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A Comparison of Finite Difference Stencils on Two Forms of the Acoustic Wave Equation

by

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A Thesis Submitted
in partial fulfillment of the requirements for the degree

Master of Arts

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A Comparison of Finite Difference Stencils on Two Forms of the Acoustic Wave Equation

Regina Shaylean Hill

Abstract

In practice, two forms of the acoustic wave equation, the velocity-stress and pressure forms, are used to simulate seismic experiments. These equations in their discrete forms lead to two families of finite difference schemes, the staggered-grid and centered difference schemes. These two difference schemes are widely used to numerically generate seismograms. Although these two difference schemes are widely used, there has been no distinction whether one is better than the other. The goal of this research is to formulate a heuristic based on computational cost and storage to determine which scheme is better than the other.
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Chapter 1

Introduction

Information about the physical properties of the Earth is useful to many scientists, including geologists and geophysicists. Scientists use changes in physical properties, such as pressure and density, to indicate the presence of natural gases, minerals, or contaminants in the Earth. For example, geophysicists can use this information to locate petroleum or natural gas reservoirs. One way to ascertain where changes in the physical properties of the Earth occur is to conduct seismic experiments. Such an experiment begins by releasing energy from an explosive or non-explosive device, referred to as the source. This energy is reflected off discontinuities in the physical properties of the subsurface, and is recorded using geophones. The recordings must then be interpreted. However, interpretation can be difficult. One reason interpretation is difficult is various subsurface scenarios can lead to the same result from a seismic experiment. Another cause for difficulty is error. Error can be introduced into seismic results from several sources, including human error and instrumentation failure. [4]. Numerical simulations are used both to help interpret and to verify the accuracy of the interpretation.

Numerical simulation is a practical way to validate the data collected through physical experimentation, provided a good model for the experiment is known. Seismic experiments are modeled by a class of partial differential equations called wave equations, which can vary in complexity, order, and dimension. The more physical attributes of the Earth the numerical simulation tries to emulate, the more complex the partial differential equations used to model the experiment. For instance, modeling
the dissipation of energy as it travels over distance, attenuation, requires a viscoelastic wave equation [5]. To make the simulation more tractable, such properties as attenuation are ignored which simplifies the problem and the model. The acoustic wave equation is the simplest wave equation which models only the simplest physical properties of the earth, such as pressure. The order of the partial differential equation refers to the highest number of partial derivatives in the equation. A first-order system of equations can be used to model a seismic experiment. This system is known as the velocity-stress form of the acoustic wave equation. The velocity-stress equations are derived from laws of physics that govern particle motion in elastic fluids. This system of equations can be manipulated to form a second-order equation known as the pressure form of the acoustic wave equation. When the proper relationship between the forcing functions in the two forms is maintained, the solutions are equivalent. Ideally, these equations would be solved in three-dimensions. Although technological advances have made three-dimensional problems solvable, two-dimensional problem are useful, consume far fewer computational resources, and are the equations studied in this thesis.

The velocity-stress and pressure forms of the acoustic wave equation in their continuous forms are difficult to solve. Given a suitable discretization method, the difficulty in solving the continuous forms of the wave equation can be alleviated in a discrete form. When choosing a discretization technique, consideration must be given to the techniques ability to model the physics of the problem. For example in the simulation of seismic experiments, the discretization must be able to propagate waves. Secondly, the technique must be able to approximate the region on which the model is being solved. Finally, the technique must give the user the ability to control the approximation error. The finite difference method accomplishes these goals. The finite difference approximations of the velocity-stress and pressure forms of the acoustic
wave equation lead to two families of finite difference schemes: the staggered grid (or leap frog) scheme and the centered difference scheme. Associated with each finite difference scheme is an order of accuracy in time and one in space. For instance, the 2-2 schemes are second-order accurate in time and in space. The velocity-stress and pressure forms of the acoustic wave equation have been studied for some time, however a distinction has not been made as to whether or not one form is superior.

The goal of this research is to determine whether preference should be given to one scheme. This will be done by comparing the storage and computational cost of computing each family of finite difference schemes. The outline for the remainder of the thesis is as follows. In Chapter 2, the forward problems of the of the velocity-stress and pressure forms are derived and a relationship between the two forms is observed, through the choice of the forcing functions. A forward problem maps an element in the model space to an element in the data space. Then in Chapter 3, the finite difference families, staggered grid and centered difference schemes, are explicitly stated to solve the discrete forward problems. Then an analysis of the storage and computational cost is given for each scheme from which they are compared. Finally, Chapter 4 describes the numerical simulation and presents the numerical results and conclusions.
Chapter 2

Forward Problem

The simulation of a seismic experiment requires solving two problems, the first of which is the forward problem. A forward problem maps an element in the model space to an element in the data space. The model space of interest is the set of all scalar-valued wave speeds \( c(x, z) \) and the data space is the set of all vector-valued pressures \( p(x, z) \). Thus to solve the forward problem associated with the simulation of a seismic experiment, a reasonable model must be developed that accepts a wave speed and returns pressure. The acoustic wave equation with constant density determines a physical property of the medium given the wave speed and is sufficient to model the forward problem. The second problem involved in simulating a seismic experiment is known as the inverse problem and is not discussed in the thesis. In this chapter, the velocity-stress and pressure forms of the acoustic wave equation are derived using the laws that describe particle motion in elastic fluids and the two equivalent forward maps are defined to model a seismic experiment in two-dimensions.

2.1 Velocity-Stress Form

The laws that govern particle motion in elastic fluids are the basis for the acoustic wave equation. They include the law of mass balance, the law of momentum balance, and the constitutive law. The law of mass balance,

\[
\frac{D\rho(x, z, t)}{Dt} = \rho(x, z, t) \left( \frac{\partial v_x(x, z, t)}{\partial x} + \frac{\partial v_z(x, z, t)}{\partial z} \right),
\]

(2.1.1)
where $\rho(x, z, t)$ is the material density and $v_x(x, z, t)$ and $v_z(x, z, t)$ are the particle velocities in the $x$ and $z$ directions, ensures that the material density does not when the material is deformed. The law of momentum balance equates the rate of change in momentum with the internal and external forces acting on the material. Mathematically, the momentum balance equations are

$$\frac{D\rho(x, z, t)v_x(x, z, t)}{Dt} = \frac{\partial p(x, z, t)}{\partial x} + f_x(x, z, t), \quad (2.1.2)$$

$$\frac{D\rho(x, z, t)v_z(x, z, t)}{Dt} = \frac{\partial p(x, z, t)}{\partial z} + f_z(x, z, t), \quad (2.1.3)$$

where $p(x, z, t)$ is the pressure (for elastic fluids the scalar pressure is a stress) and $f_x(x, z, t)$ and $f_z(x, z, t)$ are the external forces applied in the respective directions.

Finally, the constitutive law,

$$p(x, z, t) = P(\rho(x, z, t)), \quad (2.1.4)$$

relates the material density $\rho$ to pressure $p$ by a function $P$.

When the density $\rho$ is constant, the velocities $v_x$ and $v_z$ are zero, and no external forces are acting on the material, the solutions to the laws of mass and momentum balance, and the constitutive law reflect that the medium is in a steady state. Furthermore, the velocities $v_{x_0}$ and $v_{z_0}$ and the pressure $p_o$ represent a steady state solution. The steady state equations are set in motion by perturbing the external force, $f_x + \delta f_x$ and $f_z + \delta f_z$, where $\delta$ represents a small perturbation. The values $\rho$, $p$, $v_x$, and $v_z$ are then monitored as they move away from steady state, $p = p_o + \delta p$, $\rho = \rho_o + \delta \rho$, $\dot{v}_x = \dot{v}_{x_0} + \delta \dot{v}_x$, $\dot{v}_z = \dot{v}_{z_0} + \delta \dot{v}_z$. Substituting the perturbed values into (2.1.1) - (2.1.4), the laws of mass and momentum balance and the constitutive law away from equilibrium are

$$\frac{\partial(\rho_o + \delta \rho)}{\partial t} = (\rho_o + \delta \rho) \left( \frac{\partial(v_{x_0} + \delta v_x)}{\partial x} + \frac{\partial(v_{z_0} + \delta v_z)}{\partial z} \right), \quad (2.1.5)$$
\[ (\rho_o + \delta \rho) \frac{\partial (v_{zo} + \delta v_z)}{\partial t} = \frac{\partial (\rho_o + \delta \rho)}{\partial x} + f_z + \delta f_z, \]  
\[ (\rho_o + \delta \rho) \frac{\partial (v_{zo} + \delta v_z)}{\partial t} = \frac{\partial (\rho_o + \delta \rho)}{\partial z} + f_z + \delta f_z, \]  
\[ p_o + \delta p = P(\rho_o + \delta \rho). \]  

The following assumptions are made on the perturbed equations \((2.1.5)\) - \((2.1.8)\).

- The derivatives of the equilibrium solution exist and are small.
- The products of the derivative of the equilibrium solution and \(\delta\) terms are small.
- The \(\delta^2\) terms are assumed to be negligible.

Moreover, applying these assumptions and using the Taylor's Series Expansion of \((2.1.8)\) and the steady state equations \((2.1.1)\) - \((2.1.4)\), the perturbed equations \((2.1.5)\) - \((2.1.8)\) reduce to

\[ \frac{\partial \delta \rho}{\partial t} = \rho_o \frac{\partial \delta v_z}{\partial x} + \rho_o \frac{\partial \delta v_z}{\partial z}, \]  
\[ \rho_o \frac{\partial \delta v_z}{\partial t} = \frac{\partial \delta p}{\partial x} + \delta f_z, \]  
\[ \rho_o \frac{\partial \delta v_z}{\partial t} = \frac{\partial \delta p}{\partial z} + \delta f_z, \]  
\[ P(\rho_o + \delta \rho) = P(\rho_o) + \frac{\partial P}{\partial \rho}(\rho_o) \delta \rho. \]

Additionally the assumption that pressure increases as density increases is made. Therefore the derivative of pressure with respect to density is positive,

\[ \frac{\partial P}{\partial \rho} > 0. \]

Recognizing that \((2.1.8)\) is equal to \((2.1.12)\) and applying \((2.1.13)\), \(\delta \rho\) can be explicitly stated as

\[ \delta \rho = \frac{\delta p}{c^2}, \]
where \( c = \sqrt{\frac{\partial p}{\partial \rho} (\rho_0)} \) is the wave speed. Substituting (2.1.14) into (2.1.9) - (2.1.11) yields the acoustic equations of motion,

\[
\frac{1}{c^2} \frac{\partial \delta p}{\partial t} = \rho_0 \frac{\partial \delta v_x}{\partial x} + \rho_0 \frac{\partial \delta v_z}{\partial z},
\]

\[
\rho_0 \frac{\partial \delta v_x}{\partial t} = \frac{\partial \delta p}{\partial x} + \delta f_x,
\]

\[
\rho_0 \frac{\partial \delta v_z}{\partial t} = \frac{\partial \delta p}{\partial z} + \delta f_z.
\]

Equations (2.1.15) - (2.1.17) are also known as the velocity-stress form of the acoustic wave equation. This linear system of equations is used to model the forward problem in seismic simulations.

Due to computational limitations, the simulation of a seismic experiment must be defined on a region, which is defined by boundary conditions. The enforced boundary conditions are that pressure on the boundaries and outside the region are zero. The simulation must also be provided an initial condition from which to begin. The chosen initial conditions are that the pressure and velocity are both zero. The velocity-stress model, the boundary conditions, and the initial conditions define the forward problem.

Formally, the forward problem is defined as: given a model velocity, \( c(x, z) \), generate seismogram(s), of the velocity perturbations \( \delta v_x \) and \( \delta v_z \) and the pressure perturbation \( \delta p \) by applying the sampling operator \( S \) to the solution of the forward map. The sampling operator returns \( \delta v_x(x_r, z_r, t), \delta v_z(x_r, z_r, t) \) and \( \delta p(x_r, z_r, t) \), seismograms of velocity and pressure perturbations at receiver \( r \), where \((x_r, z_r)\) is the spatial location of receiver \( r \). Thus, the linear map

\[
G : c \rightarrow S \begin{bmatrix} \delta p \\ \delta v_x \\ \delta v_z \end{bmatrix} [c],
\] (2.1.18)
relates an element in the model space to an element in the data space. Solving the
following initial value problem (IVP),
\[
\begin{align*}
\frac{1}{c(x,z)^2} \frac{\partial \delta p(x,z,t)}{\partial t} &= \left( \frac{\partial \delta v_x(x,z,t)}{\partial x} + \frac{\partial \delta v_z(x,z,t)}{\partial z} \right), \\
\frac{\partial \delta v_x(x,z,t)}{\partial t} &= \frac{\partial \delta p(x,z,t)}{\partial x} + \delta f_x, \\
\frac{\partial \delta v_z(x,z,t)}{\partial t} &= \frac{\partial \delta p(x,z,t)}{\partial z} + \delta f_z, \\
\delta p(0, z, t) &= \delta p(M_L, z, t) = 0, \\
\delta p(x, 0, t) &= \delta p(x, N_L, t) = 0, \\
\delta p(x, z, 0) &= \delta v_x(x, z, 0) = \delta v_z(x, z, 0) = 0,
\end{align*}
\]
(2.1.19)
where \(0 < x < M_L, 0 < z < N_L, 0 < t < T_L, M_L\) is the length of the experiment in the
\(x\)-direction, \(N_L\) is the length of the experiment in the \(z\)-direction, and \(T_L\) is the total
simulation time, produces a solution \(\delta p, \delta v_x, \delta v_z\), and \(\delta v_z\) to the forward problem.
Hence, the velocity-stress equations are one way to model a seismic experiment.

2.2 Pressure Form

The velocity-stress system of equations has an equivalent second-order form. This
form is obtained by taking the partial derivative of (2.1.15) with respect to time,
and then using (2.1.16) and (2.1.17) to eliminate the time derivatives of \(\delta v_x\) and \(\delta v_z\).
Finally, multiplying by the constant \(\rho_0\), the resulting equation is the pressure form of
the acoustic wave equation,
\[
\frac{1}{c^2} \frac{\partial^2 \delta p(x, z, t)}{\partial t^2} - \frac{\partial^2 \delta p(x, z, t)}{\partial x^2} - \frac{\partial^2 \delta p(x, z, t)}{\partial z^2} = \nabla \cdot \delta f(x, z, t).
\]
(2.2.20)
Notice that the right-hand side of the pressure form equation is the divergence of the
forcing function in the velocity-stress form. Thus, the velocity-stress and pressure
forms are equivalent.

The pressure form can also be used to model the forward problem in the simulation
of a seismic experiment. In this case, the forward problem is: given the speed of a
wave, \(c(x, z)\) in the model space, generate a seismogram(s) of pressure perturbations,
\(\delta p\) in the data space, where \(\delta p\) is obtained by solving the pressure form of the acoustic
wave equation with boundary and initial conditions. The Initial Value Problem (IVP)
associated with the pressure form of the acoustic wave equation has homogeneous
Dirichlet boundary conditions on the spatial edges of the computational domain.
The initial conditions are zero pressure and zero change in pressure at time zero.
This problem is characterized by the linear map,
\[
F : c \to S\delta p[c].
\]
The pressure, \(\delta p\), solves
\[
\frac{1}{c^2} \frac{\partial^2 \delta p(x, z, t)}{\partial t^2} - \frac{\partial^2 \delta p(x, z, t)}{\partial x^2} - \frac{\partial^2 \delta p(x, z, t)}{\partial z^2} = \nabla \cdot \delta f, \\
\frac{\partial \delta p(x, z, 0)}{\partial t} = \delta p(x, z, 0) = 0, \\
\delta p(0, z, t) = \delta p(M_L, z, t) = 0, \\
\delta p(x, 0, t) = \delta p(x, N_L, t) = 0,
\]
where \(0 < x < M_L, 0 < z < N_L, \) and \(0 < t < T_L.\)
Chapter 3

Difference Schemes

As discussed in Chapter 2, the models used to simulate seismic experiments consist of differential equations with specified boundary and initial conditions. Furthermore, the forward models are continuous. However, these problems may be difficult to solve in their continuous forms. Given a reasonable discretization method, a discrete approximation to the continuous problem is more easily solved. Desired properties of a discretization method are: it must not hinder the modeling of the problem, it must approximate the computational domain, and it must control the approximation error.

The forward problem associated with modeling a seismic experiment requires the ability to propagate waves. The finite difference scheme uses data from previous time and spatial locations to model wave propagation. The experimental region being approximated is rectangular, easily approximated by the finite difference scheme. The approximation error of a finite difference scheme is controlled by the order of the approximation and the size of the time and spatial steps, $\Delta t$ and $\Delta x$. Hence the finite difference method is chosen to discretize the forward problems.

This chapter defines the two families of finite difference schemes that result from the discretization of the two forms of the acoustic wave equation, the staggered grid and centered difference schemes. These schemes are then formulated as matrix-vector equations to aid in their comparison to one another. Finally, the schemes are analyzed based on the amount of storage and computation required to compute them.
3.1 Staggered-Grid Scheme

When discretized the continuous forward velocity-stress problem (2.1.19) becomes

\[
\frac{\delta p_{j,k}^i}{c_{f,j,k}^2} \frac{\delta p_{j,k}^i - \delta p_{j,k}^{i-1}}{\Delta t} = \left( \frac{\delta v_{j,k}^l - \delta v_{j-1,k}^l}{\Delta x} + \frac{\delta v_{j,k}^l - \delta v_{j,k-1}^l}{\Delta z} \right),
\]

\[
\frac{\delta v_{j,k}^l}{\Delta t} = \frac{\delta p_{j+1,k}^i - \delta p_{j,k}^i}{\Delta x} + \delta f_{x,j}^{i+1},
\]

\[
\frac{\delta v_{j,k}^l}{\Delta z} = \frac{\delta p_{j,k+1}^i - \delta p_{j,k}^i}{\Delta x} + \delta f_{z,k}^{i+1},
\]

(3.1.1)

\[\delta p_{0,k}^i = \delta p_{M,k}^i = \delta p_{j,0}^i = \delta p_{j,N}^i = \delta p_{x,j,k}^0 = \delta v_{x,j,k}^0 = \delta v_{z,j,k}^0 = 0,\]

where \(\delta p_{j,k}^i = \delta p(x_j, z_k, t_i)\) and \(\delta v_{(x,z),j,k}^i = \delta v(x,z)(x_{j+1}, z_{k+1}, t_{i+1})\) for \(x_j = j\Delta x, z_k = k\Delta z, t_i = i\Delta t\) \(i = 0 \ldots T, j = 0 \ldots M, k = 0 \ldots N\). The pressures \(\delta p\) are calculated on the even nodes and velocities \(v_{x,z}\) on the odd nodes. This scheme belongs to a family of finite difference schemes known as staggered grid schemes. The above scheme is second-order accuracy in its approximation of both the time and space derivatives, \(O(\Delta t^2, \Delta x^2)\).

3.2 Centered Difference Scheme

The continuous pressure form problem (2.2.21) can also be solved in a discrete form using finite differences. The solution to the discrete forward pressure form problem using the 2-2 centered difference scheme is obtained by solving the following discrete IVP:

\[
\left( \frac{\delta p_{j,k}^{i+1} - 2\delta p_{j,k}^i + \delta p_{j,k}^{i-1}}{c_{f,j,k}^2 \Delta t^2} - \frac{\delta p_{j+1,k}^i - 2\delta p_{j,k}^i + \delta p_{j-1,k}^i}{\Delta x^2} - \frac{\delta p_{j,k+1}^i - 2\delta p_{j,k}^i + \delta p_{j,k-1}^i}{\Delta z^2} \right) = (\nabla \cdot \delta f)^i_{j,k},
\]

\[
\delta p_{0,k}^0 = \delta p_{M,k}^0 = \delta p_{j,0}^0 = \delta p_{j,N}^0 = \delta p_{x,j,k}^0 = \delta v_{x,j,k}^0 = \delta v_{z,j,k}^0 = 0,
\]

(3.2.2)

where \(\delta p_{j,k}^i = \delta p(x_j, z_k, t_i)\) for \(j = 0 \ldots M, k = 0 \ldots N, i = 0 \ldots T\). This scheme as its name implies belongs to the centered difference family of finite difference schemes. This scheme is also \(O(\Delta t^2, \Delta x^2)\) accurate.
3.3 Matrix-Vector Form

In this section, the difference schemes presented in the previous sections are formulated as matrix-vector equations. The matrix-vector equations standardize the two formulations, as well as illustrate how the difference schemes move from one time level to the next. These equations also represent the smallest amount of information that needs to be stored to calculate the next time level. The current time level is denoted the \( i \)th time level. The difference schemes are put into this form to aid in comparing them with respect to storage and computational cost.

The staggered grid scheme is a two level scheme. That is, it uses the previous time level to calculate the current time level. Therefore at any given time, two time levels are being stored: the previous time level and the time level being computed. Both levels can be stored in one vector that is overwritten. The vector contains the matrix variables \( \delta p, \delta v_x, \) and \( \delta v_z \) in vector form stacked on top of each other. In vector form, the staggered grid scheme is as follows,

\[
\begin{bmatrix}
\hat{\delta p}^i \\
\hat{\delta v}_x^i \\
\hat{\delta v}_z^i \\
\end{bmatrix} =
\begin{bmatrix}
I & H^{-1}(c, \hat{\delta v}_x^i) & H^{-1}(c, \hat{\delta v}_z^i) \\
0 & I & 0 \\
0 & 0 & I \\
\end{bmatrix}
\begin{bmatrix}
\hat{\delta p}^{i-1} \\
\hat{\delta v}_x^i \\
\hat{\delta v}_z^i \\
\end{bmatrix}, \tag{3.3.3}
\]

\[
\begin{bmatrix}
\hat{\delta p}^{i+1} \\
\hat{\delta v}_x^i \\
\hat{\delta v}_z^i \\
\end{bmatrix} =
\begin{bmatrix}
I & 0 & 0 \\
H^{+2}(\hat{\delta p}^{i+1}) & I & 0 \\
H^{+2}(\hat{\delta p}^{i+1}) & 0 & I \\
\end{bmatrix}
\begin{bmatrix}
\hat{\delta p}^{i+1} \\
\hat{\delta v}_x^{i-1} \\
\hat{\delta v}_z^{i-1} \\
\end{bmatrix}, \tag{3.3.4}
\]

where \( \hat{\delta} \) denotes the vector representation of a matrix at each time level. Each stencil operator, \( H^{-1} \) and \( H^{+2} \), represents the part of the finite difference stencil that acts on the given time level. The \( H^{-1} \) stencil, although centered around the node it approximates, is actually a backward difference scheme. Similarly, \( H^{+2} \) is a forward difference scheme centered around the midpoint it approximates. Equation (3.3.3) is updating pressures on the nodes at each time level, while the velocities dwell on the midpoints between nodes at each half time step. Since equation (3.3.4) updates
velocities on nodes at whole times the superscripts change although they refer to the same nodes and time as in equation (3.3.3). Equations (3.3.3) and (3.3.4) take the previous time level of the staggered grid to the next.

The centered difference scheme is a three-level scheme. The difference formula contains three time levels of pressure values, the previous, the current and the next time level. The centered difference scheme can be expressed as a two level scheme by creating vectors that consist of stacked time levels [7]. As with the staggered grid scheme matrix- vector form, only one vector of this form is needed to be stored, since the part of the vector that represents the oldest time level can be overwritten. For the centered difference scheme, the vector equation is

\[
\begin{bmatrix}
\hat{\delta}p^{i+1} \\
\hat{\delta}p^{i}
\end{bmatrix} = \begin{bmatrix}
H^3(c, \hat{\delta}p^i) & I \\
I & 0
\end{bmatrix} \begin{bmatrix}
\hat{\delta}p^{i} \\
\hat{\delta}p^{i-1}
\end{bmatrix},
\]

(3.3.5)

where \(H^3\) represents the part of the centered difference stencil operator that acts on the current time level. Now, written as a two level scheme, it is easily compared to the staggered grid scheme.

### 3.4 Storage and Cost

The vector form is used as a tool to compare the two schemes with respect to storage and computational cost. The amount of storage needed to store the variables in each form is determined by the number of elements in the vectors in the matrix- vector forms. The number of elements in the vectors per variable is determined by the discretization. Each variable has \(MN\) elements, where \(M\) is the number of grid points in the \(x\) direction, and \(N\) is the number of grid points in the \(z\) direction. The number of grid points, \(M\) and \(N\), depend on how fine the grid is discretized. The number of grid points in each direction also depends on the variable and where on the grid the variable is calculated. The number of grid points is determined by the grid
spacing. The discussion of grid spacing will be deferred until Chapter 4. The cost of computing the schemes consists of the number of operations needed to compute the stencil operators and update the vectors. The cost and storage requirements to compute the velocity-stress and centered difference schemes for the 2-2 and 2-4 cases are listed in Table 3.1.

To compute the staggered-grid scheme associated with the velocity-stress form of the acoustic wave equation, one vector consisting the three variables needs to be stored. Since the pressure \( \delta p \) are calculated on the nodes, there are \( MN \) pressure values to be calculated for \( T \) time steps. Moreover, the velocities \( \delta v_x \) and \( \delta v_z \) are calculated at the midpoints between nodes, requiring \( MN - M - N + 1 \) velocity values to be stored for each of the \( T - 1 \) time levels on which they are calculated. Although \( T \) time levels of pressure and \( T - 1 \) time levels of velocities need to be calculated, only one time level needs to be stored. Each successive computed time level will use and overwrite the previous time step. The number of pressure and velocity values stored depends on the order of the scheme being calculated. The total amount of storage needed to compute the staggered grid scheme is on the order of \( 3MN \) elements. The centered difference schemes compute the pressure values on all \( MN \) nodes for \( T \) time steps. Since one vector consisting of two time levels of pressure values is needed to compute the new time level, a vector of \( 2MN \) elements of pressure values is needed to be stored for the 2-2 scheme.

The computational cost of computing a finite difference scheme also depends on the desired order of accuracy of the scheme. The computational cost of the difference scheme is determined by counting the number of flops, additions/subtractions multiplications/divisions used to calculate the difference stencil at one spatial point, then multiplying it by the total number of spatial and temporal elements. The cost calculations in this section are the costs to compute the stencils \( H^{-1} \), \( H^{+2} \), and \( H^3 \).
Since the forcing function is arbitrary, the cost to compute it is not counted toward the computational cost.

In general, the cost of computing a staggered-grid scheme is given by \(2 \text{cost}(H^{-1}) + 2 \text{cost}(H^{+2})\), where \(\text{cost}(H^{-1})\) is the cost to compute stencil \(H^{-1}\) and \(\text{cost}(H^{+2})\) is the cost to compute stencil \(H^{+2}\). For the 2-2 staggered grid scheme, the cost to compute the \(H^{-1}\) stencil at one point is 4 flops. The cost to compute the \(H^{+2}\) stencil is 2 flops. Finally the constant ratio of the wave speed and time step to the spatial step, which is multiplied by the the \(H^{-1}\) stencil, requires 3 flops to compute. This constant is computed once and is multiplied by \(H^{-1}\) adding \(MNT\) flops to the total flop count. The constant multiplied by \(H^{+2}\) requires one flop to compute and adds \(MNT\) flops. Therefore, the 2-2 staggered-grid scheme requires a total of \(11MNT + 5\) flops to be computed.

The 2-4 staggered-grid scheme uses the same general formula to determine the computational cost. The \(H^{-1}\) scheme costs 12 flops per time step and spatial node to compute stencil \(H^{-1}\) and 6 flops per time step and spatial node to compute stencil \(H^{+2}\). Adding \(3MN\) flops to multiply the constants, the 2-4 staggered-grid scheme costs on the order of 27 flops. The constant multiplied by \(H^{-1}\) costs 4 flops to compute. The constant multiplied by \(H^{+2}\) costs 2 flops to compute. The total cost to compute the 2-4 staggered-grid scheme is \(27MNT + 8\) flops.

The cost to compute the next time level for the pressure form given the two previous time levels is \(\text{cost}(H^3)\), where \(\text{cost}(H^3)\) is the cost to compute the stencil \(H^3\).

### Table 3.1: Cost and Storage Equations for difference schemes

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Staggered-grid</th>
<th>Centered difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order</td>
<td>2-2</td>
<td>2-2</td>
</tr>
<tr>
<td>Storage (elements)</td>
<td>3 MN</td>
<td>3 MN</td>
</tr>
<tr>
<td>Cost</td>
<td>11MNT + 5</td>
<td>27MNT + 8</td>
</tr>
</tbody>
</table>
The 2-2 centered difference scheme requires 10 \ flops to compute \( H^3 \). The constant multiplied by \( H^3 \) costs 5 \ flops to compute. This constant is then multiplied \( MNT \) times by \( H^3 \) requiring a total of \( 11MNT + 5 \) \ flops to compute the 2-2 centered difference scheme. The 2-4 centered difference scheme requires 18 \ flops to compute \( H^3 \) and 6 \ flops to compute the constant. Hence, a total of \( 19MNT + 6 \) \ flops are required to compute the 2-4 centered difference scheme. In the next section, the numerical experiments used to compare the staggered grid and centered difference schemes are described and these cost and storage heuristics are implemented.
Chapter 4

Experiment

The two families of difference schemes discussed in the previous chapters are widely used to approximate solutions to problems modeled by partial differential equations, in particular seismic simulations. However, it has not been made clear whether one family of schemes should be preferred. This chapter begins with a description of the physical problem and two ways of modeling the physical problem are derived. Once derived the models are solved analytically, leading to an exact solution to the model. Next, the numerical solutions to the two models are explicitly stated. Finally, results are reported comparing the exact and numerical solutions. The numerical results reported here provide the objective data needed to determine a rule for selecting a scheme. The schemes are compared based on the number of floating point operations (flops) and on the storage required to compute the numerical solution to a given accuracy of the exact solution. The preferred scheme requires the least number of flops and the least storage space. When neither scheme is more efficient in both areas of comparison, a heuristic will determine which scheme to chose. The scheme with the least number of flops will be chosen when computation time is constrained and the problem is small enough for space not to be an issue. Otherwise, the scheme with the least storage will be chosen.

4.1 Physical Experiment

It is desired to ascertain the pressure profile of a given region, a seismic experiment is conducted to aid in determining this information. The region of interest is a two-
dimensional cross-section of the earth. A source is activated in the center of the region. Once energy from the source has been released, receivers record the time and amplitude of the energy that reaches them. The source is located in the center of the receivers. This arrangement of source and receivers is called a split-spread seismic experiment. The receivers are placed an equal distance apart spanning a .5 km on each side of the source. The wave speed in the experimental region is assumed constant 2.5 km/s and the receivers record for a predetermined length of time. The data recorded at the receivers is displayed graphically, the graph is called a seismogram. Seismograms are pressure values recorded at the receivers for the duration of the experiment.

This experiment is modeled analytically and numerically. The analytical model and the physical experiment are discontinuous at the source point. This discontinuity requires the seismograms to be recorded some distance from the source. Along with the source discontinuity the physical experiment and analytical model lack boundaries. That is the energy in the physical experiment and analytical model travels infinitely far away from the source in a homogeneous media. The numerical model, however does not have a source discontinuity and the implementation of boundary conditions cause reflections in the seismograms produced numerically. Therefore to compare the analytical and numerical models, seismograms will be calculated some distance from the source and before energy reflected from the boundaries in the numerical model can be detected at the given receiver.

4.2 Exact Solutions

The exact solutions are found by analytically solving the partial differential equations that model the velocity-stress and pressure forms of the acoustic wave equation.
The partial differential equations are derived from laws of physics: Mass Balance

\[
\frac{\partial \rho(x, z, t)}{\partial t} = \rho(x, z, t) \left( \frac{\partial v_z(x, z, t)}{\partial x} + \frac{\partial v_z(x, z, t)}{\partial z} \right),
\]

(4.2.1)

Momentum Balance

\[
\rho(x, z, t) \frac{\partial v_z(x, z, t)}{\partial t} = \frac{\partial p(x, z, t)}{\partial x} + f_z(x, z, t),
\]

(4.2.2)

\[
\rho(x, z, t) \frac{\partial v_z(x, z, t)}{\partial t} = \frac{\partial p(x, z, t)}{\partial z} + f_z(x, z, t),
\]

(4.2.3)

and the Constitutive law

\[
p(x, z, t) = P(\rho(x, z, t)).
\]

(4.2.4)

The equilibrium state equations (4.2.1) - (4.2.4) are perturbed by a small change to the forcing function. Under the assumptions in Chapter 2, the perturbed equations are reduced to the velocity-stress equations

\[
\frac{1}{c^2} \frac{\partial \delta p}{\partial t} = \frac{\partial \delta v_z}{\partial x} + \frac{\partial \delta v_z}{\partial z},
\]

(4.2.5)

\[
\frac{\partial \delta v_z}{\partial t} = \frac{\partial \delta p}{\partial x} + R(t) \frac{\partial \delta(x, z)}{\partial x},
\]

(4.2.6)

\[
\frac{\partial \delta v_z}{\partial t} = \frac{\partial \delta p}{\partial z} + R(t) \frac{\partial \delta(x, z)}{\partial z},
\]

(4.2.7)

where \(\delta(x, z)\) is the Dirac Delta function in two-dimensions and \(R(t)\) is a Ricker wavelet.

The seismic experiment that is being simulated in this paper is driven by a point dilatational source, that is the energy from the source is released at a fixed point equally in all directions. The source is modeled in the velocity-stress form by the gradient of the Delta function times a Ricker wavelet. The Ricker wavelet, \(R(t)\), is the second derivative of the Gaussian. The Ricker wavelet used here has a dominant frequency of 25 Hz and a maximum frequency of 55 Hz, which occurs at 2.2 times the dominant frequency.
Recalling from Chapter 2, the pressure form of the acoustic wave equation results from taking the time derivative of (4.2.5) and using equations (4.2.6) and (4.2.7) to eliminate the time derivatives of velocity in resulting equation. This process leaves $R(t)$ times the Laplacian of the Dirac Delta function on the right-hand side of the pressure form. This right-hand side is recognized as the divergence of the force vector in the velocity-stress form. Thus, the pressure form equation is

$$\frac{1}{c^2} \frac{\partial^2 \delta p}{\partial t^2} (x, z, t) - \frac{\partial^2 \delta p}{\partial x^2} (x, z, t) - \frac{\partial^2 \delta p}{\partial z^2} (x, z, t) = R(t) \nabla^2 \delta(x, z). \quad (4.2.8)$$

The exact solution to the velocity-stress form ($\delta p, \delta v_x, \delta v_z$) as given in [6] is defined as

$$\delta p = \frac{\partial u}{\partial t}, \quad \delta v_x = \frac{\partial u}{\partial x}, \quad \text{and} \quad \delta v_z = \frac{\partial u}{\partial z},$$

where $u$, the acoustic field potential, solves

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} (x, z, t) - \frac{\partial^2 u}{\partial x^2} (x, z, t) - \frac{\partial^2 u}{\partial z^2} (x, z, t) = \left( \int_{-\infty}^{t} R(t) \right) \nabla^2 \delta(x, z). \quad (4.2.9)$$

The solution to (4.2.9) is found by first solving the following two-dimensional wave equation with the delta function representing the point source on the right-hand side.

$$\frac{1}{c^2} \frac{\partial^2 u_o}{\partial t^2} (x, z, t) - \frac{\partial^2 u_o}{\partial x^2} (x, z, t) - \frac{\partial^2 u_o}{\partial z^2} (x, z, t) = \left( \int_{-\infty}^{t} R(t) \right) \delta(x, z). \quad (4.2.10)$$

The solutions to the (4.2.9) and (4.2.10) are related in the following way

$$u = \nabla^2 u_o = \left( \int_{-\infty}^{t} R(t) \right) \delta(x, z) - \frac{1}{c^2} \frac{\partial^2 u_o}{\partial t^2}. \quad (4.2.11)$$

This relationship comes from noticing that by taking the Laplacian of (4.2.10) the left and right-hand sides of (4.2.9) and (4.2.10) are equal, and if you require $u = \nabla^2 u_o$. Finally, manipulating the result of taking the Laplacian of (4.2.10) results in (4.2.11). The solution $u_o$ can be found in [9]:

$$u_o = \frac{1}{2\pi c} \left( \int_{r/c}^{t} \frac{\int_{-\infty}^{t} R(t - \tau) d\tau}{\sqrt{c^2 \tau^2 - r^2}} dr \right). \quad (4.2.12)$$
Away from the source the solution the acoustic potential is

$$u = -\frac{1}{c^2} \frac{\partial^2 u_o}{\partial t^2}. \quad (4.2.13)$$

In order to introduce as little numerical error as possible, the analytic second time derivative with respect to \( t \) is used when calculating the exact solution using the Fundamental Theorem of Calculus and the Product Rule. The solution \( u \) is defined as

$$u = -\frac{1}{2\pi c} \left( \int_{r/c}^{t} \frac{R'(t - \tau)}{\sqrt{c^2\tau^2 - r^2}} d\tau \right). \quad (4.2.14)$$

The exact pressure and velocities are found by taking the corresponding partial derivatives in time or space.

The exact solution of the pressure form is found similarly to the exact solution for the velocity-stress form in that

$$\delta p = \nabla^2 p_o = R(t)\delta(x, z) - \frac{1}{c^2} \frac{\partial^2 p_o}{\partial t^2}, \quad (4.2.15)$$

where \( p_o \) solves

$$\frac{1}{c^2} \frac{\partial^2 p_o}{\partial t^2}(x, z, t) - \frac{\partial^2 p_o}{\partial x^2}(x, z, t) - \frac{\partial^2 p_o}{\partial z^2}(x, z, t) = R(t)\delta(x, z). \quad (4.2.16)$$

The solution \( p_o \) is

$$p_o = \frac{1}{2\pi c} \left( \int_{r/c}^{t} \frac{R(t - \tau)}{\sqrt{c^2\tau^2 - r^2}} d\tau \right). \quad (4.2.17)$$

Therefore away from the source,

$$p = -\frac{1}{2\pi c} \left( \int_{r/c}^{t} \frac{R''(t - \tau)}{\sqrt{c^2\tau^2 - r^2}} d\tau \right). \quad (4.2.18)$$

The exact solutions to the velocity-stress and pressure forms are computed in Matlab. The integrals are evaluated using the Matlab function \textit{quad}. The seismograms are formed by iteratively computing the integrals \((4.2.14)\) and \((4.2.18)\) for each time step.
4.3 Numerical Solution

The exact solutions to the velocity-stress and pressure forms are used to check the validity of the numerical solutions, similar to the use of simulations of the seismic experiment to verify the interpretation of the results of the physical experiment. The finite difference schemes are arrived at by discretizing the continuous partial differential equations (4.2.5) - (4.2.8). The staggered-grid finite difference stencil introduced by Virieux [8] is used to solve the velocity-stress form of the acoustic wave equation. The 2-2 staggered-grid scheme is

\[
\frac{1}{c_{j,k}^2} \frac{\delta p_{j,k}^i - \delta p_{j,k}^{i-1}}{\Delta t} = \left( \frac{\delta v_{x,j,k}^i}{\Delta x} \right) - \left( \frac{\delta v_{x,j-1,k}^i}{\Delta x} \right) + \frac{\delta v_{z,j,k}^i - \delta v_{z,j,k-1}^i}{\Delta z},
\]

\[
\frac{\delta v_{x,j,k}^i - \delta v_{x,j,k}^{i-1}}{\Delta t} = \frac{\delta p_{j+1,k}^{i+1} - \delta p_{j,k}^{i+1}}{\Delta x} + R^{i+1} \left( \frac{\delta_{j+1,k} - \delta_{j,k}}{\Delta x} \right),
\]

\[
\frac{\delta v_{z,j,k}^i - \delta v_{z,j,k}^{i-1}}{\Delta t} = \frac{\delta p_{j,k+1}^{i+1} - \delta p_{j,k}^{i+1}}{\Delta z} + R^{i+1} \left( \frac{\delta_{j,k+1} - \delta_{j,k}}{\Delta z} \right),
\]

and the 2-4 staggered-grid finite difference stencil is

\[
\frac{1}{c_{j,k}^2} \frac{\delta p_{j,k}^i - \delta p_{j,k}^{i-1}}{\Delta t} = \left( \frac{-\delta v_{x,j+1,k}^i + 27\delta v_{x,j,k}^i + 27\delta v_{x,j-1,k}^i + \delta v_{x,j-2,k}^i}{24\Delta x} \right) + \left( \frac{-\delta v_{z,j,k+1}^i + 27\delta v_{z,j,k}^i + 27\delta v_{z,j-1,k}^i + \delta v_{z,j-2,k}^i}{24\Delta z} \right),
\]

\[
\frac{\delta v_{x,j,k}^i - \delta v_{x,j,k}^{i-1}}{\Delta t} = \left( \frac{-\delta p_{j+2,k}^{i+1} + 27\delta p_{j+1,k}^{i+1} - 27\delta p_{j,k}^{i+1} + \delta p_{j-1,k}^{i+1}}{24\Delta x} \right) + R^{i+1} \left( \frac{-\delta_{j+2,k} + 27\delta_{j+1,k} - 27\delta_{j,k} + \delta_{j-1,k}}{24\Delta x} \right),
\]
\[
\frac{\delta v^i_{j,k} - \delta v^{i-1}_{j,k}}{\Delta t} = \left( \frac{-\delta p^i_{j,k+2} + 27 \delta p^i_{j,k+1} - 27 \delta p^i_{j,k-1} + \delta p^i_{j,k-2}}{24 \Delta z} \right) + R^{i+1} \left( \frac{-\delta j_{k+2} + 27 \delta j_{k+1} - 27 \delta j_{k} + \delta j_{k-1}}{24 \Delta z} \right).
\]

The pressure form simulation results from using a centered difference stencil to solve the pressure equation. The centered difference schemes introduced by Alford et al. [1] are as follows. The 2-2 centered difference scheme

\[
\left( \frac{\delta p^{i+1}_{j,k} - 2 \delta p^i_{j,k} + \delta p^{i-1}_{j,k}}{c^2_{j,k} \Delta t^2} \right) \Delta x^2 \left( \frac{\delta p^{i+1}_{j+1,k} - 2 \delta p^i_{j+1,k} + \delta p^{i-1}_{j+1,k}}{\Delta x^2} \right) - \left( \frac{\delta p^{i+1}_{j,k+1} - 2 \delta p^i_{j,k+1} + \delta p^{i-1}_{j,k+1}}{\Delta z^2} \right) = R^i \left( \frac{\delta j_{j+1,k} + \delta j_{j-1,k}}{\Delta x^2} \right) + \left( \frac{\delta j_{j,k+1} + \delta j_{j,k-1}}{\Delta z^2} \right),
\]

and the 2-4 centered difference stencil

\[
\left( \frac{\delta p^{i+1}_{j,k} - 2 \delta p^i_{j,k} + \delta p^{i-1}_{j,k}}{c^2 \Delta t^2} \right) - \left( \frac{-\delta p^i_{j+2,k} + 16 \delta p^i_{j+1,k} - 30 \delta p^i_{j,k} + 16 \delta p^i_{j-1,k} - \delta p^i_{j-2,k}}{12 \Delta x^2} \right) - \left( \frac{-\delta p^i_{j,k+2} + 16 \delta p^i_{j,k+1} - 30 \delta p^i_{j,k} + 16 \delta p^i_{j,k-1} - \delta p^i_{j,k-2}}{12 \Delta z^2} \right) = R^i \left( \frac{-\delta j_{j+2,k} + 16 \delta j_{j+1,k} - 30 \delta j_{j,k} + 16 \delta j_{j-1,k} - \delta j_{j-2,k}}{12 \Delta x^2} \right) + \left( \frac{-\delta j_{j,k+2} + 16 \delta j_{j,k+1} - 30 \delta j_{j,k} + 16 \delta j_{j,k-1} - \delta j_{j,k-2}}{12 \Delta z^2} \right).
\]

The sources in the two families of schemes are key to the equivalence of the solutions to each form. They model the same source in different ways. The staggered-grid family uses the gradient of the Dirac Delta function, while the centered difference
family uses the Laplacian of the Dirac Delta function. In both families the time
dependent portion of the forcing function is a Ricker wavelet. The gradient and
Laplacian of the delta function on the right-hand sides are discretized using the same
finite difference scheme used on the left-hand side of the equation, keeping the order of
accuracy consistent. Simulations of the velocity-stress model and the pressure model
were coded in FORTRAN and run for 2-2 and 2-4 orders of accuracy for each family
of finite difference stencils and compared to their exact solutions.

4.4 Comparison

Many comparisons of finite difference schemes used to generate synthetic seis-
mograms have been conducted on single families of finite difference schemes, eg [2].
However, the comparisons made in this paper are across families of difference schemes.
The comparison of storage and computational cost between the two families finite dif-
ference schemes are conditioned upon the error between the exact and approximate
solutions. The finite difference approximation is required to be within 5% weighted
absolute error to the exact solution. The weighted absolute error is calculated by
taking a subset of the experimental region consisting of four points. A point was cho-
sen 100 m from the source and its three neighboring nodes furthest from the source
were used to compute the average absolute error. The absolute error is weighted by
the time step for the given scheme. Seismograms are recorded at these four points
of the exact solution and the finite difference approximation. Once averaged, the
weighted absolute error between the exact and finite difference averaged seismograms
were computed and reported in Tables 4.7 - 4.10.

In order to achieve this accuracy, the distance between nodes and the time steps
must be properly chosen. The node spacing $\Delta x$ is chosen to prevent grid dispersion,
and the time step $\Delta t$ is chosen to maintain the stability of the scheme. The node
spacing is determined by the number of grid points per wavelength $\lambda$ sampled, where $\lambda = \frac{c}{f_{\text{max}}}$ and $f_{\text{max}}$ is the maximum frequency for these experiments the maximum frequency is 55 Hz. For the 2-2 finite difference schemes, the data must be sampled on at least 10 grid points per wavelength to prevent grid dispersion, while the 2-4 schemes only require 5 grid points per wavelength to prevent grid dispersion [3].

Several values of grid points per wavelength were sampled for the 2-2 and 2-4. The time step was chosen to maintain the stability of the difference scheme. If the time step is chosen too large, the difference scheme propagates the wave too far per time step and the scheme becomes unstable. The scheme remains stable if a stability condition on the Courant Number, $\nu = \frac{c\Delta t}{\Delta x}$, is satisfied. A stable 2-2 scheme requires $\nu \leq \frac{1}{\sqrt{2}}$ while a stable 2-4 scheme requires $\nu \leq .606$ [3] for the staggered grid scheme and $\nu \leq \sqrt{\frac{2}{5}}$ for the centered difference scheme. The value of $\Delta t$ is chosen to be 90%, 95%, and 98% of the Courant number for the 2-2 staggered-grid and centered-difference schemes, while the 2-4 staggered-grid and centered-difference schemes were run at 30%, 35%, and 40% of the Courant number. Once the proper values $\Delta x$ and $\Delta t$ are determined, the ones that give the lowest absolute error, the equations in Section 3.4 are used to compare the two families of finite difference schemes with respect to storage and computational cost. The average weighted absolute errors for these stability limits for several choices of numbers of grid points per wavelength are displayed in Tables 4.1 - 4.6.

4.5 Results

The staggered-grid and centered difference schemes were implemented and tested using several number of grid points per wavelength and Courant numbers. The computational efficiency of the most accurate combination of grid points per wavelength and Courant numbers were compared using both the 2-2 and 2-4 schemes in each
family. Tables 4.1 - 4.6 compare the stability and error for several stability limits and grid points per wavelength. The stability limit that produces the most accurate approximation for the 2-2 and 2-4 schemes are plotted in figures 4.1 and 4.2. while Tables 4.7 - 4.10 compare the cost to store and compute the 2-2 and 2-4 staggered-grid and centered difference schemes that produce approximations that are within 5% of the exact solution.

![Error Graph 2-2 Schemes Stability Limit 98%](image)

**Figure 4.1: Error Graph 2-2 Schemes Stability Limit 98%**

<table>
<thead>
<tr>
<th>Table 4.1: Stability and Error 2-2 Schemes 90% of Stability Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Courant Number</td>
</tr>
<tr>
<td>Stability limit %</td>
</tr>
<tr>
<td>Relative Error</td>
</tr>
<tr>
<td>7 grid points</td>
</tr>
<tr>
<td>10 grid points</td>
</tr>
<tr>
<td>12 grid points</td>
</tr>
<tr>
<td>15 grid points</td>
</tr>
<tr>
<td>17 grid points</td>
</tr>
<tr>
<td>20 grid points</td>
</tr>
</tbody>
</table>
Table 4.2: Stability and Error 2-2 Schemes 95% of Stability Limit

<table>
<thead>
<tr>
<th></th>
<th>Staggered-grid</th>
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</thead>
<tbody>
<tr>
<td>Courant Number</td>
<td>.6718</td>
<td>.6718</td>
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<tr>
<td>Stability limit %</td>
<td>95%</td>
<td>95%</td>
</tr>
<tr>
<td>Relative Error</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 grid points</td>
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<tr>
<td>20 grid points</td>
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Table 4.3: Stability and Error 2-2 Schemes 98% of Stability Limit

<table>
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<th></th>
<th>Staggered-grid</th>
<th>Centered difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Courant Number</td>
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<td>.6930</td>
</tr>
<tr>
<td>Stability limit %</td>
<td>98%</td>
<td>98%</td>
</tr>
<tr>
<td>Relative Error</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 grid points</td>
<td>.2846</td>
<td>.2523</td>
</tr>
<tr>
<td>10 grid points</td>
<td>.1081</td>
<td>.0946</td>
</tr>
<tr>
<td>12 grid points</td>
<td>.0660</td>
<td>.0575</td>
</tr>
<tr>
<td>15 grid points</td>
<td>.0374</td>
<td>.0320</td>
</tr>
<tr>
<td>17 grid points</td>
<td>.0269</td>
<td>.0233</td>
</tr>
<tr>
<td>20 grid points</td>
<td>.0179</td>
<td>.0155</td>
</tr>
</tbody>
</table>

Table 4.4: Stability and Error 2-4 Schemes 30% of Stability Limit

<table>
<thead>
<tr>
<th></th>
<th>Staggered-grid</th>
<th>Centered difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Courant Number</td>
<td>.1818</td>
<td>.1837</td>
</tr>
<tr>
<td>Stability limit %</td>
<td>30%</td>
<td>30%</td>
</tr>
<tr>
<td>Relative Error</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 grid points</td>
<td>2.5793</td>
<td>3.8529</td>
</tr>
<tr>
<td>5 grid points</td>
<td>.1234</td>
<td>.1460</td>
</tr>
<tr>
<td>7 grid points</td>
<td>.0204</td>
<td>.0223</td>
</tr>
<tr>
<td>10 grid points</td>
<td>.0068</td>
<td>.0064</td>
</tr>
<tr>
<td>12 grid points</td>
<td>.0047</td>
<td>.0045</td>
</tr>
<tr>
<td>15 grid points</td>
<td>.0032</td>
<td>.0030</td>
</tr>
</tbody>
</table>
### Table 4.5: Stability and Error 2-4 Schemes 35% of Stability Limit

<table>
<thead>
<tr>
<th></th>
<th>Staggered-grid</th>
<th>Centered difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Courant Number</td>
<td>.2121</td>
<td>.2143</td>
</tr>
<tr>
<td>Stability limit %</td>
<td>35%</td>
<td>35%</td>
</tr>
<tr>
<td>Relative Error</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 grid points</td>
<td>2.6697</td>
<td>3.7589</td>
</tr>
<tr>
<td>5 grid points</td>
<td>.1228</td>
<td>.1438</td>
</tr>
<tr>
<td>7 grid points</td>
<td>.0204</td>
<td>.0247</td>
</tr>
<tr>
<td>10 grid points</td>
<td>.0107</td>
<td>.0102</td>
</tr>
<tr>
<td>12 grid points</td>
<td>.0075</td>
<td>.0074</td>
</tr>
<tr>
<td>15 grid points</td>
<td>.0068</td>
<td>.0064</td>
</tr>
</tbody>
</table>

### Table 4.6: Stability and Error 2-4 Schemes 40% of Stability Limit

<table>
<thead>
<tr>
<th></th>
<th>Staggered-grid</th>
<th>Centered difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Courant Number</td>
<td>.2424</td>
<td>.2449</td>
</tr>
<tr>
<td>Stability limit %</td>
<td>40%</td>
<td>40%</td>
</tr>
<tr>
<td>Relative Error</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 grid points</td>
<td>2.7170</td>
<td>3.5043</td>
</tr>
<tr>
<td>5 grid points</td>
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<td>.1430</td>
</tr>
<tr>
<td>7 grid points</td>
<td>.0331</td>
<td>.0320</td>
</tr>
<tr>
<td>10 grid points</td>
<td>.0218</td>
<td>.0155</td>
</tr>
<tr>
<td>12 grid points</td>
<td>.0118</td>
<td>.0110</td>
</tr>
<tr>
<td>15 grid points</td>
<td>.0069</td>
<td>.0069</td>
</tr>
</tbody>
</table>

### Table 4.7: Cost and Storage 2-2 Staggered-grid scheme

<table>
<thead>
<tr>
<th>Grid points per λ</th>
<th>Stability Limit 98%</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>285, 100, 205</td>
</tr>
<tr>
<td>17</td>
<td>413, 893, 089</td>
</tr>
<tr>
<td>20</td>
<td>675, 083, 205</td>
</tr>
<tr>
<td></td>
<td>Storage (elements)</td>
</tr>
<tr>
<td>326,700</td>
<td>419,628</td>
</tr>
<tr>
<td>580,800</td>
<td></td>
</tr>
</tbody>
</table>

### Table 4.8: Cost and Storage 2-2 Centered difference scheme

<table>
<thead>
<tr>
<th>Grid points per λ</th>
<th>Stability Limit 98%</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>285, 100, 205</td>
</tr>
<tr>
<td>17</td>
<td>413, 893, 089</td>
</tr>
<tr>
<td>20</td>
<td>675, 083, 205</td>
</tr>
<tr>
<td></td>
<td>Storage (elements)</td>
</tr>
<tr>
<td>217, 800</td>
<td>279,752</td>
</tr>
<tr>
<td>387, 200</td>
<td></td>
</tr>
</tbody>
</table>
Table 4.9: Cost and Storage 2-4 Staggered-grid scheme

<table>
<thead>
<tr>
<th>Grid points per $\lambda$</th>
<th>7</th>
<th>10</th>
<th>12</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost (flops)</td>
<td>270,860,444</td>
<td>790,614,008</td>
<td>1,366,181,000</td>
<td>2,666,852,108</td>
</tr>
<tr>
<td>Storage (elements)</td>
<td>71,148</td>
<td>145,200</td>
<td>209,088</td>
<td>326,700</td>
</tr>
</tbody>
</table>

Table 4.10: Cost and Storage 2-4 Centered difference scheme

<table>
<thead>
<tr>
<th>Grid points per $\lambda$</th>
<th>7</th>
<th>10</th>
<th>12</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost (flops)</td>
<td>188,803,082</td>
<td>549,920,806</td>
<td>950,792,838</td>
<td>1,858,051,806</td>
</tr>
<tr>
<td>Storage (elements)</td>
<td>47,432</td>
<td>96,800</td>
<td>139,392</td>
<td>217,800</td>
</tr>
</tbody>
</table>

Figure 4.2: Error Graph 2-4 Schemes Stability Limit 30%
4.6 Discussion

Analysis of the results of the 2-2 schemes show that the staggered-grid scheme requires 5.5% less computations than the centered differences scheme, while both schemes require the same amount of storage. The accuracy of the 2-2 schemes, Tables 4.1 and 4.3, favor the centered difference scheme. On average at 98% of the stability limit the centered difference scheme is about 13% more accurate than the staggered-grid scheme, although the same number of grid points per wavelength can be chosen to come within 5% weighted absolute error. This being the case, the advantage with respect to accuracy is minimal. Therefore, the 2-2 staggered-grid scheme is chosen when computation is expensive and when storage is limited. The 2-2 schemes prefer to be run as close to the stability limit as possible, the trials run at 98% of the stability limit produce the most accurate solutions.

The 2-4 schemes show the centered difference scheme has the advantage with respect to storage and computation. The 2-4 centered difference scheme requires 33% less storage and 30% less computation than the 2-4 staggered-grid scheme. The accuracy of the staggered-grid scheme is superior to the centered difference scheme for small numbers of grid points per wavelength. However, in most cases the small numbers of grid points per wavelength are not sufficient to come within 5% error of the exact solution. On average the 2-4 centered difference schemes are 5% more accurate than the staggered-grid schemes. Although there is no significant advantage in accuracy between the 2-4 centered difference and staggered-grid schemes, the centered difference scheme will always be chosen as it is more efficient to compute with respect to storage and computation. The 2-4 schemes produce more accurate results when run at smaller stability limits, 30%.

The results presented in this paper show that in most all cases it is more advantageous to use the centered difference scheme over the staggered-grid scheme when
numerically generating seismograms for a two-dimensional homogeneous region. The centered difference scheme is always preferred when storage space is an issue and is only slightly less computationally efficient in the 2-2 case. Finite difference generated seismograms can be very accurate in simulating seismic experiments and continue to be a popular method for verifying results from the physical experiment.
Bibliography


