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Discrete-Time Linear Periodically Time-Varying Systems: Analysis, Realization and Model Reduction

by

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Abstract

Discrete-time linear periodically time-varying (LPTV) systems, considered as a bridge between the well-studied linear time-invariant (LTI) model and the nonlinear time-varying problems in real world, have been receiving increasing attention in recent a few decades.

In this research project, we try to understand discrete-time LPTV systems both internally and externally and derive basic theories for analysis, realization and model reduction of LPTV systems. Firstly we review the system model for LPTV systems, define its transfer function matrix, Markov parameters, stability, reachability and observability. Then we emphasize on the numerically efficient and stable methods to compute LPTV system gramians and to approximate the eigenvalue decay rate.

Another main result of this thesis is Krylov-based moment matching algorithm for model reduction of LPTV systems, which is derived afterwards, and is also compared to the other approach: balancing and balanced truncation of LPTV systems.

Almost any application of discrete-time LPTV systems, including periodic digital filters and periodic control theories, demands a periodic state-space model from input-output maps. This periodic realization problem is treated at the end of the thesis with demonstration of applicable non-minimal and quasi-minimal realization methods.
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Chapter 1

Introduction

1.1 Background and Motivation

The internal expression of a discrete-time finite-dimensional explicit dynamical system that is generally nonlinear and time-varying is as follows:

\[
\Sigma := \begin{cases} 
  x(k+1) &= f(x(k), u(k), k) \\
  y(k) &= h(x(k), u(k), k) 
\end{cases} \tag{1.1.1}
\]

where \( u(k) \), \( x(k) \), \( y(k) \) are the input (excitation), an internal variable (usually the state) and the output (observation) respectively, and \( f, h \) are maps of appropriate dimensions.

A linear system, in continuous or discrete time, is a system that possesses the important property of superposition: If an input consists of the weighted sum of several signals, then the output is the superposition, i.e., the weighted sum of the response of the system to each of those signals. Therefore, in our language, linear systems are systems of linear differential or difference equations. Another important concept, time invariance, defines that the behavior and characteristics of a system are fixed over time. Externally, it means that a time shift in the input signal results in an identical time shift in the output signal. Linear time-invariant (LTI) systems are considered as a good approximation of real world problems and play a central role in control theory.
Almost every natural or technical process, however, is more or less nonlinear in nature, and nothing in reality is actually time-invariant. Correspondingly, in recent years increasing attention has been devoted to the analysis and control of discrete-time linear periodically time-varying (LPTV) systems as a step further from LTI models to the study of real world problems.

**Definition 1.1.1** A discrete-time LPTV system is a discrete-time linear system whose behavior and characteristics change periodically over time. Discrete-time LPTV systems in general have the form

$$
\Sigma := \begin{cases} 
  x(k + 1) &= A(k)x(k) + B(k)u(k) \\
  y(k) &= C(k)x(k) + D(k)u(k)
\end{cases}
$$

(1.1.2)

where the system matrices $A(k) = A(k + K) := A_k \in \mathbb{R}^{n_k \times n_k}$, $B(k) = B(k + K) := B_k \in \mathbb{R}^{n_k \times m}$, $C(k) = C(k + K) := C_k \in \mathbb{R}^{p \times n_k}$, $D(k) = D(k + K) := D_k \in \mathbb{R}^{p \times m}$ are periodic with period $K \geq 1$, and $n_k$ denotes the dimension of this LPTV system at time $k$.

**Definition 1.1.2** An alternate definition for a causal discrete-time LPTV system is in terms of superposition:

$$
y(k) = \sum_{l=-\infty}^{k} h(k, l)u(l),
$$

(1.1.3)

where $h(k, l)$ is the system output at time $k$ when the input is a unit pulse at time $l$, and for a period-$K$ system,

$$
h(k + K, l + K) = h(k, l).
$$

(1.1.4)
Externally, a discrete-time LPTV system is a discrete-time linear system where a
time shift in the input signal results in an identical time shift in the output signal if
and only if the shift is a multiple of system period.

Applications of LPTV systems can be divided into two groups. One is from lin-
earization of nonlinear systems, which is considered to be the bridge between the well
studied linear time-invariant (LTI) systems and the general nonlinear systems which
describe real world problems more authentically. The other group of applications lies
in the systems which are periodically time-varying in nature, such as those in data
sampling, multirate filters, filter banks and sampled-data feedback control systems.

1.1.1 Linearization of nonlinear systems.

Although study of nonlinear systems relies largely on computer simulations, a funda-
mental technique which gives insights is to linearize the nonlinear system around
some equilibrium trajectory. Suppose the state of a nonlinear system (1.1.1) is in the
neighborhood of some given nominal solution \( x_{\text{nom}}(k) \), and functions \( f, g \) are both
differentiable in \( x_{\text{nom}}(k) \) and nominal input \( u_{\text{nom}}(k) \). Then the time-invariant case of
nonlinear system (1.1.1) can be approximated through first order Taylor expansions
as

\[
\begin{align*}
\Delta x(k+1) &= \frac{\partial f}{\partial x}(x_{\text{nom}}(k), u_{\text{nom}}(k)) \Delta x(k) + \frac{\partial f}{\partial u}(x_{\text{nom}}(k), u_{\text{nom}}(k)) \Delta u(k) \\
\Delta y(k) &= \frac{\partial g}{\partial x}(x_{\text{nom}}(k), u_{\text{nom}}(k)) \Delta x(k) + \frac{\partial g}{\partial u}(x_{\text{nom}}(k), u_{\text{nom}}(k)) \Delta u(k)
\end{align*}
\]

(1.1.5)

The approximate system is linear time-invariant if and only if the approximation
trajectory happens to be stationary. If it is not stationary but changes with time
in a periodic manner, the approximate system after linearization will be a linear
periodically time-varying system. In general, the approximate system is a linear time-
varying system which can be treated as an extreme linear periodically time-varying system when the period goes to infinity.

1.1.2 Multirate data sampling systems.

Besides being used for analyzing nonlinear problems, LPTV systems are also suitable models for most periodic behaviors such as seasonal phenomena or rhythmic biological movements. But the major motivation for theoretical study of LPTV systems is multirate data sampling. In the signal processing area, multirate digital filters and filter banks find applications in communications, speech processing, image compression, antenna systems, analog voice privacy systems, and in the digital audio industry [33]. In control theory, multirate data sampling is largely used in multirate feedback systems, also named as sampled-data control systems in the literature.

The fundamental components in multirate data sampling are decimators and interpolators. Fig 1.1 shows block diagrams of both of them.

\[
\begin{align*}
  u(n) & \rightarrow \downarrow M \rightarrow y(n) \\
  (a) & \\
  u(n) & \rightarrow \uparrow L \rightarrow y(n) \\
  (b) & 
\end{align*}
\]

Figure 1.1 : (a) M-fold decimator. (b) L-fold interpolator.

The \textit{M}-fold decimator is characterized by the input-output relation

\[ y_D(n) = u(Mn) \quad (1.1.6) \]
which says that the output at time $n$ is equal to the input at time $Mn$. As a consequence, only the input samples with sample numbers equal to a multiple of $M$ are retained. The $L$-fold interpolator is characterized by the input-output relation

$$y_I(n) = \begin{cases} u \left( \frac{n}{L} \right) & \text{if } n \text{ is a multiple of } L \\ 0 & \text{otherwise} \end{cases} \quad (1.1.7)$$

That is, the output $y_I(n)$ is obtained by inserting $L - 1$ zero-valued samples between adjacent samples of $x(n)$.

It can be easily figured out that both decimators and interpolators are simple time-varying systems and they are not periodic in general. However, when a decimator and interpolator with the same sampling rate appear in cascade, even separated by other samplers or filters, they form a LPTV system as a whole. This is the underlying rule behind the quadrature mirror filter (QMF) banks and sampled-data control systems with a periodically time-varying sampling rate.

### 1.2 Lifting isomorphism between LPTV systems and LTI ones

#### 1.2.1 Standard lifted system constructed by Meyer and Burrus in 1975

The analysis of discrete-time LPTV systems started in late 1950s. According to [27], it was first performed in the context of sampled-data control systems. Jury and Mullin developed a technique for solving difference equations with periodically variable coefficients [18] and applied their results in an analysis of systems with a periodically varying sampling rate [19]. In an analysis of multirate feedback systems, Kranc [21] demonstrated a method for representing these systems by equivalent single rate sys-
tems. Friedland [13] observed that multirate systems belong to the class of general periodically time-varying discrete systems. Based on these achievements, in 1975, Meyer and Burrus developed a unified approach of describing discrete-time LPTV systems with LTI ones [27]. In 1990, Meyer [26] generalized this lifting algorithm to a new class of periodically shift-varying systems called \((P_i, M_j)-\)shift-varying operators, in which each entry of input and output can have different shift values. To avoid confusion of different lifted forms, the lifted system constructed by Meyer and Burrus has been named standard lifted LTI system.

**LPTV systems with period \(K = 3\)**

Before proceeding to study a general LPTV system, let's consider a simple LPTV system with period 3, whose system equations are the same as (1.1.2) with matrices

\[
\begin{bmatrix}
A_0 & B_0 \\
C_0 & D_0
\end{bmatrix}, \quad
\begin{bmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{bmatrix}, \quad
\begin{bmatrix}
A_2 & B_2 \\
C_2 & D_2
\end{bmatrix}.
\]

Departing from time 0, the input-state-output relationship is easy to derive as

\[
\begin{align*}
x(1) &= A_0 x(0) + B_0 u(0) \\
y(0) &= C_0 x(0) + D_0 u(0)
\end{align*}
\]

\[
\begin{align*}
x(2) &= A_1 x(1) + B_1 u(1) = A_1 A_0 x(0) + A_1 B_0 u(0) + B_1 u(1) \\
y(1) &= C_1 x(1) + D_1 u(1) = C_1 A_0 x(0) + C_1 B_0 u(0) + D_1 u(1)
\end{align*}
\]
\[
\begin{cases}
x(3) = A_2 x(2) + B_2 u(2) = A_2 A_1 x(0) + A_2 A_1 B_0 u(0) + A_2 B_1 u(1) + B_2 u(2) \\
y(2) = C_2 x(2) + D_2 u(2) = C_2 A_1 x(0) + C_2 A_1 B_0 u(0) + C_2 B_1 u(1) + D_2 u(2)
\end{cases}
\]

Let group input, sampled state and group output be

\[
u_0^L(0) = \begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix}, \quad x_0^L(0) = x(0), \quad y_0^L(0) = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \end{bmatrix}
\]

\[
\vdots \quad \vdots \quad \vdots
\]

We can get an LTI system which summarizes information of the LPTV system with initial state measured at time \( k = Kq \), where \( q \) is a non-negative integer:

\[
\begin{bmatrix}
A_0^L & B_0^L \\
C_0^L & D_0^L
\end{bmatrix}
= \begin{bmatrix}
A_2 A_1 A_0 & A_2 A_1 B_0 & A_2 B_1 & B_2 \\
C_0 & D_0 & 0 & 0 \\
C_1 A_0 & C_1 B_0 & D_1 & 0 \\
C_2 A_1 A_0 & C_2 A_1 B_0 & C_2 B_1 & D_2
\end{bmatrix}
\]

(1.2.8)

Similarly, we can derive the other two LTI systems which together with (1.2.8) represent our original LPTV system. They are

\[
\begin{bmatrix}
A_1^L & B_1^L \\
C_1^L & D_1^L
\end{bmatrix}
= \begin{bmatrix}
A_6 A_2 A_1 & A_6 A_2 B_1 & A_6 B_2 & B_0 \\
C_1 & D_1 & 0 & 0 \\
C_2 A_1 & C_2 B_1 & D_2 & 0 \\
C_0 A_2 A_1 & C_0 A_2 B_1 & C_0 B_2 & D_0
\end{bmatrix}
\]

(1.2.9)
\[
\begin{bmatrix}
A_1 A_0 A_2 & A_1 A_0 B_2 & A_1 B_0 & B_1 \\
C_2 & D_2 & 0 & 0 \\
C_0 A_2 & C_0 B_2 & D_0 & 0 \\
C_1 A_0 A_2 & C_1 A_0 B_2 & C_1 B_0 & D_1
\end{bmatrix}
\]

This standard lifting technique can be generalized from period-3 systems to general LPTV systems.

**General LPTV systems with period \( K \geq 1 \)**

For a general period-\( K \) LPTV system (1.1.2), the state transition matrix from time \( i \) to time \( j \) is defined by the \( n_j \times n_i \) matrix

\[
\Phi_A(j, i) := \begin{cases}
A_{j-1} \ldots A_{i+1} A_i & \text{for } j > i \\
I_{n_i} & \text{for } j = i
\end{cases}
\]

And the state transition matrix over one period beginning from time \( k \), the so called Monodromy matrix at time \( k \), is defined as

\[
\Phi_A(k + K, k) := A_{k+K-1} \ldots A_{k+1} A_k \in \mathbb{R}^{n_k \times n_k}.
\]

The eigenvalues of \( \Phi_A(k + K, k) \) are called characteristic multipliers of the system at time \( k \). Beginning from time \( k \), the lifted input, output and sampled state are defined
as

\[ u_k^L(h) := \left[ \begin{array}{c} u(k + 0 + hK) \\ u(k + 1 + hK) \\ \vdots \\ u(k + K - 1 + hK) \end{array} \right] \]

\[ x_k^L(h) = x(k + hK) \]  \hspace{1cm} (1.2.13)

\[ y_k^L(h) := \left[ \begin{array}{c} y(k + 0 + hK) \\ y(k + 1 + hK) \\ \vdots \\ y(k + K - 1 + hK) \end{array} \right] \]

Recalling definition of the Monodromy matrix (1.2.12), we have

\[ \Sigma_k^L = \begin{cases} x_k^L(h + 1) = A_k^L x_k^L(h) + B_k^L u_k^L(h) \\ y_k^L(h) = C_k^L x_k^L(h) + D_k^L u_k^L(h) \end{cases} \]  \hspace{1cm} (1.2.14)
with

\[ A_k^L = \Phi(k + K, k) = A_{k+K-1} \ldots A_{k+1} A_k \in \mathbb{R}^{n_k \times n_k}; \]

\[ B_k^L = \{ \Phi(k + K, k + j + 1)B(j + 1) \}_{j=0}^{K-1} \]
\[ = \begin{bmatrix} A_{k+K-1} \ldots A_{k+1}B_k \ldots & A_{k+K-1}B_{k+K-2} & B_{k+K-1} \end{bmatrix} \in \mathbb{R}^{n_k \times mK}; \]

\[ C_k^L = \{ C(k + j)\Phi(k + j, k) \}_{j=0}^{K-1} \]
\[ = \begin{bmatrix} C_k \\ C_{k+1}A_k \\ \vdots \\ C_{k+K-1}A_{k+K-2} \ldots A_{k+1}A_k \end{bmatrix} \in \mathbb{R}^{pK \times n_k}; \quad (1.2.15) \]

\[ D_k^L = \{ D_{i,j} \}_{i,j=0}^{K-1} \]
\[ = \begin{bmatrix} D_k \\ C_{k+1}B_k & D_{k+1} \\ \vdots & \vdots & \ddots \\ C_{k+K-1}A_{k+K-2} \ldots A_{k+1}B_k & \ldots & C_{k+K-1}B_{k+K-2} & D_{k+K-1} \end{bmatrix} \in \mathbb{R}^{pK \times mK}. \]

The system \( \Sigma_k^L = \begin{pmatrix} A_k^L \\ B_k^L \\ C_k^L \\ D_k^L \end{pmatrix} \) is equivalent to the original \( K \)-periodic system (1.1.2) in the sense that the state vector of the lifted LTI system is a time-sampled version of that of the LPTV system. With \( x_k^L(0) = x(k) \) it reproduces \( y(t) \) for \( t \geq k \) and \( x(hK + k) \) for \( h \in (Z)^+ \). The time instant \( k \) can be considered as the initial time of \( K \)-rate sampling of the state of periodic system (1.1.2).
1.2.2 Cyclic lifted system constructed by Park and Verriest in 1989 and Flamm in 1991

Meyer and Burrus’s lifting algorithm uses the period of the LPTV system as the time step of the lifted LTI systems. A consequence of this is that minimal state-space realizations of the standard lifted LTI model have for their state the state of the original system once per period. Park and Verriest [28] in 1989 and Flamm in 1991 [11] presented another time-invariant representation of a discrete-time LPTV system, in which the lifted LTI model maintains the time step of the original LPTV system, so the state of minimal realization of this model at each time step represents the state of the original system at that time step. This new algorithm is named the cyclic lifting algorithm, which implies the structure of the lifted LTI system matrices.

The advantages of the cyclic lifted LTI model over Meyer and Burrus’s standard lifted LTI model are that causality of the original LPTV system is built into its lifted LTI counterpart, and the time steps of the two are same. For understanding this algorithm, we will also examine both a simple period-3 discrete-time LPTV and general period-$K$ discrete-time LPTV systems afterwards.

**LPTV systems with period $K = 3$**

System equations of discrete-time LPTV systems (1.1.2) and system matrices of a simple LPTV system with period 3 are rewritten here as

$$
\Sigma := \begin{cases} 
  x(k+1) &= A(k)x(k) + B(k)u(k) \\
  y(k) &= C(k)x(k) + D(k)u(k)
\end{cases}
$$
\[
\begin{bmatrix}
A_0 & B_0 \\
C_0 & D_0
\end{bmatrix} \begin{bmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{bmatrix} \begin{bmatrix}
A_2 & B_2 \\
C_2 & D_2
\end{bmatrix}
\]

Suppose the input, state and output of this system are

\[
\begin{align*}
  u_0, u_1, u_2, u_3, u_4, \ldots \\
x_0, x_1, x_2, x_3, x_4, \ldots \\
y_0, y_1, y_2, y_3, y_4, \ldots
\end{align*}
\]

We construct a sequence of inputs, states and outputs as

\[
\begin{align*}
u_0^C &= \begin{bmatrix} u_0 \\ 0 \\ 0 \end{bmatrix} & u_1^C &= \begin{bmatrix} 0 \\ u_1 \\ 0 \end{bmatrix} & u_2^C &= \begin{bmatrix} 0 \\ 0 \\ u_2 \end{bmatrix} & u_3^C &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \ldots \\
x_0^C &= \begin{bmatrix} 0 \\ 0 \\ x_0 \end{bmatrix} & x_1^C &= \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} & x_2^C &= \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} & x_3^C &= \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} & \ldots \\
y_0^C &= \begin{bmatrix} y_0 \\ 0 \\ 0 \end{bmatrix} & y_1^C &= \begin{bmatrix} 0 \\ y_1 \\ 0 \end{bmatrix} & y_2^C &= \begin{bmatrix} 0 \\ 0 \\ y_2 \end{bmatrix} & y_3^C &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \ldots
\end{align*}
\]
It is easy to see that this input-state-output relationship can be realized by a LTI system composed of following system matrices:

\[
A^C = \begin{bmatrix}
  A_0 \\
  A_1 \\
  A_2 \\
  C_0 \\
  C_1 \\
  C_2 
\end{bmatrix},
\quad
B^C = \begin{bmatrix}
  B_0 \\
  B_1 \\
  B_2 \\
  D_0 \\
  D_1 \\
  D_2 
\end{bmatrix},
\quad
C^C = \begin{bmatrix}
  A \\
  0 \\
  A \\
  0 \\
  C_0 \\
  C_1 \\
  C_2 
\end{bmatrix},
\quad
D^C = \begin{bmatrix}
  B \\
  0 \\
  B \\
  0 \\
  D_0 \\
  D_1 \\
  D_2 
\end{bmatrix}
\]

This cyclic lifting technique can be generalized from period-3 systems to general LPTV systems as well.

**General LPTV systems with period** \( K \geq 1 \)

**Notation.** To simplify the presentation we introduce first some notations. For \( K \)-periodic matrix \( \{X_k\}_{k=0}^{K-1} \), we use alternatively the script notation

\[
\mathcal{X} := \text{diag}(X_0, X_1, \ldots, X_{K-1}),
\]

which associates the block-diagonal matrix \( \mathcal{X} \) to the cyclic matrix sequence \( \{X_k\}_{k=0}^{K-1} \). This notation is consistent with the standard matrix operations, for instance addition, multiplication, inversion as well as several standard matrix decompositions (Cholesky, SVD, etc.). We denote with \( \sigma \mathcal{X} \) the \( K \)-cyclic shift of the cyclic sequence \( \{X_k\}_{k=0}^{K-1} \).

\[
\sigma \mathcal{X} = \text{diag}(X_1, \ldots, X_{K-1}, X_0).
\]
By using the script notation, the periodic system (1.1.2) will be alternatively denoted by the quadruple \((\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})\).

For a general LPTV system (1.1.2), we define the cyclic shift matrix as

\[
Z_j = \begin{bmatrix}
0 & \ldots & 0 & I_j \\
I_j & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & I_j & 0 \\
\end{bmatrix}
\]

and the concatenated vector

\[
\tilde{r}(k) = \begin{bmatrix}
r^T(k - K + 1) & \ldots & r^T(k + 1) & r^T(k) \\
\end{bmatrix}.
\]

Then the cyclic lifted LTI system can be expressed as

\[
\Sigma^C = \begin{cases}
x^C(k + 1) = A^C x^C(k) + B^C u^C(k) \\
y^C(k) = C^C x^C(k) + D^C u^C(k)
\end{cases}
\]

with

\[
A^C = \begin{bmatrix}
A_1 & \ldots & A_{K-1} \\
\vdots & \ddots & \vdots \\
A_{K-1} & \ldots & A_0 \\
\end{bmatrix}, \quad B^C = \begin{bmatrix}
B_0 \\
\vdots \\
B_{K-1}
\end{bmatrix}
\]

\[
C^C = \begin{bmatrix}
C_1 & \ldots & C_{K-1} \\
\vdots & \ddots & \vdots \\
C_{K-1} & \ldots & C_0 \\
\end{bmatrix}, \quad D^C = \begin{bmatrix}
D_0 \\
\vdots \\
D_{K-1}
\end{bmatrix}
\]

and \(u^C(k) = Z_m^{k+1} \hat{u}(k), x^C(k) = Z_n^k \hat{x}(k), y^C(k) = Z_p^k \tilde{y}(k)\).
Let \( k_0 = \text{mod}(k, K) \)

\[
\begin{align*}
 u^C(k) &= \\
 &= \begin{bmatrix}
 u(k - k_0) \\
 \vdots \\
 u(k) \\
 u(k - K + 1) \\
 \vdots \\
 u(k - k_0 - 1)
\end{bmatrix}, \\
 x^C(k) &= \begin{bmatrix}
 x(k - k_0 + 1) \\
 \vdots \\
 x(k) \\
 x(k - K + 1) \\
 \vdots \\
 x(k - k_0)
\end{bmatrix}, \\
 y^C(k) &= \begin{bmatrix}
 y(k - k_0) \\
 \vdots \\
 y(k) \\
 y(k - K + 1) \\
 \vdots \\
 y(k - k_0 - 1)
\end{bmatrix}.
\end{align*}
\]

The essence of this algorithm is putting inputs, states, and outputs of the original LPTV system at cyclic places of those of the lifted LTI system. \( A^C \) has to be a block-cyclic matrix, while \( B^C, C^C, \) and \( D^C \) can be either block diagonal or block cyclic, which depends on the relative places of input, state and output of the original LPTV system in those of the cyclic lifted LTI system. Only one LTI representation is needed, and no matrix products are involved. But for the cyclic lifted LTI model, number of states \( \sum n_i \) is much larger than that of the standard lifted LTI model, which is just \( n_i \). So model reduction and the realization problem of cyclic lifted LTI model focuses on the problem of generalizing conclusions for general LTI systems without forming lifted system matrices explicitly.

1.3 Research in LPTV systems and contribution of this thesis

It has been mentioned that studies of LPTV systems started in the 1950s but were not brought to a theoretical level until the two lifting algorithms, introduced in the above section, were proposed in 1975 and late 1980s, respectively. These algorithms transform the periodic system (1.1.2) to linear time invariant (LTI) systems with
increased input and output dimensions. These lifting techniques have been very useful for many purposes, ranging from the definition of zeros and poles [7] [15], optimal $\mathcal{H}_2/\mathcal{H}_\infty$ LTI approximation of LPTV systems [8] and periodic realization problems [10]. Although these lifting techniques are useful for their theoretical insight, their sparsity and structure may not be suitable for numerical computations. Numerical methods, therefore, have recently been developed to solve LPTV system problems without computing the lifted LTI systems explicitly, by A. Varga, P. Van Dooren, J. Sreedhar, and many others in [40][35][38][39], etc., where the basic tool used is periodic Schur decomposition and its variants, in which a certain form of numerical stability and efficiency can be ensured.

Gramians still serve as very important system values in LPTV system studies, for instance computation of gramians is involved in balancing and balanced truncation of LPTV systems, which will be introduced in §3.1. In §2.3.2, I propose a way to apply modified low-rank Smith method to LPTV systems to compute approximately reachability and observability gramians in a numerically efficient manner. The decay rate of eigenvalues of the gramians are exploited in §2.4 and an eigen-decay bound and Cholesky approximation are derived.

Model reduction problems for discrete-time LPTV systems is the main topic of my graduate research. In §3.2 I present a Krylov-based moment matching algorithm for LPTV systems, which is numerically more efficient and stable compared to the existing balanced truncation approach.

Periodic realization is another topic of this thesis. Almost any application of discrete-time LPTV systems, including periodic digital filters and periodic control theories, demands a periodic state-space model from input-output maps. In the early 1990s, a few papers appeared on this topic. Sánchez, Hernández and Bru [31], Lin and King [23] solved this problem assuming that the periodic realization has constant
dimensions. However, it is well known that the discrete-time LPTV systems can have time-varying dimensions, so the minimal realization achieved in their methods is what is defined later as quasi-minimal uniform LPTV realization and does not guarantee reachability and observability. In 1995 Colaneri and Longhi [10] unified their analysis for the lifting isomorphism between discrete-time LPTV systems and their associated LTI ones in [9] and Gohberg, Kaashoek and Lerer’s results for general time-varying systems in [14] and concluded in the algorithm of minimal LPTV realization with time-varying dimensions.

This unified method, however, is not very understandable, and since model reduction of LPTV systems can be accomplished, the minimal periodic realization approach may be decomposed to two steps: realization and model reduction. An easy way of computing non-minimal periodic realization is proposed in §4.2.1, followed by introducing a quasi-minimal periodic realization algorithm proposed by other researchers.
Chapter 2

LPTV System Analysis

2.1 Transfer function matrix of lifted LTI systems

We already know that the standard lifted LTI system $\Sigma_k^L$ (1.2.14) is equivalent to the original $K$-periodic system $\Sigma$ (1.1.2) in the sense that as $x_k^L(0) = x(k)$ is satisfied, $\Sigma_k^L$ reproduces $y(t)$ for $t \geq k$. The time instance $k$ can be considered as the initial time of $K$-rate sampling, so basically we get $K$ different standard lifted LTI systems, and each of them has a transfer function matrix just as defined for ordinary LTI systems,

$$H_k^L(z) = C_k^L(zI_{n_k} - A_k^L)^{-1}B_k^L + D_k^L$$

(2.1.1)

where $I_{n_k}$ is an identity matrix of dimension equal to rank of $\Sigma$ at time $k$.

Sizes of the input and output of a LPTV system are increased by $K$ times through lifting, and the standard lifted LTI system at time $k$ summarizes all the information of the original LPTV system. So it is reasonable that there are some similarities or relationship between the $K$ standard lifted LTI systems.

**Lemma 2.1.1** The transfer matrix $H_k^L(z)$ associated with a standard lifted reformulation (1.2.14) of the periodic system (1.1.2) has the following property:

$$H_{k+1}^L(z) = \begin{bmatrix} 0 & I_{(K-1)p} \\ zI_p & 0 \end{bmatrix} H_k^L(z) \begin{bmatrix} 0 & z^{-1}I_m \\ I_{(K-1)m} & 0 \end{bmatrix}$$

(2.1.2)
Proof of this lemma can be found in Grasselli and Longhi [15]. Relationship (2.1.2) is equivalent to

$$H_k^L(z) = \begin{bmatrix} 0 & I_{(K-k)p} \\ zI_{kp} & 0 \end{bmatrix} H_0^L(z) \begin{bmatrix} 0 & z^{-1}I_{mk} \\ I_{(K-k)m} & 0 \end{bmatrix} \quad (2.1.3)$$

for any integer $k$.

As $z \to \infty$, $\|zI - A_k^L\| \to 0$, so the matrix $H_k^L(z)$ has a lower-triangular block structure given by $H_k^L(\infty) = D_k^L$. Moreover, let $\rho_k$ denote the degree of the least common multiple of all denominators of $H_k^L(z)$, it is easy to derive from the structure (2.1.3) that $|\rho_k - \rho_0| \leq 1$, for $k = 0, 1, \ldots, K - 1$.

**Definition 2.1.1** Denoted by $\chi(m, p, K)$, the class of proper rational matrices includes all the matrices formed as $H(z) = [H_{ij}(z) \in \mathbb{C}^{m \times m}]_{i,j=1}^{K}$, with $H_{ij}(\infty) = 0$, for all $i < j$.

This relationship claims that the input-output behavior of a $K$-periodic system is characterized by its standard lifted reformulation at one arbitrary initial sampling instant $k$. This result can be parallelized to Markov parameters.

For a general linear system, the output can be computed as a convolution of the input and some system values. If the linear system is discrete-time, this convolution can be expressed as matrix-vector multiplication and the entries of the infinite convolution matrix are defined as Markov parameters of general discrete-time linear
systems:

\[
\begin{bmatrix}
\vdots \\
y(-2) \\
y(-1) \\
y(0) \\
y(1) \\
\vdots \\
\end{bmatrix}
\begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdot & h(-2,-2) & h(-2,-1) & h(-2,0) & h(-2,1) & \ldots \\
\cdot & h(-1,-2) & h(-1,-1) & h(-1,0) & h(-1,1) & \ldots \\
\cdot & h(0,-2) & h(0,-1) & h(0,0) & h(0,1) & \ldots \\
\cdot & h(1,-2) & h(1,-1) & h(1,0) & h(1,1) & \ldots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
= 
\begin{bmatrix}
\vdots \\
u(-2) \\
u(-1) \\
u(0) \\
u(1) \\
\vdots \\
\end{bmatrix}.
\]

If the linear discrete-time system is causal and time-invariant, the convolution matrix turns out to be a Toeplitz matrix as

\[
\begin{bmatrix}
\vdots \\
y(-2) \\
y(-1) \\
y(0) \\
y(1) \\
\vdots \\
\end{bmatrix}
\begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdot & h_0 & 0 & 0 & 0 & \ldots \\
\cdot & h_1 & h_0 & 0 & 0 & \ldots \\
\cdot & h_2 & h_1 & h_0 & 0 & \ldots \\
\cdot & h_3 & h_2 & h_1 & h_0 & \ldots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
= 
\begin{bmatrix}
\vdots \\
u(-2) \\
u(-1) \\
u(0) \\
u(1) \\
\vdots \\
\end{bmatrix},
\]

whose Markov parameters have the following definition.

**Definition 2.1.2** For a time-invariant, causal and smooth continuous-time system or a causal linear discrete-time time-invariant system with \(m\) inputs and \(p\) outputs, the Markov parameters are defined as coefficients of the Laurent expansion of the system transfer function \(H(\xi)\) around infinity, where \(\xi\) denotes the Laplace variable for both continuous-time and discrete-time systems. Especially for a discrete-time LTI system

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix},
\]

the Markov parameters are just impulse response series of the system.
starting at time 1:

\[ h(i) = CA^{i-1}B \in \mathbb{R}^{p \times m} \text{ for } i = 1, 2, \ldots \]  

(2.1.4)

The relationship between input-output convolution of an LPTV system and its two lifted LTI representations can be explained in the following two structures, where the system period is set as 2 to make expression much simpler.

- relationship between LPTV systems and standard lifted LTI systems:

\[
\begin{bmatrix}
\vdots \\
y(-2) \\
y(-1) \\
y(0) \\
y(1) \\
\vdots 
\end{bmatrix}
= 
\begin{bmatrix}
\vdots \\
\cdots \ h(0,0) \ 0 \ 0 \ 0 \\
\cdots \ h(1,0) \ h(1,1) \ 0 \ 0 \\
\cdots \ h(0,-2) \ h(0,-1) \ h(0,0) \ 0 \\
\cdots \ h(1,-2) \ h(1,-1) \ h(1,0) \ h(1,1) \\
\vdots 
\end{bmatrix}
= 
\begin{bmatrix}
\vdots \\
u(-2) \\
u(-1) \\
u(0) \\
u(1) \\
\vdots 
\end{bmatrix}
\]

- relationship between LPTV systems and cyclic lifted LTI systems:

\[
\begin{bmatrix}
\vdots \\
o \\
y(-1) \\
y(0) \\
o \\
\vdots 
\end{bmatrix}
= 
\begin{bmatrix}
\vdots \\
\cdots \ h(0,0) \ 0 \ 0 \ 0 \\
\cdots \ 0 \ h(1,1) \ 0 \ 0 \\
\cdots \ 0 \ h(0,-1) \ h(0,0) \ 0 \\
\cdots \ h(1,0) \ 0 \ 0 \ h(1,1) \\
\vdots 
\end{bmatrix}
= 
\begin{bmatrix}
\vdots \\
o \\
u(-1) \\
u(0) \\
o \\
\vdots 
\end{bmatrix}
\]

The convolution matrices for both lifted LTI systems are block Toeplitz matrices.

According to Definition 2.1.2, Markov parameters of the standard lifted LTI sys-
tem initially sampled at \( k \) are \( Kp \times Km \) matrices \( J_k(q), q = 0, 1, \ldots, \infty \) and

\[
H_k^L(z) = \sum_{q=0}^{\infty} J_k(q)z^{-q}. \tag{2.1.5}
\]

The following result for Markov parameters of standard lifted LTI systems is simply derived from Lemma 2.1.1.

**Lemma 2.1.2** Let the Markov parameters of a standard lifted LTI system \( \Sigma_k^L \) be defined as in (2.1.5) and be partitioned as

\[
J_k(q) = \begin{bmatrix}
J_k(q)_{11} & J_k(q)_{12} \\
J_k(q)_{21} & J_k(q)_{22}
\end{bmatrix},
\]

where size of \( J_k(q)_{11} \) is \( p \times m \), size of \( J_k(q)_{22} \) is \( (K-1)p \times (K-1)m \), and \( J_k(-1) \) is assumed to be 0. Then the relationship between Markov parameters of the lifted LTI system at successive time instances is

\[
J_{k+1}(q) = \begin{bmatrix}
J_k(q)_{22} & J_k(q-1)_{21} \\
J_k(q+1)_{12} & J_k(q)_{11}
\end{bmatrix}. \tag{2.1.6}
\]

Markov parameters serve as a fundamental measure in model reduction of LTI systems using Krylov methods, which will be introduced in §3.2.1.

Similarly, the cyclic lifted LTI system \( \Sigma^C \) (1.2.18) has a transfer function matrix as

\[
H^C(z) = C^C(zI_{n_k} - A^C)^{-1}B^C + D^C, \tag{2.1.7}
\]

and the relationship between transfer function matrices of standard and cyclic lifted
LTI systems is

\[ H^C(z) = \Delta_p(z^{-1})H_0^L(z^K)\Delta_m(z) \]  \hspace{1cm} (2.1.8)

where \( \Delta_j(x) = \text{diag}\{I_j, xI_j, \ldots, x^{K-1}I_j\} \).

The relationship between Markov parameters of standard and cyclic lifted LTI systems can be derived from (2.1.8) as follows:

\[
J_q^L = (J_q^L(i,j))_{i,j=0}^{K-1} \iff \text{for } p = 0, \ldots, K-1,
\]

\[
J_{K+q+p}^C = \begin{bmatrix}
J_q^L(0,0) & J_q^L(0,K-p) & & \\
J_q^L(p,0) & J_{q+1}^L(p-1,K-1) & & \\
& \ddots & \ddots & \\
& & J_q^L(K-1,K-1-p)
\end{bmatrix}. \hspace{1cm} (2.1.9)
\]

A straightforward approach to compute the transfer function matrices (2.1.1) or (2.1.7) is to simply apply the methods for general LTI systems, for example, the pole-zero method in [34].

### 2.2 Stability, zeros and poles of LPTV systems

As well as in LTI systems, zeros and poles play an important role in the analysis and design of LPTV systems. Their definition consists of suitably extending to the periodic context the definition of those for LTI systems through either standard or cyclic lifted LTI representations.

For a standard lifted LTI system (1.2.14), its transfer function matrix is

\[ H_k^L(z) = C_k^L(zI_{n_k} - A_k^L)^{-1}B_k^L + D_k^L. \]
With rank denoted by \( r_k \), the rational function matrix \( H_k^L(z) \) can be written in the form

\[
H_k^L(z) = L(z)M(z)R(z)
\]  

(2.2.10)

where \( M(z) \) has the Smith-McMillan form,

\[
M(z) = \begin{bmatrix}
M^*(z) & 0_{r_k,m-r_k} \\
0_{p-r_k,r_k} & 0_{p-r_k,m-r_k}
\end{bmatrix}
\]  

(2.2.11)

with

\[
M^*(z) = \text{diag}\left\{ \frac{\varepsilon_1(z)}{\psi_1(z)}, \frac{\varepsilon_2(z)}{\psi_2(z)}, \ldots, \frac{\varepsilon_{r_k}(z)}{\psi_{r_k}(z)} \right\}
\]  

(2.2.12)

and \( L(z), R(z) \) being appropriate unimodular matrices [24].

**Definition 2.2.1** Given the Smith-McMillan form of the associated system matrices of a discrete-time LPTV system (2.2.11) and (2.2.12), the transmission zeros of the LPTV system at time \( k \) are zeros of the polynomial \( \eta_k(z) := \prod_{i=1}^{r_k} \varepsilon_i(z) \), while the poles of the LPTV system at time \( k \) are zeros of the polynomial \( \chi_k(z) := \prod_{i=1}^{r_k} \psi_i(z) \), both with multiplicities equal to their multiplicities in the polynomials [15].

The most recent improvement on computing transmission zeros of LPTV systems can be found in [41]. A similar process will be involved while defining other concepts of zeros, such as invariant zeros and structural zeros, for LPTV systems.

An important characteristic of a system is stability. As any discrete-time LTI system, the standard lifted LTI system (1.2.14) is asymptotically stable if and only if all eigenvalues of \( A_k^T \), also named as eigenvalues of the system at time \( k \), have
magnitude less than one. Because of equivalence between LPTV and standard lifted LTI systems, stability of a discrete-time LPTV system (1.1.2) is also decided by the eigenvalues of $A_k^L$, which is actually the monodromy matrix of a LPTV system defined in (1.2.12).

**Theorem 2.2.1** For a discrete-time LPTV system, the characteristic polynomial of $\Phi_A(k + K, k)$ is independent of $k$, i.e., let $n = \min_k \{n_k\}$, $\Phi_A(k + K, k)$ has always at least $n_k - n$ zero eigenvalues, and the remaining $n$ eigenvalues are the same for all $k$, denoted as $\{\lambda_i\}_{i=1}^n$. In addition, the LPTV system is asymptotically stable if $|\lambda_i| < 1$ for $i = 1, 2, \ldots, n$.

**Remark 2.2.1** A pole of the LPTV system $\Sigma$ at time $k$ with multiplicity $m$ is an eigenvalue of $\Sigma$ with multiplicity not less than $m$. If $\Sigma$ is reachable and observable at time $k$, its poles at time $k$ coincide with its eigenvalues, with the same multiplicity. If $\Sigma$ is reachable and observable at all times, its poles and their multiplicities are independent of time [15] [30].

### 2.3 Reachability and Observability of LPTV systems

#### 2.3.1 Definition of reachability and observability of LPTV systems and the relationship to those of lifted LTI systems

**Definition 2.3.1** The periodic system (1.1.2) is reachable at time $k$ if $\text{rank}(G_k) = n_k$, where $G_k$ is the infinite reachability matrix at time $k$,

$$G_k = \begin{bmatrix} B_{k-1} & A_{k-1}B_{k-2} & \cdots & \Phi_A(k, i + 1)B_i & \cdots \end{bmatrix}. \quad (2.3.13)$$

This system (1.1.2) is completely reachable if $\text{rank}(G_k) = n_k$ holds for all $k$. 
Definition 2.3.2 The periodic system (1.1.2) is observable at time $k$ if $\text{rank}(F_k) = n_k$, where $F_k$ is the infinite observability matrix at time $k$,

\[
F_k = \begin{bmatrix}
    C_k \\
    C_{k+1} A_k \\
    \vdots \\
    C_i \Phi_A(i, k) \\
    \vdots 
\end{bmatrix}.
\]  

(2.3.14)

This system (1.1.2) is completely observable if $\text{rank}(F_k) = n_k$ holds for all $k$.

For an asymptotically stable periodic system, the $n_k \times n_k$ reachability gramian at time $k$ is defined as

\[
P_k := \sum_{i=-\infty}^{k-1} \Phi_A(k, i + 1) B_i B_i^T \Phi_A(k, i + 1)^T = G_k G_k^T \geq 0.
\]  

(2.3.15)

with $G_k$ defined in (2.3.13).

Similarly, the $n_k \times n_k$ observability Gramian at time $k$ is defined as

\[
Q_k := \sum_{i=k}^{\infty} \Phi_A(i, k)^T C_i^T C_i \Phi_A(i, k) = F_k^T F_k \geq 0
\]  

(2.3.16)

with $F_k$ defined in (2.3.14).

Both of these infinite gramians are $K$-periodic matrices. According to the definitions, they are positive semi-definite and satisfy nonnegative (or positive) discrete periodic Lyapunov equations (PDPLEs). Numerically general, stable and efficient
methods to solve PDPLEs are of great interest in the control field [39].

\[
\begin{align*}
\text{forward-time} & \quad \sigma P &= A P A^T + B B^T \\
\text{backward-time} & \quad D &= A^T \sigma D A + C^T C.
\end{align*}
\]

(2.3.17)

Using the definitions of reachability and observability we have the following results.

**Proposition 2.3.1** The periodic system (1.1.2) is reachable at time \( k \) if \( P_k > 0 \). It is completely reachable if \( P_k > 0 \) for \( k = 0, \ldots, K - 1 \).

**Proposition 2.3.2** The periodic system (1.1.2) is observable at time \( k \) if \( Q_k > 0 \). It is completely reachable if \( Q_k > 0 \) for \( k = 0, \ldots, K - 1 \).

**Definition 2.3.3** The periodic system (1.1.2) is minimal if it is completely reachable and completely observable.

Considering the structure of standard lifted LTI systems for LPTV systems as presented in (1.2.14), we can conclude that the reachability and observability gramians of an LPTV system at time \( k \) are exactly the same as those of its standard lifted LTI representation which is initially sampled at \( k \):

\[
\begin{align*}
P_k &= P^L_{\text{mod}(k,K)} \\
Q_k &= Q^L_{\text{mod}(k,K)}
\end{align*}
\]

(2.3.18)

So we have the following results:

**Theorem 2.3.1**

1. The system (1.1.2) is reachable (observable) at time \( k \) if and only if the system \( \Sigma^L_k \) is reachable (observable).

2. The system (1.1.2) is completely reachable (observable) if and only if the system \( \Sigma^L_k \) is reachable (observable) at any \( k \).
3. The system (1.1.2) is controllable (reconstructable) at any time if and only if the system $\Sigma^L_k$ is controllable (reconstructable) at any $k$.

Part 3 of Theorem 2.3.1 comes from Lemma 2.3.1 for controllability and the similar results for reconstructability.

**Lemma 2.3.1** (a) For discrete-time LTI systems $X_{\text{reach}} \subset X_{\text{contr}}$, in particular $X_{\text{contr}} = X_{\text{reach}} + \ker(A^n)$, where $n$ is the rank of system, $X_{\text{reach}}$ and $X_{\text{contr}}$ denotes the reachable and controllable subspaces of system respectively; (b) For discrete-time period-$K$ LPTV systems, $X_{\text{reach}}^k \subset X_{\text{contr}}^k$, in particular $X_{\text{contr}}^k = X_{\text{reach}}^k + \ker(\Phi_A(k+K,k))$, where $\Phi_A(k+K,k)$ is the monodromy matrix defined in (1.2.12). $X_{\text{reach}}^k$ and $X_{\text{contr}}^k$ denote the reachable and controllable subspaces of the system at time $k$ respectively.

Similarly, discrete Lyapunov equations (Stein equations) hold for the cyclic lifted LTI representation of LPTV systems (1.2.18) as follows:

\[
\begin{align*}
P^c &= A^c P^c A^{cT} + B^c B^{cT} \\
Q^c &= A^{cT} Q^c A^c + C^{cT} C^c
\end{align*}
\]  \hspace{1cm} (2.3.19)

with

\[
\begin{align*}
P^c &= \sigma P = \text{diag}\{P_1, P_2, \ldots, P_{K-1}, P_0\} \\
Q^c &= \sigma Q = \text{diag}\{Q_1, Q_2, \ldots, Q_{K-1}, Q_0\}
\end{align*}
\]  \hspace{1cm} (2.3.20)

2.3.2 Solving for reachability and observability gramians of LPTV systems

Just as in LTI systems, reachability and observability gramians play an important role in evaluation of system norms and model reduction using balancing techniques. Most published algorithms since 1990 [35] [36] [37] [38] are on the approach of solving non-
negative (or positive) discrete periodic Lyapunov equations (PDPLEs) (2.3.17) using
the periodic Schur decomposition [4], which reduces the cyclic product $A_{K-1} \ldots A_1 A_0$
(1.2.12) to real Schur form (RSF) or extended real Schur form (ERSF) without ex-
licitly forming this product.

On the other hand, since we have constructed a relationship between the gramians
of LPTV systems and those of standard and cyclic lifted LTI systems as in (2.3.18)
and (2.3.19), it is straightforward and also promising to solve the infinite reachability
and observability gramians of lifted LTI systems for the corresponding LPTV gra-
mians. We have two lifting algorithms. The standard lifting algorithm requires matrix
multiplications and we can only get one of the $K$ LPTV gramians by solving one
standard lifted LTI system, while the cyclic lifting algorithm only forms a large order
sparse matrix and all gramians are solved based on one lifted LTI system. So we are
focusing on cyclic lifted LTI systems and computing their gramians by applying cur-
rent algorithms for time invariant Stein equations and exploiting structure of system
matrices ($A^C, B^C, C^C, D^C$) as constructed in (1.2.18).

Kressner also follows this approach in his recent paper [22]. He applies squared
Smith Iteration to the cyclic LTI representation of LPTV systems and caculates
the error bound of the approximation at $k^{th}$ step. Kressner's algorithm is designed
for dense matrix computations in parallel computing environment. Unfortunately it
does not work for sparse matrix case, because matrix multiplication destroys matrix
sparsity.

To preserve sparsity and also make use of the low-rank feature of system gramians,
I have been working on applying the modified low-rank Smith method proposed by
Gugercin et al. [16] to cyclic lifted LTI systems. The key problem is how to imple-
ment iterations while preserving the cyclic structure of system matrices.
A period-3 LPTV system is studied for example.

Objective:

To solve
\[
\begin{bmatrix}
P_1 \\ P_2 \\ P_0 
\end{bmatrix} =
\begin{bmatrix}
A_1 \\ A_2 
\end{bmatrix}
\begin{bmatrix}
P_1 \\ P_2 \\ P_0 
\end{bmatrix}
\begin{bmatrix}
A_1 \\ A_2 
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
B_0 B_0^T \\ B_1 B_1^T \\ B_2 B_2^T 
\end{bmatrix} 
\]

for \( P_1, P_2, P_0 \).

step 0:

\[
B^C =
\begin{bmatrix}
B_0 \\ B_1 \\ B_2 
\end{bmatrix}
\]

\( \text{Short SVD} \)

\[
= \begin{bmatrix}
V_0 \Sigma_0 W_0^T \\ V_1 \Sigma_1 W_1^T \\ V_2 \Sigma_2 W_2^T 
\end{bmatrix}
\]

\( \Rightarrow \hat{L}_0 =
\begin{bmatrix}
V_0 \Sigma_0 \\ V_1 \Sigma_1 \\ V_2 \Sigma_2 
\end{bmatrix}
\)

and \( \hat{P}_0 = \hat{L}_0 \hat{L}_0^T \).

step 1:

\[
A^C B^C =
\begin{bmatrix}
A_1 B_0 \\ A_2 B_1 
\end{bmatrix}
\begin{bmatrix}
A_0 B_2 
\end{bmatrix}
\begin{bmatrix}
A_0 B_2 \\ A_1 B_0 \\ A_2 B_1 
\end{bmatrix}
\begin{bmatrix}
I_m \\ I_m 
\end{bmatrix}
\]

Let \( Q =
\begin{bmatrix}
I_m \\ I_m \\ I_m 
\end{bmatrix}
\) be an orthonormal matrix,
Just as in the LTI case, we may decompose
\[
\begin{bmatrix}
A_0B_2 \\
A_1B_0 \\
A_2B_1
\end{bmatrix}
\] into two spaces, \( \text{Im}(\mathbf{V}) \) and \((\text{Im}(\mathbf{V}))\perp\) as
\[
\begin{bmatrix}
A_0B_2 \\
A_1B_0 \\
A_2B_1
\end{bmatrix} = \begin{bmatrix}
V_0 & \Gamma_0 \\
V_1 & \Gamma_1 \\
V_2 & \Gamma_2
\end{bmatrix} + \begin{bmatrix}
\hat{V}_0 \\
\hat{V}_1 \\
\hat{V}_2
\end{bmatrix} \begin{bmatrix}
\Theta_0 \\
\Theta_1 \\
\Theta_2
\end{bmatrix}
\]

Then
\[
\begin{bmatrix}
B^C & A^C B^C
\end{bmatrix} = \begin{bmatrix}
\mathbf{V} \Sigma & \mathbf{V} \Gamma + \hat{\mathbf{V}} \hat{\Theta}
\end{bmatrix}
\]

\[
\begin{bmatrix}
V_0 & \hat{V}_0 \\
V_1 & \hat{V}_1 \\
V_2 & \hat{V}_2
\end{bmatrix} \begin{bmatrix}
\Sigma_0 & \Gamma_0 \\
\Sigma_1 & \Gamma_1 \\
\Sigma_2 & \Gamma_2 \\
\Theta_0 & \Theta_1 \\
\Theta_2 & Q
\end{bmatrix} = \begin{bmatrix}
V_0 \Sigma_0 & V_0 \Gamma_0 + \hat{V}_0 \Theta_0 \\
V_1 \Sigma_1 & V_1 \Gamma_1 + \hat{V}_1 \Theta_1 \\
V_2 \Sigma_2 & V_2 \Gamma_2 + \hat{V}_2 \Theta_2
\end{bmatrix} \begin{bmatrix}
I_{3m} & Q
\end{bmatrix}
\]

where \( \hat{\mathbf{Q}} \) is also an orthonormal matrix.

The column equivalent transformation is implemented here to maintain block diagonality of the square root of gramians of the cyclic LTI representation. The value of the gramians does not change
if right multiplied by an orthonormal matrix.

\[
\begin{bmatrix}
B^C & A^C B^C
\end{bmatrix}
\]

\[
= \begin{bmatrix}
V_0 & \hat{V}_0 \\
V_1 & \hat{V}_1 \\
V_2 & \hat{V}_2
\end{bmatrix}
\begin{bmatrix}
\Sigma_0 & \Gamma_0 \\
\Theta_0 & \\
\Sigma_1 & \Gamma_1 \\
\Theta_1 & \\
\Sigma_2 & \Gamma_2 \\
\Theta_2
\end{bmatrix}
\]

where \( \begin{bmatrix}
\Sigma_i & \Gamma_i \\
\Theta_i
\end{bmatrix} \) for \( i = 0, \ldots, K - 1 \) has small size so that its singular value decomposition is easy to compute. Then a low rank square root of the system gramians \( P_1, P_2 \) and \( P_0 \) can be computed afterwards.

**step 2, 3, \ldots**

By induction, the block diagonality will be preserved at each step, although the \( Q \) and \( \hat{Q} \) matrices are different depending on the structure of \( (A^C)^i B^C \) and numbers of iterations taken.

Until now, we have successfully applied the modified low-rank Smith method to cyclic lifted LTI systems and computed controllability gramians of LPTV systems through iterations which work well for sparse matrices. We may consider this problem from another perspective and also get the same algorithm.

From Definition 2.3.15, we can express the square root of the controllability gramian of the LPTV system at time 0 as

\[
P_0 = L_0 L_0^T \text{ wth } L_0 = \begin{bmatrix}
B_2 & A_2 B_1 & A_2 A_1 B_0 & A_2 A_1 A_0 B_2 & \ldots
\end{bmatrix}
\]

This expression demonstrates directly how the low-rank Smith method can be imple-
mented to get $P_0$ after $j$ iterations. Another simplification is to compute the gramian at one time after $j$ iterations and then derive the others according to (2.3.17) instead of computing all $K$ gramians at each iteration. This algorithm and error bound is formalized in the next theorem.

**Theorem 2.3.2** Given an LPTV system (1.1.2), let $P^{(j)}_k$ denote the infinite controllability gramian at time $k$ after $j$ modified low-rank Smith iterations, which are computed as follows:

1. **computation of $\tilde{L}^{(j)}_0$, low-rank square root of $P^{(j)}_0$**

   Let $\tau > 0$ be a pre-specified tolerance value. Assume that until the $j^{th}$ step of the algorithm all the iterates $P^{(j)}_0$ satisfy $\frac{\sigma_{\text{min}} P^{(j)}_0}{\sigma_{\text{max}} P^{(j)}_0} > \tau^2$.

   $$L^{(0)} = [B_{K-1}]^{\text{ShortSVD}} V^{(0)} \Sigma^{(0)} W^{(0)} \Rightarrow \hat{L}^{(0)} = V^{(0)} \Sigma^{(0)}.$$

   Next we partition $V^{(0)}$ and $\Sigma^{(0)}$ conformably:

   $$\hat{L}^{(0)} = \begin{bmatrix} V^{(0)}_1 & V^{(0)}_2 \end{bmatrix} \begin{bmatrix} \Sigma^{(0)}_1 \\ \Sigma^{(0)}_2 \end{bmatrix} \text{ so that } \frac{\Sigma^{(0)}_2(1,1)}{\Sigma^{(0)}_1(1,1)} < \tau \ (2.3.21)$$

   Then the $1^{st}$ low-rank square root factor is approximated by

   $$\hat{L}^{(0)} = V^{(0)}_1 \Sigma^{(0)}_1.$$

2. **$j > 1$ and $j = Kl + J$ with $K$ denoting period and $J$ an integer between 0 and $K - 1$.**
Suppose the \((j - 1)^{st}\) low-rank square root factor is computed as

\[
\tilde{L}^{(j-1)} = V^{(j-1)}\Sigma^{(j-1)}.
\]

Define \(B^{(j)} = (\Phi_A(K,0))^{T}\Phi_A(K, K + 1 - J)B_{K-J}\), where \(\Phi_A(i,j)\) is the state transition matrix from time \(i\) to \(j\). Decompose \(B^{(j)}\) into the two spaces \(\text{Im}(V^{(j-1)})\) and \(\text{Im}(V^{(j-1)}))^\perp\); i.e., write

\[
B^{(j)} = V^{(j-1)}\Gamma + \tilde{V}^{(j-1)}\Theta
\]

The singular value decomposition of the small scale matrix \(\hat{S} = \begin{bmatrix} \Sigma^{(j-1)} & \Gamma \\ \Theta & \end{bmatrix} \)

is inexpensive to compute as

\[
\hat{S} = T\Sigma^{(j)}Y^T.
\]

The square root factor of \(P_0^{(j)}\) is

\[
\hat{L}^{(j)} = \tilde{V}^{(j)}\Sigma^{(j)} \text{ where } \tilde{V}^{(j)} = \begin{bmatrix} V^{(j-1)} & \tilde{V}^{(j-1)} \end{bmatrix}T.
\]

A matrix partition similar to (2.3.21) can be implemented to get a low-rank version \(\tilde{L}^{(j)}\).

2. computation of \(P_k^{(j)}\) for \(k = 1, \ldots, K - 1\) based on \(P_0^{(j)}\)

Given a low-rank square root of \(P_k^{(j)}\), \(\tilde{L}_k^{(j)} = V_k\Sigma_k\), based on the forward-time PDPLE (2.3.17), we have

\[
\hat{L}_{k+1}^{(j)} = \begin{bmatrix} B_k & A_k\tilde{L}_k^{(j)} \end{bmatrix}.
\]
When the column size of $L_k^{(j)}$ is small, which happens in most cases, the space decomposition and following steps can also be implemented to get a low-rank version $\tilde{L}^{(j)}_{k+1}$.

Example: $K = 3$, $n_0 = 110$, $n_1 = 115$, $n_2 = 120$, $m = p = 1$.

| $\tau$   | steps needed | $\frac{||P_0 - P_2||}{||P_0||}$ | $\frac{||P_1 - P_1||}{||P_1||}$ | $\frac{||P_2 - P_2||}{||P_2||}$ |
|----------|--------------|-------------------------------|-------------------------------|-------------------------------|
| $10^{-1}$ | 8            | 0.0072                        | 0.0078                        | 0.0038                        |
| $10^{-2}$ | 14           | 1.0423e - 004                 | 8.9457e - 005                 | 8.5236e - 005                 |
| $10^{-3}$ | 19           | 2.4461e - 006                 | 1.4224e - 006                 | 9.0046e - 007                 |

2.4 Eigen Decay Rates of LPTV Systems

In §2.3.2 we applied the low-rank Smith method to compute approximately the infinite grammians of discrete-time LPTV systems. The algorithm's working efficiency depends on the fact that reachability and observability grammians are normally low-rank semi-definite matrices. In the following, we will focus on the eigenvalues of LPTV system grammians, to approximate them and put a bound on how fast they decay.

2.4.1 Up to date study of eigenvalue decay rates for grammians of continuous-time LTI systems

The two Lyapunov equations associated with a continuous-time LTI system $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ are

\[ AP + PA^T = -BB^T \quad (2.4.22) \]
\[ A^TQ + QA = -C^TC \quad (2.4.23) \]
An interesting feature of these two equations is that the right hand side (RHS) has low rank, which results in the low-rank characteristic of the solutions \( P \) and \( Q \) in most cases. Below are the main theorems for the decay rate of eigenvalues of \( P \) and \( Q \) [42]. Results will be presented for \( P \), while those for \( Q \) are similar. The system is assumed to be asymptotically stable, completely reachable and observable.

**Theorem 2.4.1** For a continuous-time LTI system with stable and diagonalizable matrix \( A \), the eigenvalues of the solution \( P \) of the Lyapunov equation (2.4.22) satisfy the following decay-rate bound:

\[
\frac{\lambda_{mk+1}(P)}{\lambda_1(P)} \leq \kappa^2(X) \left\{ \max_{1 \leq i \leq n} \prod_{i=1}^{k} \left| \frac{\tilde{\tau}_i - \lambda_i}{\tilde{\tau}_i^* + \lambda_i} \right| \right\}^2
\]  

(2.4.24)

where

\[
\{\tilde{\tau}_1, \tilde{\tau}_2, \ldots, \tilde{\tau}_k\} = \arg \min_{\tau_1, \ldots, \tau_k \in \mathbb{C}} \left\{ \max_{1 \leq i \leq n} \prod_{i=1}^{k} \left| \frac{\tau_i - \lambda_i}{\tau_i^* + \lambda_i} \right| \right\}
\]  

(2.4.25)

The \( \lambda_i \)'s are eigenvalues of \( A \). \( X \) is right eigenvector matrix of \( A \). \( \kappa(X) \) is 2-norm condition number of \( X \), which is the ration of largest to smallest singular value in the singular value decomposition of \( X \). \( m \) is the input size of this LTI system and \( k \) is any nonnegative integer such that \( mk < n \).

The minimax problem (2.4.25) in Theorem 2.4.1 has not yet been solved for general \( A \), so mostly we refer to other suboptimal shifts, one of which uses the eigenvalues of \( A \) as the shifts, and we get

\[
\frac{\lambda_{mk+1}(P)}{\lambda_1(P)} \leq \kappa^2(X) \left\{ \max_{1 \leq i \leq n} \prod_{i=1}^{k} \left| \frac{\lambda_i - \lambda_i}{\lambda_i^* + \lambda_i} \right| \right\}^2
\]  

(2.4.26)
We may find that for some real models, (2.4.26) already gives a good bound for the actual eigen-decay rate [42].

Now we introduce the theorem which approximates the eigen-decay rate by the diagonal Cholesky factors of the Cauchy kernel for system grammians [3].

**Lemma 2.4.1** For a stable and minimal continuous-time single-input-single-output (SISO) LTI system \( \begin{pmatrix} A & b \\ c & \end{pmatrix} \), let \( A \) have eigenvalues \( \{\lambda_k\}_{k=1}^n \) and right eigenvector matrix \( X \). The Cauchy matrix \( C := \left\{ \frac{-1}{\lambda_i + \lambda_j} \right\} \) is positive definite, and thus it has Cholesky factorization

\[
C = L\Delta L^*,
\]

where \( \Delta = \text{diag}\{\delta_1, \delta_2, \ldots, \delta_n\} \) with Cholesky factors

\[
\delta_k = \frac{-1}{2\text{real}(\lambda_k)} \prod_{j=1}^{k-1} \left| \frac{\lambda_k - \lambda_j}{\lambda_k^* + \lambda_j} \right|^2
\]

When the \( \lambda_k \)'s are ordered in “Cholesky ordering”, which means

\[
\lambda_k = \arg \max_{\lambda \in \{\lambda_k, \lambda_{k+1}, \ldots, \lambda_n\}} \left\{ \frac{-1}{2\text{real}(\lambda)} \prod_{j=1}^{k-1} \left| \frac{\lambda - \lambda_j}{\lambda^* + \lambda_j} \right|^2 \right\}
\]

the eigen-decay rate \( \frac{\lambda_k(\mathcal{P})}{\lambda_1(\mathcal{P})} \) can be approximated by \( \frac{\delta_k}{\delta_1} \) with

\[
\mathcal{P} = X_b CX_b^* = X_b L\Delta LX_b^*, \quad X_b = X \text{diag}(X^{-1}b).
\]

**Theorem 2.4.2** Let \( \mathcal{P} = \sum_{j=1}^n \delta_j z_j z_j^* \) solve the SISO case of the Lyapunov equation (2.4.22), and define \( \mathcal{P}_k = \sum_{j=1}^k \delta_j z_j z_j^* \), with \( \delta_1 \geq \delta_2 \geq \cdots \geq \delta_n > 0 \), and \( z_j = X_b L e_j \),
then

$$\|P - P_k\|_2 \leq (n - k)^2 \delta_{k+1} (\kappa_2(X)\|B\|_2)^2.$$  (2.4.31)

For a multi-input-multi-output (MIMO) system with $B = [b_1, b_2, \ldots, b_m]$ where $b_i \in \mathbb{R}^n$, the reachability gramian can be computed as the sum of those of the SISO systems

$$\begin{pmatrix} A & b_i \\ C & \end{pmatrix}$$

for $i = 1, \ldots, m$. The Cauchy kernel, which depends only on the matrix $A$, is the same for all SISO systems so that

$$P_i = X_iCX_i^* = X_iL\Delta L^*X_i^*,$$

where $X_i = X\text{diag}(X^{-1}b_i)$  (2.4.32)

Let

$$Z_j = [X_1Le_j, X_2Le_j, \ldots, X_mLe_j].$$  (2.4.33)

We have following result for continuous-time MIMO LTI systems [3].

**Theorem 2.4.3** Let $P = \sum_{j=1}^n \delta_j Z_j Z_j^*$ solve the MIMO case of the Lyapunov equation (2.4.22), and define $P_{km} = \sum_{j=1}^k \delta_j Z_j Z_j^*$. The eigen-decay rate $\frac{\lambda_{km}(P)}{\lambda_1(P)}$ can be approximated by $\frac{\delta_k}{\delta_1}$. If $\frac{\delta_{k+1}}{\delta_1} < \epsilon$, then $P_{km}$ is an approximation to $P$ of rank at most $km$ which satisfies

$$\|P - P_{km}\|_2 \leq \epsilon \delta_1 m(n - k)^2 (\kappa_2(X)\|B\|_2)^2$$  (2.4.34)

with $\delta_1 \approx \|P\|_2$. 
2.4.2 Bilinear transformation between continuous-time and discrete-time LTI systems

Later we will focus on discrete-time LPTV systems, so now we need to derive the decay rate bound for discrete-time LTI systems.

Given a discrete-time system \( \begin{pmatrix} A_d & B_d \\ C_d & D_d \end{pmatrix} \), using the bilinear transformation \( s = \frac{z - 1}{z + 1} \), we can get its continuous-time counterpart as

\[
\Sigma_c = \begin{bmatrix}
A_c = (A_d + I)^{-1}(A_d - I) & B_c = \sqrt{2}(A_d + I)^{-1}B_d \\
C_c = \sqrt{2}C_d(A_d + I)^{-1} & D_d = D_c - C_d(A_d + I)^{-1}B_d
\end{bmatrix}.
\] (2.4.35)

It can be proved that the bilinear transformation preserves gramians, i.e. \( P_c = P_d \) and \( Q_c = Q_d \). So to gain the insight about eigen decay rate of discrete-time LTI systems, we only need to study that of its continuous-time counterpart.

We also assume \( A_d \) is diagonalizable, and it has eigenvalue decomposition \( A_d = X_d\Lambda_d X_d^{-1} \). Then

\[
A_c = (X_d\Lambda_d X_d^{-1} + I)^{-1}(X_d\Lambda_d X_d^{-1} - I)
= [X_d(\Lambda_d + I)X_d^{-1}]^{-1}[X_d(\Lambda_d - I)X_d^{-1}]
= X_d(\Lambda_d + I)^{-1}(\Lambda_d - I)X_d^{-1}.
\] (2.4.36)

Thus the eigenvector matrices of \( A_c \) and \( A_d \) are the same, while the eigenvalues of \( A_c \) are \( \lambda_c = \frac{\lambda_d + 1}{\lambda_d - 1} \), with \( \lambda_c \) and \( \lambda_d \) denoting eigenvalues of \( A_c \) and \( A_d \), respectively. The relationship (2.4.36) applies even when some of \( \lambda_d \)'s are zeros, which gives those of their counterpart \( \lambda_c = -1 \).
2.4.3 Eigen-decay rate and error bound of discrete-time LPTV system gramians

A straightforward approach to the study the eigen-decay rate of discrete-time LPTV systems is to applying those algorithms for general LTI systems to the lifted LTI representations. Most algorithms introduced earlier in this section primarily depends on the eigenvalues of matrix $A$. Cyclic lifted LTI systems (1.2.18), therefore, are not suitable, because eigenvalues of $A^C$ are not ready for analyzing when the size of system states varies with time. However, the standard lifted LTI model (1.2.14) has many good characteristics in terms of identifying eigenvalue decay rate of system gramians. Firstly, according to Theorem 2.2.1, non-zero eigenvalues of all lifted LTI system matrices $\{A_k^L\}_{k=0}^{K-1}$ are the same. Secondly, reachability and observablity gramians of LPTV and standard lifted LTI systems are the same, as stated in (2.3.18). In the following, we will study the eigen-decay rate and error bound of discrete-time LPTV system gramians in the context of standard lifted LTI representations. To simplify notation, we will assume $n_0 = \min_{i=0}^{K-1}(n_i)$, i.e., for a LPTV system, the number of states reaches its minimum at time $Kk$, where $K$ is the system period, and $k$ is any non-negative integer.

**Lemma 2.4.2** Assume the monodromy matrix (1.2.12) at time 0 is diagonalizable as $A_0^L = X_0 \Lambda_0 X_0^{-1}$. Then the monodromy matrix at another time, $\{A_k^L\}_{k=1}^{K-1}$ are all diagonalizable

$$A_k^L = X_k \Lambda_k X_k^{-1}$$ (2.4.37)

with $\Lambda_k = \begin{bmatrix} \Lambda_0 & \mathbf{0}_{n_k-n_0} \\ \mathbf{0}_{n_0} & \end{bmatrix}$ and $X_k = \begin{bmatrix} \Phi_A(k, 0) X_0 & Y_k \end{bmatrix}$ where $Y_k$ denotes an orthonormal basis of $\ker(A_k^L)$. 
Lemma 2.4.2 assures that Theorem 2.4.1 applies for each standard lifted LTI system.

**Theorem 2.4.4** For a discrete-time LPTV system which has diagonalizable monodromy matrices, the eigenvalues of the system gramians at time \( k \) satisfy the following decay-rate bound:

\[
\frac{\lambda_{Km_{j}+1}(P_{k})}{\lambda_{1}(P_{k})} \leq \kappa^{2}(X_{k}) \left\{ \max_{1 \leq l \leq n_{k}} \prod_{i=1}^{j} \left| \frac{\tau_{i} - \frac{\lambda_{1}+1}{\lambda_{i}-1}}{\tau_{i}^{*} + \frac{\lambda_{1}+1}{\lambda_{i}-1}} \right| \right\}^{2} \tag{2.4.38}
\]

where

\[
\{\hat{\tau}_{1}, \hat{\tau}_{2}, \ldots, \hat{\tau}_{j}\} = \arg \min_{\tau_{1}, \ldots, \tau_{j} \in \mathbb{C}} \left\{ \max_{1 \leq l \leq n_{k}} \prod_{i=1}^{j} \left| \frac{\tau_{i} - \frac{\lambda_{1}+1}{\lambda_{i}-1}}{\tau_{i}^{*} + \frac{\lambda_{1}+1}{\lambda_{i}-1}} \right| \right\} \tag{2.4.39}
\]

The \( \lambda_{i} \)'s are eigenvalues of \( A_{k} \), \( X_{k} \) is a right eigenvector matrix of \( A_{k} \), \( m \) is the input size of this LTI system and \( j \) is any nonnegative integer such that \( K_{mj} \leq \min_{k=0}^{K-1} \{n_{k}\} \).

When \( n_{0} = \min\{n_{k}\}_{k=0}^{K-1} \), eigenvalues of \( A_{k}^{T} \) for \( k = 1, \ldots, K - 1 \) contain eigenvalues of \( A_{k} \) and zeros, so that \( \left\{ \max_{1 \leq l \leq n_{k}} \prod_{i=1}^{j} \left| \frac{\tau_{i} - \frac{\lambda_{1}+1}{\lambda_{i}-1}}{\tau_{i}^{*} + \frac{\lambda_{1}+1}{\lambda_{i}-1}} \right| \right\} \) are the same for \( k = 1, \ldots, K - 1 \). Besides this, since \( X_{k} = \begin{bmatrix} \Phi_{A}(k,0)X_{0} & Y_{k} \end{bmatrix} \), the 2-norm condition number of the \( X_{k} \)'s are of the same order. The eigenvalue decay bound for the system gramians at different times are very close, which complies to the numerical simulations.

Similarly, Theorem 2.4.3 can be applied to approximate eigenvalue decay rate of LPTV system gramians.

**Theorem 2.4.5** Let \( P^{(k)} = \sum_{i=1}^{n_{k}} \delta_{i}^{(k)} Z_{i}^{(k)*} Z_{i}^{(k)*} \) denote the reachability gramian of an LPTV system (1.1.2) at time \( k \), and define \( P_{Km_{j}}^{(k)} = \sum_{i=1}^{j} \delta_{i}^{(k)} Z_{i}^{(k)*} Z_{i}^{(k)*} \), the eigenvalue decay rate \( \frac{\lambda_{Km_{j}}(P^{(k)})}{\lambda_{1}(P^{(k)\prime})} \) can be approximated by \( \frac{\delta_{i}^{(k)}}{\delta_{i}^{(k)\prime}} \) with \( \delta_{i}^{(k)} \) computed as in Lemma 2.4.1. If \( \frac{\delta_{i}^{(k)}}{\delta_{i}^{(k)\prime}} < \epsilon \), then \( P_{Km_{j}}^{(k)} \) is an approximation to \( P \) of rank at most \( Km_{j} \), which
satisfies

\[ \| P^{(k)} - P_{km}^{(k)} \|_2 \leq \epsilon \delta_1^{(k)} K m (n_k - j)^2 (\kappa_2 X_k) \| B_k^L \|_2^2 \]  

(2.4.40)

with \( \delta_1^{(k)} \approx \| P^{(k)} \|_2 \).

These approximants only depend on the eigenvalues of standard lifted LTI matrices \( A_k^L \); accordingly they are identical for system gramians at different times \( k \).

In many numerical experiments and real word problems, the eigenvalues of reachability (observability) gramians at different times \( k \) are very close in magnitude. This is a direct result from discrete periodic Lyapunov equations (PDPLE) (2.3.17). Example: \( K = 3, n_0 = 110, n_1 = 115, n_2 = 120, m = p = 1 \).

![Figure 2.1: Poles of LPTV systems](image-url)
Figure 2.2: Eigenvalue decay rate, bound and approximation of $\mathcal{P}$
Chapter 3

Model Reduction of LPTV Systems

Let the general dynamical system (1.1.1) be interpreted as an input-output map:

$\Sigma: \ u \rightarrow y,$

The goal of model reduction is to compute a system with much smaller size:

$\hat{\Sigma}: \ u \rightarrow \hat{y}$

such that 1. the approximation error is small, and there exists a global error bound on $\|y - \hat{y}\|$, 2. system properties, like stability and passivity are preserved, and 3. the procedure be numerically stable and computationally efficient with respect to storage and arithmetic operations [16].

3.1 Balancing-related Model Reduction of LPTV systems

This section is mainly based on A. Varga [39], where circular similarity transformation plays a key role in balancing related model reduction. In the following parts, we will demonstrate that circular similarity transformations are actually the soul of many operations on LPTV systems, such as moment matching using Krylov methods and periodic realizations. Script notation (1.2.16) and (1.2.17) takes into account the structure of circular transformations to make their representations much simpler.
Definition 3.1.1 Let $\mathcal{I}$ be the script notation of a $K$-periodic invertible matrix $T_k$.

Two LPTV systems $\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \tilde{\mathcal{C}} \end{pmatrix}$ and $\begin{pmatrix} \mathcal{A} & \tilde{\mathcal{B}} \\ \mathcal{C} & \tilde{\mathcal{C}} \end{pmatrix}$ related by the transformation

$$
\begin{pmatrix} \mathcal{A} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{C}} \end{pmatrix} = \begin{pmatrix} \sigma \mathcal{I}^{-1} \mathcal{A} \mathcal{I} & \sigma \mathcal{I}^{-1} \tilde{\mathcal{B}} \\ \mathcal{C} \mathcal{I} & \mathcal{C} \tilde{\mathcal{I}} \end{pmatrix}
$$

(3.1.1)

are called Lyapunov-similar and (3.1.1) is called a Lyapunov similarity transformation.

The gramians of the two Lyapunov-similar LPTV systems have the following relationship in script notation:

$$
\tilde{\mathcal{P}} = \mathcal{I}^{-1} \mathcal{P} \mathcal{I}^{-T}
$$

(3.1.2)

$$
\tilde{\mathcal{Q}} = \mathcal{I}^{T} \mathcal{Q} \mathcal{I}
$$

(3.1.3)

For a completely reachable and completely observable LPTV system, $\mathcal{I}$ can be determined such that the transformed reachability and observability gramians are equal and diagonal, and thus the transformed periodic system is balanced. The diagonal elements of the balanced gramians are called Hankel singular values and are positive square-roots of the eigenvalues of the product $\mathcal{P} \mathcal{Q}$, just as those defined for LTI systems. Similarly, the maximum of the Hankel singular values is the Hankel-norm of the LPTV system (1.1.2)

3.1.1 Square-root balancing and balanced truncation of LPTV systems

Let $\mathcal{P} = \mathcal{I}^{T} \mathcal{P}$ and $\mathcal{Q} = \mathcal{R}^{T} \mathcal{Q}$ be the Cholesky factorizations of reachability and observability gramians, respectively. For a minimal system, in analogy to the balancing
transformation of LTI systems, we can use the singular value decomposition

\[
\mathcal{R} \mathcal{T}^T = \mathcal{U} \Sigma \mathcal{V}^T
\]  
(3.1.4)

and the balancing transformation matrix \( \mathcal{T} \) and its inverse \( \mathcal{T}^{-1} \) is computed as

\[
\mathcal{T} = \mathcal{T}^T \Sigma^{-1/2}, \quad \mathcal{T}^{-1} = \Sigma^{-1/2} \mathcal{V}^T \mathcal{R}.
\]  
(3.1.5)

It is simple to prove that the LPTV system \( \left( \begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{E} & \mathcal{F} \end{array} \right) \) transformed by a Lyapunov transformation (3.1.1) from the original LPTV system \( \left( \begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{E} & \mathcal{F} \end{array} \right) \) is balanced.

It is important to note that each diagonal block of \( \mathcal{T} \) is nothing but the balancing transformation matrix of a standard lifted LTI system (1.2.14), which reveals the fact that this circular balancing algorithm is based on the standard lifting algorithm (1.2.14), though it does not construct the lifted system matrices (1.2.15) directly.

Assume we have a minimal and balanced LPTV system and the gramians at different times are partitioned as

\[
\mathcal{P}_k = \mathcal{Q}_k = \Sigma_k = \begin{bmatrix} \Sigma_{k,1} & 0 \\ 0 & \Sigma_{k,2} \end{bmatrix}
\]  
(3.1.6)
and the LPTV system matrices are partitioned accordingly as

\[
A_k = \begin{bmatrix}
A_{k,11} & A_{k,12} \\
A_{k,21} & A_{k,22}
\end{bmatrix},
\]

\[
B_k = \begin{bmatrix}
B_{k,1} \\
B_{k,2}
\end{bmatrix},
\]

\[
C_k = \begin{bmatrix}
C_{k,1} & C_{k,2}
\end{bmatrix}.
\]

(3.1.7)

We have the following theorem for balanced truncation of LPTV systems.

**Theorem 3.1.1** Given an asymptotically stable and minimal LPTV system \( \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} \end{bmatrix} \), let

\[
\begin{bmatrix}
A_r & B_r \\
C_r & C_1
\end{bmatrix} = \begin{bmatrix}
A_{11} & B_1 \\
C_1 & C_1
\end{bmatrix}
\]

be the reduced LPTV system of order \( r \) resulted from the partition introduced above. Then \( \begin{bmatrix} \mathcal{A}_r & \mathcal{B}_r \\ \mathcal{C}_r \end{bmatrix} \) is also asymptotically stable.
and minimal. If we define the periodic error system

\[
\begin{pmatrix}
A_{err} \\
B_{err} \\
C_{err}
\end{pmatrix}
\]

as

\[
A_{err,k} = \begin{bmatrix}
A_k & 0 \\
0 & A_{k,11}
\end{bmatrix},
\]

\[
B_{err,k} = \begin{bmatrix}
B_k \\
B_{k,1}
\end{bmatrix},
\]

\[
C_{err,k} = \begin{bmatrix}
C_k & -C_{k,1}
\end{bmatrix}
\]

its $\mathcal{H}_\infty$-norm is bounded as

\[
|\Sigma_{err}|_{\mathcal{H}_\infty} \leq 2 \sum_{k=0}^{K-1} tr(\Sigma_{k,2}).
\]

(3.1.8)

Here we implement this algorithm in a very simple period-2 discrete-time LPTV system whose state size is not a constant, i.e., $A_k$ matrices of this system are rectangular instead of being square.

- original period-2 LPTV systems

\[
\begin{pmatrix}
A_0 & B_0 \\
C_0 & D_0
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
1 & 0 \\
1 & 1
\end{pmatrix}; \quad
\begin{pmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} & 0 & 1 \\
0 & 1 & 2
\end{pmatrix}
\]
• Monodromy matrix

when \( j \) is even, \( \Phi_A(j + K, j) = A_1 A_0 = \frac{1}{2} \)

when \( j \) is odd, \( \Phi_A(j + K, j) = A_0 A_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \)

characteristic multipliers are \( \frac{1}{2} \) and \( 0 \) \( \Rightarrow \) asymptotically stable

• completely reachable and completely observable

reachability matrix \( G_0 = [1] \quad G_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \)

observability matrix \( F_0 = [1] \quad F_1 = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \)

• reachability and observability gramians

\[
P_0 = \frac{5}{3} \quad P_1 = \begin{pmatrix} \frac{8}{3} & \frac{5}{3} \\ \frac{5}{3} & \frac{5}{3} \end{pmatrix}
\]

\[
Q_0 = \frac{8}{3} \quad P_1 = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{pmatrix}
\]
• balanced system

\[
\begin{pmatrix}
\tilde{A}_0 & \tilde{B}_0 \\
\tilde{C}_0
\end{pmatrix} = \begin{pmatrix}
-0.8598 & -0.4445 \\
0.1790 & -0.7299 \\
0.8891 & \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
\tilde{A}_1 & \tilde{B}_1 \\
\tilde{C}_1
\end{pmatrix} = \begin{pmatrix}
-0.6584 & -0.3695 & 1.1247 \\
-0.9178 & 0.5589 & \\
\end{pmatrix}
\]

Just as for LTI systems, it is possible to determine directly the matrices of the reduced system \((A_r, B_r, C_r)\) using already truncated transformation matrices.

Let us write the singular value decomposition (3.1.4) at each time instant \(k\) in the partitioned form

\[
R_k \Sigma_k^T = \begin{bmatrix}
U_{k,1} & U_{k,2}
\end{bmatrix} \begin{bmatrix}
\Sigma_{k,1} & 0 \\
0 & \Sigma_{k,2}
\end{bmatrix} \begin{bmatrix}
V_{k,1} \\
V_{k,2}
\end{bmatrix}^T
\]  
(3.1.10)

where \(\Sigma_{k,1} \in \mathbb{R}^{r_k \times r_k}, U_{k,1} \in \mathbb{R}^{n_k \times r_k}, V_{k,1} \in \mathbb{R}^{n_k \times r_k}\), and \(\sigma_{\min}(\Sigma_{k,1}) > \sigma_{\max}(\Sigma_{k,2})\).

From the above decomposition, with \(\tilde{\Sigma}_1 = \text{diag}(\Sigma_{0,1}, \ldots, \Sigma_{K-1,1})\), the truncated transformation matrices are defined as

\[
\mathcal{L} = \tilde{\Sigma}_1^{-1/2} \Psi_1 \tilde{\mathcal{R}}
\]

\[
\mathcal{J} = \mathcal{P}^T \Psi_1 \tilde{\Sigma}_1^{-1/2}
\]  
(3.1.11)

Then the matrices of the reduced system can be computed as

\[
\begin{pmatrix}
\tilde{A}_r & \tilde{B}_r \\
\tilde{C}_r
\end{pmatrix} = \begin{pmatrix}
\sigma \mathcal{L} \mathcal{A} \mathcal{J} & \sigma \mathcal{L} \\
\epsilon \mathcal{J}
\end{pmatrix}
\]  
(3.1.12)
The computation of the reduced model relies exclusively on square-root information (the Cholesky factors of gramians), which leads to a guaranteed enhancement of the overall numerical accuracy, especially when Hankel singular values of the system decay fast.

### 3.1.2 Balancing-free square-root balancing method

Parallel to LTI systems, we have several algorithms for balancing and balanced truncation of LPTV systems, which although in theory identical, in practice yield algorithms with quite different numerical properties. In this subsection, we will introduce the balancing-free square-root balancing method, for example.

If the original system is poorly balanced, the balancing truncation matrices \( \mathcal{L} \) and \( \mathcal{U} \) can be ill-conditioned. To avoid potential accuracy loss, we do not constrain the realization to be balanced and use alternatively a balancing-free approach to compute the two truncation matrices. Let QR-decompositions of these matrices be

\[
\mathcal{H}^T \mathcal{V}_1 = \tilde{\mathcal{H}} \mathcal{X} \\
\mathcal{R}^T \mathcal{V}_1 = \tilde{\mathcal{R}} \mathcal{Y}
\]  

(3.1.13)

where \( \mathcal{X} \) and \( \mathcal{Y} \) are nonsingular matrices and \( \tilde{\mathcal{H}} \) and \( \tilde{\mathcal{R}} \) are matrices with orthonormal columns. We define the corresponding \( \tilde{\mathcal{L}} \) as

\[
\tilde{\mathcal{L}} = (\tilde{\mathcal{U}}^T \tilde{\mathcal{F}})^{-1} \tilde{\mathcal{F}}^T.
\]  

(3.1.14)

Then the LPTV system

\[
\begin{pmatrix}
    \mathcal{A} & \mathcal{B} \\
    \mathcal{E} & \mathcal{C}
\end{pmatrix} = 
\begin{pmatrix}
    \sigma \tilde{\mathcal{L}} \tilde{\mathcal{A}} \tilde{\mathcal{F}} & \sigma \tilde{\mathcal{L}} \mathcal{B} \\
    \mathcal{E} \tilde{\mathcal{F}} & \mathcal{C} \tilde{\mathcal{F}}
\end{pmatrix}.
\]
is a minimal realization of the system \((\mathcal{A}, \mathcal{B}, \mathcal{C})\), but it is not balanced.

### 3.2 Krylov-Based Model Reduction of LPTV Systems

An alternative approach to implement model reduction besides balanced truncation is to match the moments of the full-order systems and consequently approximate the full-order transfer function at single or multiple frequencies to a certain order. Krylov methods, the Arnoldi and two-sided Lanczos are numerically efficient algorithms to accomplish this goal. When the frequency around which the moments are to be matched is in the neighbourhood of infinity, the problem is called partial realization and the moments are literally the Markov parameters of systems defined in Definition 2.1.2. In this part, we will first have a brief review of the main results of partial realization for LTI systems, then define and solve the similar problem for discrete-time LPTV systems based on lifting algorithms: standard lifting (1.2.14) and cyclic lifting (1.2.18). The fact that the system size is much larger than input and output sizes is assumed hereafter.

#### 3.2.1 Krylov-based moment matching methods for LTI systems

Given an single-input-single-output (SISO) LTI system \(\Sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\), recall the QR factorization of the rank-\(l\) reachability matrix \(\mathcal{R}_l \in \mathbb{R}^{n \times l}\)

\[
\mathcal{R}_l = VU,
\]

where \(V \in \mathbb{R}^{n \times l}\), \(V^*V = I_l\), and \(U \in \mathbb{R}^{l \times l}\) is block upper triangular. The Arnoldi's factorization \(AV = VH + fe_l^T\) with the columns of \(V\) forming an orthonormal basis for \(\mathcal{R}_l\), \(H \in \mathbb{R}^{l \times l}\) a block upper Hessenberg matrix, \(f \in \mathbb{R}^n\) and \(e_l\) denoting last
column of identity matrix $I$, is a numerically efficient and stable method to compute $V$.

**Theorem 3.2.1** Let the reduced-order system be computed as

$$
\hat{\Sigma} = \begin{pmatrix} V^*AV & V^*b \\ cV & d \end{pmatrix}.
$$

(3.2.16)

Then the first $l$ Markov parameters of the full-order and reduced-order systems are the same.

Given an single-input-single-output (SISO) LTI system $\Sigma = \begin{pmatrix} A & b \\ c & d \end{pmatrix}$, let $\mathcal{O}_l$, $\mathcal{R}_l$, $\mathcal{H}_l$ denote the order-$l$ observability matrix, reachability matrix and Hankel matrix respectively. Assuming that $\det(\mathcal{H}_l) \neq 0$, for $i = 0, \ldots, l$, we compute the LU decomposition of $\mathcal{H}_l$ as

$$
\mathcal{H}_l = LU
$$

(3.2.17)

with $L$ lower-triangular and $U$ upper-triangular.

Define the maps $\pi_L$ and $\pi_U$ as

$$
\pi_L := L^{-1}\mathcal{O}_l \quad \text{and} \quad \pi_U := \mathcal{R}_lU^{-1},
$$

(3.2.18)

which can be computed in a numerically efficient and stable manner using the two-sided Lanczos factorization as:

$$
AV = VT + fe_i^T
$$

$$
A^TW = WT^T + ge_i^T
$$

(3.2.19)
with $\pi_U = V \in \mathbb{R}^{n \times l}$, $\pi_L = W \in \mathbb{R}^{n \times l}$ forming orthonormal bases for $\mathcal{R}_l$ and $\mathcal{O}_l$ respectively, $f, g \in \mathbb{R}^n$, $V^TW = I_l$, $V^Tg = 0$, $W^Tf = 0$, and $e_l$ denoting the last column of the identity matrix $I_l$.

**Theorem 3.2.2** Let the reduced-order system be computed as:

$$
\hat{\Sigma} = \begin{pmatrix} 
\pi_L A \pi_U & \pi_L b \\
\pi_U & 0 \\
\end{pmatrix}.
$$

(3.2.20)

*Then the first 2l Markov parameters of the full-order and reduced-order systems are the same.*

**Remark 3.2.1**

1. The multi-input-multi-output (MIMO) case with $m$ inputs and $p$ outputs of Theorem 3.2.1 is based on the block Arnoldi algorithm [5], where an extra QR factorization is involved in each Arnoldi iteration. The reduced-order system has size $r$, which is the rank of the order-$l$ reachability matrix of the system and $r \leq lm$.

2. To avoid the unwanted “serious breakdowns” or “serious near-breakdowns” in the two-sided Lanczos iterations, the look-ahead algorithm [12] is included by relaxing the vector-wise biorthogonality of the Lanczos basis vectors to a cluster-wise biorthogonality in situations of breakdowns or near-breakdowns [1].

3. The multi-input-multi-output (MIMO) case with $m$ inputs and $p$ outputs ($m \neq p$ in general) of Theorem 3.2.2 demands a vector-wise Lanczos algorithm which deals with different beginning vector sizes and includes a deflation procedure to detect and delete linearly dependent vectors in the left and right block Krylov subspaces respectively [1].
4. The number of operations and storage space needed to compute a reduced system of order \( k \), given an order-\( n \) full order dense system using the Lanczos or Arnoldi factorizations is \( O(n^2k) \) and \( O(nk^2) \), as opposed to the \( O(n^3) \) operations and the \( O(n^2) \) storage space needed for the SVD-based methods, respectively. The Krylov-based model reduction is numerically stable and efficient compared to SVD-based model reduction, for example, balanced truncation, but the reduced-order system might be unstable and no global error bound is presently assured [2].

3.2.2 Krylov-based moment matching algorithms for discrete-time LPTV systems through study of standard lifted LTI systems

As elucidated several times so far, a discrete-time LPTV system (1.1.2) and its standard lifted LTI representation (1.2.14) are equivalent in the sense that the input-output map of the standard lifted LTI system is nothing but a grouped version of that of the original LPTV system. Thus if the moments of the lifted LTI representations of the full-order and reduced-order LPTV systems are matched around single or multiple frequencies to a certain order, the input-output map of the full-order LPTV system is accordingly approximated by that of the reduced-order LPTV system with a small error and is achieved in a numerically stable and efficient way as in §3.2.1.

We also choose Markov parameters as the starting point. Once we obtain the theory for Markov parameters, the problem of matching other moments or including the restarting algorithm to preserve stability will be simply a step further in this direction.

At the beginning of §3.1, we proposed that the circular transformations play a key role in model reduction of LPTV systems and subsequently demonstrated it in the Lyapunov similarity transformation (3.1.1) together with notation (1.2.16) and
(1.2.17). To make it clear why this works, we compare the input-output maps of a discrete-time LTI system and a general discrete time-varying system.

- **discrete-time LTI systems**

\[
y(t) = C(A^{t-t_0}x_0 + \sum_{j=t_0}^{t-1} A^{t-1-j}Bu(j)) + Du(t) \text{ for } t \geq t_0, \quad (3.2.21)
\]

- **discrete time-varying systems**

\[
y(t) = C_t(A_{t-1} \ldots A_{t_0}x_0 + \sum_{j=t_0}^{t-1} A_{t-1} \ldots A_{j+1}B_ju(j)) + D_tu(t) \text{ for } t \geq t_0. \quad (3.2.22)
\]

For discrete-time LPTV systems, the \( A \) matrices vary over the time and they are not even square, with the column size of \( A_{i+1} \) equal to the row size of \( A_i \). And the matrix product involved in the external description (3.2.22) gives us the hint of transforming each \( A_i \) to \( \hat{A}_i = V_{i+1}^*A_iV_i \) with reduced row and column sizes. Following this approach, we can generalize the Arnoldi-based moment-matching methods for LTI systems to LPTV ones.

Take a period-2 discrete-time LPTV system for example.

- **LPTV system:**

\[
\begin{pmatrix}
A_0 & B_0 \\
C_0 & D_0
\end{pmatrix};
\begin{pmatrix}
A_1 & B_1 \\
C_1 & D_1
\end{pmatrix}
\]
• Standard lifted LTI systems:

\[
\begin{pmatrix}
A_0^L & B_0^L \\
C_0^L & D_0^L
\end{pmatrix}
= \begin{pmatrix}
A_1A_0 & A_1B_0 & B_1 \\
C_0 & D_0 & 0 \\
C_1A_0 & C_1B_0 & D_1
\end{pmatrix};
\]

(3.2.23)

\[
\begin{pmatrix}
A_1^L & B_1^L \\
C_1^L & D_1^L
\end{pmatrix}
= \begin{pmatrix}
A_0A_1 & A_0B_1 & B_0 \\
C_1 & D_1 & 0 \\
C_0A_1 & C_0B_1 & D_0
\end{pmatrix}
\]

\(l\)-step Arnoldi iterations are implemented to get the first \(l\) Markov parameters of each lifted LTI system (3.2.23) matched as in Theorem 3.2.1. Let \(V_k(l)\) denote the orthonormal basis of the rank-\(l\) reachability matrix of the \(k^{th}\) standard lifted LTI system \(\Sigma_k^L = \begin{pmatrix}
A_k^L & B_k^L \\
C_k^L & D_k^L
\end{pmatrix}\) such that

\[
R_k^L(l) = [B_k^L \ A_k^L \ B_k^L \ \ldots \ (A_k^L)^{l-1} B_k^L ] = V_k(l)U_k(l).
\]

(3.2.24)

Because \(V_k^L\) is unitary, the above QR factorization will consequently decide the following expression

\[
V_k(l)V_k^*(l)R_k^L(l) = R_k^L(l).
\]

(3.2.25)
We construct a reduced-order LPTV system through circular projections:

\[
\begin{pmatrix}
\hat{A}_0 & \hat{B}_0 \\
\hat{C}_0 & \hat{D}_0
\end{pmatrix} = \begin{pmatrix}
\frac{V_1^*(l)A_0V_0(l)}{C_0V_0(l)} & \frac{V_1^*(l)B_0}{D_0}
\end{pmatrix};
\]

\[
\left(\begin{array}{c}
\hat{A}_1 \\
\hat{C}_1
\end{array}\right) = \begin{pmatrix}
\frac{V_1^*(l)A_1V_1(l)}{C_1V_1(l)} & \frac{V_0^*(l)B_1}{D_1}
\end{pmatrix}.
\]

Then we get two reduced-order LTI systems lifted from (3.2.26) as

\[
\begin{pmatrix}
\hat{A}_0^L & \hat{B}_0^L \\
\hat{C}_0^L & \hat{D}_0^L
\end{pmatrix} = \begin{pmatrix}
\frac{V_0^*(l)A_0V_0(l)V_0^*(l)A_0V_0(l)}{C_0V_0(l)} & \frac{V_0^*(l)A_0V_0(l)V_0^*(l)B_0}{D_0} & \frac{V_0^*(l)B_0}{D_0}
\end{pmatrix};
\]

\[
\left(\begin{array}{c}
\hat{A}_1^L \\
\hat{C}_1^L
\end{array}\right) = \begin{pmatrix}
\frac{V_1^*(l)A_0V_0(l)V_0^*(l)A_1V_1(l)}{C_0V_0(l)V_0^*(l)A_0V_0(l)} & \frac{V_0^*(l)A_0V_0(l)V_0^*(l)B_0}{D_0}
\end{pmatrix}.
\]

Then we compare the Markov parameters of the full-order lifted LTI systems (3.2.23) and the reduced-order lifted LTI systems (3.2.27) for \(l = 1\) and 2 respectively. We will only examine the lifted LTI system initially sampled at time 0 for simplicity and will explain why the same result holds for the lifted LTI systems initially sampled at other times afterwards.

- \(l = 1\), equation (3.2.25) is rewritten as follows:

\[
V_0(1)V_0^*(1) \begin{bmatrix} A_1 & B_1 \end{bmatrix} = \begin{bmatrix} A_1B_0 & B_1 \end{bmatrix};
\]

\[
V_1(1)V_1^*(1) \begin{bmatrix} A_0 & B_0 \end{bmatrix} = \begin{bmatrix} A_0B_1 & B_0 \end{bmatrix}.
\]
The first Markov parameters $h_1$ and $\hat{h}_1$:

- the full-order lifted LTI system

$$h_1 = \begin{bmatrix} C_0 \\ C_1 A_0 \end{bmatrix} \begin{bmatrix} A_1 B_0 & B_1 \end{bmatrix} = \begin{bmatrix} C_0 A_1 B_0 & C_0 B_1 \\ C_1 A_0 A_1 B_0 & C_1 A_0 B_1 \end{bmatrix}; \quad (3.2.29)$$

- the reduced-order lifted LTI system

$$\hat{h}_1 = \begin{bmatrix} C_0 V_0(1) \\ C_1 V_1(1) V_1^*(1) A_0 V_0(1) \end{bmatrix} \begin{bmatrix} V_0^*(1) A_1 V_1(1) V_1^*(1) B_0 & V_0^*(1) B_1 \\ C_0 V_0(1) V_0^*(1) A_1 V_1(1) V_1^*(1) B_0 \\ C_1 V_1(1) V_1^*(1) A_0 V_0(1) V_0^*(1) B_1 \end{bmatrix} = \begin{bmatrix} C_0 V_0(1) V_0^*(1) A_1 V_1(1) V_1^*(1) B_0 & C_0 V_0(1) V_0^*(1) B_1 \\ C_1 V_1(1) V_1^*(1) A_0 V_0(1) V_0^*(1) A_1 V_1(1) V_1^*(1) B_0 & C_1 V_1(1) V_1^*(1) A_0 V_0(1) V_0^*(1) B_1 \end{bmatrix}; \quad (3.2.30)$$

We can see that with only equations (3.2.28), i.e., we get the circular projection matrices $V_0(l)$ and $V_1(l)$ as the orthonormal bases of $B_0^L$ and $B_1^L$ respectively, we can only guarantee that the $(1,1)$, $(1,2)$, $(2,2)$ blocks of $h_1$ and $\hat{h}_1$ are equal.

- $l = 2$, equation (3.2.25) are rewritten as follows:

$$V_0(2) V_0^*(2) \begin{bmatrix} A_1 B_0 & B_1 & A_1 A_0 A_1 B_0 & A_1 A_0 B_1 \end{bmatrix} = \begin{bmatrix} A_1 B_0 & B_1 & A_1 A_0 A_1 B_0 & A_1 A_0 B_1 \end{bmatrix}; \quad (3.2.31)$$

$$V_1(2) V_1^*(2) \begin{bmatrix} A_0 B_1 & B_0 & A_0 A_1 A_0 B_1 & A_0 A_1 B_0 \end{bmatrix} = \begin{bmatrix} A_0 B_1 & B_0 & A_0 A_1 A_0 B_1 & A_0 A_1 B_0 \end{bmatrix}. \quad (3.2.32)$$

The first Markov parameters $h_1$ and $\hat{h}_1$:

- the full-order lifted LTI system

$$h_1 = \begin{bmatrix} C_0 \\ C_1 A_0 \end{bmatrix} \begin{bmatrix} A_1 B_0 & B_1 \end{bmatrix} = \begin{bmatrix} C_0 A_1 B_0 & C_0 B_1 \\ C_1 A_0 A_1 B_0 & C_1 A_0 B_1 \end{bmatrix}; \quad (3.2.32)$$
- the reduced-order lifted LTI system

\[
\dot{h}_1 = \begin{bmatrix}
C_0 V_0(2) \\
C_1 V_1(2) V_1^*(2) A_0 V_0(2)
\end{bmatrix}
\begin{bmatrix}
V_0^*(2) A_1 V_1(2) V_1^*(2) B_0 \\
C_0 V_0(2) V_1^*(2) B_1
\end{bmatrix}
\begin{bmatrix}
V_1(2) V_1^*(2) A_0 V_0(2) V_0^*(2) A_1 V_1(2) V_1^*(2) B_0 \\
C_1 V_1(2) V_1^*(2) A_0 V_0(2) V_0^*(2) B_1
\end{bmatrix}
\]

(3.2.33)

We can see that all the blocks of the first Markov parameters of the full-order lifted LTI systems are matched, by running the Arnoldi iteration for two steps, i.e., calculating calculate \( V_0(2) \) and \( V_1(2) \) as the orthonormal bases of \( \begin{bmatrix} A_0^L & B_0^L \\ B_0^L \end{bmatrix} \) and \( \begin{bmatrix} A_1^L & B_1^L \\ B_1^L \end{bmatrix} \) respectively. And it can be easily derived that with \( l = 2 \), only the diagonal and upper triangular blocks of the second Markov parameters of the full-order and reduced-order standard lifted LTI systems are matched.

By recalling Lemma 2.1.2, we can draw the conclusion that if the 1st to \((q - 1)\)th Markov parameters and the main diagonal and upper triangular blocks of the \( q \)th Markov parameters of the standard lifted LTI system initially sampled at time 0 are matched, so are those Markov parameters of the standard lifted LTI systems initially sampled at other times. Here we generalize the above derivation to a general discrete-time LPTV system.

**Theorem 3.2.3** Given a discrete-time LPTV system (1.1.2) with period \( K \geq 1 \), its associated standard lifted LTI representations \( \Sigma^L_k = \begin{pmatrix} A^L_k & B^L_k \\ C^L_k & D^L_k \end{pmatrix} \) are derived in (1.2.14). The l-step Arnoldi method

\[
A^L_k V_k(l) = V_k(l) H_k(l) + f_k(l) e_k^T,
\]

(3.2.34)

with \( V_k(l) \in \mathbb{R}^{n_x \times lKm} \), \( V_k(:, 1 : Km) \) being an orthonormal basis for column space of
\( B_k^l, V_k^T(l) f_k = 0 \) and \( H_k \) being upper Hessenberg, is implemented for each standard lifted LTI system.

Let

\[
\mathcal{V} := \text{diag}\{V_0(l), V_1(l), \ldots, V_{K-1}(l)\},
\]

\[
\sigma \mathcal{V} := \text{diag}\{V_1(l), \ldots, V_{K-1}(l), V_0(l)\},
\]

and let circular projection be indicated by script notation (1.2.16) as

\[
\begin{pmatrix}
\mathcal{A} & \mathcal{B} \\
\mathcal{C} & \mathcal{D}
\end{pmatrix}
= 
\begin{pmatrix}
\sigma \mathcal{V}^* \mathcal{A} \mathcal{V} & \sigma \mathcal{V}^* \mathcal{B} \\
\mathcal{C} \mathcal{V} & \mathcal{D}
\end{pmatrix}.
\]

(3.2.35)

Then the first \( l - 1 \) Markov parameters of the full-order and reduced-order standard lifted LTI systems are identical, as are the main diagonal and upper-triangular blocks of the \( l \)th Markov parameters of two systems. This means that the diagonal and first \( K(l-1) \) sub-diagonal blocks in the convolution matrix of the full-order LPTV system are matched.

proof: For a discrete-time LPTV system (1.1.2) with period \( K \geq 1 \), its associated
standard lifted LTI systems are $\Sigma^L_k = \begin{pmatrix} A^L_k \mid B^L_k \\ C^L_k \mid D^L_k \end{pmatrix}$ with

$$A^L_k = A_{k+K-1} \cdots A_{k+1} A_k$$
$$B^L_k = \begin{bmatrix} A_{k+K-1} \cdots A_{k+1} B_k & \cdots & A_{k+K-1} B_{k+K-2} & B_{k+K-1} \end{bmatrix}$$
$$C^L_k = \begin{bmatrix} C_k \\ C_{k+1} A_k \\ \vdots \\ C_{k+K-1} A_{k+K-2} \cdots A_{k+1} A_k \end{bmatrix}.$$

So the $l^{th}$ Markov parameters of the full-order standard lifted LTI system initially sampled at time $k$ are

$$h^L_k(l) = C^L_k (A^L_k)^{l-1} B^L_k.$$

$h^L_k(l)$ is a $K \times K$ block matrix corresponding to the lifting algorithm. Each block has dimension $p \times m$ and the $(i, j)$ block for $i, j = 0, \ldots, K - 1$ is

$$h^L_k(l)_{i,j} = C_{k+i} A_{k+i-1} \cdots A_k (A_{k+K-1} \cdots A_k)^{l-1} A_{k+K-1} \cdots A_{k+j+1} B_{k+j}.$$

The order-$l$ controllability matrix of the $k^{th}$ standard lifted LTI system is

$$R^L_k(l) = \begin{bmatrix} B^L_k & A^L_k B^L_k & \cdots & (A^L_k)^{l-1} B^L_k \end{bmatrix}.$$
QR factorization of $R_k^L(l)$ as:

$$R_k^L(l) = V_k^L(l)U \Rightarrow V_k^L(l)(V_k^L(l))^*R_k^L(l) = R_k^L(l) \quad (3.2.36)$$

i.e., $V_k^L(l)(V_k^L(l))^*(A_k^L)^qB_k^L = (A_k^L)^qB_k^L$ for $q = 0, 1, \ldots, l - 1$,

\[
\text{i.e., } V_k^L(l)V_k^L(l)^*(A_{k+K-1}A_k) = (A_{k+K-1}A_k)^q \begin{bmatrix}
A_{k+K-1} & A_{k+1}B_k & \ldots & B_{k+K-1} \\
A_{k+K-1} & A_{k+1}B_k & \ldots & B_{k+K-1}
\end{bmatrix}
\]

for $q = 0, \ldots, l - 1$, and $k = 0, \ldots, K - 1$. The $l^{th}$ Markov parameter $\hat{h}_k^L(l)$ of the $k^{th}$ LTI system lifted from the reduced-order LPTV system (3.2.35) is also a $K \times K$ block matrix whose $(i, j)$ block is

\[
\hat{h}_k^L(l) = C_{k+i}V_k^L(l)\frac{V_k^L(L)A_{k+1}V_k^L(L)\ldots V_k^L(L)A_kV_k^L(L)A_{k+K-1}V_k^L(L)\ldots V_k^L(L)A_kV_k^L(L)}{V_k^L(L)A_{k+1}V_k^L(L)\ldots V_k^L(L)A_kV_k^L(L)A_{k+K-1}V_k^L(L)\ldots V_k^L(L)A_kV_k^L(L)}B_k^L.
\]

The first underbraced part has $i$ $V^*AV$ ingredients, and the second underbraced part has $K - j - 1$ $V^*AV$ ingredients. To make use of (3.2.36) to eliminate all the $V$ matrices, $i$ and $j$ have to conform to the relationship $i + (K - j - 1) \leq K - 1$, which is $i \leq j$, so that the diagonal and upper-triangular blocks of $h_k^L(l)$ and $\hat{h}_k^L(l)$ are equal. If the Arnoldi algorithm is implemented for a step further, the requirement for $i$ and $j$ becomes $i + (K - j - 1) \leq (K - 1) + K$ and it holds for all $0 \leq i, j \leq K - 1$ naturally.

The theory for Lanczos-based moment matching problems is similar.

**Theorem 3.2.4** Given a single-input-single-output (SISO) discrete-time LPTV system (1.1.2) with period $K \geq 1$, its associated standard lifted LTI representation
\[ \Sigma_k^L = \begin{pmatrix} A_k^L & B_k^L \\ C_k^L & D_k^L \end{pmatrix} \] is derived in (1.2.14), with \( \pi_k^L \) and \( \pi_k^U \) as the maps constructed in (3.2.2) for the \( k^{th} \) standard lifted LTI system.

Let

\[ \Pi_U = \text{diag}\{\pi_k^0, \pi_k^1, \ldots, \pi_k^{K-1}\}, \]
\[ \sigma \Pi_L = \text{diag}\{\pi_k^1, \ldots, \pi_k^{K-1}, \pi_k^0\}. \]

The circular projection is indicated by the script notation (1.2.16) as

\[ \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} \sigma \Pi_L \mathcal{A} \Pi_U & \sigma \Pi_L \mathcal{B} \\ \mathcal{C} \Pi_U & \mathcal{D} \end{pmatrix}. \]

Then the first \( 2l - 1 \) Markov parameters of full-order and reduced-order standard lifted LTI systems are identical, as are the main diagonal and upper-triangular blocks of the \( 2l^{th} \) Markov parameters of the two systems, which means the diagonal and first \( K(2l-1) \) sub-diagonal blocks in the convolution matrix of the full-order LPTV system are matched.

**Proof:** We will follow the system setup used in the proof of Theorem 3.2.3. Compared to the Arnoldi method, the two-sided Lanczos method works on both the controllability matrix and the observability matrix. For the \( k^{th} \) full-order standard lifted LTI system, \( l \) two-sided Lanczos iterations construct

\[ V_k^L(l)W_k^L(l)R_k^L(l) = R_k^L(l), \]
\[ O_k^L(l)V_k^L(l)W_k^L(l) = O_k^L(l), \]

where \( R_k^L(l) \) and \( O_k^L(l) \) are the order-\( l \) controllability and observability matrix repre-
tively. The $q^{th}$ Markov parameter $\hat{h}_k^L(l) = \hat{C}_k^L(\hat{A}_k^L)^q \hat{B}_k^L$ of the LTI system lifted from reduced-order LPTV systems (3.2.4) is still a $K \times K$ block matrix whose $(i, j)$ block is

\[
\hat{h}_k^L(l)_{ij} = C_{k+i+1}V_{k+i+1}^L(l)W_{k+i+1}^L(l)A_{k+i+1}V_{k+i+1}^L(l) \ldots W_{k+j+1}^L(l)A_{k+j+1} \ldots A_kV_k^L(l) \nonumber
\times W_k(l)A_{k+j+1}W_{k+j+1}^L(l)A_{k+j+1} \ldots W_{k+j+1}^L(l)A_{k+j+1}V_{k+j+1}^L(l)W_{k+j+1}^L(l)B_{k+j}.
\]

Still we are seeking for requirements on $i, j$ so that the $V$ and $W$ matrices can be eliminated. By counting the number of $A$ matrices involved in $\hat{h}_k^L(l)$, the requirement turns out to be

\[
i + K(q - 1) + (K - j - 1) - 1 \leq 2 \times (K - 1 + K(l - 1))
\]

and this requirement can be simplified to be $i \leq K(2l - q) + j - 1$, i.e.,

- $q \leq 2l - 1 \Rightarrow 0 \leq i, j \leq K - 1$

- $q = 2l \Rightarrow i \leq j$

The proof is complete.

**Remark 3.2.2** 1. Theorem 3.2.3 and 3.2.4 offer a promising new approach to accomplish model reduction of LPTV systems. Real algorithms to implement Krylov factorization, like the block Arnoldi method and the vector-wise Lanczos method including deflation-excluding and look-ahead components, as mentioned in Remark 3.2.1, are also involved. And future work in this direction lies in rationally interpolating the LPTV systems, preserving stability and passivity, and achieving a local or global error bound for LPTV Krylov model reduction.

2. The orthonormal bases in $V$ and $W$, both in Theorem 3.2.3 and 3.2.4, can have different sizes, and accordingly, the number of the state variables of the reduced-order LPTV system changes with time in a periodic manner.
3. The number of operations and storage space needed to compute a reduced-order LPTV system of size $l$ using Lanczos or Arnoldi factorizations, given a size-$n$ full order dense LPTV system, is $O(Kn^2 l)$ and $O(Knl^2)$, with $K$ denoting the system period, as opposed to the $O(Kn^3)$ operations and $O(Kn^2)$ storage space needed for the SVD-based methods for LPTV systems [39], respectively. Even though the system matrices of the standard lifted LTI system are largely involved in the algorithm derivation, there is no need to formulate them explicitly in computation.

4. Krylov-based model reduction is numerically stable and efficient compared to SVD-based model reduction, but the reduced system can be unstable and no global error bound is presently assured [2].

3.2.3 Krylov-based moment matching algorithms for discrete-time LPTV systems through study of cyclic lifted LTI systems

As mentioned earlier, the cyclic lifted LTI representation (1.2.18) summarizes all the information of the discrete-time LPTV system (1.1.2) in just one LTI model, thus everything that works for general LTI systems is theoretically applicable to the cyclic LTI system. In this section, we will first analyze the cyclic LTI representation lifted from the reduced-order LPTV system computed in Theorem 3.2.3, then start Krylov method over based on the cyclic LTI representation itself.

**Proposition 3.2.1** Let $\Sigma$ denote a full-order LPTV system (1.1.2) and $\hat{\Sigma}$ denote its reduced-order counterpart computed in Theorem 3.2.3, while $\Sigma^C$ and $\hat{\Sigma}^C$ are the cyclic LTI representations lifted from $\Sigma$ and $\hat{\Sigma}$ respectively. Then the first $Kl$ Markov parameters of $\Sigma^C$ and $\hat{\Sigma}^C$ are the same.
Proof of this proposition is just a step further after proving Theorem 3.2.3 and only involves counting how many \( V_i V_i^* \) factors in the blocks of reduced-order Markov parameters can be cancelled.

This result corresponds to the relationship between transfer function matrices of the standard and cyclic lifted LTI representations of a general LPTV system which is rewritten here:

\[
H^C(z) = \Delta_p(z^{-1})H_0^L(z^K)\Delta_m(z),
\]

where \( \Delta_j(x) = \text{diag}\{I_j, xI_j, \ldots, x^{K-1}I_j\} \). Let \( H_0^L(z) \) and \( H^C(z) \) be expanded in Laurent series as

\[
\begin{cases}
H_0^L(z) = D_0^L + \sum_{q=1}^{\infty} J_q z^{-q} \Rightarrow H_0^L(z^K) = D_0^L + \sum_{q=1}^{\infty} J_q z^{-Kq} \\
H^C(z) = D^C + \sum_{q=1}^{\infty} J_q^C z^{-q}.
\end{cases}
\]  

(3.2.38)

We can derive the relationship between Markov parameters of the standard and cyclic lifted LTI systems as

\[
J_q^C = (J_q^L(i,j))_{i,j=0}^{K-1} \iff \quad \text{for } p = 0, \ldots, K - 1,
\]

\[
J_{K+q+p}^C = 
\begin{bmatrix}
J_q^L(p,0) & & & & J_q^L(0,K-p) \\
& \ddots & & & \\
& & \ddots & & J_q^L(p-1,K-1) \\
& & & J_q^L(K-1,K-1-p)
\end{bmatrix},
\]

(3.2.39)

such that the first \( l - 1 \) Markov parameters and the upper-triangular and diagonal blocks of the \( l^{th} \) Markov parameters of \( \Sigma_0^L \) are matched if and only if the first \( Kl \) Markov parameters of \( \Sigma^C \) are matched.
To make this correspondence more obvious, we will take a closer look at an LPTV system example with \( K = 2, l = 2, n \gg m = p = 1 \) and let \( \Sigma^L \) and \( \hat{\Sigma}^L \) be the standard LTI representations lifted from \( \Sigma \) and \( \hat{\Sigma} \) respectively. According to Theorem 3.2.3, the first Markov parameters of \( \Sigma^L \) are matched, as are the main diagonal and upper triangular parts of the second Markov parameters. Expanding the transfer function of any LTI system \( H(z) \) in Laurent series for infinitely large \( z \) as in the definition of Markov parameters:

\[
H(z) = D + CBz^{-1} + CABz^{-2} + \cdots + CA^{k-1}Bz^{-k} + \cdots, \quad (3.2.40)
\]

we get

\[
\hat{H}^L(z) = \hat{D}_0 + \hat{H}^L(1)z^{-1} + \hat{H}^L(2)z^{-2} + \cdots
\]

\[
\Rightarrow \hat{H}^L(z^2) = \hat{D}_0 + \hat{H}^L(1)z^{-2} + \hat{H}^L(2)z^{-4} + \cdots
\]

\[
= \begin{bmatrix} \hat{D}_0 \\ \hat{C}_1\hat{B}_0 \hat{D}_1 \end{bmatrix} + \begin{bmatrix} \hat{h}^L_{11} & \hat{h}^L_{12} \\ \hat{h}^L_{21} & \hat{h}^L_{22} \end{bmatrix} z^{-2} + \begin{bmatrix} \hat{h}^L_{11} & \hat{h}^L_{12} \\ \hat{h}^L_{21} & \hat{h}^L_{22} \end{bmatrix} z^{-4} + \cdots
\]

so that

\[
\hat{H}^C(z) = \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \hat{H}^L(z^2) \begin{bmatrix} 1 \\ z \end{bmatrix}
\]

\[
= \begin{bmatrix} D_0 \\ z^{-1}C_1B_0 \end{bmatrix} + \begin{bmatrix} h^{L(1)}_{11} & h^{L(1)}_{12} \\ h^{L(1)}_{21} & h^{L(1)}_{22} \end{bmatrix} z^{-2} + \begin{bmatrix} h^{L(2)}_{11} & h^{L(2)}_{12} \\ h^{L(2)}_{21} & h^{L(2)}_{22} \end{bmatrix} z^{-4} + \cdots
\]

\[
= \begin{bmatrix} D_0 \\ D_1 \end{bmatrix} + \begin{bmatrix} h^{L(1)}_{12} \\ C_1B_0 \end{bmatrix} z^{-1} + \begin{bmatrix} h^{L(1)}_{11} \\ h^{L(1)}_{22} \end{bmatrix} z^{-2} + \begin{bmatrix} h^{L(2)}_{11} \\ h^{L(2)}_{22} \end{bmatrix} z^{-4} + \begin{bmatrix} x \end{bmatrix} z^{-5}.
\]
This demonstrates why the first $Kl = 4$ Markov parameters of $\Sigma^C$ are matched while the 5th is not.

Since the cyclic LTI representation of an LPTV system is an LTI system, Krylov-based model reduction can be accomplished as discussed in Theorem 3.2.1 and Theorem 3.2.2. However, this approach is not applicable because the reduced-order $A^C$, $B^C$, $C^C$ matrices lose the form of cyclic LTI representations. In the following, we will study how to overcome this by state transformations.

The QR factorization of the rank-$l$ reachability matrix of a cyclic lifted LTI system is

$$R^C_l = \begin{bmatrix} B^C & A^C B^C & \cdots & (A^C)^{l-1}B^C \end{bmatrix} = VR. \quad (3.2.41)$$

From observation, $V$ can be diagonalized through column transformation as $V_{gonal}^D = VP^T$, where $V^D = diag\{V^D_0, V^D_1, \ldots, V^D_{k-1}\}$, and each block $V^D_k$ is an ortho basis of $\tilde{G}_k$, which contains the first $l$ blocks of $G_k$, the infinite reachability matrix at time $k$ defined in (2.3.13). $P$ is an orthonormal matrix. The reduced-order system $\hat{\Sigma}^C$ computed below then matches first $l$ Markov parameters of the full-order cyclic LTI system:

$$\hat{\Sigma}^C = \begin{pmatrix} \hat{A}^C & \hat{B}^C \\ \hat{C}^C & \hat{D}^C \end{pmatrix} = \begin{pmatrix} V^*A^CV & V^*B^C \\ C^CV & D^C \end{pmatrix} = \begin{pmatrix} P^*V^D A^C V^D P & P^*V^D B^C \\ C^CV^D P & D^C \end{pmatrix} \quad (3.2.42)$$

Since $V^D$ is block diagonal, the system $\hat{\Sigma}^C = \begin{pmatrix} \hat{A}^C & \hat{B}^C \\ \hat{C}^C & \hat{D}^C \end{pmatrix} := \begin{pmatrix} V^D A^C V^D & V^D B^C \\ C^C V^D & D^C \end{pmatrix}$ maintains the structure of being as a cyclic lifted LTI representation and has the same transfer function matrix as $\hat{\Sigma}^C$.

**Theorem 3.2.5** Given a discrete-time LPTV system (1.1.2) and an integer $1 \leq l \ll n$, let $V^D_k$ denote the orthonormal basis of the first $l$ blocks of the reachability matrix
of the LPTV system (2.3.13) at time $k$. Define a reduced-order LPTV system as

$$
\begin{pmatrix}
\bar{A}_k^L & \bar{B}_k^L \\
\bar{C}_k^L & \bar{D}_k^L
\end{pmatrix} =
\begin{pmatrix}
V_k^D & \bar{A}_k^L V_{k-1}^D \\
C_k^L V_{k-1}^D & D_k^L
\end{pmatrix}.
$$

(3.2.43)

Let $\Sigma^C$ and $\bar{\Sigma}^C$ denote the cyclic LTI representations lifted from the full-order and reduced-order LPTV systems respectively. Then the first $l$ Markov parameters of $\Sigma^C$ and $\bar{\Sigma}^C$ are the same.

Remark 3.2.3 This approach is equivalent to that in Proposition 3.2.1 in terms of the number of operations and storage needed to match the first $l$ Markov parameters of the full-order cyclic lifted LTI systems.
Chapter 4

Periodic Realization

In this part of the thesis, we will assume the input-output relationship, i.e., the external expression of a linear discrete-time system is given, and the problem is to determine whether there exists a periodic realization and, if so, how to compute it.

4.1 Existence of realization of LPTV systems

Definition 4.1.1 1. Given a transfer function matrix $H(z) \in \mathbb{C}^{pK \times mK}$, the quadruple $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is a period-$K$ LPTV realization of $H(z)$ if, for at least one integer $k$,

$$\Omega^k(z, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \equiv H(z),$$

where $\Omega^k(z, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is a class of proper rational matrices as defined in Definition 2.1.1.

2. A $K$-periodic realization of $H(z) \in \mathbb{C}^{pK \times mK}$ with dimension $\{n_k\}_{k=0}^{K-1}$ is minimal if $n_k \leq \bar{n}_k$ for all $k \in [0, K - 1]$, where $\{\bar{n}_k\}_{k=0}^{K-1}$ is the dimension of any other $K$-periodic realization of $H(z) \in \mathbb{C}^{pK \times mK}$.

3. A $K$-periodic realization of $H(z) \in \mathbb{C}^{pK \times mK}$ with dimension $\{n_k\}_{k=0}^{K-1}$ is quasi-minimal if, for at least one time $\bar{k} \in [0, K - 1]$, $n_{\bar{k}} \leq \bar{n}_{\bar{k}}$, where $\{\bar{n}_k\}_{k=0}^{K-1}$ is the dimension of any other $K$-periodic realization of $H(z) \in \mathbb{C}^{pK \times mK}$.
4. A $K$-periodic realization of $H(z) \in \mathbb{C}^{pK \times mK}$ with dimension $\{n_k\}_{k=0}^{K-1}$ is uniform if all the $n_k$'s for $k \in [0, K-1]$ are equal.

In §2.3.1 we established the correspondence between reachability and observability of a discrete-time LPTV system and its standard lifted LTI representation, which determines the following statement naturally.

**Lemma 4.1.1**  
1. A $K$-periodic realization of $H(z) \in \mathbb{C}^{pK \times mK}$ is quasi-minimal if and only if its standard lifted LTI system $\Sigma_k^L$ at at least one initial sampling time instant $k$, is minimal.

2. A $K$-periodic realization of $H(z) \in \mathbb{C}^{pK \times mK}$ is minimal if and only if its standard lifted LTI systems at all initial sampling time instants, are minimal.

Before quoting the well-recognized theory on the existence of a $K$-periodic realization, we will firstly define the Hankel matrix of the standard lifted LTI systems.

**Definition 4.1.2** The Hankel matrix of order $r$, associated with a rational transfer function matrix $H(z)$, denoted by $\Phi^L_k(r)$, is defined as

\[
\Phi^L_k(r) = \begin{bmatrix}
J^L_k(1) & J^L_k(2) & \cdots & J^L_k(r) \\
J^L_k(2) & J^L_k(3) & \cdots & J^L_k(r+1) \\
\vdots & \vdots & \ddots & \vdots \\
J^L_k(r) & J^L_k(r+1) & \cdots & J^L_k(2r-1)
\end{bmatrix},
\]

where $J^L_k(q)$ for $q \in \mathbb{Z}^+$ are the Markov parameters associated with $H(z) \in \mathbb{C}^{pK \times mK}$, i.e., the coefficients of the Laurent expansion of $H(z)$ around infinity.

As is well known from the realization theory of LTI systems\cite{17,20,32}, the rank of $\Phi^L_k(r)$ does not increase for $r \geq \rho_k$, with $\rho_k$ being the degree of the least common
multiple of all denominators of $H(z)$. And we already know that $|\rho_k - \rho_0| \leq 1$, for $k = 0, 1, \ldots, K - 1$, so letting $\bar{\tau} = \rho_0 + 1$, it follows that for all $\tau \geq \bar{\tau}$ and $k = 0, 1, \ldots, K - 1$, $\text{rank}(\Phi_k^L(\tau)) = \text{rank}(\Phi_k^L(\bar{\tau}))$.

**Theorem 4.1.1** Given a transfer function matrix $H(z) \in \mathbb{C}^{p \times m \times K}$,

1. there exists a $K$-periodic minimal realization $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ if and only if $H(z) \in \chi(p, m, K)$;

2. there exists a $K$-periodic uniform minimal realization $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ if and only if $H(z) \in \chi(p, m, K)$ and the Hankel matrix $\Phi_k(\rho_0 + 1)$ has constant rank for $k = 0, 1, \ldots, K - 1$.

### 4.2 Construction of minimal/quasi-minimal periodic realizations

In the following we will demonstrate approaches of periodic realization on our simple period-3 discrete-time LPTV systems and then present the general algorithm for any period-$K$ discrete-time LPTV systems.

#### 4.2.1 An easy way to consider periodic realization problems

Suppose we are given a transfer function $H(z) \in \chi(p, m, K)$ which is defined as a class of proper rational matrices $H(z) = \{H_{ij}(z) \in \mathbb{C}^{p \times m}\}_{i,j=1}^{K}$, with $H_{ij}(\infty) = 0$, for $i < j$ and $i, j = 1, \ldots, K$. And we have the information that it is the transfer function matrix of the standard lifted LTI representation of a period-$K$ discrete-time LPTV system. Thus we can get an LTI realization using Silverman's realization method [32]. Besides this, taking into account the relationship between the standard lifted LTI systems noted in (2.1.3), we can compute the other two transfer functions and
the other two LTI realizations similarly. Thus we end up with three quadruples

\[
\begin{bmatrix}
  A_0^L & B_0^L \\
  C_0^L & D_0^L
\end{bmatrix};
\begin{bmatrix}
  A_1^L & B_1^L \\
  C_1^L & D_1^L
\end{bmatrix};
\begin{bmatrix}
  A_2^L & B_2^L \\
  C_2^L & D_2^L
\end{bmatrix}.
\]

Considering the standard lifted LTI form

\[
\begin{bmatrix}
  A_0^L & B_0^L \\
  C_0^L & D_0^L
\end{bmatrix} = \begin{bmatrix}
  A_2 A_1 A_0 & A_2 A_1 B_0 & A_2 B_1 & B_2 \\
  C_0 & D_0 \\
  C_1 A_0 & C_1 B_0 & D_1 \\
  C_2 A_1 A_0 & C_2 A_1 B_0 & C_2 B_1 & D_2 
\end{bmatrix}
\]

\[
\begin{bmatrix}
  A_1^L & B_1^L \\
  C_1^L & D_1^L
\end{bmatrix} = \begin{bmatrix}
  A_0 A_2 A_1 & A_0 A_2 B_1 & A_0 B_2 & B_0 \\
  C_1 & D_1 \\
  C_2 A_1 & C_2 B_1 & D_2 \\
  C_0 A_2 A_1 & C_0 A_2 B_1 & C_0 B_2 & D_0 
\end{bmatrix}
\]

\[
\begin{bmatrix}
  A_2^L & B_2^L \\
  C_2^L & D_2^L
\end{bmatrix} = \begin{bmatrix}
  A_1 A_0 A_2 & A_1 A_0 B_2 & A_1 B_0 & B_1 \\
  C_2 & D_2 \\
  C_0 A_2 & C_0 B_2 & D_0 \\
  C_1 A_0 A_2 & C_1 A_0 B_2 & C_1 B_0 & D_1 
\end{bmatrix},
\]

we can get \( \begin{bmatrix} B_0 \\ C_0 \\ D_0 \end{bmatrix}, \begin{bmatrix} B_1 \\ C_1 \\ D_1 \end{bmatrix}, \begin{bmatrix} B_2 \\ C_2 \\ D_2 \end{bmatrix} \) by partitioning the system matrices of the LTI systems, and the partitioned block size only depends on the period \( K \). And the beauty of this method is that the dimensions of the minimal LPTV realizations are just the dimensions of those minimal LTI realizations.
Let us partition the $C^L$ matrices as

$$
C^L_0 = \begin{bmatrix}
C_{01} \\
C_{02} \\
C_{03}
\end{bmatrix}, \quad C^L_1 = \begin{bmatrix}
C_{11} \\
C_{12} \\
C_{13}
\end{bmatrix}, \quad C^L_2 = \begin{bmatrix}
C_{21} \\
C_{22} \\
C_{23}
\end{bmatrix}.
$$

We need solve following equations to get the $A$ matrices of original LPTV system:

$$
C_1A_0 = C_{02},
C_2A_1 = C_{12},
C_0A_2 = C_{22}.
$$

The only problem is that there are other constraints on the $A$ matrices and we need to prove that such $A$ solutions do exist.

**Lemma 4.2.1 Matrix equations**

$$
UX = \overline{U}, \quad XV = \overline{V} \tag{4.2.2}
$$

where $U \in \mathbb{R}^{m_1 \times n_1}$, $\overline{U} \in \mathbb{R}^{m_1 \times n_2}$, $V \in \mathbb{R}^{n_2 \times m_2}$, $\overline{V} \in \mathbb{R}^{n_1 \times m_2}$, has a solution $X \in \mathbb{R}^{n_1 \times n_2}$ if and only if

i) $\text{range}(\overline{U}) \subseteq \text{range}(U)$, $\text{range}(\overline{V}^T) \subseteq \text{range}(V^T)$

$$
\tag{4.2.3}
$$

and ii) $U\overline{V} = \overline{U}V$

Proof of this lemma can be found in [25].
It is easy to check that the two properties are both satisfied in this situation, so solutions always exist.

A disadvantage of this method is also obvious. It is not numerically efficient, especially when the period of the system gets very large.

### 4.2.2 Quasi-minimal uniform periodic realization

Just as in the last part, we are given a transfer function matrix \( H(z) \in \chi(p, m, K) \) which is in the class of proper rational matrices \( H(z) = \{ H_{ij}(z) \in \mathbb{C}^{p \times m} \}_{i,j=1}^{K} \), with \( H_{ij}(\infty) = 0 \), for \( i < j \) and \( i, j = 1, \ldots, K \). And we have the information that \( H(z) \) is the transfer function of the standard lifted LTI representation of a period-\( K \) discrete-time LPTV system. So we can get an LTI realization using Silverman’s realization method [32]. The difference is that in this part we only get one LTI realization and solve more sophisticated equations to get the original LPTV systems. This idea of quasi-minimal uniform periodic realization was firstly brought up by Lin and King [23]. For the minimal periodic realization, we refer the readers to Colaneri and Longhi [10].

Based on

\[
\begin{bmatrix}
A^L & B^L \\
C^L & D^L
\end{bmatrix} =
\begin{bmatrix}
A_0 & B_0 & B_1 & B_2 \\
C_0 & D_{00} & 0 & 0 \\
C_1 & D_{10} & B_{11} & 0 \\
C_2 & D_{20} & B_{21} & B_{22}
\end{bmatrix}
\begin{bmatrix}
A_2A_1A_0 & A_2A_1B_0 & A_2B_1 & B_2 \\
C_0 & D_0 & 0 & 0 \\
C_1A_0 & C_1B_0 & D_1 & 0 \\
C_2A_1A_0 & C_2A_1B_0 & C_2B_1 & D_2
\end{bmatrix}
\]
For a general period-\(K\) realization problem, we need to solve the following equations

\[
\begin{align*}
A_{K-1}A_{K-2}\ldots&\ldots A_1A_0 = \overline{A} \\
A_{K-2}A_k\ldots&\ldots A_{i+1}B_i = \overline{B}_i, \quad \text{for } i = 0,1,\ldots,K-2 \\
C_iA_{i-1}\ldots&\ldots A_1A_0 = \overline{C}_i, \quad \text{for } i = 1,\ldots,K-1 \\
C_iA_{i-1}\ldots&\ldots A_{j+1}B_j = \overline{D}_{ij}, \quad \text{for } i = 1,\ldots,K-1, j = 0,\ldots,i-1
\end{align*}
\] (4.2.4)

to get periodic system matrices.

**Theorem 4.2.1** Equation (4.2.4) has a solution \(\{A_k\}_{k=0}^{K-1}, \{B_k\}_{k=0}^{K-1}, \{C_k\}_{k=0}^{K-1}\) if and only if for \(i = 0,\ldots,K-2,\)

\[
\rho(K_i) \leq n,
\] (4.2.5)

where \(\rho(K_i)\) denotes the rank of \(K_i\) and

\[
K_i = \begin{bmatrix}
\overline{A} & \overline{B}_0 & \ldots & \overline{B}_i \\
\overline{C}_{K-1} & \overline{D}_{K-1,0} & \ldots & \overline{D}_{K-1,i} \\
\vdots & \vdots & \ddots & \vdots \\
\overline{C}_{i+1} & \overline{D}_{i+1,0} & \ldots & \overline{D}_{i+1,i}
\end{bmatrix}.
\] (4.2.6)

Proof of this theorem can be found in [23]. Also, if \(\overline{A}\) is not singular, then (4.2.4) has a solution if and only if

\[
\overline{D}_{ij} = \overline{C}_i(\overline{A})^{-1}\overline{B}_j
\] (4.2.7)

for all \(i = 0,\ldots,K-1, j = 0,\ldots,i-1\). If condition (4.2.5) is not satisfied, i.e., \(\rho(K_i) > n\) for some \(i\), then any \(K\)-periodic uniform quasi-minimal realization of \(H(z)\) must be of dimension \(\max_{0 \leq i \leq K-1} \rho(K_i)\). The following is an algorithm for computing
a $K$-periodic uniform quasi-minimal realization of a given transfer function matrix
$H(z) \in \mathbb{R}(z)^{pK \times mK}$.

1. Find a minimal realization $\{\overline{A}, \overline{B}, \overline{C}, \overline{D}\}$ of $G(z)$, where $\overline{A} \in \mathbb{R}^{n \times n}$;

2. For $i = 0, \ldots, K - 2$, form the matrix $K_i$ as defined in (4.2.6);

3. Compute $\overline{n} = \max\{\max_{0 \leq i \leq K - 1} \rho(K_i), n\}$;

4. If $\overline{n} > n$, then expand size of $\overline{A}$, row size of $\overline{B}$, and column size of $\overline{C}$ to $\overline{n}$ as

$$\overline{A} := \begin{pmatrix} 0 & 0 \\ 0 & \overline{A} \end{pmatrix}, \quad \overline{B} := \begin{bmatrix} 0 & \overline{B} \end{bmatrix}, \quad \overline{C} := \begin{bmatrix} 0 \\ \overline{C} \end{bmatrix}$$

and form the corresponding $K_i$ for $i = 0, \ldots, K - 2$;

5. Decompose $K_i$ into $U_i, V_i$ with $\rho(K_i) = \rho(U_i) = \rho(V_i)$;

6. Obtain $A_{K-1}, A_0, \{B_i\}_{i=0}^{K-2}$, and $\{C_i\}_{i=1}^{K-1}$;

7. For $i = 1, \ldots, K - 2$, solve equations to get $A_i$.

To make this algorithm clear, we write down $K_i$ and their decompositions for a period-3 LPTV system.

$$K_0 = \begin{bmatrix} \overline{A} & \overline{B}_0 \\ \overline{C}_2 & \overline{D}_{20} \\ \overline{C}_1 & \overline{D}_{10} \end{bmatrix} = \begin{bmatrix} A_2A_1A_0 & A_2A_1B_0 \\ C_2A_1A_0 & C_2A_1B_0 \\ C_1A_0 & C_1B_0 \end{bmatrix} = \begin{bmatrix} A_2A_1 \\ C_2A_1 \end{bmatrix} \begin{bmatrix} A_0 & B_0 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} \overline{A} & \overline{B}_0 & \overline{B}_1 \\ \overline{C}_2 & \overline{D}_{20} & \overline{D}_{21} \end{bmatrix} = \begin{bmatrix} A_2A_1A_0 & A_2A_1B_0 & A_2B_1 \\ C_2A_1A_0 & C_2A_1B_0 & C_2B_1 \end{bmatrix} = \begin{bmatrix} A_2 \\ C_2 \end{bmatrix} \begin{bmatrix} A_1A_0 & A_1B_0 \\ B_1 \end{bmatrix} \quad (4.2.8)$$
Example: Let

\[
H(z) = \frac{1}{z-1} \begin{bmatrix}
z + 2 & 4 & 1 \\
6z & 3z + 5 & 2 \\
9z & z + 11 & z + 2
\end{bmatrix}
\]

We wish to obtain a 3-periodic uniform quasi-minimal realization for \( H(z) \). Compute a minimal LTI realization as

\[
\bar{A} = 1, \bar{B} = \begin{bmatrix} 3 & 4 & 1 \end{bmatrix}, \bar{C} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \bar{D} = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 3 & 0 \\ 9 & 1 & 1 \end{bmatrix}.
\]

By step 2 and 3, form \( K_0 \) and \( K_1 \). By computations, \( \rho(K_0) = 1 \) and \( \rho(K_1) = 2 > 1 \).

By step 4, let

\[
\bar{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 4 & 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 3 & 0 \\ 9 & 1 & 1 \end{bmatrix}.
\]

Based on (4.2.8), form the corresponding \( K_0 \) and \( K_1 \). Obtain

\[
D_0 = 1, D_1 = 3, D_2 = 1, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
By step 5, decompose $K_0$, $K_1$ as

$$K_0 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 3 & 6 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} 0 & 0 \\ 1 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

By step 6,

$$A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix} , A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} , B_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} , B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 2 & 4 \end{bmatrix} , C_2 = \begin{bmatrix} 3 & 1 \end{bmatrix} .$$

By step 7, obtain $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. 
Chapter 5

Conclusions: Summary and Future Work

The main contributions of this thesis may be summarized in three parts.

(1) We surveyed the existing literature in the area of linear periodically time-varying systems since 1950s in terms of system analysis, model reduction and realization and their applications. References are also provided for the interested readers.

(2) We applied modified low-rank Smith method to LPTV systems to compute approximately reachability and observability gramians in a numerically efficient manner. The decay rate of eigenvalues of the gramians are also exploited and an eigen-decay bound and Cholesky approximation are derived.

(3) We present a Krylov-based moment matching algorithm for LPTV systems, which is numerically more efficient and stable compared to the existing balanced truncation approach.

Future work: Large-scale real-world problems are to be computed to test the algorithms derived in this thesis. For the computation of system gramians part, error bounds are to be computed for the approximations. For the model reduction part, the expansion of other Krylov-based methods, such as rational interpolation, is worth of attentions. Moreover, the general linear time-varying systems, which are LPTV systems with period goes to infinity, are of great interest to people in this area and are considered as the next step after the study of LPTV systems.
Bibliography


