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Decoupled Space Station / Shuttle Analysis in the Presence of
Non-Classical Damping and Geometric Non-Linearities

by

James F. Brusoe

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE
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Abstract

Decoupled Space Station / Shuttle Analysis in the Presence of
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Dynamic analysis methods for computationally demanding systems are examined. Component Mode Synthesis methods are detailed and their limitations established. Decoupled analysis is offered as a solution for problems that traditional methods are incapable of addressing, such as systems with non-classical damping or geometric non-linearities. The methodology of decoupled analysis is detailed along with three modifications to the basic decoupled analysis method. This method is then applied to two large problems involving the International Space Station and the Space Shuttle.
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1 Introduction

A major problem facing structural dynamicists today is the size of the structures being studied. For example, the model of the International Space Station (ISS) contains tens of thousands of degrees of freedom. The difficulty in analyzing such bulky models leads the analyst to seek model reduction methods.

The objective of model reduction methods is to reduce the degrees of freedom of a model while keeping the accuracy of the original model. A widely accepted method of model reduction is to delete the modes from the dynamic representation that produce a negligible effect on the loads and deflections found by the model. The result is replacing the full modal matrix with a truncated modal matrix to reduce the computations necessary to produce an accurate solution.

Another difficulty facing structural dynamicists is that the different components of a structure may be constructed and analyzed by different groups. For example, the ISS is being constructed by the United States in conjunction with fifteen other countries from around the world. The ideal analytical method would include the ability to independently develop structural models for the component parts that can then be assembled into a model that accurately calculates the dynamic behavior of the entire structure. A method capable of accomplishing this task is the Component Mode Synthesis (CMS) method.

In CMS, the dynamic models for the separate structures are developed independently. Model reduction methods, such as truncation of modes, are applied to the substructures. The substructure models are then assembled into a system model, which may be reduced once again by truncating the system modes. In the case of the ISS, a
sample model containing tens of thousands of degrees of freedom may be reduced to a model containing a few hundred degrees of freedom.

Despite the use of reduction and synthesis techniques, the final system may still be quite large. An ideal alternative to CMS would be a method where the system model need not be assembled at all. One such alternative to CMS is the decoupled analysis method. In this method, only a representation for the junction between components needs to be assembled. The junction motion is solved for and then applied separately to the components. This way, the components are loaded in a realistic manner that reflects the condition of being attached to the entire system without having to solve the entire system as a single unit. This significantly reduces the computational time necessary to produce an accurate solution.

Truncation of the modal matrix in both CMS and decoupled analysis has the unwanted effect of stiffening the structure. Deleting modes that would otherwise participate, albeit insignificantly, in the description of the deflection of a system results in the system's deflection being somewhat underestimated. The accuracy of the model could be improved by retaining the effects of the truncated modes. Of course, the modes were originally dropped to reduce the size of the model. The residual flexibility method manages to retain the effects of the truncated modes while still reducing the size of the system by condensing the truncated modes into a fewer number of modes.

An alternative to truncating the mode sets is to generate mode sets that are more efficient in describing the likely displacement of the system. One such way is to generate Lanczos vectors, which are based on the spatial loading of the structure. The static deflection of the system under its spatial loading is used to formulate the first Lanczos
vector. An algorithm that includes orthogonalizing the vector in relation to the previous Lanczos vectors produces additional vectors. With this method a set of vectors, smaller in dimension than the original non-truncated modal matrix, is assembled to accurately describe the motion of the system with less computational effort.

This paper is divided into five chapters. Chapter One is the introduction. Chapter Two describes the CMS method. In particular, the Craig-Bampton method is fully described. Its ease of use and gains in efficiency have established it as an industry standard. Chapter Three covers decoupled analysis. This method’s ability to avoid the synthesis of a system-sized model has made it a very useful tool for analysis, particularly for the analysis of payloads in launch vehicles. The formulation of the decoupled analysis method using the Newmark integrator is covered, along with an alternate formulation based on linear interpolation of the excitation forces. In addition, Chapters Two and Three include descriptions of both the residual flexibility method and the Lanczos vectors formulation applied to the two dynamic analysis methods. Chapter Four presents two large sample problems involving the ISS and the Shuttle that demonstrate the usefulness of the decoupled analysis method. Chapter Five completes the paper with a discussion of the conclusions.

2 Component-Mode Synthesis

2.1 Background

The history of the CMS method can be traced back to the early 1960’s with papers by Przemieniecki [30] and Hurty [15]. Przemieniecki was motivated to break up the
analysis of an aircraft structure into components because “different types of analysis have to be used on different components, or because the capacity of the digital computer is not adequate to cope with the analysis of the complete structure.” These same motivations carry through to the present day despite modern computing capabilities. Additional motivation is provided by projects that span several organizations. Components may be built and analyzed separately by the organizations responsible for certain parts, and this analysis may be passed on to a separate organization responsible for combining the component models from all sources into a single system model. This is the case with the ISS project.

Przemieniecki’s static substructuring method was adapted to dynamic analysis by Hurty. Craig and Bampton [7] simplified Hurty’s representation with their realization that rigid body modes and constraint modes could be treated the same, as opposed to Hurty’s special treatment of rigid body modes. The Craig-Bampton method has become an industry standard because of its ease of application. The Craig-Bampton method will be covered further in Section 2.2.1.

Though the Craig-Bampton method is widely used, many improvements have been suggested. Most of these center on ways to adjust the truncated modal set to help it more closely resemble the actual behavior of the component. Benfield and Hruda [3] suggested modifying a substructure’s modal representation by the inertia and stiffness of adjacent components. The drawback is that this eliminates the advantage of separate analysis of components. MacNeal [19] and Rubin [31] suggested that the effects of the truncated modes be summarized and included in the component representation. Doing so reduces the stiffening of the structure caused by truncation. Rubin described a method
that includes the second order approximation of the residual flexibility of the truncated modes, improving on MacNeal's first order approximation. This residual flexibility method will be covered further in Section 2.2.2.

Hintz [14] developed a criterion for assessing a component's representation. He asserted that the representation should consist of statically complete sets of modes. These sets of modes are able to completely describe the deflection caused by static loading of the component. Wilson, Yuan, and Dickens [38] turned Hintz's reasoning around by developing modes based on the distribution of the static load. Their method incorporates Ritz vectors as a substitute for the normal modes, which avoids solving a computationally demanding eigenproblem. They have also shown that a smaller set of Ritz vectors can often display the same accuracy as a larger set of normal modes, leading to more gains in computational efficiency.

In the field of applied mathematics, Lanczos [16] developed an efficient method for solving eigenproblems in 1950. The method did not see much use until some problems with the method's numerical stability had been worked out [27,13,32,28]. After this mathematical work, the method appeared in structural dynamics applications [37,24,25,4,18,6,5,26]. An advantage of Lanczos vectors over Ritz vectors is seen when applying the vectors to CMS and decoupled analysis methods. While the Ritz vectors produce fully populated matrices in the system equations, the Lanczos vectors produce only narrowly banded equations. The fully populated system equations produced with Ritz vectors require solving a system-sized eigenproblem in order to solve the system. In contrast, the narrow band of the Lanczos-derived equations makes direct integration of the equations more efficient and bypasses the need to solve a system-sized eigenproblem.
The Lanczos method’s usefulness is now being seen in CMS methods [17] and decoupled analysis methods [35]. The CMS formulation using Lanczos vectors will be covered in Section 2.2.3.

2.2 Methodology

2.2.1 Craig-Bampton

The Craig-Bampton method starts from the premise that the behavior of the whole system can be estimated by the behavior of the subsystems, which are described by a truncated set of their natural modes with fixed interfaces. Convergence toward the system solution is aided by the use of constraint modes (defined later in this section). Dynamic systems are often represented in matrix form with the equation

\[ M \ddot{x} + C \dot{x} + K x = f \]  

(2.1)

where \( M \), \( C \), and \( K \) are the system’s mass, damping, and stiffness matrices respectively; \( f \) is the excitation vector; \( x \) is the physical displacement vector; and a dot above a variable represents differentiation with respect to time. Also, bold small case variables denote vectors and bold upper case variables denote matrices. CMS methods use partitioning to break the system down into smaller subsystems, or components, to aid in the problem solution. Denoting a representative component with the superscript \( \alpha \), the component equation of motion becomes

\[ M^\alpha \ddot{x}^\alpha + C^\alpha \dot{x}^\alpha + K^\alpha x^\alpha = f^\alpha \]  

(2.2)

where \( x^\alpha \) is the physical displacement vector of the component. The dimension of \( x^\alpha \) is \( n \times 1 \) with \( n \) being the degrees of freedom of the subsystem. The displacement vector may be partitioned into interior \((i)\) and junction \((j)\) degrees of freedom.
\[ \mathbf{x} = \begin{bmatrix} x_i \\ x_j \end{bmatrix}. \quad (2.3) \]

The junction degrees of freedom in the Craig-Bampton method include all degrees of freedom at which the component connects to other components. The interior degrees of freedom are those that are completely within the component and do not interface with other components. Note that two connected components will share their boundary degrees of freedom between them.

Applying this partitioning to the mass and stiffness matrices produces

\[
\mathbf{M}^a = \begin{bmatrix} \mathbf{M}_{ii} & \mathbf{M}_{ij} \\ \mathbf{M}_{ji} & \mathbf{M}_{jj} \end{bmatrix}^a
\]

and

\[
\mathbf{K}^a = \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ij} \\ \mathbf{K}_{ji} & \mathbf{K}_{jj} \end{bmatrix}^a.
\]

The interior partitions of these matrices can be formed into the following eigenproblem:

\[
| -\omega^2 \mathbf{M}^a_n + \mathbf{K}^a_n | = 0
\]

where \( \omega \), the square roots of the eigenvalues, are the component natural frequencies.

Solving for the eigenvalues and arranging them into a diagonal matrix produces the natural frequency matrix

\[
\Lambda^a_{nn} = \begin{bmatrix}
\omega_1^2 & 0 & 0 & 0 \\
0 & \omega_2^2 & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & \omega_n^2
\end{bmatrix}^a.
\]

(2.7)
Usually the frequencies in the matrix are arranged in ascending order, with
\[ \omega_1 < \omega_2 < \ldots < \omega_n. \] The eigenvectors can be found from the equation
\[
\left(-\omega^2 M_n + K_n\right)\phi = 0
\] (2.8)
where \( \phi \) is the eigenvector, also referred to as the modal shape vector. The modal shape vectors are then assembled into the modal matrix
\[
\Phi_n^a = \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_n \end{bmatrix}. \] (2.9)
The modal matrix can then be normalized with respect to the mass matrix such that
\[
[\Phi_n^a]^T M_n \Phi_n^a = I_{nn}
\] (2.10)
where \( I_{nn} \) is an \( n \times n \) identity matrix and the superscript \( T \) represents the transpose of the matrix. The component natural frequencies in the Craig-Bampton method are derived from an eigenproblem that only uses the interior degrees of freedom of the component.
This formulation results in the junction degrees of freedom being, in effect, fixed in place during the natural frequency calculation. Therefore, the Craig-Bampton method can be classified as a fixed-interface method. The junction movement is included in the constraint modes (derived later in this section) that are based on unit displacements of the junction degrees of freedom.

A key step in traditional CMS methods is the reduction of the component models by dropping modes corresponding to higher frequencies from the modal matrix. If the cutoff frequency is set at \( \omega_k \), then the partition of the modal matrix is
\[
\Phi_n^a = \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_k & \phi_{k+1} & \phi_{k+2} & \ldots & \phi_n \end{bmatrix},
\] (2.11)
where the vertical dashed line shows the partitioning. Dropping the modes corresponding to frequencies above the cutoff produces the matrix of kept modes

\[ \Phi_k^a = [\phi_1 \phi_2 \ldots \phi_k]. \]  

(2.12)

So far, the modal representation of the component is based solely on the interior partitions of the mass and stiffness matrices. In order to include the effects of connectivity with other components, the constraint matrix can be included in the component representation. Constraint modes are the modes associated with fixing in place all of the junction degrees of freedom of the component, save for one that is subjected to a unit displacement. These modes can be calculated using the following:

\[ \Psi_c^a = \{ -K_n^{-1}K_o \}_f^a \]  

(2.13)

where \( \Psi \) is used to denote a general modal matrix and the subscript c specifies a constraint modal matrix. Together, the normal modes and the constraint modes form a statically complete representation of the component as specified by Hintz, with the computational savings coming from dropping certain higher frequency modes deemed to have a negligible effect on the solution. The kept normal modes and constraint modes are assembled into the following reduced component mode matrix:

\[ \Psi^a = \begin{bmatrix} \Phi_k^a & \Psi_c^a \\ 0 & I \end{bmatrix}. \]  

(2.14)

The dimension of the kept modal matrix is \( i \times k \), the constraint matrix is \( i \times j \), the null matrix is \( j \times k \), and the identity matrix is \( j \times j \). The null matrix represents the constrained condition of the junction degrees of freedom corresponding to the fixed-interface normal modes. The identity matrix represents the unit displacements of the junction degrees of
freedom corresponding to the constraint modes. The reduced component matrix can then be used to create these generalized mass and stiffness matrices:

\[ \tilde{M}^\alpha = [\Psi^\alpha]^T M^\alpha \Psi^\alpha \]  \hspace{1cm} (2.15)

and

\[ \tilde{K}^\alpha = [\Psi^\alpha]^T K^\alpha \Psi^\alpha \]  \hspace{1cm} (2.16)

where the hat denotes the generalized reduced form of the matrices. The generalized mass and stiffness matrices for all of the components can then be assembled into these system mass and stiffness matrices:

\[ \tilde{M}^{\text{sys}} = \begin{bmatrix} \tilde{M}^\alpha & 0 & 0 \\ 0 & \tilde{M}^\beta & 0 \\ 0 & 0 & \ddots \end{bmatrix} \]  \hspace{1cm} (2.17)

and

\[ \tilde{K}^{\text{sys}} = \begin{bmatrix} \tilde{K}^\alpha & 0 & 0 \\ 0 & \tilde{K}^\beta & 0 \\ 0 & 0 & \ddots \end{bmatrix} \]  \hspace{1cm} (2.18)

The generalized displacement vector that corresponds to the above system matrices is

\[ \mathbf{p} = \begin{bmatrix} q_k^\alpha \\ x_j^\alpha \\ q_k^\beta \\ x_j^\beta \\ \vdots \end{bmatrix} \]  \hspace{1cm} (2.19)

Note that the degrees of freedom denoted with a q in Equation (2.19) correspond to the generalized modal displacements of the interior of a given component and the
degrees of freedom denoted with an $x$ are physical displacements at the junctions of components.

The next step is to remove the dependent degrees of freedom from the generalized displacement vector. It was previously noted that two adjacent components share the same junction degrees of freedom. This redundancy carries through into Equation (2.19). The vector containing only the linearly independent degrees of freedom of the system is

$$q = \begin{bmatrix} q_k \\ x_j \end{bmatrix}.$$  \hspace{1cm} (2.20)

The task is then to create a transformation matrix that converts $q$ to $p$. Calling this transformation matrix by $S$, the transformation equation appears as:

$$p = Sq.$$ \hspace{1cm} (2.21)

Once formed, the transformation matrix can be applied to the system mass and stiffness matrices formed above to remove the dependent degrees of freedom:

$$M^{sys} = S^T\tilde{M}^{sys}S$$ \hspace{1cm} (2.22)

and

$$K^{sys} = S^T\tilde{K}^{sys}S.$$ \hspace{1cm} (2.23)

This also reduces the dimension of the system, making the system equations easier to solve. The natural frequencies of the system can be found by again forming an eigenproblem, this time using the system mass and stiffness matrices:

$$[-\omega^2 M^{sys} + K^{sys}] = 0.$$ \hspace{1cm} (2.24)

Solving this equation gives the natural frequencies of the system, which can be assembled into the following matrix:
\[ \Lambda^{\text{sys}} = \begin{bmatrix} \left( \omega_1^{\text{sys}} \right)^2 & 0 & 0 & 0 \\ 0 & \left( \omega_2^{\text{sys}} \right)^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \left( \omega_n^{\text{sys}} \right)^2 \end{bmatrix}^{\text{sys}} \] 

(2.25)

where the frequencies are once again arranged such that \( \omega_1^{\text{sys}} < \omega_2^{\text{sys}} < \ldots < \omega_n^{\text{sys}} \). The accompanying eigenvectors are then found with

\[ \left( -\left( \omega^{\text{sys}} \right)^2 M^{\text{sys}} + K^{\text{sys}} \right) \phi^{\text{sys}} = 0 \] 

(2.26)

and the system modal matrix is then

\[ \Phi^{\text{sys}} = \begin{bmatrix} \phi_1^{\text{sys}} & \phi_2^{\text{sys}} & \ldots & \phi_n^{\text{sys}} \end{bmatrix}. \] 

(2.27)

The system modal matrix transforms the coordinates from the \( q \) coordinates to the \( z \) coordinates, which are the system principle coordinates:

\[ q = \Phi^{\text{sys}} z. \] 

(2.28)

The system equations can then be decoupled by using the system modal matrix:

\[ \tilde{M}^{\text{sys}} = \left( \Phi^{\text{sys}} \right)^T M^{\text{sys}} \Phi^{\text{sys}} \] 

(2.29)

and

\[ \tilde{K}^{\text{sys}} = \left( \Phi^{\text{sys}} \right)^T K^{\text{sys}} \Phi^{\text{sys}} \] 

(2.30)

where the hat denotes the reduced form of the system matrices.

Up until this point, the damping matrix has been ignored. In practice, it is often ignored until this point because of the difficulty in determining an accurate damping matrix expression. When the system equations are decoupled the assumption of modal damping can be applied. Modal damping can be implemented by using
$$\tilde{C}^{\text{sys}} = \begin{bmatrix} 2\zeta_1 \omega_1^{\text{sys}} & 0 & 0 & 0 \\ 0 & 2\zeta_2 \omega_2^{\text{sys}} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 2\zeta_n \omega_n^{\text{sys}} \end{bmatrix}$$

(2.31)

as an estimate for the system damping matrix. In this equation, \( \zeta_i \) is set to reflect the level of damping the given system mode experiences. An alternative to the modal damping assumption is to create the following component damping matrices that are linear combinations of the mass and stiffness matrices:

$$C^a = a_0 M^a + a_i K^a$$

(2.32)

where \( a_0 \) and \( a_i \) are constants scaled to represent the component’s level of damping. This is called the Rayleigh damping assumption. With this assumption the modal matrices that decouple the mass and stiffness matrices also decouple the damping matrix. The difficulty with this method is that the starting damping matrix can be hard to determine.

The final item to formulate in creating the system equations of motion is the excitation vector. This is done by assembling the excitation vector from the component interior and junction degrees of freedom:

$$f = \begin{bmatrix} f_1^a \\ f_2^a \\ \vdots \\ f_1^\beta \\ f_2^\beta \\ \vdots \end{bmatrix}$$

(2.33)

Then the system excitation vector that is compatible with the reduced equations can be found with
\[ \hat{f}^{y5} = \left[ \Phi^{y5} \right]^T S^T \Psi^T f \]  \hspace{1cm} (2.34)

where

\[ \Psi = \begin{bmatrix} \Psi^\alpha & 0 & 0 \\ 0 & \Psi^\beta & 0 \\ 0 & 0 & \ddots \end{bmatrix}. \]  \hspace{1cm} (2.35)

The final decoupled system equation is

\[ \tilde{M}^{y5} \ddot{z} + \tilde{C}^{y5} \dot{z} + \tilde{K}^{y5} z = \hat{f}^{y5}. \]  \hspace{1cm} (2.36)

The displacements for each degree of freedom can be solved using methods for single degree of freedom systems. Then the solution in the original coordinates can be recovered by reversing all of the transformations used in getting to the decoupled system equation:

\[ x = \Psi S \Phi^{y5} z. \]  \hspace{1cm} (2.37)

When using numerical methods where the excitation and response is divided up into discrete time intervals, this transformation can become computationally expensive. If the displacements at only a few of the original degrees of freedom are needed, then the above equation may be partitioned and only the calculations for the desired points carried out, reducing the computational expense.

### 2.2.2 Residual Flexibility

The residual flexibility method attempts to account for the flexibility that is lost when the modes are truncated. Truncating modes has the tendency to stiffen the structure, leading to solutions with possibly less displacement than would be expected in reality. The residual flexibility method generates residual modes that, similarly to
constraint modes in the Craig-Bampton method aid in convergence to the solution by accounting for effects that are lost due to the method's representation of the component.

While constraint modes combined with the fixed-interface normal modes form a statically complete set, residual modes combined with free-interface normal modes also form a statically complete set [14].

The residual flexibility method is based on a free-interface modeling of the components. The free-interface condition means that during the solution of the component natural frequencies and mode shapes, the junction degrees of freedom are not constrained against motion but instead are free to displace. Including the junction degrees of freedom in the eigenproblem derives the free-interface component natural frequencies and mode shapes. The eigenvalue problem is then

\[ \left| -\omega^2 M^\alpha + K^\alpha \right| = 0 \]  

(2.38)

and the eigenvectors can be found from the equation

\[ \left( -\omega^2 M^\alpha + K^\alpha \right) \phi = 0. \]  

(2.39)

The free-interface modal matrix is then normalized with respect to the mass matrix by ensuring that

\[ \left[ \Phi^\alpha_n \right]^T M^\alpha \Phi^\alpha_n = I_{nn}. \]  

(2.40)

The modal matrix is partitioned as before.

\[ \Phi^\alpha_n = \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_k & \phi_{k+1} & \phi_{k+2} & \ldots & \phi_n \end{bmatrix}. \]  

(2.41)

with the dropped modes being denoted as the dropped modal matrix,

\[ \Phi^\alpha_d = \begin{bmatrix} \phi_{k-1} & \phi_{k-2} & \ldots & \phi_n \end{bmatrix}. \]  

(2.42)
The natural frequency matrix is also partitioned.

\[ \Lambda_n^a = \begin{bmatrix} \Lambda_{kk}^a & 0 \\ 0 & \Lambda_{dd}^a \end{bmatrix}. \] (2.43)

Note that since this formulation assumes unconstrained interfaces, it is possible that the component may have rigid body modes, which are easily identified as modes with natural frequencies of zero (or near zero if using a computer, with resultant numerical error). These modes must be included in the kept partitions of Equations (2.41) and (2.43). The reason for this is apparent in Equation (2.44), shown below.

Once the modal matrix and the natural frequency matrix are partitioned, the residual flexibility matrix is calculated with

\[ G_d^a = \Phi_d^a [\Lambda_{dd}^a]^{-1} [\Phi_d^a]^T \] (2.44)

where \( G \) denotes the flexibility matrix and the subscript \( d \) denotes the residual portion. The inverse of the dropped natural frequency matrix is not possible if modes with zero for natural frequencies are kept in the matrix, which explains why rigid body modes must be partitioned into the kept natural frequency matrix.

Next, the applied force matrix, \( F_j \), is assembled:

\[ F_j^a = \begin{bmatrix} 0_j \\ I_j \end{bmatrix}. \] (2.45)

This force matrix represents applying a unit force to each generalized junction coordinate successively. The residual modes represent the displacement due to the truncated modes that results from application of unit forces to the junction coordinates:

\[ \Psi_d^a = G_d^a F_j^a. \] (2.46)
The reduced component modal matrix is then assembled from the kept modes and the residual modes:

\[ \Psi^a = [\Phi^a \quad \Psi^a]. \]  \hspace{1cm} (2.47)

The generalized component mass and stiffness matrices are composed as they were in the Craig-Bampton procedure at this point using

\[ \tilde{M}^a = [\Psi^a]^T M^a \Psi^a \]  \hspace{1cm} (2.48)

and

\[ \tilde{K}^a = [\Psi^a]^T K^a \Psi^a. \]  \hspace{1cm} (2.49)

The generalized mass and stiffness matrices for all of the components can then be assembled into system mass and stiffness matrices:

\[ \tilde{M}^{\text{sys}} = \begin{bmatrix} \tilde{M}^a & 0 & 0 \\ 0 & \tilde{M}^\beta & 0 \\ 0 & 0 & \ddots \end{bmatrix} \]  \hspace{1cm} (2.50)

and

\[ \tilde{K}^{\text{sys}} = \begin{bmatrix} \tilde{K}^a & 0 & 0 \\ 0 & \tilde{K}^\beta & 0 \\ 0 & 0 & \ddots \end{bmatrix}. \]  \hspace{1cm} (2.51)

At this point, the procedure differs from the Craig-Bampton method in assembling the transformation matrix. Previously, the junction degrees of freedom were included in the final linearly independent system coordinates. Because of the formulation of the residual modes, the junction degrees of freedom no longer need to be included in the final formulation. This can be seen by looking at the residual partition of the generalized equations of motion.
\begin{equation}
\Psi_d^T M \Psi_d \ddot{p}_d + \Psi_d^T C \Psi_d \dot{p}_d + \Psi_d^T K \Psi_d p_d = \Psi_{jd}^T f_j \tag{2.52}
\end{equation}

where the component designation, \(\alpha\), is temporarily dropped for clarity. The matrix \(\Psi_{jd}^T\) is the transformation of the lower partition of \(\Psi_d\) when partitioned similarly to Equation (2.45). The above equation may be approximated with the pseudo-static equation

\begin{equation}
\Psi_d^T K \Psi_d p_d = \Psi_{jd}^T f_j. \tag{2.53}
\end{equation}

Then by application of Equations (2.44) and (2.46), Equation (2.53) becomes

\begin{equation}
F_j^T \Phi_d \Lambda^{-1}_{dd} \Phi_d^T K \Phi_d \Lambda^{-1}_{dd} \Phi_d^T F_j = F_j^T \Phi_d \Lambda^{-1}_{dd} \Phi_{jd}^T F_j. \tag{2.54}
\end{equation}

This equation can be simplified by realizing that

\begin{equation}
\Phi_d^T K \Phi_d = \Lambda_{dd}, \tag{2.55}
\end{equation}

\begin{equation}
F_j^T \Phi_d = \Phi_{jd}. \tag{2.56}
\end{equation}

and

\begin{equation}
\Phi_d^T F_j = \Phi_{jd}^T. \tag{2.57}
\end{equation}

With these, Equation (2.54) can be rearranged to form the equation

\begin{equation}
\Phi_{jd} \Lambda^{-1}_{dd} \Phi_{jd}^T \left( p_d - f_j \right) = 0. \tag{2.58}
\end{equation}

By inspection it can be seen that for this to make sense, \(d\) must equal \(j\). This is assured by having a residual mode for each junction mode. Since the rigid body modes are partitioned out of the residual frequency matrix, it is apparent that the portion outside the parentheses on the left hand side of Equation (2.58) is nonsingular. This allows the equation to be reduced to

\begin{equation}
p_d = f_j. \tag{2.59}
\end{equation}
This simplifies the system by reducing the size of the equation set. The displacement vector that corresponds to the system matrices of Equations (2.50) and (2.51) is

\[
\mathbf{p} = \begin{bmatrix}
\mathbf{p}_1^a \\
\mathbf{p}_1^b \\
\vdots
\end{bmatrix}.
\]  

(2.60)

Due to the formulation of the residual modes, the independent system coordinates are represented by the vector.

\[
\mathbf{q} = \begin{bmatrix}
\mathbf{p}_j^a \\
\mathbf{p}_j^b \\
\vdots
\end{bmatrix}.
\]  

(2.61)

The junction degrees of freedom can be left out because of the equality stated in Equation (2.59) and the formulation of the residual modes using \( \mathbf{F}_j \) as shown in Equation (2.46). Use of the residual mode formulation makes the junction degrees of freedom linearly dependent on the interior degrees of freedom.

The method then proceeds the same as the Craig-Bampton method, with the assembly of the transformation matrix using

\[
\mathbf{p} = \mathbf{S}\mathbf{q}
\]  

(2.62)

followed by assembly of the system matrix, decoupling, and solution of the system.

### 2.2.3 Lanczos Vectors

The Lanczos vectors formulation seeks to avoid solving the eigenproblem by using an alternative algorithm to assemble the modal matrix. The formulation is often load-dependent, using the static response to the loading vector as a seed vector. Craig
and Hale demonstrated methods of eliminating the load dependence from the formulation [10].

The load-dependent formulation fits conceptually with Hintz's criterion for the modal representation of a component. It has been shown that the algorithm converges to the fundamental eigenvector as long as

\[ r_0^T M \phi_1 = 0 \]  
(2.63)

where \( r_0 \) is the seed vector and \( \phi_1 \) is the fundamental eigenvector [29]. A straightforward way to fulfill this requirement is by setting

\[ r_0 = K^{-1} f, \]  
(2.64)

where \( f \) is the spatial distribution of the loading vector. The original loading vector can be expressed using the spatial distribution and a scalar function of time, \( g(t) \):

\[ f = f_s g(t). \]  
(2.65)

One option is to seed the algorithm with a constraint mode [10]. This might prove especially useful when the component is subjected to forces transmitted through the component's interface instead of to exterior forces.

Two different algorithms for formulating Lanczos vectors will be given in the following sections. The first algorithm, detailed in 2.2.3.1, is a general method for single component problems. It will also be used later in the development of the decoupled method using Lanczos vectors. The second algorithm, shown in 2.2.3.2, is designed specifically for multiple component problems, and is thus applicable to CMS methods. This second algorithm partitions the matrices into interior and junction partitions and
performs the Lanczos algorithm to the interior degrees of freedom while preserving the separation of the junction.

2.2.3.1 Single Component Load-Dependent Lanczos Algorithm

The seed vector is equated to the static response from the spatial load:

\[ r_0 = K^{-1}f_s. \]  
(2.66)

The vector \( v_0 \) is initialized with

\[ v_0 = 0. \]  
(2.67)

Then the following five equations are repeated for \( i = 1, 2, \ldots, k \):

\[ \beta_i^2 = r_{i-1}^T M r_{i-1}, \]  
(2.68)

\[ v_i = \frac{1}{\beta_i} r_{i-1}, \]  
(2.69)

\[ \bar{r}_i = K^{-1} M v_i, \]  
(2.70)

\[ \alpha_i = \bar{r}_i^T M v_i, \]  
(2.71)

and

\[ r_i = \bar{r}_i - \alpha_i v_i - \beta_i v_{i-1}. \]  
(2.72)

Note that in the last step, Equation (2.72), the vector is orthogonalized with only the two preceding vectors. It has been shown that in exact math, this theoretically ensures orthogonality with all previously calculated vectors [23]. Now the Lanczos vectors are assembled into the matrix:

\[ V = \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix}. \]  
(2.73)

Also, a tri-diagonal matrix can be formed with the following:
\[ T = V^T M K^{-1} M V. \]  
(2.74)

Alternately, the tri-diagonal matrix can be assembled with less computational effort by using intermediate calculations from the above algorithm:

\[ T = \begin{bmatrix}
\alpha_1 & \beta_2 & 0 & 0 & 0 \\
\beta_2 & \alpha_2 & \beta_3 & 0 & 0 \\
0 & \beta_3 & \alpha_3 & \ddots & 0 \\
0 & 0 & \ddots & \ddots & \beta_k \\
0 & 0 & 0 & \beta_k & \alpha_k
\end{bmatrix}. \]  
(2.75)

Now the Lanczos matrix can be used to form a transformation

\[ u = Vp \]  
(2.76)

from the original physical coordinates to a new set of generalized coordinates. The Lanczos transformation cannot be applied directly to the system equations. Instead, the undamped system equations need to be premultiplied by \( MK^{-1} \):

\[ MK^{-1}\ddot{u} + MK^{-1}Ku = MK^{-1}f. \]  
(2.77)

With this preparation, the transformation can then be applied to produce

\[ T\ddot{p} + Ip = V^T MK^{-1} f. \]  
(2.78)

As described in Section 2.3.1, making a Rayleigh damping assumption allows the transformation to be applied to the damping matrix as well. With the following assumption,

\[ C = a_0 M + a_1 K. \]  
(2.79)

the system equation can be converted into

\[ T\ddot{p} + (a_0 T + a_1 I)p + Ip = V^T MK^{-1} f. \]  
(2.80)
Note that these matrices are narrow-banded and that no eigenproblems are solved to arrive at this point. Numerical integration schemes that can take advantage of this tri-diagonal structure can be used to efficiently solve for the displacements without any further manipulation.

2.2.3.2 Multiple Component Load-Dependent Lanczos Algorithm

For the multiple component case, it is once again necessary to partition the excitation vector into its spatial distribution and an accompanying function of time:

\[
\begin{bmatrix}
M_{ii} & M_{ij} \\
M_{ij}^T & M_{jj}
\end{bmatrix}
\begin{bmatrix}
\dot{u}_i \\
\dot{u}_j
\end{bmatrix}
+ \begin{bmatrix}
K_{ii} & K_{ij} \\
K_{ij}^T & K_{jj}
\end{bmatrix}
\begin{bmatrix}
u_i \\
u_j
\end{bmatrix}
= \begin{bmatrix}
f_i \\
f_j_{E,s}
\end{bmatrix} \text{g(t)} + f_{j,L} + f_{j,NL} \tag{2.81}
\]

where the E.s subscript denotes that the force vector represents the spatial distribution of the externally applied forces. \(f_{j,L}\) is the linear force transmitted through the junction and \(f_{j,NL}\) is the non-linear force transmitted through the junction. The desired tri-diagonal structure of the solution when the Lanczos algorithm is applied can only be obtained in the multiple component case when the problem is converted to standard form. The standard form causes the algorithm to converge to the highest eigenmode instead of the lowest [17]. Thus the algorithm needs to be carried out using the inverse of the system matrices in order to force convergence to the lowest eigenmode, which is of greater importance in structural dynamic systems. An efficient method for this is to break the stiffness matrix into its Cholesky factors [29,17]

\[
K_{ii} = L_{ii}L_{ii}^T \tag{2.82}
\]

and then form the operator
\[ \overline{M} = L^{-1}_{ii} M_{ii} L^T_{ii}. \]  

(2.83)

The eigenvalues of \( \overline{M} \) are the inverses of the generalized eigenvalues of \( K_{ii} \) and \( M_{ii} \).

This ensures that the algorithm converges to the lowest modes. This decomposition also affects the starting vector:

\[ r_0 = L^{-1}_{ii} f_{E,E}. \]  

(2.84)

where \( f_{E,E} \) is the interior partition of the spatial distribution of the external force. The initialization of the Lanczos vector remains the same as before, with

\[ v_0 = 0. \]  

(2.85)

Then the following five equations are repeated for \( i = 1, 2, \ldots, k \):

\[ \beta_i = r_{i-1}^T r_{i-1}, \]  

(2.86)

\[ v_i = \frac{1}{\beta_i} r_{i-1}, \]  

(2.87)

\[ r_i = \overline{M} v_i, \]  

(2.88)

\[ \alpha_i = v_i^T r_i, \]  

(2.89)

and

\[ r_i = r_i - \alpha_i v_i - \beta_i r_{i-1}. \]  

(2.90)

The \( V \) and \( T \) matrices are formed from the algorithm as in the single component case with Equations (2.73) and (2.75).

The above algorithm ignores the boundary elements. A modified constraint matrix is developed to aid in the completeness of the set:

\[ \Psi^\prime_{ij} = -[L_{ii}]^T L^{-1}_{ii} K_{ij}. \]  

(2.91)
This constraint matrix and the Lanczos matrix are then assembled into the transformation matrix

\[
\mathbf{u} = \begin{bmatrix} \mathbf{L}_{ii}^{-1} & \mathbf{V} \\ \mathbf{0} & \mathbf{I}_{jj} \end{bmatrix} \mathbf{V} \mathbf{\Psi}_{ij} \mathbf{p},
\]  

(2.92)

which transforms the component physical coordinates to the generalized coordinates.

This transformation can be applied to the component equations to get

\[
\begin{bmatrix} \mathbf{T}_{kk} & \mathbf{M}_{ij}^* \\ \mathbf{M}_{ij}^* & \mathbf{K}_{ij} \end{bmatrix} \begin{bmatrix} \mathbf{\bar{p}}_k \\ \mathbf{\bar{p}}_j \end{bmatrix} + \begin{bmatrix} \mathbf{I}_{kk} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{ij} \end{bmatrix} \begin{bmatrix} \mathbf{p}_k \\ \mathbf{p}_j \end{bmatrix} = \begin{bmatrix} \mathbf{f}_k^* \\ \mathbf{f}_j^* \end{bmatrix}_{E,s} \mathbf{g}(t) + \mathbf{f}_{j,L} + \mathbf{f}_{j,NL}
\]  

(2.93)

where

\[
\mathbf{M}_{ij}^* = \mathbf{M}_{ij} + \mathbf{\Psi}_{ij}^T \mathbf{M}_{ii} \mathbf{\Psi}_{ij} + \mathbf{M}_{ij}^T \mathbf{\Psi}_{ij} + \mathbf{\Psi}_{ij}^T \mathbf{M}_{ij}.
\]  

(2.94)

\[
\mathbf{M}_{kj}^* = \mathbf{V}_k^T \mathbf{L}_{ii}^{-1} \left( \mathbf{M}_{ij} + \mathbf{M}_{ii} \mathbf{\Psi}_{ij} \right),
\]  

(2.95)

\[
\mathbf{K}_{ij}^* = \mathbf{K}_{ij} - \mathbf{K}_{ii}^T \mathbf{K}_{ii}^{-1} \mathbf{K}_{ij},
\]  

(2.96)

\[
\mathbf{f}_k^* = \mathbf{V}^T \mathbf{L}_{ii}^{-1} \mathbf{f}_{i,E,s},
\]  

(2.97)

and

\[
\mathbf{f}_j^* = \mathbf{f}_{j,E,s} + \mathbf{\Psi}_{ij}^T \mathbf{f}_{i,E,s}.
\]  

(2.98)

Note that except for the fully populated junction partitions, the interior partitions are tightly banded. This structure can be readily preserved when synthesizing the system level equations. Numerical algorithms that exploit this tri-diagonal form can be utilized to efficiently calculate the displacements without having to perform an eigensolution.
3 Decoupled Analysis

3.1 Background

Spanos et al [33] proposed an alternative method in which the components are never combined into a system-sized problem. Instead, the method uses the junction degrees of freedom from the component representations to derive junction-sized sets of equations. These equations are then solved for the junction motions, which are then applied to the components separately. This separate integration of the component motions leads to the description of this method as decoupled analysis. Spanos et al [34] then refined the decoupled analysis to allow for local non-linearities between components. This greatly enhanced the utility of the method and spurred its use in aerospace applications. Decoupled analysis will be covered further in Section 3.2.

Spanos and Majed [36] improved the method through application of residual flexibility concepts to the decoupled analysis problem. The use of residual flexibility in decoupled analysis will be covered in Section 3.2.1.

An alternate formulation that applies the Lanczos algorithm to decoupled analysis has been developed. This formulation avoids the need to solve eigenproblems and results in a tri-diagonal form for the equations. Application of the Lanczos algorithm to decoupled analysis is described in a paper by Spanos and Lavelle [35] where the benefits of the tri-diagonal form are surmised. In Section 3.2.2, the development of the method will be shown and the methods of numerical integration that exploit the tri-diagonal form of the equations will be developed.
Another alternate formulation for decoupled analysis has been developed that reformulates the method to solve for the resulting motion explicitly instead of using a numerical integrator. An explicit expression has the advantage that the time step used may be larger than is possible when using numerical integration. Any possible increase in time step size is problem-dependent. This reformulation is detailed in Section 3.2.3.

3.2 Methodology

Decoupled analysis uses the same first coordinate transformation as the CMS method. Equations (2.15) and (2.16) can be expanded to show that

\[ \tilde{M}^a = \begin{bmatrix} \Psi^a \end{bmatrix}^T M^a \Psi^a = \begin{bmatrix} 1 & \tilde{M}^a_j \\ \tilde{M}^a_i & \tilde{M}^a \\ - \tilde{M}^a_i & \tilde{M}^a \\ \end{bmatrix} \]  

(3.1)

and

\[ \tilde{K}^a = [\Psi^a]^T K^a \Psi^a = \begin{bmatrix} \Lambda^a_k & 0 \\ 0 & \tilde{K}^a_j \end{bmatrix}. \]  

(3.2)

Equation (2.14) will be repeated here for clarity:

\[ \Psi^a = \begin{bmatrix} \Phi^a_k & \Phi^a_c \\ 0 & 1 \end{bmatrix}. \]  

(3.3)

Using the above expressions, the component Equation (2.2) from the CMS method can be rewritten to show the interior composition of the matrices:

\[ \begin{bmatrix} 1 & \tilde{M}^a_j \\ \tilde{M}^a_i & \tilde{M}^a \\ \end{bmatrix} \begin{bmatrix} \dot{q}^a_i \\ \dot{q}^a_j \end{bmatrix} + \begin{bmatrix} \Lambda^a_k & 0 \\ 0 & \tilde{K}^a_j \end{bmatrix} \begin{bmatrix} \dot{x}^a_i \\ \dot{x}^a_j \end{bmatrix} = \begin{bmatrix} \Phi^a_k & 0 \\ \Phi^a_c & 1 \end{bmatrix} f^a. \]  

(3.4)

This equation can then be partitioned. The top partition.
\[ \ddot{q}_i^\alpha + \Lambda_i^\alpha q_i^\alpha = \begin{bmatrix} \Phi_k^\alpha \\ \mathbf{0} \end{bmatrix}^T \mathbf{f}^\alpha - \tilde{M}_i^\alpha \ddot{x}_j^\alpha. \] (3.5)

describes the behavior of the component's interior. Damping can be included by making the assumption of component modal damping:

\[ \tilde{C}_i^\alpha = \begin{bmatrix} 2\zeta_1 \omega_1 & 0 & 0 & 0 \\ 0 & 2\zeta_2 \omega_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 2\zeta_j \omega_j \end{bmatrix}. \] (3.6)

Then Equation (3.5) can be rewritten as

\[ \ddot{q}_i^\alpha + \tilde{C}_i^\alpha \dot{q}_i^\alpha + \Lambda_i^\alpha q_i^\alpha = \begin{bmatrix} \Phi_k^\alpha \\ \mathbf{0} \end{bmatrix}^T \mathbf{f}^\alpha - \tilde{M}_i^\alpha \ddot{x}_j^\alpha. \] (3.7)

The bottom partition of Equation (3.4) produces the component's junction equation:

\[ \tilde{M}_i^\alpha \ddot{x}_i^\alpha + \tilde{K}_i^\alpha x_i^\alpha = \begin{bmatrix} \Psi_i^\alpha \\ \mathbf{I} \end{bmatrix}^T \mathbf{f}^\alpha - \tilde{M}_i^\alpha \ddot{q}_i^\alpha. \] (3.8)

A junction equation can also be derived for the component that shares this junction:

\[ \tilde{M}_j^\beta \ddot{x}_j^\beta + \tilde{K}_j^\beta x_j^\beta = \begin{bmatrix} \Psi_j^\beta \\ \mathbf{I} \end{bmatrix}^T \mathbf{f}^\beta - \tilde{M}_j^\beta \ddot{q}_j^\beta. \] (3.9)

These two junction equations are then combined. Some simplifications in the combined equation can be made by first expanding the excitation force into the following:

\[ \mathbf{f}^\alpha = \mathbf{f}_E^\alpha + \mathbf{f}_{iL}^\alpha + \mathbf{f}_{iNL}^\alpha. \] (3.10)

where \( \mathbf{f}_E^\alpha \) is the force applied externally to the component, \( \mathbf{f}_{iL}^\alpha \) is the linear force transmitted through the junction, and \( \mathbf{f}_{iNL}^\alpha \) is the non-linear force transmitted through the
juncture. Realizing that the forces transmitted through the junction must be equal and opposite, the following equalities are clear:

\[ f^\alpha_{j,L} = -f^\beta_{j,L} \]  \hspace{1cm} (3.11)

\[ f^\alpha_{j,NL} = -f^\beta_{j,NL}. \]  \hspace{1cm} (3.12)

Since the junction degrees of freedom are shared between the two adjacent components,

\[ x^\alpha_j = x^\beta_j = x_j. \]  \hspace{1cm} (3.13)

With these equalities, the two junction equations can be combined and simplified:

\[
\begin{aligned}
\left( \ddot{\mathbf{x}}^\alpha_j + \ddot{\mathbf{x}}^\beta_j \right) &= \left( \begin{bmatrix} \Psi^\alpha_c \\mathbf{I} \end{bmatrix}^T \left( f^\alpha_E + f^\alpha_{j,NL} \right) - \ddot{\mathbf{x}}^\alpha_j \ddot{\mathbf{q}}^\alpha_i \right. \\
&\left. + \begin{bmatrix} \Psi^\beta_c \\mathbf{I} \end{bmatrix}^T \left( f^\beta_E + f^\beta_{j,NL} \right) - \ddot{\mathbf{x}}^\beta_j \ddot{\mathbf{q}}^\beta_i \right)
\end{aligned}
\]  \hspace{1cm} (3.14)

The eigenproblem associated with this equation uses \((\ddot{\mathbf{x}}^\alpha_j + \ddot{\mathbf{x}}^\beta_j)\) and \((\dddot{\mathbf{x}}^\alpha_j + \dddot{\mathbf{x}}^\beta_j)\) to form the junction natural frequencies, \(\Lambda_{ij}\), and modal matrix, \(\Phi_{ij}\). The transformation is then

\[ x_j = \Phi_{ij} q_j. \]  \hspace{1cm} (3.15)

Applying the transformation to the combined junction equation and adding damping produces

\[
\ddot{q}_j + \overline{C}_{ij} \ddot{q}_j + \Lambda_{ij} q_j =
\Phi_{ij}^T \begin{bmatrix} \Psi^\alpha_c \\mathbf{I} \end{bmatrix}^T \left( f^\alpha_E + f^\alpha_{j,NL} \right) - \ddot{\mathbf{x}}^\alpha_j \ddot{\mathbf{q}}^\alpha_i \\
\left. + \begin{bmatrix} \Psi^\beta_c \\mathbf{I} \end{bmatrix}^T \left( f^\beta_E + f^\beta_{j,NL} \right) - \ddot{\mathbf{x}}^\beta_j \ddot{\mathbf{q}}^\beta_i \right)
\]  \hspace{1cm} (3.16)

where \(\overline{C}_{ij}\) is formulated similarly to Equation (3.6) using the junction natural frequencies and assumed damping rates.
Numerical integration of this junction equation can be done with the Newmark algorithm [34]. In the case where there are non-linear elements in the junctions, the Newmark algorithm produces equations that must be iterated to reach convergence at each time step.

3.2.1 Residual Flexibility

In Section 2.2.2, the residual modes were based on a first-order approximation that did not include inertial or damping effects. The application of residual modes to decoupled analysis requires the use of a second-order approximation. Rewriting Equation (2.46) with the excitation vector \( f \) replacing the specialized vector \( F_i \) gives

\[
\mathbf{u}^{(1)}_d = \mathbf{G}_d \mathbf{f}
\]

(3.17)

where the superscript denotes the first order approximation [19]. The second order approximation [31], which accounts for the damping and inertial effects, is given by

\[
\mathbf{u}^{(2)}_d = \left( \mathbf{G}_d - \mathbf{B}_d p - \mathbf{H}_d p^2 \right) \mathbf{f}
\]

(3.18)

where \( p \) represents the differential operator. The damping matrix, \( \mathbf{B}_d \), and the inertia matrix, \( \mathbf{H}_d \), are defined by the equations

\[
\mathbf{B}_d = \mathbf{G}_d \mathbf{C}_d \mathbf{G}_d
\]

(3.19)

and

\[
\mathbf{H}_d = \mathbf{G}_d \mathbf{M}_d \mathbf{G}_d .
\]

(3.20)

The same partitioning of the equations of motion that took place in the previous section is repeated in forming the residual flexibility representation in this section. Starting with the component equations of motion,
\[ M^a \ddot{u}^a + C^a \dot{u}^a + K^a u^a = f^a. \]  

(3.21)

the component eigenproblem is solved assuming free interfaces. Unlike the previous section where the use of constraint modes prevented this step from fully diagonalizing the equations of motion, the free interface assumption leads to a fully decoupled version. The transformation can be partitioned to show the interior and junction degrees of freedom

\[
\begin{bmatrix}
  u^a_i \\
  u^a_j
\end{bmatrix} = \begin{bmatrix}
  \Phi^a_{ik} \\
  \Phi^a_{jk}
\end{bmatrix} q^a_k. \tag{3.22}
\]

The bottom partition is then

\[ u^a_j = \Phi^a_{jk} q^a_k. \tag{3.23} \]

The second order residual flexibility approximation for the displacement due to the deleted modes in equation (3.18) can be added to the above equation for the junction displacement:

\[
 u^a_j = \Phi^a_{jk} q_k + (\mathbf{G}^a_{ji,r} - \mathbf{B}^a_{ji,r} p - \mathbf{H}^a_{ji,r} p^2) f^a_E + (\mathbf{G}^a_{ji,r} - \mathbf{B}^a_{ji,r} p - \mathbf{H}^a_{ji,r} p^2) f^a_j. \tag{3.24}
\]

In the above equation, the excitation vector is partitioned into the externally applied forces, \( f^a_E \), and the forces transmitted through the junction, \( f^a_j \). Recalling that the junction displacements for adjacent components are equal and opposite, the junction displacements for the \( \beta \) component can be expressed as

\[
 u^\beta_j = \Phi^\beta_{jk} q_k + (\mathbf{G}^\beta_{ji,r} - \mathbf{B}^\beta_{ji,r} p - \mathbf{H}^\beta_{ji,r} p^2) f^\beta_E - (\mathbf{G}^\beta_{ji,r} - \mathbf{B}^\beta_{ji,r} p - \mathbf{H}^\beta_{ji,r} p^2) f^\beta_j. \tag{3.25}
\]

Equations (3.24) and (3.25) can be joined using the displacement compatibility

\[ u^a_i = u^\beta_j \tag{3.26} \]

to give
\[(G_{ij}^{\alpha,\beta} - B_{ij}^{\alpha,\beta} p - H_{ij}^{\alpha,\beta} p^2) f_j = \Phi_{jk}^{\beta} q_k^{\alpha} - \Phi_{jk}^{\alpha} q_k^{\beta} + (G_{jE,r}^{\beta} - B_{jE,r}^{\beta} p - H_{jE,r}^{\beta} p^2) f_E^a - (G_{jE,r}^{\alpha} - B_{jE,r}^{\alpha} p - H_{jE,r}^{\alpha} p^2) f_E^a. \tag{3.27}\]

Equation (3.27) represents a junction-sized set of differential equations. Studies in Component-Mode Synthesis using Rubin's residual flexibility representation of a component suggest that the contribution of the second order effects to the residual displacements may be approximated by the first order terms with an acceptable level of accuracy [36]. Using this assumption reduces Equation (3.27) to

\[G_{ij}^{\alpha,\beta} f_j = \Phi_{jk}^{\beta} q_k^{\alpha} - \Phi_{jk}^{\alpha} q_k^{\beta} + G_{jE,r}^{\beta} f_E^a - G_{jE,r}^{\alpha} f_E^a. \tag{3.28}\]

As in the previous section, solution of the problem commences with application of the Newmark algorithm [23]. The details of application are available in the literature [36].

### 3.2.2 Lanczos Vectors

The representation of components using Lanczos vectors often leads to dimensionally more compact representations. The tri-diagonal form that these representations have requires careful attention to preserve this structure when the decoupled analytical method is formulated.

Equation (2.93) can be rewritten as

\[
\begin{bmatrix}
T_{kk} & M_{kj}^* \\
M_{kj} & K_{jj}
\end{bmatrix}
\begin{bmatrix}
\dot{p}_k \\
\dot{p}_j
\end{bmatrix}
+ \begin{bmatrix}
I_{kk} & 0 \\
0 & K_{jj}
\end{bmatrix}
\begin{bmatrix}
p_k \\
p_j
\end{bmatrix}
= \begin{bmatrix}
V^T L_i^{-1} & 0 \\
\Psi_{ij} & I_{jj}
\end{bmatrix}
\begin{bmatrix}
f_i \\
f_j
\end{bmatrix}
+ \begin{bmatrix}
0 \\
f_{j,L} + f_{j,NL}
\end{bmatrix} \tag{3.29}
\]

where the excitation vector is reassembled from its spatial and temporal components. As in the previous developments of decoupled analysis, Equation (3.29) can be partitioned. The top partition,
The Lanczos algorithm is then applied to this junction equation. To facilitate the use of the algorithm, a mass matrix and stiffness matrix needs to be formed:

$$\tilde{M}_\mu = \left( M_{\mu}^{*\alpha} + M_{\mu}^{*\beta} \right)$$

(3.35)

and

$$\tilde{K}_\mu = \left( K_{\mu}^{*\alpha} + K_{\mu}^{*\beta} \right).$$

(3.36)
The seed vector is formulated from the spatial distributions of the external loads, transformed by the constraint matrices:

$$
\mathbf{r}_0 = \mathbf{K}^{-1}_y \left( \begin{bmatrix} \Psi^a_{ij} \\ \mathbf{f}^a_{E,s} \\ \mathbf{F}^a_{E,s} \\ \mathbf{M}_y \end{bmatrix} \right) + \left( \begin{bmatrix} \Psi^b_{ij} \\ \mathbf{f}^b_{E,s} \\ \mathbf{F}^b_{E,s} \\ \mathbf{M}_y \end{bmatrix} \right),
$$

(3.37)

The Lanczos algorithm can then be applied to produce the transformation

$$
\mathbf{p}_j = \mathbf{W}_y \mathbf{q}_j,
$$

(3.38)

where \( \mathbf{W}_y \) is the Lanczos matrix produced by the mass matrix, stiffness matrix, and seed vector shown above. The \( \mathbf{T}_y \) matrix is assembled from the Lanczos algorithm intermediate values, as in Equation (2.75). The junction equation is pre-mounted by \( \mathbf{M}_y \mathbf{K}^{-1}_y \) and then the transformation from Equation (3.38) is applied. The result is

$$
\mathbf{T}_y \ddot{\mathbf{q}}_j + (b_y \mathbf{T}_y + b_y \mathbf{F}_y) \dot{\mathbf{q}}_j + \mathbf{F}_y = \mathbf{W}_y^T \mathbf{M}_y \mathbf{K}^{-1}_y
$$

$$
\left[ \begin{bmatrix} \Psi^a_{ij} \\ \mathbf{f}^a_{E,s} + \mathbf{f}^a_{j,NL} - \mathbf{M}_y^a \mathbf{p}^a_k \\ \mathbf{f}^b_{E,s} + \mathbf{f}^b_{j,NL} - \mathbf{M}_y^b \mathbf{p}^b_k \end{bmatrix} \right],
$$

(3.39)

which displays all of the desired tri-diagonal properties.

Numerical integration methods can then be used with this equation to solve for the junction and component displacements. The first step in most numerical methods is to break time up into discrete units:

$$
t_n = nh; \quad n = 0, 1, 2, \ldots,
$$

(3.40)

where \( h \) is the spacing between consecutive time points. A common numerical integration method that will be employed here is the Newmark algorithm [23]. At time \( t_{n+1} \) the junction generalized displacements and velocities are approximated as

$$
\mathbf{q}_{j,n+1} = \mathbf{q}_{j,n} + h \ddot{\mathbf{q}}_{j,n} + \alpha_2 \dot{\mathbf{q}}_{j,n} + \alpha_0 \mathbf{q}_{j,n-1}
$$

(3.41)
and
\[ \ddot{q}_{j,n+1} = \dot{q}_{j,n} + \alpha_2 \ddot{q}_{j,n} + \alpha_1 \dddot{q}_{j,n+1} \]  
(3.42)

where
\[ \alpha_0 = \alpha h^2. \]  
(3.43)
\[ \alpha_1 = \beta h. \]  
(3.44)
\[ \alpha_2 = (1 - \beta) h. \]  
(3.45)

and
\[ \alpha_3 = \left( \frac{1}{2} - \alpha \right) h^2. \]  
(3.46)

The \( \alpha \) and \( \beta \) terms are the Newmark quadrature parameters.

The Lanczos junction Equation (3.39) at time \( t_{n+1} \) is

\[
T_{ji} \ddot{q}_{j,n+1} + (b_0 T_{ji} + b_1 I_{ji}) \dot{q}_{j,n+1} + I_{ji} q_{j,n+1} = W_{ji}^T \tilde{M}_{ji} \tilde{K}_{ji}^{-1} \begin{bmatrix} \Psi_{ij}^\alpha \ T_{i,n+1}^a + f_{i,n+1}^a - \left[ M_{kj}^* \right]^T \tilde{p}_{k,n+1}^a + \begin{bmatrix} \Psi_{ij}^\beta \\ I_{ij} \end{bmatrix}^T f_{E,n+1}^\beta + f_{j,n+1}^\beta - \left[ M_{kj}^* \right]^T \tilde{p}_{k,n+1}^\beta \end{bmatrix}. 
\]
(3.47)

Substituting Equations (3.41) and (3.42) into (3.39) gives

\[
T_{ji} \ddot{q}_{j,n+1} + (b_0 T_{ji} + b_1 I_{ji}) \left( \dot{q}_{j,n} + 2 \ddot{q}_{j,n} + 3 \dddot{q}_{j,n+1} \right) \\
+ I_{ji} \left( q_{j,n} + h \dot{q}_{j,n} + \alpha_2 \ddot{q}_{j,n} + \alpha_1 \dddot{q}_{j,n+1} \right) = W_{ji}^T \tilde{M}_{ji} \tilde{K}_{ji}^{-1} \begin{bmatrix} \Psi_{ij}^\alpha \ T_{i,n+1}^a + f_{i,n+1}^a - \left[ M_{kj}^* \right]^T \tilde{p}_{k,n+1}^a + \begin{bmatrix} \Psi_{ij}^\beta \\ I_{ij} \end{bmatrix}^T f_{E,n+1}^\beta + f_{j,n+1}^\beta - \left[ M_{kj}^* \right]^T \tilde{p}_{k,n+1}^\beta \end{bmatrix}. 
\]
(3.48)

This equation can be simplified to
\[ \bar{M}_{\alpha} \bar{q}_{j,n+1} = W_{\alpha}^{T} \bar{M}_{\alpha} \bar{K}_{\alpha}^{l} + \left( \left[ \Psi_{ij}^{\alpha} \right]^{T} f_{E,n+1}^{\alpha} + f_{j,NL,n+1}^{\alpha} \bar{p}_{k,n+1}^{\alpha} \right) + \left( \left[ \Psi_{ij}^{\alpha} \right]^{T} f_{E,n+1}^{\beta} + f_{j,NL,n+1}^{\beta} \right) \bar{p}_{k,n+1}^{\beta} \] 

(3.49)

where

\[ \bar{M}_{\alpha} = T_{ij} + (b_{0} T_{ij} + b_{1} I_{ij}) \alpha_{i} + I_{ij} \alpha_{0} \]  

(3.50)

and

\[ f_{v,j,n} = -(b_{0} T_{ij} + b_{1} I_{ij}) \left( \bar{q}_{j,n} + \alpha_{i} \bar{q}_{j,n} \right) - \left( \bar{q}_{j,n} + h \bar{q}_{j,n} + \alpha_{i} \bar{q}_{j,n} \right). \]  

(3.51)

This same process can be repeated for the component interior equations. At time \( t_{n+1} \), Equation (3.30) can be expressed as

\[ T_{\alpha}^{\alpha} \dot{p}_{k,n+1}^{\alpha} + (a_{0} T_{\alpha,k}^{\alpha} + a_{1} I_{\alpha,k}) \dot{p}_{k,n+1}^{\alpha} + p_{k,n+1}^{\alpha} = \left[ V_{\alpha}^{\alpha} \right]^{T} \left[ L_{\alpha,i}^{\alpha} \right]^{-1} f_{i,E,n+1}^{\alpha} - \left[ M_{\alpha}^{\alpha} \right]^{T} \ddot{p}_{k,n+1}^{\alpha}. \]  

(3.52)

The component interior displacements and velocities at \( t_{n+1} \) are

\[ p_{k,n+1}^{\alpha} = p_{k,n}^{\alpha} + h \dot{p}_{k,n}^{\alpha} + \alpha_{i} \ddot{p}_{k,n}^{\alpha} + \alpha_{0} \dddot{p}_{k,n+1}^{\alpha} \]  

(3.53)

and

\[ \dot{p}_{k,n+1}^{\alpha} = \dot{p}_{k,n}^{\alpha} + \alpha_{i} \ddot{p}_{k,n}^{\alpha} + \alpha_{0} \dddot{p}_{k,n+1}^{\alpha}. \]  

(3.54)

Substituting (3.53) and (3.54) into (3.52) gives

\[ T_{\alpha}^{\alpha} \dot{p}_{k,n+1}^{\alpha} + (a_{0} T_{\alpha,k}^{\alpha} + a_{1} I_{\alpha,k}) \left( \dot{p}_{k,n}^{\alpha} + \alpha_{i} \ddot{p}_{k,n}^{\alpha} + \alpha_{0} \dddot{p}_{k,n+1}^{\alpha} \right) + p_{k,n+1}^{\alpha} + h \dot{p}_{k,n}^{\alpha} + \alpha_{i} \ddot{p}_{k,n}^{\alpha} + \alpha_{0} \dddot{p}_{k,n+1}^{\alpha} \]  

\[ = \left[ V_{\alpha}^{\alpha} \right]^{T} \left[ L_{\alpha,i}^{\alpha} \right]^{-1} f_{i,E,n+1}^{\alpha} - \left[ M_{\alpha}^{\alpha} \right] \dddot{p}_{k,n+1}^{\alpha}. \]  

(3.55)

Rearranging and simplifying gives
\[ \overline{M}_{kk}^a \ddot{p}_{k,n-1} = f_{v,n}^a + \left[ V^a \right]^T \left[ L_n^a \right]^{-1} f_{i,E,n-1}^a - M_{kj}^a \ddot{p}_{j,n+1} \]  
(3.56)

where

\[ \overline{M}_{kk}^a = T_{kk}^a + (a_0^a T_{kk}^a + a_1^a I_{kk}) \alpha_1 + I_{kk} \alpha_0 \]  
(3.57)

and

\[ f_{v,n}^a = -(a_0^a T_{kk}^a + a_1^a I_{kk}) (p_{k,n}^a + \alpha_2 \dot{p}_{k,n}^a) - (p_{k,n}^a + h \dot{p}_{k,n}^a + \alpha_3 \ddot{p}_{k,n}^a). \]  
(3.58)

Note that \( \overline{M}_{kk}^a \) is tri-diagonal. This is important because the inverse of this matrix is needed in order to solve (3.56) for the interior accelerations. Inverting a tri-diagonal equation is much simpler than inverting a fully populated matrix (approximately 2n operations versus \( n^3/6 \) operations.) Solving Equation (3.56) for the interior accelerations gives

\[ \ddot{p}_{k,n+1} = \left( \overline{M}_{kk}^a \right)^{-1} \left\{ f_{v,n}^a + \left[ V^a \right]^T \left[ L_n^a \right]^{-1} f_{i,E,n-1}^a - M_{kj}^a \ddot{p}_{j,n+1} \right\}. \]  
(3.59)

This equation can now be substituted into Equation (3.49):

\[ \overline{M}_{ji} \ddot{q}_{j,n+1} = f_{v,j,n} + W_{ji}^T \overline{M}_{ji} \ddot{q}_{j,n} \]

\[ \begin{cases} 
\left[ \Psi_{ji}^a \right]^T f_{E,n+1}^a + f_{j,NL,n+1}^a - \left[ M_{kj}^a \right]^T \left( \overline{M}_{kk}^a \right)^{-1} \left\{ f_{v,n}^a + \left[ V^a \right]^T \left[ L_n^a \right]^{-1} f_{i,E,n-1}^a - M_{kj}^a \ddot{p}_{j,n+1} \right\} 
\end{cases} \]

\[ + \begin{cases} 
\left[ \Psi_{ji}^b \right]^T f_{E,n+1}^b + f_{j,NL,n+1}^b - \left[ M_{kj}^b \right]^T \left( \overline{M}_{kk}^b \right)^{-1} \left\{ f_{v,n}^b + \left[ V^b \right]^T \left[ L_n^b \right]^{-1} f_{i,E,n-1}^b - M_{kj}^b \ddot{p}_{j,n+1} \right\} 
\end{cases} \]  
(3.60)

The next task is to get \( \ddot{p}_{j,n+1} \) to the left hand side of the equation:
\[
\overline{\mathbf{M}}_j \ddot{q}_{j,n+1} - \mathbf{W}^T_j \overline{\mathbf{M}}_j \overline{\mathbf{K}}^{-1}_j \left( \left[ \mathbf{M}^{\alpha}_{kj} \right]^T \left[ \overline{\mathbf{M}}^{\alpha}_{kk} \right]^{-1} \mathbf{M}^{\alpha}_{kj} \right) \dot{p}_{j,n+1}^{\alpha} + \mathbf{W}^T_j \overline{\mathbf{M}}_j \overline{\mathbf{K}}^{-1}_j \left( \left[ \mathbf{M}^{\alpha}_{kj} \right]^T \left[ \overline{\mathbf{M}}^{\beta}_{kk} \right]^{-1} \mathbf{M}^{\beta}_{kj} \right) \dot{p}_{j,n+1}^{\beta} = \mathbf{f}_{v,j,n} + \mathbf{f}_{E,j,n+1}^{\alpha} + \mathbf{f}_{j,NL,n+1}^{\alpha} - \left[ \mathbf{M}^{\alpha}_{kj} \right]^T \left[ \overline{\mathbf{M}}^{\alpha}_{kk} \right]^{-1} \left\{ \mathbf{f}_{v,n}^{\alpha} + \left[ \mathbf{V}^{\alpha} \right]^T \left[ \mathbf{L}^{\alpha}_{nn} \right]^{-1} \mathbf{f}_{i,E,n+1}^{\alpha} \right\} + \mathbf{W}^T_j \overline{\mathbf{M}}_j \overline{\mathbf{K}}^{-1}_j \left[ \mathbf{M}^{\beta}_{kj} \right]^T \left[ \overline{\mathbf{M}}^{\beta}_{kk} \right]^{-1} \left\{ \mathbf{f}_{v,n}^{\beta} + \left[ \mathbf{V}^{\beta} \right]^T \left[ \mathbf{L}^{\beta}_{nn} \right]^{-1} \mathbf{f}_{i,E,n+1}^{\beta} \right\}.
\]

(3.61)

Now the transformation from the component junction coordinates to the generalized junction coordinates

\[
\ddot{p}_{j,n+1}^{\beta} = \mathbf{W}^T_j \ddot{q}_{j,n+1}^{\beta}
\]

(3.62)

can be used. A further simplification can be realized by noting that

\[
\ddot{q}_{j,n+1}^{\alpha} = \ddot{q}_{j,n+1}^{\beta} = \ddot{q}_{j,n+1}.
\]

(3.63)

Substituting these two equations into (3.61) and rearranging produces

\[
\left\{ \overline{\mathbf{M}}_j - \mathbf{W}^T_j \overline{\mathbf{M}}_j \overline{\mathbf{K}}^{-1}_j \left( \left[ \mathbf{M}^{\alpha}_{kj} \right]^T \left[ \overline{\mathbf{M}}^{\alpha}_{kk} \right]^{-1} \mathbf{M}^{\alpha}_{kj} + \left[ \mathbf{M}^{\beta}_{kj} \right]^T \left[ \overline{\mathbf{M}}^{\beta}_{kk} \right]^{-1} \mathbf{M}^{\beta}_{kj} \right) \mathbf{W}^T_j \right\} \ddot{q}_{j,n+1} = \mathbf{f}_{v,j,n} + \mathbf{f}_{E,j,n+1}^{\alpha} + \mathbf{f}_{j,NL,n+1}^{\alpha} - \left[ \mathbf{M}^{\alpha}_{kj} \right]^T \left[ \overline{\mathbf{M}}^{\alpha}_{kk} \right]^{-1} \left\{ \mathbf{f}_{v,n}^{\alpha} + \left[ \mathbf{V}^{\alpha} \right]^T \left[ \mathbf{L}^{\alpha}_{nn} \right]^{-1} \mathbf{f}_{i,E,n+1}^{\alpha} \right\} + \mathbf{W}^T_j \overline{\mathbf{M}}_j \overline{\mathbf{K}}^{-1}_j \left[ \mathbf{M}^{\beta}_{kj} \right]^T \left[ \overline{\mathbf{M}}^{\beta}_{kk} \right]^{-1} \left\{ \mathbf{f}_{v,n}^{\beta} + \left[ \mathbf{V}^{\beta} \right]^T \left[ \mathbf{L}^{\beta}_{nn} \right]^{-1} \mathbf{f}_{i,E,n+1}^{\beta} \right\}.
\]

(3.64)

Solving for the junction accelerations gives

\[
\ddot{q}_{j,n+1} = \overline{\mathbf{M}}^{-1}_j \left\{ \mathbf{f}_{v,j,n} + \mathbf{W}^T_j \overline{\mathbf{M}}_j \overline{\mathbf{K}}^{-1}_j \left[ \mathbf{M}^{\alpha}_{kj} \right]^T \left[ \overline{\mathbf{M}}^{\alpha}_{kk} \right]^{-1} \left\{ \mathbf{f}_{v,n}^{\alpha} + \left[ \mathbf{V}^{\alpha} \right]^T \left[ \mathbf{L}^{\alpha}_{nn} \right]^{-1} \mathbf{f}_{i,E,n+1}^{\alpha} \right\} \right. + \mathbf{W}^T_j \overline{\mathbf{M}}_j \overline{\mathbf{K}}^{-1}_j \left[ \mathbf{M}^{\beta}_{kj} \right]^T \left[ \overline{\mathbf{M}}^{\beta}_{kk} \right]^{-1} \left\{ \mathbf{f}_{v,n}^{\beta} + \left[ \mathbf{V}^{\beta} \right]^T \left[ \mathbf{L}^{\beta}_{nn} \right]^{-1} \mathbf{f}_{i,E,n+1}^{\beta} \right\} \right\}.
\]

38
where

$$\overline{M}_j = \left\{ \overline{M}_j - W_j^T \overline{M}_j \overline{K}_j^{-1} \left[ \left[ M_{k_i}^a \right]^T \left[ \overline{M}_{k_k}^a \right]^{-1} M_{k_i}^a + \left[ M_{k_i}^b \right]^T \left[ \overline{M}_{k_k}^b \right]^{-1} M_{k_i}^b \right] W_j \right\} \right\} (3.66)$$

Equation (3.65) can be used to solve for the junction accelerations in the next time step. The junction velocities and displacements can be found once the accelerations are known with Equations (3.41) and (3.42). The interior accelerations can also be found after the junction is solved for the next time step:

$$\ddot{p}_{k,n+1}^a = \left[ \overline{M}_{kk}^a \right]^{-1} \left\{ f_{v,n}^a + \left[ V_{ji}^a \right]^T \left[ L_{ij}^a \right]^{-1} f_{i,E,n+1}^a - M_{kj}^a W_j \ddot{q}_{j,n+1} \right\}$$

(3.67)

The interior velocities and displacements are then found with Equations (3.53) and (3.54).

All of the quantities on the right hand side of Equation (3.65) are known explicitly for each time step when the system is completely linear. Systems with non-linearities, $f_{j,NL,n+1}^a \neq 0$, need to be solved iteratively. Assuming that the nonlinear force is a function of position and velocity, the estimate

$$f_{NL,n+1}^a = f_{NL} (x_n + h\dot{x}_n, \dot{x}_n + h\ddot{x}_n)$$

(3.68)

can be used in the right hand side of Equation (3.65). Calculations are then carried out and checked against the non-linear force estimate. When two iterations change the non-linear estimate by less than a set amount, convergence to the solution can be assumed.

### 3.2.3 Explicit Solution

An alternative to numerical integration with the Newmark algorithm or to formulation with Lanczos vectors is an exact solution that uses piecewise-linear interpolation of the excitation. In some cases such exact solutions may allow for an
increase in the size of the time step that can be used in the step-by-step calculation of the response. In numerical integration, the upper bound on the size of the time step is set by the tendency of the integration algorithm to diverge as the time step size grows. In an exact solution for a linear problem, the size of the time step is limited only by the user’s desire to sample the data frequently enough so as not to miss the peak values. In problems that include a local non-linearity, the size of the time step is also limited by the need to update the non-linear force often enough to ensure adequate agreement between the actual non-linear function and the piecewise-linear approximation.

Derivation of an exact solution for the decoupled method using the Craig-Bampton component representation begins with Equation (3.7). In place of the Newmark Equations (3.41) and (3.42), the displacement and velocity at the next time step for component \( \alpha \) can be expressed as

\[
q_{i,n+1}^{\alpha} = A^{\alpha} p_{i,n}^{\alpha} + B^{\alpha} p_{i,n+1}^{\alpha} + C^{\alpha} q_{i,n}^{\alpha} + D^{\alpha} q_{i,n}^{\alpha}
\]

(3.69)

and

\[
\dot{q}_{i,n+1}^{\alpha} = A''^{\alpha} p_{i,n}^{\alpha} + B''^{\alpha} p_{i,n+1}^{\alpha} + C''^{\alpha} q_{i,n}^{\alpha} + D''^{\alpha} q_{i,n}^{\alpha}.
\]

(3.70)

The terms for the diagonal matrices \( A \) through \( D' \) are computed with the following equations:

\[
a_{i} = \frac{1}{k_i \omega_{i,d} h} \left\{ e^{-\beta_i h} \left( \left( \frac{\omega_{i,d}^2 - \beta_i^2}{\omega_i^2} - \beta_i \right) \sin \omega_{i,d} h - \left( \frac{2\omega_{i,d} \beta_i}{\omega_i^2} \right) \cos \omega_{i,d} h \right) + \frac{2\omega_{i,d} \beta_i}{\omega_i^2} \right\}
\]

(3.71)

\[
b_{i} = \frac{1}{k_i \omega_{i,d} h} \left\{ e^{-\beta_i h} \left( -\left( \frac{\omega_{i,d}^2 - \beta_i^2}{\omega_i^2} \right) \sin \omega_{i,d} h - \left( \frac{2\omega_{i,d} \beta_i}{\omega_i^2} \right) \cos \omega_{i,d} h \right) + \omega_{i,d} h + \frac{2\omega_{i,d} \beta_i}{\omega_i^2} \right\}
\]

(3.72)
\[ c_{ii} = e^{-\theta_{i,h}} \left[ \cos \omega_{i,d} h + \left( \frac{\beta_i}{\omega_{i,d}} \right) \sin \omega_{i,d} h \right] \] (3.73)

\[ d_{ii} = \left( \frac{1}{\omega_{i,d}} \right) e^{-\theta_{i,h}} \sin \omega_{i,d} h \] (3.74)

\[ a'_{ii} = \frac{1}{k_\eta \omega_{i,d} h} \left( e^{-\theta_{i,h}} \left[ (\beta_i + \omega_i^2 h) \sin \omega_{i,d} h + \omega_{i,d} \cos \omega_{i,d} h \right] - \omega_{i,d} \right) \] (3.75)

\[ b'_{ii} = \frac{1}{k_\eta \omega_{i,d} h} \left[ -e^{-\theta_{i,h}} (\beta_i \sin \omega_{i,d} h + \omega_{i,d} \cos \omega_{i,d} h) + \omega_{i,d} \right] \] (3.76)

\[ c'_{ii} = -\left( \frac{\omega_i^2}{\omega_{i,d}} \right) e^{-\theta_{i,h}} \sin \omega_{i,d} h \] (3.77)

\[ d'_{ii} = e^{-\theta_{i,h}} \left[ \cos \omega_{i,d} h - \frac{\beta_i}{\omega_{i,d}} \sin \omega_{i,d} h \right] \] (3.78)

where the simplifying variables.

\[ \beta_i = \zeta_i \omega_i \] (3.79)

and

\[ \omega_{i,d} = \omega_i \sqrt{1 - \zeta_i^2} \] (3.80)

are used.

The applied forces at this and the next time step, \( p_{i,n}^a \) and \( p_{i,n-1}^a \), are equivalent to the right hand side of Equation (3.7).

\[ p_{i,n}^a = \begin{bmatrix} \Phi_k^a \end{bmatrix}^T f_n^a - \tilde{M}_n^a \dot{x}_{j,n}^a \] (3.81)

and
\[
P_{i,n+1}^\alpha = \begin{bmatrix} \Phi_k^\alpha \\ 0 \end{bmatrix}^T f_{n+1}^\alpha - \tilde{M}_{ij}^\alpha \ddot{x}_{j,n+1}^\alpha. \tag{3.82}
\]

Equation (3.7) can then be rewritten using the above equations to solve explicitly for the component acceleration at the next time step. First, substitute Equations (3.69) through (3.82) into (3.7):

\[
\ddot{q}_{i,n+1}^\alpha + \ddot{C}_i^\alpha \left[ A'^\alpha \left( \begin{bmatrix} \Phi_k^\alpha \\ 0 \end{bmatrix}^T f_n^\alpha - \tilde{M}_{ij}^\alpha \ddot{x}_j^\alpha \right) + B'^\alpha \left( \begin{bmatrix} \Phi_k^\alpha \\ 0 \end{bmatrix}^T f_{n+1}^\alpha - \tilde{M}_{ij}^\alpha \ddot{x}_{j,n+1}^\alpha \right) + \right] + C'^\alpha q_{i,n}^\alpha + D''^\alpha \ddot{q}_{i,n}^\alpha \\
+ \Lambda_k^\alpha \left[ A^\alpha \left( \begin{bmatrix} \Phi_k^\alpha \\ 0 \end{bmatrix}^T f_n^\alpha - \tilde{M}_{ij}^\alpha \ddot{x}_j^\alpha \right) + B^\alpha \left( \begin{bmatrix} \Phi_k^\alpha \\ 0 \end{bmatrix}^T f_{n+1}^\alpha - \tilde{M}_{ij}^\alpha \ddot{x}_{j,n+1}^\alpha \right) + \right] + C^\alpha q_{i,n}^\alpha + D^\alpha \ddot{q}_{i,n}^\alpha \\
= \begin{bmatrix} \Phi_k^\alpha \\ 0 \end{bmatrix}^T f_{n+1}^\alpha - \tilde{M}_{ij}^\alpha \ddot{x}_{j,n+1}^\alpha. \tag{3.83}
\]

Terms in the above equation that are not iterated upon within a given time step can be assigned to a variable representing a virtual force.

\[
f_{i,v}^\alpha = \ddot{C}_i^\alpha \left[ A'^\alpha \left( \begin{bmatrix} \Phi_k^\alpha \\ 0 \end{bmatrix}^T f_n^\alpha - \tilde{M}_{ij}^\alpha \ddot{x}_j^\alpha \right) + B'^\alpha \left( \begin{bmatrix} \Phi_k^\alpha \\ 0 \end{bmatrix}^T f_{n+1}^\alpha \right) + C'^\alpha q_{i,n}^\alpha + D''^\alpha \ddot{q}_{i,n}^\alpha \\
+ \Lambda_k^\alpha \left[ A^\alpha \left( \begin{bmatrix} \Phi_k^\alpha \\ 0 \end{bmatrix}^T f_n^\alpha - \tilde{M}_{ij}^\alpha \ddot{x}_j^\alpha \right) + B^\alpha \left( \begin{bmatrix} \Phi_k^\alpha \\ 0 \end{bmatrix}^T f_{n+1}^\alpha \right) + \right] + C^\alpha q_{i,n}^\alpha + D^\alpha \ddot{q}_{i,n}^\alpha \right] - \begin{bmatrix} \Phi_k^\alpha \\ 0 \end{bmatrix}^T f_{n+1}^\alpha. \tag{3.84}
\]

Using Equation (3.84) and solving Equation (3.83) for \( \ddot{q}_{i,n+1}^\alpha \), the acceleration of the component at the next time step is

\[
\ddot{q}_{i,n+1}^\alpha = (\ddot{C}_i^\alpha B'^\alpha + \Lambda_k^\alpha B^\alpha - \ddot{C}_i^\alpha B'^\alpha - I_{ii}^\alpha \tilde{M}_{ij}^\alpha \ddot{x}_{j,n+1}^\alpha - f_{i,v}^\alpha. \tag{3.85}
\]

An expression for the acceleration at the next time step for component \( \beta \) can be derived in a similar way to produce
\[ \ddot{q}_{j,n+1} = (\dddot{C}_\beta^b B^{\beta} + \Lambda_k^{\beta} B^{\beta} - \dddot{I}_n^b) \dddot{M}_{j,n+1}^{\beta} - \dddot{f}_{i,\chi}^{\beta} \]  

(3.66)

where \( \dddot{f}_{i,\chi}^{\beta} \) is the virtual force applied to component \( \beta \).

Similarly to Equations (3.69) and (3.70), the junction displacement and velocity for the next time step can be expressed as

\[ q_{j,n+1} = A p_{j,n} + B p_{j,n+1} + C q_{j,n} + D \dot{q}_{j,n} \]  

(3.77)

and

\[ \dot{q}_{j,n+1} = A'p_{j,n} + B'p_{j,n+1} + C'q_{j,n} + D'\dot{q}_{j,n} \]  

(3.87)

where

\[ p_{j,n} = \Phi_y^T \left( \Psi_c^\alpha T f_{E,n}^\alpha + \Psi_c^\beta T f_{E,n}^\beta + f_{j,NL,n}^\alpha - \dddot{M}_{j,n}^{\alpha} q_{i,n}^{\beta} - \dddot{M}_{j,n}^{\beta} q_{i,n}^{\alpha} \right) \]  

(3.89)

and

\[ p_{j,n+1} = \Phi_y^T \left( \Psi_c^\alpha T f_{E,n+1}^\alpha + \Psi_c^\beta T f_{E,n+1}^\beta + f_{j,NL,n+1}^\alpha - \dddot{M}_{j,n+1}^{\alpha} q_{i,n+1}^{\beta} - \dddot{M}_{j,n+1}^{\beta} q_{i,n+1}^{\alpha} \right). \]  

(3.90)

Now Equation (3.16) can be solved for \( \dddot{q}_{j,n+1} \) by using Equations (3.85) through (3.90).

First, rewrite Equation (3.16) for time \( t_{n+1} \):

\[ \dddot{q}_{j,n+1} + \dddot{C}_j^{\beta} \dddot{q}_{j,n+1} + \Lambda_{j} \dddot{q}_{j,n+1} = \]  

\[ \Phi_y^T \left[ \left( f_{E,n+1}^\alpha + f_{j,NL,n+1}^\alpha \right) - \dddot{M}_{j,n+1}^{\alpha} q_{i,n+1}^{\beta} + \left( f_{E,n+1}^\beta + f_{j,NL,n+1}^\beta \right) - \dddot{M}_{j,n+1}^{\beta} q_{i,n+1}^{\alpha} \right]. \]  

(3.91)

Then Equations (3.85) through (3.90) can be substituted into Equations (3.91) and (3.15) used to restate all time \( t_{n+1} \) accelerations in terms of \( \dddot{q}_{j,n+1} \). With additional algebraic simplifications, the generalized junction acceleration can be expressed in terms of all known quantities with the exception of the non-linear force, \( f_{j,NL,n+1} \):
\[
\ddot{q}_{j,n} = \overline{M}^{-1}\left(\left[I_{ij} - \overline{C}_{ij} B' - \Lambda_{ij} B\right]\Phi_{ij}^{T}\left[\overline{M}_{ij}^{a} T f_{a}^{a} + \Psi_{c}^{\alpha T} f_{E,n+1}^{\alpha} + \overline{M}_{ij}^{B T} f_{v}^{B} + \Psi_{c}^{B T} f_{E,n+1}^{B} + f_{j,NL,n+1}\right]
\right. \\
\left. + \left(\overline{C}_{ij} A' + \Lambda_{ij} A\right)\Phi_{ij}^{T}\left[\overline{M}_{ij}^{a} T q_{i,n}^{a} - \Psi_{c}^{a T} f_{E,n}^{a} + \overline{M}_{ij}^{B T} q_{i,n}^{B} - \Psi_{c}^{B T} f_{E,n}^{B} - f_{j,NL,n}\right]
\right) \\
- \left(\overline{C}_{ij} C' + \Lambda_{ij} C\right)\ddot{q}_{j,n} - \left(\overline{C}_{ij} D' + \Lambda_{ij} D\right)\dot{q}_{j,n}
\] (3.92)

where

\[
\overline{M} = I_{ij} + \left(I_{ij} - \overline{C}_{ij} B' - \Lambda_{ij} B\right)\Phi_{ij}^{T}\left[\overline{M}_{ij}^{a} T \left(\overline{C}_{ii}^{a} B'^{a} + \Lambda_{ii}^{a} B^{a} - I_{ii}^{a}\right)\overline{M}_{ij}^{a}
\right. \\
\left. + \overline{M}_{ij}^{B T} \left(\overline{C}_{ii}^{B} B'^{B} + \Lambda_{ii}^{B} B^{B} - I_{ii}^{B}\right)\overline{M}_{ij}^{B}\right] \Phi_{ij}
\] (3.93)

When compared to the Newmark solution [35], this explicit formulation requires more calculations per time step. The efficiency of the explicit method lies in the possibility of using larger time steps in the calculation that would cause the numerical integrator to diverge from the solution. The only constraint on the size of the time step when using the explicit formulation comes from the approximation of the linear and non-linear forces as piecewise-linear excitations. The time step must be kept sufficiently small so that the piecewise-linear approximation matches the true behavior of the forces to an acceptable level.

### 4 Sample Problems

Perhaps the most useful property of the decoupled analysis method is its ability to efficiently solve dynamic problems with local non-linearities or non-classical damping. The decoupled analysis method has already proven its utility in Shuttle-payload systems [34]. Additional applications of the method involving the International Space Station and the Shuttle will be shown in this chapter.
4.1 Unlocked Beta Gimbal

While the ISS is currently under construction, the engineering analysis of it is by no means complete. Teams of engineers are currently involved in the verification of the safety of operating procedures and contingency scenarios. One example of these efforts involves the electromagnetic motor responsible for turning the large United States photovoltaic arrays toward the sun (see Figure 4.1.1). This motor is commonly called the beta gimbal.

The beta gimbal was equipped with a mechanical locking mechanism capable of grasping the base of the mast of the photovoltaic arrays. With this locking mechanism engaged, the sensitive components of the gimbal are protected from the loads that are induced by the Shuttle’s reaction control thrusters impinging on the arrays and by the Shuttle docking to the station. If the locking mechanism is not engaged, the force needed to prevent the arrays from turning out of alignment when struck by forces can only be supplied by the electromagnetic resistance of the gimbal and friction.

The flight rules previously stipulated that whenever the Shuttle is performing operations in proximity of the ISS, the locking mechanism must be engaged. The safety of this system was verified using traditional linear methods. Concerns about the reliability of the locking mechanism, specifically the ability to unlock the mechanism after locking, allowing the arrays to turn toward the sun and generate electricity for the station, prompted the desire to verify the safety of the system without the locking mechanism engaged. Traditional linear methods are incapable of modeling the system
without the locking mechanism engaged because the resistance of the gimbal to the forces is highly non-linear.

This scenario drove the development of decoupled analysis tools to verify the safety of the system when the locking mechanism is not engaged. The method detailed in Section 3.2 was applied to the problem.

The particular scenario chosen for inclusion here involves the forces induced by the reaction control system (RCS) thrusters of the Shuttle impinging upon a U.S. photovoltaic array (see Figure 4.1.1). The figure shows an early configuration of the Station scheduled for November of 2000. This will be the first time the Shuttle performs proximity operations near the Station with the large United States photovoltaic arrays deployed.

Figure 4.1.1
First Shuttle Operations Near ISS with Photovoltaic Arrays Deployed
The photovoltaic array assembly was broken into an alpha component and a beta component for the analysis. The alpha component consisted of a U.S. photovoltaic array from where it connects to the beta gimbal. This structure was reduced to a Craig-Bampton model with 102 degrees of freedom. Of these, six degrees of freedom were kept in physical space and the remaining 96 were generalized interior modes, with the highest interior mode reaching 9.9 Hz. The six degrees of freedom that were kept in physical space represented the physical attachment point to the beta gimbal.

The beta component consisted of the structure from the photovoltaic array attachment point to the connection point of the beta gimbal assembly to the main truss of the Station. The model was grounded at the connection point to the main truss because the truss is much stiffer than the items included in the problem, allowing it to be modeled as a fixed interface with little loss of fidelity. The beta component consisted of six physical degrees of freedom and 50 generalized modes.

The alpha and beta components were tied together in the traditional linear analysis way for five of the six degrees of freedom at the interface. The sixth DOF, the US photovoltaic array torsion DOF, was not tied together to allow insertion of the servo motor non-linearity. The servomotor was modeled as having a rotational stiffness of 650 in-lb/rad and a non-classical rotational damping of 10,000 in-lb/rad/sec. It is this damping that makes decoupled analysis critical for the efficient solution of this problem.

A great efficiency of the decoupled analysis method is that all of the degrees of freedom of the two components do not need to be included in the junction representation. In this case, including all of the degrees of freedom in the junction would produce a
problem with 158 degrees of freedom that would need to be iterated upon at each time step to reach convergence at the interface of the two components. In the current analysis, very accurate results were found using one-fourth of this total. A total of 30 of the 102 degrees of freedom of the photovoltaic array mass and stiffness matrices were included in the junction representation. Of these 30 degrees of freedom, six were the physical degrees of freedom necessary for connecting the solar arrays. Whereas the original photovoltaic array representation had a cutoff frequency of 9.9 Hz, the highest mode included in the junction representation was 0.43 Hz.

The junction representation was thus made up of 36 modes from the U.S. photovoltaic array and six modes from the beta gimbal totaling 42 degrees of freedom. The interface between the alpha and beta components consisted of a single grid point. At this connection point, five of the six degrees of freedom were directly connected. This results in the reduction of the junction size from 42 to 37 degrees of freedom. An eigensolution was then applied to this reduced problem to produce a decoupled set of 37 degrees of freedom that were iterated to reach conversion at the non-linear interface. It is critical to the efficiency of the method that this eigensolution only needs to be performed once due to the computational expense of eigensolutions.

This decoupled model was subjected to an excitation force of four pulses from the Shuttle RCS thrusters upon the array. These pulses were timed 13.8 seconds apart to maximize the response of the array.
4.1.1 Results

The time histories of 16 different load indicators located on the U.S photovoltaic array were output. The longerons are the axial structural elements of the array mast responsible for absorbing bending loads; the battens are the braces between the longerons designed to absorb torsion loading. The gimbal forces and moments are from the shaft connecting the array to the beta gimbal.

The accuracy of the decoupled representation was assessed by changing the soft spring located between the two components from 650 in-lb/rad to a hard spring (1,000,000 in-lb/rad). This change effectively removes the non-linearity of the beta gimbal from the problem. The expected accuracy of the decoupled model with the non-linearity could then be assessed based upon how well the decoupled model with the hard spring matched the results from a completely linear model without a non-linearity between components. The linear model represents the array with the locking mechanism engaged.

Figures 4.1.2 through 4.1.17 show the results of this assessment. The greatest accuracy is found in those indicators with responses dominated by lower frequencies. For example, the longeron responses are due mainly to the low frequency bending moments of the photovoltaic array. The accuracy of the decoupled model with the hard beta gimbal spring can be assessed by comparing the peak forces and moments of the indicators. All longeron and batten peak responses are within 5% of the linear case. Some higher frequency content is seen in the shear and axial peak responses of the gimbal shaft. These results vary by no more than 10%. A summary of the percent
differences between the decoupled model with the hard beta gimbal spring and the linear model is presented in Figures 4.1.16 and 4.1.17.

The response of the U.S. photovoltaic array when the locking mechanism is not engaged was assessed from the decoupled model with the soft beta gimbal spring inserted between the components. Note that this soft spring also allows the non-classical damper to affect the response of the system.

Figures 4.1.18 through 4.1.33 show the effect of the soft spring and non-classical damper on the system. Figures 4.1.18 through 4.1.21 show that the longeron responses are mostly unaffected. This is to be expected since the unlocked gimbal is more able to rotate but still maintains the stiffness of the structure for bending. Thus the bending moments, which are the main component of the longeron responses, are the same for both an unengaged locking mechanism and an engaged locking mechanism. The battens, shown in Figures 4.1.22 through 4.1.25, respond to torsion loading. The torsion of the structure is greatly affected by the unlocking of the gimbal. As can be expected, the softening of the gimbal torsion spring results in a decrease in the torsion loads seen by the battens. The gimbal loads, Figures 4.1.26 through 4.1.31, are largely unaffected in all but one case. That case, shown in Figure 4.1.31, is the gimbal torsion. A summary of the percent changes in the peak responses between the locked and unlocked beta gimbal cases can be seen in Figures 4.1.32 and 4.1.33.

Unlocking the gimbal greatly reduces the torsion. As a result, the entire U.S. photovoltaic array is freer to rotate about its longitudinal axis. In certain cases this
rotation can exceed 10°. The implications of this rotation are discussed in the next section.
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.2 Longeron #1 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Engaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.3 Longeron #2 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Engaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with Non-Classically Damped Beta Gimbal

Figure 4.1.4 Longeron #3 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Engaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.5 Longeron #4 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Engaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.6 x-axis Batten #1 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Engaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.7 x-axis Batten #2 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Engaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with Non-Classically Damped Beta Gimbal

Figure 4.1.8 y-axis Batten #1 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Engaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.9  y-axis Batten #2 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Engaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with Non-Classically Damped Beta Gimbal

Figure 4.1.10 Gimbal Shaft Moment #1 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Engaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.11  Gimbal Shaft Moment #2 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Engaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.12 Gimbal Shaft Shear #1 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Engaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.13  Gimbal Shaft Shear #2 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Engaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.14 Gimbal Shaft Axial Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Engaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with Non-Classically Damped Beta Gimbal

Figure 4.1.15 Gimbal Shaft Torque Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Engaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.16 Percent Change in Peak Responses for Longerons and Battens
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Engaged
Figure 4.1.17  Percent Change in Peak Responses for Gimbal Shaft Forces and Moments  
Comparison of Decoupled and Linear Methods  
Beta Gimbal Locking Mechanism Engaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.18  Longeron #1 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Disengaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with Non-Classically Damped Beta Gimbal

Figure 4.1.19 Longeron #2 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Disengaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.20 Longeron #3 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Disengaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

![Graph showing force vs. time with Decoupled and Linear markers.]

Figure 4.1.21 Longeron #4 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Disengaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.22  x-axis Batten #1 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Disengaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with Non-Classically Damped Beta Gimbal

Figure 4.1.23  x-axis Batten #2 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Disengaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with Non-Classically Damped Beta Gimbal

Figure 4.1.24  y-axis Batten #1 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Disengaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with Non-Classically Damped Beta Gimbal

Figure 4.1.25 y-axis Batten #2 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Disengaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with Non-Classically Damped Beta Gimbal

Figure 4.1.26 Gimbal Shaft Moment #1 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Disengaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.27 Gimbal Shaft Moment #2 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Disengaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with
Non-Classically Damped Beta Gimbal

Figure 4.1.28  Gimbal Shaft Shear #1 Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Disengaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with Non-Classically Damped Beta Gimbal

Figure 4.1.29 Gimbal Shaft Shear #2 Response Comparison of Decoupled and Linear Methods Beta Gimbal Locking Mechanism Disengaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with Non-Classically Damped Beta Gimbal

Figure 4.1.30 Gimbal Shaft Axial Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Disengaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with Non-Classically Damped Beta Gimbal

![Graph of force vs. time showing decoupled and linear response](image)

**Figure 4.1.31** Gimbal Shaft Torque Response
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Disengaged
Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with Non-Classically Damped Beta Gimbal

Figure 4.1.32 Percent Change in Peak Responses for Longerons and Battens
Comparison of Decoupled and Linear Methods
Beta Gimbal Locking Mechanism Disengaged

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Decoupled Analysis Sample Problem #1
Shuttle RCS Thruster Impinging U.S. Photo-Voltaic Array with Non-Classically Damped Beta Gimbal

Figure 4.1.33 Percent Change in Peak Responses for Gimbal Shaft Forces and Moments Comparison of Decoupled and Linear Methods Beta Gimbal Locking Mechanism Disengaged
4.1.2 Discussion

Based upon the model demonstrated in this sample problem, it has been
determined that the U.S. photo-voltaic arrays do not need to have the beta gimbal locking
mechanism engaged during approach, docking, and separation of the Shuttle. The
correlation between the decoupled model with the hard spring and the linear model is
convincing and the results of the decoupled model with the soft spring and non-classical
damper match what would be expected.

One concern with the analysis stems from the higher rotations of the array that are
seen when the beta gimbal locking mechanism is not engaged. This higher rotation may
reasonably be expected to change the nature of the excitation. With the array turned by
any significant amount, the way in which the Shuttle RCS thruster exhaust impinges upon
the array would change. This change in excitation could be included while employing the
decoupled model but would slow down the computation. Note that the increase in
necessary computations to include this effect would be the same between decoupled
analysis and linear analysis.

4.2 Orbiter Docking System Freeplay

On missions to the ISS, the Shuttle is equipped with a docking system to permit
pressurized access from the Shuttle crew cabin to the interior of the Station. This Orbiter
Docking System (ODS) is held in place in the Shuttle payload bay by a set of five
trunnions. On early flights to the Station and during earlier flights to the Russian Mir
station, there was evidence of freeplay between the Shuttle and the stations. The freeplay
was most pronounced in the pitch direction of the Shuttle. Analysis of the geometric
specifications of the ODS shows that there may be approximately 0.11 degrees of freeplay between the Shuttle and the Station in the pitch direction of the Shuttle.

The existence of this freeplay called into question the pre-flight analyses designed to determine the maximum structural forces that the mated system would experience. It was also worrisome from a controls standpoint since the mated attitude control parameters were based on a system pitch mode of 0.24 Hz, whereas actual flight data showed a pitch mode of 0.16 Hz. These two concerns could not be addressed with the standard linear dynamic model of the system.

The decoupled analysis method was applied to the problem in an effort to improve on the accuracy of the previous linear analyses. The interface between the Shuttle and the Station is reduced to a single point and is assumed linear in all three translational directions as well as the yaw and roll directions (see Figure 4.2.1). In the pitch direction, a 0.11-degree gap is inserted. This geometric non-linearity has the desired effect of adding freeplay to the interface but lacks the ability to supply contact forces. These contact forces would tend to dissipate the low-frequency energy into higher frequencies each time the gap is closed. Although this effect is lost in the current application, the results are still found to more accurately match actual data gathered during the early missions than the linear model.
A non-classical damper is also inserted in conjunction with the gap. This damper is intended to model the effects of the trunnion pins sliding in their sockets. The amount of damping produced by this sliding was not known. A parametric study was performed to find a likely range of values for this damping.

The Shuttle model used in the analysis is a Craig-Bampton representation with the six degrees of freedom of the docking point of the ODS kept in the physical boundary along with three rotational degrees of freedom associated with the Shuttle Inertial Maneuvering Unit (IMU). The IMU is responsible for providing the Shuttle crew with
attitude data. The IMU data is also downloaded during flights and in this case is used for model verification. These nine physical degrees of freedom along with 56 generalized modes constitute the mass and stiffness matrices used to represent the Shuttle in this analysis. The frequency cutoff for the Shuttle model is 11.6 Hz.

The Station model has a total of 114 degrees of freedom with only six kept physical. Those six are the docking point for the Shuttle. The other 108 degrees of freedom are all generalized modes with a frequency cutoff of 9.4 Hz.

Efficiency was gained by including a subset of the modes from each model in the junction representation. The nine physical modes of the Shuttle model along with the lowest nine generalized modes were included in the junction. The highest frequency of the nine generalized modes was 1.1 Hz. The six physical modes of the Station model along with the lowest 16 generalized modes with a frequency cutoff of 0.97 Hz were included in the junction representation.

The decoupled model of the Shuttle mated to the Station was then subjected to a simulated Shuttle reboost. This is a series of Shuttle RCS firings designed to boost the Station into a higher orbit above the Earth. This is done periodically to keep the Station from slowly losing altitude and eventually burning up in the Earth’s atmosphere. The time history of the Shuttle IMU pitch attitude was recorded during these reboost firings on a recent mission. These time histories provide an opportunity to verify the results from the decoupled analysis method against real flight data on a large aerospace structure.

In order to determine the amount of damping caused by trunnion friction as well as verify the geometric estimate of the freeplay, a series of analyses were performed while varying these two constants. The output from the various runs were compared to
the real flight data to find the best fit. The next section shows that the two variables have separate effects on the results, ensuring the uniqueness of the solution.

4.2.1 Results

The pitch attitude output from the Shuttle IMU is compared for flight data and model results in Figures 4.2.2 through 4.2.11.

Figure 4.2.2 shows the results from the linear study compared against the flight data. The linear model was processed with a typical 1% modal damping. The fundamental frequency of the linear model is 0.24 Hz. Both the damping and the fundamental frequency are significantly different from the flight data.

Figures 4.2.3 through 4.2.5 show the results from 0.05 degrees of freeplay. The small amount of freeplay has a slight effect on the frequency of the output. The three figures show progressively larger amounts of trunnion friction damping.

Figures 4.2.6 through 4.2.8 have 0.11 degrees of freeplay. This is the amount found in the geometric analysis of the Shuttle ODS system. It can be seen that the increase in gap size has dropped the fundamental frequency of the system. Figure 4.2.7 is the closest match found in this study to the flight data with a freeplay of 0.11 degrees and a damping of $1 \times 10^6$ in-lb/rad/sec. The poor correlation at the lower amplitudes evident in Figure 4.2.7 suggests that a qualitative change in the system behavior below a certain response amplitude may be driving the results at this level. The good correlation at the higher amplitude is significant. The area of greatest engineering interest from a loads and controls standpoint is in this higher region that the linear model fails to match. This correlation is discussed further in the next section.
Figures 4.2.9 through 4.2.11 show the analytical results of having 0.15 degrees of freeplay. The significant change is in the fundamental frequency that continues to drop as the freeplay amount increases.
Decoupled Analysis Sample Problem #2
Shuttle Reboost of International Space Station with
Freeplay and Non-Classical Damping

Figure 4.2.2 IMU Pitch Response
Comparison of Linear Results (smooth line) and Flight Data (jagged line)
No Freeplay and 1% Modal Damping
Decoupled Analysis Sample Problem #2
Shuttle Reboost of International Space Station with
Freeplay and Non-Classical Damping

Figure 4.2.3 IMU Pitch Response
Comparison of Decoupled Results (smooth line) and Flight Data (jagged line)
0.05° of Freeplay and Low Damping
Figure 4.2.4 IMU Pitch Response
Comparison of Decoupled Results (smooth line) and Flight Data (jagged line)
0.05° of Freeplay and Medium Damping
Decoupled Analysis Sample Problem #2
Shuttle Reboost of International Space Station with Freeplay and Non-Classical Damping

Figure 4.2.5  IMU Pitch Response
Comparison of Decoupled Results (smooth line) and Flight Data (jagged line)
0.05° of Freeplay and High Damping
Decoupled Analysis Sample Problem #2
Shuttle Reboost of International Space Station with
Freeplay and Non-Classical Damping

Figure 4.2.6 IMU Pitch Response
Comparison of Decoupled Results (smooth line) and Flight Data (jagged line)
0.11° of Freeplay and Low Damping
Decoupled Analysis Sample Problem #2
Shuttle Reboost of International Space Station with
Freeplay and Non-Classical Damping

Figure 4.2.7 IMU Pitch Response
Comparison of Decoupled Results (smooth line) and Flight Data (jagged line)
0.11° of Freeplay and Medium Damping

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Decoupled Analysis Sample Problem #2
Shuttle Reboost of International Space Station with
Freeplay and Non-Classical Damping

Figure 4.2.8 IMU Pitch Response
Comparison of Decoupled Results (smooth line) and Flight Data (jagged line)
0.11° of Freeplay and High Damping
Decoupled Analysis Sample Problem #2
Shuttle Reboost of International Space Station with
Freeplay and Non-Classical Damping

Figure 4.2.9 IMU Pitch Response
Comparison of Decoupled Results (smooth line) and Flight Data (jagged line)
0.15° of Freeplay and Low Damping
Decoupled Analysis Sample Problem #2
Shuttle Reboost of International Space Station with Freeplay and Non-Classical Damping

Figure 4.2.10 IMU Pitch Response
Comparison of Decoupled Results (smooth line) and Flight Data (jagged line)
0.15° of Freeplay and Medium Damping
Decoupled Analysis Sample Problem #2
Shuttle Reboost of International Space Station with
Freeplay and Non-Classical Damping

![Graph of IMU Pitch Response](image)

**Figure 4.2.11** IMU Pitch Response
Comparison of Decoupled Results (smooth line) and Flight Data (jagged line)
0.15° of Freeplay and High Damping
4.2.2 Discussion

The good correlation between the decoupled analysis results and the flight data at higher amplitudes has decreased the uncertainty in these analyses. The good correlation has led to the use of this tool to aid in making controls decisions as well as certifying that the peak loads in the presence of freeplay are within an acceptable percent of those predicted by the linear model with no freeplay.

The poor correlation at the lower amplitudes evident in Figure 4.2.7 suggests that a qualitative change in the system behavior below a certain response amplitude may be driving the results at this level. The good correlation at the higher amplitude is significant in that the area of greatest engineering interest from a loads and controls standpoint is in this higher region. Figure 4.2.2 shows that the linear model failed to match at any amplitude.

The low amplitude region of the flight data shows a continued decrease in the fundamental frequency while the decoupled model shows a convergence toward the linear model frequency of 0.24 Hz. This is due to stiction in the pitch freeplay in the decoupled method that is not apparent in the flight data. Future work includes modifying the application of the trunnion friction to more closely match the behavior seen in the flight data.
5 Conclusion

While the size of engineering projects continue to expand, time and money allocated for accurate engineering design and analysis does not. Projects like the ISS demand intelligent model reduction methods that maintain accuracy while decreasing the size of the problem to a manageable level. CMS methods have been used for the majority of this work. The Craig-Bampton method described in Section 2.2.1 remains the most widely used tool, with the residual flexibility and Lanczos vectors methods are finding more occasional use (though the math behind the Lanczos vectors formulation has become the new standard tool for solving eigenproblems in all methods).

As the analysis of the ISS has progressed to more specific scenarios, non-linearities present in the system can no longer be ignored. The decoupled analysis methods are finding increased utility as the complexities of the problems increase. The method described in Section 3.2 and using a Newmark integrating scheme is the most widely used decoupled analysis method because of its efficiency and accuracy while the residual flexibility, Lanczos vectors, and explicit formulations find more limited use.

The sample problems described in Sections 4.1 and 4.2 demonstrate the utility of the decoupled analysis method on large aerospace structures. In particular, the data available for the ODS freestyle analysis of Section 4.2 offered an opportunity to verify the accuracy of the decoupled method against real flight data. Through both of the complex problems shown here, the decoupled method has demonstrated indispensable utility and notable accuracy.
6 Bibliography


