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Deterministic and Stochastic Analysis of Nonlinear Systems with Biot Hysteretic Damping

by

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Abstract

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Time domain analysis of nonlinear systems with hysteretic damping is conducted. Specifically, the visco-elastic model proposed by Biot is examined. This hysteretic element represents an integral transform in the time domain. Thus, it yields integro-differential equations when it is incorporated into the system dynamics models. Two numerical methods are proposed to solve these equations. The first method approximates the kernel of this integral transform by a sum of exponentials making the computational cost minimal. The second method uses digital filters designed to match the transfer function, real and imaginary parts, of the Biot hysteretic element. These techniques are employed in calculating the response of a single-degree-of-freedom (SDOF) system with hysteretic damping and nonlinear stiffness subjected to deterministic, seismic, and random excitation. The method of statistical linearization is used to estimate the variance of the response of the SDOF system subjected to white noise. The accuracy of the results is verified by pertinent Monte Carlo studies. The presented approaches can be extended to treat multi-degree-of-freedom (MDOF) systems with hysteretic behavior.
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CHAPTER 1
Introduction

1.1 General remarks

Almost in every natural or engineering system there is an energy dissipation or damping mechanism. An engineer thus needs a simple mathematical model to account for this phenomenon. The most widely used damping model is the ideal viscous damper, introduced by Rayleigh [31]. The force-displacement relation for this damper is

\[ f(t) = c \frac{dx}{dt}, \]  

where \( f(t) \) is the force, and \( x(t) \) is the displacement. If \( x(t) \) has the following form

\[ x(t) = A \sin(\omega t), \] 

equation (1.1) yields

\[ f(t) = cA\omega \cos(\omega t). \] 

The dissipated energy per cycle of duration \( T = \frac{2\pi}{\omega} \) is equal to

\[ E_T = \int_{x(0)}^{x(T)} f(t) dt = \int_0^{2\pi/\omega} cA^2 \omega^2 \left( \frac{2\pi}{\omega} - \int_0^{2\pi/\omega} \cos^2(\omega t) dt \right) = \pi c|\omega|A^2. \] 

The energy dissipation per cycle is proportional to the excitation frequency and the square of the amplitude of the vibration.

A useful measure the dissipated energy per cycle is the loss factor \( \eta \). It indicates the energy loss relative to the energy that is stored as potential energy in other elements such as springs. It is defined as

\[ \eta = \frac{W}{2\pi V}, \] 

where \( W \) is the energy loss per cycle, and \( V \) is the peak potential energy.
The ideal viscous damper is a quite useful model. However, it is inappropriate for a considerable class of engineering systems. Experiments in the 1920's [14,42] showed that for many engineering materials the internal damping is such that the energy loss per cycle is almost independent of the frequency of the applied strain and proportional to the square of the strain amplitude. This fact necessitates the use of a damping model with loss modulus independent of the frequency of the excitation. This kind of damping is referred to as \textit{ideal hysteretic damping} [3,41]. Early linear hysteretic damping models included the complex stiffness [37] and the frequency dependent damping models [24]. These models have serious mathematical flaws. The first yields complex valued response to real valued input and the second is a non-causal model, that is, the output at a given instant depends on future input. The first successful linear hysteretic damping model was the Biot element [2]. These models are analyzed in Chapter 2.

The preceding models are important. Nevertheless, they are just a small part of the effort to understand the ubiquitous phenomenon of hysteresis. Hysteretic damping is just one aspect of this subject. Hysteresis is a type of behavior of systems with memory, that is, the output of the system at a given instant depends on the history of motion of the system and it is reflected by quite distinctive \textit{hysteretic loops} [23]. These loops are created by drawing diagrams of the output versus the input of a hysteretic system. Figure 1.1a depicts a hysteretic loop of a SDOF system with linear hysteretic damping, and figure 1.1b shows a more general nonlinear hysteretic loop.
1.2 Thesis organization

Chapter 2 presents certain models of linear hysteretic damping that are used in the literature. In Chapter 3 some significant concepts of linear system theory are presented such as the frequency and time domain analyses. Numerical methods for time domain integration of differential equations are presented in Chapter 4. In Chapter 5 random vibration issues, and the Monte Carlo method are discussed. In Chapter 6 a SDOF system with hysteretic damping is used in a series of simulations. It consists of a mass connected to a Biot hysteretic element and it is subjected to random and deterministic excitations.
This case is linear and both time domain and frequency domain methods are used to calculate the response. To the same system, a spring with nonlinear (cubic) stiffness is appended and the method of statistical linearization is applied to estimate the variance of the output of the system for white noise input. Monte Carlo simulations that use the proposed time domain solution techniques are performed to assess the accuracy of the statistical linearization. In Chapter 7 the results of the preceding chapters are evaluated and concluding remarks are incorporated.
CHAPTER 2
Mathematical Models of Hysteretic Damping

2.1 Definition of the problem

In order to create a linear model for hysteretic damping, the problem is first examined using the frequency domain description of linear systems. The goal is to create a linear damping element that reflects the experimental observation that in many materials the energy consumption per cycle of oscillation is approximately constant over a wide range of frequencies of oscillation.

Every linear system is uniquely described by the frequency response function (FR), usually denoted by $H(\omega)$. The frequency response function represents the response of the system to a sinusoidal excitation, when the system reaches steady-state. Consider, for example, an excitation of the form

$$x(t) = X \sin(\omega_0 t + \varphi) \quad \text{for} \quad -\infty < t < \infty. \quad (2.1)$$

The response of the system is a sinusoid which differs from $x(t)$ in both amplitude and phase. The amplitude of $x(t)$ is multiplied by the modulus of the FR evaluated for $\omega = \omega_0$, $|H(\omega_0)|$, and the new phase is equal to the phase of $x(t)$, $\varphi$ plus the argument of the complex number $H(\omega_0)$, $\angle H(\omega_0)$. The symbols $|H(\omega)|$ and $\angle H(\omega)$ are usually called the amplitude and phase response of the system, respectively, see Chapter 3. The function $H(\omega_0)$ can be written as

$$H(\omega_0) = Ae^{i\theta} = H_R + iH_I, \quad (2.2)$$

where $A = |H(\omega_0)|$, $\theta = \angle H(\omega_0)$, and $H_R$ and $H_I$ are the real and imaginary parts of $H(\omega_0)$, respectively. Thus, the response of the system is
\[ f(t) = AX \sin(\omega_0 t + \varphi + \theta) = H_R X \sin(\omega_0 t + \varphi) + H_I X \cos(\omega_0 t + \varphi), \]  
(2.3)

where \( x(t) \) is the displacement and \( f(t) \) is the induced force. The energy consumption over a period \( T = \frac{2\pi}{\omega_0} \) is equal to

\[ E_T = \int_{x(0)}^{x(T)} f(t) \dot{x}(t) dt = \int_0^T f(t)x(t)dt, \]  
(2.4)

where \( \dot{x}(t) \) is the velocity of motion. From equations (2.3) and (2.4) it is trivial to derive that the energy consumption is

\[ E_T = \pi X^2 H_I. \]  
(2.5)

Equation (2.5) shows that the energy dissipation over one cycle depends only on the imaginary part of the frequency response of the system.

This observation alone leads to the early linear hysteretic damping models explained below. It is noted that the ideal viscous damper has frequency response equal to \( H_{vd}(\omega) = i\omega \). Thus, one can readily see why the energy dissipation per cycle of this damping model is proportional to the frequency of oscillation.

The preceding remarks lead one to state the problem of seeking a linear system with energy dissipation per cycle of oscillation independent of the frequency of oscillation or equivalently to seek a linear system having a FR function with constant imaginary part for \( \omega > 0 \).

2.2 Linear hysteretic damping models

An obvious way of achieving constant energy dissipation per cycle is to introduce the notion of complex stiffness \( k = a + ib \). The restoring force in this model is
\[ f(x) = (a + ib)x. \]  \hspace{1cm} (2.6)

The frequency response of this 'complex spring' is \( H_{cs}(\omega) = a + ib \) which has a constant imaginary part as needed from the analysis in section 2.1. This model was introduced by Soroka [37]. A quite similar model of complex stiffness was introduced by Myklestad [25] and the restoring force is equal to

\[ f(x) = e^{ig}x. \]  \hspace{1cm} (2.7)

The problem with the complex stiffness model is that it yields complex response to real excitation. A property from the linear system theory can be invoked to elucidate not only this case but also other cases of linear systems with complex valued parameters. Specifically, a system with FR given by \( H(\omega) \) has a real valued response when its excitation is real, if and only if \(|H(\omega)|\) is an even function and \( \angle H(\omega) \) is an odd function. This clearly shows the inappropriateness of the complex stiffness model and leads one to introduce the so-called 'ideal hysteretic damping element' [10].

Another early model of hysteretic damping involves the frequency dependent damping (FDD). The fact that the FR of the ideal viscous damper is \( H_{vd}(\omega) = ic\omega \) and the need for a constant imaginary part of the FR for energy dissipation per cycle independent of frequency, lead one to adopt the frequency dependent damping parameter

\[ c(\omega) = \frac{h}{\omega}. \]  \hspace{1cm} (2.8)

The restoring force is equal to

\[ f(x) = \frac{h}{\omega} x + kx. \]  \hspace{1cm} (2.9)

This model had been used by Mindlin [24] and Bishop [3] and it is attributed to Collar.
In equation (2.9) the interpretation of the frequency term, ω, is obvious when the motion is sinusoidal but it is quite obscure when arbitrary oscillations are considered.

Another model of linear hysteretic damping that uses the Hilbert transform is referred to as 'ideal hysteretic damping' by Crandall [10].

The Hilbert transform of a signal \( x(t) \) is defined as

\[
\hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} \, d\tau.
\]  

(2.10)

In the frequency domain equation (2.10) is equivalent to multiplication of the Fourier Transform of \( x(t) \), \( X(\omega) \) with the function \( isgn(\omega) \), where \( sgn(\omega) \) is the signum function.

The ideal hysteretic element has a restoring force defined as

\[
f(t) = k(x(t) + \eta \dot{x}(t)),
\]  

(2.11)

where \( k \) is the stiffness parameter and \( \eta \) is the loss factor of the element. The FR of this element is equal to

\[
H_h(\omega) = k[1 + \eta sgn(\omega)].
\]  

(2.12)

This model is actually a correction of the models of complex stiffness and frequency dependent damping described earlier. If the frequency dependent damping model is rewritten as

\[
c(\omega) = \frac{h}{|\omega|},
\]  

(2.13)

the FDD model is equivalent to the ideal hysteretic damping model. The use of the signum function in the imaginary part of the FR remedies the problem of complex valued output for real input. The problem with this model is that it is a non-causal system [8], which means that the response of the system depends not only on the past of the input but
also on its future. This fact makes the model physically unrealizable. Figure 2.1a shows a SDOF system with ideal hysteretic damping and with FR equal to

\[ H_{bs} = \frac{1}{-\omega^2 + \omega_0^2[1 + i\eta \text{sgn}(\omega)]}, \]  

(2.14)

where \( \omega_0 \) is the natural frequency of the system, and \( \eta \) is the loss factor of the hysteretic damper. In figure 2.1b the impulse response of the system of figure 2.1a is depicted, and it clearly shows the non-causality of the system since it has non-zero values for \( t < 0 \).

**Figure 2.1a**
SDOF system with ideal hysteretic damping

**Figure 2.1b**
Impulse response of a SDOF system with ideal hysteretic damping
Natural frequency \( \omega_0 = 1 \), loss factor \( \eta = 1 \)
The pathological nature of the ideal hysteretic damper is discussed in the papers of Caughey [5], Crandall [8,9], Inaudi and Kelly [12]. Some attempts have been made to solve this problem. Bishop and Price [4] proposed the band-limited hysteretic damper, but Crandall [10] proved that it is non-causal. Inaudi and Kelly [12] developed a time domain iterative method for calculating the response of systems with ideal hysteretic damping. The analysis of convergence of the iterative algorithm was performed only for the linear case and it imposed an upper limit to the value of the loss factor, η, for the convergence of the algorithm. It is noted that for the linear case the calculation of the response using the FFT, see Chapter 3, is an easier and more accurate method and it works for every value of η. A meaningful time domain solution technique would be one that produces reliable results for linear and, most importantly, for nonlinear problems, and it does not introduce spurious constraints to the values of the parameters of the system.
2.3 The Biot hysteretic element

The causality requirement was met for the first time in the visco-elastic model invented by Biot [2] and illustrated in figure 2.2.

![Diagram of Biot's visco-elastic element]

Figure 2.2
Biot's visco-elastic element

It consists of a spring of stiffness $k$ connected in parallel with a large number of spring-dashpot combinations. The $j$th spring and dashpot induces a force

$$f_j = \beta_j (x - \dot{x}_j) = k_j x_j .$$

(2.15)

The above equation is a differential equation with respect to $\dot{x}_j$ and can be written as

$$\dot{x}_j + r_j x_j = \dot{x},$$

(2.16)

where

$$r_j = \frac{k_j}{\beta_j} .$$

(2.17)
It is assumed that before the initial time \( t_0 \) the displacement \( x(t) \) is zero. The solution of equation (2.16) is

\[
x_j = \int_{t_0}^{t} e^{-r_j(t-\tau)} x(\tau) d\tau.
\]  

(2.18)

Equations (2.15), (2.17) and (2.18) yield

\[
f_j = \beta_j r_j \int_{t_0}^{t} e^{-r_j(t-\tau)} x(\tau) d\tau.
\]  

(2.19)

The total force \( f(t) \) is equal to the sum of the components \( f_j \) plus the spring force \( kx \).

Thus,

\[
f(t) = kx + \sum_{j=1}^{N} b_j r_j \int_{t_0}^{t} e^{-r_j(t-\tau)} x(\tau) d\tau.
\]  

(2.20)

Equation (2.20) can be seen as a weighted sum with weights \( b_j r_j \). If the number of spring-dashpot elements is allowed to tend to infinity, the summation is replaced by integration.

In this case the weights is replaced by a weighting function \( g(r) \). That is,

\[
f(t) = kx + \lim_{N \to \infty} \sum_{j=1}^{N} \beta_j r_j \int_{t_0}^{t} e^{-r_j(t-\tau)} x(\tau) d\tau = kx + \int_{t_0}^{t} e^{-r(t-\tau)} x(\tau) d\tau.
\]  

(2.21)

Biot defined \( g(r) \) as

\[
g(r) = \begin{cases} 
\frac{2k \eta}{\pi r} & r > \varepsilon \\
0 & r < \varepsilon
\end{cases}
\]  

(2.22)

where \( \eta \) is the loss factor of the Biot model, and \( \varepsilon \) is a constant.

The response of the 'Biot spring' can be expressed as
\[ f(t) = k \left[ x + \frac{2\eta}{\pi} \int_0^t \int_{t_0}^t e^{-r(t-\tau)} \dot{x}(\tau) d\tau dr \right]. \] (2.23)

If the order of integration is changed, equation (2.23) becomes

\[ f(t) = k \left[ x + \frac{2\eta}{\pi} \int_{t_0}^t E_i[\varepsilon(t-\tau)] \dot{x}(\tau) d\tau dr \right], \] (2.24)

where \( E_i(\alpha) \) is the exponential integral \([1]\)

\[ E_i(\alpha) = \int_{\alpha}^{\infty} \frac{e^{-x}}{x} dx. \] (2.25)

The Biot hysteretic element was applied by Caughey \([5]\) to transient and steady-state vibrations using analytical tools.

### 2.3.1 Steady state vibration of the Biot element

Consider a sinusoidal input for the Biot model. Further, since the goal is to calculate the steady state behavior of the element, assume that \( t_0 = -\infty \).

Thus,

\[ x(t) = X \sin(\omega t), \] (2.26)

and from equation (2.23)

\[
\begin{align*}
  f(t) &= Xk \left[ \sin(\omega t) + \frac{2\eta}{\pi} \int_0^t \int_{t_0}^t \frac{e^{-r(t-\tau)}}{r} \omega \cos(\omega t) d\tau dr \right] = \\
  &= Xk[\sin(\omega t) + \frac{2\eta}{\pi} \int_{t_0}^t \frac{\omega \cos(\omega t) + \omega^2 \sin(\omega t)}{r(r^2 + \omega^2)} dr] = \\
  &= Xk[(1 + \frac{2\eta}{\pi} \ln(1 + \frac{\omega^2}{\varepsilon^2})/2) \sin(\omega t) + \frac{2\eta}{\pi} \frac{1}{\varepsilon} \tan^{-1} \frac{\omega}{\varepsilon} \cos(\omega t)].
\end{align*}
\] (2.27)
By comparing equations (2.26) and (2.27) with equations (2.1), (2.2) and (2.3) it is readily seen that the frequency response of the Biot element is

\[ H_{\text{Biot}}(\omega) = k\left[1 + \frac{2\eta}{\pi} \ln \sqrt{1 + \frac{\omega^2}{\varepsilon^2}} + i\frac{2\eta}{\pi} \tan^{-1} \frac{\omega}{\varepsilon}\right]. \] (2.28)

From equations (2.5) and (2.28), the energy dissipation per oscillation cycle is

\[ E_{\text{biot}} = 2\eta k X^2 \tan^{-1} \frac{\omega}{\varepsilon}, \] (2.29)

which is practically constant for \( \frac{\omega}{\varepsilon} > 10 \).

![Figure 2.3](image)

Energy dissipation per cycle for the Biot visco-elastic model

2.3.2 Vibration of a SDOF system with Biot’s spring.

Consider the system of figure 2.4.
The equation of motion can be written as

$$m\ddot{x} + k\left[ x + \frac{2\eta}{\pi} \int_{t_0}^{t} E_i[\varepsilon(t - \tau)]\dot{x}(\tau)d\tau \right] = P(t); \quad x(t_0) = x_0, \dot{x}(t_0) = v_0 \quad (2.30)$$

or in the more convenient form

$$\ddot{x} + \omega_n^2 \left[ x + \frac{2\eta}{\pi} \int_{t_0}^{t} E_i[\varepsilon(t - \tau)]\dot{x}(\tau)d\tau \right] = R(t), \quad (2.31)$$

where \( R(t) = \frac{P(t)}{m}, \omega_n = \sqrt{\frac{k}{m}}. \)

The integro-differential equation (2.31) can be solved by numerical integration; obviously, there exists some difficulty because of its mathematical form. The usual numerical integration algorithms (Runge-Kutta, Predictor-Corrector methods etc.) do a fixed number of function evaluations per time step. This holds for the non-adaptive versions of the algorithms. Usually, the cost of these function evaluations is constant, or it has small variations, depending on the values of the independent variables. This means that, in most cases, the cost of solving an ODE is \( O(N) \) where \( N \) is the number of samples of the time interval for which the solution is calculated. When one applies the same
methods to equation (2.31), the computational cost per time step is dominated by the evaluation of an integral from $t_0$ to $t$. This usually takes $O(n)$ operations where $n$ is the number of samples in the interval $[t_0,t]$. This means that, under the assumptions made earlier, the total computational cost is $O(N^2)$ which is unacceptable in many cases. In Chapter 6 two methods are proposed to circumvent this problem and reduce the cost to $O(n)$.

A time domain numerical solution is unavoidable when the system is nonlinear. In the case of linear systems, however, an inexpensive and accurate method of calculating the response is the use of the frequency domain representation. It is trivial to verify that the equivalent form of equation (2.31) in the frequency domain is

$$X(\omega) = H_B(\omega)R(\omega),$$

(2.32)

where

$$H_B(\omega) = \frac{1}{-\omega^2 + \omega_n^2 \left[ 1 + \frac{2\eta}{\pi} \left( \ln \sqrt{1 + \frac{\omega^2}{\epsilon^2}} + i \tan^{-1} \frac{\omega}{\epsilon} \right) \right]}$$

(2.33)

is the FR of the system of figure 2.4 and $R(\omega)$ and $X(\omega)$ are the Fourier transforms of $R(t)$ and $x(t)$, respectively. Figure 2.5 depicts the amplitude of $H_B(\omega)$ for different values of the loss factor $\eta$. 
Figure 2.5
Amplitude response of SDOF system with Biot's hysteretic element
CHAPTER 3
Linear systems theory

3.1 System classification

A system can be viewed as a black box, see figure 3.1, that receives a certain input, or excitation, and produces an output, or response. One has to refine this broad definition in order to describe better the type of systems that exist in engineering. There are deterministic and stochastic, linear and nonlinear, continuous time and discrete time, time invariant and time variant systems. The following presentation focuses on linear, deterministic, time invariant (LTI) systems in continuous as well as in discrete time.

A system is linear if and only if the superposition principle holds. The superposition principle states that when the input of a system is a linear combination of two or more signals, then the output is the linear combination of the outputs of the system to each one of the individual inputs, see figure 3.2.

![Figure 3.1](image)

**Figure 3.1**
Graphical representation of a linear system.
A system is stochastic if its parameters such as mass, stiffness etc. are random variables. In any other case it is called deterministic. Systems whose parameters are defined over a continuous range of time are continuous time systems. On the other hand, a system whose parameters are defined only at discrete time instants $t_0, t_1, ..., t_k, ...$, is a discrete time system. Usually, these time instants are equally spaced. In that case the lag $t_{k+1} - t_k$ is denoted by $T$ and is referred to as the sampling interval. A time invariant system is one that does not change with time. In mathematical terminology, if the input is $x(t)$ and the corresponding output is $X(t)$, then the input $x(t-l)$ produces output $X(t-l)$.

### 3.2 Linear system description using the impulse response.

The problem addressed in this section is the calculation of the response of a system to an arbitrary excitation. The answer to this question requires the introduction of the impulse function. It is defined by the equations

$$\int_{-\infty}^{\infty} \delta(t) dt = 1, \quad \text{and} \quad \delta(t) = 0 \quad \text{for} \quad t \neq 0. \quad (3.1)$$
The impulse function has the well-known sampling property
\[ \int_{-\infty}^{\infty} \delta(t) \varphi(t) \, dt = \varphi(0), \] (3.2)
where \( \varphi(t) \) is a function continuous at \( t=0 \). This is proved by considering the impulse function as the limit of a rectangular pulse with unit area, see figure 3.3. Equation (3.2) can be readily generalized as
\[ \int_{-\infty}^{\infty} \delta(t-a) \varphi(t) \, dt = \varphi(a). \] (3.3)

The significance of the impulse function to the study of LTI systems comes from the representation of an arbitrary input as a sum of shifted impulses and from the superposition principle. If a LTI system is excited by an impulse function \( \delta(t) \), the output is \( h(t) \) which is referred to as the impulse response of the system. Since the system is time invariant, a shifted impulse will induce a shifted version of \( h(t) \), see figure 3.4. Also, by using equation (3.3) an arbitrary input \( x(t) \) can be represented as an infinite sum of shifted impulses. That is,
\[ x(t) = \int_{-\infty}^{\infty} \delta(t-\tau)x(\tau) \, d\tau. \] (3.4)
From the superposition principle and the preceding discussion it is seen that the output $y(t)$ of the system with input $x(t)$ is simply

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)x(\tau)d\tau = h(t) * x(t). \quad (3.5)$$

The preceding equation is often called the Duhammel Integral. The ‘$*$’ symbol represents the convolution operation which is defined as

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(t - \tau)f_2(\tau)d\tau. \quad (3.6)$$

Convolution is commutative and satisfies the distributive and associative laws. The concept of the impulse response is explained previously for continuous time systems only, but it can be easily converted to apply to discrete time systems. In the discrete time case things are somewhat easier, since there is no dealing with functions like the
continuous time impulse function, which is described adequately only by using the theory of distributions [16].

An impulse in the discrete time is defined as

$$\delta(nT) = \delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases},$$  \hspace{1cm} (3.7)

where $T$ is the sampling interval.

![Graph of impulse function](image)

Figure 3.5
The impulse function in discrete-time

A signal $x(n) \equiv x(nT)$ can be written as a sum of impulses. That is,

$$x(n) = \sum_{k=-\infty}^{\infty} \delta(n-k)x(k).$$  \hspace{1cm} (3.8)

The impulse response of a discrete time LTI system is denoted by $h(n)$. With reasoning similar to the discussion for the continuous time case, the response of the discrete time LTI system, $y(n)$, to an arbitrary input, $x(n)$, is
\[ y(n) = \sum_{k=\infty}^{\infty} h(n-k)x(k). \]  

(3.9)

This is the convolution of the sequences \( y(n) \), \( x(n) \).

![Graph A](image1)

![Graph B](image2)

**Figure 3.6**  
Impulse responses of discrete time LTI systems  
a) Causal system  
b) Non-causal system

A concept essential in LTI systems in both continuous, as well as in discrete time, is the *causality* property. A system is *causal* if its impulse response is zero for negative time values.

\[ h(t) = 0 \text{ for } t < 0 \text{ (continuous time)}, \]

\[ h(n) = 0 \text{ for } n < 0 \text{ (discrete time)}. \]  

(3.10)
A system that fails to satisfy equation (3.10) is called non-causal. The importance of this property is clear from equations (3.5) and (3.9). If a system is causal then these equations become

\[ y(t) = \int h(t - \tau)x(\tau)d\tau, \quad (3.11) \]

and

\[ y(n) = \sum_{k=\to}^{\to} h(n - k)x(k). \quad (3.12) \]

Equations (3.11) and (3.12) do not involve the values of the input beyond the current time instant. This is not the case in equations (3.5) and (3.9) where the value of the output of the system at a given instant \( t \) or \( n \) depends not only on the past of the input, but also on its future. Clearly, physically realizable systems have the causality property. Non-causal systems can be implemented on a digital computer with stored data as input. If the computer is working in real-time mode, an acceptably small delay can be introduced so that the non-causal system can be implemented. Figures 3.6a,b show impulse responses of a causal and a non-causal LTI systems, respectively. It is noted that a signal that is zero for \( t>0 \) (\( n>0 \)) is called anticausal.

3.3 Transform description methods for LTI systems.

3.3.1 The Laplace transform, stability of continuous-time LTI systems.

The Laplace transform is a widely utilized tool for analyzing LTI continuous time systems. The bilateral Laplace transform of a signal \( x(t) \) is defined as

\[ X(s) = \mathcal{L}(x) = \int_{-\infty}^{\infty} x(t)e^{-st}dt. \quad (3.13) \]

The set of values of \( s \) on the complex plane (s-plane) for which the integral in (3.13) is convergent, is the region of convergence (ROC).
For the functions encountered in engineering there is almost always a non-empty ROC.

The signal \( x(t) \) can be calculated from \( X(s) \) using the Inverse Laplace transform

\[
x(t) = \mathcal{L}^{-1}(X) = \frac{1}{2\pi i} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds.
\]

(3.14)

The integration is performed along a vertical line that belongs to the ROC of the integral in equation (3.13). The Laplace transform has a useful property; the Laplace transform of the convolution of two functions is equal to the product of the Laplace transforms of the individual functions. Thus, the application of the Laplace transform to both parts of (3.5) yields

\[
Y(s) = H(s)X(s),
\]

(3.15)

where \( Y(s) = \mathcal{L}(y) \), \( X(s) = \mathcal{L}(x) \), and \( H(s) = \mathcal{L}(h) \) which is referred to as the transfer function (TF) of the system. The transfer function of a LTI system not only allows one to calculate the response of the system to any input, but also gives information about the system itself.

The continuous time LTI systems encountered in practice are usually described by differential equations of the form

\[
b_n \frac{d^n x}{dt^n} + b_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + b_1 \frac{dx}{dt} + b_0 x =
\]

\[
= a_m \frac{d^m y}{dt^m} + a_{m-1} \frac{d^{m-1} y}{dt^{m-1}} + \ldots + a_1 \frac{dy}{dt} + y.
\]

(3.16)

The Laplace transform has the important property [15], for quiescent initial conditions,

\[
\mathcal{L} \left( \frac{d^2 x}{dt^2} \right) = s^2 \mathcal{L}(x).
\]

(3.17)

By taking the Laplace transform of both sides of equation (3.16) and using equation (3.17) one obtains
\[(b_n s^n + b_{n-1} s^{n-1} + \ldots + b_0)X(s) = (a_m s^m + \ldots + a_1 s + 1)Y(s) \Leftrightarrow \]

\[\Leftrightarrow \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{n} b_k s^k}{1 + \sum_{k=1}^{m} a_k s^k}.\]

(3.18)

From equations (3.15) and (3.18), the transfer function of the system described by equation (3.16) is a rational expression of s. That is,

\[H(s) = \frac{\sum_{k=0}^{n} b_k s^k}{1 + \sum_{k=1}^{m} a_k s^k}.\]

(3.19)

The ROC of \(H(s)\) is a half-plane on the s-domain. This is due to the fact that the system described by (3.16) is a causal system. The boundary of the ROC is a vertical line. The ROC is the half-plane to the right of this line whose abscissa is equal to the maximum real part of the poles of (3.19), see figure 3.7. For anticausal systems the ROC of the TF is still a half plane but it lies on the left of the boundary vertical line. The abscissa of the boundary is equal to the minimum real part of the poles of the TF of the anticausal system, see figure 3.8.
Figure 3.7
The impulse response and the ROC of the TF of a causal system

Figure 3.8
The impulse response and the ROC of the TF of an anticausal system
In the general case where the impulse response has both anticausal and causal parts, the ROC of the TF is a stripe, see figure 3.9. The poles on the right half plane are the poles of the Laplace transform of the anticausal part, and the poles in the left half plane are the poles of the Laplace transform of the causal part of $h(t)$.

![ROC of the TF of a non-causal system](image)

**Figure 3.9**
The ROC of the TF of a non-causal system

The location of the poles gives information about the behavior of the system. Specifically, it determines the *stability* of the system. A LTI system is stable if for any bounded input, the output is also bounded. This kind of system is referred to as *BIBO stable*, where BIBO stands for ‘Bounded Input – Bounded Output’. It can be proved [15,30] that a LTI system is BIBO stable if and only if the ROC of the TF contains the imaginary axis. This means, for a causal system, that all its poles belong to the left half-
plane. For a generic, non-causal, system all poles due to the causal part of the impulse response must lie on the left half plane (LHP) and all poles due to the anticausal part of the impulse response must lie on the right half plane (RHP).

3.3.2. The Z-transform, stability of discrete-time systems.

The Z-transform is the counterpart of the Laplace transform in the discrete domain. The Z-transform of a signal \( x(n) \) is defined as

\[
X(z) = \mathcal{Z}(x) = \sum_{n=-\infty}^{\infty} x(n)z^{-n},
\]

where \( z \) is a complex number. The set of values of \( z \) for which the sum in equation (3.20) converges is the region of convergence of the Z-transform of \( x \). Inversely, \( x(n) \) can be calculated from \( X(z) \) using the equation

\[
x(n) = \mathcal{Z}^{-1}(X) = \frac{1}{2\pi i} \oint_C X(z)z^{n-1},
\]

where \( C \) is a closed curve that contains the origin and lies in the ROC of \( X(z) \). The application of the Z-transform to both sides of (3.9) yields

\[
Y(z) = H(z)X(z),
\]

where \( Y(z) = \mathcal{Z}(y) \), \( X(z) = \mathcal{Z}(x) \), and \( H(z) = \mathcal{Z}(h) \) which is the transfer function of the discrete time system with impulse response \( h(n) \). In the continuous time, one deals primarily with systems described by differential equations, see equation (3.16). In the discrete time, difference equations are encountered of the form

\[
b_q x(n - q) + b_{q-1} x(n - q + 1) + \ldots + b_1 x(n - 1) + b_0 x(n) =
\]

\[
= a_p y(n - p) + a_{p-1} y(n + p + 1) + \ldots + a_1 y(n - 1) + y(n).
\]
From equation (3.23) it is readily seen that the current value of the output $y(n)$ depends on previous values of the output, $y(n-1),...,y(n-p)$ and on the current and previous values of the input $x(n),...,x(n-q)$. Therefore, it is a causal LTI system. If equation (3.23) is changed by incorporating future values of the input or the output, $y(n+1), y(n+2),...,x(n+1),...$ a non-causal system results. This system, of course, is physically unrealizable.

The $Z$-transform has the property

$$Z(x(n-k)) = z^k Z(x(n)).$$  \hspace{1cm} (3.24)

By taking the $Z$-transform of both sides of (3.23) and using (3.24) one derives the equation

$$(b_q z^{-q} + b_{q-1} z^{-q+1} + ... + b_0)X(z) = (a_p z^{-p} + ... + a_1 z^{-1} + 1)Y(z) \Leftrightarrow$$

$$\Leftrightarrow \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 + \sum_{l}^{p} a_l z^{-l}}.$$ \hspace{1cm} (3.25)

The transfer function of the system described by (3.23) is given by the equation

$$H(z) = \frac{\sum_{k=0}^{q} b_k z^{-k}}{1 + \sum_{l}^{p} a_l z^{-l}}.$$ \hspace{1cm} (3.26)

The ROC of $H(z)$ is the complex ($z$-) plane except a circular disk centered at the origin. This is due to the causality property of the system, see figure 3.10. For an anticausal system, the ROC of the TF is a disk centered at the origin, see figure 3.11. In general, when a signal has a causal as well as an anticausal part, the ROC of its $Z$-transform is a
circular ring centered at the origin, see figure 3.12. The radii of these circles depend on the location of the poles of the TF.

**Figure 3.10**
The ROC of the TF of a causal discrete-time LTI system

**Figure 3.11**
The ROC of the TF of an anticausal system
Figure 3.12
The ROC of the TF of a non-causal system

For a non-causal system the radius of the inner circle is the maximum modulus of the poles of $H_c(z)$, where $H_c(z)$ is the Z-transform of the causal part of the impulse response $h(n)$, see figure 3.12. That is,

$$H_c(z) = \sum_{k=0}^{\infty} h(k)z^{-k}.$$  \hspace{1cm} (3.27)

The radius of the outer circle is the minimum modulus of the poles of $H_{ac}(z)$ defined by the equation

$$H_{ac}(z) = \sum_{k=0}^{\infty} h(k)z^{-k},$$  \hspace{1cm} (3.28)

and which is associated with the anticausal part of $h(n)$.

A discrete time LTI system is BIBO stable if any bounded input yields a bounded output. It can be proved [30] that the preceding statement is true, if and only if the ROC of the TF of the system contains the unit circle. Intuitively, this corresponds to an impulse response described by a sum of exponentially decaying sinusoids.
3.3.3 Frequency Response

In this subsection the notion of the frequency response of a stable LTI system is introduced. This implies, for a continuous time system, that the ROC of the TF of the system includes the imaginary axis of the complex plane. This allows the evaluation of the Laplace transform in equation (3.13) for \( s = j\omega \) which yields the equation

\[
H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} \, dt.
\]  

(3.29)

The impulse response \( h(t) \) can be evaluated by equation (3.14). That is,

\[
h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)e^{j\omega t} \, d\omega.
\]  

(3.30)

From equations (3.29) and (3.30) it is readily seen that the functions \( h(t) \) and \( H(j\omega) \) are a Fourier transform pair. If a system with impulse response \( h(t) \) has input \( x(t) \) and the corresponding output is \( y(t) \), equation (3.15) yields

\[
Y(j\omega) = H(j\omega)X(j\omega),
\]  

(3.31)

where \( Y(j\omega) \) and \( X(j\omega) \) are the Fourier transforms of \( y(t) \) and \( x(t) \) respectively. Equation (3.31) leads one to adopt the notion of the frequency response of a LTI system. The Fourier transform of a signal \( x(t) \) is given by

\[
X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt.
\]  

(3.32)

Inversely,

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} \, d\omega.
\]  

(3.33)

From equation (3.33) it is readily seen that \( x(t) \) signal is represented by an infinite sum of complex sinusoids of circular frequency \( \omega \) and amplitude \( |X(j\omega)| \Delta\omega \), where \( \Delta\omega \) is let to tend to zero. On the other hand, equation (3.31) shows that every infinitesimal frequency
component of $x(t)$ is multiplied by $H(i\omega)$. The function $H(i\omega)$ represents the change in
amplitude and phase of every frequency component of the input and it is referred to as the
frequency response of the system. If the input is a sinusoid of infinite duration, that is,
$x(t) = \sin \omega t$, then the output can be expressed as
\[ y(t) = |H(i\omega)|\sin[\omega t + \angle H(i\omega)]. \] (3.34)
The functions $|H(i\omega)|$ and $\angle H(i\omega)$ are called amplitude response and phase response
of the LTI system, respectively. For systems with real valued impulse response the
amplitude response is an even function of frequency and the phase response is an odd
function of frequency $\omega$. In other words
\[ H(-i\omega) = H^{*}(i\omega), \] (3.35)
where the star symbol represents complex conjugation.

In the discrete time case the stability condition is that the unit circle belongs in the
ROC of the $Z$-transform of the impulse response of the LTI system. This allows the use
of $z = e^{i\omega}$ for calculating the integral in equation (3.20). This is simply the Discrete Time
Fourier Transform (DTFT) of $h(n)$,
\[ H(e^{i\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-i\omega n}. \] (3.36)
The Inverse Discrete Time Fourier Transform provides $h(n)$ from $H(e^{i\omega})$. Specifically,
\[ h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{i\omega})e^{i\omega n}. \] (3.37)
The function $H(e^{i\omega})$ is referred to as the frequency response of the discrete-time LTI
system with impulse response $h(n)$. The reasoning behind this definition is analogous to
that for the continuous time case. The function $H(e^{j\omega})$ is periodic with period $2\pi$. This is due to the periodicity of the complex exponentials $e^{j\omega n}$ that are involved in equation (3.36). Finally, it is noted that the relation in the frequency domain between the input $x(n)$ and the output $y(n)$ of a system with impulse response $h(n)$ is given by the equation

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \quad (3.38)$$

where $X(e^{j\omega})$ and $Y(e^{j\omega})$ are the DTFTs of $x(n)$ and $y(n)$.

### 3.4 The sampling theorem

The continuous time analysis of dynamic systems is quite versatile. However, in many cases analytical methods are not sufficient and one has to use numerical algorithms on a digital computer which deals only with discrete time signals of finite length. Thus, a methodology for converting a continuous time signal to a discrete time signal is needed. This methodology should allow one to assess the accuracy of the conversion so that the numerical results are meaningful. A practical way of representing a continuous time signal $x_a(t)$ in a digital form is by sampling it. First the sampling interval $T$ must be selected and then the discrete time signal

$$x(n) = x_a(nT), -\infty < n < \infty, \quad (3.39)$$

is formed. In this case uniform sampling is used since the time interval between two successive samples is constant. It is noted that in a real digital computer, quantizing of $x(n)$ is performed in parallel with the sampling procedure, since there is only finite arithmetic accuracy.
There is a straightforward connection between the two signals $x_a(t)$ and $x(n)$ in the frequency domain. Let $X_a(i\omega)$ be the Fourier transform of $x_a(t)$ and $X(e^{i\omega})$ be the DTFT of $x(n)$. Then the following relation holds [30]

$$X(e^{i\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(i\omega - \frac{2\pi k}{T}). \quad (3.40)$$

The preceding equation shows that the spectrum of $x(n)$ is a summation of shifted and scaled replicas of the spectrum of the analog signal $x_a(t)$. In figure 3.13 the effect of sampling to the spectrum of $x(n)$ is shown for various values of the sampling frequency $1/T$.

(a)

(b)

(c)

**Figure 3.13**

The effect of sampling of a continuous-time signal

a) Spectrum of the original signal
b) Spectrum of the sampled signal with sampling interval $T<\pi/\Omega$

c) Spectrum of the sampled signal with sampling interval $T>\pi/\Omega$
The frequency content of a discrete time signal is confined in the interval \([0, \pi]\). The values of \(X(e^{j\omega})\) in \([-\pi, 0]\) are a mirror of the interval \([0, \pi]\) with an additional complex conjugation. Outside the interval \([-\pi, \pi]\) there is repetition of the same pattern, since \(X(e^{j\omega})\) is a periodic function of \(\omega\). By examining figure 3.13b in the interval \([0, \pi]\) it is seen that if one chooses a sampling interval that is smaller than \(\pi/\Omega\) then there is no distortion to the original spectrum, except from a scaling and stretching operation. In figure 3.13c the sampling interval is greater than \(\pi/\Omega\) and there is a distortion called aliasing. Aliasing is due to the mapping of high frequencies to lower frequencies via sampling. This is demonstrated in figure 3.14. The quantity \(\Omega\) is referred to as the bandwidth of the original signal \(x_a(t)\). The sampling theorem, which follows, states a condition for perfect reconstruction of the original analog signal from its samples.

**Sampling theorem.** A continuous-time signal with zero frequency content above \(\Omega\) rad/sec (bandlimited signal with bandwidth \(\Omega\)) can be uniquely recovered from its samples provided that the sampling interval \(T \leq \pi/\Omega\) secs.

The frequency \(1/T = \Omega/\pi\) is called Nyquist rate. If the condition of the sampling theorem is met, the original signal can be reconstructed by the formula

\[
x_s(t) = T \sum_{n=-\infty}^{\infty} x_a(nT) \frac{\sin(\pi t/T - \pi n)}{\pi t/T - \pi n}.
\] (3.41)
3.5 The DFT algorithm, computation of the response of a LTI system using the frequency domain representation

The frequency domain description of a discrete time signal $x(n)$ is its DiscreteTime Fourier Transform $X(e^{i\omega})$, as explained in subsection 3.3.3. This is a continuous function of frequency, hence it is not a convenient representation of $x(n)$ by a digital computer. This problem can be circumvented by sampling $X(e^{i\omega})$ in the frequency domain. The key assumption is that $x(n)$ has finite duration, which is true for all real computational schemes. Suppose that $x(n)$ is non-zero for $0 \leq n \leq N - 1$. The idea is to sample $X(e^{i\omega})$ at $N$ equidistant points in the interval $[0,2\pi]$ that is, $X(k) \equiv X(e^{i2\pi k/N})$ for $0 \leq k \leq N - 1$. One may be interested in representing the
signal $x(n)$ by the $N$ samples in the frequency domain ($X(k)$). This is feasible by using the **Discrete Fourier Transform** (DFT) and its inverse transform (IDFT). Specifically,

$$\text{DFT}(x): \quad X(k) = X(e^{i2\pi k/N}) = \sum_{n=0}^{N-1} x(n)e^{i2\pi kn/N} \quad k = 0, 1, \ldots, N-1. \quad (3.42)$$

$$\text{IDFT}(X): \quad x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)e^{i2\pi kn/N} \quad n = 0, 1, \ldots, N-1. \quad (3.43)$$

The computational cost of the DFT is $O(N^2)$ since for every $k$ there are $O(N)$ computations and this is done for $N$ values of $k$. A cost of order $O(N^2)$ is not satisfactory for most practical applications. Fortunately, there is an alternative, recursive algorithm, for evaluating the sums of equation (3.41) that achieves $O(N\log N)$ calculations for $N=2^l$.

This is the **Fast Fourier Transform** (FFT) which is described, along with some alternative versions of the algorithm, in [30].

The problem addressed is the calculation of the response of a continuous time LTI system to a bandlimited excitation. Consider a LTI system with impulse response $h(t)$ and frequency response $H(i\omega)$. The input to this system is a finite duration signal $x(t)$ with Fourier transform $X(i\omega)$, and the corresponding output is denoted by $y(t)$ which has Fourier transform $Y(i\omega)$. The signals $x(t)$ and $y(t)$ are related in the frequency domain by equation (3.31). On a digital computer $x(t)$ and $y(t)$ are represented by their samples.

Considering the sampling theorem, one chooses a sufficiently small sampling interval $T$ so that $x(t)$ is represented without distortion by $x(n) \equiv x(nT)$. This can be done only if $x(t)$ is bandlimited. The goal is to calculate the samples of $y(t)$, $y(n) \equiv y(nT)$. First, the sequence $\{x(n)\}$ must be truncated, and only the samples from the interval where $x(t)$ is non-zero are kept. It is noted that, since $x(t)$ has finite duration, its spectrum is not
bandlimited. This is due to the uncertainty principle [27,30]. This introduces a distortion which can be arbitrarily small for sufficiently small $\tau$. The preceding remarks establish that the sequence of length $N$ \{x(n)\} represents adequately $x(t)$. This makes possible the calculation of the DFT of \{x(n)\}, \{X(k)\}. The connection between $X(i\omega)$ and \{X(k)\} is established by the sampling theorem. The spectrum of the discrete time signal $x(n)$ is $X(e^{i\omega})$ which is, from (3.40), equal to a sum of scaled and stretched copies of $X(i\omega)$.

By ignoring the effect of aliasing, one obtains the equation

$$X(e^{i\omega}) = \begin{cases} 
\frac{1}{T} X(i\omega T) & 0 \leq \omega \leq \pi \\
\frac{1}{T} X^*(i[2\pi - \omega]T) & \pi < \omega \leq 2\pi 
\end{cases} \quad (3.44)$$

The numbers \{X(k)\} are the samples of $X(e^{i\omega})$ at $\omega_k = 2\pi k / N$.

Equation (3.31) can be rewritten

$$Y(i\omega) = H(i\omega)X(i\omega) \quad , -\Omega \leq \omega \leq \Omega$$

or,

$$\frac{1}{T} Y(e^{i\omega}) = \tilde{H}(i\omega) \frac{1}{T} X(e^{i\omega}) \quad , 0 \leq \omega \leq 2\pi \quad , (3.45)$$

where

$$\tilde{H}(iw) = \begin{cases} 
H(iw/T) & , 0 \leq w \leq \pi \\
H(i[w - 2\pi]/T) & , \pi < w \leq 2\pi 
\end{cases} \quad (3.46)$$

Equation (3.46) correlates the spectra of the discrete-time signals $y(n)$, $x(n)$. Equation (3.45) holds for every value of the frequency variable in the interval $[0,2\pi]$, thus it is also true for $\omega_k = 2\pi k / N$, $k=0,1,\ldots,N-1$. This means that

$$Y(k) = \tilde{H}(2\pi k / N)X(k) \quad k = 0,1,\ldots,N-1 \quad , (3.47)$$
where \{Y(k)\} and \{X(k)\} are the DFT's of \{y(n)\} and \{x(n)\} respectively. Finally the samples of the output \{y(n)\} are computed by taking the IDFT of \{Y(k)\} computed by equation (3.47) with the aid of equation (3.46). It is noted that the computational cost of this procedure is \(O(N\log N)\), whereas the computation of the response by calculating the convolution integral (3.11) takes typically \(O(N^2)\) operations.
CHAPTER 4
Numerical Methods

4.1 Preliminary remarks.

Systems of ordinary differential equations are commonly used in engineering. The general form of these systems of equations is

$$\dot{x} = f(x, t), \quad x(t_0) = x_0,$$

(4.1)

where \( x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \) is the state variable of the system, and \( x_1, x_2, \ldots, x_n \) are referred to as the states of the system. The vector \( x_0 \) contains the initial conditions at \( t = t_0 \). Equation (4.1) describes lumped parameter models as opposed to distributed parameter models which are described by partial differential equations. In this thesis 2\(^{nd}\)-order differential equations are encountered. That is,

$$m \ddot{x} + g(x, x, t) = F(t).$$

(4.2)

Differential equations of 2\(^{nd}\) or higher order can be written in the standard form of (4.1). Specifically, by introducing a new state variable \( x_1 \), equation (4.2) can be written in the equivalent form

$$\dot{x} = x_1,$$

(4.3)

$$\dot{x}_1 = F(t) - g(x, x_1, t),$$
where $\mathbf{x} = \begin{bmatrix} x \\ x_1 \end{bmatrix}$ is the state vector.

In this chapter numerical methods for solving the one-dimensional case of equation (4.1) are examined. These methods can be extended to treat higher order systems. The standard form used throughout the remainder of the chapter is

$$\dot{x} = f(x, t), \quad x(t_0) = x_0. \quad (4.4)$$

### 4.2 Self-starting methods

The goal is to evaluate the solution of equation (4.4) $x(t)$ by starting at $t_0$ and continuing stepwise to compute approximate values of $x(t)$ at equally spaced points $t_1 = t_0 + h$, $t_2 = t_0 + 2h$, ..., where $h$ is a constant called the step size. The relation between the samples of solution $x(t)$ evaluated at two consecutive time steps is

$$x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} f(x, t) \, dt, \quad (4.5)$$

where $x_k = x(t_k) = x(t_0 + kh)$. All the algorithms described in this section calculate the integral in equation (4.5) with various levels of trade-off between computational cost and accuracy of the approximate solution. A possible approximate solution method is described by the equation

$$x_{n+1} = x_n + hf(x_n, t_n). \quad (4.6)$$

This is Euler's method and the idea is to approximate the integral in equation (4.5) by $hf(x_n, t_n)$, see figure 4.1a.
In figures 4.1b, 4.1c the ideas behind the Modified Euler and Heun methods are illustrated.

Modified Euler method: $x_{n+1} = x_n + hf(x_n + \frac{1}{2} hf_n, t_n + \frac{1}{2} h)$, \hspace{1cm} (4.7)

where $f_n = f(x_n, t_n)$.

Here the *midpoint rule* is used, but instead of $\dot{x}(t_n + \frac{1}{2} h) = f(x(t_n + \frac{1}{2} h), t_n + \frac{1}{2} h)$ the approximation $\dot{x}(t_n + \frac{1}{2} h) = f(x_n + \frac{1}{2} hf_n, t_n + \frac{1}{2} h)$ is used.

Heun's method: $x_{n+1} = x_n + \frac{1}{2} h[f(x_n + hf_n, t_n + h) + f(x_n, t_n)]$. \hspace{1cm} (4.8)

In this method the *trapezoidal rule* is employed to approximate the area under $\dot{x}(t) = f(x, t)$. Figures 4.1a, b, c show various schemes to approximate this integral. This is
possible only if the values of the real solution are known. This is the reason for using approximations of the slope of the solution \( \dot{x}(t) = f(x,t) \) by evaluating \( f(x,t) \) at proper points. For example, in Heun’s method the value \( \dot{x}(t_{n+1}) = f(x_{n+1}, t_n + h) \) that is needed for evaluating the area of the trapezoid in figure 4.1c is approximated by \( f(x_n + hf_n, t_n + h) \).

The presented methods are all self-starting methods. This term applies to all numerical methods for initial value problems (IVP) where all slope estimates are calculated after \( x_n \) is obtained. This means that with only the initial conditions at \( t_0 \) the algorithm can start and calculate the solution for the entire interval of interest. As for the accuracy of the presented methods, Euler’s method is a first-order method since the error of the approximate solution at each step is \( O(h^2) \). The modified Euler and Heun methods are second-order methods since the error at each step is \( O(h^3) \). This can be proved, for Euler’s method, by expressing (4.5) in a different form by using the Taylor expansion,

\[
x(t_n + h) = x(t_n) + h \dot{x}(t_n) + \frac{h^2}{2!} \ddot{x}(t_n) + \ldots
\]

(4.8)

In Euler’s method, the preceding series is truncated beyond the 2nd term. Thus, the error is of the form \( \frac{h^2}{2!} \ddot{x}(\xi) \), where \( \xi \in [t_n, t_n + h] \), which is \( O(h^3) \).

The next method uses more evaluations of \( f(x,t) \) in the interval \([t_n, t_n+h]\) in order to integrate the slope \( \dot{x}(t) \) with cubic accuracy. In other words, the integral in (4.5) is calculated exactly if \( \dot{x}(t) \) is a cubic polynomial in the interval of integration. This method is the Fourth-Order Runge-Kutta (RK4) method and the error per step is \( O(h^5) \). It requires 4 estimates of the slope per step.
4th-Order Runge-Kutta (RK4) method:

\[ x_{n+1} = x_n + \frac{h}{6} \left[ s_1 + 2s_2 + 2s_3 + s_4 \right], \]

where

\[ s_2 = f(x_n + \frac{h}{2}s_1, t_n + \frac{h}{2}), \]

\[ s_3 = f(x_n + \frac{h}{2}s_2, t_n + \frac{h}{2}), \]

\[ s_4 = f(x_n + hs_3, t_n + h). \]

\[ (4.9) \]

### 4.3 Predictor-Corrector methods

Predictor-Corrector (multistep) methods use previous slope estimates for evaluating \( x_{n+1} \). These methods are not self-starting because they cannot be used at \( t=t_0 \).

The basic scheme used in multistep methods is a two step strategy: In the first step, a predicted value of \( x_{n+1} \) is computed

\[ p_{n+1} = x_n + h g_1(f_k, f_{k+1}, \ldots, f_n) \quad k \leq n, \]

\[ (4.10) \]

where \( f_j = f(x_j, t_j) \). The function \( g_1 \) is an integration formula of \( \dot{x}(t) \) in the interval \([t_n, t_n+h]\) whose input is previous values of slope estimates, \( f_j \).

In the second step a corrected approximation of \( x_{n+1} \) is calculated using previous values \( f_j \) and the predicted value \( p_{n+1} \). That is,

\[ c_{n+1} = x_n + h g_2\left(f_{k+1}, f_{k+2}, \ldots, f_n, f(p_{n+1}, t_n + h)\right), \]

\[ (4.11) \]

where \( g_2 \) is an integration formula of \( \dot{x}(t) \) that uses previous and the predicted estimates of the slope, \( f_{k+1}, \ldots, f_n, f(p_{n+1}, t_n + h) \).
If polynomial interpolation is used for the evaluation of the desired integrals, the functions \( g_1 \) and \( g_2 \) are just a linear combination of their operands. This is done by the **Fourth-Order Adams Predictor-Corrector method (APC4)**. It fits a cubic polynomial to the points \((t_{k-3}, f_{k-3}), \ldots, (t_n, f_n)\) so that the predicted value of \( x_{n+1} \), \( p_{n+1} \), is calculated with accuracy \( O(h^5) \). The same procedure is followed for calculating the corrected estimate \( c_{n+1} \), see figure 4.2.

\[
\text{Adams Predictor: } p_{n+1} = x_n + \frac{h}{24} [-9f_{n-3} + 37f_{n-2} - 59f_{n-1} + 55f_n], \quad (4.12a)
\]

\[
\text{Adams Corrector: } c_{n+1} = x_n + \frac{h}{24} [f_{n-2} - 5f_{n-1} + 19f_n + 9f(p_{n+1}, t_n + h)]. \quad (4.12b)
\]

![Figure 4.2](image-url)

**Figure 4.2**

a) Adams Predictor

b) Adams Corrector
CHAPTER 5
Stochastic Processes and Random Vibrations

Random (stochastic) vibration theory uses concepts of stochastic process theory to determine the statistical properties of the response of engineering systems subjected to random excitation. Examples of systems excited by random input are buildings with wind loading, marine structures in waves, vibrating aircraft due to turbulent flow. For linear systems the estimation of the statistics of the response to random excitation is straightforward. Difficulties arise when one deals with nonlinear systems. The method of statistical linearization [33] is examined in this chapter. It is a widely utilized method for estimating the statistical parameters of the response of nonlinear systems to random excitation. An especially useful tool in the analysis of randomly excited systems is the *Monte Carlo simulation* technique. It is a statistical sampling method which involves the generation of a great number of realizations of a stochastic input and the subsequent calculation of the response using deterministic methods. The statistics of the response are estimated from the set of computed response records. It provides a reliable reference point for every approximation technique such as the statistical linearization.

5.1 Random variables

The basis of stochastic process theory is the probability theory, the main idea of which is the concept of a *random variable*. A random variable $\eta$ is a number that characterizes a certain event of an experiment. There are *continuous* and *discrete* random variables. The set of events for a continuous random variable is uncountably infinite and it results from measurements of physical quantities. On the other hand, a
A discrete random variable corresponds to a set of events that has distinct elements. In this chapter only continuous random variables are examined.

The mathematical description of a continuous random variable is its \emph{probability distribution function}, defined by

\[ F(x) = P(\eta \leq x). \tag{5.1} \]

If the set of events contains all \( \mathbb{R} \), the probability distribution function \( F(x) \) tends to zero when \( x \to -\infty \) and to one when \( x \to \infty \). The derivative of \( F(x) \), \( f(x) \), is called the \emph{probability density function} (pdf) of \( \eta \). Thus, the probability that \( \eta \) is less or equal to \( x \) is

\[ P(\eta \leq x) = \int_{-\infty}^{x} f(x_1) dx_1. \tag{5.2} \]

The extension of the preceding definitions for the case of a vector of random variables \( \eta = [\eta_1, \eta_2, \ldots, \eta_n]^T \) leads to the introduction of the joint distribution function, which is defined by the equation

\[ F(\mathbf{x}) = P(\eta \leq \mathbf{x}). \tag{5.3} \]

The joint probability density function is given by

\[ f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n) = \frac{\partial^n F(x_1, x_2, \ldots, x_n)}{\partial x_1 \partial x_2 \cdots \partial x_n}. \tag{5.4} \]

The probability that \( \eta \) belongs to a region \( R \) can be expressed as

\[ P(\eta \in R) = \int \ldots \int f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n. \tag{5.5} \]

The pdf describes totally the statistical properties of a random variable. There are some important quantities that partially describe a random variable such as the \emph{mean} and the \emph{variance}. The mean, or expected value, of a random variable is the average of its values
in a infinite number of experiments. The variance is a measure of the dispersion around the mean value. Specifically, the mean value $\mu$ is defined by

$$\mu = E[\eta] = \int_{-\infty}^{\infty} x f(x) \, dx,$$

(5.6)

and the variance is defined by

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx.$$  

(5.7)

The quantity $\sigma$ is referred to as the standard deviation of $\eta$. It is noted that $\mu$ and $\sigma^2$ are the expected values of appropriate functions of the random variable $x$. In general, the expected value of a function $g(\eta)$ is defined as

$$E\{g(\eta)\} = \int_{-\infty}^{\infty} g(x) f(x) \, dx.$$  

(5.9)

The $n$-th moment of $\eta$ is given by

$$E\{\eta^n\} = \int_{-\infty}^{\infty} x^n f(x) \, dx.$$  

(5.10)

An important probability distribution is the Gaussian distribution. For the one-dimensional case it is defined by the equation

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(\tau - \mu)^2}{2\sigma^2}} \, d\tau,$$

(5.11)

where $\mu$ and $\sigma^2$ are the mean and variance of the Gaussian random variable.

The joint pdf for a $n$-dimensional Gaussian variable $\eta = [\eta_1 \quad \eta_2 \quad \cdots \quad \eta_n]^T$ is given by

$$f(x) = \frac{1}{(2\pi)^{n/2} |C|^{1/2}} e^{-\frac{1}{2}(x-m)^T C^{-1} (x-m)},$$

(5.12)
where \( C \) is the covariance matrix whose elements are \( C_{ij} = \mathbb{E}\{(\eta_i - m_i)(\eta_j - m_j)\} \), and \( \mathbf{m} = \mathbb{E}\{\eta\} \) is the mean value vector.

A quite important property of Gaussian random variables is that their sum is Gaussian as well. This must be taken into account when one deals with linear systems, since their output is a weight sum (or integral) of its input at several time instants. The ubiquity of the Gaussian distribution is due to the Central Limit Theorem, which states that the pdf of a sum of \( N \) independent random variables with arbitrary pdf's tends to a Gaussian pdf when \( N \) tends to infinity. In physical systems random phenomena are usually a combination of many unrelated effects. For example, thermal noise in electric circuits is the combination of the random motion of a large number of electrons, therefore it is Gaussian. Another useful property of a Gaussian random variable is the relation

\[
\mathbb{E}\{g(\eta)\eta\} = \mathbb{E}\{\eta \eta^T\} \mathbb{E}\{\nabla g(\eta)\},
\]

where \( \nabla \) is the gradient operator

\[
\nabla = \begin{bmatrix} \frac{\partial}{\partial \eta_1} & \frac{\partial}{\partial \eta_2} & \cdots & \frac{\partial}{\partial \eta_n} \end{bmatrix}^T.
\]

5.2 Stochastic Processes

A random variable is a mapping of events to numerical values. Every time an experiment is performed a number is obtained. This concept can be generalized so that in every experiment a function \( \eta_k(\tau) \) is obtained where \( \tau \) is a parameter. Usually this parameter is time. A stochastic process \( \eta(t) \) is the 'ensemble' of the infinite number of possible sample functions \( \eta_k(t) \), see figure 5.1. The particular function \( \eta_n(t) \) obtained after each experiment is called a 'realization' of the stochastic process \( \eta(t) \).
A stochastic process $\eta(t)$ at a given time instant $t_k$ is a random variable described by its own pdf. The process itself is described completely by the joint probability distribution of the infinite number of random variables each one of which corresponds to a different time instant. Specifically,

$$F(x, t) = P(\eta(t) \leq x)$$

(5.15)
The function $F(x,t)$ is referred to as the distribution of $\eta(t)$. The probability density of $\eta(t)$ is given by the equation

$$f(x,t) = \frac{\partial F(x,t)}{\partial x}.$$  \hfill (5.16)

Although $F(x,t)$ describes totally the stochastic process $\eta(t)$, in practice only certain average values such as the expected value of $\eta(t)$ and $\eta^2(t)$ are needed. These functions of time, along with the autocorrelation function, are defined by the equations:

Mean value : \quad $\mu(t) = E\{\eta(t)\} = \int_{-\infty}^{\infty} xf(x,t)dx$, \hfill (5.17)

Autocorrelation : \quad $R(t_1,t_2) = E\{\eta(t_1)\eta(t_2)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1x_2f(x_1,x_2,t_1,t_2)dx_1dx_2$. \hfill (5.18)

The expected value of $\eta^2(t)$ is equal to $R(t,t)$ and it is referred to as the average power of $\eta(t)$ at time $t$. The autocorrelation function is a measure of the statistical dependence of the values of the stochastic process at time instants $t_1, t_2$.

A special case of random process is the **Gaussian process**. The joint pdf of a Gaussian process $\eta(t)$ at time instants $t_1, t_2, \ldots, t_n$ is a $n$-dimensional Gaussian density described by equation (5.12). This means that the statistical properties of a Gaussian process are determined solely by its mean value $\mu(t) = E\{\eta(t)\}$ and by its covariance function

$$c(t_1,t_2) = E\{|\eta(t_1) - \mu(t_1)|\eta(t_2) - \mu(t_2)|\} = R(t_1,t_2) - \mu(t_1)\mu(t_2)$$. \hfill (5.19)

A significant class of stochastic processes is the class of **stationary processes**. The statistics of a strictly stationary stochastic process are invariant to a time shift. This means that the mean value $\mu(t)$ is a constant $\mu$ and that the autocorrelation function $R(t_1,t_2)$ is a function of the time lag $\tau = t_1 - t_2$, denoted by $R(\tau)$. A stationary process in the
**wide sense** is one whose only the mean value \( \mu(t) \) is constant and the autocorrelation function \( R(t_1, t_2) \) depends only on the difference \( \tau = t_1 - t_2 \). The average power of a stationary stochastic process is equal to

\[
E[|\eta(t)|^2] = R(0).
\]

(5.20)

A property of a subclass of stationary random processes is **ergodicity.** Ergodicity allows the estimation of statistical parameters of a random process from a single realization of the process. This can be done by **time averaging.** Consider a realization \( \eta_1(t) \) of an ergodic process \( \eta(t) \). An estimation of the mean value of \( \eta(t) \) \( \mu \) is given by

\[
\mu_1 = \lim_{T \to \infty} \frac{1}{T} \int_0^T \eta_1(t) dt.
\]

(5.21)

This quantity can be viewed as a sample of the random variable

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \eta(t) dt.
\]

(5.22)

The ergodicity property is then defined in the **mean square sense** as

\[
\lim_{T \to \infty} E \left( \left| \frac{1}{T} \int_0^T \eta(t) dt - \mu \right|^2 \right).
\]

(5.23)

Another concept related to a stationary stochastic process is the **power spectrum** \( S(\omega) \). It is related to the autocorrelation function \( R(\tau) \) by the Wiener-Khinchin equations

\[
S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{-i\omega \tau} d\tau,
\]

(5.24a)

and

\[
R(\tau) = \int_{-\infty}^{\infty} S(\omega) e^{i\omega \tau} d\omega.
\]

(5.24b)

By setting \( \tau = 0 \) equation (5.24b) one obtains
\[ R(0) = \mathbb{E}[|\eta(t)|^2] = \int_{-\infty}^{\infty} S(\omega) \, d\omega. \] (5.25)

Equation (5.25) shows that the power spectrum \( S(\omega) \) is the distribution of the average power of \( \eta(t) \) along the frequency axis.

### 5.3 Linear systems with stochastic input

Consider a LTI system with impulse response \( h(t) \) and frequency response \( H(\omega) \). The output of this system to an arbitrary input is given by the convolution integral, see equation 3.5. The problem addressed in this section is the determination of the statistical properties of the output when the input is a stochastic process, see figure 5.2.

![Figure 5.2](image)

A LT1 system with stochastic input

A convenient approach for solving this problem involves the concept of the power spectrum. The relation of the power spectra of the input and the output is given by the equation

\[ S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega), \] (5.26)

where \( S_{xx} \) and \( S_{yy} \) are the power spectra of the input and the output [28].

Consider a white noise random process \( x(t) \). A white noise process is one whose power spectrum is constant, taken equal to 1 for convenience. In the time domain, the
autocorrelation function is a delta function at \( \tau = 0 \). If the system of figure 5.2 is excited with \( x(t) \), the power spectrum of the output is

\[
S_{yy}(\omega) = |H(\omega)|^2.
\]  

(5.27)

### 5.3.1 Modeling of random processes on a digital computer

It is important in some cases to simulate stationary random processes by a digital computer. The capacity to generate realizations of a random process with a given power spectrum is needed. Digital filters can be used to generate samples of the required random process realizations. Their input is a discrete time signal generated by sampling a white noise process. The basis of this solution method is equation (5.27) and the sampling theorem. First a desired power spectrum \( S_d(\omega) \) is selected, and a sampling interval \( T \) is selected so that the effect of aliasing is negligible. Then, a discrete time filter must be designed so that the squared modulus of its FR matches \( S_d(\omega) \). The input is simply a sequence of independently generated random samples with proper variance and mean value zero. The output is a sequence of samples of a realization of a random process with power spectrum approximately equal to \( S_d(\omega) \). This procedure leads to a set of linear equations for the filter parameters when an all-pole (AR) filter is considered.

The transfer function of the AR filter is

\[
\hat{H}(z) = \frac{G}{1 + \sum_{k=1}^{p} a_k z^{-k}}.
\]  

(5.28)

The parameters \( a_k \) are determined so that an appropriate optimality criterion is satisfied.

That is,
\[ E \equiv \int_{-\omega_b}^{\omega_b} \frac{S_d(\omega)}{|H(e^{i\omega T})|^2} d\omega = \text{minimum}, \quad (5.29) \]

where \( \omega_b = \pi/T \) is called cutoff frequency and it is double the sampling circular frequency. The constant \( G \) is determined by the equation

\[ \int_{-\omega_b}^{\omega_b} S_d(\omega) d\omega = \int_{-\omega_b}^{\omega_b} |H(e^{i\omega T})|^2 d\omega. \quad (5.30) \]

Equation (5.29) leads to a set of linear algebraic equations which are referred to as the Yule-Walker equations [38]

\[
\begin{bmatrix}
R_0 & R_1 & \ldots & R_{p-1} \\
R_1 & R_0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & R_1 \\
R_{p-1} & R_{p-2} & \ldots & R_0
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_p
\end{bmatrix}
=
\begin{bmatrix}
R_1 \\
R_2 \\
\vdots \\
R_p
\end{bmatrix},
\]

where \( R_k \) are the values of the autocorrelation function \( R(\tau) \) at lags \( \tau = kT \).

\[ R_k = \int_{-\omega_b}^{\omega_b} S_d(\omega)e^{i\omega kT} d\omega, \quad k = 0, 1, \ldots \quad (5.32) \]

The solution of the system (5.31) yields values of the parameters \( a_1, \ldots, a_p \). These values are used in equation (5.30) to calculate the \( G \) constant. It can be proved that \( G \) is given by

\[ G = \frac{T}{2\pi} \left[ R_0 + \sum_{k=1}^{p} a_k R_k \right]^{1/2}. \quad (5.33) \]

A time series can be generated by the recursive equation,

\[ y_n = y(nT) = Gw_n - \sum_{k=1}^{p} a_k y_{n-k}. \quad (5.34) \]
The sequence \( \{y_n\} \) contains the samples of a realization of the target process. The sequence \( \{w_n\} \) are samples of a realization of a clipped white noise process. The power spectrum of a clipped white noise process is constant in \([-\omega_b, \omega_b]\) and 0 outside that interval. Since the variance of a zero mean random process is given by equation (5.25), the noise deviates \( \{w_n\} \) have variance equal to \( 2\omega_b \). The synthesis of the sequence \( \{w_k\} \) is done by random number generation algorithms. An example of the AR filter approach is shown in figure 5.3.

\[
S(\omega) = \frac{1 + \left( \frac{\omega}{\omega_b} \right)^2}{\left( 1 - \left( \frac{\omega}{\omega_b} \right)^2 \right)^2 + \left( 2\zeta_b \frac{\omega}{\omega_b} \right)^2}, \text{ for } \omega_b=2, \zeta_b=0.2.
\]

**Figure 5.3**
AR approximation of the Kanai-Tajimi spectrum [38],
5.4 The method of statistical linearization

It has been shown in the previous sections that a tool for the computation of the
statistics of the response of a LTI system to stationary stochastic excitation is the input-
output relation of power spectra expressed by equation (5.26). However, for a nonlinear
system excited by a stationary random process there is no exact solution for most cases.
The method of statistical linearization [33] is introduced and analyzed in conjunction with
an example. It is a method for replacing the nonlinearity in a system by an equivalent, in
some sense, linear element.

The system under consideration is a SDOF system with nonlinear stiffness and it
is described by the following 2\textsuperscript{nd} order differential equation

\begin{equation}
\ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 [x + eg(x)] = f(t), \tag{5.35}
\end{equation}

where \( \omega_n \) is the natural frequency of the system, \( \zeta \) is the damping ratio, \( f(t) \) is the
excitation, \( g(x) \) is an odd non-linear function of \( x \), and \( e \) is a parameter indicating the
severity of the nonlinearity. It is assumed that \( f(t) \) is a zero-mean stationary Gaussian
process with power spectrum \( S_f(\omega) \). The goal is to derive an equivalent linear equation

\begin{equation}
\ddot{x} + 2\zeta \omega_n \dot{x} + \omega_{eq}^2 x = f(t), \tag{5.36}
\end{equation}

where \( \omega_{eq} \) is selected such as a certain minimization criterion is met. This criterion is the
minimization of the expected value of the error \( \delta = \omega_n^2 [x + eg(x)] - \omega_{eq}^2 x \) in the least
square sense. Thus

\begin{equation}
E(\delta^2) = \text{minimum} \Rightarrow \\
\Rightarrow \frac{d}{d\omega_{eq}^2} E(\delta^2) = 0. \tag{5.37}
\end{equation}

From equation (5.37) one derives the equation
\[
\omega_{eq}^2 = \omega_n^2 \left( 1 + e ^ {\frac{E[xg(x)]}{E[x^2]}} \right) = \omega_n^2 \left( 1 + e ^ {\frac{E[xg(x)]}{\sigma_x^2}} \right). \tag{5.38}
\]

where \( \sigma_x^2 \) is the variance of the output \( x \). It is assumed that the input has zero-mean. The output, as well, is a zero-mean process, and \( \sigma_x^2 = E \{ (x(t) - \mu)^2 \} = E \{ x(t)^2 \} \). In evaluating the expected value of \( xg(x) \) in equation (5.38) the pdf of \( x(t) \) is required which is unknown. The key idea is to approximate the process \( x(t) \) by a Gaussian process and, thus, describe its probability density in terms of its variance \( \sigma_x^2 \) solely. This approximation leads to a simplified version of equation (5.38) by using the result of equation (5.13). That is,

\[
\omega_{eq}^2 = \omega_n^2 \left( 1 + e \frac{dg}{dx} \right). \tag{5.39}
\]

Having determined \( \omega_{eq} \), one can proceed by calculating the variance of the response, involved in equation (5.38), by using the equation of the equivalent linear system described by equation (5.36). This can be achieved by employing equation (5.26) which yields

\[
S_x(\omega) = |H(\omega)|^2 S_f(\omega) = \left| \frac{1}{-\omega^2 + \omega_{eq}^2 + i2\zeta\omega_n\omega} \right|^2 S_f(\omega), \tag{5.40}
\]

where \( H(\omega) \) is the frequency response of the equivalent system. Finally, one derives

\[
\sigma_x^2 = \int S_x(\omega) d\omega = \int \left| \frac{1}{-\omega^2 + \omega_{eq}^2 + i2\zeta\omega_n\omega} \right|^2 S_f(\omega) d\omega. \tag{5.41}
\]

Equations (5.41) and (5.38) reflect nonlinear equations involving the unknowns \( \omega_{eq}^2 \) and \( \sigma_x^2 \), and can be solved iteratively.

This technique is next demonstrated for the estimation of the variance of the response of a SDOF system with cubic stiffness. Consider a SDOF system described by the equation
\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 (x + ex^3) = f(t). \]  

(5.42)

For this case equations (5.39) and (5.41) can be written as

\[ \omega_{eq}^2 = \omega_n^2 \left( 1 + eE(3x^3) \right) = \omega_n^2 \left( 1 + 3e\sigma_x^2 \right), \]  

(5.43)

\[ \sigma_x^2 = \int \frac{1}{-\omega^2 + \omega_{eq}^2 + i2\zeta \omega_n} \left| S_r(\omega) \right|^2 d\omega = \int \frac{1}{-\omega^2 + \omega_n^2 (1 + 3e\sigma_x^2) + i2\zeta \omega_n} \left| S_r(\omega) \right|^2 d\omega, \]  

(5.44)

where \( E(x^2) = \sigma_x^2. \)

Equation (5.44) is a nonlinear algebraic equation with \( \sigma_x^2 \) as the unknown. It is an equation of the form

\[ X = F(X, e), \]  

(5.45)

and it can be solved by the following iterative algorithm.

1. **Step 1:** \( X_0 = \) Initial Approximation, \( \text{tol} = \) Tolerance of the Solution.

2. **Step 2:** \( X_{n+1} = F(X_n, e). \)

3. **Step 3:** Repeat Step 2 until \( |X_{n+1} - X_n| \leq \text{tol}. \)
5.5 The Monte Carlo Method

The Monte Carlo method is a numerical method for solving problems by simulating experiments having inherent randomness. A large number of experiments is required so that the estimations of the unknown statistical quantities can be reliable. The results have always some degree of uncertainty and confidence intervals for its values are calculated. First a confidence level \( p \) is specified. An interval is calculated, in which the true value of the estimated parameter belongs with probability \( p \). This gives more information about the unknown parameter than just its estimated value.

The Monte Carlo method has great versatility and it is applied to diverse problems in various scientific disciplines. Although it is a stochastic method, it can be used, as well, to solve certain deterministic problems. This can be done by inventing a stochastic model whose random variables have statistical parameters associated with the unknown quantities of the original problem. An example of this approach is the evaluation of a definite integral using the Monte Carlo method [35].

The first step of the Monte Carlo method is the generation of random numbers. A random variable is described by its probability distribution function. Thus one needs computer generated samples that conform with this particular distribution. It turns out that the only needed algorithm is one that generates samples of a uniform random variable in the interval \([0,1]\). A uniform random variable \( \eta \) has probability distribution equal to

\[
F_{\eta}(x) = \begin{cases} 
0 & , x < 0 \\
 x & , 0 \leq x \leq 1 \\
 1 & , 1 \leq x 
\end{cases} \quad (5.46)
\]
Consider a random variable $y$ with probability distribution $F_y(x)$. A new random variable $z$ is defined as a function of $y$. That is,

$$z = F_y(y).$$  \hspace{1cm} (5.47)

The probability distribution of $z$, $F_z(x)$ is given by the equation

$$F_z(x) = P[z \leq x] = P[F_y(y) \leq x] = P[y \leq F_y^{-1}(x)] = F_y[F_y^{-1}(x)] = x.$$  \hspace{1cm} (5.48)

Inverting $F_y(x)$ is possible since $F_y(x)$ is a monotonous function in $\mathbb{R}$. Equation (5.48) describes a method for generating samples of a random variable with a given probability distribution using samples of a uniformly distributed random variable; if $\{u_1, \ldots, u_n\}$ are uniformly distributed samples, the numbers $w_i = F_y^{-1}(u_i)$ are samples of a random variable with probability distribution $F(x)$. Algorithms that generate samples of a random variable are called pseudorandom number generators. This is due to the fact that the 'random' numbers are generated by an algorithm that produces always the same results when the same seed number is used. The seed number is a starting point for the algorithm. The simplest case of an random number generator is given by the equation

$$w_{n+1} = F(w_n).$$  \hspace{1cm} (5.49)

where $F$ is an appropriate function [35].

The iterative algorithm of equation (5.49) produces a periodic sequence $\{w_i\}$ since the computer has finite digit length. The goal is to make this period as large as possible and to have a uniform as possible spread of the generated numbers.

The second step of the method is the repeated simulation of the experiments using the sequence of pseudorandom numbers from step 1. In problems of dynamics the objective is to determine the statistics of the response of a given system to random excitation. In
this case the sequence \( \{w_i\} \) is used to generate several time histories for which the corresponding responses are calculated. The time histories used to excite the system are generated so that they are compatible with a given power spectrum. The AR algorithm described in section 5.4 can be employed for this purpose.

The third step is to use statistical methods for estimating the unknown statistical quantities, based on the set of computed response records. Consider, for example, estimating the mean value \( \mu \) of the response of a system to random excitation using the Monte Carlo method. This can be done by running the simulation for each one of the excitation records long enough for the system to reach steady-state. It is assumed that this happens before the time instant \( t_s \). An estimate of the mean value \( \mu \) is then obtained by the equation

\[
\bar{\mu} = \frac{1}{N} \sum_{k=1}^{N} x_k(t_s),
\]

where \( x_k(t), k=1,\ldots,N \) are the calculated response records. The quantity \( \bar{\mu} \) is called the \textit{sample mean} and it is a random variable with mean value \( E(\bar{\mu}) = \mu \), and variance given by \( E((\bar{\mu} - \mu)^2) = \frac{\sigma_x^2}{N} \), where \( \mu \) and \( \sigma_x^2 \) are the true mean value and variance of the response of the system. Because of the central limit theorem, \( \bar{\mu} \) has approximately a Gaussian distribution which is described only by \( \mu \) and \( \sigma_x^2 \). This enables the calculation of confidence intervals for the mean estimator \( \bar{\mu} \). A well known relation is the \textit{three sigma} rule which applies to a Gaussian distribution. It states that for a Gaussian random variable \( x \) with mean \( m \) and standard deviation \( \sigma \) the following relation holds

\[
P\{|x - m| < 3\sigma\} = 0.997.
\]
Applying the preceding equation for the random variable $\bar{\mu}$ yields the equation

$$
P\left( \left| \frac{1}{N} \sum_{k=1}^{N} x_k(t_s) - \mu \right| \leq \frac{3\sigma_x}{\sqrt{N}} \right) = 0.997. \tag{5.52}$$

Equation (5.52) provides an indication of the expected error of the estimation of the mean as the number of simulations $N$ increases. In order to gain one digit in accuracy the number of simulations must be multiplied by 100. This fact confines the use of the Monte Carlo method to estimations with confidence intervals of 90% to 95%. In the presented analysis the variance of the response is assumed known. Usually this is not the case, and $\sigma^2_x$ must be estimated. An estimator of the variance is given by the equation

$$
\overline{\sigma^2_x} = \frac{1}{N} \sum_{k=1}^{N} \left( x_k(t_s) - \bar{\mu} \right)^2, \tag{5.53}
$$

which is referred to as the *sample variance*. The derivation of confidence intervals for the mean and the variance in this case can be found in [15].
CHAPTER 6
Numerical Simulations

In this chapter numerical simulation results are presented for the SDOF system displayed in fig. 6.1. It consists of a mass attached to a Biot hysteretic element in parallel with a cubic stiffness spring.

![Figure 6.1](image)

SDOF system with cubic stiffness with Biot hysteretic damping

The equation of motion of the system is

$$\ddot{x} + \omega_n^2 \left[ x + \varepsilon (x) + \frac{2\eta}{\pi} \int_0^t \int_0^\tau e_i [e(t - \tau)] \dot{x}(\tau)d\tau \right] = P(t), \quad (6.1)$$

where $\omega_n = \sqrt{\frac{k}{m}}$ is the natural frequency of the system, $E_i(a)$ is the exponential integral defined in chapter 2, and $\eta$ is the loss factor. It is convenient to write equation (6.1) in the dimensionless form

$$\ddot{x} + 4\pi^2 \left[ x + \varepsilon x + \frac{2\eta}{\pi} \int_0^t \int_0^\tau e_i [e'(t' - \tau)] \dot{x}(\tau)d\tau \right] = \hat{P}(t'), \quad (6.2)$$

where
\[ t' = \frac{t}{T_n}, \quad T_n = \frac{2\pi}{\omega_n}, \]  
\[ \varepsilon' = \varepsilon T_n. \]

It is noted that in equation (6.2) the differentiation is performed with respect to the dimensionless time variable \( t' \).

### 6.1 Frequency domain analysis

When the nonlinearity parameter \( \varepsilon \) in equation (6.2) is set equal to zero, the system becomes linear and the response to an arbitrary input can be calculated by using the frequency domain description of the system. The frequency response of the system is

\[
H_B(\omega) = \frac{1}{-\omega^2 + 4\pi^2 \left[ 1 + \frac{2\eta}{\pi} \left( \ln \sqrt{1 + \frac{\omega^2}{\varepsilon'^2} + i \tan^{-1} \frac{\omega}{\varepsilon'}} \right) \right]}. \tag{6.5}
\]

An approach for computing the response using equation (6.5) can be based on the FFT algorithm. The procedure is analyzed in section 3.5 and it is summarized in equations (3.45), (3.46) and (3.47). The only quantity that needs to be calculated is the sampling interval \( T \). It is selected such as \( |H_B(\omega)| = 0 \) for \( |\omega| > \omega_b \), where \( \omega_b = \frac{\pi}{T} \) is the cutoff frequency. Figure 6.2 shows the amplitude response \( |H_B(\omega)| \) for various values of \( \eta \). A reasonable cutoff frequency is \( \omega_b = 15\omega_n \) which corresponds to a sampling interval \( T = 1/15 \). It is reminded that \( T \) is normalized with respect to the natural period of the system \( T_n \).
The use of FFT for the computation of the response of the system is demonstrated for two different sampling values. The input is the continuous time signal

$$x(t) = \sin \pi t' + \sin 2\pi t',$$

and the loss factor $\eta$ is set equal to 0.5.
(a) Transient response computed using the FFT
(b) Absolute difference of responses calculated with various sampling frequencies.
The excitation shown in figure 6.3a has finite length. Thus, its spectrum has infinite support [27,30]. This introduces an error due to aliasing in the calculation of the response. From figure 6.3b, it is readily seen that this error is quite small. The comparison is performed with respect to the calculation using a sampling interval of length $T_s=1/100$ which almost eliminates the effect of aliasing. It is noted that the absolute error is nearly zero when the system reaches steady state. For the time intervals where the transient behavior dominates, the error is higher. This is expected because the high frequency components dominate the transient behavior of the system, and these high frequencies are affected more by aliasing.

6.2 Time domain numerical analysis

The time domain numerical schemes that are used for solving equation (6.2) are the 4th order Runge-Kutta (RK4) and the 4th order Adams Predictor-Corrector (APC4) methods described in Chapter 4.

First, equation (6.2) is written in the state-space form

$$
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= P(t') - 4\pi^2 \left[ x_1 + ex_1^3 + \frac{2\eta}{\pi} \int_0^{t'} E_i [e^i(t'-\tau)] x_2 d\tau \right], \\
\end{align*}
$$

(6.7)

where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the state, and $x_1$, $x_2$ are the state variables which represent displacement and velocity accordingly. The time variable $t'$ is discretized at equally spaced points, $t'_n = nh$, $n = 0, 1, \ldots, N$, where $h$ is the time step used in the differential...
equation solver. A difficulty in solving equation (6.7) is encountered in the calculation of the integral \( \int_0^{\tau_n} \zeta(t_n' - \tau) x_2(\tau) \, d\tau \) at every time instant \( t_n \). The function \( \zeta(t_n' - \tau) \) tends to infinity when \( \tau \to t_n \), thus the integral is improper. For calculating such an integral, the function \( E_1(x) \) is sampled at equally spaced points. That is,

\[
g_k = E_1(\varepsilon' k h), \quad k=1,\ldots,N.
\] (6.8)

The value of \( g_0 \) can be calculated by polynomial extrapolation using the values \( g_1, g_2, \ldots \), see figure 6.4a. Another approach is to require that the integral \( \int_0^{\varepsilon' h} E_1(x) \, dx \) is equal to an estimate of the integral that uses the sample values \( g_0, g_1 \), etc. If for example, the trapezoidal rule is employed, see figure 6.4b, \( g_0 \) is defined by the equation

\[
\frac{g_0 + g_1}{2} \varepsilon' h = \int_0^{\varepsilon' h} E_1(x) \, dx.
\] (6.9)

Since \( \varepsilon' h \) is small, the following approximation is valid [1]

\[
E_1(x) = -\ln(x) - \gamma,
\] (6.10)

where \( \gamma \) is the Euler constant, \( \gamma = 0.5772 \ldots \), and \( 0 < x \leq \varepsilon' h \). The preceding approximation and integration by parts are used to evaluate both sides of equation (6.9). Equations (6.9) and (6.10) yield

\[
g_0 = -\ln(\varepsilon' h) + 2 - \gamma.
\] (6.11)

This definition of \( g_0 \) is used in all simulations in this thesis. It is possible to use a higher order integration rule instead of the trapezoidal rule for better accuracy. In that case, \( g_0 \) is defined by the equation
\[ w_0 g_0 + w_1 g_1 + \ldots + w_m g_m = \int_0^h E_1(x) dx, \tag{6.12} \]

where \( g_1, \ldots, g_m \) are given by equation (6.8) and the weights \( \{w_k\} \) depend on the quadrature rule that is selected. The samples \( g_0, g_1, \ldots, g_n \) defined by equations (6.8) and (6.11) can be used in the integration formula

\[ \int_0^{t_n} E_1(x) x_2(x) dx = \sum_{k=0}^n w_k g_{n-k} x_2(kh). \tag{6.13} \]

If the trapezoidal rule is employed, the weights \( w_k \) are

\[ w_0 = 1/2, w_1 = \ldots = w_{n-1} = 1, w_n = 1/2. \tag{6.14} \]

The calculation of the sum in (6.13) takes \( cn \) calculations. This must be done for every \( n=1,\ldots,N \). This means that the total cost is

\[ c \sum_{k=1}^N n = c \frac{N(N+1)}{2} = O(N^2), \]

which is undesirable for many practical cases. In order to reduce the cost to \( O(N) \), two methods are introduced. The first calculates the integral in equation (6.7) recursively. The second uses digital filters whose frequency response matches the frequency response of the linear system described, in the time domain, by the integral under consideration.
Figure 6.4
a) Calculation of $g_0$ by polynomial extrapolation
b) Calculation of $g_0$ by equalizing the areas under $y_1$ and $y_2$
6.2.1 Recursive computation of a time dependent integral using Prony’s method.

Consider the integral

\[ I(t') = \int_0^{t'} E_i[\epsilon'(t' - \tau)] x_2(\tau) d\tau = \int_0^{t'} F(t', \tau) x_2(\tau) d\tau. \]  \hspace{1cm} (6.15)

The preceding equation yields

\[ I(t' + \Delta t') = \int_0^{t' + \Delta t'} F(t' + \Delta t', \tau) x_2(\tau) d\tau. \]  \hspace{1cm} (6.16)

The difficulty in calculating \( I(t' + \Delta t') \) using the previous value \( I(t') \) is that the integrand changes along with the upper limit of integration. The function \( F(t' + \Delta t', \tau) \) can be expanded in a Taylor series. That is,

\[ F(t' + \Delta t', \tau) = F(t', \tau) + \Delta t \frac{\partial F(t', \tau)}{\partial t'} + \frac{(\Delta t')^2}{2!} \frac{\partial^2 F(t', \tau)}{\partial t'^2} + ... \]  \hspace{1cm} (6.17)

Then, one can write

\[ I(t' + \Delta t') = \int_0^{t' + \Delta t'} F(t' + \Delta t', \tau) x_2(\tau) d\tau + \int_0^{t'} F(t', \tau) x_2(\tau) d\tau = \int_0^{t'} \frac{\partial F(t', \tau)}{\partial t'} x_2(\tau) d\tau + I(t') + \Delta t \int_0^{t'} \frac{\partial^2 F(t', \tau)}{\partial t'^2} x_2(\tau) d\tau + ... \]  \hspace{1cm} (6.18)

Equation (6.18) shows that the problem of having to evaluate an integral over the entire time interval \([0,t]\) is still present. Specifically, the integral \( \int_0^{t'} \frac{\partial F(t', \tau)}{\partial t'} x_2(\tau) d\tau \) needs to be evaluated. It is noted that in this particular case \( F(t, \tau) \) cannot even be expanded to a Taylor series as in equation (6.17) because it is discontinuous. The preceding analysis is performed in order to show that the problem exists even if \( F(t', \tau) \) is a continuous function that approximates \( E_i[\epsilon'(t' - \tau)] \) in the interval of integration.
A solution to this problem is to approximate the function \( y = E_i(x) \) by a sum of exponentials using Prony’s method which is described in Appendix I.

Specifically, set

\[
E_i(x) = \sum_{i=1}^{p} b_i e^{a_i x}, \quad (6.19)
\]

where \( a_k \) and \( b_k \) are appropriate constants to be determined.

Certain auxiliary functions are defined by the formula

\[
I_k(t') = \int_{0}^{t'} e^{a_k [e^{t'+\Delta t}-e^t]} x_2(\tau) d\tau, \quad k = 1, \ldots, p. \quad (6.20)
\]

Combining equations (6.15), (6.19) and (6.20) yields

\[
I(t' + \Delta t') = \int_{t'}^{t' + \Delta t'} \! E_i[e^{t'+\Delta t'} - e^t] x_2(\tau) d\tau + \int_{t'}^{t' + \Delta t'} \! \sum_{k=1}^{p} b_k e^{a_k [e^{t'+\Delta t'} - e^t]} \int_{0}^{t'} e^{a_k [e^{t'-\tau} - e^t]} x_2(\tau) d\tau =
\]

\[
= \int_{t'}^{t' + \Delta t'} \! E_i[e^{t'+\Delta t'} - e^t] x_2(\tau) d\tau + \sum_{k=1}^{p} b_k e^{a_k \Delta t'} I_k(t') = \quad (6.21)
\]

Similarly, a recursive formula is derived for every \( I_k(t') \),

\[
I_k(t' + \Delta t') = \int_{t'}^{t' + \Delta t'} \! e^{a_k [e^{t'+\Delta t'} - e^t]} x_2(\tau) d\tau + e^{a_k \Delta t'} I_k(t'), \quad k = 1, \ldots, p. \quad (6.22)
\]

Equations (6.21) and (6.22) can be written in a form suitable for the differential equation solver. Specifically,

\[
I(n+1) \equiv I(t'),_{t=(n+1)h} = \int_{nh}^{(n+1)h} E_i[e^{t'-\tau}] x_2(\tau) d\tau + \sum_{k=1}^{p} b_k e^{a_k \tau} I_k(n), \quad (6.23a)
\]

\[
I_k(n+1) \equiv I_k(t'),_{t=(n+1)h} = \int_{nh}^{(n+1)h} e^{a_k [e^{(n+1)h-t}-e^{nh}]} x_2(\tau) d\tau + e^{a_k \tau} I_k(n), \quad k = 1, \ldots, p. \quad (6.23b)
\]
The integrals in equation (6.23) are evaluated over an interval of constant length, thus the computational cost is constant. In the simulations that follow, these integrals are calculated using the trapezoidal rule. Specifically, from equations (6.8), (6.10), (6.13) and (6.23) the following approximations are derived

\[
\int_{nh}^{(n+1)h} E_i[e'(t_{n+1} - \tau)] x_2(\tau) d\tau = \frac{h}{2} \left( g_0 x_2[(n + l)h] + g_1 x_2(nh) \right), \tag{6.24}
\]

and

\[
\int_{nh}^{(n+1)h} e^{2 \lambda e'[z(n+1)h - \tau]} x_2(\tau) d\tau = \frac{h}{2} \left[ x_2[(n + l)h] + e^{2 \lambda e^h} x_2(nh) \right]. \tag{6.25}
\]

It is seen that, the developed method is based on calculating the integral of equation (6.15) by using a linear combination of a family of integrals \( I_k(n) \) which are updated at each step by relying on equations (6.23b) and (6.25). Tables I and II contain sets of coefficients \( \{a_k, b_k\} \) which approximate the function \( E_i(x) \) with various levels of accuracy. In all presented simulations, the coefficients of Table I are used. Figure 6.5 shows the response of the system described by equation (6.2) which is calculated with the RK4 method with step size and sampling interval given by \( h=T=1/100 \). The benchmark for the accuracy of the method is the computed response by using the FFT algorithm, see section 6.1. The system parameters are \( \eta=0.5, c=0 \).
Figure 6.5
Calculation of the response using the proposed time domain technique

6.2.2 Digital filter design for time-domain integration

In section 6.2.1 a numerical method is proposed for evaluating recursively the time-dependent integral $I(t)$ given by equation (6.15). In this section the use of digital filters for approximating $I(t)$ is examined. Consider the integral

$$ I_n = I(t_n) = \int_0^{t_n} E_1[\dot{e}^T(t_n - \tau)]x_1(\tau) \, d\tau, \quad t_n = nh, \quad n = 0, \ldots, N. \quad (6.26) $$

The goal is to design a digital filter whose output $\hat{I}_n$ approximates $I_n$. The formula for calculating $\hat{I}_n$ is given by the equation

$$ \hat{I}_n = \sum_{k=0}^{d} b_k x_1(t_{n-k}) - \sum_{k=1}^{d} a_k \hat{I}_{n-k}, \quad (6.27) $$

where $x_1(t_{n-k})$ are the current and previous samples of the displacement $x_1(t)$. 
The design of the filter can be performed in the frequency domain [29]. The integral in equation (6.15) leads in the frequency domain to the equations

\[ I(\omega) = H_1(\omega)X_1(\omega), \]

\[ H_1(\omega) = \ln \sqrt{1 + \frac{\omega^2}{\varepsilon^2} + i \tan^{-1} \frac{\omega}{\varepsilon}}, \]  

where \( I(\omega) \), and \( X_1(\omega) \) are the Fourier transforms of \( I(t) \), and \( x_1(t) \), respectively. The objective is to design a digital filter whose frequency response matches the \( H_1(\omega) \) in the least square sense. That is,

\[ \int_{-\omega_d}^{\omega_d} \left| H_d(e^{j\omega T_s}) - H_1(\omega) \right|^2 d\omega = \text{minimum}, \]  

where \( H_d(e^{j\omega}) = \sum_{k=0}^{q} b_k e^{-j\omega k} \) is the frequency response of the digital filter, \( \omega_d \) is the cutoff frequency used in the design, and \( T_s = \frac{\pi}{\omega_d} \) is the corresponding sampling interval.

It is noted that the transfer function \( H_1(\omega) \) depends only on the variable \( \sigma = \omega / \varepsilon' \). This fact is used in Appendix II so that the designed filters can be utilized in the simulations of the Biot element with various parameters. The design procedure in subsection 5.3.1 is referred to an AR (Auto-Regressive) filter whose amplitude response matches a target function. In that case only the modulus of the frequency response is taken into account.

The problem addressed in this section regards the matching of both the modulus and the phase of the transfer function \( H_d(\omega) \) with those of the corresponding target transfer function. For this purpose, nonlinear optimization algorithms [22,11] are used to minimize the pertinent error function given by equation (6.29).
First, a long AR filter that matches the modulus of $H_d(\omega)$ is designed using the algorithm described in section 5.3.1. This AR filter is used to determine the coefficients of a shorter ARMA filter using a method that matches the transfer functions of both filters to a certain exponent of $z$. This is the Power Order Matching method described in [39].

The coefficients of this ARMA filter are used as initial conditions for an optimization algorithm that minimizes a discretized version of the square error described by equation (6.29). That is,

$$E = \sum_{k=0}^{M} |H_d(e^{i\omega_k T_d}) - H_1(\omega_k)|^2,$$

(6.30)

where $\{\omega_k\}$ are equally spaced point in the $[0, \omega_d]$ interval. It is noted that the used algorithm is a constraint optimization algorithm. The constraint is that all the filter poles must lie inside the unit circle so that the filter is stable, see section 3.3.2.

The filter coefficients of Table IIIb are incorporated into a 4th order Adams Predictor-Corrector (APC4) solver for time domain simulations of the SDOF system under consideration. Figure 6.6 compares the response of the linear system determined by the preceding approach with the response obtained using the FFT approach. The parameters are: $\eta=0.5$, $\varepsilon=0$, $T_d=h=T=1/100$. 
Figure 6.6
Response calculated using digital filter simulation of the Biot element
6.3 Response of a SDOF system with Biot damping to seismic excitation

The configuration examined in this section is shown in figure 6.7. It involves the original system of figure 6.1 attached to a base which corresponds to the soil. This configuration allows seismic records as excitation of the system.

![Diagram of SDOF system with Biot damping](image)

**Figure 6.7**
SDOF system with Biot damping for seismic response simulation

The equation of motion of this system is

\[
\ddot{y} + \omega_n^2 \left[ (y - z) + e(y - z)^3 + \frac{2\eta}{\pi} \int_{t_0}^{t} E_1(\varepsilon(t - \tau))[y(\tau) - \dot{z}(\tau)]d\tau \right] = 0. \tag{6.31}
\]

This equation is selected instead of the dimensionless version (6.2) because real seismic data records are used and the time variable cannot be scaled. By setting \(x = y - z\), equation (6.31) becomes

\[
\ddot{x} + \omega_n^2 \left[ x + e x^3 + \frac{2\eta}{\pi} \int_{t_0}^{t} E_1(\varepsilon(t - \tau))x(\tau)d\tau \right] = -\ddot{z}. \tag{6.32}
\]
An acceleration record from the El Centro earthquake\textsuperscript{1} is used, see figure 6.8. Two values for the natural frequency are used, $\omega_n=1$ and $\omega_n=10$. The damping coefficient is $\eta=0.2$ and $\eta=1.2$. The calculations are made for both the linear and the nonlinear case. The parameter $\varepsilon$ of the Biot hysteretic element is $\varepsilon=0.1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{el_centro_acceleration_record}
\caption{El Centro acceleration record (1940), normalized with respect to the peak acceleration}
\end{figure}

\textsuperscript{1} Imperial Valley Aftershock, May 19, 1940.
Figure 6.9
Responses of the SDOF system subjected to seismic excitation
$\omega_n=1$, $\eta=0.2$

Figure 6.10
Responses of the SDOF system subjected to seismic excitation
$\omega_n=5$, $\eta=0.2$
Figure 6.11
Responses of the SDOF system subjected to seismic excitation
$\omega_n=1$, $\eta=1.2$

Figure 6.12
Responses of the SDOF system subjected to seismic excitation
$\omega_n=5$, $\eta=1.2$
6.4 Random vibration analysis

In this section, the system of figure 6.1 is subjected to white noise, and the statistical linearization method is employed to estimate the variance of the output. Monte Carlo simulations to assess the accuracy of the method. The equation of motion of the system is

\[ \ddot{x} + \omega_n^2 \left[ x + 4x^3 + \frac{2\eta}{\pi} \int_0^t E_1[\epsilon(t - \tau)] x(\tau) d\tau \right] = w(t), \]  \hspace{1cm} (6.33)

where \( w(t) \) denotes a white noise stochastic process, that is, its power spectrum is equal to \( S_{ww}(\omega) = 1, -\infty < \omega < \infty \). This equation can be written in the dimensionless form

\[ \ddot{\hat{x}} + 4\pi^2 \left[ \hat{x} + 4\hat{x}^3 + \frac{2\eta}{\pi} \int_0^{t'} E_1[\epsilon'(t' - \tau)] \dot{x}(\tau) d\tau \right] = \hat{w}(t'), \]  \hspace{1cm} (6.34)

where \( t' = t/T_n, T_n = 2\pi/\omega_n \), \( \epsilon' = \epsilon T_n \), \( \hat{\hat{x}} = \frac{\hat{x}}{\sigma} \), \( \hat{\epsilon} = \epsilon \sigma^2 \). \hspace{1cm} (6.35) (6.36) (6.37) (6.38)

The variable \( \sigma \) in equation (6.37) is the standard deviation of the response of the linearized system (\( \epsilon = 0 \)). The variance of the response of the linearized system \( \sigma^2 \) is given by the equation

\[ \sigma^2 = \int [H_b(\omega)]^2 S_{ww}(\omega) d\omega = \int [H_b(\omega)]^2 d\omega, \]  \hspace{1cm} (6.39)

where \( H_b(\omega) \) is the frequency response of the linear system and it is equal to
6.4.1 Application of the Statistical Linearization Method.

The goal is to estimate the variance $\sigma^2_x$ of the response of the nonlinear system as a function of the nonlinearity coefficient $\varepsilon$. The method of statistical linearization described in section 5.4 is employed. The system described by equation (6.33) is associated with an equivalent linear system whose equation of motion is given by the equation

$$\ddot{x} + \omega_n^2 x + \omega_n^2 \frac{2n}{\pi} \int_{t_0}^{t} E_r[\varepsilon(t - \tau)]x(\tau)d\tau = w(t).$$  

Following the procedure described by equations (5.39-5.44), an algebraic equation with unknown $\sigma^2_x$ is derived. Specifically,

$$\sigma^2_x = \int \left[ \frac{1}{-\omega^2 + \omega_n^2 \left[ 1 + 3\varepsilon^2 \sigma^2_x + \frac{2n}{\pi} \ln \sqrt{1 + \frac{\omega^2}{\varepsilon^2} + i \tan^{-1} \frac{\omega}{\varepsilon}} \right]} \right]^2 d\omega.$$  

Equation (6.42) can be solved by the iterative algorithm described in section 5.4. The variance $\sigma^2_x$ is then normalized with respect to $\sigma^2$ so that all quantities are dimensionless. That is,

$$\hat{\sigma}_x = \frac{\sigma_x}{\sigma}.$$  

The statistical linearization method is applied for two configurations of the system. For the first one, the loss coefficient is $\eta = 0.2$ and the dimensionless Biot element constant
is $\varepsilon' = 0.1$. For the second case the coefficients are $\eta = 1.2$ and $\varepsilon' = 0.1$. Figure 6.13 shows the normalized standard deviation of the response $\sigma_\hat{x}$ versus the normalized nonlinearity coefficient $\varepsilon$.

**Figure 6.13**

Estimation of the dimensionless variance of the response as a function of the dimensionless nonlinearity parameter.
6.4.2 Monte Carlo simulations

In order to experimentally extract information about the statistics of the system response, Monte Carlo studies must be performed, see section 5.5. First, a large number of excitation records are required. The results presented here are produced by calculating the response of the system to 300 excitation records. Each record \( w_i \) is a realization of a zero-mean clipped noise process. The corresponding power spectrum is given by the equation

\[
P_{ww}(\omega) = \begin{cases} 
1 & ,|\omega| \leq \frac{\pi}{T} \\
0 & ,|\omega| > \frac{\pi}{T}
\end{cases}
\] (6.43)

where \( T \) is the sampling interval. Equation (5.25) yields that the variance of \( w_i \) is equal to \( 2\pi/T \). The sampling interval \( T \) and the time step for the Runge-Kutta solver \( h \) are \( T=h=1/100 \). Each excitation record has duration equal to 30. The estimation of the variance of the output is made by using the samples at the beginning of the 30\(^{th} \) period so that the transient component of the solution does not affect the results. Three hundred samples are used in equation (5.53) to estimate the variance of the output. This procedure is performed for various values of the loss factor \( \eta \) and the nonlinearity coefficient \( \hat{e} \). The Biot element coefficient is \( \hat{e}'=0.1 \).

The results of the Monte Carlo simulation are shown in figure 6.14. The sample variance of the output is compared with the estimate of the variance by the statistical linearization method. The results show that the statistical linearization method may slightly underestimate the variance of the response for certain values of \( \hat{e} \), but overall predicts reliably its qualitative and quantitative features.
Figure 6.14
Estimation of the variance of the response using the Monte Carlo Method.
CHAPTER 7
Concluding Remarks

In the present study the dynamic response of linear models of hysteretic damping has been examined in conjunction with either linear or nonlinear stiffness elements. Certain models encountered in the literature have been discussed. The frequency domain representation of LTI systems has been reviewed along with important issues such as causality and stability. Biot's hysteretic damping model has been reviewed, since it is an established model conforming with the causality requirement for physically realizable systems.

Both time and frequency domain methods have been employed to calculate the response of a SDOF system consisting of a mass attached to a Biot hysteretic element. The frequency domain solution methods provide a reliable means for determining the response of a linear system. When nonlinearities are present, time domain techniques are almost mandatory. The incorporation of the Biot element in system dynamics models yields integro-differential equations. The integration of these equations with traditional methods is computationally expensive. Two methods have been introduced to minimize the computational burden. The first uses a recursive algorithm to calculate the time dependent integral present in the equation of motion of the system. The second uses digital filters to approximate the integral under consideration. The filter is designed in the frequency domain by matching the frequency response of the Biot element in both amplitude and phase.

The presented integration methods have made feasible the calculation of the response of the SDOF system to both deterministic and stochastic excitations. These
Numerical simulations have been performed both for the linear model and for the system with cubic nonlinearity.

The method of statistical linearization has been employed to estimate the variance of the response of the system to white noise. The estimation has been made for a range of values of the parameter that controls the severity of the nonlinearity. Monte Carlo studies have been performed to assess the accuracy of the statistical linearization method. It has been found that the theoretical results slightly underestimate the observed measured variance of the response. This trend has also been observed in previous results presented in [33]. Nevertheless, the method predicts, overall, reliably the statistical characteristics of the response.

The presented methods can be applied in a straightforward manner to multi-degree-of-freedom (MDOF) systems which incorporate Biot hysteretic elements. It is expected that future work will concentrate on more advanced hysteresis models which can account for nonlinearities [23]. Nonlinear hysteretic phenomena are commonly encountered in engineering applications, and the herein developed numerical scheme may serve as a useful auxiliary instrument for mathematical treatment of hysteresis.
Bibliography


APPENDIX I
Prony's method

The goal is to approximate a function \( y = f(x) \) by a sum of exponentials, that is,

\[
\hat{y} = \sum_{k=1}^{p} b_k e^{a_k x}.
\]

An appropriate criterion for the accuracy of the approximation is the mean square error over the interval of interest. That is,

\[
E = \int_{a}^{b} |y - \hat{y}|^2 \, dx. \tag{A.1}
\]

In chapter 6 the function \( y = E_i(x) \) must be approximated in an interval of the form [0, X]. The coefficients \( \{a_k, b_k\} \) must minimize the function

\[
E(a_1, \ldots, a_p, b_1, \ldots, b_p) = \int_{0}^{X} \left( E_i(x) - \sum_{k=1}^{p} b_k e^{a_k x} \right)^2 \, dx. \tag{A.2}
\]

A similar minimization criterion can be written in discrete form by using the \( \{g_k\} \) coefficients described by equations (6.8) and (6.9) which are used to approximate the function \( E_i(x) \) in the discrete time. That is,

\[
E(a_1, \ldots, a_p, b_1, \ldots, b_p) = \sum_{n=0}^{N} \left( g_n - \sum_{k=1}^{p} b_k e^{a_k n \Delta x} \right)^2, \tag{A.3}
\]

where \( \Delta x \) is the interval between two successive samples.

The optimization problems described by equations (A.2) and (A.3) are highly nonlinear and the use of a computationally expensive algorithm is required. Further, there is no guarantee that it will converge to the optimum values of the coefficients \( \{a_k, b_k\} \), especially when the number of coefficients is large. Prony's method solves the discretized problem described by (A.3) in a suboptimal way by solving systems of linear equations and by performing a polynomial factorization.
First, it is assumed that the number of data points is equal to the number of exponential parameters used. The following function is defined

\[ y(n) = \sum_{k=1}^{p} b_k z_k^{n-1}. \]  

(A.4)

In this case \( y(n) \) can be exactly equal to the target sequence \( \{g_k\} \). This is due to the fact that exactly 2p samples \( g_0, \ldots, g_{2p-1} \) are used to derive an exact exponential model that uses 2p parameters \( b_1, \ldots, b_p, z_1, \ldots, z_p \). The system of p equations that follows from (A.4), for \( 1 \leq n \leq p \), is written in a matrix form as

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 & b_1 \\
z_1 & z_2 & \cdots & z_p & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
z_1^{p-1} & z_2^{p-1} & \cdots & z_p^{p-1} & b_p
\end{bmatrix}
= \begin{bmatrix}
g_0 \\
g_1 \\
\vdots \\
g_{p-1}
\end{bmatrix}.
\]  

(A.5)

This system involves only the first p samples. The last p samples are used to determine the \( \{z_k\} \) sequence. Consider a polynomial \( F(z) \) with roots \( z_k \). That is,

\[ F(z) = (z - z_1)(z - z_2) \cdots (z - z_p). \]

(A.6)

This polynomial can be written in the expanded form

\[ F(z) = \sum_{i=0}^{p} h_i z^{p-i}. \]

(A.7)

where \( h_0 = 1 \).

The next step is to form the sum,

\[
\sum_{i=0}^{p} h_i g_{n-i-1} = \sum_{i=0}^{p} h_i y(n-i) = \sum_{i=0}^{p} h_i \sum_{k=1}^{p} b_k z_k^{n-i-1}, \quad p + 1 \leq n \leq 2p.
\]  

(A.8)

Replacing \( z^{n-i-1} \) by \( z_k^{p-i-1} z_k^{p-i} \), equation (A.8) becomes

\[
\sum_{i=0}^{p} h_i g_{n-i-1} = \sum_{k=1}^{p} b_k z_k^{p-i-1} \sum_{i=0}^{p} h_i z_k^{p-i} = 0.
\]  

(A.9)
The sum $\sum_{i=0}^{p} h_i z_k^{p-i}$ in equation (A.9) is simply the polynomial $F(z)$ evaluated at the roots $z = z_k$, thus, its value is equal to zero. Equation (A.9) is a set of linear equations with unknowns the parameters $\{h_i\}$,

$$\begin{bmatrix}
g_{p-1} & g_{p-2} & \cdots & g_0 & h_1 \\
g_p & g_{p-1} & \cdots & g_1 & h_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
g_{2p-2} & g_{2p-3} & \cdots & g_{p-1} & h_p
\end{bmatrix} = -\begin{bmatrix}
g_p \\
g_{p+1} \\
\vdots
\end{bmatrix}. \quad (A.10)$$

Prony's method can be summarized in three steps. First, the solution of equation (A.10) yields the coefficients of the polynomial $F(z)$ defined by (A.6). Second, the roots of this polynomial $\{z_k\}$, are calculated. Finally, the $\{b_k\}$ coefficients are calculated by solving the system of (A.5). The $\{a_k\}$ coefficients are related to the $\{z_k\}$ roots by the equation

$$z_k = e^{a_k \Delta x}, \quad (A.11)$$

which is solved to obtain $\{a_k\}$. It is noted that the roots of the resulting polynomial are assumed real. In the general case where some roots are complex numbers, instead of a sum of exponentials, a sum of exponentials and exponentially decaying sinusoids is obtained.

In the beginning of the preceding analysis it is assumed that the number of parameters is equal to the number of samples. In practice, the number of samples is larger than the number of parameters. This leads to an extension of Prony's method which solves the problem in the mean square sense. Specifically, the procedure remains the same except that the linear systems of (A.5) and (A.10) become overdetermined and they are solved in the least square sense. The system that corresponds to (A.5) for the new formulation of the problem is
\[ Z^H Z b = Z^H g, \]  
(A.12)

where \( Z(i,j) = [z_j^{i-1}] \) is a \( N \times p \) matrix, \( N \) is the number of samples, \( \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_p \end{bmatrix} \) and 

\[ \mathbf{g} = \begin{bmatrix} g_0 \\ \vdots \\ g_N \end{bmatrix} \]

is a vector containing all \( N \) samples. It is reminded that \( Z^H \) represents the transpose and complex conjugate of the matrix \( Z \), which is equal to the transpose \( Z^T \) if all roots \( \{z_k\} \) are real. Similarly, the system which corresponds to (A.10) is

\[ G^T G \mathbf{h} = -G^T \mathbf{g}_1, \]  
(A.13)

where \( G = \begin{bmatrix} g_{p-1} & g_{p-2} & \cdots & g_0 \\ g_p & g_{p-1} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ g_{N-1} & g_{N-2} & \cdots & g_{N-p} \end{bmatrix} \) is a \( (N-p) \times p \) matrix, \( \mathbf{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_p \end{bmatrix} \), and \( \mathbf{g}_1 = \begin{bmatrix} g_p \\ \vdots \\ g_N \end{bmatrix} \).

The preceding procedure is applied to approximate the function \( E_i(x) \) by using the sample values \( \{g_k\} \) defined by equations (6.8) and (6.11) with \( \Delta x = 0.01 \) in the interval \([0,10]\). The number of samples is \( N = 1001 \) and the number of exponentials is \( p = 12 \). The coefficients \( \{a_k,b_k\} \) are tabulated in Tables I. It is noted that neither \( N \) nor \( p \) can be large because the linear systems in equations (A.12) and (A.13) become ill-conditioned. Table II contains two sets of coefficients which approximate \( y = E_i(x) \) in several intervals to achieve better accuracy. The use of different approximations of \( y = E_i(x) \) in different intervals can be incorporated to the algorithm described by (6.23), with an increase in computational cost. However, the computational cost of the algorithm remains \( O(N) \).
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**Table I**

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Target function: y = E_i(x)
Total number of samples: 1001
Total square error: 1.5454e-05

**Table II**
APPENDIX II

Digital Filter design for simulation of the Biot element

As explained in subsection 6.2.2, the task is to calculate the digital filter coefficients $a_1, \ldots, a_p, b_0, b_1, \ldots, b_q$ such as the expression

$$E = \sum_{k=0}^{M} \left| H_d(e^{j\omega_k T_d}) - H_t(\omega_k) \right|^2$$

(A.14)

is minimized with $H_d(e^{j\omega}) = \sum_{k=0}^{q} b_k e^{-j\omega k}$ being the frequency response of the digital filter, and $H_t(\omega)$ being the target response function given by equation (6.28). Since the objective is to match both the real and imaginary parts of $H_d(e^{j\omega T_d})$ and $H_t(\omega)$, the error function

$$E' = \sum_{k=0}^{M} \left( \text{Re}\{H_d(e^{j\omega_k T_d})\} - \text{Re}\{H_t(\omega_k)\} \right)^2 w_1 + \sum_{k=0}^{M} \left( \text{Im}\{H_d(e^{j\omega_k T_d})\} - \text{Im}\{H_t(\omega_k)\} \right)^2 (1 - w_1)$$

(A.15)

can be also used, where $w_1$ is a weight number.

The procedure for designing the filter is explained in section 6.2.2. The constraint optimization algorithm that is used is explained in [22,11]. It is a Sequential Quadratic Programming (SQP) method. This class of iterative algorithms solve a quadratic programming problem at each iteration. That is, they approximate the error function with a quadratic form and they find its minimum. When the problem has constraints, the quadratic form approximates the corresponding Lagrangian function [15].
The filter coefficients given in the following tables refer to three different combinations of sampling frequency - ε′ parameter pairs and the two different error measures given by equations (A.14) and (A.15). It is readily seen from equation (6.28) that \( H_\epsilon(\omega) \) is a function of the ratio \( \frac{\omega}{\epsilon} \). This means that instead of selecting different pairs of cutoff frequency \( \omega_d \) and the parameter \( \varepsilon' \), it is preferable to design filters which correspond to a certain value of the product \( T_d\varepsilon' = \frac{\pi\varepsilon'}{\omega_d} \). In this manner, the same filter can be used for different configurations of the sampling interval and of the coefficient \( \varepsilon' \) as long as their product is in agreement with the specifications of the filter.

\[
\begin{array}{|c|c|}
\hline
\{a_k\} & \{b_k\} \\
\hline
-0.8095 & 6.3557 \\
-1.1677 & -6.7286 \\
0.8736 & -6.3688 \\
0.2529 & 7.0754 \\
-0.1465 & 0.7919 \\
-1.1255 & 0.7129 \\
\hline
\end{array}
\]

\[
\text{Error function used: } E = \sum_{k=0}^{M} |H_d(e^{i\omega_k T_d}) - H_i(\omega_k)|^2
\]

Number of sample points \( \omega_k \): 256

\[
\begin{array}{|c|c|}
\hline
\{a_k\} & \{b_k\} \\
\hline
0.1043 & 6.3545 \\
-1.5498 & -1.3268 \\
-0.1274 & -10.0571 \\
0.5665 & 1.6273 \\
0.0286 & 3.7782 \\
-0.3688 & 0.2843 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\{a_k\} & \{b_k\} \\
\hline
0.3630 & 7.0489 \\
-1.5042 & 0.5696 \\
-0.4550 & -11.3215 \\
0.5251 & -0.8736 \\
0.1074 & 4.3290 \\
0.2843 & 0.2843 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\{a_k\} & \{b_k\} \\
\hline
0.0706 & 7.7276 \\
-0.7412 & 3.8753 \\
-0.8569 & -7.0645 \\
-0.1137 & -6.1273 \\
0.1538 & 0.2169 \\
1.6456 & 1.6456 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\{a_k\} & \{b_k\} \\
\hline
0.0074 & 7.7384 \\
-1.5676 & -1.9336 \\
-0.0087 & -12.1497 \\
0.5817 & 2.4090 \\
0.0017 & 4.5178 \\
-0.5627 & -0.5627 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\{a_k\} & \{b_k\} \\
\hline
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\hline
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\]

\[
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-0.0087 & -12.1497 \\
0.5817 & 2.4090 \\
0.0017 & 4.5178 \\
-0.5627 & -0.5627 \\
\hline
\end{array}
\]
Figures A.1 and A.2 show the accuracy of frequency response of the designed filters with respect to the target function $H_I(\omega)$, for $T_d e^\prime = 1/500$.

**Figure A.1**
Approximation accuracy of an ARMA(5,5) filter designed with error criterion given by (A.14)
Figure A.2
Approximation accuracy of an ARMA(5,5) filter designed with error criterion given by (A.15)

In figure (A.2) the approximation of the imaginary part of the frequency response is quite satisfactory. This is due to the weight used in the optimization procedure. Clearly, in this case the quality of the approximation of the real part deteriorates. It is noted, however, that it is the imaginary part of the frequency response which is directly related to the energy dissipation per cycle of oscillation as explained in Chapter 2. Although the achieved matching is not ideal, the ARMA modeling offers a viable alternative to the simulation of Biot-type hysteretic systems.