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Transport Equation for Disordered Interacting electrons in 2D and Magnetic Metal

by

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ABSTRACT

Transport Equation for Disordered Interacting electrons in 2D and Magnetic Metal

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Jun Sun

We develop a transport formalism for interacting electrons in the presence of quenched disorder. Quantum effects on transport, due both to quantum interference and interaction effects, are incorporated through non-analytic terms in the irreducible interactions and appropriate contributions to the electron self-energy. Perturbatively, our approach recovers the standard results on quantum corrections to the Drude conductivity. We argue that the strong coupling fixed point is a magnetic metal beyond perturbation theory. Extensions of the theory are outlined.
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Chapter 1
Introduction and Motivation

The ground state phase of disordered interacting electrons in 2D is a long-standing problem, many work has been done in this field[1, 2, 3, 4, 5]. The Hamiltonian for such a system are composed of three parts, as shown in the following equation.

\[ H = H_0 + H_I + H_e \]

\[ H_0 = \sum_k \epsilon_k C_k^+ C_k \]

\[ H_I = \sum_{k,q} u(q) C_{k+q}^+ C_k \rho(q) \]

\[ H_e = \frac{1}{2} \sum_{k,k',q} v(q) C_{k+q}^+ C_{k'-q}^+ C_k C_k \]

(1.1)

The first part \( H_0 \) is just the kinetic energy of free electrons in 2D. The second part \( H_I \) is the energy due to scattering by impurities. The last part \( H_e \) is the electron-electron interaction. If only the impurity effect and the kinetic energy are taken into account, then an Anderson insulator will be obtained[2]. On the other hand, if only \( H_e \) is considered, apart from the kinetic energy, then, in the weak interaction limit, the system is just normal Fermi Liquid; and in the strong interaction limit, it leads to Wigner crystal [6], which is an insulator as well. It is very natural to ask that what the phase is, when both interaction and impurity effect are in presence. It is believed up to 1990’s that it is an insulating phase as well.
In 1994, Kravchenko et. al. found that the temperature dependence of resistivity for high mobility Si MOSFET is metallic-like in relatively higher density, as shown in Fig.B.1. The 2D system is located on the depletion and/or inversion layer in Si MOSFET, with the depletion layer width much less than the mean free path. Such metallic behavior was confirmed by other groups as well, using different materials and devices. Another interesting phenomenon observed in such experiments is that the magnetoresistivity is dramatically increased in the presence of parallel magnetic field, as shown in Fig.B.2. Aparently, the parallel magnetic field only couples to the spin fluctuation as opposed to the orbital fluctuation as in Quantum Hall Effect. The unexpected metal-insulator transition (MIT) and its sensitivity to magnetic field greatly excites us and leads us to the current research.

Our work is based on many preceededent theories in this field. The following is a brief review of what has been done in this field.

Fermi-liquid theory for interacting electrons weakly scattering off dilute impurities was derived by Betbeder-Matibet and Nozieres (B-MN hereafter)[1]. This theory is valid only when the scattering rate

\[ 1/\tau = 2\pi n_i|T|^2 N_F << E_f \]  

(1.2)

or when the mean-free path \( \ell \) satisfies \( k_f \ell >> 1 \). Here \( n_i \) is the density of impurities, \( |T|^2 \) is the average of the squared scattering matrix for the impurities, and \( N_F \) is the density of states at the Fermi-level. To this order, the essential difference between the
problem with and without impurities is that in the former, the Fermi-surface is not defined in momentum space. But since the energy \( \omega \) is conserved in the collisions, the Fermi-surface in energy space at \( \omega = \mu \), continues to be well defined. This introduces some technical differences in the derivation of Fermi-liquid theory compared to the pure case.

Altshuler, Aronov and co-workers have discovered that to order \( 1/k_f \ell \), logarithmic singularities appear in the scattering amplitude at low energies and momenta of impure interacting electrons in 2- dimensions[2]. The reason for the singularity is that the density correlation function for impure electrons is diffusive. The energy of the fluctuations \( \epsilon \) is therefore related to their momentum \( q \) as \( \epsilon \sim q^2 \). Recall that in the density correlation function for non-interacting electrons in the pure limit \( \epsilon \) scales as \( q \). For any interaction in the pure limit logarithmic singularities then arise only for one dimension. For the diffusive propagator the logarithmic singularity appears perturbatively in the interaction for \( d = 2 \). In two dimensions, to the same order in \( 1/k_f \ell \), non-interacting electrons also have the so-called weak-localization singularities which are also \( \sim ln\omega \)[2]. So the Fermi-surface is not well-defined even in \( \omega \)-space.

The most complete work for this problem is due to Finkelstein[3], who carried out a perturbative scaling analysis treating interaction effects to infinite orders and disorder effects to first order. As the temperature is lowered or, equivalently, the spatial scale is increased, the triplet interaction is found to scale towards infinity, the
conductivity to a finite value, and the electron density of states towards a pseudo-gap form. The run-away flow of the triplet amplitudes implies that the scaling analysis eventually fails at sufficiently low temperatures. The nature of the strong coupling fixed point remains an open problem.

In this thesis, we generalize the usual transport equation approach for impure interacting electrons to incorporate the singular effects associated with both diffusion and quantum interference. As with B-MN, we work with the standard vector vertex, which describes the coupling of a physical electron to an external vector field. The advantage of working with such a quantity is that, we do not need to know the elementary excitations of the system. In addition, there are Ward identities associated with these vector vertices that can be taken advantage of. Unlike B-MN, however, we find that the irreducible interactions have to be taken to be non-analytic in order to incorporate the diffusive and quantum interference effects.

Formally, this problem has its novel aspects. Singularities arise from irreducible vertices rather than from \((GG)_{\text{coherent}}\) as they do in Landau-Pomeranchuk instabilities or SDW,CDW or superconductivity. Such non-Fermi-liquids may be ubiquitous. The methods developed here may be useful in such problems, also can treat modification of landau-pomeranchuk singularities by disorder in \(d=3\), metal-insulator transitions, etc.

Physically, our chief motivation is to address the nature of the strong coupling fixed
point hinted by the weak-coupling scaling approach of Finkelstein. In this paper, we argue that the fixed point is a magnetic metal through applying the transport equation we developed. In addition, we extend our theory to the case where parallel magnetic field is in presence, which only couples to the spin of the system.

The remainder of this paper is organized as follows. In Chapter 2, we constructed the general Transport Equation for disordered interacting electrons in 2D. In Chapter 3, we apply the theory to the case where density of states (DOS) can be assumed as a constant, and test the validity of the Equation perturbatively. In chapter 4, we address the the ground state of such a 2D system in the critical regime by using our Transport Equation beyond perturbation, where DOS has a psuedo-gap. The extension of our theory is also outlined in this chapter. We summarize our conclusions in Chapter 5. In Appendix A, the calculation details for Ward Identity and conductivity are given and all the diagrams for this thesis are shown in Appendix B.
Chapter 2
Transport Equation

2.1 Green’s functions, irreducible interaction, and vertex equations

2.1.1 Single particle Green’s function

The disorder – averaged Green’s function $G(k, \omega)$ may be expressed in terms of the non-interacting eigenvalues of the pure system $\epsilon(k)$ and the self-energy $M(k, \omega)$:

$$G(k, \omega) = \frac{1}{\omega - \epsilon(k) - M(k, \omega)}.$$ \hspace{1cm} (2.1)

The chemical potential $\mu$ has been chosen zero. In $d = 2$ and for finite disorder, $M(k, \omega)$ has singularities for $\omega \to \mu$, so that $G(k, \omega)$ has branch-cuts rather than poles. Nevertheless, causality requires that

$$\text{Im} M(k, \omega) \sim \text{sgn}(\omega - \mu).$$ \hspace{1cm} (2.2)

We make extensive use of this property. It is standard to separate $G$ into a ‘coherent’ part and an ‘incoherent’ part for $\omega$ close to $\mu$. In the presence of singular diffusive corrections, we will no longer make such a separation.

2.1.2 Product of Two Green’s functions

Just as in the derivation of Fermi-liquid theory, it is important to separate the singularities in the products of the Green’s function $G(p + \omega/2)G(p - \omega/2)$. Here
and subsequently the following notation is used

\[ p = (k, \omega); \]
\[ \omega = (q, \epsilon), \]

so that in the particle-hole Green's function, see Fig.B.3, the total incoming momentum-energy is \( \omega \). The product can be separated as

\[ G(p + \omega/2)G(p - \omega/2) = \Phi_{inc}(p, \omega) + Q(p, \omega)y(\omega, \epsilon), \]  

(2.4)

where

\[ y(\omega, \epsilon) = \begin{cases} 
1, & \text{if } |\omega - \mu| < \epsilon/2; \\
0, & \text{if } |\omega - \mu| > \epsilon/2. 
\end{cases} \]  

(2.5)

Then the second part in Eq.(2.4) is non-zero only if the incoming particle and hole have energies on opposite sides of the chemical potential \( \mu \). For \( \epsilon \to 0 \), we can set \( y(\omega, \epsilon) = \epsilon \delta(\omega) \). For later purposes we also define \( \bar{y}(\omega, \epsilon) \), which is the complement of \( y(\omega, \epsilon) \); i.e. it is non-zero only if the incoming energies are on the same side of \( \mu \). Obviously,

\[ \bar{y}(\omega, \epsilon) + y(\omega, \epsilon) = 1 \]
\[ \bar{y}(\omega, \epsilon)y(\omega, \epsilon) = 0. \]  

(2.6)

In the decomposition Eq.(2.4)

\[ \Phi_{inc}(p, \omega) = \left( \frac{1}{-\epsilon_+ + \omega_+ - \text{Re} M_+ + i\text{Im} M} \right) \left( \frac{1}{-\epsilon_- + \omega_- - \text{Re} M_- + i\text{Im} M} \right) \]
\[ Q(p, \omega) = \left( \frac{1}{-\epsilon_+ + \omega_+ - \text{Re} M_+ + i \text{Im} M} \right) \left( \frac{1}{-\epsilon_- + \omega_- - \text{Re} M_- + i \text{Im} M} \right) \]
\[ - \left( \frac{1}{-\epsilon_+ + \omega_+ - \text{Me} M_- + i \text{Im} M} \right) \left( \frac{1}{-\epsilon_- + \omega_- - \text{Me} M_- + i \text{Im} M} \right) \]

(2.7)

As BMN pointed out, due to the sign change of imaginary part of the self-energy on fermi surface, the first term of Q has two poles in the complex \( \epsilon_k \) plane, located on opposite sides of the real-axis. Once the diffusion correction is included, the second term of Q as well as \( \phi_{\text{inc}} \) also contain have two poles on the opposite sides of the real-axis in the complex \( \epsilon_k \) plane.

### 2.1.3 Irreducible Interactions

In the Fermi liquid case, the irreducible interactions \( I(p, p', \omega) \) are analytic. Once the diffusive and interference singularities are included, this is no longer the case: the value of \( I(p, p', \omega) \) in the limit \( \omega_i \rightarrow \mu \) depends on whether \( \omega_i \) approaches \( \mu \) from the same or different sides of \( \mu \). We find it useful to separate the irreducible interactions \( I(p, p', \omega) \) into three parts,

\[ I(p, p', \omega) = I_1(p, p', \omega) + I_2(p, p', \omega)\delta(\omega - \omega') + I_3(p, p', \omega) \]

(2.8)

as perturbatively illustrated in Fig.B.7. \( I_2 \) is the part in which no interaction line connects the particle and the hole-lines directly, so that the energy of the particle and the hole lines is separately conserved. Therefore, \( I_2 \) has a \( \delta \) function attached to it.
The remaining part of the irreducible interaction contains interaction lines that connect the particle and the hole-lines directly. While the separation of this remaining part into $I_1$ and $I_3$ is somewhat arbitrary, $I_1$ is taken to be analytic while $I_3$ contains non-analytic contributions.

To treat the non-analyticities of $I_2$ and $I_3$, we further decompose them into

\begin{equation}
I_2(p, p', \varpi) = I_{21}(p, p', \varpi)\gamma(\omega, \epsilon) + I_{22}(p, p', \varpi)\bar{\gamma}(\omega, \epsilon) \tag{2.9}
\end{equation}

\begin{equation}
I_3(p, p', \varpi) = I_{31}(p, p', \varpi)\gamma(\omega, \epsilon)\gamma(\omega', \epsilon) + I_{32}(p, p', \varpi)\gamma(\omega, \epsilon)\bar{\gamma}(\omega', \epsilon) + I_{33}(p, p', \varpi)\bar{\gamma}(\omega, \epsilon)\gamma(\omega', \epsilon) + I_{34}(p, p', \varpi)\bar{\gamma}(\omega, \epsilon)\bar{\gamma}(\omega', \epsilon) \tag{2.10}
\end{equation}

The $\gamma$ and $\bar{\gamma}$ factors take care of how $\omega_i$ approach $\mu$. The remaining factors, $I_{21}$, $I_{22}$, $I_{31}$, $I_{32}$, $I_{33}$, and $I_{34}$, are all analytic functions.

2.1.4 Vertex Equations

The vertex for scattering is given in terms of the irreducible interactions $I$ and the single-particle Green's functions $G$ by, see Fig.??

\begin{equation}
\gamma(p, p', \varpi) = I(p, p', \varpi) + \sum_{p''} I(p, p'', \varpi)G(p'' - \varpi/2)G(p'' + \varpi/2)\gamma(p, p'', \varpi) \tag{2.11}
\end{equation}

Given an external perturbation $\lambda_\mu(K)$, where $K = (k, \sigma)$, the vertex to the external perturbation $\Lambda(P, \varpi)$ is given by

\begin{equation}
\Lambda_\mu = \lambda_\mu + \lambda_\mu GG\gamma \tag{2.12}
\end{equation}
which can be rewritten using Eq. (2.11) for $\gamma$ as

$$\Lambda_{\mu} = \lambda_{\mu} + \Lambda_{\mu} G G I.$$  \hspace{1cm} (2.13)

The correlation functions $S_{\mu\nu}(\sigma)$, see Fig.B.5, are given by

$$S_{\mu\nu} = \lambda_{\mu} G G \Lambda_{\nu}.$$  \hspace{1cm} (2.14)

So far, we give the vertex equations in general. Nevertheless, in the current case, where the irreducible interaction has a strong singularity $\delta(\omega - \omega')$, the vertex as the product of Green functions has nonanalyticity in the vicinity of Fermi surface as well. Therefore, we make the decomposition.

$$\Lambda_{\mu} = \Lambda_{\mu 1} + \Lambda_{\mu 2} y (\omega + \epsilon)$$  \hspace{1cm} (2.15)

In next section, we will take advantage of this, though seemingly it makes the problem complicated.

2.2 Ward Identities and limiting form of Correlation functions

2.2.1 Ward Identities

The limiting forms of the vertex functions, $\Lambda_{\mu}(p, r)$, where $r = \frac{q}{\epsilon}$, as well as the correlation functions $S_{\mu\nu}(r)$ for $r = 0$ and $r = \infty$ will now be considered. The discussion of BMN about the Ward identities connecting the limiting forms of the
Vertex functions to the derivatives of the self-energy carries through, even for the cases that the irreducible vertices have singularities provided that these singularities at $\epsilon = 0$ are functions of $(\omega - \omega')$, and at $q = 0$ are functions of $(k - k')$. As in BMN, not all the Ward-identities for the pure system are preserved for the problem with disorder.

The scalar vertex $\Lambda_4(p, 0)$ for $q = 0$ in the limit $\epsilon \to 0$, coupling to density fluctuations follows

$$\Lambda_4(p, 0) = 1 - \frac{\partial M(p)}{\partial \omega}.$$  \hspace{1cm} (2.16)

The vector vertex in the opposite limit, for $\epsilon = 0$ and the limit $q_\alpha \to 0$, coupling to velocity in the $\alpha$-th direction follows

$$\Lambda_\alpha(p, \infty) = \frac{k_\alpha}{m} + \frac{\partial M(p)}{\partial k_\alpha} \equiv \dot{z}(p) \frac{k_\alpha}{m},$$ \hspace{1cm} (2.17)

where $\dot{z} = 1 + \frac{\partial M(p)}{\partial k}$ is the ratio of the bare density of states to the renormalised density of states $\nu_0/\nu$. Therefore

$$\Lambda_\alpha(p, \infty) = \frac{k_\alpha}{m} \left(\frac{\nu_0}{\nu}\right)$$ \hspace{1cm} (2.18)

Just as we mentioned in previous section, in the case of non-Fermi liquid, the derivative of self-energy will be singular, so does the limiting form of the vertices. Moreover, at different limits, the scalar vertex is one order more singular than the vector vertex.
Due to the impurities current is not conserved. Therefore no Ward-identity exists for the current vertex, \( \Lambda_\alpha(p, 0) \), i.e. the vertex for \( q = 0 \) in the limit \( \epsilon \to 0 \). However, it is easy to see from the general equation for the vertex in the following that, if the irreducible vertex does not provide a singularity stronger than \( \epsilon^{-1} \), then

\[
\Lambda_\alpha(p, \infty) = \Lambda_\alpha(p, 0). \tag{2.19}
\]

In the pure system Galilean invariance enforces that \( \Lambda_\alpha(p, \infty) \sim p_\alpha/m \) while \( \Lambda_\alpha(p, 0) \sim p_\alpha/m^* \). In the impure system the two as proved as follows are equal.

We start with the general ward identity, which is required by gauge invariance[6].

\[
\epsilon \cdot \Lambda_4(k, \omega; q, \epsilon) - q_\alpha \cdot \Lambda_\alpha(k, \omega; q, \epsilon) = \epsilon - q_\alpha \cdot k_\alpha - (M(k + q, \omega + \epsilon) - M(k, \omega)) \tag{2.20}
\]

Apparently, we could obtain the two aforementioned ward identities from this by choosing either the static limit, i.e., \( \epsilon = 0, q \to 0 \), or the dynamic limit, i.e. \( q = 0, \epsilon \to 0 \). In order to arrive at Eq.(2.17), one need to assume \( \Lambda_4 \) is no more singular than \( \epsilon^{-1} \) in the static limit. Similarly, to deduce Eq.(2.16), it is assumed that \( \Lambda_\alpha \) has singularity softer than \( q^{-1} \). The two assumptions are easily to been seen satisfied through analysing the singularity of the irreducible interaction vertices.

Now, if we take derivative with respect to \( q_\beta \) of Eq.(2.20) without approaching any limit, and then let \( q = 0 \), we reach
\[ \Lambda_1(k, \omega; q = 0, \epsilon \to 0) = (1 + \frac{\partial M}{\partial \epsilon_k}) \Lambda^0 \] (2.21)

\[ \Lambda_2(k, \omega; q = 0, \epsilon) y(\omega, \epsilon) = \epsilon \frac{\partial \Lambda_4(k, \omega; q, \epsilon)}{\partial q} \bigg|_{q=0} \] (2.22)

In fact, there is one more term in the derivative, \( q_a \partial \Lambda_{\alpha} / \partial q_\beta \), which we believe is negligible as long as the \( \partial \Lambda_{\alpha} / \partial q_\beta \) has no singularity in \( q \) stronger than \( q^{-1} \). In the end, we let \( \epsilon \to 0 \), and we arrive at that the vector vertex is invariant in different limiting cases provided that \( \Lambda_4 \) is no more singular than \( \epsilon^{-1} \), which we used in deducing the first ward identity.

In the perturbation approach, where both interaction and the disorder\((1/g)\) are taken to the first order, ward identities, i.e. Eq.(2.16), Eq.(2.17), and Eq.(2.22) are proved explicitly by computing both sides to the leading singularity, the details are shown in the appendix.

### 2.2.2 Correlation Functions

**Density-density correlation function**

Given that the Ward-identity Eq.(2.16) holds, the procedure in [5] (Page 277) may be used to prove that the limiting form of the correlation function \( S_{44}(r = 0) \) is always zero.

\[ S_{44}(0) = 0. \] (2.23)
This follows directly from the Gauge Invariance or equivalently the continuity equation.

**Current-Current Correlation Functions**

For pure Fermi-liquids as well as for those in which the singular effects of the impurities are neglected, the dynamical vertices are calculated by first deriving an equation for the static vertices. This procedure is not possible when the irreducible interactions are singular. To proceed, we first define auxiliary vertices, \( \tilde{A} \) and Interactions \( \tilde{\gamma} \) which satisfy

\[
\tilde{A}(k, \omega; q, \epsilon) = \lambda(q, \epsilon) + \int d{k'}d{\omega'} I(k, \omega, k', \omega'; q, \epsilon) \Phi_{inc}(k', \omega') \tilde{A}(k, \omega; q, \epsilon)
\]

\[
\tilde{\gamma} = I + \int I \Phi_{inc} \tilde{\gamma}
\]

which leads to

\[
\tilde{\Lambda} = \lambda + \lambda \Phi_{inc} \tilde{\gamma}
\]

The actual vertex satisfies,

\[
\Lambda = \tilde{\Lambda} + \tilde{\gamma} Q y(\omega', \epsilon) \Lambda
\]

as shown in Fig.B.6.

We also define auxiliary Correlation functions

\[
\tilde{\mathcal{S}}_{\alpha\beta} = \lambda_{\alpha} \Phi_{inc} \tilde{\Lambda}_{\beta}
\]
in terms of which the actual correlation function is given by, as shown in Fig.B.6,

\[ S_{a\beta} = \tilde{S}_{a\beta} + \tilde{\Lambda}_a Q_y(\omega, \epsilon) \Lambda_\beta \]  

(2.28)

Now we make the decomposition as we do with the actual vertex,

\[ \tilde{\Lambda} = \tilde{\Lambda}_1 + \tilde{\Lambda}_2 y(\omega, \epsilon) \]  

(2.29)

The auxilliary reducible interaction vertices share the same non-analytic form as the original irreducible interaction vertices. Now, we plug them into Eq.(2.26) and note step function in it, we may rewrite the equation as

\[
\begin{align*}
\Lambda_1 + \Lambda_2 y(\omega, \epsilon) &= \tilde{\Lambda}_1 + \tilde{\Lambda}_2 y(\omega, \epsilon) \\
&+ \int dk'd\omega'[\tilde{\gamma}_1 + \tilde{\gamma}_{31} y(\omega, \epsilon) + \tilde{\gamma}_{33} g(\omega, \epsilon)]Q(k', \omega'; q, \epsilon)y(\omega', \epsilon)(\Lambda_1 + \Lambda_2) \\
&+ \int dk'\tilde{\gamma}_{21} y(\omega, \epsilon)Q(k', \omega; q, \epsilon)(\Lambda_1 + \Lambda_2)
\end{align*}
\]  

(2.30)

Then, we find the relations between the actual vertices and the auxilliary vertices.

\[
\begin{align*}
\Lambda_2 &= \tilde{\Lambda}_2 + \int dk'd\omega' (\tilde{\gamma}_{31} - \tilde{\gamma}_{33}) Q_y(\omega', \epsilon)(\Lambda_1 + \Lambda_2) \\
&+ \int dk'\tilde{\gamma}_{21} Q(\Lambda_1 + \Lambda_2) \\
&= \tilde{\Lambda}_2 + \int dk'\tilde{\gamma}_{21} Q(\Lambda_1 + \Lambda_2) + O(\epsilon) \\
\Lambda_1 &= \tilde{\Lambda}_1 + \int dk' (\tilde{\gamma}_1 + \tilde{\gamma}_{33}) Q(\Lambda_1 + \Lambda_2) \epsilon
\end{align*}
\]  

(2.31)
DC Conductivity

The DC conductivity is related to current-current correlation function by the Kubo formula,

\[
\sigma = \lim_{\epsilon \to 0} \frac{i}{\epsilon} (S_{\mu\nu}(q = 0, \epsilon) + \frac{n_0}{m} \delta_{\mu\nu})
\]  

(2.33)

where, \( n_0 \) is electron density and the second term come from the difference between the canonical momentum and real momentum and plays trivial role here.

Inserting the expression for auxilliary vertices into the current-current correlation function, or Eq.(2.28), and using the aforementioned Kubo formula, we obtain the DC-conductivity at zero temperature:

\[
\sigma = \sigma_1 + \sigma_2
\]

(2.34)

where

\[
\sigma_1 = -Im \lim_{\epsilon \to 0} \lim_{q \to 0} \int dk d\omega \Lambda^0 \frac{\partial}{\partial \epsilon} [\Phi_{inc} \tilde{A}_1]
\]

\[
+ -Im \lim_{\epsilon \to 0} \lim_{q \to 0} \int dk \Lambda^0 \Phi_{inc} \tilde{A}_2
\]

\[
\sigma_2 = Im \lim_{\epsilon \to 0} \lim_{q \to 0} \int dk (Q \tilde{\gamma}_{21} - 1) Q (\Lambda_1 + \Lambda_2)^2
\]

(2.35)

where, \( \Lambda^0 \) is the bare vector vertex. Eq. (2.35) is the key result of our approach. It will be used to address the nature of the the strong coupling limit hinted at the
weak-coupling scaling analysis of Finkelstein in the last section. In next section, however, we will show that our formalism recover the standard results when disorder and interactions are treated to the leading orders of the perturbation theory.
Chapter 3
Perturbative Results

We now apply this formalism to a perturbation case, where both the interaction and disorder (1/g) are taken to the first order. We will consider the case of short-range interactions. The self-energy in this case and in the vicinity of \( \omega = 0 \) is shown in Fig.B.4 or

\[
M(k, \omega) = -\frac{i}{2\tau} \text{sgn}(\omega) - \frac{V_1}{2g^2} \ln \left| \frac{1}{\omega \tau} \right| G_0(k, \omega; -\text{sgn}(\omega)) + \frac{iV_1 k^2 G_0^3}{2g^2} \omega \ln \left| \frac{1}{\omega \tau} \right|
\]

\[
+ \frac{iV_1}{2g^2} \ln \left| \frac{1}{\omega \tau} \right| \text{sgn}(\omega) + \frac{3V_1}{2g} \omega \ln \left| \frac{1}{\omega \tau} \right|
\]

\[
= -\frac{i}{2\tau} \text{sgn}(\omega) - \frac{V_1}{2g^2} \ln \left| \frac{1}{\omega \tau} \right| G_0(k, \omega; -\text{sgn}(\omega))
\]

\[
+ \frac{iV_1}{2g^2} \ln \left| \frac{1}{\omega \tau} \right| \text{sgn}(\omega) + O(\omega)
\]  

(3.1)

where, \( V_1 = V_0 N_F \), \( g = 4\pi^2 \sigma_0 \), and \( G_0 \) is the non-interacting Green function in the presence of disorder (and with disorder treated in Born approximation). The existence of \( G_0 \) in self-energy actually gives extra pole in \( \epsilon_k \) space. This is an important distinction diffusive corrections bring in.

We also need to know what the irreducible interaction vertices look like perturbatively, in order to compute the conductivity. To the first order in interaction and 1/g, we give all the diagrams of irreducible interaction in Fig.B.7. Where, the diffuson, \( I^d_{21} \), and the cooperon, \( I^c_{21} \), are well known particle-hole propagators[5]. Here, we only give the expression in the regime \( \theta(-\omega(\omega + \epsilon)) = 1 \). Hereafter, the definition of \( y(\omega, \epsilon) \)
would be changed to \( y(\omega, \epsilon) = \theta(-\omega(\omega + \epsilon)) \) for simplicity.

\[
I_{21}^\delta(p, p'; \omega) = \frac{1}{2\pi N_F T^2 (-i |\epsilon| + Dq^2)} \quad (3.2)
\]

\[
I_{21}^\epsilon((p, p'; \omega) = \frac{1}{2g_T^2} ln[\frac{1}{\epsilon_T}] \delta(K + K' + q) \quad (3.3)
\]

The diffusion-corrected interaction shown in Fig.B.7 can, as well be given, but we only give here those will be used in our further calculation. They are as follows.

We will compute the vector vertices, therefore only the p-wave component of the irreducible interaction is to our concern, denoted by \( l = 1 \).

\[
I_{22,l=1}^{--;--}(k, k', \omega; q, \epsilon) = \frac{(G_{0+}(k, \omega)G_{0+}(k', \omega)(\frac{k_x+q_x}{m})\frac{k'_x+q_x}{m})}{2\pi N_F T^4} \cdot \frac{-V_i}{8gD} ln[\frac{1}{-(\omega + \epsilon_T)}] \\
I_{34,l=1}^{--;++}(k, k', \omega; q, \epsilon) = \frac{G_{0+}(k, \omega)G_{0-}(k, \omega)(\frac{k_x+q_x}{m})\frac{k'_x}{m}}{2\pi N_F T^4} \cdot \frac{V_i}{8gD} ln[\frac{1}{-\omega_T}] \delta(k' - k)^{l=1} \quad (3.4)
\]

The two terms in \( I_{34} \) corresponds to the two diagrams in Fig.B.7. The RHS of \( I_{22} \) and the second term in \( I_{34} \), in fact need to be doubled due to interchanging the particle and hole line in the diagram of the vertices. The \( \delta \) function regarding \( \omega \) and \( \omega' \) in the above expression, means that the logarithmic corrections comes from scattering, where the energy change is small but does make the particle propagating on the opposite side of fermi surface.

We will first find out the relationship between \( \tilde{\Lambda}_\alpha \) and \( \Lambda_\alpha \).
\[ \tilde{\Lambda}_\alpha = \Lambda_\alpha - \int dk'd\omega'[I_{31}(\omega, \epsilon) + I_{33}\tilde{y}(\omega, \epsilon) + I_{21}\delta(\omega - \omega')]_{t=1}^{(1)} Q^{(0)} y(\omega', \epsilon)\Lambda_\mu^{(0)} + O(V^2), \]

where (1) means both interaction and disorder are taken to first order. Since we are dealing with vector vertices, only the p-wave component of the interaction vertex contributes. Furthermore, we are interested only in first order of interaction, 1/g and logarithmic singularities. Taking all of this into account, we find that

\[ \tilde{\Lambda}_\alpha = \Lambda_\alpha - y(\omega, \epsilon) \int dk'I_{21, t=1}^c Q^{(0)} \Lambda_\alpha^{(0)} + O(V^2) \]

where, \( I_{21, t=1}^c \) represents the cooperon in \( I_{21} \), which makes a contribution in the order of 1/g ln.

We then solve for \( \Lambda_\alpha \) to first order of \( V_1 \) and 1/g:

\[
\Lambda_\alpha = \Lambda_\alpha^{(0)} + I_{t=1}^{(1)} G G \Lambda_\alpha^{(0)} \\
= \Lambda_\alpha^{(0)} + \tilde{y}(\omega, \epsilon)\Lambda_{\alpha 34} + \tilde{y}(\omega, \epsilon)\Lambda_{\alpha 22} + y(\omega, \epsilon)\Lambda_{\alpha 21}^{c} \]

where,

\[
\Lambda_{\alpha 34} = \int dk'd\omega' I_{34}\tilde{y}G_0 G_0 \Lambda_\alpha^{(0)} \]
\[ \Lambda_{a22} = \int d\mathbf{k}' I_{22} \tilde{y} G_0 G_0 \Lambda^{(0)}_a \] (3.9)

\[ \Lambda_{a21}^c = \int d\mathbf{k}' I_{21}^c G_0 G_0 \Lambda^{(0)}_a \] (3.10)

The reason we neglect some terms in Eq.(3.7) from line one to line three is due to that in the limit of \( q \to 0 \), the p-wave components of those interactions do not give contribution as singular as logarithmic. The diagrams for \( \Lambda_{a34}, \Lambda_{a22} \) and \( \Lambda_{a21}^c \) are shown in Fig.B.8, and details are given in appendix. We, then, have the explicit expression for \( \tilde{\Lambda}_1 \) and \( \tilde{\Lambda}_2 \) perturbatively.

\[ \tilde{\Lambda}_1 = \Lambda_1 = \Lambda^{(0)}_a + \Lambda_{a34} + \Lambda_{a22} \]

\[ \tilde{\Lambda}_2 = \Lambda_2 - \int d\mathbf{k}' I_{21}^c Q^{(0)} \Lambda^{(0)}_a = \Lambda_{a21}^c (\Phi_{inc}) - \Lambda_{a34} - \Lambda_{a22} \] (3.11)

where, \( \Lambda_{a21}^c (\Phi_{inc}) \) is defined as in Eq.(3.10), only that now the two Green’s function involved are of the same sign. Bearing in mind that \( \gamma_{21} \) perturbatively is just \( I_{21}^c \), and inserting these results into Eq.(2.35), we find that

\[ \sigma_1 = \frac{V_i}{g} \sigma_0 \ln \left[ \frac{1}{\epsilon_T} \right] \]

\[ \sigma_2 = \sigma_0 - \frac{2V_i}{g} \sigma_0 \ln \left[ \frac{1}{\epsilon_T} \right] - \frac{1}{g} \sigma_0 \ln \left[ \frac{1}{\epsilon_T} \right] \] (3.12)

Details of the calculation is shown in Appendix A.3.

This result is in agreement with the known work of Altshuler and Aronov[2]. To first order in \( 1/g \) and interaction, the cooperon correction and the diffusion-corrected
interaction are both taken into account with no magnetic field or impurity in presence.

The gauge invariance is still satisfied, as long as vertices have both diffusion-corrected interaction and cooperon at the same time are screened out. This point results from that in order for the ward identity holds, one has to make correction to vertices in consistent with the self-energy correction.
Chapter 4
Magnetic metal and Magneto-resistance

While considerable progress has been made in the understanding of the metal-insulator transition in two dimensions [9, 10, 11, 12, 13, 14, 15], the subject has remained controversial. A very basic issue concerns whether or not electron interactions play an important role. In this context, transport in a magnetic field applied parallel to the plane has emerged as a key measurement. There are two significant features in the phenomenology. First, a strong parallel field suppresses the metallic behavior[10, 13, 16, 17, 18, 19, 20, 21]. Second, for a small parallel field the metallic behavior appears to persist[18, 16, 17]. In this paper, we show that both features can be understood when interaction effects are taken into account. Our results also provide further support for a picture of the metal-insulator transition proposed earlier[22].

Theoretically, the nature of the strong coupling fixed point in interacting disordered electrons in two dimensions has been a long-standing open problem. The results of the one-loop renormalization group (RG) analysis[4, 23] include a) the triplet interaction amplitude is scaled to infinity; b) the single-particle density of states develops a pseudo-gap, going to zero at the Fermi energy; and c) the conductivity is scaled to a finite value. The divergence of the triplet amplitude occurs at a finite length scale,
which has been interpreted as signaling the formation of local moments at such length scales. Since local moments are ordinarily in favor of localization, a natural question is whether the strong coupling fixed point corresponds to a metal or an insulator.

The crucial observation we make here are two-folds. First, a pseudo-gap in the density of states implies that the electron self-energy is singular. This singularity, in turn, dictates that certain effective interactions between the electrons as well as the coupling of the electrons to an external electromagnetic field are also singular. To address the nature of the strong coupling fixed point, then, requires a systematic treatment of these singularities. In fact, we will show that these singularities cancel out with each other and leave a finite conductivity. Second, once the local moments have formed, the exchange coupling among them is expected to lead to a magnetic ordering. Such an ordering will have a finite rigidity against a small Zeeman coupling. The system will then remain metallic for a finite range of Zeeman coupling.

When the density of states vanishes at the chemical potential, the electron self-energy \( \Sigma(k, \omega) \) is singular. Due to the diffusive origin, we expect that the most singular term of \( \Sigma(k, \omega) \) takes the following form,

\[
\Sigma(k, \omega) = f(k)(\omega - \mu)^{-\alpha}
\]  

(4.1)

Here \( \mu \) is the chemical potential and \( f(k) \) varies smoothly with \( k \). The exponent \( \alpha \) characterizes the way the density of states goes to zero. The one-loop analysis[26] gives \( \alpha = 1 \), but our result will not be sensitive to the specific value of \( \alpha \) (so long as
it is positive).

With such a singular self-energy and the expression for conductivity as shown in Eq. (2.35), we can analyse whether it gives a metallic phase ($\sigma(T = 0) \neq 0$) or an insulating phase ($\sigma(T = 0) = 0$).

We will first focus on $\sigma_2$. With a singular self-energy, Eq. (4.1), the expression for $Q$ simplifies to

$$Q(k\omega; q = 0e \to 0) \simeq 2i \frac{1}{\Sigma(k, \omega)} \text{Im} \frac{1}{\Sigma(k, \omega)}$$  \hspace{1cm} (4.2)

If one recall the Ward Identity, Eq. (2.22), then it is obvious that the singularity in $Q$ cancel out the singularity in $\Lambda_1^2$, leading to a finite conductivity contribution. $\int d\mathbf{k}'Q_{\tilde{2}1}$ is only a factor of the order of unit.

Secondly, we take a closer look at the first term in $\sigma_1$ is zero due to the fact that its corresponding correlation function is lack of nonanalyticity, therefore it makes no contribution to conductivity. Which is demonstrated for perturbative case as shown in appendix.

At last, all the other terms are related to $\Lambda_2$, which comes from the derivative of scalar vertex, having different structure than $\Lambda_1$, therefore one has no reason to expect them to cancel out the contribution due to $\Lambda_1$.

Sum all in one, we end up with a finite conductivity at zero temperature. Or, the groud state of disordered interacting electeons in 2D at the strong coupling fixed point is a metal.
In addition to providing an understanding of the origin of the metallic phase at zero temperature, our results also allow us to address the transport properties in a parallel magnetic field[29]. A small Zeeman coupling, meaning that the Zeeman coupling is much less than the thermal fluctuation, will leave the ordering of the local moments intact. The system, then, remains in the strong coupling regime and the conductivity remains finite.

A large Zeeman coupling greater than the thermal fluctuation, on the other hand, starts to gap out two of the three triplet channels: Only the longitudinal \( S_z = 0 \) part of the spin diffusion remains important, together with the charge diffusion. When the screening length is shorter than the mean free path, this problem was already considered in detail in Refs. [4, 23] (see also Ref. [25]). The triplet interaction amplitude no longer grows to infinity (and neither does the energy renormalization factor). The system becomes an insulator.

When the Zeeman coupling is even larger than the energy scale corresponding to the scattering rate due to impurity, the two triplet channels with \( |S_z| = 1 \), are completely gapped out, and magnetoresistivity saturates.

Combining the results in the two limits gives rise to the schematic picture shown in Fig.E.10. The system turns from being metallic at small Zeeman couplings to being insulating at large Zeeman couplings. The details of how the system evolves between the two regimes go beyond the scope of this paper.
Chapter 5
Summary and Outlook

We constructed a generalized transport equation for disordered interacting electrons in 2D. The singularity due to interaction and weak-localization is incorporated into the irreducible interaction and consistently into the self-energy.

The theory is applied first perturbatively to the leading order in interaction and disorder as a confirmation of its validity. The known perturbative result is recovered.

We apply the transport equation to the critical regime, where a pseudo-gap in density of states is found on the Fermi surface. The singular self-energy implied by the pseudo-gap leads to a infinitesimal Green function, meaning that the propagating probability for single particle is infinitesimal. Nevertheless, the same singularity comes into vector vertex as well, leading to a singular vertex, meaning that the current carried by each particle is singular. The cancelation of the two singularity gives the finite conductivity.

The theory is extended to the case where parallel magnetic field exists. As the magnetic field is small, the system remains metallic; when a large magnetic field comparing with the thermal fluctuation is in presence, it becomes insulating.

Many questions are still open. For instance, can we say more about $\Lambda_2$? Can we say more about how the system evolves as the parallel magnetic field increases? We
believe these can be addressed if we can express other physical quantities within this frame, such as spin susceptibility, charge polarization.
References


20. A. A. Shashkin, S. V. Kravchenko, and T. M. Klapwijk, cond-mat/0009180.


28. This is supported by the one-loop RG result of Refs. [4, 23], which indeed has a singular self-energy but a finite conductivity. Note that the singlet-only problem considered in Refs. [4, 23] is very different. From the RG analysis, the pseudogap will not be developed until the system has already scaled to the strongly localized regime. In that case, our formulation no longer applies.

29. A parallel field also induces some orbital effect, due to the finite confinement length in the third direction (S. Das Sarma and E. H. Hwang, Phys. Rev. Lett. 84, 5596 (2000)). This effect, however, has been demonstrated to be small in Si-MOSFETs (Ref. [15]).


32. See also P. Schwab and C. Castellani, Phys. Rev. Lett. 84, 4779 (2000); Q. Si and C. M. Varma, 84, 4780 (2000).

Appendix A
Details of Calculation

A.1

In this appendix we show that, to the order of $\frac{1}{g} \ln$ and $\frac{\nu}{g} \ln$, a) both the Ward identity for the vector vertex, Eq. (2.17), and that for the scalar vertex, Eq. (2.16), are valid; and b) the vector vertex in the static limit ($\epsilon \to 0$ first and then $q \to 0$) and the dynamic limit ($q \to 0$ first and then $\epsilon \to 0$) are equal to each other [Eq. (2.19)].

We start with the Ward identity for the vector vertex, Eq. (2.17), as well as the equality of the vector vertex in the two limits, Eq. (2.19). To the order $\frac{1}{g} \ln$ and $\frac{\nu}{g} \ln$, the vector vertex can be explicitly calculated and has the following form,

$$\Lambda_\alpha(k, \omega; q, \epsilon) = \Lambda_\alpha^{(0)} + \int d\kappa' d\omega' \left[ (I_{34}\bar{y}(\omega', \epsilon) + I_{22}\delta(\omega - \omega')) \bar{y}(\omega, \epsilon) ight. \\
\left. + I_{21}^c \delta(\omega - \omega') y(\omega, \epsilon) \right] \Lambda_\alpha^{(0)} + O.T. \quad (A.1)$$

This is just Eq.(3.7) without letting $q \to 0$. O.T. stands for other terms which are non-zero for generic $q$ and $\epsilon$ but vanish in either the static limit or the dynamic limit. The first three correction terms are illustrated in Fig.B.8. They correspond to $\Lambda_{a34}$, $\Lambda_{a22}$, and $\Lambda_{c21}^c$ introduced in Eq. (3.10). Note that $I_{21}^c$ here means only the Cooperon contribution. Written explicitly, the three correction terms are
\[ \Lambda_{a22} = \int (d\Omega) \int (dk') \int (dQ) \frac{k_x}{m} G_{0-}(k', \omega) G_{0-}(k + q, \omega + \epsilon) \]
\[ \cdot G_{0+}(k' + q + Q, \omega + \epsilon + \Omega) G_{0+}(k + q + Q, \omega + \epsilon + \Omega) \]
\[ \cdot \frac{iV \theta(-\omega(\omega + \Omega))\theta(-\omega(\omega + \epsilon + \Omega))\theta(-\omega(\omega + \epsilon + \Omega))}{2\pi N_{F} \tau^{4}(-i\Omega + DQ^{2})^{2}(-i(\epsilon + \Omega) + D(q + Q)^{2})} \]
\[ \Lambda_{a34} = \int (d\Omega) \int (dk') \int (dQ) \frac{k_x}{m} G_{0-}(k', \omega) G_{0+}(k' + Q, \omega + \Omega) \]
\[ \cdot G_{0+}(k' + q + Q, \omega + \epsilon + \Omega) G_{0+}(k + q + Q, \omega + \epsilon + \Omega) \]
\[ \cdot \frac{iV \theta(-\omega(\omega + \Omega))\theta(-\omega(\omega + \epsilon + \Omega))\theta(-\omega(\omega + \epsilon + \Omega))}{2\pi N_{F} \tau^{4}(-i\Omega + DQ^{2})^{2}(-i(\epsilon + \Omega) + D(q + Q)^{2})} \]
\[ + \int (d\Omega) \int (dQ) \frac{iV}{(-i\Omega + DQ^{2})^{2} \tau^{2}} \frac{k_x}{m} G_{0+}(k + q + Q, \omega + \epsilon + \Omega) \]
\[ \cdot G_{0+}(k + Q, \omega + \Omega) \theta(-\omega(\omega + \Omega)) \theta(-\omega(\omega + \epsilon + \Omega)) \]
\[ \Lambda_{a21} = \int (dQ) G_{0+}(k + Q, \omega + \epsilon) G_{0-}(k + q + Q, \omega) \frac{1/2\pi N_{F} \tau^{2}}{\frac{-i\epsilon + DQ^{2}}{}} \]

(A.2)

\( G_0 \) is still the Green function for the noninteracting impurity system. Without losing generality, we choose \( \omega \) negative. We leave the expression as it is without making any computation, so that when the limits are taken, we can see clearly that the result are not sensitive to how the limits are approached. This point is actually easy to see, because neither the Green functions nor the corrected interaction (the fraction terms above) are sensitive to the commutation of the limits. Therefore, the results we will obtain at either limit are the same. Or, in mathematical language
\[ \Lambda_\alpha(k, \omega; q \to 0, \epsilon = 0) = \Lambda_\alpha(k, \omega; q = 0, \epsilon \to 0). \]  
(A.3)

In both the static and dynamic limit, the sum of \( \Lambda_{\alpha 22} \) and the first term of \( \Lambda_{\alpha 34} \) is proportional to \( \epsilon \). (More generally, this linear relationship occurs so long as \( q \) goes to zero.) Therefore, it goes to zero either way. \( \Lambda_{\alpha 21}^\epsilon \) is proportional to \( y(\omega, \epsilon) \), so it vanishes in the dynamic limit. In addition, the factor multiplying \( y(\omega, \epsilon) \) is only logarithmically singular in terms of \( \epsilon \). Therefore, \( \Lambda_{\alpha 21}^\epsilon \) vanishes even in the static limit. So, in the end, only the second term of \( \Lambda_{\alpha 34} \) gives rise to a finite contribution.

In addition, the result is the same for the static and dynamic limit. Two conclusions follow. First, we have proven the equality of the vector vertex in the two limits, as specified by Eq. (2.19). Second, the perturbative expression for the vector vertex for both the static and dynamic limit is given by

\[
\Lambda_\alpha(q = 0, \epsilon = 0) = \Lambda_\alpha^0 + \int (d\Omega) \int (d) \frac{iV}{(-i\Omega + DQ^2)^2 \tau^2 \epsilon} k_x \cdot G_{0+}(k + Q, \omega + \Omega) \theta(-\omega(\omega + \Omega)) \]  
(A.4)

Recall the expression for self energy as in Eq.(3.1), take derivative with respect to \( k_x \) and we find exactly the same result as the correction obtained above. Therefore Eq.(2.17) is proved perturbatively.

We now work on the Ward Identity defined in Eq.(2.16), which basically connects
the derivative of self energy with respect to frequency to the scalar vertex in the
dynamic limit. We will prove it in different regimes, \( \theta(-\omega(\theta + \omega)) \) and \( \theta(-\omega(\theta + \omega)) \),
becuse different diagrams make the correction in the two opposite regimes. First,
in the \( \theta(-\omega(\theta + \omega)) \) regime, let us focus on the correction to scalar vertex to the
first order in interactin and \( 1/g \), which to the zeroth order in \( \epsilon \), are composed of the
diagrams(a-f,A-E) in Fig.B.9. One point worth to be opointed out is that both
diffuson and cooperon exists only in the vicinity of fermi surface in the frequency
space, or \( |\omega| < \frac{1}{\tau} \), and this actually gives the cutoff in infinity. In the computation,
only the leading singularity of \( \delta(\omega)ln\left[\frac{1}{\omega \tau}\right] \) are kept, and terms with only a \( \delta \) function
or a \( ln\left[\frac{1}{\omega \tau}\right] \) are dropped for not as singular as the former term, which can easily been
seen after integrating them over \( \omega \). We give the final result of the diagramms as
follows, assuming \( \omega \to 0^- \) which is actually a result of the assumption \( \epsilon > 0 \).

\[
\begin{align*}
a + b + c + d + e &= \frac{-2iV_i}{g\tau} ln\left[\frac{-1}{\omega \tau}\right] \delta(\omega) \\
c' + d' + e' &= 0 \\
e'' &= \frac{iV_i}{2g\tau^3} G_{0+}G_{0-} ln\left[\frac{-1}{\omega \tau}\right] \delta(\omega) \\
f &= \frac{iV_i}{g\tau} ln\left[\frac{-1}{\omega \tau}\right] \delta(\omega) \\
A + B &= A' + B' = A'' + B'' = 0 \\
C + D &= C' + D' = C'' + D'' = 0
\end{align*}
\]
\[ E = \frac{i}{\tau} \delta(\omega) \quad (A.5) \]

The RHS, which involves self energy derivative, can be easily obtained from taking derivative of Eq.(3.1) with respect to \( \omega \) and only keeping the most singular term, which is to the order of \( \delta(\omega)ln[\frac{1}{-\omega^2}] \). One should be careful with the derivative and it is easier to perform by using the definition \( \frac{M(\omega+\epsilon)-M(\omega)}{\epsilon} \) at \( \theta(-\omega(\theta + \omega)) \). It is worth to point out that the only thing involves in derivative is the sign function and the logarithmic function is treated as \( \omega \) independent for as \( \omega \to 0 \) the thermal fluctuation will come in as the cutoff. We show the result as follows,

\[ \frac{\partial M}{\partial \omega} = \frac{-i}{\tau} \delta(\omega) + \frac{iV_1}{g\tau} ln\left[\frac{-1}{\omega\tau}\right] \delta(\omega) + \frac{-iV_1}{2g\tau^2} G_{0+} G_{0-} ln\left[\frac{-1}{\omega\tau}\right] \delta(\omega) \quad (A.6) \]

Therefore, Eq.(2.16) is satisfied at \( \theta(-\omega(\theta + \omega)) \).

On the opposite regime, the correction diagrams(g-j) are as shown in Fig.B.9, where the most sigular term is proportional to \( \omega^{-1} \).

\[ g + h = 0 \]

\[ i = \frac{-V_1}{2g\tau^2} G_{0+}(k,0) \frac{1}{\omega} \]

\[ j = \frac{iV_1}{2g\tau} sgn(\omega) \frac{1}{\omega} \quad (A.7) \]

The derivative of self energy involves the logarithmic terms only because the sign of \( \omega + \epsilon \) and \( \omega \) are the same, and therefor gives singularity of \( \omega^{-1} \). It is easy to see it
is just the opposite of the sum of the two terms above. So, Eq.(2.16) holds as well in 
\( \theta(\omega(\omega + \epsilon)) \).

A.2

In this appendix, we will prove the relationship between the dynamic component of the vector vertex and the derivative of the scalar vertex, as specified by Eq.(2.22). We first compute contributions to the LHS with leading order in \( \epsilon \), i.e., zeroth order. In order to do that, let’s revisit Eq.(3.11) and write down the perturbative expression for \( \Lambda_2 \).

\[
\Lambda_2 = \Lambda_{a21}^c - \Lambda_{a34} - \Lambda_{a22}
\]  

(A.8)

The corresponding correction diagrams for \( \Lambda_2 \) defined as above are shown in Fig.B.8. One should be aware that the diagrams involved \( I_{34} \) and \( I_{22} \) do NOT exist at \( y(\omega, \epsilon) \), on the contrary they exist only at \( \bar{y} \). The reason they appears in \( \lambda_2 \) lies in that we rewrite \( \bar{y} \) as \( 1 - y \), and the \( y(\omega, \epsilon) \) part leads to \( \Lambda_2 \). So,

\[
\Lambda_2 = \frac{kV}{2gm^2} \ln\left[ -\omega \right] G_{0-}^2 + \frac{-ik}{2gm^2} \ln\left[ \frac{1}{\epsilon} \right] G_{0+}G_{0-} + O(\epsilon).
\]  

(A.9)

where the contribution actually comes from the third and fourth diagram in Fig.B.8. The contribution from the first two diagrams is proportional to \( \epsilon \), therefore dropped.

Now that the RHS of Eq.(2.22) involves first order derivative of scalar vertex with
respect to \( q \) at \( q = 0 \), we need to look for diagrams which is odd function of \( q \). Also, the scalar vertex to our interest should have a \( y(\omega, \epsilon) \) as a factor and apart form the \( y \) function be of the order of \((-1)\) in \( \epsilon \). Therefore, we end up with diagram \( e'' \), \( A, B, C, \) and \( D \) as shown in Fig.B.9, which all have a Green function left unintegrated on the particle line, and it is in fact this Green function that gives the odd dependence in \( q \). In fact, diagram \( B \) with the unintegrated Green Function on the hole line also contributes due to the special structure of cooperon. Calculation shows that

\[
\frac{\partial e''}{\partial q} \bigg|_{q=0} = \frac{kV_1}{2gm\epsilon^2} \ln\left[\frac{1}{-\omega_T}\right] G_{0^-} y(\omega, \epsilon)
\]

\[
\frac{\partial (A' + B')}{\partial q} \bigg|_{q=0} = \frac{-ik}{2gm\epsilon^2} \ln\left[\frac{1}{\epsilon_T}\right] G_{0^+} G_{0^-} y(\omega, \epsilon)
\]

\[
C' + D' = 0
\]

(A.10)

Comparing the result with \( \Lambda_2 \), we just proved Eq.(2.22).

A.3

In this section, the details of computing correction to conductivity will be shown.

Now that we have the expression for \( \tilde{\Lambda}_\alpha, \Lambda_\alpha \) and Green function in first order of interaction and \( 1/g \), the current-current correlation function can be written as

\[
S_{aa} = S_{1,aa} + S_{2,aa}
\]

\[
S_{1,aa} = \Lambda^{(0)}_\alpha \Phi_{inc} \tilde{\Lambda}_1 + y(\omega, \epsilon) \Lambda^{(0)}_\alpha \Phi_{inc} \tilde{\Lambda}_2
\]
\begin{align*}
S_{2,a\alpha} &= \left[ \Lambda_\alpha^{(0)} + \Lambda_\alpha^{(20)} \right] - \int dk' I_{20} Q^{(0)} \Lambda_\alpha^{(0)} \right) Q_y(\omega, \epsilon) \Lambda_\alpha^{(0)} + \Lambda_\alpha^{(0)} \right) G_0(\omega + \epsilon) G_0(\omega) J_{21} G_0(\omega + \epsilon) G_0(\omega) Q_y(\omega, \epsilon) \Lambda_\alpha^{(0)} \\
&= \Lambda_\alpha^{(0)} Q_y(\omega, \epsilon) \Lambda_\alpha^{(0)} + \Lambda_\alpha^{(0)} G_0(\omega + \epsilon) G_0(\omega) J_{21} G_0(\omega + \epsilon) G_0(\omega) Q_y(\omega, \epsilon) \Lambda_\alpha^{(0)}
\end{align*}
(A.11)

where \( \tilde{\Lambda}_1 \) and \( \tilde{\Lambda}_2 \) are defined in Eq.(3.11). One may find that in the calculation with \( I_{21} \) involved, the Green functions are directly reduced to zeroth order in interaction. The reason for that lies in, first, calculation dealing with cooperon and diffusion-corrected interaction present at the same time is inconsistent with the self-energy we considered, therefore violate the gauge invariance. \( S_1 \) and \( S_2 \) lead to the contribution of \( \sigma_1 \) and \( \sigma_2 \) respectively.

Recall that we mentioned previously that the correction to self-energy has an extra pole in it, therefore the Normal Fermi-liquid theory treatment doesn’t apply here. Fortunately, if we in first order of perturbation take \( G = G^0 + G^0 \delta M G^0 \), where \( \delta M \) is the sum of the last two terms in \( M \), then the extra pole will be treated correctly and give right correction to conductivity. The reason to justify such an operation lies in that \( \frac{\nu_1}{g} \) is viewed as small quantity. Keeping all the terms with logarithmic singularity and to first order in \( V_1 \) and \( 1/g \), we have

\begin{align*}
S_{1,a\alpha} &= S_{11} + S_{12} \\
S_{11} &= \frac{-n_0}{m}
\end{align*}
\[ S_{12} = -\frac{iV_1}{g} \sigma_0 \epsilon \ln \left[ \frac{1}{\epsilon r} \right] \]

\[ S_{2aa} = S_{21} + S_{22} \]

\[ S_{21} = -i \epsilon \sigma_0 + \frac{2iV_1}{g} \epsilon \sigma_0 \ln \left[ \frac{1}{\epsilon r} \right] \]

\[ S_{22} = \frac{i}{g} \epsilon \sigma_0 \ln \left[ \frac{1}{\epsilon r} \right] \]

(A.12)

where, \( \sigma_0 \) is the conductivity of a pure noninteracting electron system. \( S_{11} \) and \( S_{12} \) correspond to the first and second terms in \( S_1 \), and \( S_{21} \) and \( S_{22} \) are defined similarly.

Plug these expression back into Eq.(2.33), we obtain the conductivity in first order perturbation at zero temperature.

\[ \sigma_{aa} = \sigma_0 (1 - \frac{V_1}{g} \ln \left[ \frac{1}{\epsilon r} \right] - \frac{1}{g} \ln \left[ \frac{1}{\epsilon r} \right]) + O\left( \frac{V}{g^2}, \frac{1}{g^2} \right) \]  

(A.13)

We give further detailed calculation in the following. Plugging in the expression for \( \tilde{\Lambda}_1 \), we have

\[ S_{11} = \Lambda_{\alpha}^{(0)} \phi_{inc} \Lambda_{\alpha}^{(0)} + \Lambda_{\alpha}^{(0)} \phi_{inc} \Lambda_{\alpha 34} + \Lambda_{\alpha}^{(0)} \phi_{inc} \Lambda_{\alpha 22} \]  

(A.14)

The first term is in fact

\[ \Lambda_{\alpha}^{(0)} G(\omega + \epsilon) G(\omega; \text{sgn}(\omega + \epsilon)) \Lambda_{\alpha}^{(0)} = A + 2B \]

\[ A = \Lambda_{\alpha}^{(0)} G_0(\omega + \epsilon) G_0(\omega; \text{sgn}(\omega + \epsilon)) \Lambda_{\alpha}^{(0)} \]

\[ B = \Lambda_{\alpha}^{(0)} G_0^2(\omega + \epsilon) G_0(\omega; \text{sgn}(\omega + \epsilon)) \delta M(\omega + \epsilon) \Lambda_{\alpha}^{(0)} \]  

(A.15)

where, in first order of interaction, \( G = G_0 + G_0 \delta M G_0 \).
\[ A = (-i) \int_{-\epsilon}^{\infty} (d\omega) \int (dk) \left( \frac{k_x}{m} \right)^2 \frac{1}{\omega + \epsilon - \varepsilon_k + \alpha + \frac{i}{2\tau}} \frac{1}{\omega - \varepsilon_k + \alpha + \frac{i}{2\tau}} \]
\[ + \frac{n_0}{m} \]  
\[ \text{(A.16)} \]

where \( n_0 \) is the carrier's density. In fact, an extra prefactor of \((-i)\) is due to the fact that the correlation function makes a fermion loop.

Recall the expression for \( \delta M \), which is the self-energy due to diffusion-corrected interaction, we have

\[ B = (-i) \int_{-\epsilon}^{\frac{1}{\tau}} (d\omega) \int (dk) \left( \frac{k_x}{m} \right)^2 G_0^+ G_0^- \left( -\frac{V_1}{2g\tau^2} \right) \ln \left[ \frac{1}{(\omega + \epsilon)\tau} \right] \]
\[ + (-i) \int_{-\frac{1}{\tau}}^{-\epsilon} (d\omega) \int (dk) \left( \frac{k_x}{m} \right)^2 G_0^- G_0^+ \left( -\frac{V_1}{2g\tau^2} \right) \ln \left[ -\frac{1}{(\omega + \epsilon)\tau} \right] \]
\[ = 0 \]  
\[ \text{(A.17)} \]

In the calculation above, the second term of \( \delta M \) is also plugged in, yet it makes no contribution due to the fact that all the Green functions involved are on the same side of fermi surface[2]. The zero, in the end, results from that the integration ranges from the lower cutoff to upper cutoff, leaving no nonanalyticity. And for the same reason the contributions from the last two terms in \( S_{11} \) are zero.

As for \( S_{12} \),

\[ S_{12} = -\Lambda^{(0)}_{\alpha} \phi_{incy}(\omega, \epsilon) \Lambda_{\alpha 34} - \Lambda^{(0)}_{\alpha} \phi_{incy}(\omega, \epsilon) \Lambda_{\alpha 22} + \Lambda^{(0)}_{\alpha} \phi_{incy}(\omega, \epsilon) \Lambda_{\alpha 21} (\Phi_{inc}) \]  
\[ \text{(A.18)} \]
The first term can be broken into two parts, corresponding to $\Lambda_{\alpha 34,1}$ and $\Lambda_{\alpha 34,2}$ respectively, which are second and third diagram in Fig.B.8. We find

\[ \Lambda^{(0)}_{\alpha \phi in\chi y}(\omega, \epsilon)\Lambda_{\alpha 34,1} = \frac{-2iV_i}{g} \sigma_0 \epsilon ln[\frac{1}{\epsilon_T}] \]
\[ \Lambda^{(0)}_{\alpha \phi in\chi y}(\omega, \epsilon)\Lambda_{\alpha 34,2} = \frac{2iV_i}{g} \sigma_0 \epsilon ln[\frac{1}{\epsilon_T}] \] (A.19)

Therefore we reach the result of zero contribution from the first term.

\[ \Lambda^{(0)}_{\alpha \phi in\chi y}(\omega, \epsilon)\Lambda_{\alpha 22} = \Lambda^{(0)}_{\alpha \phi in\chi \bar{y}}(\omega, \epsilon)\Lambda^{(22)}_{\alpha} \]
\[ = \frac{-iV_i}{g} \sigma_0 \epsilon ln[\frac{1}{\epsilon_T}] \] (A.20)

The second term therefore makes contribution of $\frac{-V_i}{g} \sigma_0 \epsilon ln[\frac{1}{\epsilon_T}]$. The last term in $S_{12}$ is zero for that the four Green's fuctions have poles on the same side of the complex plane of $\varepsilon_k$, and their integration range in frequency space is finite. At this point, we may give the analytic expression for $S_1$, the same as shown in Eq.(A.12).

As for $S_2 = S_{21} + S_{22}$, we have, first, the expression for

\[ S_{21} = \Lambda^{(0)}_{\alpha G_0(\omega + \epsilon)G_0(\omega)I_{21}G_0(\omega + \epsilon)G_0(\omega)\Lambda^{(0)}} \]
\[ = \frac{i}{g} \epsilon_0 \epsilon ln[\frac{1}{\epsilon_T}] \] (A.21)
then the analytic expression for

\[ S_{22} = Λ(0)_α Q y(ω, ε) Λ(0)_α \]

\[ = Λ(0)_α G_0(ω + ε) G_0(ω) y(ω + ε) Λ(0)_α \]

\[ + 2Λ(0)_α G_0^2(ω + ε) δ M(ω + ε) G_0(ω) y(ω + ε) Λ(0)_α \]

\[ - 2Λ(0)_α G_0^2(ω + ε) δ M(ω + ε) G_0(ω; sgn(ω + ε)) y(ω + ε) Λ(0)_α \]

\[ = F + 2G - 2H \quad (A.22) \]

where, F, G and H are the three terms in second line.

\[ F = (-i) \int_{-ε}^{0} (dω) \int (dk) \frac{k_ε^2}{m} \frac{1}{ω + ε + ε_k + \frac{i}{2τ}} \frac{1}{ω - ε_k - \frac{i}{2τ}} = (-i)σ_0 ε \]

\[ G = (-i) \int_{-ε}^{0} (dω) \int (dk) \frac{k_ε^2}{m} G_{0+}^2 G_{0-} (-\frac{V_1}{2gτ^2} ln[\frac{-1}{ωτ}]G_{0-} + \frac{iV_1}{2gτ} ln[\frac{1}{ωτ}]) \]

\[ = \frac{iV_1}{2g} σ_0 ε ln[\frac{1}{ετ}] \]

\[ H = (-i) \int_{-ε}^{0} (dω) \int (dk) \frac{k_ε^2}{m} G_{0+}^3 (-\frac{V_1}{2gτ^2} ln[\frac{-1}{ωτ}]G_{0-}) \]

\[ = \frac{-iV_1}{2g} σ_0 ε ln[\frac{1}{ετ}] \quad (A.23) \]

Therefore, in the end, we obtain the final expression for the full correlation function.

\[ S = (-i)σ_0 ε (1 - \frac{1 + V_1}{g} ln[\frac{1}{ετ}]) + \frac{-n_0}{m} \quad (A.24) \]
Appendix B
Figures
Figure B.1  Temperature dependence of resistivity for different carriers' densities. By Kravchenko et. al.
Figure B.2  Resistivity grows dramatically as the parallel magnetic field increases. By Simonian et. al.

Figure B.3  The product of two Green's functions. The upper and lower propagators are one particle and one hole, respectively.
Figure B.4  Self-energy of single particle to the first order in interaction and disorder. Only exchange channel is considered.

Figure B.5  Transport Equation in general. $\gamma$ is the interaction vertex. $\Lambda$ is the field vertex. $S_{\mu\nu}$ is the current-current correlation function.
Figure B.6  Isolation of the nonanalyticity in vertex and transport equation.
Figure B.7  Irreducible interaction vertices to the first order in interaction and disorder. Only those have logarithmic singularity are kept.
Figure B.8  Vector vertices which are nonzero at either the static or the dynamic limit. The first two diagrams also imply the diagrams obtained by exchanging the particle and hole line. The second and third one both belongs to $\Lambda_{34}$. 
Figure B.9  Diagrams for scalar vertices to the first order in interaction and disorder. Diagrams linear in $\epsilon$ or higher are neglected.
Figure B.10  Schematics of the resistivity as a function of parallel field at different temperatures. The system is metallic at small fields, but becomes insulating at high fields. The dashed lines represent interpolations between the two limits.