Wave propagation in randomly layered media an application to time reversal

by

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Abstract

Wave propagation in randomly layered media with an application to time-reversal

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Fernando González del Cueto

We describe the propagation of acoustic waves through randomly layered media over distances much larger than the typical wavelength of a pulse that is emitted from a point source. The layered medium is modeled by a smooth reference background modulated by fast random small-scale variations. Using asymptotic methods, we arrive to the O’Doherty-Anstey (ODA) formula which describes the coherent part of the pulse in a deterministic expression up to a small random time correction.

An application on time-reversal is presented, where a pulse is sent through the medium, recorded in a small window, time-reversed, and then sent back towards the source. The striking phenomenon of enhanced refocusing occurs, where the randomness in the medium actually improves the spatial refocusing around the initial source.
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Chapter 1

Introduction

Wave propagation is a classical topic and there exists a vast literature that studies both the theoretical and numerical aspects of the problem. However, most of it deals with known (deterministic) media. In many important applications, the properties of the medium are only known on the larger scale, while the small scale fluctuations are unknown. For example, in radar we know that the atmosphere is mostly homogeneous, but there are fast and strong small fluctuations due to, for instance, rain drops. In geophysics, the large-scale profile of the Earth’s subsurface is rather well known, but not at all in a pointwise manner.

The question then is, how to deal with such inhomogeneities that we know are present yet are impossible to specify precisely. Can we make quantitative or qualitative statements about how waves propagate in such media? Moreover, can we use our characterization to reliably solve inverse problems in such media? This is a hard problem, but recent progress has been made for weakly fluctuating isotropic media in [7, 12, 15, 16, 3]. Here we talk about the randomly layered medium, which is often relevant in geophysics where there is a rich wave propagation theory developed in [2, 9, 20, 18].

So far, the inverse problem has not been addressed properly. While our future goal is to study it, we present the relevant theory for the forward problem that explores
in a quantitative way the behaviour of waves in the randomly layered medium.

In the rest of this chapter we present relevant notation that is used throughout the thesis. In Chapter 2, we present the founding notions behind propagation of waves in deterministic homogeneous and inhomogeneous media.

Then, in Chapter 3 we derive the O’Doherty-Anstey (ODA) formula, which describes the propagation of an acoustic wave in a randomly layered medium. This theory is applied in Chapter 4, in an experiment known as time-reversal.

1.1 Notation

We present a brief description of the mathematical notation used in this work.

- Variables in normal roman typeface are scalar-valued quantities, e.g.
  \[ a \in \mathbb{R}. \]

- Variables in roman boldface are vector-valued quantities, e.g.
  \[ \mathbf{x} \in \mathbb{R}^3. \]

- Often we will separate a three-dimensional variable in a two-dimensional lateral variable \( \mathbf{x} \in \mathbb{R}^2 \) and a one-dimensional depth variable \( z \in \mathbb{R} \),
  \[ (\mathbf{x}, z) \in \mathbb{R}^3. \]

- \( |\mathbf{x}| \) denotes the Euclidean norm of a vector \( \mathbf{x} \).

- Given functions \( f(t) \) and \( g(t) \), we will denote the convolution of \( f \) with \( g \) with an asterisk:
  \[ [f \ast g](\tau) = \int_{-\infty}^{\infty} f(t)g(\tau - t) \, dt \quad (1.1) \]
We work extensively with some physical quantities. Here we list the symbols that will represent them:

\( c \): speed of sound in the medium.

\( \gamma \): slowness. It is the reciprocal of the speed of sound, \( \gamma = 1/c \).

\( \omega \): the frequency. It is \( 2\pi \) times the "usual" frequency.

\( \lambda \): wavelength.

\( k \): wavenumber. It obeys the following relation

\[
    k = \frac{\omega}{c} = \omega \gamma = \frac{2\pi}{\lambda}.
\]  \hspace{1cm} (1.2)

Often, for a certain material property, such as the speed of sound, we will refer to its reference (average) value with subindex 0. For instance, \( c_0 \) will denote the reference speed of sound of a certain medium.

### 1.1.1 Fourier transform

In the present text, the following convention for the Fourier transform will be used. Given a function \( f(t) \), \( \hat{f}(\omega) \) denotes the Fourier transform of \( f \) with respect to \( t \), and it is defined as

\[
    \hat{f}(\omega) = \mathcal{F}\{f\}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt.  \hspace{1cm} (1.3)
\]

and the inverse Fourier transform is

\[
    f(t) = \mathcal{F}^{-1}\{\hat{f}\}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega t} \, d\omega.  \hspace{1cm} (1.6)
\]
As we will be interested in high frequency regimes\textsuperscript{1}, we may use an $\varepsilon$-scaled Fourier transform:

\[
\hat{f}^\varepsilon(\omega) = \int_{-\infty}^{\infty} f(t) e^{i \omega t \varepsilon} \, dt,
\]

\[
= \hat{f}\left(\frac{\omega}{\varepsilon}\right),
\]

and its corresponding inverse transformation,

\[
f(t) = \frac{1}{2\pi\varepsilon} \int_{-\infty}^{\infty} \hat{f}^\varepsilon(\omega) e^{-i \omega t \varepsilon} \, d\omega.
\]

\textsuperscript{1}section 2.3.2, page 10.
Chapter 2

Wave propagation in a deterministic medium

Wave propagation in the Earth’s subsurface is often simulated by solving the elastic wave equation. The wavefield, in this case, is described by a three-component displacement vector that includes two types of waves, compressional (P-waves) and shear (S-waves).

Frequently geophysicists have resorted to the acoustic wave approximation to model P-wave propagation [5]. The wavefield in acoustic media is described by a scalar quantity, thereby making the problem significantly simpler, theoretically and less expensive computationally.

In this chapter, we shall review briefly the acoustic equations (§2.1) and discuss their solution in a homogenous medium (§2.2) and in a known (deterministic) heterogeneous medium (§2.3). This will be a basic foundation for the material covered in the next chapter.
2.1 The acoustic wave equation

When sound propagates through a medium, occupying some finite or infinite domain \( \Omega \subseteq \mathbb{R}^3 \), a material particle located at \( x \in \Omega \), moves with velocity \( u(t, x) \), at time \( t \). The mass density at \( x \) is \( \rho(t, x) \) and the motion is governed by the equations of conservation of mass and momentum.

The equation of conservation of momentum, with a forcing term \( F(t, x) \) is

\[
\rho(t, x)u_t(t, x) + \rho(t, x)(u(t, x) \cdot \nabla)u(t, x) + \nabla p(t, x) = F(t, x),
\]

(2.1)

where \( p \) denotes the acoustic pressure. The equation for the conservation of mass is

\[
\rho_t(t, x) + \nabla \cdot (\rho(t, x)u(t, x)) = 0.
\]

(2.2)

Also, we assume that the medium is not moving so \( \rho \) is independent of time. In many important applications, the displacements are small, so it is reasonable to linearize the equations. We then obtain the linearized equations of acoustics\(^1\):

\[
\rho(x)\frac{\partial u}{\partial t}(t, x) + \nabla p(t, x) = F(t, x), \quad t \geq 0,
\]

(2.3a)

\[
\frac{1}{K(x)}\frac{\partial p}{\partial t}(t, x) + \nabla \cdot u(t, x) = 0, \quad x \in \Omega,
\]

(2.3b)

where

\[
K(x) = (c(x))^2 \rho(x) > 0
\]

(2.4)

is the bulk modulus of the material and \( c(x) \) is the local speed of sound. The reciprocal of the speed of sound is called the slowness

\[
\gamma(x) = \frac{1}{c(x)} = \sqrt{\frac{\rho(x)}{K(x)}}.
\]

(2.5)

Both equations (2.3) can be combined into a second-order differential equation

\[
\frac{1}{K(x)} \frac{\partial^2 p}{\partial t^2} + \nabla \cdot \left( \frac{1}{\rho(x)} (F(t, x) - \nabla p(t, x)) \right) = 0.
\]

(2.6)

\(^1\)For a more detailed derivation, see [17, pages 101-102]
In many geophysical applications, the change in density $\rho$ is negligible compared to the change in bulk modulus $K$. Hence, in this work it will be assumed constant. Then, (2.6) results in the usual wave equation\(^2\)

$$\gamma^2(x) \frac{\partial^2 p}{\partial t^2}(t, x) - \Delta p(t, x) = -\nabla \cdot F(t, x), \quad t \geq 0, \quad x \in \Omega, \quad (2.7)$$

and Fourier-transforming (2.7) with respect to $t$, the *Helmholtz or reduced wave equation* is obtained:

$$\omega^2 \gamma^2(x) \hat{p}(\omega, x) + \Delta \hat{p}(\omega, x) = \nabla \cdot \hat{F}(\omega, x), \quad \omega \in \mathbb{R}, \quad x \in \Omega. \quad (2.8)$$

Of particular interest is the case of a point source

$$F(t, x) = f(t) \delta(x - x_s)e, \quad (2.9)$$

where $e \in \mathbb{R}^3$ is a unit directivity vector, $x_s \in \Omega$ is the source position and $f(t) \in C_0(\mathbb{R})$ is a pulse-shape function with compact support.

### 2.2 Wave propagation in a homogeneous medium

Let us consider first the simplest scenario: a three-dimensional homogeneous medium with constant speed $c_0$ and with no boundaries. The origin of the system of coordinates is the source point $x_s$.

The Helmholtz equation (2.8) simplifies to

$$(\Delta + k^2)\hat{p} = \nabla \cdot \hat{F}, \quad (2.10)$$

where

$$k = \frac{\omega}{c_0} = \gamma_0 \omega \quad (2.11)$$

is the wave number.

\(^2\)u may also satisfy the wave equation if the field is assumed to be irrotational [17, pages 102].
We can solve (2.10) using the outgoing free-space Green's function $\hat{G}(\omega, x; x')$ satisfying
\[(\Delta + k^2) \hat{G}(\omega, x; x') = -\delta(x - x'), \quad x, x' \in \mathbb{R}^3\] (2.12)
and the outgoing radiation condition\(^3\)
\[
\lim_{r \to \infty} r (G_r + i k^2 G) = 0, \quad r = |x - x'|.
\] (2.13)
This Green's function\(^4\) is
\[
\hat{G}(\omega, x; x') = \frac{e^{ik|x-x'|}}{4\pi|x-x'|},
\] (2.14)
and the solution of (2.10) follows from Green's identity as
\[
\hat{p}(\omega, x) = \int \hat{G}(\omega, x; x') \hat{f}(\omega) \nabla \cdot (\delta(x')e) \, dx',
\] (2.15)
where the divergence of the generalized function $\delta(x)e$ must be regarded in terms of distribution theory.

In time domain, the pressure is obtained by the convolution theorem:
\[
p(t, x) = \mathcal{F}^{-1}\{\hat{p}(\omega, x)\}
\] (2.16)
\[
= \mathcal{F}^{-1} \left\{ \int \hat{G}(\omega, x; x') \hat{f}(\omega) \nabla \cdot (\delta(x')e) \, dx' \right\}
\] (2.17)
\[
= \int \mathcal{F}^{-1} \left\{ \hat{G}(\omega, x; x') \hat{f}(\omega) \right\} \nabla \cdot (\delta(x')e) \, dx'
\] (2.18)
\[
= \int [f(\cdot) \ast G(\cdot, x; x')] (t) \nabla \cdot (\delta(x')e) \, dx'.
\] (2.19)

2.3 Wave propagation in heterogeneous media

In the case a heterogeneous medium, where the slowness $\gamma$ is a function of $x$, it is not possible to give an explicit formula such as (2.15). Numerical alternatives are

---

\(^3\)See appendix A for more details.

\(^4\)The derivation of this Green's function is included in appendix A, page 38.
then used, such as finite elements[13], integral equation approaches[10], ray-tracing techniques [20], etc.

We lay out the setup of the problem in §2.3.1. We are interested in a high frequency regime introduced in §2.3.2 and we use the asymptotic WKB method described in §2.3.3.

2.3.1 Problem setup

In a geophysical context, consider a simple seafloor model, where the homogeneous upper halfspace \((z < 0)\) corresponds to the ocean and the lower homogeneous halfspace is the surface of the Earth’s crust. A sketch is shown in picture 2.1. We will probe the heterogeneous ground with a point source located slightly above the surface.

![Diagram](image)

\(z < 0 \quad \text{homogeneous halfspace} \)

\(z > 0 \quad \text{heterogeneous halfspace} \)

point source

Figure 2.1: The space is divided in two halfspaces: an upper homogeneous halfspace and a lower heterogeneous halfspace.

The variations in the medium affect the speed at which the wave propagates. Therefore, we model the heterogeneity of the medium through the slowness \(\gamma = \rho/K\)
(recall $\rho$ is assumed constant):

$$
\gamma^2(z) = \begin{cases} 
\gamma_0^2 & \text{if } z \leq 0 \\
\gamma_1^2(z) & \text{if } z \geq 0
\end{cases}
$$

(2.20)

where $\gamma_0$ is constant.

Although the general case of a discontinuity at $z = 0$ can be handled with some extra work\cite{2}, we take for simplicity the matched case

$$
\gamma_0 = \gamma_1(0).
$$

(2.21)

### 2.3.2 The high frequency regime

Let us suppose that the propagation distance is $L > 0$ and a typical frequency is $\omega_0$. Then, scaling the variables

$$
\ddot{x} = \frac{x}{L}, \quad \dot{\omega} = \frac{\omega}{\omega_0},
$$

transform the homogeneous Helmholtz equation into

$$
\left( \Delta_x + \left( \frac{L}{\lambda_0} \right)^2 \frac{(2\pi c_0)^2}{c^2} \right) \hat{\omega} \hat{p} = 0,
$$

(2.22)

where $\lambda_0 = 2\pi c_0/\omega_0$ is the wavelength.

We take a propagation distance $L$ that is much larger than the typical wavelength and we define a small parameter $\varepsilon$

$$
\frac{1}{\varepsilon} = \frac{L}{\lambda_0} \gg 1.
$$

(2.23)

The scaled Helmholtz equation is

$$
\left( \Delta_x + \frac{\omega^2}{\varepsilon^2} \gamma^2 \right) \hat{p} = 0,
$$

(2.24)

and we think of $L$ as being of $O(1)$ and the typical wavelength $\lambda_0$ of $O(\varepsilon)$.

For this high frequency regime, we use the pulse shape function

$$
\varepsilon^{1/2} f\left( \frac{t}{\varepsilon} \right),
$$

(2.25)
and the amplitude is chosen as $\varepsilon^{1/2}$ to achieve an energy of $O(1)$.

The $\varepsilon$-scaled Fourier transform\(^5\) is then

$$
\mathcal{F} \left\{ \varepsilon^{1/2} f \left( \frac{t}{\varepsilon} \right) \right\} = \varepsilon^{1/2} \int_{\mathbb{R}} f \left( \frac{t}{\varepsilon} \right) e^{i \varepsilon t} \, dt
$$

\hspace{1cm}

$$
= \varepsilon^{3/2} \int_{\mathbb{R}} f(t') e^{i \varepsilon t'} \, dt' \quad t' = t / \varepsilon
$$

\hspace{1cm}

$$
= \varepsilon^{3/2} \hat{f}(\omega).
$$

### 2.3.3 The WKB approximation

Clearly, the solution of equation (2.24) depends on $\omega / \varepsilon$, so we seek a high-frequency approximation using the WKB method. Namely, we assume that the solution depends exponentially on the fast variation and is of the form

$$
\hat{p}^\varepsilon(x, z, \omega) = A(x, z) e^{i \varepsilon \tau(x, z)},
$$

where $\tau(x, z)$ is a phase function and $A(x, z)$ is the amplitude function.

In Appendix B (page 41) we derive the leading order WKB approximation for the Green’s function satisfying

$$
\left( \Delta + \frac{\omega^2}{\varepsilon^2} \gamma^2 \right) G = -\delta(x) \delta(z - z_0),
$$

and it is given by

$$
G \sim \frac{1}{4\pi} \sqrt{\frac{d\Omega}{da(\sigma)}} \frac{\gamma_0}{\gamma(\sigma)} e^{i \varepsilon \tau}.
$$

Here, $d\Omega$ is an element of solid angle of the initial directions of the ray about the ray path passing through $(x, z)$ and $da$ is the associated element of area on the phase front. The ray passing through $(x, z)$ is denoted by $\Gamma$ and is shown by a line in figure 2.2

On the other hand, $\hat{p}^\varepsilon$ must satisfy

$$
\left( \Delta + \frac{\omega^2}{\varepsilon^2} \gamma^2 \right) \hat{p}^\varepsilon = \varepsilon^2 \hat{f}(\omega) \delta(x) \delta'(z - z_0),
$$

\(^5\)defined in (1.7), page 4
so a leading order approximation for $p$ can be found as

$$
p^e = -\varepsilon^2 \hat{f}(\omega) \frac{\partial}{\partial z} G$$

$$= -\varepsilon^2 \hat{f}(\omega) \frac{\partial}{\partial z} \left\{ \sqrt{\frac{d\Omega}{da(\sigma)} \frac{\gamma_0}{\gamma(\sigma)}} e^{i\varepsilon \tau} \right\}$$

$$= -\frac{\varepsilon^2 \hat{f}(\omega)}{4\pi} \left\{ \frac{\partial}{\partial z} \sqrt{\frac{d\Omega}{da(\sigma)} \frac{\gamma_0}{\gamma(\sigma)}} e^{i\varepsilon \tau} + \frac{i}{\varepsilon \tau z} \sqrt{\frac{d\Omega}{da(\sigma)} \frac{\gamma_0}{\gamma(\sigma)}} e^{i\varepsilon \tau} \right\}$$

$$\approx -\frac{\varepsilon^2 \hat{f}(\omega)}{4\pi} \left\{ + \frac{i}{\varepsilon \tau} \sqrt{\frac{d\Omega}{da(\sigma)} \frac{\gamma_0}{\gamma(\sigma)}} e^{i\varepsilon \tau} \right\}$$

Now, performing an inverse transform, we can recover $p(x, z, t)$,

$$p(x, z, t) = \frac{1}{2\pi \varepsilon} \int -\frac{\varepsilon^2 \hat{f}(\omega)}{4\pi} \left\{ + \frac{i}{\varepsilon \tau} \sqrt{\frac{d\Omega}{da(\sigma)} \frac{\gamma_0}{\gamma(\sigma)}} e^{i\varepsilon \tau} \right\} \, d\omega$$

$$= \frac{\tau z}{4\pi} \sqrt{\frac{d\Omega}{da(\sigma)} \frac{\gamma_0}{\gamma(\sigma)}} f'(t - \tau)$$

and $f'$ appears as we recall that $\mathcal{F}[f'] = i\omega \mathcal{F}[f]$. Then, we do a change of variable $t = \tau + \varepsilon s$ and we get an expression for the pressure, in a window centered at $\tau$ and of size $\varepsilon$.

$$p(x, z, \tau + \varepsilon s) \sim \frac{\tau z}{4\pi} \sqrt{\frac{d\Omega}{da(\sigma)} \frac{\gamma_0}{\gamma(\sigma)}} f'(s).$$

In the next chapter we will be more interested in the strong small-scale fluctuations of the medium, rather than the smooth large-scale variations. For this reason, let us simplify the problem by taking the background slowness constant, $\gamma \equiv \gamma_0$. In that case, the approximation (2.39) is reduced to

$$p(x, z, \tau_1 + \varepsilon s) \sim \frac{\gamma_0}{4\pi r} \cos(\theta) f'(s),$$

with $r$ is the distance from the source to the observation point $(x, z)$. 
Figure 2.2: Rays associated with propagation from a point source.
Chapter 3

Wave propagation in random media

In a typical geophysical application, the velocity profile of the medium is similar to the one depicted in Figure 3.1. This motivates the description of the variations in the medium in two separate profiles: a small-scale fluctuation on top of a large-scale profile.

![Velocity profile](image)

Figure 3.1: A sample velocity profile of a geophysical medium.

We recall the scalings described in the previous chapter (subsection 2.3.2), where we postulated that the propagation distance is of order 1 and the typical wavelengths are of order \( \varepsilon \).
Now, we have a third scale, that describes the inhomogeneities (fast fluctuations) in the medium. Depending on this scale, we distinguish different regimes. We are interested in the case of microscale of $O(\varepsilon^r)$ (with $r \geq 1$) so that there is strong interaction of waves with the medium.

When the microscale is much smaller than $\varepsilon$, one may think that simple effective medium theory applies, but this is true only for short propagation distances, i.e. a few wavelengths [2]. Here, the propagation distance $L = O(1)$ so even in the case of wavelengths much larger than the microscale, the inhomogeneities affect significantly the signal.

In this chapter we focus on an intermediate scaling, where the length scale of the small-scale fluctuations is of order $\varepsilon^2$. In this particular scaling regime, we can actually describe the propagating pulse with good accuracy. This is, in fact, the regime considered in the O’Doherty-Anstey theory [18].

In §3.1 we discuss the modeling of the fluctuations as a random process. In §3.2 we identify two relevant regimes, for which the theory presented in §3.3 works.

### 3.1 Random media model

In usual geophysical applications, the medium is not know with minute detail. Even if the large-scale velocity profile can be estimated, information about the small-scale features is usually unknown and often impossible to obtain. Because of this uncertainty, it is plausible to model the small-scale fluctuations in the slowness as a random process:

$$[\text{slowness}] = [\text{background slowness}] \cdot (1 + [\text{fast fluctuation}])$$  \hspace{1cm} (3.1)

A random process $\nu$ models our uncertainty of these fast fluctuations. We assume that $\nu$ satisfies the following properties:

**Zero mean:** The expectation of $\nu$, $E[\nu]$, is zero. This means that $\nu$ models the deviation from the average profile $\gamma_1$. 
Stationary or statistically homogeneous: The correlation function

\[ C(x_0, x_1) = E[\nu(x_0)\nu(x_1)] \]

depends only on the difference \( x_1 - x_0 \). In other words, it is translation invariant, as the correlation only depends on the distance between the points and not the points themselves.

Ergodicity: Also called the “mixing” property, it fundamentally means that there are no long-range correlations. We assume that the correleation function \( C \) is compactly supported.

Boundedness: \( \nu \) is strictly bounded by some quantity that ensures that the fluctuating slowness remains always positive.

3.1.1 Layered media

In certain scales, the inhomogeneities of the Earth have a layer-like structure: the changes are more rapid with respect to depth than with respect to the transversal plane. We can take advantage of this property and include it in our model by making the medium independent of \( x \). From now on, the slowness will be a function of depth only:

\[ \gamma(x, z) = \gamma(z) \]

3.2 Scalings

We consider wave propagation in layered media in two regimes: the weakly heterogeneous regime and the strongly heterogeneous regime. We focus on these regimes because they can be described by the ODA theory, as we explain in the remainder of this chapter.
3.2.1 Weak medium fluctuations

The weakly heterogeneous regime is important in earth phenomena [2] where the size of the fluctuations is small. It is relevant in the context of wave propagation in the ocean where the variations occur on the scale of kilometers, the propagation length is typically of the order of hundreds of kilometers and the microscale is of order of meters. In terms of the small parameter $\varepsilon$, we have that $L = \mathcal{O}(1)$, $\lambda = \mathcal{O}(\varepsilon^2)$ and the microscale is $\mathcal{O}(\varepsilon^2)$, as well. The strength of the fluctuation is $\mathcal{O}(\varepsilon)$; this means that scattering occurs about every wavelength, but we take the strength of the fluctuations as $\mathcal{O}(\varepsilon)$ so over distance $L = \mathcal{O}(1)$ we observe an $\mathcal{O}(1)$ effect on the signal.

The slowness $\gamma$ is given by

$$
\gamma_{\text{weak}}^2(z) = \begin{cases} 
\gamma_0^2 & \text{if } z \text{ is in homogeneous region}, \\
\gamma_1^2(z) \left(1 + \varepsilon \nu \left(\frac{z}{\varepsilon^2}\right)\right) & \text{otherwise.}
\end{cases}
$$

The function $\gamma_1(z)$ is responsible for the smooth, large-scale description of the medium, while $\nu$ accounts for the rapid fluctuations. We remind the reader of the simplifying assumption that there are no interfaces, so $\gamma_1(0) = \gamma_0$.

3.2.2 Strong medium fluctuations

The strongly heterogeneous regime is relevant in the context of reflection seismology, in a region with strong variation in the earth parameters [2]. The incident pulse is typically tens of meters wide, which is large relative to the strong fine-scale medium fluctuations that are on the scale of meters, but small relative to the distance traveled by the pulse, which is on the scale of kilometers.

In this case, the slowness $\gamma$ is modeled as

$$
\gamma_{\text{strong}}^2(z) = \begin{cases} 
\gamma_0^2 & \text{if } z \text{ is in homogeneous region}, \\
\gamma_1^2(z) \left(1 + \nu \left(\frac{z}{\varepsilon^2}\right)\right) & \text{otherwise.}
\end{cases}
$$
Akin to the weakly heterogeneous media, the length scale of the fluctuations is $\mathcal{O}(\varepsilon^2)$, but the magnitude is large, $\mathcal{O}(1)$. However, the wavelength is now $\mathcal{O}(\varepsilon)$, so averaging occurs and as in the previous regime, the random medium effect on the signal is $\mathcal{O}(1)$ over propagation distances $L = \mathcal{O}(1)$.

Table 3.1 summarizes the scalings in both regimes.

<table>
<thead>
<tr>
<th></th>
<th>Weak</th>
<th>Strong</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transmission length</td>
<td>$\mathcal{O}(1)$</td>
<td>$\mathcal{O}(1)$</td>
</tr>
<tr>
<td>Wavelength</td>
<td>$\mathcal{O}(\varepsilon^2)$</td>
<td>$\mathcal{O}(\varepsilon)$</td>
</tr>
<tr>
<td>Magnitude of the fluctuations</td>
<td>$\mathcal{O}(\varepsilon)$</td>
<td>$\mathcal{O}(1)$</td>
</tr>
<tr>
<td>Size of microscale</td>
<td>$\mathcal{O}(\varepsilon^2)$</td>
<td>$\mathcal{O}(\varepsilon^2)$</td>
</tr>
<tr>
<td>Scaling of the shape pulse function</td>
<td>$\varepsilon^2 f\left(\frac{t}{\varepsilon^2}\right)$</td>
<td>$\varepsilon f\left(\frac{t}{\varepsilon}\right)$</td>
</tr>
</tbody>
</table>

Table 3.1: Comparison between the strong and weak regimes.

**Note.** Fortunately, the analysis of the wave propagation in both the strong and weak regimes is essentially the same. For this reason, we work only with the strong model from now on.

Also, although it is certainly possible to use an arbitrary smooth function $\gamma_1(z)$, it is often assumed to be constant $\gamma_1 = \gamma_0$ to simplify the analysis. This simplification allows us to focus on the random fluctuations and avoid the additional work that comes from choosing an arbitrary background function $\gamma_1(z)$.

### 3.3 Wave propagation in strongly layered media

Consider a heterogeneous slab $z \in [0, L]$, as shown in figure 3.2. The upper ($z \leq 0$) and the lower ($z \geq L$) halfspaces are homogeneous. A point source is located just above the surface $z = 0$. 
Figure 3.2: Sketch of the setup of a medium with a layered heterogeneous slab.

If the medium were homogeneous, the pulse emitted by the point source would spread in spherical fronts. In the randomly layered heterogeneous medium, the propagation and the shape of the pulse is more difficult to characterize. The first step in such characterization is plane wave decomposition.

3.3.1 Plane wave decomposition

Upon excitation, the point source generates a spherical wave. It is possible, and we will show next how, to decompose this single spherical wave into a family of unidirectional plane waves and deal with them individually. Then we can synthesize them and recover the resulting wave.

Let us briefly review the representation of a plane wave in a homogeneous medium.
3.3.1.1 Plane waves in a homogeneous medium

In the homogeneous case, an oblique plane wave with shape $f$ traveling with lateral slowness $\gamma_x$ and vertical slowness $\gamma_z$, has the representation

$$w(t, x, z; \gamma_x, \gamma_z) := f \left( t - \begin{pmatrix} \gamma_x \\ \gamma_z \end{pmatrix} \cdot \begin{pmatrix} x \\ z \end{pmatrix} \right)$$

$$= f(t - \gamma_x \cdot x - \gamma_z z)$$

or, in terms of the $\hat{w}(\omega)$,

$$\frac{1}{2\pi\varepsilon} \int \hat{w}(\omega)e^{-i\omega(t-\gamma_x x + i\gamma_z z)} d\omega.$$  

(3.6)

Naturally, because the medium is homogeneous, the magnitude of $(\gamma_x, \gamma_z)$ is the same as the reference slowness $\gamma_0$, that is

$$|\gamma_x|^2 + \gamma_z^2 = \gamma_0^2.$$  

(3.7)

Then, let us write $\gamma_z$ as a function of $\gamma_x$. As this quantity is used frequently, we introduce the shorthand for the norm of the lateral slowness,

$$|\gamma| := |\gamma_x|,$$  

(3.8)

and with this shorthand we can write the vertical slowness $\gamma_z$ as

$$\gamma_z(|\gamma|) = \sqrt{\gamma_0^2 - |\gamma|^2}.$$  

(3.9)

We neglect the evanescent waves so we deal only with real wave numbers. That is, $0 < |\gamma| < \gamma_0$.

3.3.1.2 Plane waves in the strongly layered medium

Now let us go back to the strongly layered medium. We exploit the independence of the medium on the transversal space $x$, and as we have done in the previous section we apply a Fourier transform not only in time $t$ but in space $x$ as well.
First, we denote by $u := u \cdot e_3$ the vertical component of the displacement vector. Then, we apply an $\varepsilon$-scaled Fourier transform to $u$ and $p$ with respect to $x$ and time $t$:

$$\hat{u}^\varepsilon(\omega, \gamma_x, z) = \iint e^{i\varepsilon(t-\gamma_x x)} u(t, x, z) \, dt \, dx$$

$$\hat{p}^\varepsilon(\omega, \gamma_x, z) = \iint e^{i\varepsilon(t-\gamma_x x)} p(t, x, z) \, dt \, dx.$$  \hspace{1cm} (3.10a)

In terms of the Fourier coefficient\(^1\) $\hat{p}^\varepsilon$, we can synthesize the pressure as

$$p^\varepsilon(t, x, z) = \frac{1}{(2\pi)^2 \varepsilon} \iint e^{-i\varepsilon(t-\gamma_x x)} \hat{p}^\varepsilon(\omega, \gamma_x, z) \frac{\omega^2}{\varepsilon^2} \, dt \, dx.$$  \hspace{1cm} (3.11)

With the Fourier transforms (3.10), the acoustic equations become

$$-i\omega \frac{\varepsilon}{\varepsilon^2} \hat{u}^\varepsilon + \frac{\partial \hat{p}^\varepsilon}{\partial z} = 0$$

$$-i\omega \frac{\gamma^2_x}{\varepsilon^2} \hat{p}^\varepsilon - \frac{\partial \hat{u}^\varepsilon}{\partial z} = 0,$$  \hspace{1cm} (3.12a)

with slowness $\gamma^2_x$ equal to

$$\gamma^2_x = \gamma^2_x(\gamma) \left( 1 + \frac{\gamma^2_0}{\gamma^2_x(\gamma)} \nu \left( \frac{z}{\varepsilon^2} \right) \right).$$  \hspace{1cm} (3.13)

This shows that, with a fixed mode $\gamma_x$, the problem has been transformed in a one-dimensional wave propagation problem with constant density $\rho$ and the mode-dependent slowness (3.13). In fact, it is only dependent of the length of the mode, $|\gamma| \equiv |\gamma_x|$.

### 3.3.2 Up- and down-going waves

When a wave propagates in a smoothly heterogeneous medium, there are no abrupt changes in the slowness that would cause reflections. The wave is modified in a “gentle” way. In contrast, in a strongly layered medium, things are rather different: about every $\varepsilon^2$ distance, the wave encounters a reflective layer where part of the wave

\(^1\)Note the presence of the factor $\omega^2$ due to the specific choice of Fourier variables.
is transmitted and the rest is reflected back. This is precisely what the up and down wave decomposition gives us.

The pressure and displacement fields solve (3.12) and are decomposed as

\[ p^\varepsilon = \frac{1}{2} \sqrt{I(|\gamma|)} \left( A^\varepsilon e^{i\frac{\varepsilon}{2} \gamma z} - B^\varepsilon e^{-i\frac{\varepsilon}{2} \gamma z} \right) \] (3.14a)

\[ u^\varepsilon = \frac{1}{2} \sqrt{I(|\gamma|)} \left( A^\varepsilon e^{i\frac{\varepsilon}{2} \gamma z} + B^\varepsilon e^{-i\frac{\varepsilon}{2} \gamma z} \right) \] (3.14b)

where

\[ I(|\gamma|) = \frac{\rho}{\gamma_x(|\gamma|)} \] (3.15)

is the acoustic impedance (independent of \( \omega \) and \( z \)), and \( A^\varepsilon \) and \( B^\varepsilon \) are unknown amplitude functions of \( \omega, |\gamma| \), and \( z \). \( A^\varepsilon \) corresponds to the wave that goes through or “downward”, whereas \( B^\varepsilon \) corresponds to the wave that is reflected back, or “upward”.

If we were in an homogeneous medium, \( A^\varepsilon \) and \( B^\varepsilon \) would be constant, independent of depth because waves move undisturbed. In our scattering random environment, these amplitudes depend on \( z \) and naturally, they are random quantities.

As the pulse is impinging on the random slab from \( z < 0 \), we can formulate the following boundary condition:

\[ A^\varepsilon(\omega, \gamma_x, z = 0) = \hat{\phi}(\varepsilon, \omega, \gamma_x), \] (3.16)

where \( \hat{\phi}(\varepsilon, \omega, \gamma_x) \) is obtained by matching the incident pressure field at \( z = 0 \) with the solution (2.40) in the homogeneous medium (for \( z < 0 \), and has the appropriate scaling with respect to \( \varepsilon \).

Additionally, as there is no energy coming in the slab from the lower homogeneous halfspace \( z > L \), we have another boundary condition:

\[ B^\varepsilon(\omega, \gamma_x, z = L) = 0. \] (3.17)

When expressions (3.14) are substituted in (3.12), we get the equations for the transformed waves:

\[
\frac{\partial}{\partial z} \begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix} = \begin{bmatrix} \frac{i \omega}{2 \varepsilon} \gamma_x(\gamma |\gamma|) \nu_{\gamma |\gamma|} \left( \frac{z}{\varepsilon^2} \right) & \left( e^{2i \frac{\varepsilon}{2} \gamma z} - e^{-2i \frac{\varepsilon}{2} \gamma z} \right) \\ e^{2i \frac{\varepsilon}{2} \gamma z} \nu_{\gamma |\gamma|} \left( \frac{z}{\varepsilon^2} \right) & -1 \end{bmatrix} \begin{bmatrix} A^\varepsilon \\ B^\varepsilon \end{bmatrix},
\] (3.18)
where
\[ \nu_{\gamma\lambda}(\frac{z}{\varepsilon^2}) := \left(\frac{\gamma_0}{\gamma_z(\gamma\lambda)}\right)^2 \nu(\frac{z}{\varepsilon^2}). \] (3.19)

Equations (3.18) along with the boundary conditions (3.16) and (3.17) form a two-point boundary ordinary differential equation. This problem can be solved using the method of propagators, that considers the initial value problem
\[ \frac{d}{dz} \mathbb{P}^\varepsilon_{(\omega,\gamma\lambda)}(z) = \frac{1}{\varepsilon} H(\omega,\gamma\lambda) \left(\frac{z}{\varepsilon}, \nu(\frac{z}{\varepsilon^2})\right) \mathbb{P}^\varepsilon_{(\omega,\gamma\lambda)}(0, z), \quad \text{for } z > 0 \] (3.20a)
\[ \mathbb{P}^\varepsilon_{(\omega,\gamma\lambda)}(z = 0) = I \] (3.20b)
Using the special form of \( H \),
\[ H(\omega,\gamma\lambda) \left(\frac{z}{\varepsilon}, \nu(\frac{z}{\varepsilon^2})\right) := \frac{i\omega}{2} \gamma_z(\gamma\lambda) \nu_{\gamma\lambda}(\frac{z}{\varepsilon^2}) \begin{bmatrix} 1 & -e^{-\frac{2i\varepsilon}{\varepsilon^2} \gamma_\alpha z} \\ e^{\frac{2i\varepsilon}{\varepsilon^2} \gamma_\alpha z} & -1 \end{bmatrix}, \] (3.21)
it can be shown that \( \mathbb{P}^\varepsilon \) is of the form
\[ \mathbb{P}^\varepsilon_{(\omega,\gamma\lambda)}(z) = \begin{pmatrix} \alpha^\varepsilon_{(\omega,\gamma\lambda)}(z) \\ \beta^\varepsilon_{(\omega,\gamma\lambda)}(z) \end{pmatrix} = \begin{pmatrix} \frac{\alpha^\varepsilon_{(\omega,\gamma\lambda)}(z)}{\alpha^\varepsilon_{(\omega,\gamma\lambda)}(0)} \\ \frac{\beta^\varepsilon_{(\omega,\gamma\lambda)}(z)}{\beta^\varepsilon_{(\omega,\gamma\lambda)}(0)} \end{pmatrix}. \] (3.22)

Then, the solution of (3.18), at \( z = L \), is given by the propagator as
\[ \begin{bmatrix} A^\varepsilon(z) \\ B^\varepsilon(z) \end{bmatrix} = \mathbb{P}^\varepsilon_{(\omega,\gamma\lambda)}(z) \begin{bmatrix} A^\varepsilon(0) \\ B^\varepsilon(0) \end{bmatrix}. \] (3.23)
So, at \( z = L \), using the boundary condition (3.17) we can derive both \( A^\varepsilon \) and \( B^\varepsilon \). Nevertheless, we are only interested in knowing \( A^\varepsilon \) as it quantifies the amplitude of the transmitted wave through the slab, at \( z = L \):
\[ A^\varepsilon(\omega, \gamma\sigma, L) = \phi(\varepsilon, \omega, \gamma\sigma) \mathcal{T}^\varepsilon_{(\omega,\gamma\lambda, L)} \] (3.24)
where \( \mathcal{T}^\varepsilon_{(\omega,\gamma\lambda, L)} \) is called the transmission coefficient and is given by
\[ T^\varepsilon_{(\omega,\gamma\lambda, L)} := \frac{1}{\alpha^\varepsilon_{(\omega,\gamma\lambda)}(L)}. \] (3.25)
Additionally, using again the special form of $H$ we derive the identity

$$|\alpha^\varepsilon_{(\omega, |\gamma|)}(z)|^2 - |\beta^\varepsilon_{(\omega, |\gamma|)}(z)|^2 = 1,$$

for any $z \in [0, L]$ \hfill (3.26)

and consequently, the total energy conservation formula

$$|A^\varepsilon(\omega, \gamma_\omega, L)|^2 + |B^\varepsilon(\omega, \gamma_\omega, 0)|^2 = |\hat{\phi}(\varepsilon, \omega, \gamma_\omega)|^2,$$

and this can be seen as $|A^\varepsilon|^2$ is the energy of the transmitted wave through the slab, $|B^\varepsilon|^2$ is the energy of the reflected waves at the surface and $|\hat{\phi}|^2$ is the energy of the incident wave.

### 3.3.3 Integral representation of the transmitted wave

Gathering all our results, we write the transmitted pressure as

$$p^\varepsilon(t, x, L)$$

$$= \frac{1}{(2\pi\varepsilon)^3} \iiint \frac{\sqrt{I(|\gamma|)}}{2} e^{-i\frac{\pi}{\varepsilon}(t-\gamma_\omega x - \gamma_t L)} \hat{\phi}(\omega, \gamma_\omega, L) \omega^2 \, d\gamma_\omega \, d\omega$$

$$= \frac{1}{(2\pi\varepsilon)^3} \iiint \frac{\sqrt{I(|\gamma|)}}{2} e^{-i\frac{\pi}{\varepsilon}(t-\gamma_\omega x - \gamma_t L)} \frac{\epsilon^2 \hat{\phi}(\omega, |\gamma|)}{\alpha^\varepsilon_{(\omega, |\gamma|)}(L)} \omega^2 \, d\gamma_\omega \, d\omega. \hfill (3.28)$$

This is called the \textit{forward model} because it takes the source and we relate it to the signal that arrives at $z = L$.

### 3.3.4 Limiting process

Let us remark that to get $p^\varepsilon$ from (3.28), we must solve (3.20) and obtain $\alpha^\varepsilon$ at $z = L$. However, $\nu$ is not known so it is impossible to solve for $\alpha^\varepsilon$. Nevertheless, one can give a very explicit characterization of $p^\varepsilon$ for $\varepsilon \to 0$, using two asymptotic methods.

The first method deals with the randomness of the problem and achieves the averaging over a wavelength, known as a diffusion limit (§3.3.4.1) and is similar in spirit to the Central Limit Theorem [19].
The second method is a high frequency one: we exploit the fact that the wavelength is much smaller than the propagation distance, \( \lambda = \mathcal{O}(\varepsilon) \ll L = \mathcal{O}(1) \). This method is known as the method of stationary phase §3.3.4.2.

### 3.3.4.1 Diffusion limit

We are interested in calculating moments of \( p^\varepsilon \) as \( \varepsilon \to 0 \), so we need expectations of products of \( T^\varepsilon_{(\omega_k, \gamma_k, L)} \) for different frequencies \( \{\omega_k\}_{k=1}^n \) and wave vectors \( \{\gamma_k\}_{k=1}^n \)

\[
\lim_{\varepsilon \to 0} E\left[ T^\varepsilon_{(\omega_1, \gamma_1, L)} \cdots T^\varepsilon_{(\omega_n, \gamma_n, L)} \right].
\] (3.29)

The asymptotic tool needed for calculating (3.29) is the diffusion limit theorem derived by Papanicolaou, Stroock and Varadhan [8]:

**Theorem 3.3.1.** Let \( (\nu(z)) \) be a homogeneous ergodic (mixing) Markov process and let \( P(z, h, \nu, x), Q(z, h, \nu, x) \) be two smooth, bounded functions from \( \mathbb{R} \times \mathbb{R} \times S \times \mathbb{R}^d \) to \( \mathbb{R}^d \). Both functions are periodic in \( h \), with period \( T_0 \), independent of \( z, \nu \) and \( x \). Also, \( E[P] = 0 \) with respect to the measure of \( \nu \).

If \( X^\varepsilon(z) \) solves the stochastic ODE:

\[
\frac{dX^\varepsilon(z)}{dz} = \frac{1}{\varepsilon} P(z, \frac{z}{\varepsilon}, \nu(z/\varepsilon^2), X^\varepsilon(z)) + Q(z, \frac{z}{\varepsilon}, \nu(z/\varepsilon^2), X^\varepsilon(z))
\] (3.30a)

\[
X^\varepsilon(0) = x(0)
\] (3.30b)

as \( \varepsilon \to 0 \), then \( X^\varepsilon(z) \) converges weakly in distribution to \( X(z) \), an Itô diffusion process with infinitesimal generator:

\[
\mathcal{L}f(x) = \int_0^\infty du \int_0^{T_0} dh \ E\left[ P(z, h, \nu(0), x) \cdot \nabla_x \left[ P(z, h, \nu(u), x) \right] \cdot \nabla_x \right] f(x)
\] (3.31)

\[
+ \frac{1}{T_0} \int_0^{T_0} dh \ E\left[ Q(z, h, \nu(0), x) \cdot \nabla_x \right] f(x).
\] (3.32)

Now, we need to apply this theorem to calculate (3.29) and we note that Theorem 3.3.1 says that if we wish to calculate moments of \( X^\varepsilon \) as \( \varepsilon \to 0 \), we may as well use the limit process \( X \) instead. We recall [19] that the meaning of the generator is
\[ \mathcal{L}f(x) = \lim_{\delta \to 0} \frac{E[f(X(\delta)) \mid X(0) = x] - f(x_0)}{\delta}. \quad (3.33) \]

However, it is not straightforward, because \( \tilde{\alpha} \) satisfies a complex system of stochastic differential equations instead of a real one, as is assumed in the theorem. The algebra involved in transforming the complex system (3.20) into one of the form (3.30) is complicated but manageable.

Indeed, for linear \( P(\cdot) \) in \( X^\varepsilon \), and \( Q \equiv 0 \) the above theorem can be rewritten as a corollary (see Clouet & Fouque [9])

**Corollary 3.3.1.** Let \( \nu(z) \) be the Markov process satisfying the same assumptions as in Theorem 3.3.1 and suppose that \( X^\varepsilon \) solves the stochastic ODE (3.30). Then, as \( \varepsilon \to 0 \), \( X^\varepsilon \) converges in distribution to \( X \), an Itô diffusion process, and the generator of the limit process becomes

\[ \mathcal{L}f(x) = \text{tr} \left[ (Px \mid (Px)^T)(z) \nabla^2 f(x) \right] + \left[ (P \mid P)(z)x \right]^T \nabla f(x), \quad (3.34) \]

where we use the notation

\[ \langle M \mid N \rangle(z) := \int_0^\infty du \frac{1}{T_0} \int_0^{T_0} dh \, E \left[ M(z, h, \nu(0), x)N(z, h, \nu(0), x) \right]. \quad (3.35) \]

for arbitrary matrices \( M \) and \( N \). In fact, one can write

\[ dX(z) = \sum_k P_k^z(z)X(z)dB_k(z) + (\langle P \mid P \rangle(z))X(z)z \]

\[ X_0 = x \quad (3.36a) \]

\[ X_0 = x \quad (3.36b) \]

where matrices \( P_z^k \) satisfy

\[ \langle P \mid P \rangle_z = \frac{1}{2} \sum_k (P_z^k x)(P_z^k x)^T, \quad x \in \mathbb{R}^d \quad (3.37) \]

which is, essentially, taking the square root of the diffusion matrix.
Now, here are the steps: To compute (3.29), we need to define a $2-n$ dimensional propagator

$$P^e(\omega_1, \ldots, \omega_n) = \begin{pmatrix}
P^e(\omega_1) \\
\vdots \\
P^e(\omega_n)
\end{pmatrix}$$

(3.38)

where each diagonal block is given by (3.22), then we can use (3.20) to derive the equation for $P^e$. This is a complex system, that we rewrite in real form, by separating the real and imaginary parts.

Then, we can apply result (3.36), for $X(z)$. In particular, we get an equation for the limit $\alpha(z, \omega_i), i = 1, \ldots, n$. Now $T^e = 1/\tilde{\alpha}(L)$, so using Ito calculus, we get the stochastic equation for the limit $T$, which is

$$dT(\omega, |\eta|) = -\omega \frac{\gamma^2(\eta)}{2} T(\omega, |\eta|) dz + i\omega \gamma_z(\eta) \sqrt{\frac{\ell(\eta)}{2}} T(\omega, |\eta|) dW(z)$$

(3.39)

where $W(z)$ is standard Brownian motion and the coefficient $\ell(\eta)$ is given by

$$\ell(\eta) = \int_0^\infty E[\nu(0) \nu(\eta,s)] ds = \left(\frac{\gamma_0}{\gamma_z(\eta)}\right)^4 \bar{\ell}$$

(3.40)

$$\bar{\ell} = E[\nu(0) \nu(s)]$$

(3.41)

where $\bar{\ell}$ is called the correlation length, and is dependent on $\nu$. See [2] for ways of estimating $\bar{\ell}$ from the measured reflected or transmitted pressure.

Again, using Ito calculus [11], we obtain an explicit solution to (3.39):

$$\tilde{T}(\omega, |\eta|)(L) = \exp \left( i\omega \gamma_z(\eta) \sqrt{\frac{\ell(\eta)}{2}} W(L) - \omega^2 \gamma_z(\eta)^2 \frac{\ell(\eta)}{4} L \right)$$

(3.42)

Then, using this result and substituting in (3.28), we obtain

$$\tilde{p}(t, x, L) = \frac{1}{(2\pi \epsilon)^3} \int \int \int T(\omega, |\eta|)(L) e^{-i\epsilon(t-x-y-z)L} \tilde{T}(\omega, |\eta|)(L) \epsilon^2 \tilde{\phi}(\omega, |\eta|) \omega^2 \, d\gamma_x \, d\omega$$

(3.43)

### 3.3.4.2 Stationary phase

In this section, we apply the method of stationary phase [4] to evaluate the slowness $\gamma_x$ in (3.43).
We recall that the main contribution in this integral comes from the stationary points [6, equation 8.4.44] of the phase

\[ \theta = t - \gamma_x \cdot x - \gamma_L. \] (3.44)

From our expression (3.43), the stationary point must solve

\[ \nabla_{\gamma_x} \theta = -x + \frac{L}{\gamma_z(\gamma)} \gamma_x = 0. \] (3.45)

The solution of (3.45), denoted by \( \gamma_x^{sp} \), satisfies

\[ \gamma_x^{sp} := \frac{x}{L} \gamma_z(\gamma_{sp}), \] (3.46)

where \( |\gamma|_{sp} \) follows the same convention that we had, and is defined as

\[ |\gamma|_{sp} := |\gamma_x^{sp}|. \] (3.47)

As \( \gamma_z(\gamma) := \sqrt{\gamma_0^2 - |\gamma|^2} \), we can easily find\(^2\) that

\[ \frac{|\gamma|_{sp}^2}{\gamma_0^2} = \frac{|x|^2}{|x|^2 + L^2}, \] (3.48)

and substituting in (3.46) we obtain the stationary point

\[ \gamma_x^{sp} = \gamma_0 \frac{x}{\sqrt{|x|^2 + L^2}}. \] (3.49)

Finally, the value of the phase at the stationary point is given by,

\[ \theta(t_0, \gamma_x^{sp}, x) = t_0 - \gamma_0 \sqrt{|x|^2 + L^2}, \] (3.50)

and we choose \( t_0 \) to cancel it:

\[ t_0 = \gamma_0 \sqrt{|x|^2 + L^2}. \] (3.51)

Observe that this corresponds to choosing \( t_0 \) to be the travel time from the source point at the origin to the point of observation \((x, L)\) under the constant effective

\(^2\)Also recalling the assumption of the lack of evanescent waves.
medium slowness $\gamma_0$. Thus if we focus our attention near the coherent arrival of the pressure field, we can define a window $t = t_0 + \varepsilon \sigma$, and evaluate (3.43) using the stationary phase at $t = t_0 + \varepsilon \sigma$.

Substituting $|\eta_{sp}|^2$ in (3.40) we obtain

$$\ell(|\eta_{sp}|) = \left(1 + \frac{|x|^2}{L^2}\right)\tilde{\ell}.$$  \hspace{1cm} (3.52)

and substituting (3.52) in (3.42) we get

$$\tilde{T}_{(\omega, |\eta_{sp}|)}(L) = \exp \left( i\omega \gamma_0 \sqrt{\frac{\tilde{\ell}}{2}} \sqrt{1 + \frac{|x|^2}{L^2}} W(L) - \omega^2 \frac{\gamma_0 \tilde{\ell}}{4} \left(1 + \frac{|x|^2}{L^2}\right)L \right).$$  \hspace{1cm} (3.53)

and therefore we also get an expression for the pressure

$$p(\sigma, x, L) = \frac{\gamma_0^2}{8\pi^2 |x|^2 + L^2} \int \frac{\sqrt{I_{|\eta_{sp}|}}}{2} \exp \left\{ i\omega \sigma + i\omega \gamma_0 \sqrt{\tilde{\ell}/2} \sqrt{1 + \frac{|x|^2}{L^2}} W(L) \right. \\
- \omega^2 \frac{\gamma_0}{4} \left(1 + \frac{|x|^2}{L^2}\right)L \left. \right\} i\omega \hat{\phi}(\omega, k_{sp}) \, d\omega.$$  \hspace{1cm} (3.54)

3.3.4.3 O’Doherty-Anstey formula

Now let us compare the result (3.54) with the pressure field in a homogeneous medium, which is given by setting $\tilde{\ell} = 0$,

$$p_0(\sigma, x, L) := \frac{\gamma_0^2}{8\pi^2 |x|^2 + L^2} \int e^{i\omega \sigma} i\omega \hat{\phi}(\omega, k_{sp}) \, d\omega.$$  \hspace{1cm} (3.55)

We observe that the random medium effect is twofold. First, we have random correction to the arrival time $\varepsilon \theta$ (3.56c)

$$p(\sigma, x, L) = [p_0(\cdot, x, L) * \mathcal{N}_{D(L,x)}](\sigma - \theta(L, x))$$  \hspace{1cm} (3.56a)

$$D_{(L,x)} := \gamma_0 \frac{\tilde{\ell}}{2} \left(1 + \frac{|x|^2}{L^2}\right)L$$  \hspace{1cm} (3.56b)

$$\theta(L, x) := \gamma_0 \sqrt{\frac{\tilde{\ell}}{2}} \left(1 + \frac{|x|^2}{L^2}\right)W(L)$$  \hspace{1cm} (3.56c)

$$\mathcal{N}_D(s) := \frac{1}{\sqrt{2\pi D}} e^{-s^2/2D}. \hspace{1cm} (3.56d)$$
Second, we have a spreading of the pulse, due to the Gaussian (3.56d).

Finally, we note that if we center our window at the random time \( t_0 + \varepsilon \theta \), \( p \) is given by a deterministic quantity (3.56a),

\[
\bar{a}_0(\cdot, x, L) * \mathcal{N}_{D(L, x)}.
\]

This is often referred to as stabilization of the front.
Chapter 4

Application: Time Reversal in layered media

Consider the same setup as in 3.3. The problem involves the following steps:

1. A point source is located in the halfspace \( z < 0 \) and it generates an acoustic pulse that is impinging on the heterogeneous slab \( 0 < z < L \). The wave field is subject to multiple scattering in the slab, before it hits the time-reversal mirror (TRM) at depth \( z = L \), (see Figure 4.1) where it is recorded in a time window. Since we are interested in the coherent part of the signal, we choose the time window as \( t = t_0 + \varepsilon \sigma \), where \( t_0 \) is the deterministic arrival time and \( \sigma \) is the scaled time offset. The window size is scaled by \( \varepsilon \), to be in agreement with the support of the pulse. The spatial size of the TRM is also small, \( \mathcal{O}(\varepsilon) \).

2. The signal captured at the mirror is reversed in time and then, reemitted.

3. We are interested in the refocusing of the signal near the source, at \( z = 0 \).

Because the wave equation is time reversible, it is natural to expect good refocusing, if we capture the transmitted pressure field on a large aperture mirror, in a long time interval. However, here both the time window and the aperture are \( O(\varepsilon) \), so it is not clear that any refocusing can be observed.
Figure 4.1: Setup of the time-reversal experiment
4.1 Time-reversal of the transmitted front

The time and spatial support of the TRM are defined, on the $\varepsilon$ scale, by $G_1(\sigma)$ and $G_2(x')$, respectively, so the signal recorded is given by

$$y(\sigma, x') = A(t_0' + \varepsilon \sigma, \varepsilon x', L) G_1(\sigma) G_2(x').$$

Now, we time reverse, and obtain

$$\psi^\varepsilon(\sigma, x') = y(-\sigma, x') = p(t_0' - \varepsilon \sigma, \varepsilon x', L) G_1(-\sigma) G_2(x')$$

(4.1)

or, in the Fourier domain,

$$\hat{\psi}(\omega, k, \varepsilon) = \frac{1}{(2\pi)^3} \int \int \int \left\{ \frac{\sqrt{I(|k'|)}}{2} \hat{T}_{(\omega',|k'|)}(L) \hat{\phi}(\omega', k', \varepsilon) \hat{G}_1(\omega - \omega'). \right\}$$

$$\left[ \int \int G_2(x') e^{-i(\omega k + \omega' k')} dx' \right] e^{i \varepsilon |t_0' - \gamma_s(|k'|)| L} \omega^2 \right\} d\omega' d\mathbf{k}'. \quad (4.2)$$

Next, we back-propagate $\hat{\psi}^\varepsilon(\omega, x)$ to plane $z = 0$, at offset $x$ and observe the field in the window $t_1 + \varepsilon \sigma$. This is the quantity of interest and it is given by

$$S^\varepsilon_L(t_1 + \varepsilon \sigma, x) = \frac{1}{(2\pi)^3} \int \int \int \int \int \int e^{-i \varepsilon (t_1 + \varepsilon \sigma - \omega, x - \gamma_s(|k'|) L) T_{(\omega,|k'|)}(L) \varepsilon^2 \hat{\psi}(\omega, k, \varepsilon) \omega^2} d\omega d\mathbf{k}, \quad (4.3)$$

as follows from (**3.42), with $\phi$ replaced by $\psi^\varepsilon$. Equivalently, using (4.1) in (4.3), we obtain

$$S^\varepsilon_L(t_1 + \varepsilon \sigma, x) =$$

$$\frac{1}{(2\pi)^6} \int \int \int \int \int \int e^{-i \varepsilon (t_1 + \varepsilon \sigma - \omega, x - \gamma_s(|k'|) L) \hat{T}_{(\omega,|k'|)}(L) \hat{\phi}(\omega', |k'|, \varepsilon) \hat{G}_1(\omega - \omega').}$$

$$\left[ \int \int G_2(x') e^{-i(\omega k + \omega' k') \cdot x'} dx' \right] e^{i \varepsilon |t_0' - \gamma_s(|k'|)| L} \omega^2 \right\} d\omega d\mathbf{k} \quad (4.4)$$

Next, using a argument of moments, similar to that in the ODA theory (see §3.3.4.3) gives that, as $\varepsilon \to 0$, we can replace $T^e$ by $\tilde{T}$ (3.42) and, using a stationary
phase approximation we get that the main contribution to the integrals over $k$ and $k'$ comes from

$$k = \frac{x}{\sqrt{|x|^2 + L^2}} \quad k' = 0.$$  \hspace{1cm} (4.5) (4.6)

Moreover, the remaining integrals over $\omega$ and $\omega'$ give that $S_L^p$ is large for

$$t_1 = k^{sp} \cdot x + \gamma_z(k^{sp}) L = \gamma_0 \sqrt{|x|^2 + L^2} \quad t'_1 = \gamma_0 L.$$  \hspace{1cm} (4.7) (4.8)

Then, after some straightforward algebra, we obtain

$$S_L^p(t_1 + \varepsilon \sigma, x) \approx \frac{\gamma_0^4}{(2\pi\varepsilon)^4 L \sqrt{|x|^2 + L^2}} \int e^{-i\omega \sigma \phi(\omega', 0, \varepsilon)} \hat{G}_1(\omega - \omega') e^{-i(\omega \sigma(x) - \omega \sigma(x, \omega))} e^{-(\omega D_{(t,0)}^2 - (\omega D_{(t,x)}^2)^2} \hat{G}_2(-\omega k^{sp}) \omega \omega' \, d\omega \, d\omega'. \quad (4.9)$$

Note that if the mirror were large in space, for instance, $G_2(x) = 1$, then $\hat{G}_2(k) = \delta(k)$ and so, by (4.5), only $x = 0$ would give a nonzero field $S_L^p$. In this case, the spatial focusing would be virtually perfect. However, we calculated $S_L^p$ for a small mirror and, if there were no fluctuations, the formula would reduce to

$$S_L^p(t_1 + \varepsilon \sigma, x) \approx \frac{\gamma_0^4}{(2\pi\varepsilon)^4 L \sqrt{|x|^2 + L^2}} \int e^{-i\omega \sigma \phi(\omega', 0, \varepsilon)} \hat{G}_1(\omega - \omega') \hat{G}_2(-\omega k^{sp}) \omega \omega' \, d\omega \, d\omega'. \quad (4.10)$$

Thus, in homogeneous media, there is rather poor spatial focusing, due to the decay of the amplitude as $(L^2 + |x|^2)^{-1/2}$ and no time focusing. Nevertheless, in random media, we have the extra factors in the expression (4.9) of $S_L^p$, and they are responsible for obtaining refocusing, in spite of the small size of the mirror, as we show next.
First, let us adopt the notation:

$$H(s, x; \sigma, L) = \int \int e^{-i(\omega' + \omega s)} \tilde{G}_1(\omega - \omega') \tilde{G}_2(-\omega k^{sp}) \left\{ e^{-i(\omega' \theta_{(L,0)} - \omega \theta_{(L,x)})} - (\omega' D_{(L,0)})^2 - (\omega D_{(L,x)})^2 \right\} \omega \omega' \ d\omega \ d\omega'$$ \hspace{0.5cm} (4.11)

and forget factors such as $(2\pi\varepsilon)^{-4}$, by using the symbol $\sim$ to denote equality, up to scaling factors. We also define

$$r^2 = |x|^2 + L^2$$ \hspace{1cm} (4.12)

and rewrite (4.9) as

$$S^s_L(\sigma, x) \sim \frac{1}{r} \int \phi(s, \varepsilon) H(s, x; \sigma, L) \ ds,$$ \hspace{1cm} (4.13)

where

$$\phi(s, \varepsilon) = \frac{1}{2\pi} \int \tilde{\phi}(\omega', 0, \varepsilon) e^{i\omega' s} \ ds.$$ \hspace{1cm} (4.14)

Now, recalling (3.56), we have

$$D^2_{(L,0)} = \frac{\bar{\ell}_0^2 L}{2} ; \hspace{1cm} D^2_{(L,x)} = \frac{\bar{\ell}_0^2 L}{2} \left( 1 + \frac{|x|^2}{L^2} \right)$$ \hspace{1cm} (4.15)

$$\theta_{(L,0)} = \gamma_0 W(L) \sqrt{\bar{\ell}} \frac{L}{2} ; \hspace{1cm} \theta_{(L,x)} = \gamma_0 W(L) \sqrt{\bar{\ell}} \left( 1 + \frac{|x|^2}{L^2} \right).$$ \hspace{1cm} (4.16)

and (4.11) becomes

$$H(s, x; \sigma, L) = H_0(s + \theta_{(L,0)}, x; \sigma - \theta_{(L,x)}, L) *_{\sigma} N_{D_{(L,0)}} *_{\sigma} N_{D_{(L,x)}}$$ \hspace{1cm} (4.17)

where $*_{\omega}$ denotes convolution with respect to variable $w$, and $H_0$ is the deterministic expression

$$H_0(s, x; \sigma, L) = \int \int e^{-i(\omega' + \omega s)} \tilde{G}_1(\omega - \omega') \tilde{G}_2(-\omega k^{sp}) \omega \omega' \ d\omega \ d\omega'.$$ \hspace{1cm} (4.18)

We note that the key to focusing is the Gaussian $N_{D_{(L,x)}}$ so let us take care of the $s$ part first.
Let $\tilde{\omega} = \omega - \omega'$ and integrate over $\tilde{\omega}$. Then $H_0$ becomes

$$H_0(s, x; \sigma, L) \sim [G_1(s) + iG_1'(s)]$$

(4.19)

and, consequently,

$$H(s, x; \sigma, L) \sim \left[ G_1'(s + \theta_{(L,0)}) \frac{\partial}{\partial s} + G_1(s + \theta_{(L,0)}) \frac{\partial^2}{\partial s^2} \right]$$

$$\cdot \int G_2\left( \frac{|k|}{|k'|} \right) \frac{1}{|k'|} N_{D(L,x)}(s + \sigma + \theta_{(L,0)} - \theta_{(L,x)} - v) \, dv$$

$$\star \sigma \, N_{D(L,0)}$$

(4.20)

This is the time reversed, backpropagated field in the random medium. Now, to make the refocusing transparent, suppose that spatial window $G_2$ were a Gaussian

$$G_2(v) = N_{D_a}(v) ,$$

of standard deviation $a$. Then, (4.20) can be evaluated explicitly and we obtain that, it is approximately of the form

$$S^\varepsilon_L(\sigma, x) \sim \frac{1}{r} \int \phi(s, \varepsilon) \left[ G_1'(s + \theta_{(L,0)}) \frac{\partial}{\partial s} + G_1(s + \theta_{(L,0)}) \frac{\partial^2}{\partial s^2} \right]$$

$$\frac{1}{\sqrt{D^2_{(L,x)} + D^2_{(L,0)} + a^2}} \exp \left\{ -\frac{(s + \sigma + \theta_{(L,0)} - \theta_{(L,x)})^2}{2(D^2_{(L,x)} + D^2_{(L,0)} + a^2)} \right\} \, ds .$$

(4.21)

(4.22)

Thus, recalling definitions (3.56), we note that we have focusing in space, because the magnitude of $S^\varepsilon_L$ is like $(L^2 + |x|^2)^{-2}$. We also have focusing in time, because of the Gaussian in (4.22). Note also that the focusing in both space and time gets better as the correlation length increases so, the stronger the fluctuations, the better focus we get.

Finally, we note the random shift in the focusing time, due to the phase $\theta_{(L,0)} - \theta_{(L,x)}$ in (4.22), but this is small, when |x| is small. In fact, |x| is small because of the spatial focusing explained above.
4.1.1 Conclusions

We have obtained a precise description of the transmitted coherent field and also of the time-reversed and back-propagated field. Our main interest has been in characterizing the spatial focusing properties of this field. We have shown that randomness improves the focusing. The reason for this improvement is a spreading in time of the coherent field. This spreading increases with the lateral offset from the source point. In the layered case, the random fluctuations in the medium create randomness in the travel time of the coherent field, and the main effect of time-reversal is to compensate for this random time shift.
Appendix A

Derivation of the 3D homogeneous free-space Green’s function

Let us consider the three-dimensional free-space homogeneous Helmholtz equation,

\[(\Delta + k^2) \hat{p} = 0.\] (A.1)

To solve the equation, we will find the corresponding Green’s function \(G(x; x')\):

\[(\Delta + k^2) G(x; x') = -\delta(x - x').\] (A.2)

However, the differential operator does not depend on \(x'\), so we can reduce our equation to

\[(\Delta + k^2) G(x) = -\delta(x),\] (A.3)

which strongly suggests a spherical symmetry. Therefore, we look for a solution of the form \(G(x) = G(r)\),

\[(\Delta + k^2) G(r) = \frac{\delta(r)}{4\pi r},\] (A.4)

where \(r\) is the Euclidean norm of \(x\), \(r = |x|_2\). It is important to remark that in spherical coordinates, \(\delta(x)\) becomes \(\delta(r)/(4\pi r)\),
As $G$ does not depend on the angular variables, the Laplacian of $G$ is
\[
\Delta G(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial G}{\partial r} \right). \tag{A.5}
\]

So the new problem becomes
\[
\left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial G}{\partial r} \right) + k^2 \right) G(r) = -\frac{\delta(r)}{4\pi r} \tag{A.6}
\]
and with a little algebraic manipulation, it can be rewritten as
\[
\frac{1}{r} \left( \frac{\partial^2}{\partial r^2} (rG) + k^2 (rG) \right) = -\frac{\delta(r)}{4\pi r}. \tag{A.7}
\]

For $r > 0$, the delta function vanishes, so we have
\[
\left( \frac{\partial^2}{\partial r^2} (rG) + k^2 (rG) \right) = 0, \tag{A.8}
\]
hence the natural ansatz to use is $(rG) = Ae^{ikr} + Be^{-ikr}$, or equivalently,
\[
G = \frac{A}{4\pi r} e^{ikr} + \frac{B}{4\pi r} e^{-ikr}. \tag{A.9}
\]
The $1/4\pi$ coefficient is introduced to simplify the forthcoming calculations.

These two terms correspond to a superposition of two waves: a wave moving away from the origin and another wave moving towards the origin. The physical solution is an outgoing wave. The mathematical statement of this physical interpretation corresponds to the three-dimensional Sommerfeld radiation condition
\[
\lim_{r \to \infty} r \left( G_r + ik^2 G \right) = 0. \tag{A.10}
\]

Substituting the ansatz (A.9) into this limit, and find that $B \equiv 0$.

Now we seek the value of $A$. We redefine $G$, extending it to admit all values of $r \in \mathbb{R}$:
\[
G(r) = A \frac{H(r)}{4\pi r} e^{ikr}, \tag{A.11}
\]
where $H$ is the Heaviside function.
It can be shown [1], using that the generalized function corresponding to the Laplacian of $G$ is

\[
\Delta G = \frac{-A k^2 e^{ikr}}{4\pi r} - H(r) - A \frac{\delta(r)}{4\pi r} \tag{A.12}
\]

\[
= -k^2 G - A \frac{\delta(r)}{4\pi r} \tag{A.13}
\]

that is,

\[
(\Delta + k^2)G = -A \frac{\delta(r)}{4\pi r}, \tag{A.14}
\]

which means that $A$ must be equal to 1, giving us the complete expression for $G$

\[
G(r) = \frac{e^{ikr}}{4\pi r}, \tag{A.15}
\]

or back in Cartesian coordinates,

\[
G(x; x') = \frac{e^{i\omega |x-x'|}}{4\pi |x-x'|} \tag{A.16}
\]
Appendix B

The WKB approximation

In this part we will explain the procedure for obtaining the leading-order approximation for the solution of the reduced wave equation (2.24),

$$\Delta \hat{\varphi} + \left( \frac{\omega}{\epsilon} \right)^2 \gamma^2(z) \hat{\varphi} = 0.$$  \hfill (B.1)

We will suppose that the solution depends exponentially on the fast variation \( \frac{\omega}{\epsilon} \) and is of the form

$$\hat{\varphi} = A e^{i \frac{\omega}{\epsilon} \tau}.$$  \hfill (B.2)

where \( \tau = \tau(x, z) \) is a phase function and the amplitude function \( A = A(x, z) \) is a series expansion of powers of a small perturbation \( \frac{\epsilon}{\omega} \),

$$A = A_0 + \frac{\epsilon}{\omega} A_1 + \frac{\epsilon^2}{\omega^2} A_2 + \cdots.$$  \hfill (B.3)

We find the expressions for \( \nabla u \) and \( \Delta u \):

$$\nabla u = \nabla A e^{i \frac{\omega}{\epsilon} \tau} + i \frac{\omega}{\epsilon} e^{i \frac{\omega}{\epsilon} \tau} \nabla \tau$$ \hfill (B.4)

$$\Delta u = \nabla \cdot \left( \nabla A e^{i \frac{\omega}{\epsilon} \tau} + i \frac{\omega}{\epsilon} e^{i \frac{\omega}{\epsilon} \tau} \nabla \tau \right)$$ \hfill (B.5)

$$= e^{i \frac{\omega}{\epsilon} \tau} \Delta A + \nabla A \cdot \nabla (e^{i \frac{\omega}{\epsilon} \tau}) + i \frac{\omega}{\epsilon} A e^{i \frac{\omega}{\epsilon} \tau} \Delta \tau + \nabla \tau \cdot \nabla (i \frac{\omega}{\epsilon} A e^{i \frac{\omega}{\epsilon} \tau})$$

$$= e^{i \frac{\omega}{\epsilon} \tau} \left\{ \Delta A + i \frac{\omega}{\epsilon} \left( 2 \nabla A \cdot \nabla \tau + A \Delta \tau \right) - \frac{\omega^2}{\epsilon^2} (A \nabla \tau \cdot \nabla \tau) \right\}.$$  \hfill (B.6)
and we substitute them into (2.24),

\[
e^{i \frac{t}{\varepsilon}} \left\{ \Delta A + i \frac{\omega}{\varepsilon} \left(2 \nabla A \cdot \nabla \tau + A \Delta \tau\right) - \frac{\omega^2}{\varepsilon^2} \left(A \nabla \tau \cdot \nabla \tau - A \gamma^2\right) \right\} = 0, \tag{B.7}
\]

the exponential drops out,

\[
\Delta A + i \frac{\omega}{\varepsilon} \left(2 \nabla A \cdot \nabla \tau + A \Delta \tau\right) - \frac{\omega^2}{\varepsilon^2} \left(A \nabla \tau \cdot \nabla \tau - A \gamma^2\right) = 0. \tag{B.8}
\]

Now, we substitute \( A = \sum_{n=0}^{\infty} (\varepsilon/\omega)^n A_n \) and match terms with respect to the orders of \( \frac{\varepsilon}{\omega} \) to get the following leading order approximations:

### B.1 Eikonal equation

Matching the \( \mathcal{O} \left((\frac{\varepsilon}{\omega})^2\right) \) terms results in \( |\nabla \tau|^2 A_0 + \gamma^2 A_0 = 0 \).

As \( A_0 \) is nontrivial, we have

\[
|\nabla \tau|^2 = \gamma^2(z), \tag{B.9}
\]

the eikonal equation.\(^1\)

The eikonal equation is a first-order nonlinear partial differential equation for \( \tau(x, z) \). It is possible to construct its solutions by applying the general theory of characteristics of first-order partial differential equations but it is possible to exploit the special form of the eikonal equation and use a simplified procedure [14, page 5].

The level surfaces defined by \( \tau = \) constant are called wavefronts and the curves orthogonal to them are called rays. Rays are the characteristic curves of the differential equation.

We write a ray in terms of a parameter \( \sigma \),

\[
\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) = \Gamma(\sigma). \tag{B.10}
\]

\(^1\)eikonal comes from Greek \( \text{εικόνα} \) meaning image, and this equation appears often in geometrical optics.
The condition of orthogonality translates into the existence of a function $\lambda = \lambda(\Gamma(\sigma))$ such that
\[
\frac{\partial \Gamma_j}{\partial \sigma} = \lambda \frac{\partial \tau}{\partial \Gamma_j},
\]  
(B.11)
because the gradient of a function is always orthogonal to the level surfaces.

It can be shown that $\lambda = \gamma^{-1}$ or $\lambda = 1$ are the only choices for the orthogonality condition to be satisfied. It is convenient to choose $\lambda = \gamma^{-1}$ because $\sigma$ becomes the arc length along the ray.

So we get,
\[
\frac{\gamma}{\partial \sigma} \left( \frac{\partial \Gamma_j}{\partial \sigma} \right) = \frac{\partial}{\partial \Gamma_j} \left( \frac{\gamma^2}{2} \right), \quad j = 1, 2, 3; \quad (B.12a)
\]
\[
\sum_{j=1}^{3} \left( \frac{\partial \Gamma_j}{\partial \sigma} \right)^2 = 1. \quad (B.12b)
\]

These are called the ray equations. Note that they do not depend on $\tau$. The rays are determined only by $\gamma$, once the initial values are specified.

To solve the eikonal equation for $\tau$, we note that (B.9) and (B.11) yield, for the derivative of $\tau$ along a ray, the result
\[
\frac{\partial}{\partial \sigma} \tau(\Gamma(\sigma)) = \nabla \tau \cdot \frac{\partial \Gamma}{\partial \sigma} = \lambda |\tau|^2 = \lambda \gamma^2, \quad (B.13)
\]
and integrating with respect to $\sigma$ we obtain
\[
\tau(\sigma) = \tau(\sigma_0) + \int_{\sigma_0}^{\sigma} \gamma(\sigma') \, d\sigma', \quad (B.14)
\]
where $\tau(\sigma)$ stands for $\tau(\Gamma(\sigma))$ and $\gamma(\sigma)$ stands for $\gamma(\Gamma(\sigma))$.

**B.2 Transport equations**

The equation resulting from the matching of $O(\frac{w}{\varepsilon})$ terms is
\[
2 \nabla A_0 \cdot \nabla \tau + A_0 \Delta \tau + A_1 |\nabla \tau|^2 + A_1 \gamma^2 = 0. \quad (B.15)
\]
Due to the eikonal equation, the last two terms cancel out and it reduces to

\[ 2 \nabla A_0 \cdot \nabla \tau + A_0 \Delta \tau = 0. \quad (B.16) \]

Similarly, using the same procedure with the $\mathcal{O}(1)$ terms, we obtain

\[ \Delta A_0 + 2 \nabla A_1 \cdot \nabla \tau + A_1 \Delta \tau = 0. \quad (B.17) \]

Both (B.16) and (B.17) are known as transport equations.

In the previous subsection, the rays were used to obtain the solution $\tau$ of the eikonal equation. They can also be used to solve these transport equations.

Keller [14] shows that for a point source, there is an expression for $A_k$:

\[ A_k(\sigma) = \tilde{A}_k(0) \sqrt{\frac{d\Omega}{da(\sigma)}} \frac{\gamma(0)}{\gamma(\sigma)} \quad (B.18) \]

where

\[ \tilde{A}_k(0) = \lim_{\sigma_0 \to 0} \sigma_0 A_k(\sigma_0) \quad (B.19) \]

where $d\Omega$ is the solid angle of the initial directions of a ray that passes through $(x, z)$ and $da$ is the associated element of area on the phase front. See picture 2.2, page 13.

To obtain the correct initial conditions, we match the approximation with the homogeneous free-space Green's function in a neighborhood about the point source, which lies in the homogeneous halfspace. Accordingly, we choose $\tau = 0$ at the source $\sigma_0 = (0)$.

So, if $G$ solves

\[ \left( \Delta + \frac{\omega^2}{c^2} \gamma^2 \right) G = -\delta(x)\delta(z - z_a) \quad (B.20) \]

and its leading approximation is

\[ G \sim \frac{1}{4\pi} \sqrt{\frac{d\Omega}{da(\sigma)}} \frac{\gamma(0)}{\gamma(\sigma)} e^{i \psi \tau}. \quad (B.21) \]
So if $\hat{p}^\tau$ solves

$$\left(\Delta + \frac{\omega^2}{\varepsilon^2} \gamma^2\right) \hat{p}^\tau = \hat{f} \ \delta(x) \delta'(z - z_s),$$  \hspace{1cm} (B.22)

then the Green's function can be used to find the leading-order approximation for $p$ (as $\varepsilon \to 0$):

$$p(x, z, \tau + \varepsilon s) \sim \frac{\tau_s}{4\pi} \sqrt{\frac{d\Omega}{da(\sigma)} \frac{\gamma(0)}{\gamma(\sigma)}} f'(s).$$  \hspace{1cm} (B.23)

An illustrative diagram is shown in figure 2.2, page 13.
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