) \right] \] (4.15)

4.2.3 Application to a SDOF dynamical system

Consider a SDOF system, which is subjected to monochromatic excitation and has a nonlinear restoring force \( f(x, \dot{x}, t) \). The equation of motion for this system can be written as in Eq. (4.16).

\[ m \ddot{x}(t) + C \dot{x}(t) + K x(t) + f(x, \dot{x}, t) = F(t), \quad F(t) = F_o \cos(\omega t) \] (4.16)

where \( C \) represents the damping, \( K \) the stiffness, \( \omega \) the excitation frequency and \( m \) is the mass (assume \( m=1 \)). Following the EL procedure, the nonlinear term can be replaced with the sum shown in Eq. (4.17).

\[ f(x, \dot{x}) = C_{eq} \dot{x} + k_{eq} x + \delta(x, \dot{x}) \] (4.17)

Thus, by replacing the original nonlinear differential Eq. (4.16), the equivalent linear presented in Eq. (4.18) is derived.

\[ \ddot{x} + (C + C_{eq}) \dot{x} + (k + k_{eq}) x + \delta(x, \dot{x}) = F(t) \] (4.18)
Figure 4.5: A SDOF system with a non-linear (hysteretic) component driven by a trigonometric excitation

The error term $\delta(x,\dot{x})$ is minimized with respect to $C_{eq}$ and $k_{eq}$ in a temporal averaging setting as shown Eq. (4.19), where $T$ is one period of excitation.

$$C_{eq} = \frac{\int_0^T f(x,\dot{x})\dot{x}dt}{\int_0^T \dot{x}^2dt} \quad \text{and} \quad k_{eq} = \frac{\int_0^T f(x,\dot{x})xdt}{\int_0^T x^2dt} \quad (4.19)$$

The response of the system is assumed to be of the form shown in Eq. (4.20).

$$x(t) = A\cos(\theta(t)), \quad \theta(t) = \omega t - \phi \quad (4.20)$$

In this case Eq. (4.19) yields the results presented in Eq. (4.21).
\[ C_{eq} = -\frac{1}{\omega A \pi} \int_{0}^{2\pi} f(A, \theta) \sin(\theta(t)) d\theta \quad \text{and} \quad k_{eq} = \frac{1}{A \pi} \int_{0}^{2\pi} f(A, \theta) \cos(\theta(t)) d\theta \quad \text{(4.21)} \]

Based now on Eqs. (4.12), (4.14), (4.15) and (4.21) the expressions for the equivalent damping and stiffness terms shown in Eqs. (4.22) and (4.23) are obtained.

\[ C_{eq} = -\frac{1}{\omega A \pi} \left\{ \int_{0}^{\pi} \left[ kA \cos(\theta) + \Phi(\alpha_{N-1}, \beta_{N-1}) - H(\alpha_{N-1}, 2f_{y, min} - A \cos(\theta)) \right] \sin(\theta) d\theta + \right\} \]
\[ + \int_{0}^{2\pi} (kA \cos(\theta) + H,A \cos(\theta) - \beta_{N-1} - 2f_{y, min}) \Phi(A \cos(\theta), \beta_{N-1}) \sin(\theta) d\theta \]
\[ \quad \text{(4.22)} \]

\[ k_{eq} = \frac{1}{A \pi} \left\{ \int_{0}^{\pi} \left[ kA \cos(\theta) + \Phi(\alpha_{N-1}, \beta_{N-1}) - H(\alpha_{N-1}, 2f_{y, min} - A \cos(\theta)) \right] \cos(\theta) d\theta + \right\} \]
\[ + \int_{0}^{2\pi} (kA \cos(\theta) + H,A \cos(\theta) - \beta_{N-1} - 2f_{y, min}) \Phi(A \cos(\theta), \beta_{N-1}) \cos(\theta) d\theta \]
\[ \quad \text{(4.23)} \]

Due to the pseudo-harmonic behavior of the response the dominant extreme values of the input can be set equal to the amplitude of the response. Hence, by also assuming \( f_{y, min} = 0 \) (it is explained later why), Eqs. (4.22) and (4.23) result in Eqs. (4.24) and (4.25).

\[ C_{eq} = -\frac{1}{\omega A \pi} \left\{ \int_{0}^{\pi} \left[ kA \cos(\theta) + \Phi(A, -A) - \Phi(A, A \cos(\theta)) \right] \sin(\theta) d\theta + \right\} \]
\[ + \int_{0}^{2\pi} (kA \cos(\theta) + \Phi(A \cos(\theta), -A)) \sin(\theta) d\theta \]
\[ \quad \text{(4.24)} \]

\[ k_{eq} = \frac{1}{A \pi} \left\{ \int_{0}^{\pi} \left[ kA \cos(\theta) + \Phi(A, -A) - \Phi(A, A \cos(\theta)) \right] \cos(\theta) d\theta + \right\} \]
\[ + \int_{0}^{2\pi} (kA \cos(\theta) + \Phi(A \cos(\theta), -A)) \cos(\theta) d\theta \]
\[ \quad \text{(4.25)} \]
Thus, one obtains Eq. (4.26).

\[ C_{eq} = \frac{Ak^2}{3\pi\omega f_{y,\text{max}}} \quad \text{and} \quad k_{eq} = k - \frac{Ak^2}{4f_{y,\text{max}}} \] (4.26)

Hence, the steady-state amplitude of the response can be computed by means of classical linear vibrations theory and this yields Eq. (4.27).

\[ A_{ss} = \frac{F_0}{\sqrt{\left(-\omega^2 + (k + k_{eq})\right)^2 + (C + C_{eq})^2 \omega^2}} \] (4.27)

The response of the SDOF system can be computed also by direct integration of Eq. (4.16) by means of an initial value problem solver. For this purpose, a 4th order Runge-Kutta scheme was used. Relevant numerical results are presented next and they are compared to the analytical approximation for a range of values of the excitation frequency.
4.2.4 Results

Clearly, numerical results can be obtained for given values of the stiffness, the damping and the force coefficients in Eq. (4.16). However, when this model is applied to study the behavior of a real system, one should specify the properties of the Jenkin’s elements according to the shape of the hysteretic loop observed experimentally. Moreover, due to lack of experimental data, such selection of the model parameters is at the point arbitrary. For the present model, a particularly simple representation of the system is obtained when $f_{y,\min} = 0$. In this case, the equivalent damping and stiffness terms are given in Eq. (4.26). Further, substituting in Eq. (4.18) and using Eq. (4.27), one obtains the analytical solution. The numerical results are compared with the analytical solution for various values of $r$, defined in Eq. (4.28).

$$r = \frac{F_0}{(f_{y,\max} + f_{y,\min})/2} \quad (4.28)$$

Fig. 4.6 presents the steady-state response amplitude of the SDOF system for a range of excitation frequencies. It is obvious that the system’s response curve exhibits softening behavior, which is typical of most hysteretic models. The above results derived using the Preisach formalism are in very good agreement with the method used by Iwan in reference [25] and its accuracy must be compared to that of the latter method. What is novel though, in the present study is the numerical solution which is obtained using the expression for the non-linear restoring force of the system defined using the CPM and by
direct numerical integration of Eq. (4.16). The computed numerical results are in excellent agreement with the corresponding analytical ones for different values of \( r \).

Figure 4.6: The steady-state amplitude \( A \) for a range of frequencies (\( k = C = 0 \) and \( f_{y_{\text{min}}} = 0 \))

Figure 4.7: The steady-state amplitude \( A \) for a range of frequencies (\( k = 16, C = 0.08 \) and \( f_{y_{\text{min}}} = 0 \))
Figure 4.8: Representative response of the SDOF system and corresponding hysteretic loop both computed numerically for $\omega = 3$ rad/s and $r = 0.4$ ($k = C = f_{y,\text{min}} = 0$)

Similar results have been derived using other values of $f_{y,\text{min}}$. Again, the numerical and the analytical results are in excellent agreement. Furthermore, the response amplitude in this case exhibits softening behavior as well. However, this is not the case when the SDOF system has damping and stiffness coefficients different than zero (Fig 4.7). In this case, the corresponding curve of the steady-state amplitude versus the excitation frequency has a sharp peak at one particular frequency, whose value depends on the choice of the system parameters as well as the properties of the Jenkin's elements. Finally, the shape of the hysteretic loops changes according to the values of these parameters. In Fig 4.9, the effect of the choice of $f_{y,\text{min}}, f_{y,\text{max}}, k, C$ and $\omega$ is presented by giving three characteristic types of loops that are obtained numerically for this system, and exhibit in general a nonlinear kind of hysteresis as it was expected.
Figure 4.9: Representative hysteric loops ($\omega = 3$ rad/s and $r = 0.8$ in all cases):
i) $k = C = f_{y,\text{min}} = 0$, ii) $k = 4$, $C = 0.04$ and $f_{y,\text{min}} = 0$, iii) $k = 4$, $C = 0.04$, $f_{y,\text{min}} = 2$
4.3 Case 2: SMA system

The Classical Preisach model (CPM), in the form presented in Chapter 3 has been successfully implemented in the case of the steady-state response of the IM. It was found that the Preisach formalism can be efficiently implemented in order to describe the non-linear forces in both analytical and numerical calculations. However, it was pointed out that one possible disadvantage of this approach lies on the identification procedure. Subsequently, there are possible errors in the numerical computations (although in the case of the IM did not affect the agreement of the analytical and numerical results) and therefore many researchers in the field have tried to formulate new ways to identify the hysteresis loops using the experimental data. The approach proposed in references [31,32], which involves the fitting of a known function to the experimental data by minimizing the errors in a least square sense, has been already implemented in the case of SMA devices. The target of this section is to use this alternative method in order to compute the steady-state dynamic response of a SMA system subjected to monochromatic excitation.

4.3.1 The Shape Memory Alloys

Shape Memory Alloys (SMA) [1,60] constitute a class of metal compounds, such as NiTiNOL and CuZnAl, which possess the capability to sustain and recover relatively large strains (up to 10%) without undergoing plastic deformation. These unique material characteristics are due, to a great extent, to the capacity of the material to undergo internal crystalline transformations in the presence of external applied stress and/or
changes in temperature. Moreover, the increasing use of SMA in systems as sensors or actuators created the necessity to model efficiently their dynamical behavior under deterministic and random loads. In this regard, many constitutive models have been proposed in the literature. Among all, the Preisach model seems quite promising.

![Graph](image.png)

**Figure 4.10:** Example of a force–displacement plot of an SMA device.

### 4.3.2 The Preisach formalism in the case of a SMA system

In order to model more efficiently the hysteretic behavior of SMA systems, a modification of the CPM has been proposed [22,28,46]. The main feature of the new model reduces to the properties of the hysteretic operators used (Fig. 4.11), which in this case are defined as in Eq. (4.29).

\[
\hat{\gamma}_{a,\beta} = \begin{cases} 
+1 & \text{if } u > \alpha \text{ and increasing or } u > \beta \text{ and decreasing} \\
0 & \text{if } u < \beta \text{ and decreasing or } u < a \text{ and increasing}
\end{cases}
\]  

(4.29)
The input-output relationship is given by the same equation as in the CPM that is as in Eq. (4.30).

\[ f(t) = \int \int_{\alpha \geq \beta} \mu(\alpha, \beta) \gamma_{\alpha, \beta} u(t) d\alpha d\beta = \Gamma u(t) \]  \hspace{1cm} (4.30)

However, the Preisach plane and consequently the support of the Preisach function are expressed differently as shown in Fig. 4.12. Clearly, based on these modifications at any instant of time the domain defined by the limiting triangle \( T \) can be subdivided into two sub-domains: \( S^{(I)}(t) \) that includes all the operators whose output is equal to \( I \) and \( S^{(0)}(t) \), which similarly consists of the operators with zero output.

\[ \hat{r}_{\alpha, \beta} u \]

\[ 0 \]

\[ \beta \]

\[ \alpha \]

\[ u \]

**Figure 4.11:** A typical hysteresis operator used in the case of SMA systems

Now one can easily verify the result shown in Eq. (4.31).

\[ f(t) = \int \int_{\alpha \geq \beta} \mu(\alpha, \beta) \gamma_{\alpha, \beta} u(t) d\alpha d\beta \Rightarrow f(t) = \int \int_{S^{(I)}(t)} \nu(\alpha, \beta) d\alpha d\beta \]  \hspace{1cm} (4.31)
Therefore, knowing the evolution of the sub-domain \( S^{(n)}(t) \) is sufficient to determine the restoring force due to the SMA device.

![Diagram](image)

**Figure 4.12:** The Preisach plane and a typical interface \( L(t) \) for an SMA system

### 4.3.3 Identification procedure

In the case of SMA systems the authors in references [31,32] have implemented a new identification procedure in order to determine the Preisach function \( \mu(\alpha, \beta) \). The approach is based on a *least-squares fitting* of the parameters of a known statistical distribution to a major hysteresis loop obtained by the experimental data. The distribution chosen is a *bivariate Gaussian distribution* of the form:

\[
\xi(\alpha, \beta) = \frac{1}{2\pi\sigma_\alpha\sigma_\beta\sqrt{1-\tau^2}} \times \exp\left[ -\frac{1}{2(1-\tau^2)} \left( \frac{\alpha - \mu_\alpha}{\sigma_\alpha} \right)^2 - 2\tau \frac{\alpha - \mu_\alpha}{\sigma_\alpha} \times \frac{\beta - \mu_\beta}{\sigma_\beta} + \left( \frac{\beta - \mu_\beta}{\sigma_\beta} \right)^2 \right] \quad (4.32)
\]
where $\mu_\alpha, \mu_\beta, \sigma_\alpha, \sigma_\beta$ and the correlation coefficient $\tau$ are the five parameters that must be determined in the fitting process. Furthermore, it is assumed that the non-linear restoring force of the SMA device, consists of two parts as it is often considered in hysteretic studies: a no-memory part $f_p(t)$, which stands for the "backbone" of the hysteretic loop and a purely hysteretic part $f_H(t)$, which represents the rate-independent but memory dependent nonlinear force. Thus, the total restoring force that an SMA device could contribute to the system where it is incorporated is given in Eq. (4.33).

$$f_{SMA}(t) = f_p(t) + f_H(t)$$  \hspace{1cm} (4.33)

It is also assumed that $f_p(t)$ is a polynomial of the form shown in Eq. (4.34).

$$f_p(t) = \sum_{i=1}^{N} c_i x^i$$  \hspace{1cm} (4.34)

The hysteretic component $f_H(t)$, can be computed using the Preisach formalism as presented in Eq. (4.35).

$$f_H(t) = \iint_{s^{\partial}(t)} \xi(\alpha, \beta) d\alpha d\beta$$  \hspace{1cm} (4.35)

Thus, the total force assumes the form of Eq. (4.36).

$$f_{SMA}(t) = \sum_{i=1}^{N} c_i x^i + c_{N+1} \iint_{s^{\partial}(t)} \xi(\alpha, \beta) d\alpha d\beta$$  \hspace{1cm} (4.36)
Therefore, there are in total $N+6$ parameters to be determined through a minimization of the form given in Eq. (4.37), where $\lambda$ is a small positive number.

$$\sum_{n=1}^{m} |f_{\text{exp}}(n) - f_{\text{SMA}}(n)|^2 < \lambda \tag{4.37}$$

The identification becomes easier if it assumed that $\tau = 0$ and if the case of symmetrical loops centered at the origin is considered, this results in the additional assumptions shown in Eq. (4.38).

$$\sigma_\alpha^2 = \sigma_\beta^2 \quad \text{and} \quad \mu_\alpha = \mu_\beta \tag{4.38}$$

### 4.3.4 Numerical implementation

Assuming $N=3$ in equation (4.36) and using the Preisach plane of Fig. 4.12, the non-linear restoring force due to the SMA device can be written as in Eq. (4.39) for the ascending status and as in Eq. (4.40) for the descending one.

$$f_{\text{SMA}}(t) = c_1 x(t) + c_2 x^2(t) + c_3 x^3(t) + c_4 [F(x(t), \beta_{n-1}) + \sum_{j=1}^{n-1} F(\alpha_j, \beta_{j-1}) - F(\alpha_j, \beta_j)] \tag{4.39}$$

$$f_{\text{SMA}}(t) = c_1 x(t) + c_2 x^2(t) + c_3 x^3(t) + c_4 [F(\alpha_n, \beta_{n-1}) - F(\alpha_n, x(t)) + \sum_{j=2}^{n-1} F(\alpha_j, \beta_{j-1}) - F(\alpha_j, \beta_j)] \tag{4.40}$$
\[ F(\alpha_j, \beta_j) = \int_{\beta_j}^{\alpha_j} \int_{\beta_j}^{\alpha_j} \xi(\alpha, \beta) \, d\alpha d\beta \]  

(4.41)

Given that the Preisach function is given in Eq. (4.30) and because of the assumptions shown in Eq. (4.38), Eq. (4.41) yields Eq. (4.42), where \(\text{erf}(\cdot)\) denotes the error function.

\[ F(\alpha_j, \beta_j) = \frac{1}{8} \left[ \text{erf} \left( \frac{\alpha_j - \mu}{\sqrt{2}\sigma} \right) - \text{erf} \left( \frac{\beta_j - \mu}{\sqrt{2}\sigma} \right) \right] \]  

(4.42)

Thus, Eqs. (4.40) and (4.41) can be used in order to compute numerically the response of this hysteretic system.

4.3.5 Application to a SDOF dynamical system

Consider a SDOF system, which is subjected to monochromatic excitation, and has a nonlinear restoring force due to a SMA device \(f_{\text{SMA}}(x, \dot{x}, t)\).

![Figure 4.13: A SDOF system with a SMA device driven by a monochromatic excitation](image)
The equation of motion for this system can be written as shown in Eq. (4.43), where $C$ represents the damping, $K$ the stiffness, $\omega$ the excitation frequency and $m$ is the mass (assume $m=1$). In the same equation, $\alpha$ is an adjusting parameter that can take the values 0 and 1 and is used in order to include or remove the stiffness term from the equation of motion.

$$m \ddot{x}(t) + C \dot{x}(t) + \alpha K x(t) + f_{SM4}(x, \dot{x}, t) = F(t), \quad F(t) = F_0 \cos(\omega t) \quad (4.43)$$

The undetermined parameters can be derived by implementing the identification procedure described earlier. For the system of interest the following values are obtained based on a nonlinear least squares fitting involving 305 data points:

$$c_1 = 51.384, \quad c_2 = -24.49, \quad c_3 = 5.63, \quad c_4 = -137.16, \quad \mu = 2.56 \quad \text{and} \quad \sigma = 1.51.$$

Thus, Eqs. (4.39), (4.40) and (4.41) can next be used in order to compute numerically the response of the SDOF system. Further, the same Runge-Kutta scheme used in the case of the IM with time step $dt = 0.01$ sec is applied for a range of excitation frequencies.

Furthermore, an analytical solution for the steady-state amplitude of the response can be obtained by using the equivalent linearization procedure as in the case of the Iwan model. In particular, using equation (4.21) it is possible to derive Eqs. (4.44) and (4.45) which provide the equivalent damping and stiffness terms.
\[ C_n = -\frac{c_t}{\omega A\pi} \left( \int_0^{\pi/2} (F(A,0) - F(A,A\cos(\theta))) \sin(\theta) d\theta - \int_{\pi/2}^{\pi} (F(-A\cos(\theta),0)) \sin(\theta) d\theta \right) \]
\[ - \int_\pi^{3\pi/2} (F(A,0) - F(A,-A\cos(\theta))) \sin(\theta) d\theta + \int_{3\pi/2}^{2\pi} F(A\cos(\theta),0) \sin(\theta) d\theta \] (4.44)

\[ K_n = c_i + \frac{3c_t}{4} A^2 + \frac{c_t}{A\pi} \left( \int_0^{\pi/2} (F(A,0) - F(A,A\cos(\theta))) \cos(\theta) d\theta - \int_{\pi/2}^{\pi} (F(-A\cos(\theta),0)) \cos(\theta) d\theta \right) \]
\[ - \int_\pi^{3\pi/2} (F(A,0) - F(A,-A\cos(\theta))) \cos(\theta) d\theta + \int_{3\pi/2}^{2\pi} F(A\cos(\theta),0) \cos(\theta) d\theta \] (4.45)

Due to the presence of the error function the evaluation of the equivalent terms in this case does not result into a compact expression as in the first case examined in section 4.2. In particular, Eq. (4.45) can only be evaluated numerically in an iterative scheme where there is an initial guess for the value of the amplitude. On the contrary, a closed form solution for the damping term can be derived for any value of the excitation frequency. Therefore, by using again Eq. (4.27), it is possible to derive the analytical approximation of the amplitude of the response. The results, for both values of the parameter \( \alpha \) and for a wide range of excitation frequencies are shown in Fig. 4.14 and 4.15. Furthermore, Fig. 4.16 presents typical responses and hysteresis loops for both values of the parameter \( \alpha \). In the case that \( \alpha = 1 \), that is when there is a spring on the oscillator, the response and the corresponding force-response loop for the SDOF system and for a corresponding oscillator with no SMA device added to it are shown in this figure in order to demonstrate the effect of the device on the system’s response.
Figure 4.14: The steady-state amplitude $A$ for a range of frequencies and $\alpha = 0$

($k = 16$, $C = 0.08$ and $F_0 = 1$)

Figure 4.15: The steady-state amplitude $A$ for a range of frequencies and $\alpha = 1$

($k = 16$, $C = 0.08$ and $F_0 = 1$)
Figure 4.15: Representative response of the SDOF system and corresponding hysteretic loop both computed numerically for $\omega = 9$ rad/s and $\alpha = 0$ (Case 1), $\alpha = 1$ (Case 2), ($k = 16$, $C = 0.4$ and $F_0 = 1$). In Case 2, the response and the loop are compared to the ones (given on the right) obtained when no SMA device exists on the oscillator.
Chapter 5

Random response of systems with hysteresis using the Preisach formalism

5.1 Introduction

In Chapter 4, the response of two systems with hysteresis when subjected to deterministic loading was presented and the relevant results showed excellent agreement between the analytical and numerical calculations. The next step of the analysis of systems endowed with the property of hysteresis is to study their response under random excitation [37,44, 70,71,74,75]. To this aim, the method of Stochastic Averaging (SA) presented in Chapter 2 will be implemented in the case that the non-linear term is modeled by the IM (case 1 of Chapter 4) in order to analytically derive the stationary probability density function of the response envelope. Finally, the analytical results will be compared with a pertinent Monte Carlo study.

5.2 Response of a SDOF system subjected to Gaussian excitation

Consider again the SDOF system of Fig. 4.5 where the trigonometric excitation $F(t)$ has been replaced for the purposes of the current analysis by a Gaussian White Noise Process (GWNP) with spectral density $S$ constant in the whole frequency range $(-\infty, \infty)$. The governing equation of motion for this system can be written as in Eq. (5.1), where $\varepsilon$ is a parameter that quantifies the effect of the nonlinear term $f(x, \dot{x}, t)$. 
\[ m \ddot{x}(t) + C \dot{x}(t) + K x(t) + \varepsilon f(x, \dot{x}, t) = w(t) \]  

(5.1)

In general, \( \varepsilon \) can take any value, but for the needs of the SA procedure this parameter must be small compared to the viscous and stiffness terms. Further, it is obvious that if \( \varepsilon = 0 \), the non-linear Eq. (5.1) becomes linear.

The assumptions specified by Eqs. (2.34) and (2.35) are necessary for the application of the SA method. Based on the smallness of the damping and the spectral density of the excitation, the response of the oscillator during one cycle of excitation will be pseudo-harmonic as shown in Eq. (5.2), where \( a(t) \) and \( \phi(t) \) are slowly varying functions of time.

\[ x(t) = a(t) \cos(\omega_n t + \phi(t)) \]  

(5.2)

\[ \dot{x}(t) = -a(t) \omega_n \sin(\omega_n t + \phi(t)) \]  

(5.3)

In the equations above, \( \omega_n \) stands for the natural frequency of the non-linear system (5.1), which in general for \( \varepsilon \) different than zero differs from the angular frequency defined in Eq. (5.4)

\[ \omega_o = \sqrt{\frac{K}{m}} \]  

(5.4)

By solving Eqs. (5.2) and (5.3) with respect to the unknown functions \( a(t) \) and \( \phi(t) \), it is straightforward to show the following equations.
\[ a(t) = x^2 + \frac{\dot{x}^2}{\omega_n^2} \quad (5.5) \]

\[ \phi(t) = \tan^{-1}\left( \frac{\dot{x}}{\omega_n x} \right) - \omega_n t \quad (5.6) \]

The amplitude of the response, as presented in Eq. (5.5) can be regarded also as an envelope in the sense that it forms a boundary for the values that the response may take.

Furthermore, as in the deterministic case, the non-linear term \( f(x, \dot{x}, t) \), can be replaced by the expression shown in Eq. (5.7), where \( C_{eq} \) and \( k_{eq} \) are parameters to be determined.

\[ f(x, \dot{x}) = C_{eq} \dot{x} + k_{eq} x \quad (5.7) \]

It has been proved in Chapter 4 that in the case that the error resulting from this substitution is minimized in a mean square sense, the equivalent stiffness and damping terms are given in Eq. (4.25). This corresponds to the case that the nonlinearity of the system is of a hysteretic type and the IM is used to model its effects on the response of the oscillator. Now, by introducing the parameter \( H \) defined in Eq. (5.8), the terms resulting from the EL substitution take the form of Eq. (5.9)

\[ H = \frac{k^2}{f_{y,\text{max}}} \quad (5.8) \]

\[ C_{eq} = \frac{a H}{3\pi \omega_n} \quad \text{and} \quad k_{eq} = k - \frac{a H}{4} \quad (5.9) \]
Based on Eqs. (5.7) and (5.9), it is now possible to define the equivalent linear system of
the original non-linear given in Eq. (5.1) as shown in Eq. (5.10), where the natural
frequency is defined in Eq. (5.11).

\[ m \ddot{x}(t) + (C + \varepsilon C_{eq}) \dot{x}(t) + (K + \varepsilon k_{eq}) x(t) = w(t) \]  \hspace{1cm} (5.10)

\[ \omega_n^2 = \frac{K + \varepsilon k_e(a)}{m} \]  \hspace{1cm} (5.11)

Since it is assumed that the equivalent linear system is a good approximation of the
corresponding non-linear one, (5.2) could also be considered as an approximation of the
solution of the linear oscillator. Thus, it is appropriate now to use standard results from
linear random vibration theory in the case that the response under Gaussian excitation is
considered. There are readily available results in this area that will be used whenever is
needed. Reference [58] shows that for a weakly damped linear oscillator, which is excited
by a stationary wide-band random process, the amplitude of its response is uncoupled
from the phase. By applying this result to the system described in Eq. (5.10), the 1st order
differential equation that governs the amplitude of the response is presented in Eq. (5.12).

\[ \ddot{a}(t) = -\frac{C + \varepsilon C_e(a)}{2m} a + \frac{\pi S}{2am^2\omega_n^2} + \frac{(\pi S)^{1/2}}{m\omega_n} \eta(t) \]  \hspace{1cm} (5.12)

In Eq. (5.12) \( \eta(t) \) is a delta correlated process with zero mean and its autocorrelation
function is the Dirac delta function \( \delta(t) \). In reference [58] it is proved that for this choice
of \( \eta(t) \) the response envelope \( a(t) \) can be viewed as a Markovian process. Thus, its transition pdf is governed by the FPE, which is associated with Eq. (5.12). Moreover, if only the linear part of the non-linear system is considered, then from standard linear vibration theory it is known that the variance of the stationary response is given by Eq. (5.13).

\[
\sigma^2 = \frac{\pi S}{2 \zeta \omega_0^3 m^2} \tag{5.13}
\]

\[
2 \zeta \omega_0 = \frac{C}{m} \tag{5.14}
\]

Hence, in the case that the power spectral density of the Gaussian excitation is the one presented in Eq. (5.15), then \( \sigma^2 = 1 \).

\[
S = \frac{2 \zeta \omega_0^3}{\pi} \tag{5.15}
\]

If it also assumed that \( m = 1 \), then Eq. (5.12) can be written in the form of Eq. (5.16).

\[
\dot{a}(t) = -\zeta \omega_0 (a - \frac{1}{a} \left( \frac{\omega_o}{\omega_n} \right)^2) - \varepsilon \frac{C(\omega)}{2} + \left( 2 \zeta \omega_o \right)^{1/2} \frac{\omega_o}{\omega_n} \eta(t) \tag{5.16}
\]

The FPE that corresponds to Eq. (5.16), according to the literature, is given next in Eq. (5.17).
\[
\frac{\partial p(a,t)}{\partial t} = \frac{\partial}{\partial a} \left\{ p(a,t) \left[ \zeta \omega_o(a - \frac{1}{a} \omega_o) \right] + \varepsilon \frac{\omega_e(a)}{2} \right\} + \\
+ \frac{\partial}{\partial a} \left[ \zeta \omega_o \left( \frac{\omega_o}{\omega_n} \right)^2 \frac{\partial p(a,t)}{\partial a} + \frac{1}{2} \zeta \omega_o p(a,t) \frac{\partial}{\partial a} \left( \frac{\omega_o}{\omega_n} \right)^2 \right]
\]
(5.17)

In Eq (5.17) \( p(a,t) \) is the pdf of the response amplitude \( a(t) \). In the case that \( \varepsilon = 0 \), equation (5.17) is simplified in the form of shown in Eq. (5.18).

\[
\frac{\partial p(a,t)}{\partial t} = \frac{\partial}{\partial a} \left[ \zeta \omega_o(a - \frac{1}{a} p(a,t)) \right] + \zeta \omega_o \frac{\partial^2 p(a,t)}{\partial a^2}
\]
(5.18)

Solving Eq. (5.18) for \( \frac{\partial p(a,t)}{\partial t} = 0 \), the stationary pdf of the response amplitude is found to be a Rayleigh distribution of the form given in Eq. (5.19).

\[
p_s(a,t) = \lim_{t \rightarrow \infty} p(a,t) = a \exp(-a^2/2)
\]
(5.19)

This is a key result of linear random vibration theory and it will serve as a limit case for the general system with \( \varepsilon \) different than zero.

When \( \varepsilon \neq 0 \), Eq. (5.11) can be written in the form shown in Eq. (5.20).

\[
\omega_o^2 = \omega_n^2 + \varepsilon \omega_e^2(a)
\]
(5.20)
where $\omega_0^2$ has been defined in Eq. (5.4) and $\omega^2_0(a)$ is given in Eq. (5.21).

$$\omega^2_0(a) = \frac{k_0(a)}{m}$$  \hspace{1cm} (5.21)

From Eqs. (5.20) and (5.21) it is straightforward to show the result of Eq. (5.22)

$$\frac{\omega^2_0}{\omega^2_0} = 1 - \varepsilon \left( \frac{\omega^2_0(a)}{\omega^2_0} \right)$$  \hspace{1cm} (5.22)

Eq. (5.22) is correct up to $O(\varepsilon)$. Thus, Eq. (5.17) can be written in the form shown in Eq. (5.23).

$$\frac{\partial p(a,t)}{\partial t} = \frac{\partial}{\partial a} \left[ p(a,t) \left( \zeta \omega_0 (a - \frac{1}{a}) \right) \right] + \zeta \omega_0 \frac{\partial^2 p(a,t)}{\partial a^2}$$

$$+ \varepsilon \left[ \frac{\partial}{\partial a} \left[ \frac{\zeta \omega^2_0(a)}{\omega_0} \left( \frac{p(a,t)}{a} - \frac{\partial p(a,t)}{\partial a} \right) \right] + \frac{p(a,t) C_\lambda(a) a}{2} - \frac{1}{2} \zeta \frac{p(a,t)}{\omega_0} \frac{d}{da} \left[ \omega^2_0(a) \right] \right]$$  \hspace{1cm} (5.23)

Following reference [58], equation (5.23) for $\frac{\partial p(a,t)}{\partial t} = 0$, yields the result shown in Eq. (5.24).

$$p_\lambda(a,t) = \lim_{t \to \infty} p(a,t) = \frac{G}{\sqrt{\Delta}} \exp \left( 2 \int_\Delta \frac{\Lambda(a)}{\Delta} da \right)$$  \hspace{1cm} (5.24)
\[-\Lambda(a) = -\zeta \omega_o \left( a + \frac{1}{a} \left( \frac{\omega_o}{\omega_n} \right)^2 \right) - \varepsilon \frac{C_e(a)}{2} a \] 

(5.25)

\[\Delta = 2\zeta \omega_o \left( \frac{\omega_o}{\omega_n} \right)^2\] 

(5.26)

\(G\) is a constant that can be determined by the requirement of Eq. (5.27).

\[\int_0^\infty p_x(a)da = 1\] 

(5.27)

Therefore, the stationary pdf of the response envelope of the non-linear oscillator when subjected to white noise excitation has been determined analytically using the SA method.

5.3 Results

Based on Eq. (5.24), it is possible to compute the stationary pdf of the response amplitude for various values of \(\varepsilon\). Pertinent results are shown in Fig. 5.4 and are compared with data derived from a Monte Carlo study that involves 1000 response samples. Each individual response was computed numerically using a 4th order Runge-Kutta integration scheme with a time step \(dt = 0.01\) and using the damping ratio value \(\zeta = 0.01\), as well as the following values for the rest of the system parameters:
\[ K = 100, \; k = 120, \; f_{y, \text{max}} = 80 \]

In figures 5.1, 5.2 and 5.3 the response of the hysteretic system for various values of \( \varepsilon \), as computed numerically, is shown. It is obvious that in the case that \( \varepsilon = 0 \), the system behaves linearly and for increasing values of \( \varepsilon \), the loop formed by the total restoring force and the response of the oscillator becomes more non-linear.

In Fig. 5.4, it should be noted that as \( \varepsilon \) decreases the stationary pdf of the response envelope approaches the Rayleigh distribution as it was expected. In all the considered cases, the reliability of the analytical results is assessed by the Monte Carlo simulation data.
Figure 5.1: White Noise excitation, response of the non-linear system, hysteresis loop and phase diagram

for $\varepsilon = 0$
Figure 5.2: White Noise excitation, response of the non-linear system, hysteresis loop and phase diagram for $\varepsilon = 0.5$
Figure 5.3: White Noise excitation, response of the non-linear system, hysteresis loop and phase diagram for $\varepsilon = 1$
Figure 5.4: Stationary probability density function of the response envelope computed using equation (5.24) and comparison with the results of a Monte Carlo study for three values of $\varepsilon$: 0, 0.5 and 1
Chapter 6

Concluding remarks

Structural components used in various engineering applications often exhibit hysteretic behavior. Therefore, modeling the contribution of hysteretic type nonlinearities in the response of structural systems under specified loads is a useful step towards a more efficient design strategy. In this context, the main goal of the thesis is to present a modeling procedure that can be efficiently implemented in the case of mechanical systems that include hysteretic components and can produce results both analytically and numerically.

For this purpose, the thesis has been organized in a manner that the essential information about how this modeling can be accomplished are presented in a coherent way to connect the general phenomenon of hysteresis with the theory of mechanical vibrations. Thus, in Chapter 1, the phenomenon has been defined and its main characteristics have been presented. Moreover, it has been clarified that there are two main types of hysteretic nonlinearities, whose difference lies on their memory formation mechanism. Therefore, there is hysteretic behavior with local and hysteretic behavior with nonlocal memory. The case of hysteretic nonlinearities with nonlocal memory is the kind on which the proposed modeling procedure is focusing, since this case presents a more complex behavior which most of the existing models in the literature fail to capture. On the contrary, the Classical Preisach Model (CPM) is an alternative mathematical model that presents unique
features due to its basic principles and manages to cope with nonlocal memories in a very efficient way.

Chapter 2, which has intentionally been set before the presentation of the CPM serves as the framework in which hysteresis has been viewed. Moreover, it is noted that the present analysis aims to model the hysteretic dynamic behavior of mechanical systems, which are subjected to various loads, both deterministic and random. Therefore, the essential mathematical information that should be added in the thesis comprises the theory of vibrations. In the deterministic case, results from both linear and nonlinear systems have been given in the context of what is presented in Chapter 4. In particular, the method of Equivalent Linearization (EL) of nonlinear systems subjected to deterministic loads has been explicitly presented in view again of the results, which have been derived in Chapter 4. Furthermore, in the stochastic case the theory of random variables and stochastic processes has been considered as the necessary step to connect the reader who has no connection with the subject with the analysis of Chapter 5 where results based both on linear and non-linear random vibration theory are used. More specifically, Chapter 2 focuses in a standard method for solving the stochastic differential equations that arise in this class of problems: the Stochastic Averaging (SA) method. This method is the basic mathematical scheme used in Chapter 5.

In Chapter 3, the CPM has been presented in a detailed way to demonstrate two main features that are responsible for its wide acceptance by many fields of science that deal with hysteresis: its memory formation mechanism and the identification procedure. These properties are used to derive the kernel of the model, which is the Preisach function.
addition, necessary mathematical conditions that must be satisfied in order to use the model as well as ways to implemented it numerically have been given since this model and its successful analytical and numerical implementation is the "backbone" of the thesis.

*Chapter 4* has focused on the first application of the CPM in mechanical systems. To this aim, two cases have been examined. The first case comprises a SDOF oscillator that has a non-linear component whose behavior has been modeled using the procedure proposed by *Iwan*. For this model, the Preisach function has been explicitly derived as an example of how this can be accomplished in the case of a hysteretic study, and the CPM is being used along with the EL method to derive analytically the *steady-state amplitude* of the system's response when the excitation is a trigonometric function. The results for a wide range of excitation frequencies have been compared with the corresponding results from a numerical integration scheme that is based on a 4th order *Runge-Kutta* solver. The numerical and analytical results have been found in excellent agreement. Furthermore, the effect of some of the parameters of the *Iwan model* and the oscillator on the shape of the hysteretic loops has been examined. It has been shown that the *Iwan model* and the CPM offer considerable flexibility, through a suitable choice of these parameters to control the shape of the hysteretic loops. The second case presented in *Chapter 4*, includes one more application of the CPM in real mechanical systems. More specifically, the case of a SMA system has been considered along with an oscillator similar to the one of the first case. The second case has presented some key differences in the modeling procedure with respect to the first. The goal for this second system has been again the steady state amplitude of the response. This has been numerically accomplished based on
an identification procedure, which consists of fitting a known statistical distribution to the existing experimental data. The distribution selected is a bivariate Gaussian distribution and the amplitude of the response is computed for a range of excitation frequencies. The excitation is again periodic as in the first case.

Finally, Chapter 5 has presented the results of the modeling procedure in the case of a SDOF oscillator that comprises a non-linear hysteretic term and is subjected to white noise Gaussian excitation with specified spectral density. The goal is to derive the stationary probability density function of the response envelope. This has been accomplished by using the SA method presented in Chapter 2. In this context, the response has been viewed as a Markov process and the associated Fokker Planck equation is solved analytically for the stationary case. The results have been compared with a pertinent Monte Carlo study that was based on the direct numerical integration of the non-linear stochastic differential equation. Sample records of the excitation have been generated repeatedly in order to obtain an ensemble of response functions, which were sufficient to provide information about its pdf. It has been found that the analytical and the Monte Carlo results are in good agreement. Further work on the theme of the thesis may focus on the application of the developed methodology for multi-degree-of-freedom (mdof) systems exhibiting hysteresis and are subjected to deterministic transient and stochastic non-stationary excitations.
List of references


