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Analysis of Nonlinear Structural Systems Accounting for both System Parameter Uncertainty and Excitation Stochasticity

by

Prashanth K. Vijalapura

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE
Master of Science

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May, 1998
Analysis of Nonlinear Structural Systems Accounting for both System Parameter Uncertainty and Excitation Stochasticity

Abstract
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Prashanth K. Vijalapura

The inherent uncertainties in nature and the imperfect state of our knowledge manifest themselves as uncertainties in loading, material or system parameters and modeling the behavior of any structural system. Limiting ourselves to only loading and system parameter uncertainties, the dynamic loading to a structure can be modeled as a random function while the system or material parameters can be modeled as random fields. Through a process of discretization, these random functions and random fields can be lumped into a vector of random variables that completely describe the uncertainties in loading and material parameters. These uncertainties result in a finite probability of failure or of the structural system not performing as intended. A powerful method to compute the failure probability is via the time history of the mean rate of out-crossing the “safe domain” by the structural system. Computing the mean out-crossing rate at any instant of time amounts to solving iteratively a constrained optimization problem. Each iteration of the constrained optimization problem requires the gradients of the constraints with respect to the vector of uncertain parameters and hence the response sensitivities with respect to loading and material parameters since the constraints are constituted in terms of response quantities.

In the first phase of this study, response sensitivities with respect to various loading and system parameters and out-crossing rates for white noise base excitation are computed for several phenomenological (rule-based) single-degree-of-freedom (SDOF) systems with
deterministic and uncertain system properties. The second phase of the study focuses on mechanics-based plasticity models of SDOF and multi-degree-of-freedom (MDOF) structural systems and the derivation of their response sensitivity equations. Finally, several realistic structural examples are considered to illustrate the application of the response sensitivity computation algorithms developed.
Acknowledgments

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CHAPTER 1 Introduction

1.1 General Remarks

The design of any structural system is performed in the face of numerous uncertainties. These uncertainties are attached to the loading or excitation, the material and geometric parameters of the structural system, and the mathematical models used to predict the response of the structural system. Even under the assumption of exact modeling of the structural behavior, there is a finite probability of failure or of not performing as intended due to loading and material parameter uncertainties. Therefore, it is essential to develop a rational framework to model the uncertainties of both the loading and material parameters and compute the resulting probability of failure or its complement, the reliability of the structural system.

Structural reliability analysis requires the following two-step approach: (1) developing a mathematical framework to evaluate the probability of failure due to the uncertainties in both loading and material parameters, and (2) developing supporting tools to compute various ingredients such as response sensitivities that are needed for performing the reliability analysis. For large and complex structural systems or even simple single-degree-of-freedom (SDOF) systems incorporating nonlinear, hysteretic behavior, and subjected to dynamic earthquake loading, the solution for response can be obtained numerically only as no closed form analytical solutions are available. Therefore, reliability analysis of such systems must be performed in conjunction with a numerical scheme for response computation such as the finite element method.

With ever increasing availability of computational power, rigorous reliability analysis
tools are becoming easier to implement. The growing realization of the importance of probabilistic methods in design and analysis of structural systems (e.g., performance-based engineering design) have made the reliability analysis tools indispensable. The reliability of complex structural systems can be evaluated by linking the modern methods of reliability analysis and powerful numerical methods for system response evaluation such as the finite element method.

The present study is motivated by the above considerations. Its objectives and scope are outlined in the next section.

1.2 Objectives and Scope of Study

The two main objectives of the present study are:

(a) to develop routines for computation of the response and exact response sensitivities for various SDOF and multi-degree-of-freedom (MDOF) nonlinear, hysteretic dynamic systems, and

(b) to perform the complete reliability analysis of SDOF dynamic systems with uncertain system (material) parameters and subjected to stochastic loading.

The probability of failure of a dynamic system can be obtained indirectly through the time history of the rate at which the system out-crosses the safe domain and fails. This out-crossing rate formulation for evaluating the probability of failure requires the computation of response sensitivities with respect to the various material and loading parameters which contribute to the uncertainty of the system behavior. Thus, the goals are firstly to develop efficient and accurate algorithms for response sensitivity calculations and secondly to compute the out-crossing rates for failure probability calculations.
1.3 Organization of Thesis

The thesis is organized as follows:

In chapter 2, the numerical schemes for computing system response and response sensitivities with respect to loading and material parameters for so-called phenomenological (rule-based) SDOF models are presented. Specifically, the Duffing oscillator, the Bouc-Wen oscillator and the rule-based bilinear hysteretic oscillator models are considered. The dynamic loading function for these oscillators is defined in the form of a train of impulses occurring at constant intervals of time. Numerical examples for computing response sensitivities and with respect to various loading and material parameters are then presented and their comparison with finite difference results for the above oscillators.

In chapter 3, the complete framework to compute the probability of failure due to system parameter and loading uncertainties in terms of the out-crossing rates is described in detail. Specifically, the loading function is assumed to be random and is modeled as a white noise process. The effects on the mean out-crossing rate time history and hence the probability of failure for a white noise base excitation when system uncertainties are included are demonstrated through application examples.

In chapter 4, numerical schemes to compute the response of mechanics based plasticity models of dynamic systems are described. The finite element method and plasticity theory together with computational aspects in plasticity formulations are reviewed. The complete sets of equations for integrating the equation of motion using the Newmark-β method incorporating the J₂ plasticity model and the cap plasticity model are described.

In chapter 5, the algorithms for response sensitivity calculation for dynamic systems with
mechanics based models are presented. Two alternative methods to constitute the response sensitivity equations are presented. In formulating the sensitivity equations, questions regarding continuity of response sensitivities in time arise. A mathematical description of these discontinuities (if any) together with a physical interpretation of their cause in terms of switching between material states of the underlying plasticity model are presented. As application examples, the complete set of sensitivity equations for the uniaxial $J_2$ plasticity model and the multi-axial, multi-surface cap plasticity model, obtained by differentiating exactly the numerical scheme for the response calculation are derived. These sensitivities obtained by differentiating exactly the numerical scheme for response are compared with finite difference results for various loading and material sensitivity parameters using SDOF and MDOF systems.

Finally, the summary and conclusions of this study are presented in chapter 6.
PART I: PHENOMENOLOGICAL CONSTITUTIVE STRUCTURAL MODELS
CHAPTER 2 Dynamic Response Sensitivity Analysis

2.1 Introduction

Dynamic response sensitivity refers to the sensitivity of a response quantity of a dynamical system to various parameters that control the response. The response sensitivities are important in themselves, as they provide valuable indices to understand the behavior of complex dynamic systems. Of equal importance is their indispensability in a majority of algorithms used for fundamental engineering problems such as structural optimization, structural reliability and system identification.

In general, the various parameters controlling the dynamic response are the loading and system parameters. In this context, interest lies in computing the sensitivities with respect to these parameters.

This chapter deals with computing the displacement response sensitivities with respect to loading and system parameters. These quantities are derived analytically for the Duffing oscillator, Bouc-Wen hysteretic oscillator and the rule-based bilinear hysteretic oscillator. These analytical response sensitivities are compared with those obtained by forward finite difference in several application examples provided at the end of the chapter.

2.2 Duffing Oscillator

The single-degree-of-freedom(SDOF) Duffing oscillator is a nonlinear, stiffening type elastic oscillator. The equation of motion for the Duffing oscillator subjected to support excitation is given by

\[ \ddot{u}(t) + 2\xi \omega \dot{u}(t) + \omega^2 (1 + \gamma u^2(t)) u(t) = -\ddot{u}_g(t) \]  \hspace{1cm} (2.1)

where \( u(t) \) denotes the displacement response of the oscillator with respect to its support,
and $\omega$ and $\xi$ are the natural circular frequency and damping ratio of the SDOF oscillator; these two parameters are selected as system parameters for response sensitivity calculations. Parameter $\gamma$ describes the extent to which the oscillator is nonlinear, and $\ddot{u}_g(t)$ denotes the support acceleration time history. Notice that the case $\gamma = 0$ corresponds to a linear elastic oscillator. Therefore, response and response sensitivities for a linear elastic system are computed by substituting $\gamma = 0$ in the formulations and results presented in the following sections.

2.2.1 Computation of the Nonlinear Elastic Response

The response of the Duffing oscillator is computed through numerical integration of the equation of motion (2.1). For this purpose, the Newmark-$\beta$ method with parameters $\alpha = 0.5$ and $\beta = 0.25$ is used. These values for $\alpha$ and $\beta$ correspond to the constant acceleration method which is unconditionally stable for linear systems (Chopra 1995). The oscillator is assumed to be initially (i.e., at time $t_0$) at rest.
The interpolations for the velocity and displacement responses at time step \( t_{i+1} \) are given by:

\[
\begin{align*}
    u_{i+1} &= 0.25(\ddot{u}_{i+1} + \ddot{u}_i)\Delta t^2 + \dot{u}_i \Delta t + u_i \\
    \dot{u}_{i+1} &= 0.5(\ddot{u}_{i+1} + \ddot{u}_i)\Delta t + \dot{u}_i \\
    \ddot{u}_{i+1} &= -\ddot{u}_g(t_{i+1}) - 2\xi \omega \dot{u}_{i+1} - \omega^2(1 + \gamma u_{i+1}^2)u_{i+1}
\end{align*}
\] (2.2)

In the above equations, \( \Delta t \) denotes the constant time step used in integrating the equation of motion. For an input \( -\ddot{u}_g(t) \) modeled as a train of impulses \( \sum_{0}^{n} f_i \delta(t_{i} - t) \) (see Sec. 3.2) occurring at discrete times \( t_{i+1} = i\Delta t \), \( i = 0, 1, 2, \ldots \), the velocity at \( t_i^+ \) (just after application of the impulse \( f_i \delta(t_{i} - t) \)) is obtained exactly as: \( \dot{u}_i^+ = \dot{u}_i + f_i \). Thus, the forcing function value \( -\ddot{u}_g(t_i) \) is translated into an equivalent update of the velocity response at \( t_i^+ \) and the equation of motion is integrated from \( t_i^+ \) to \( t_{i+1}^+ \) as if the problem is one of free vibration. Equation (2.2) can be written equivalently as:

\[
\begin{align*}
    u_{i+1} &= 0.25(\ddot{u}_{i+1}^+ + \ddot{u}_i)\Delta t^2 + \dot{u}_i^+ \Delta t + u_i \\
    \dot{u}_{i+1}^+ &= 0.5(\ddot{u}_{i+1}^+ + \ddot{u}_i)\Delta t + \dot{u}_i^+ \\
    \ddot{u}_{i+1}^+ &= -2\xi \omega \dot{u}_{i+1}^+ - \omega^2(1 + \gamma u_{i+1}^2)u_{i+1}
\end{align*}
\] (2.3)

The quantities \( u_{i+1}^+ \), \( \dot{u}_{i+1}^+ \) and \( \ddot{u}_{i+1}^+ \) are solved for, using a predictor-corrector iterative method. Then, \( \dot{u}_{i+1}^+ \) is updated as mentioned earlier, namely:

\[
\dot{u}_{i+1}^+ = \dot{u}_i + f_{i+1}
\] (2.4)

Finally, \( \ddot{u}_{i+1}^+ \) is found from the equation of dynamic equilibrium as:
\[ u_i^{+} = -2\xi \omega u_i^{+} - \omega^2 (1 + \gamma u_i^2) u_i^{+} \quad \text{or} \quad (2.5) \]

\[ u_i^{-} = u_i^{-} - 2\xi \omega f_i^{-} \quad \text{(2.6)} \]

The relative displacement response \( u(t) \) does not change from \( t_i^{-} \) to \( t_i^{+} \) i.e.,

\[ u_i^{+} = u_i^{-} = u_i^{-} \quad \text{.} \]

The initial conditions are given as:

\[ u(0) = 0 \]
\[ \dot{u}(0^+) = f_0 \]
\[ \ddot{u}(0^+) = -2\omega \xi \dot{u}(0^+) \quad \text{(2.7)} \]

### 2.2.2 Computation of Gradients of the Nonlinear Elastic Response

The numerical scheme with equivalent velocity updates, used for computing the response of the Duffing oscillator to support excitation modeled as a train of impulses is differentiated exactly with respect to the sensitivity parameter \( x \), which is either a system\( (\omega \) or \( \xi \)) or a loading parameter \( (f_i, i = 0, 1, 2, \ldots) \). This yields a set of gradient equations which are solved exactly step-by-step, similar to the response equation.

The gradient equations for the sensitivity parameter \( 'x' \) are:

\[ \frac{\partial u_i^{-}}{\partial x} = 0.25 \left( \frac{\partial \dot{u}_i^{-}}{\partial x} + \frac{\partial \ddot{u}_i^{-}}{\partial x} \right) \Delta t^2 + \frac{\partial \ddot{u}_i^{-}}{\partial x} \Delta t + \frac{\partial u_i^{-}}{\partial x} \quad \text{(2.8)} \]

\[ \frac{\partial u_i^{+}}{\partial x} = 0.5 \left( \frac{\partial \dot{u}_i^{+}}{\partial x} + \frac{\partial \ddot{u}_i^{+}}{\partial x} \right) \Delta t + \frac{\partial \ddot{u}_i^{+}}{\partial x} \quad \text{(2.9)} \]

\[ \frac{\partial \dot{u}_i^{+}}{\partial x} = -2\xi \frac{\partial \omega}{\partial x} u_i^{+} - 2\xi \omega \frac{\partial u_i^{+}}{\partial x} - 2\omega \xi \frac{\partial u_i^{+}}{\partial x} - 2\omega \frac{\partial \omega}{\partial x} (1 + \gamma u_i^2) u_i^{+} \]

\[ -\omega^2 (1 + 3\gamma u_i^2) \frac{\partial u_i^{+}}{\partial x} \quad \text{(2.10)} \]
It is noticed that when the sensitivity parameter is a loading parameter \((x = f_j)\), the partial derivatives \(\frac{\partial u_i}{\partial f_j}, \frac{\partial \dot{u}_i}{\partial f_j}\) and \(\frac{\partial \ddot{u}_i}{\partial f_j}\) are zero for \(j > i\). Furthermore, the derivatives of system parameters \(\omega\) and \(\xi\) with respect to loading parameters are zero and vice versa.

Substituting the expressions for \(\frac{\partial u_i^{+1}}{\partial x}\) and \(\frac{\partial \dot{u}_i^{+1}}{\partial x}\) from (2.9) and (2.10) in (2.8) results in a linear algebraic equation in \(\frac{\partial u_i^{+1}}{\partial x}\), which is solved exactly in a single iteration. Then, \(\frac{\partial u_i^{+1}}{\partial x}\) and \(\frac{\partial \dot{u}_i^{+1}}{\partial x}\) are easily obtained using (2.9) and (2.10). The updates for \(\frac{\partial u_i^{+1}}{\partial x}\) and \(\frac{\partial \dot{u}_i^{+1}}{\partial x}\) are obtained from (2.4) as:

\[
\frac{\partial u_i^{+1}}{\partial x} = \frac{\partial u_i^{-1}}{\partial x} + \frac{\partial f_i^{+1}}{\partial x} \tag{2.11}
\]

and from (2.6) as:

\[
\frac{\partial u_i^{+1}}{\partial x} = \frac{\partial u_i^{-1}}{\partial x} - 2\xi \omega \frac{\partial f_i^{+1}}{\partial x} - 2\frac{\partial \xi}{\partial x} \omega f_i^{+1} - 2\frac{\partial \omega}{\partial x} f_i^{+1} \tag{2.12}
\]

2.3 Bouc-Wen Oscillator

The Bouc-Wen oscillator belongs to the class of nonlinear, hysteretic oscillators. The equation of motion for the Bouc-Wen oscillator subjected to support excitation is given by:

\[
\ddot{u}(t) + 2\xi \omega \dot{u}(t) + \omega^2 (\alpha u(t) + (1 - \alpha)z(t)) = -\ddot{u}_e(t) \tag{2.13}
\]
\[ \ddot{z}(t) = -\gamma|\dot{u}(t)||z(t)|^{n-1}z(t) - \beta|z(t)|^n\dot{u}(t) + A\dot{u}(t) \quad (2.14) \]

where the support/ground acceleration \( \ddot{u}_g(t) \) is modeled as a train of impulses

\[ \sum_{0}^{n} -f_i \delta(t_i - t) \]

as described in the previous section. State variable \( 'z(t)' \) is a hysteretic displacement evolving according to (2.14) and defining the hysteretic component of the restoring force. Parameters \( \omega \) and \( \xi \) are the initial natural circular frequency and the damping ratio of the system. Parameter \( \alpha \) determines the extent to which the restoring force is contributed by the hysteretic variable \( z(t) \). Parameters \( \gamma \), \( \beta \), \( n \) and \( A \) are shape parameters that can be adjusted to match an observed hysteresis loop, see Fig. 2.2. In this study, a value of \( n = 1 \) is assumed. The system is assumed to be initially at rest (i.e., at time \( t_0 = 0 \)). For the response sensitivity analysis, the system parameters considered are \( \omega \), \( \xi \) and \( \gamma \), while the loading parameters are \( f_i \), \( i = 0, 1, 2, ... \).

Fig. 2.2  Restoring Force versus Displacement for the Bouc-Wen Oscillator
2.3.1 Computation of the Nonlinear Hysteretic Response

The Newmark-β method with parameters \( \alpha = 0.5 \) and \( \beta = 0.25 \) is used to integrate the nonlinear equation of motion (2.13) of the Bouc-Wen oscillator. The interpolations for displacement and velocity responses and the equation of dynamic equilibrium at time \( t_{i+1} = i\Delta t \), are given by:

\[
\begin{align*}
  u_{i+1} &= 0.25(\ddot{u}_{i+1} + \dot{u}_i)\Delta t^2 + \dot{u}_i\Delta t + u_i \\
  \dot{u}_{i+1} &= 0.5(\dddot{u}_{i+1} + \ddot{u}_i)\Delta t + \ddot{u}_i \\
  \dddot{u}_{i+1} &= -\dddot{u}_p(t_{i+1}) - 2\xi\omega u_{i+1} - \omega^2(\alpha u_{i+1} + (1 - \alpha)z_{i+1})
\end{align*}
\]  

(2.15)

Since the support/ground excitation is modeled as a train of impulses, the velocity response at \( t_i^+ \) (just after application of the impulse \( f_i\delta(t_i - t) \)) is obtained exactly as \( u_i^+ = u_i^- + f_i \), as previously. Thus, the excitation is converted into an equivalent update of the velocity response at time \( t_i^+ \) and the oscillator is in free vibration from \( t_i^+ \) to \( t_{i+1}^+ \). For free vibration, equations (2.15) become:

\[
\begin{align*}
  u_{i+1} &= 0.25(\ddot{u}_{i+1} + \dot{u}_i^+)\Delta t^2 + \dot{u}_i^+\Delta t + u_i \\
  \dot{u}_{i+1} &= 0.5(\dddot{u}_{i+1} + \ddot{u}_i^+)\Delta t + \ddot{u}_i^+ \\
  \dddot{u}_{i+1} &= -2\xi\omega u_{i+1}^+ - \omega^2(\alpha u_{i+1} + (1 - \alpha)z_{i+1})
\end{align*}
\]  

(2.16)

Consider the following numerical integration scheme for the hysteretic variable \( z(t) \).

\[
z_{i+1} = 0.5(\dot{z}_i^- + \dot{z}_i^+)\Delta t + z_i
\]  

(2.17)

Substituting (2.14) evaluated at \( t_{i+1}^+ \) in (2.17) gives:

\[
z_{i+1} = (0.5\dot{z}_i^+\Delta t + z_i) + 0.5\Delta t(-\gamma|\dot{z}_{i+1}^-|z_{i+1}^- - \beta|z_{i+1}^-|\dot{u}_{i+1}^- + A\dddot{u}_{i+1}^-)
\]  

(2.18)

The initial conditions at time \( t_0^+ = 0^+ \) are:
\[ u(0) = 0 \]
\[ \dot{u}(0^+) = f_0 \]
\[ \ddot{u}(0^+) = -2\omega \xi \dot{u}(0^+) \]
\[ z(0) = 0 \]
\[ \dot{z}(0^+) = A \dot{u}(0^+) \]

Expressing \( \ddot{u}_{i+1} \) in terms of \( u_{i+1} \) in (2.16)_1 and substituting for \( \ddot{u}_{i+1} \) in terms of \( u_{i+1} \) in (2.16)_2, \( u_{i+1} \) can be expressed in terms of \( \dot{u}_{i+1} \) as:

\[ u_{i+1} = (\dot{u}_{i+1} - (0.5 \dot{u}_i^+ \Delta t + \dot{u}_i^+)) \frac{\Delta t}{2} + 0.25 \dot{u}_i^+ \Delta t^2 + \dot{u}_i^+ \Delta t + u_i \]  

(2.20)

Substituting for \( \ddot{u}_{i+1} \) from (2.16)_3 in (2.16)_1 and \( u_{i+1} \) from (2.16)_1 in (2.16)_2, a single equation in terms of \( z_{i+1} \) and \( \dot{u}_{i+1} \) is obtained.

\[ a \dot{u}_{i+1} + b z_{i+1} = 2 \frac{k_2}{\Delta t} + \omega^2 \alpha (0.5 k_2 \Delta t - k_1) \]  

(2.21)

where

\[ a = 2/\Delta t + 2\xi \omega + 0.5 \omega^2 \alpha \Delta t \]
\[ b = \omega^2 (1 - \alpha) \]
\[ k_1 = 0.25 \dot{u}_i^+ \Delta t^2 + \dot{u}_i^+ \Delta t + u_i \]
\[ k_2 = 0.5 \dot{u}_i^+ \Delta t + \dot{u}_i^+ \]

Equations (2.18) and (2.21) are solved for \( z_{i+1} \) and \( \dot{u}_{i+1}^+ \) using a simple predictor-corrector method and choosing \( \dot{u}_i^+ \) and \( z_i \) as the initial predictors. Once \( \dot{u}_{i+1}^+ \) is obtained, \( \ddot{u}_{i+1}^+ \) is determined from (2.4) and \( u_{i+1} \) is determined from (2.20). According to (2.14) evaluated at \( t_{i+1}^+ \) and (2.16)_3 the updates for \( \dot{z}_{i+1}^+ \) and \( \dot{u}_{i+1}^+ \) are as follows:
\[ z_{i+1}^+ = -\gamma|u_{i+1}^+|z_{i+1}^+ - \beta|u_{i+1}^+|u_{i+1}^+ + A\dot{u}_{i+1}^+ \]
\[ \dot{u}_{i+1}^+ = -2\xi\omega u_{i+1}^+-\omega^2(\alpha u_{i+1}^+ + (1-\alpha)z_{i+1}) \]  
(2.22)

Notice that \( z_{i+1} = z_{i+1}^+ = z_{i+1} \) and \( u_{i+1} = u_{i+1}^+ = u_{i+1} \).

In the computation of the hysteretic response, the time at which the interpolated velocity changes sign is determined exactly. The change in sign or zero-crossing of the velocity response within a time step \( (t_i^+, t_{i+1}^-) \) is identified by the first corrector for velocity at \( t_{i+1} \), obtained at the end of one iteration. Defining \( \delta t \) \( (0 \leq \delta t \leq \Delta t) \), the local time within a time step at which the velocity vanishes, using the subscript \( 'j' \) to denote various response quantities at that instant, and using the same numerical integration scheme as in any typical step, we find:

\[ \dot{u}_j = 0 \]
\[ \dot{z}_j = 0 \]
\[ z_j = (z_i + 0.5\delta t) \]
\[ b z_j = \ddot{u}_i + 2\dot{u}_i / \delta t - \omega^2 \alpha u_i - 0.5 \dot{u}_i^+ \omega^2 \delta t \alpha \]

(2.23)

The expression for \( z_j \) in (2.23)_3 is substituted in (2.23)_4 which yields the following algebraic equation quadratic in \( \delta t \).

\[ (\omega^2 \alpha \dot{u}_i + b \ddot{z}_j) \delta t^2 + 2(b z_i + \omega^2 \alpha u_i - \ddot{u}_i) \delta t - 4 \ddot{u}_i = 0 \]

(2.24)

Among the two possible solutions for \( \delta t \), the solution that lies in the interval \( (0, \Delta t) \) is chosen.

After (2.24) is solved for \( \delta t \), the quantities at \( t_{i+1} \) are obtained as for any typical time step using a predictor-corrector scheme with \( (\Delta t - \delta t) \) used instead of \( \Delta t \) in (2.18) and (2.21)
and in the definition of the variables \(a, b, k_1\) and \(k_2\), and choosing \(\hat{u}_j\) and \(z_j\) as initial predictors for \(\hat{u}_{i+1}^+\) and \(z_{i+1}^+\). Replacing \(\Delta t\) with \((\Delta t - \delta t)\), subscript ‘i’ with ‘j’ and substituting \(\hat{u}_j = 0\) in (2.20), and using \(\hat{u}_{i+1}^+ = 0.5(\hat{u}_{i+1}^- + \hat{u}_j)(\Delta t - \delta t)\) from (2.16)\(\text{2}\), the displacement response \(u_{i+1}^-\) is obtained as:

\[
u_{i+1}^- = 0.5(\hat{u}_{i+1}^-)(\Delta t - \delta t) + u_j
\]

(2.25)

According to (2.14) evaluated at \(t_{i+1}^-\) and (2.16)\(\text{3}\) the updates for \(z_{i+1}^+\) and \(\hat{u}_{i+1}^+\) are as follows:

\[
\begin{align*}
z_{i+1}^+ &= -\gamma|\hat{u}_{i+1}^+||z_{i+1}^- - \beta|z_{i+1}^-|\hat{u}_{i+1}^- + \beta u_{i+1}^+ \\
\hat{u}_{i+1}^+ &= -2\xi \omega \hat{u}_{i+1}^- - \omega^2(\alpha u_{i+1}^- + (1 - \alpha)z_{i+1}^-)
\end{align*}
\]

(2.26)

### 2.3.2 Computation of Gradients of the Nonlinear Hysteretic Response

This section deals with the computation of the exact gradients with respect to both loading and system parameters of the nonlinear hysteretic response obtained using the numerical scheme presented in the previous section. The system sensitivity parameters considered are \(\omega, \xi\) and \(\gamma\), while the loading parameters are the magnitudes of the impulses \(f_i, i = 0, 1, 2, \ldots\). Differentiating (2.21), the numerical response equation with respect to the sensitivity parameter ‘\(x\)’ which is either a system or a loading parameter we have:

\[
\frac{\partial \hat{u}_{i+1}^-}{\partial x} + \hat{u}_{i+1}^- \frac{\partial a}{\partial x} + z_{i+1}^+ \frac{\partial b}{\partial x} + b \frac{\partial z_{i+1}^+}{\partial x} = \frac{2}{\Delta t} \frac{\partial k_2}{\partial x} + 2\omega \alpha(0.5k_2\Delta t - k_1) \frac{\partial \omega}{\partial x}
\]

\[
+ \omega^2 \alpha \left( \frac{\Delta t}{2} \frac{\partial k_1}{\partial x} - \frac{\partial k_1}{\partial x} \right)
\]

(2.27)

where
\[
\frac{\partial a}{\partial x} = \left( 2x \frac{\partial \omega}{\partial x} + 2\omega \frac{\partial x}{\partial x} + \frac{\partial \omega}{\partial x} \alpha \Delta t \right)
\]  

(2.28)

\[
\frac{\partial b}{\partial x} = 2\omega \frac{\partial \omega}{\partial x} (1 - \alpha)
\]  

(2.29)

\[
\frac{\partial k_1}{\partial x} = \frac{\partial u_i^+ \Delta t}{\partial x} + \frac{\partial u_i^+}{\partial x} \Delta t + \frac{\partial u_i}{\partial x}
\]  

(2.30)

\[
\frac{\partial k_2}{\partial x} = \frac{\partial u_i^+ \Delta t}{\partial x} + \frac{\partial u_i^+}{\partial x}
\]  

(2.31)

To differentiate (2.18) with respect to \(x\), the term \(|z_{i+1}|\) has to be smoothened. This smoothening is done in such a way that the system behavior is not altered and continuous differentiability of \(|z_{i+1}|\) is ensured. For this purpose, the absolute-value term \(|z_{i+1}|\) is replaced by the following smooth function \(\psi(z_{i+1})\):

\[
\psi(z_{i+1}) = \begin{cases} 
  z_{i+1} & \text{for } \varepsilon \leq |z_{i+1}| \\
  \frac{\varepsilon}{2} + \frac{1}{2\varepsilon} z_{i+1}^2 & \text{for } \varepsilon \geq |z_{i+1}|
\end{cases}
\]  

(2.32)

In the limit when \(\varepsilon \to 0\), the smooth function \(\psi(z_{i+1})\) asymptotically approaches \(|z_{i+1}|\).

Smoothening is not performed for \(|\dot{u}_{i+1}|\), since the time at which the velocity vanishes is exactly located and the sign of the velocity is taken care of accordingly, carrying the discontinuity in the derivative \(\frac{\partial |\dot{u}_{i+1}|}{\partial x}\) along.

Differentiating (2.18) with respect to \('x'\)

\[
\frac{\partial z_{i+1}}{\partial x} = \left( \frac{\partial z^+_i \Delta t}{\partial x} + \frac{\partial z_i}{\partial x} \right) - \frac{\partial \gamma |\dot{u}_{i+1}| |z_{i+1}| \Delta t}{\partial x} - \frac{\partial |\dot{u}_{i+1}|}{\partial x} \frac{\partial z_{i+1} \Delta t}{\partial x} - \gamma |\dot{u}_{i+1}| \frac{\partial z_{i+1} \Delta t}{\partial x}
\]  

\[
- \beta \frac{\partial \psi(z_{i+1})}{\partial x} \dot{u}_{i+1} \frac{\partial \psi(z_{i+1})}{\partial x} - \beta \frac{\partial \psi(z_{i+1})}{\partial x} \frac{\partial \dot{u}_{i+1} \Delta t}{\partial x} + A \frac{\partial \dot{u}_{i+1} \Delta t}{\partial x}
\]  

(2.33)
Now, (2.27) and (2.33) are solved for \( \frac{\partial z_{i+1}}{\partial x} \) and \( \frac{\partial u_{i+1}}{\partial x} \) using a predictor-corrector method. Differentiating (2.20), the response gradient \( \frac{\partial u_{i+1}}{\partial x} \) is obtained as:

\[
\frac{\partial u_{i+1}}{\partial x} = \left( \frac{\partial u_{i+1}}{\partial x} - \left( 0.5 \frac{\partial u_{i+1}^*}{\partial x} \Delta t + \frac{\partial u_{i+1}^*}{\partial x} \right) \frac{\Delta t}{2} + 0.25 \frac{\partial u_{i+1}^*}{\partial x} \Delta t^2 + \frac{\partial u_{i+1}^*}{\partial x} \Delta t + \frac{\partial u_{i+1}}{\partial x} \right)
\]

(2.34)

The updating from \( t_i^* \) to \( t_{i+1}^* \) is performed as follows, see (2.4) and (2.22):

\[
\begin{align*}
\frac{\partial u_{i+1}^*}{\partial x} &= \frac{\partial u_{i+1}}{\partial x} + \frac{\partial f_{i+1}}{\partial x} \\
\frac{\partial u_{i+1}^*}{\partial x} &= -2\xi \frac{\partial \omega}{\partial x} \frac{\partial z_{i+1}}{\partial x} - 2\omega \frac{\partial z_{i+1}}{\partial x} - 2\xi \frac{\partial \omega}{\partial x} - 2\omega \frac{\partial \omega}{\partial x} (\alpha u_{i+1} + (1 - \alpha) z_{i+1}) \\
&\quad - \omega \left( \alpha \frac{\partial u_{i+1}}{\partial x} + (1 - \alpha) \frac{\partial z_{i+1}}{\partial x} \right) \\
\frac{\partial z_{i+1}}{\partial x} &= -\frac{\partial \psi}{\partial x} \frac{\partial u_{i+1}}{\partial x} \frac{\partial z_{i+1}}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial u_{i+1}}{\partial x} \frac{\partial z_{i+1}}{\partial x} \\
&\quad - \beta \frac{\partial \psi}{\partial x} \frac{\partial u_{i+1}}{\partial x} \frac{\partial z_{i+1}}{\partial x} + \beta \psi (z_{i+1}) \frac{\partial u_{i+1}}{\partial x} + \frac{\partial u_{i+1}^*}{\partial x}
\end{align*}
\]

(2.35)

The time interval \( (t_i^*, t_{i+1}^*) \) in which the velocity vanishes is handled separately as was done for the response calculation. In this case, \( \delta t \) (the time from the beginning of the regular time step until the zero-crossing of the velocity) is also a function of the sensitivity parameter \( 'x' \) and the derivative of \( \delta t \) with respect to \( 'x' \) is needed to update the gradient of the state variables at \( t_{i+1}^* \).

\( \frac{\partial (\delta t)}{\partial x} \) is found by differentiating (2.24) with respect to \( 'x' \) as:
\[
\frac{\partial (\delta t)}{\partial x} = \left( -\left( 2 \omega \omega + \omega^2 \alpha \frac{\partial \omega}{\partial x} + \frac{\partial b}{\partial x} \frac{\partial \omega}{\partial x} \right) (\delta t)^2 + 2 \left( 2 \omega (1 - \alpha) \frac{\partial \omega}{\partial x} + \omega^2 (1 - \alpha) \frac{\partial z_j}{\partial x} + 2 \omega \frac{\partial \omega}{\partial x} \alpha u_i + \omega^2 \alpha \frac{\partial u_i}{\partial x} - \frac{\partial \omega}{\partial x} \right) \delta t + \frac{4 \partial u_i}{\partial x} \right) 2(\omega^2 \alpha u_i + b z_i \delta t) + 2(b z_i + \omega^2 \alpha u_i - u_i) \right) \delta t \]
\]

Denoting with subscript 'j' the quantities at time \( t_i + \delta t \), we have \( \frac{\partial u_j}{\partial x} = 0 \), since \( \dot{u}_j = 0 \), and \( \frac{\partial z_j}{\partial x} = 0 \), since \( \dot{z}_j = 0 \). Differentiating equations (2.20) and (2.23) with respect to 'x', after replacing \( \Delta t \) with \( \delta t \) and accounting for \( \delta t \) as a function of 'x', we have

\[
\frac{\partial u_j}{\partial x} = -0.5 \frac{\partial (0.5 \dot{u}_i \delta t + \dot{u}_i)}{\partial x} \delta t - 0.5 (0.5 \dot{u}_i \delta t + \dot{u}_i) \frac{\partial (\delta t)}{\partial x} + \frac{\partial (0.25 \dot{u}_i (\delta t)^2 + \dot{u}_i \delta t + u_i)}{\partial x} \\
\frac{\partial z_j}{\partial x} = \frac{\partial z_i}{\partial x} + 0.5 \frac{\partial z_j}{\partial x} \delta t + 0.5 \dot{z}_j \frac{\partial (\delta t)}{\partial x} \]
\]

Differentiating the equation of dynamic equilibrium at \( t_i + \delta t \) with respect to x, we have:

\[
\frac{\partial \dot{u}_j}{\partial x} = -\omega^2 \left( \alpha \frac{\partial u_j}{\partial x} + (1 - \alpha) \frac{\partial z_j}{\partial x} \right) - 2 \omega \frac{\partial \omega}{\partial x} (\alpha u_j + (1 - \alpha) z_j) 
\]

Using \( \frac{\partial}{\partial x} (\Delta t - \delta t) = -\frac{\partial (\delta t)}{\partial x} \), the gradients are updated from \( t_j \) to \( t_{i+1} \) by solving the following two coupled algebraic equations for \( \frac{\partial \dot{u}_j}{\partial x} \) and \( \frac{\partial z_j}{\partial x} \) iteratively. These two equations are obtained by differentiating with respect to 'x' equations (2.21) and (2.18), respectively, as applied to the time step \( (t_j, t_{i+1}) \) of size \( (\Delta t - \delta t) \).
\[
(2 + 2\xi\omega(\Delta t - \delta t) + 0.5\omega^2\alpha(\Delta t - \delta t)^2)\frac{\partial u_{i+1}^-}{\partial x} + \omega^2(1 - \alpha)(\Delta t - \delta t)\frac{\partial z_{i+1}^-}{\partial x} = \\
\left(2\xi\omega\frac{\partial(\delta t)}{\partial x} + \frac{\partial z_{i+1}^-}{\partial x} + \omega^2\alpha(\Delta t - \delta t) - 2\xi\frac{\partial \omega}{\partial x}(\Delta t - \delta t) - 2\omega\frac{\partial \xi}{\partial x}(\Delta t - \delta t) - \omega\frac{\partial \omega}{\partial x} \alpha(\Delta t - \delta t)^2\right)\frac{\partial \tilde{u}_{i+1}^-}{\partial x} \\
+ \omega^2(1 - \alpha)z_{i+1}^- \frac{\partial \delta t}{\partial x} - 2\omega \frac{\partial \omega}{\partial x}(1 - \alpha)z_{i+1}^- (\Delta t - \delta t)
\]

(2.39)

\[
-\omega^2\alpha \frac{\partial u_j^-}{\partial x}(\Delta t - \delta t) - 2\omega \frac{\partial \omega}{\partial x} \alpha u_j(\Delta t - \delta t)
\]

\[
\frac{\partial z_{i+1}^-}{\partial x} = \frac{\partial z_j}{\partial x} - 0.5 \frac{\partial \delta t}{\partial x} (-\gamma |\tilde{u}_{i+1}^-| z_{i+1}^- - \beta |z_{i+1}^-| \tilde{u}_{i+1}^- + A \dot{u}_{i+1}^-) + \\
0.5(\Delta t - \delta t) \left( -\frac{\partial \gamma}{\partial x} |\tilde{u}_{i+1}^-| z_{i+1}^- - \frac{\partial |\tilde{u}_{i+1}^-|}{\partial x} z_{i+1}^- \right) + \\
0.5(\Delta t - \delta t) \left( -\gamma |\tilde{u}_{i+1}^-| \frac{\partial z_{i+1}^-}{\partial x} - \beta \frac{\partial \psi(z_{i+1}^-)}{\partial x} \dot{u}_{i+1}^- - \beta \psi(z_{i+1}^-) \frac{\partial u_{i+1}^-}{\partial x} + A \frac{\partial \tilde{u}_{i+1}^-}{\partial x} \right)
\]

(2.40)

The gradient of the displacement response \(\frac{\partial u_{i+1}^-}{\partial x}\), obtained by differentiating (2.25) is given by:

\[
\frac{\partial u_{i+1}^-}{\partial x} = \frac{\partial u_j}{\partial x} + 0.5 \frac{\partial \tilde{u}_{i+1}^-}{\partial x}(\Delta t - \delta t) - 0.5 \tilde{u}_{i+1}^- \frac{\partial (\delta t)}{\partial x}
\]

(2.41)

\(\frac{\partial \tilde{u}_{i+1}^-}{\partial x}, \frac{\partial \dot{u}_{i+1}^-}{\partial x}, \) and \(\frac{\partial z_{i+1}^-}{\partial x}\) are updated as in (2.35).

Here again, it is to be noted that the derivatives of the system variables with respect to the loading variables are zero and vice versa. Also, \(\frac{\partial u_i}{\partial f_j} = 0\) for \(j > i\).

2.4 Rule-Based Bilinear Hysteretic Model

The equation of motion of a SDOF oscillator with a rule-based bilinear hysteretic restor-
The restoring force is given by:

\[ m \ddot{u} + c \dot{u} + R(\dot{u}, u) = -m \ddot{u}_g(t) \]  \hspace{1cm} (2.42)

in which \( R(\dot{u}, u) \) denotes the restoring force which depends on whether the state of the system is elastic or plastic; \( m \) and \( c \) are the mass and damping coefficient of the system.

The support/ground excitation \( \ddot{u}_g(t) \) is modeled as an impulse train \( \sum_{0}^{n} -f_i \delta(t - t_i) \) as in the previous sections. The restoring force model is mathematically defined as follows:

**Elastic state:**

\[ R(\dot{u}, u) = (k_2 - k_1)u_c + k_1u \]  \hspace{1cm} (2.43)

**Plastic state:**

\[ R(\dot{u}, u) = (k_1 - k_2) \text{sgn}(\ddot{u})u_y + k_2u \]  \hspace{1cm} (2.44)

where the symbols \( u_y, u_c, k_1, \) and \( k_2 \) are defined in Fig. 2.3

![Restoring Force versus Displacement](image)

**Fig. 2.3** Restoring Force versus Displacement
From Fig. 2.3, it is clear that \( k_1 \) and \( k_2 \) denote the pre-yield and post-yield stiffness, respectively. Displacement \( u_y \) and \( R_y = k_1 u_y \) are referred to as the yield displacement and yield strength respectively, while \( u_c \) denotes the position of the center of yielding. In integrating the equation of motion the system is assumed to be initially (i.e., at time \( t_0' = 0' \) at rest.

### 2.4.1 Computation of the Nonlinear Hysteretic Response

As in the Bouc-Wen model, the Newmark-\( \beta \) method with parameters \( \alpha = 0.5 \) and \( \beta = 0.25 \) is used to integrate the equation of motion of the bilinear hysteretic SDOF oscillator. The interpolations for velocity and displacement and the equation of dynamic equilibrium at discrete time \( t_{i+1} = i\Delta t \) are:

\[
\begin{align*}
    u_{i+1} & = 0.25(\ddot{u}_{i+1} + \dot{u}_i)\Delta t^2 + \dot{u}_i \Delta t + u_i \\
    \dot{u}_{i+1} & = 0.5(\ddot{u}_{i+1} + \dot{u}_i)\Delta t + \dot{u}_i \\
    m\dddot{u}_{i+1} & = -m\ddot{u}(t_{i+1}) - c\dot{u}_{i+1} - R(\dot{u}_i, u_{i+1})
\end{align*}
\]  

(2.45)

The restoring force \( R(\dot{u}_i, u_{i+1}) \) is given by (2.43) or (2.44) depending on whether the system is in an elastic or plastic state.

Similar to the Bouc-Wen model, the impulse train input is converted into an equivalent update of the velocity response as \( \dot{u}^*_{i+1} = \dot{u}^*_{i} + f_i \). Equations (2.45) for free vibration expressed at time \( t_{i+1} \) become:
\[ u_{i+1} = 0.25(\ddot{u}_{i+1} + \ddot{u}_i^+)\Delta t^2 + \dot{u}_i^+ \Delta t + u_i \]
\[ \dot{u}_{i+1} = 0.5(\ddot{u}_{i+1} + \ddot{u}_i^+)\Delta t + \dot{u}_i^+ \] (2.46)
\[ m\dddot{u}_{i+1} = -c\dddot{u}_{i+1} - R(\ddot{u}_i, u_{i+1}) \]

The equivalent initial conditions are:
\[ \begin{align*}
  u(0) &= 0 \\
  \dot{u}(0^+) &= f_0 \\
  \ddot{u}(0^+) &= \frac{c}{m} \dot{u}(0^+) 
\end{align*} \] (2.47)

Equations (2.46) are solved iteratively using a predictor-corrector algorithm to obtain \(u_{i+1}, \dot{u}_{i+1}^+, \text{ and } \ddot{u}_{i+1}^+\). The velocity update from \(t_i\) to \(t_{i+1}\) is performed as:
\[ \dot{u}_{i+1}^+ = \dot{u}_{i+1}^0 + f_{i+1} \] (2.48)

The corresponding update for the acceleration response is:
\[ \dddot{u}_{i+1}^+ = \frac{1}{m}(-c\dddot{u}_{i+1}^0 - R(\ddot{u}_i, u_{i+1})) \] (2.49)

In the computation of the response, the local time \(\delta t \ (0 \leq \delta t \leq \Delta t)\) within a regular time step \((t_i, t_{i+1})\), at which the system becomes plastic from an elastic state is determined exactly. The local time \(\delta t\) within a regular time step \((t_i, t_{i+1})\) at which the interpolated velocity response vanishes is also determined exactly. The quantities at those exact yielding and elastic unloading events are denoted generically with the subscript 'j'.

A yielding event (i.e. transition point from elastic to plastic state) within a regular time step is detected by solving (2.46) with elastic restoring force given by (2.43) and comparing the displacement obtained \(u_{i+1}\) with \(u_{ei} + \text{sgn}(\ddot{u}_i^+)u_y\), the threshold displacement for yielding, accounting for the current center of yielding \(u_{ei}\) and the yielding direction.
Expressing (2.46) for time $t_i^* + \delta t$ instead of $t_{i+1}^*$, solving (2.46)$_1$ for $\bar{u}_j$ in terms of $u_j$ and substituting in (2.46)$_2$ to express $\dot{u}_j$ also in terms of $u_j$, and substituting these expressions for $\ddot{u}_j$ and $\dot{u}_j$ in (2.46)$_3$ yields:

$$4mu_j - m\delta t^2 \ddot{u}_i^+ - 4m\dot{u}_i^+ \delta t - 4m_1 c \delta t - 2cu_i \delta t - c\dot{u}_i^+ \delta t^2 + R(\dot{u}_i^+, u_j) \delta t^2 = 0 \quad (2.50)$$

Since $u_j = u_{ci} + \text{sgn}(\dot{u}_i^+) u_y$ and $R(\dot{u}_i^+, u_j)$ can be obtained from (2.43), equation (2.50) has only $\delta t$ as the unknown. This equation, quadratic in $\delta t$ is solved for $\delta t$ and the root that lies in the interval $(0, \Delta t)$ is chosen.

The variable $u_c$ denotes the current position of the center of yielding. This history dependent variable is updated at time $t_{i+1}$ only if the state of the system at that time is plastic. The updating rule is given by:

$$u_{c,i+1} = u_c + (u_{i+1} - u_c - \text{sgn}(\dot{u}_i) u_y) \quad (2.51)$$

where subscript ‘j’ indicates that the most recent yielding event occurred in the regular interval $(t_j, t_{j+1})$. The system has remained in the plastic state from time $t_j + \delta t$ to $t_{i+1}$.

The term in parenthesis on the right hand side of (2.51) denotes the displacement increment from $t_j + \delta t$ to $t_{i+1}$, when the state of the system is plastic. The value of $u_c$ initially (i.e. at $t_0^* = 0^*$) is zero.

An elastic unloading event within a regular time step $(t_i^*, t_{i+1}^*)$ is detected from the corrector for $u_{i+1}$ at the end of the first iteration. The local time $\delta t$ at which the numerically interpolated velocity response vanishes is determined exactly. Expressing (2.46) for the
time \( t_i^* + \delta t \) instead of \( t_{i+1}^* \), and using (2.46)\(_1\) and (2.46)\(_2\) to express \( u_j \) and \( \dot{u}_j \) in terms of the quantities at \( t_i \) knowing that \( \dot{u}_j = 0 \), we have:

\[
m\left(-2\frac{\dot{u}_i^*}{\delta t} - \dot{u}_i^*\right) = -(k_1 - k_2) \text{sgn}(\dot{u}_i^*)u_y + k_2(0.5\dot{u}_i^*\delta t + u_i) \tag{2.52}
\]

Equation (2.52), quadratic in \( \delta t \) is solved for \( \delta t \) and the root that lies in the interval \((0, \Delta t)\) is chosen.

The response quantities \( u_{i+1}, \dot{u}_{i+1}^* \) and \( \ddot{u}_{i+1}^* \) at the end of \( t_{i+1}^* \) of the regular time step are obtained from (2.46), by replacing \( \Delta t \) with \( \Delta t - \delta t \) and subscript ‘i’ with ‘j’, and solving for them iteratively.

### 2.4.2 Computation of Gradients of the Nonlinear Hysteretic response

The gradients of the nonlinear hysteretic response are computed with respect to the loading parameters \( f_i, \ i = 0, 1, 2, ..., \) describing the train of impulses \( \sum_{i=0}^{a} f_i \delta(t - t_i) \) and the parameters characterizing the system. The system sensitivity parameters considered here are \( m, c \) and \( u_y \).

Differentiating with respect to the sensitivity parameter ‘x’, the numerical integration scheme (2.46) used for computing the response of the system, we have:

\[
\frac{\partial u_{i+1}}{\partial x} = 0.25\left(\frac{\partial \dot{u}_{i+1}^*}{\partial x} + \frac{\partial \ddot{u}_i^*}{\partial x}\right)\Delta t^2 + \frac{\partial \dot{u}_i^*}{\partial x} \Delta t + \frac{\partial u_i}{\partial x} \tag{2.53}
\]

\[
\frac{\partial \dot{u}_{i+1}^*}{\partial x} = 0.5\left(\frac{\partial \ddot{u}_{i+1}^*}{\partial x} + \frac{\partial \dot{u}_i^*}{\partial x}\right)\Delta t + \frac{\partial \ddot{u}_i^*}{\partial x} \tag{2.54}
\]
\[
\frac{\partial u_i^{+}}{\partial x} + 1 = \frac{1}{m} \left( -\frac{\partial c}{\partial x} u_i^{+} + \frac{\partial m}{\partial x} u_i^{+} + \frac{\partial u_i^{+}}{\partial x} - \frac{\partial R(u_i^{+}, u_{i+1})}{\partial x} \right) \tag{2.55}
\]

The gradient of the restoring force is given from (2.43) and (2.44):

Elastic state:

\[
\frac{\partial}{\partial x} R(u_i^{+}, u_{i+1}) = (k_2 - k_1) \frac{\partial u_{\varepsilon i}}{\partial x} + k_1 \frac{\partial u_i^{+}}{\partial x} \tag{2.56}
\]

Plastic state:

\[
\frac{\partial}{\partial x} R(u_i^{+}, u_{i+1}) = k_2 \frac{\partial u_i^{+}}{\partial x} + (k_1 - k_2) \text{sgn}(u_i^{+}) \frac{\partial u_i^{+}}{\partial x} \tag{2.57}
\]

In the derivation of equation (2.57), the term \(\frac{\partial (\text{sgn}(u_i^{+}))}{\partial x}\) vanishes. When the current state of the system is plastic, a change of sign in the velocity response corresponds a change of state of the system from plastic to elastic. Therefore, as long as the state of the system is plastic (without changing to elastic) the sign of the velocity response does not change and hence the partial derivative \(\frac{\partial (\text{sgn}(u_i^{+}))}{\partial x}\) is zero.

Equations (2.53), (2.54) and (2.55) are solved for the displacement, velocity and acceleration gradients at \(t_{i+1}^{\ast}\) using an iterative predictor-corrector. From (2.48) and (2.46)

applied at \(t_{i+1}^{\ast}\), the following derivative updates from \(t_{i+1}^{\ast}\) to \(t_{i+1}^{\ast}\) are made

\[
\frac{\partial u_i^{+}}{\partial x} + 1 = \frac{\partial u_i^{+}}{\partial x} + \frac{\partial f_{i+1}}{\partial x}
\]

(2.58)

A regular interval \((t_i^{\ast}, t_{i+1}^{\ast})\) in which the system changes state from elastic to plastic is
handled separately for the response gradient calculations. In this case, \( \delta t \), the time from the beginning of the time step to the yielding event is also a function of 'x' and derivatives of \( \delta t \) with respect to 'x' must be determined in order to update the state variables at \( t_{i+1} \).

Differentiating (2.50) with respect to 'x' we find the following expression for \( \frac{\partial (\delta t)}{\partial x} \):

\[
\frac{\partial (\delta t)}{\partial x} = \frac{1}{[-2m\delta t\ddot{u}_i^+ - 4m\dot{u}_i^+ - 2c\dot{u}_i^+ \delta t + 2cu_j - 2cu_i + 2R(\dot{u}_i^+, u_j)\delta t]} \times \\
\left( \begin{align*}
    m\delta t^2 \frac{\partial \dot{u}_i^+}{\partial x} + 4m\delta t \frac{\partial u_i^+}{\partial x} + 4m \frac{\partial u_i}{\partial x} + c\delta t^2 \frac{\partial u_i^+}{\partial x} + 2c\delta t \frac{\partial u_i}{\partial x} - 2c\delta t \frac{\partial u_i^c}{\partial x} \\
    -4m\delta t \frac{\partial u_i^c}{\partial x} - k_i \frac{\partial u_i}{\partial x} \delta t^2 - 4m \frac{\partial m}{\partial x} \dot{u}_j \frac{\partial m}{\partial x} + 4u_i^+ \delta t \frac{\partial m}{\partial x} + 4u_i^+ \frac{\partial m}{\partial x} \\
    -2u_j \delta t \frac{\partial c}{\partial x} + 2u_i \delta t \frac{\partial c}{\partial x} + \ddot{u}_j^+ \delta t^2 \frac{\partial c}{\partial x} - 4m \text{sgn}(u_i^+) \frac{\partial u_j^+}{\partial x} - 2c \text{sgn}(u_i^+) \delta t \frac{\partial u_j^+ y}{\partial x} - k_i \text{sgn}(u_i^+) \delta t^2 \frac{\partial u_j^+ y}{\partial x}
\end{align*} \right)
\] (2.59)

Once \( \frac{\partial (\delta t)}{\partial x} \) is known, the response derivatives at \( t_i + \delta t \), namely \( \frac{\partial u_i}{\partial x}, \frac{\partial \dot{u}_i}{\partial x} \) and \( \frac{\partial \ddot{u}_i}{\partial x} \) are found as solutions of the following linear system of equations obtained by differentiating (2.46) with respect to 'x' expressed at \( t_j \) instead of \( t_{i+1} \).

\[
\frac{\partial u_j}{\partial x} = 0.25 \left( \frac{\partial \ddot{u}_i}{\partial x} + \frac{\partial u_i^+}{\partial x} \right) \delta t^2 + 0.5(\ddot{u}_j + \dot{u}_j^+) \frac{\partial (\delta t)}{\partial x} + \frac{\partial u_i^+}{\partial x} \delta t + \dot{u}_j \frac{\partial (\delta t)}{\partial x} + \frac{\partial u_i}{\partial x} \] (2.60)

\[
\frac{\partial \dot{u}_j}{\partial x} = 0.5 \left( \frac{\partial \ddot{u}_i}{\partial x} + \frac{\partial u_i^+}{\partial x} \right) \delta t + 0.5(\ddot{u}_j + \dot{u}_j^+) \frac{\partial (\delta t)}{\partial x} + \frac{\partial u_i^+}{\partial x} \] (2.61)

\[
\frac{\partial \ddot{u}_j}{\partial x} = \frac{1}{m} \left( -\frac{\partial c}{\partial x} \ddot{u}_j - \frac{\partial m}{\partial x} \dot{u}_j - c \frac{\partial \ddot{u}_j}{\partial x} - \frac{\partial}{\partial x} R(\dot{u}_i^+, u_j) \right) \] (2.62)

where
\[
\frac{\partial}{\partial x} R(\tilde{u}_i^+, u_j) = (k_2 - k_1) \frac{\partial u_{c,i}^+}{\partial x} + k_1 \frac{\partial u_j}{\partial x}
\]  
(2.63)

The gradient of \( u_{c,i} \), \( \frac{\partial u_{c,i}}{\partial x} \) in (2.63) is updated only when the state of the system is plastic.

It remains unchanged between a unloading event and the following yielding event. From (2.51):

\[
\frac{\partial u_{c,i}}{\partial x} = \frac{\partial u_{i+1}}{\partial x} - \text{sgn}(\tilde{u}_j) \frac{\partial u_j}{\partial x}
\]  
(2.64)

Using \( \frac{\partial}{\partial x}(\Delta t - \delta t) = -\frac{\partial(\delta t)}{\partial x} \), the response gradients at \( t_{i+1}^+ \) are determined by solving the following gradient equations derived from (2.46).

\[
\frac{\partial u_{i+1}}{\partial x} = 0.25 \left( \frac{\partial \tilde{u}_i^+}{\partial x} + \frac{\partial \tilde{u}_j}{\partial x} \right) (\Delta t - \delta t)^2 - 0.5(\tilde{u}_i^+ + \tilde{u}_j)(\Delta t - \delta t) \frac{\partial(\delta t)}{\partial x} \\
+ \frac{\partial}{\partial x}(\tilde{u}_j)(\Delta t - \delta t) - \tilde{u}_j \frac{\partial(\delta t)}{\partial x} + \frac{\partial u_j}{\partial x}
\]  
(2.65)

\[
\frac{\partial \tilde{u}_i^+}{\partial x} = 0.5 \left( \frac{\partial \tilde{u}_i^+}{\partial x} + \frac{\partial \tilde{u}_j}{\partial x} \right) (\Delta t - \delta t) - 0.5(\tilde{u}_i^+ + \tilde{u}_j) \frac{\partial(\delta t)}{\partial x} + \frac{\partial u_j}{\partial x}
\]  
(2.66)

\[
\frac{\partial \tilde{u}_j^+}{\partial x} = \frac{1}{m} \left( - \frac{\partial c}{\partial x} \tilde{u}_i^+ - \frac{\partial m}{\partial x} \tilde{u}_i^+ - c \frac{\partial \tilde{u}_i^+}{\partial x} - \frac{\partial}{\partial x} R(\tilde{u}_j, u_{i+1}) \right)
\]  
(2.67)

where

\[
\frac{\partial}{\partial x} R(\tilde{u}_j, u_{i+1}) = k_2 \frac{\partial u_{i+1}}{\partial x} + (k_1 - k_2) \text{sgn}(\tilde{u}_j) \frac{\partial u_j}{\partial x}
\]  
(2.68)

which corresponds to the system being in the plastic range.

A time interval \((t_i^+, t_{i+1}^+)\) in which elastic unloading occurs is again handled separately.

Differentiating (2.46)\(_2\) expressed at \( t_i + \delta t \), with respect to \( 'x' \), knowing \( \tilde{u}_j = 0 \) we
obtain:

\[
\frac{\partial \ddot{u}_i}{\partial x} = -\left(0.5(\dddot{u}_j + \dddot{u}_i^+)\frac{\partial (\delta t)}{\partial x} + 0.25\left(\frac{\partial \ddot{u}_j}{\partial x} + \frac{\partial \ddot{u}_i^+}{\partial x}\right)(\delta t)^2\right)
\]  

(2.69)

Differentiating (2.52) and substituting for \(\frac{\partial \ddot{u}_i}{\partial x}\) given by (2.69) we obtain an expression for \(\frac{\partial (\delta t)}{\partial x}\):

\[
\frac{\partial (\delta t)}{\partial x} = \frac{1}{\left[0.5(\dddot{u}_j + \dddot{u}_i^+) - 0.25\frac{k_2}{m}\delta t \dddot{u}_i^+\right]} \times \\
\left[0.25 \frac{k_2}{m} \delta t \left(\frac{\partial \ddot{u}_i}{\partial x} + 2 \frac{\partial u_i}{\partial x}\right) + \frac{0.5 \ddot{u}_i \delta t \delta m}{m} \left\{-0.5 \delta t \frac{\partial \ddot{u}_i^+}{\partial x} - \frac{\partial \ddot{u}_i}{\partial x}\right\} + 0.5 \left(\frac{k_1 - k_2}{m}\right) \text{sgn}(\dddot{u}_i^+) \delta t \frac{\partial u_i}{\partial x}\right]
\]  

(2.70)

Once \(\frac{\partial (\delta t)}{\partial x}\) is known, the displacement, velocity and acceleration response gradients at \(t_j = t_i + \delta t\) are obtained by solving iteratively the following three gradient equations.

\[
\frac{\partial u_j}{\partial x} = 0.25 \left(\frac{\partial \ddot{u}_j}{\partial x} + \frac{\partial \ddot{u}_i^+}{\partial x}\right)(\delta t)^2 + 0.5(\dddot{u}_j + \dddot{u}_i^+) \delta t (\frac{\partial (\delta t)}{\partial x} + \frac{\partial \ddot{u}_i^+}{\partial x} \delta t + \dddot{u}_i^+ \frac{\partial (\delta t)}{\partial x} + \frac{\partial \ddot{u}_i}{\partial x})
\]  

(2.71)

\[
\frac{\partial \ddot{u}_j}{\partial x} = 0.5 \left(\frac{\partial \ddot{u}_j}{\partial x} + \frac{\partial \ddot{u}_i^+}{\partial x}\right)(\delta t) + 0.5(\dddot{u}_j + \dddot{u}_i^+) \frac{\partial (\delta t)}{\partial x} + \frac{\partial \ddot{u}_i^+}{\partial x}
\]  

(2.72)

\[
\frac{\partial \dddot{u}_j}{\partial x} = \frac{1}{m}\left(-\frac{\partial c}{\partial x}u_j - \frac{\partial m}{\partial x} \ddot{u}_j - c \frac{\partial \ddot{u}_j}{\partial x} - \frac{\partial R(u_i^+, u_j)}{\partial x}\right)
\]  

(2.73)

where

\[
\frac{\partial R(u_i^+, u_j)}{\partial x} = k_2 \frac{\partial u_j}{\partial x} + (k_1 - k_2) \text{sgn}(\dddot{u}_i^+) \frac{\partial u_i}{\partial x}
\]  

(2.74)
Using $\frac{\partial}{\partial x}(\Delta t - \delta t) = -\frac{\partial(\delta t)}{\partial x}$, the updating of response gradients from $t_j$ to $t_{i+1}^*$ is performed by solving iteratively the following three gradient equations.

\[
\frac{\partial u_{i+1}}{\partial x} = 0.25 \left( \frac{\partial u_{i+1}}{\partial x} + \frac{\partial \bar{u}_j}{\partial x} \right) (\Delta t - \delta t)^2 - 0.5(\bar{u}_{i+1} + \bar{u}_j)(\Delta t - \delta t) \frac{\partial(\delta t)}{\partial x} + \frac{\partial}{\partial x}(\bar{u}_j)(\Delta t - \delta t) - \bar{u}_j \frac{\partial(\delta t)}{\partial x} + \frac{\partial u_j}{\partial x},
\]

(2.75)

\[
\frac{\partial \bar{u}_{i+1}}{\partial x} = 0.5 \left( \frac{\partial \bar{u}_{i+1}}{\partial x} + \frac{\partial \bar{u}_j}{\partial x} \right) (\Delta t - \delta t) - 0.5(\bar{u}_{i+1} + \bar{u}_j) \frac{\partial(\delta t)}{\partial x} + \frac{\partial \bar{u}_j}{\partial x},
\]

(2.76)

\[
\frac{\partial \bar{u}^+_i}{\partial x} = \frac{1}{m} \left( -\frac{\partial c}{\partial x} \bar{u}_{i+1} - \frac{\partial m}{\partial x} \bar{u}_{i+1}^+ - \frac{\partial \bar{u}_{i+1}^+}{\partial x} - \frac{\partial}{\partial x} R(\bar{u}_{i+1}^+, u_{i+1}) \right)
\]

(2.77)

where

\[
\frac{\partial}{\partial x} R(\bar{u}_i^+, u_{i+1}) = (k_2 - k_1) \frac{\partial u_{ci}}{\partial x} + k_1 \frac{\partial u_{i+1}}{\partial x}
\]

(2.78)

For both yielding and elastic unloading events, updates of the response gradients from $t_{i+1}^*$ to $t_{i+1}^+$ are obtained as:

\[
\frac{\partial u_{i+1}^+}{\partial x} = \frac{\partial u_{i+1}^-}{\partial x} + \frac{\partial f_{i+1}}{\partial x},
\]

(2.79)

\[
\frac{\partial \bar{u}_{i+1}^+}{\partial x} = \frac{1}{m} \left( -\frac{\partial c}{\partial x} \bar{u}_{i+1}^+ - \frac{\partial m}{\partial x} \bar{u}_{i+1}^+ - \frac{\partial \bar{u}_{i+1}^+}{\partial x} - \frac{\partial}{\partial x} R(\bar{u}_{i+1}^+, u_{i+1}) \right)
\]

where

\[
\frac{\partial}{\partial x} R(\bar{u}_{i+1}^+, u_{i+1}) = (k_2 - k_1) \frac{\partial u_{cj}}{\partial x} + k_1 \frac{\partial u_{i+1}}{\partial x}.
\]
2.5 Application Examples

The following sections provide application examples of the response gradient computation for the various SDOF models considered in the previous sections. The examples include the linear elastic SDOF oscillator, the Duffing oscillator, the Bouc-Wen oscillator and the rule based bilinear hysteretic oscillator. The support/ground excitation considered for the above cases is a train of impulses at equally spaced time intervals.

The displacement response sensitivity results obtained by solving the sensitivity equations are compared with forward finite difference results. Both system and loading sensitivity parameters are considered. For all the examples presented in this section, the Newmark-β with the parameters $\alpha = 0.5$ and $\beta = 0.25$ is used to integrate the equation of motion and compute the response sensitivities.

2.5.1 Linear Elastic SDOF Oscillator

A linear elastic SDOF oscillator is subjected to a train of 499 impulses $\sum_{n=0}^{498} f_n \delta(t - t_n)$ at every 0.005 sec as shown in Fig. 2.4. The equation of motion is integrated with a constant time step of 0.0003125 sec. The SDOF oscillator considered has a natural circular frequency $\omega = 20 \text{ rad/sec}$ and a damping ratio $\xi = 0.1$

In the previous sections, describing the numerical schemes to integrate the equation of motion and compute the response sensitivities, $\Delta t$ represented the constant time interval between two adjacent loading impulses and the constant time step for integrating the equation of motion. However, when the time interval between the occurrence of consecutive
Fig. 2.4 Impulse Train Excitation to Linear Oscillator

impulses is further divided into ‘m’ equal subintervals and each of these subintervals is used as time step for integrating the equation of motion and computing response gradients, an equivalent free vibration problem is solved for each of these sub time intervals. The updates of response quantities and their derivatives due to the loading impulses are now performed only every m subintervals.

The implementation of the displacement response sensitivity calculation with respect to both loading and system parameters is verified through comparison with finite difference computation of the response gradients. Computation of response sensitivities with respect to loading parameters $f_1$ and $f_{200}$, and system parameters $\omega$ and $\xi$ are shown in the figures below.
Fig. 2.5 Displacement Response Sensitivity with respect to Loading Variable $f_1$

Fig. 2.6 Displacement Response Sensitivity with respect to Loading Variable $f_{200}$
Fig. 2.7 Displacement Response Sensitivity with respect to Natural Frequency $\omega$

Fig. 2.8 Displacement Response Sensitivity with respect to Damping Ratio $\xi$

The sensitivities of the displacement response, $u(t)$, with respect to the loading parameters
$f_1$ and $f_{200}$ are compared with finite difference results in Fig. 2.5 and Fig. 2.6. For greater clarity, a zoom-view for each case is provided. In Fig. 2.5, as the parameter increment is decreased the sign of error between finite difference and analytical result changes indicating that there exists a parameter increment for which the error is zero. In Fig. 2.6, decreasing the parameter increment improves the match between analytical and finite difference results in the beginning and worsens it later. This is due to the so called “step-size dilemma” (Haftka and Gurdal 1993). If the parameter increment selected is too large, the truncation errors govern. On the other hand, if the parameter increment is too small, the round off errors govern. In some cases there may not be any size of the parameter increment which yields an acceptable error. Thus, computing gradients analytically is advantageous in two respects; they are computationally less expensive and accurate.

As above, the sensitivities of the displacement response, $u(t)$, with respect to the natural circular frequency $\omega$ and $\xi$ are compared with finite difference results in Fig. 2.7 and Fig. 2.8 respectively. These sensitivity results indicate that, as the parameter increment is decreased, the finite difference results converge to the analytical result obtained by solving sensitivity equations, thus verifying the implementation of the gradient computation algorithm.

**2.5.2 Duffing Oscillator**

A Duffing SDOF oscillator is subjected to the same train of impulses at every 0.005 sec as in the previous section, see Fig. 2.4. The properties of the oscillator considered are:
\[ \omega = 20 \text{ [rad/sec]} \]
\[ \xi = 0.1 [-] \]
\[ \gamma = 0.2 \text{ [in}^{-2}\text{]} \]

The equation of motion is integrated with a constant time step at every 0.0003125 sec. The response sensitivities with respect to the system parameters \( \omega \) and \( \xi \), and the loading parameters \( f_1 \) and \( f_{200} \) are computed. As for the case of a linear elastic oscillator, an irregular impulse train loading is used in order to check the stability and robustness of the numerical schemes to integrate the equation of motion and compute the response sensitivities.

The displacement response sensitivities with respect to various parameters are shown in the figures below.

![Graph showing displacement response sensitivities](image)

**Fig. 2.9** Displacement Response Sensitivity with respect to Loading Variable \( f_1 \)
Fig. 2.10 Displacement Response sensitivity with respect to Loading Variable $f_{200}$

Fig. 2.11 Displacement Response Sensitivity with respect to Natural Frequency $\omega$
Fig. 2.12 Displacement Response Sensitivity with respect to Damping Ratio $\xi$

In Fig. 2.9 and Fig. 2.10 the sensitivities of the displacement response, $u(t)$, with respect to the loading variables $f_1$ and $f_{200}$ are compared with finite results. For the range of parameter increments considered, decreasing the size of parameter increment improves the match between analytical gradients and those computed using finite difference. Similarly, in Fig. 2.11 and Fig. 2.12 where the response sensitivities with respect to $\omega$ and $\xi$ are computed, decreasing the size of parameter increment decreases the error between the analytical gradients and those computed using finite difference, thus verifying the implementation of the gradient computation algorithm.

It is interesting to compare Fig. 2.5 and Fig. 2.9 that show the displacement response sensitivities with respect to the variable $f_1$ of a linear elastic oscillator and a Duffing oscillato-
tor to the same impulse train excitation. The displacement sensitivity for the Duffing oscillator matches closely with that of the linear elastic oscillator in the beginning, when the behavior of the Duffing oscillator is close to being linear. However, when the response of the Duffing oscillator starts to become highly nonlinear, as in the later half, the displacement sensitivities are very different from that for the linear elastic oscillator.

2.5.3 Bouc-Wen SDOF Oscillator

A Bouc-Wen hysteretic oscillator with the following properties

\[ \omega = 3\pi \text{ [rad/sec]} \]
\[ \alpha = 0.6 \text{ [-]} \]
\[ \gamma = 2.581081 \text{ [in}^{-1}\text{]} \]
\[ A = 1.0 \text{ [-]} \]
\[ \beta = 2.581081 \text{ [in}^{-1}\text{]} \]
\[ \xi = 0.05 \text{ [-]} \]

is subjected to a train of impulses at every 0.025 sec as shown in Fig. 2.13. The time step for the Newmark-\(\beta\) integration of the equation of motion is 0.0015625 sec. The input of impulse train is shown in Fig. 2.13. The response to this input is shown in Fig. 2.14. The effect of \(\gamma\) as a shape parameter for the hysteresis loop is illustrated in Fig. 2.15. It is observed that \(\gamma\) controls the threshold hysteretic displacement at which yielding occurs.
**Fig. 2.13** Impulse Train Excitation to Bouc-Wen Oscillator

**Fig. 2.14** Hysteretic Displacement versus Displacement for Bouc-Wen Oscillator
Fig. 2.15 Effect of $\gamma$ on Shape of Hysteresis Loops for Bouc-Wen Oscillator

The computed response sensitivities with respect to various loading and system parameters are shown in the figures below.

Fig. 2.16 Displacement Response Sensitivity with respect to Loading Variable $f_1$
Fig. 2.17 Displacement Response Sensitivity with respect to Loading Variable $f_{200}$

Fig. 2.18 Displacement Response Sensitivity with respect to Natural Frequency $\omega$
Fig. 2.19  Displacement Response Sensitivity with respect to Damping Ratio $\xi$

Fig. 2.20  Displacement Response Sensitivity with respect to Model Parameter $\gamma$
The analytical gradients of the displacement response with respect to loading and system parameters and their comparison with finite difference results are shown in the figures above. In each case presented above, it is seen that the finite difference gradients converge asymptotically to the analytical gradients as the size of parameter increment is decreased, thus verifying the implementation of the gradient computation scheme.

2.5.4 Rule-Based Bilinear Hysteretic Oscillator

A bilinear hysteretic oscillator with a rule-based restoring force formulation is subjected to a train of impulses at every 0.025 sec as shown in Fig. 2.21. The properties of the bilinear hysteretic oscillator considered here are:

\[
\begin{align*}
    m & = 1.00 \text{ [lb]} & \text{mass} \\
    c & = 1.256637 \text{ [s}^{-1}\text{]} & \text{damping coefficient} \\
    k_1 & = 39.4784 \text{ [lb/in]} & \text{pre-yield stiffness} \\
    k_2 & = 0.01k_1 \text{ [lb/in]} & \text{post-yield stiffness} \\
    u_y & = 0.2 \text{ [in]} & \text{yield displacement}
\end{align*}
\]

The equation of motion is integrated using the Newmark-\(\beta\) with a constant step size of \(\Delta t = 0.025\) sec. The force-displacement response to the impulse train in Fig. 2.21 is shown in Fig. 2.22, which indicates multiple yielding and elastic unloading events of the system. The sensitivities of the displacement response, \(u(t)\), with respect to both loading parameters and system parameters are compared with those obtained through finite difference in Figs. 2.23 to 2.27.
**Fig. 2.21** Impulse Train Excitation to Bilinear Hysteretic Oscillator

**Fig. 2.22** Restoring Force versus Displacement
Fig. 2.23  Displacement Response Sensitivity with respect to Loading Variable $f_1$

Fig. 2.24  Displacement Response Sensitivity with respect to Loading Variable $f_{200}$
Fig. 2.25  Displacement Response Sensitivity with Respect to Mass Parameter $m$

Fig. 2.26  Displacement Response Sensitivity with Respect to Damping Coefficient $c$
Fig. 2.27 Displacement Response Sensitivity with Respect to Yield Displacement $u_y$

The comparison between analytical and finite difference gradients of the displacement response shown in the figures above indicate that in each case the finite difference gradient converges to the analytical gradient as the parameter increment is decreased, thus validating the analytical gradient computation scheme and its computer implementation.

2.6 Conclusions

This chapter discusses numerical schemes to compute the response quantities and sensitivities with respect to both loading and system parameters. The linear elastic, Duffing nonlinear elastic, Bouc-Wen hysteretic, and rule-based bilinear hysteretic SDOF oscillators are considered. The analytical gradient computation schemes for displacement response sensitivities were validated with finite difference results through application examples for
each of the four oscillator models. In each case, the input excitation in each case was modeled as an irregular train of impulses to check the stability and robustness of the numerical scheme used to integrate the equation of motion and to compute the exact sensitivities of the numerical response.

The computational schemes presented here to determine the exact sensitivities of the numerical response of SDOF dynamic systems with respect to both loading and system parameters provide a necessary ingredient of structural reliability, structural optimization and system identification.
CHAPTER 3 Calculation Of Out-Crossing Rates

3.1 Introduction

In structural reliability, the probability that the response of a structure subjected to stochastic loading exits a safe domain during a time interval is of great interest. The safe domain is defined in terms of one or more limit state functions, expressed in terms of critical response quantities. These response quantities in turn are functions of various parameters defining the system and describing the stochastic loading. An upper bound failure probability in a given time interval due to both system and loading uncertainties can be obtained through the computation of the mean rate at which the system out-crosses the safe domain as a function of time (Lin 1967).

3.2 Discrete Random Loading Model

In the present study, a white noise is considered for stochastic loading. In order to simulate numerically the response of a dynamic system excited by white noise it is necessary to define a discrete white noise process. A single realization of a discrete white noise is obtained by generating statistically independent random variables at equal time intervals. In particular if the random variables are normally distributed, the discrete white noise obtained approximates a Gaussian white noise. Two discretization methods for a Gaussian white noise process are presented in the following sections.

3.2.1 Discretization of White Noise with Random Pulse Train

Consider a zero-mean white noise process denoted by \( W(t) \) and with autocorrelation function:
\[ E[W(t_1)W(t_2)] = 2\pi \phi_0 \delta(t_1 - t_2) \]  \hspace{1cm} (3.1)

where \( \delta(.) \) denotes the Dirac delta function.

Define \( f_i \) as the integral of \( W(t) \) in the time interval \( [t_i, t_{i+1}) \). This integral can be expressed as the limit of the sum \( \sum_{t_{i+1}}^{t_i} W(t_j) \delta t_j \) where \( Max(\delta t_j) \to 0 \). If the white noise is Gaussian, then each of the random variables \( W(t_j) \) is normally distributed. The summation being a linear operator, the resulting variable \( f_i \) is also normally distributed. From the above definitions and properties of a white noise process, it follows that \( f_j \) and \( f_k \) are zero mean and statistically independent random variables \( (k \neq j) \). The variance of \( f_i \) is calculated as

\[ E(f_i^2) = \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} E[W(t_1)W(t_2)] dt_1 dt_2 \]  \hspace{1cm} (3.2)

![Fig. 3.1 Realization of a Random Impulse Discretization for White Noise](image)

Substituting (3.1) in (3.2) we get:
\[ E[f_i^2] = 2\pi\phi_0 \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \delta(t_1 - t_2) dt_1 dt_2 \]
\[ E[f_i^2] = 2\pi\phi_0 \int_{t_i}^{t_{i+1}} dt \]
\[ E[f_i^2] = 2\pi\phi_0 (t_{i+1} - t_i) \]  
(3.3)

The stationary response to the discrete impulse train defined by \( f_j, j = \ldots, -2, -1, 0, 1, 2, \ldots \) shown in Fig. 3.1 is computed as follows. The random impulse train is mathematically expressed as \( \sum_{i=0}^{n} f_i \delta(t - t_i) \). Notice that: \( \int \sum_{t_i}^{t_{i+1}} f_i \delta(t - t_i) dt = f_i \). The displacement response of a linear SDOF oscillator to an impulse train excitation is given in closed form as:

\[ u(t) = \sum_{i=0}^{n} f_i h(t - t_i) \]  
(3.4)

where \( h(t - t_i) \) denotes the impulse response function given by:

\[ h(t - t_i) = \frac{1}{\omega_d} \sin(\omega_d(t - t_i)) e^{-\frac{\xi}{\omega_d}(t - t_i)} H(t - t_i) \]  
(3.5)

In the above equation \( H(.) \) represents the Heaviside unit step function. The stationary mean square relative displacement response of a linear SDOF oscillator excited by the above random impulse train is given by:

\[ E[u^2(t)] = \sum_{i=0}^{n} \sum_{j=0}^{n} E[f_i f_j] h(t - t_i) h(t - t_j) \]
\[ E[u^2(t)] = \sum_{i=0}^{n} E[f_i^2] h^2(t - t_i) \]  
(3.6)

\[ E[u^2(t)] = 2\pi\phi_0 \left\{ \sum_{i=0}^{n} h^2(t - t_i) \right\} \Delta t \]
for \( t_n \leq t \leq t_{n+1} \).

In the limit, when \( t \to \infty \), \( n \to \infty \), and \( \Delta t \to 0 \) the summation in (3.6) can be replaced by the integral:

\[
E[u^2(t)] = 2\pi \phi_0 \int_0^\infty h^2(\tau) d\tau
\]

(3.7)

Integrating (3.7) using the expression for \( h(\tau) \) given in (3.5) gives:

\[
E[u^2(t)] = 2\pi \phi_0 \left( \frac{1}{1-\xi^2} \left( \frac{1}{2\xi \omega_3} - \frac{\xi}{2\omega_3} \right) \right)
\]

\[
E[u^2(t)] = \frac{\pi \phi_0}{2\xi \omega_3}
\]

(3.8)

which is exactly the mean squared stationary relative displacement response of a linear SDOF oscillator subjected to white noise ground motion. Thus, this consistency validates the above white noise discretization as a random pulse train.

3.2.2 Linearity Interpolated Discrete Gaussian white noise

The linear interpolation for discrete Gaussian white noise assumes that the random excitation process is linearly varying between the values \( f_i \) of the process at discrete times \( t_i \), \( i = -2, -1, 0, 1, 2, \ldots \) as shown in Fig. 3.2. These discrete random process values are characterized as statistically independent normal random variables with zero mean and variance equal to \( \sigma_f^2 \).
For the discretization of a stationary Gaussian white noise process \( W(t) \), \( \alpha_t \) shown in Fig. 3.2 is a uniformly distributed random variable between 0 and \( \Delta t \). The autocorrelation function \( R_f(\tau) \) for the linearly interpolated discrete white noise can be derived as (Clough and Penzien 1982):

\[
R_f(\tau) = \left(1 - \frac{|\tau|}{\Delta t}\right) \sigma_f^2
\]  

(3.9)

where \( \tau = t_2 - t_1 \) for any two arbitrary points in time \( t_1 \) and \( t_2 \), assuming that \( t_1 < t_2 \). In the limit as \( \Delta t \to 0 \), \( R_f(\tau) \) approaches the value \( (\sigma_f^2 \Delta t) \delta(\tau) \) in which \( \delta(\cdot) \) denotes the Dirac delta function. From (3.1), the autocorrelation function of a white noise is given exactly by \( 2\pi \phi_0 \delta(\tau) \). Hence, equivalence between the discrete-time white noise \( \{f_i\} \) and the continuous-time white noise \( W(t) \) is given by:
\[ \frac{\sigma^2 \Delta t}{2\pi} = \phi_0 \]  

(3.10)

In the present study, linear discrete white noise realizations are generated by fixing \( \alpha_r = 0 \). This relaxes the condition of stationarity, but it has negligible effects as shown by the comparison of simulated mean squared displacement and velocity responses with their theoretical counterparts in Appendix A. Piecewise linear white noise discretization shall be used for all plasticity based formulations of dynamic structural systems.

3.3 Probability of Failure Within a Time Interval

Failure of a dynamic system within a time interval (0,T) is the complement of the event that the system starts within a safe domain and has zero out-crossing of the safe domain defined by \( G(u(x, t)) > 0 \) during this period of time. This gives a basis for calculating the probability of failure \( P_f(T) \) within a time interval (0,T) through out-crossing rate calculation. Symbol \( u \) above denotes a scalar dynamic response of the system, while vector \( x \) contains all loading and system parameters which are modeled as random variables.

Thus,

\[ P_f(T) = 1 - P[G(u(x, 0)) > 0 \cap N(T) = 0] \]  

(3.11)

in which \( G(u) \) defines the state of the system, whether it fails or survives depending on whether \( G(u(x, t)) < 0 \) or \( G(u(x, t)) > 0 \). For structural reliability problems, the event \( \{ G(u(x, 0)) > 0 \} \) can be assumed to have 100% probability of occurrence; that is, the system is assumed to be within the safe domain initially (at time \( t = 0 \)). Hence, (3.11) can be simply written as (Lin 1967):

\[ P_f(T) = 1 - P(N(T) = 0) \]
\[ \sum_{n=1}^{\infty} P(N(T) = n) \leq \sum_{n=1}^{\infty} nP(N(T) = n) = E[N(T)] \]

Therefore, an upper bound of the failure probability \( P_f(T) \) is given by:

\[ P_f(T) \leq E(N(T)) = \int_0^T v(t) dt \quad (3.12) \]

where \( v(t) \) denotes the mean rate of out-crossing the safe domain or the mean out-crossing rate. The upper bound becomes tighter when \( P_f(T) \) is small and the probability of two or more outcrossing events is negligible compared to that of only one out-crossing in the time interval \((0,T)\).

### 3.4 Out-Crossing Rate as a Parallel System Sensitivity Measure

The out-crossing rate may be computed as the zero level down-crossing rate of the scalar process \( G(u(x,t)) \). Using Rice’s formula (Rice 1944), namely

\[ v(t) = \int_{-\infty}^{0} |\dot{G}| f_{GG}(G=0, \dot{G}) d\dot{G} \quad (3.13) \]

where \( f_{GG}(G=0, \dot{G}) \) is the joint probability density function of \( G(u(x,t)) \) and its time derivative \( \dot{G}(u(x,t)) \). This joint density function is extremely difficult or practically impossible to determine for a general nonlinear inelastic system and hence the Rice’s formula cannot be applied here.

Alternatively, the mean out-crossing rate can be computed from the limit formula (Hagen
et al. 1991):

\[ v(t) = \lim_{\delta t \to 0} \frac{P[(g_1 < 0) \cap (g_2 < 0)]}{\delta t} \quad (3.14) \]

where \( g_1(x) = -G(u(x, t)) \) and \( g_2(x) = G(u(x, t + \delta t)) \). The numerator is the probability that the system is in the safe domain at time \( t \) and in the unsafe domain at time \( (t + \delta t) \). For sufficiently small \( \delta t \), at most one out-crossing event is possible, and in the limit \( \delta t \to 0 \) the right hand side of (3.14) approaches the theoretical mean out-crossing rate. Thus, a numerical approximation of the out-crossing rate can be obtained by considering a finite but small \( \delta t \), computing the numerator of (3.14) and dividing it by \( \delta t \). The evaluation of the numerator corresponds to solving a time-invariant parallel system reliability problem of two components with limit state functions \( g_1 \) and \( g_2 \), respectively, at a specific time. For this purpose, the first-order reliability method (Ditlevsen and Madsen 1996) will be used.

3.5 FORM Approximation for Failure Probability

Calculation of the numerator in (3.14) involves the computation of the probability integral

\[ P = \int_{F} f(x) \, dx \quad (3.15) \]

where \( x \) denotes a set of random variables with the joint probability density function \( f(x) \), and \( F \) represents the domain of intersection of two events, i.e.,

\[ F \equiv (g_1 < 0) \cap (g_2 < 0) \quad (3.16) \]

A FORM approximation of the probability \( P \) is obtained as described in the following.
The random variables contained in the vector $\mathbf{x}$ are transformed to the standard normal variates through a one-to-one transformation $\mathbf{y} = \mathbf{y}(\mathbf{x})$ which is nonlinear in general. The standard normal variables $\mathbf{y}$ have the following joint probability density function:

$$f(\mathbf{y}) = (2\pi)^{-\frac{n}{2}} \exp(-\mathbf{y}'\mathbf{y})$$

in which $n$ denotes the number of random variables.

The limit-state surfaces $g_1(\mathbf{x}) = 0$ and $g_2(\mathbf{x}) = 0$ become in the standard normal space $g_1(x(y)) = h_1(y) = 0$ and $g_2(x(y)) = h_2(y) = 0$, respectively. These limit-state surfaces are linearized at points $\mathbf{y}_1^*$ and $\mathbf{y}_2^*$ in the standard normal space. The choice of these expansion points $\mathbf{y}_1^*$ and $\mathbf{y}_2^*$ will be described later. The linearized limit-state functions scaled by the norm of their gradient at $\mathbf{y}_1^*$ and $\mathbf{y}_2^*$ can be written as:

$$\frac{1}{\|\nabla h_i\|} h_i(y)_{\text{linearized}} = \frac{1}{\|\nabla h_i\|} h_i(y_i^*) + \frac{\nabla h_i^T(y_i^*)}{\|\nabla h_i\|} (\mathbf{y} - \mathbf{y}_i^*) , \quad i = 1, 2 \quad (3.17)$$

Equation (3.17) can be more compactly written as:

$$\frac{1}{\|\nabla h_i\|} h_i(y)_{\text{linearized}} = \beta_i - \alpha_i \mathbf{y} , \quad i = 1, 2 \quad (3.18)$$

where $\beta_i = \frac{h_i(y_i^*) - \nabla h_i^T(y_i^*)}{\|\nabla h_i\|} , \quad i = 1, 2$, is the distance from the origin to the hyperplane defined by the linearized limit-state function, while $\alpha_i = \frac{-\nabla h_i^T(y_i^*)}{\|\nabla h_i^T(y_i^*)\|} , \quad i = 1, 2$, is the unit normal vector of the hyperplane pointing towards the failure domain. When the lin-
earization point is selected on the limit-state surface, \( h_i(y^*) = 0 \) we have \( \beta_i = \alpha_i y_i^* \).

A first-order approximation of the probability \( P \) is obtained by using the linearized limit-state functions in place of the actual nonlinear limit-state functions. The probability content in the failure domain defined by the intersection of the two linearized limit-state surfaces is given by (Madsen et al. 1986):

\[
P = \Phi(-\beta_1)\Phi(-\beta_2) + \int_0^{\rho_1} \phi_2(-\beta_1, -\beta_2, \rho) d\rho
\]

(3.19)

where \( \rho_{12} = \alpha_1 \alpha_2^T \) is the correlation coefficient between the two linearized failure modes and \( \phi_2(v_1, v_2, \rho) \) denotes the bi-variate normal joint PDF with zero means, unit variances and correlation coefficient \( \rho \), i.e.,

\[
\phi_2(v_1, v_2, \rho) = \frac{1}{2\pi(1 - \rho^2)} \exp\left( -\frac{v_1^2 + v_2^2 - 2\rho v_1 v_2}{2(1 - \rho^2)} \right)
\]

(3.20)

### 3.5.1 Choice of Linearization Points

The linearization point(s) for the limit state functions are chosen in the domain \( F \) defined by the intersection of the two componental failure domains \( h_i(y) \leq 0 \) and \( h_2(y) \leq 0 \), and such that the probability density at the linearization point(s) in the standard normal space is maximum. By the property of the standard normal space which has radially exponentially decaying with the square of the distance from the origin joint density function, finding the point(s) for linearization within \( F \), with maximum probability density reduces to solving the following constrained optimization problem:
Minimize \( \|y\| \)
subject to: \((g_1(x) \leq 0)\) and \((g_2(x) \leq 0)\)

\begin{equation}
(3.21)
\end{equation}

![Diagram](image)

**Fig. 3.3** Case 1: Single Linearization Point

The objective function to be minimized being a vector norm, it is a convex function of the variables defining the standard normal space. Clearly, the objective function has an unconstrained minimum at the origin. For a convex objective function, it is well known that the constrained minimum lies on the active constraint(s) and hence on the boundary of \( F \) which provides a good check for the minimization results.

If the minimization solution point lies on the intersection of limit-state surfaces \((h_1 = 0\) and \(h_2 = 0)\) the linearization points for both the limit state functions coincide and are denoted by \(y^*\) called the design point as shown in Fig. 3.3. If the solution point lies on only one of the two limit-state surfaces i.e. satisfies only \(h_1(y^*) = 0\) for say, a lineariza-
ation point for the other limit-state function, here \( h_2(y) \), is obtained by solving a second constrained minimization problem:

\[
\begin{align*}
\text{Minimize} & \quad \|y\| \\
\text{subject to:} & \quad h_2(y) = 0 \\
& \quad h_1(y) \leq 0
\end{align*}
\] (3.22)

This case is illustrated in Fig. 3.4.

The limit-state functions \( h_1(y) \) and \( h_2(y) \) are functions of the random vector \( x = \begin{bmatrix} f^T & v^T \end{bmatrix}^T \) and the one-to-one nonlinear transformation between \( x \) and \( y \). The components \( f_i \) of the vector \( f \) represent the random loading variables, while the components \( v_i \) of the vector \( v \) represent the system variables. By definition, we have that \( g_1(x) = G(u(x, t)) \) and \( g_2(x) = G(u(x, t + \delta t)) \). Writing a first-order approximation for \( g_2(x) \)
\[ g_2(x) = G(u(x, t)) + \nabla_u G(u(x, t)) \dot{u}(x, t) \dot{\delta} t \]  

(3.23)

As already explained, the determination of the linearization point(s) for the two limit-state functions involves the solution of one or two constrained optimization problems. The most efficient algorithms for this purpose require the computation of gradients of the objective function and of the constraints. Hence it is necessary to compute the gradients of the objective function \( \|y\| \), and of the constraints \( g_1(x) \) and \( g_2(x) \) with respect to \( x \).

The gradients of the objective function \( \|y\| = (y^T \cdot y)^{\frac{1}{2}} \) is obtained using the chain rule of differentiation.

\[ \frac{\partial \|y\|}{\partial x} = \frac{\partial \|y\|}{\partial y} \frac{\partial y}{\partial x} = \frac{y}{\|y\|} J_y, x \]  

(3.24)

in which \( J_y, x \) denotes the jacobian of the transformation from the \( x \) space to the \( y \) space.

In the present work, the limit-state function \( G(x) \) is taken as

\[ G(x, t) = u(x, t) - \sigma \]  

(3.25)

where \( u(x, t) \) is a displacement response quantity and \( \sigma \) represents a deterministic threshold or critical value whose exceedance defines "failure". Hence computing gradients with respect to \( x \) of the constraint functions \( g_1(x) \) and \( g_2(x) \) amounts to computing response sensitivities with respect to \( x \), i.e., \( \frac{\partial u(x, t)}{\partial x} \) and \( \frac{\partial \dot{u}(x, t)}{\partial x} \). More specifically,
\[
\frac{\partial g_1(x)}{\partial x} = - \frac{\partial G(u(x, t))}{\partial u} \frac{\partial u(x, t)}{\partial x} = - \frac{\partial u(x, t)}{\partial x}
\]

\[
\frac{\partial g_2(x)}{\partial x} = \frac{\partial G(u(x, t))}{\partial u} + \frac{\partial}{\partial x} \left[ \nabla_u G(u(x, t)) \dot{u}(x, t) \delta t \right]
\]

\[
= \frac{\partial u(x, t)}{\partial x} + \frac{\partial}{\partial x} \left[ \dot{u}(x, t) \delta t \right]
\]

\[
= \frac{\partial u(x, t)}{\partial x} + \frac{\partial \dot{u}(x, t)}{\partial x} \delta t
\]

(3.26)

Exact and efficient computation of these response gradients with respect to both loading and system parameters is the subject of chapter 2.

3.6 Code Implementation of Mean Out-Crossing Rate Computation

A schematic representation of the various steps to compute the out-crossing rates and the different computing tools used in each step is shown in Fig. 3.5. As discussed in the previous section, it is necessary to determine the linearization points for the two limit-state functions \( h_1(y(x)) \) and \( h_2(y(x)) \) to compute the out-crossing rates. It was shown in the previous section that finding these linearization points in the standard normal space amounts to solving one or two constrained optimization problems. Most of the algorithms to solve the constrained optimization problem(s) require the evaluation of the constraints or the limit-state functions and their derivatives with respect to the variables defining the standard normal space in which the optimization is performed. For this purpose, computer programs are developed in Fortran 77 for each of the SDOF models. to evaluate the two limit-state functions, and their gradients with respect to variables defining the physical space. Several modules from the reliability analysis software STRAP, written in Fortran 77, are used to perform the transformation of the variables from the physical space to the standard normal space and vice-versa. These modules are also used to evaluate the jaco-
bian of the transformation to transform the gradients of the limit-state functions with respect to variables in the physical space to the gradients with respect to variables defining the standard normal space. The constrained optimization problem in the standard normal space is solved using the optimization toolbox in MATLAB. For each iteration of the optimization algorithm, MATLAB writes the current values of the variables in the standard normal space onto a file. Then, it calls a Fortran executable program (comprising of the STRAP modules to perform the transformation and SDOF modules to compute the response and response gradients) which reads these variables in the standard normal space, transforms them into variables in the physical space and computes the jacobian of the transformation from the standard normal space to the physical space. The variables in the physical space and the jacobian matrix are used to compute the values of the limit-state functions and their derivatives with respect to variables in the standard normal space which are outputted in separate files. These in turn are read by MATLAB to proceed with the next iteration of the optimization algorithm. After convergence is achieved, Mathematica is used to evaluate the bi-variate normal integral in (3.19), knowing the linearization points and the gradients of the limit-state functions at the linearization points. The out-crossing rates are computed using (3.14).
Fig. 3.5  Flow Chart Diagram of the Code Implementation to Compute Out-Crossing Rates
3.7 Application Examples

3.7.1 Linear Elastic SDOF Oscillator

A linear elastic SDOF oscillator with a natural circular frequency $\omega = 20$ rad/sec and damping ratio $\xi = 0.1$ is considered. The input excitation is assumed to be of the form of an impulse train $\sum_{i=0}^{n} f_i \delta(t - t_i)$ with constant time interval $t_i - t_{i-1} = \Delta t$. The oscillator is assumed to be initially (at time $t_0 = 0$) at rest. The time increment used to evaluate the mean out-crossing rate in (3.14) is taken as $\delta t = 0.0001$. For the above system parameters, the theoretical, mean square stationary displacement response to white noise excitation with intensity $\phi_0 = 100$, is from (3.8) $\sigma_u^2 = (0.44311346)^2 \text{[}in^2\text{]}$. The variance $\sigma_f^2$ of each of the statistically independent normal random variables used for the white noise discretization is, from (3.3) is $\sigma_f^2 = 200\pi \Delta t = (1.77245)^2 \text{[}in^2/s^2\text{]}$ for $\Delta t = 0.005$ sec.

The effect of system uncertainty is examined by modeling $\omega$ and $\xi$ as lognormal random variables with mean values equal to the deterministic values specified above and coefficient of variation equal to 0.05 for $\omega$ and 0.20 for $\xi$.

A convergence study was performed for the numerical mean out-crossing rate for a threshold level corresponding to three times the theoretical standard deviation of the displacement response $\Gamma = 3\sigma_u$, varying $\Delta t$, the time step used for the discretization of white noise and $\delta t$, the time increment used in evaluating the mean out-crossing rate in (3.14). A constant time step to $\Delta t/32$ was used to integrate the equation of motion using the New-
mark-$\beta$ method. The mean out-crossing rate was evaluated at $t = 2.5$ sec when the system response has almost reached stationarity as seen in Fig. 3.6. The results obtained are compared in Table 3.1 with the theoretical result of (Lin 1967):

$$v(\Gamma^+) = \frac{\omega}{2\pi} \exp \left( \frac{-\Gamma^2}{2\sigma_u^2} \right)$$

(3.27)

**Table 3.1** Convergence of Mean Out-crossing Rate with $\Delta t$ ($\Gamma = 3\sigma_u$, $\delta t = 0.0001$ sec)

<table>
<thead>
<tr>
<th>$\Delta t$ [sec]</th>
<th>Stationary Out-Crossing Rate [1/sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.075</td>
<td>$15.87 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.05</td>
<td>$8.25 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.01</td>
<td>$3.74 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.005</td>
<td>$3.52 \times 10^{-2}$</td>
</tr>
<tr>
<td>Exact</td>
<td>$3.53 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

In producing the results presented in Table 3.2, a constant time step of $\Delta t = 0.005$ was chosen to discretize the white noise excitation and a constant time interval of $\Delta t/32$ was used in integrating the equation of motion using the Newmark-$\beta$ method. Again the out-crossing rates were evaluated at $t = 2.5$ sec. It is important to recognize that as the time increment $\delta t$ becomes smaller and smaller, the cross modal correlation coefficient $\rho_{12}$ between the two linearized limit-state functions $(g_1)_{linearized}$ and $(g_2)_{linearized}$ become closer and closer to -1.0. The smallest value of $\delta t = 0.0001$ sec used here has led to val-
ues of $\rho_{12}$ in the range $|1 - \rho_{12}| < 10^{-7}$. For these values of $\rho_{12}$, sophisticated numerical integration schemes with automatic step size refinement as in Mathematica (Wolfram 1996) have to be used.

**Table 3.2** Convergence of Mean Out-crossing Rate with $\delta t$ ($\Gamma = 3\sigma_u$, $\Delta t = 0.005$ sec)

<table>
<thead>
<tr>
<th>$\delta t$ [sec]</th>
<th>Stationary Out-crossing Rate [1/sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>5.26173</td>
</tr>
<tr>
<td>0.001</td>
<td>$3.36 \times 10^{-1}$</td>
</tr>
<tr>
<td>0.0001</td>
<td>$3.52 \times 10^{-2}$</td>
</tr>
<tr>
<td>Exact</td>
<td>$3.53 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

The above reliability-based numerical method for computing mean out-crossing rates has the considerable advantage of providing the solution with a desired rich information as by-product. The "design-point" excitation corresponding to out-crossing rate computation at time ‘t’ gives the realization of the input excitation among all possible random realizations of the input excitation that has the highest likelihood to produce an out-crossing event the time ‘t’ for both deterministic and uncertain systems. For an uncertain system the “design-point” gives the most likely values of the system parameters in conjunction with the most likely realization of the input excitation to produce an out-crossing event at time ‘t’. The “design-point” system and loading parameters are expected to produce one out-crossing at time ‘t’ as the value chosen for $\delta t$ is too small to allow more than one out-crossing. When two different linearization points are used for the two limit-state functions, the “design-
point" refers to the most likely one, i.e., the one closest to the origin in the standard normal space.

One of the important ingredients in the solution process is the computation of gradients with respect to the vector $\mathbf{x}$ of both loading and system parameters modeled as random variables. The response gradients with respect to both loading and system parameters at the design point, i.e., \[ \left. \frac{\partial u(\mathbf{x}, t)}{\partial \mathbf{x}} \right|_{\mathbf{x} = \mathbf{x}^*} \], provide considerable insight into the behavior of the system for given loading randomness and system parameter uncertainty. These response gradients evaluated at the design point and normalized with respect to the standard deviation of the respective loading and system variables, i.e., \[ \left. \frac{\partial u(\mathbf{x}, t)}{\partial x_i} \right|_{\mathbf{x} = \mathbf{x}^*} \times \sigma_{x_i} \], give insight into the extent to which the various loading and system parameters play a role in producing an out-crossing event at time $'t'$.

The time evolution of the mean out-crossing rate to the limit-state function $G(\mathbf{x}, t) = u(\mathbf{x}, t) - 3\sigma_u$ is shown in Fig. 3.6 for both deterministic and uncertain system. It is observed that the mean out-crossing rate increases significantly (about 60% after stationarity is reached) due to system parameter uncertainty. To understand this result better the values of $\omega$ and $\xi$ at the design point $\mathbf{x}^*$ are plotted as a function of time in Fig. 3.7 and Fig. 3.8 and compared with the values of $\omega$ and $\xi$ for the deterministic system.
**Fig. 3.6** Mean Out-Crossing Rate for Deterministic and Uncertain Linear Elastic SDOF System

**Fig. 3.7** Natural Circular Frequency $\omega$ at the Design Point
These “design point” values of $\omega$ and $\xi$ indicate the likelihood of lower values of $\omega$ and $\xi$ have produced the significant increase in the mean out-crossing rate for the uncertain system. Also of interest are the “design-point” excitation time history and the displacement response time history which are plotted in Fig. 3.9 and Fig. 3.10 respectively for the
uncertain system and for out-crossing level of $\Gamma = 3\sigma_u$ and time $t = 2.5 \text{sec.}$

**Fig. 3.10** "Design Point" Displacement Response Time History

**Fig. 3.11** "Design Point" Displacement Response Sensitivity with respect to Loading Variables
From Fig. 3.10, it is clear that the displacement time history at the design point corresponds to the first outcrossing at the time of interest $t = 2.5 \text{ sec}$. Further, it is clear that only one out-crossing in the small time increment $\delta t$ occurs at the design point.

An interesting observation can be made from Fig. 3.9. Among all possible realizations (theoretically infinitely many) of the discrete white noise, the one shown in Fig. 3.9 has the maximum likelihood of producing an out-crossing at $t = 2.5 \text{ sec}$. The impulses creating the "design point" excitation structure themselves into an harmonic type excitation of increasing amplitude when approaching the time of interest $t = 2.5 \text{ sec}$ and of frequency slowly varying around the design point frequency $\omega^*$, thus creating a resonance like condition. Notice that the magnitudes of the impulses at low values of time are small as their contribution to the "design point" response after a relatively long time is small and their most probable/likely values are around zero, the mean value of the loading random variables $f$. This is also apparent from the plot of the sensitivity of $u(t = 2.5 \text{sec})$ with respect to loading variables at the design point as shown in Fig. 3.11. In the process of finding the design point, the excitation time history evolves from a very erratic/random initial trial realization into a well ordered "design point" excitation. The "design-point" excitation achieves an optimum compromise between the two conflicting objectives of maximum probability/likelihood and most efficient excitation to produce the out-crossing event.

The effect of system uncertainty on mean out-crossing rates is examined for a different threshold level. Thus, the limit state function $G(x, t) = u(x, t) - 2.4\sigma_u$ is considered here.
Fig. 3.12 Mean Out-Crossing Rate for Deterministic and Uncertain Linear Elastic SDOF System

From Fig. 3.12 it appears that system uncertainty has a lesser role to play although still significant (~25% increase) for this lower response threshold value.

3.7.2 Duffing Oscillator

The properties of the Duffing oscillator considered here are

\[
\begin{align*}
\omega &= 20 \ [\text{rad/sec}] \\
\xi &= 0.1 \ [-]
\end{align*}
\]

The stationary mean-squared relative displacement response, \( \sigma_u^2 \), for a linear elastic SDOF oscillator (\( \gamma = 0 \)) with the above deterministic system properties to white noise of intensity \( \phi_0 = 100 \), given by (3.8), is \( \sigma_u^2 = (0.4431134)^2 \cdot in^2 \). The variance \( \sigma_f^2 \) of each normal loading random variable given by (3.3) for \( \Delta t = 0.005 \ sec \) is
\[ \sigma_t^2 = (1.77245)^2 \left[ in^2/s^2 \right] \] Various degrees of nonlinearity of the Duffing oscillator are considered by selecting the parameter \( \gamma \): \( \gamma = 0.039/\sigma_u^2 \), \( \gamma = 0.078/\sigma_u^2 \), \( \gamma = 0.118/\sigma_u^2 \) and \( \gamma = 0.157/\sigma_u^2 \). The limit-state function considered here is again \( G(x, t) = u(x, t) - 3\sigma_u \). For the examples presented in this section, a time interval for the white noise discretization of white noise of \( \Delta t = 0.005 \text{ sec} \) was used throughout and a time increment of \( \delta t = 0.0001 \text{ sec} \) was used in (3.14) to evaluate the out-crossing rate. Furthermore, the equation of motion was integrated using the Newmark-\( \beta \) algorithm with a constant time step of \( \Delta t/32 \).

The figures that follow illustrate (1) the accuracy of prediction of the reliability-based method used in this study to compute out-crossing rates, (2) the effect of system uncertainty, (3) the design point excitation and displacement response time histories at \( t = 2.5 \text{ sec} \), and (4) the design point system variables for the displacement threshold level indicated above. The probabilistic and physical interpretation of “design point” excitation and system variables closely follow that given in Sec. 3.7.1. The numerically obtained mean out-crossing rates for the deterministic system are compared with the corresponding theoretical, exact stationary mean out-crossing rate (Lin 1967).

The stationary mean out-crossing rate for a given threshold, \( \Gamma = 3\sigma_u \) in the present case is given as (Lin 1967):

\[ \nu(\Gamma^+) = \frac{C}{2\pi} \frac{\phi_0}{\omega} \exp \left( -\frac{2\xi}{\pi\phi_0} \left( \frac{1}{2} + \gamma \frac{\Gamma^4}{4} \right) \right) \quad (3.28) \]

in which ‘\( C \)’ is a normalization constant such that:
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{u, \dot{u}} dud\dot{u} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C \exp \left( -\frac{2\xi \omega^3}{\pi \phi_0} \left( \frac{u^2}{2} + \frac{\dot{u}^2}{2} + \frac{\gamma \dot{u}^4}{4} \right) \right) dud\dot{u} = 1 \]  

(3.29)

where \( f_{u, \dot{u}} \) denotes the joint probability density function of displacement and velocity at an arbitrary point in time. For a Duffing oscillator, it can be shown that the stationary displacement and velocity response processes are statistically independent for Gaussian-white noise excitation (Lin 1967). Hence, the marginal densities for \( u \) and \( \dot{u} \) can be written in the form

\[ f_u = C_1 \exp \left( -\frac{2\xi \omega^3}{\pi \phi_0} \left( \frac{u^2}{2} + \frac{\gamma \dot{u}^4}{4} \right) \right) \]

\[ f_{\dot{u}} = C_2 \exp \left( -\frac{2\xi \omega^3}{\pi \phi_0} \left( \frac{\dot{u}^2}{2} \right) \right) \]

(3.30)

The two-fold integral given in (3.29) reduces into the product of two single-fold integrals. Knowing that

\[ C_2 \int_{-\infty}^{\infty} \exp \left( -\frac{2\xi \omega^3}{\pi \phi_0} \left( \frac{\dot{u}^2}{2} \right) \right) d\dot{u} = C_2 \sqrt{\frac{2\pi}{\frac{2\xi \omega^3}{\pi \phi_0}}} \]

(3.31)

and \( C = C_1 C_2 \), \( C \) can be found from the relation:

\[ C \int_{-\infty}^{\infty} \exp \left( -\frac{2\xi \omega^3}{\pi \phi_0} \left( \frac{u^2}{2} + \frac{\gamma \dot{u}^4}{4} \right) \right) du = \sqrt{\frac{\xi \omega^3}{\pi \phi_0}} \]

(3.32)

The integral in (3.32) is computed numerically using Mathematica (Wolfram 1996). For the values of the nonlinearity parameter \( \gamma \) considered here, the theoretical “exact” stationary mean out-crossing rates for the threshold level \( \Gamma = 3\sigma_u \) are given in Table 3.3. In the case of uncertain system, the system parameters \( \omega \) and \( \xi \) are modeled as lognormal random variables with the same parameters as in Sec. 3.7.1.
Table 3.3  Theoretical Values of Mean Out-Crossing Rate for Duffing Oscillator

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$v(3\sigma^+)$ [1/sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.039/\sigma_u^2$</td>
<td>0.0163832</td>
</tr>
<tr>
<td>$0.078/\sigma_u^2$</td>
<td>0.00739674</td>
</tr>
<tr>
<td>$0.117/\sigma_u^2$</td>
<td>0.00339505</td>
</tr>
<tr>
<td>$0.157/\sigma_u^2$</td>
<td>0.00150772</td>
</tr>
</tbody>
</table>

Fig. 3.13  Mean Out-Crossing Rates for Deterministic and Uncertain Duffing Oscillator
\[ \gamma = \frac{0.039}{\sigma_u^2} \]

Threshold = 3\( \sigma_u \)

**Fig. 3.14** "Design-Point" Excitation for Duffing Oscillator at \( t = 2.5 \text{sec} \)

\[ \gamma = \frac{0.039}{\sigma_u^2} \]

Threshold = 3\( \sigma_u \)

---

**Fig. 3.15** Initial Natural Circular Frequency \( \omega \) at the Design Point

---

**Fig. 3.16** Damping Ratio \( \xi \) at the Design Point
\[ \gamma = \frac{0.039}{\sigma_u^2} \]
Threshold = $3\sigma_u$

Fig. 3.17 “Design Point” Displacement Response Time History

\[ \gamma = \frac{0.078}{\sigma_u^2} \]
Threshold = $3\sigma_u$

Fig. 3.18 Mean Out-Crossing Rates for Deterministic and Uncertain Duffing Oscillator

\[ \gamma = \frac{0.078}{\sigma_u^2} \]
Threshold = $3\sigma_u$

Fig. 3.19 Initial Natural Circular Frequency $\omega$ at the Design Point
Fig. 3.20 Damping Ratio $\xi$ at the Design Point

Fig. 3.21 Mean Out-Crossing Rates for Deterministic and Uncertain Duffing Oscillator

Fig. 3.22 Initial Natural Circular Frequency $\omega$ at the Design Point
Fig. 3.23 Damping Ratio $\zeta$ at the Design Point

Fig. 3.24 Mean Out-Crossing Rates for Deterministic and Uncertain Duffing oscillator

Fig. 3.25 Initial Natural Circular Frequency $\omega$ at the Design Point
Fig. 3.26 Damping Ratio $\xi$ at the Design Point

From the figures above, it is clear that the mean out-crossing rates decreases with increasing system nonlinearity, which is very intuitive since the Duffing oscillator has a stiffening type of restoring force nonlinearity. Further, the accuracy of the approximation of the mean out-crossing rate obtained using the reliability-based method degrades slightly with increasing nonlinearity.

To examine the dependence of the effect of system uncertainty on the displacement threshold level, an example with $G(x, t) = u(x, t) - 2.4\sigma_u$ and $\gamma = 0.039/\sigma_u^2$ is considered. Uncertain parameters $\omega$ and $\zeta$ remain the same as before.

Fig. 3.27 Mean Out-Crossing Rates for Deterministic and Uncertain Duffing Oscillator
As expected, for both cases of deterministic and uncertain system, the mean out-crossing rate increases for this lower threshold. The effect of system uncertainty on the mean out-crossing rate (~25% increase) for threshold = 2.4σ is not as dramatic as in the earlier case (~100% increase) for a threshold of 3σ_u.

3.7.3 Bouc-Wen SDOF Oscillator

The properties of the Bouc-Wen hysteretic oscillator considered here are, see Sec. 2.3:

\[ \omega = 3\pi \text{ [rad/sec]} \]
\[ \xi = 0.05 [-] \]
\[ \gamma = \frac{1}{2\sigma_u} = 2.581081 \text{ [1/in]} \]
\[ \beta = \frac{1}{2\sigma_u} = 2.581081 \text{ [1/in]} \]
\[ n = 1 \]
\[ A = 1.0 [-] \]

The stationary mean-squared relative displacement response, \( \sigma_u^2 \), of a linear elastic SDOF oscillator with the above parameter values for \( \omega \) and \( \xi \) and for white noise excitation of intensity \( \phi_0 = 1.0 \) is, according to (3.8) is \( \sigma_u^2 = (0.19371)^2 \text{ in}^2 \). The standard deviation \( \sigma_f \) of each loading normal random variable given by (3.3) is \( \sigma_f = 0.39633 \text{ [in/s]} \), for \( \Delta t = 0.025 \text{ sec} \). The limit state function considered here is again \( G(x, t) = u(x, t) - 3\sigma_u \). All results presented in this section were obtained using \( \Delta t = 0.025 \text{ sec} \) for white noise discretization, \( \delta t = 0.0001 \text{ sec} \) as the time increment for evaluating the mean out-crossing rates using (3.14), and \( \Delta t/32 \) as the constant time step to integrate nonlinear inelastic equation of motion using Newmark-β method.
The effect of system uncertainty is examined by modeling $\omega$, $\xi$, and $\gamma$ as random variables. While $\omega$ and $\xi$ are treated as lognormal variables, $\gamma$ is given a gamma distribution, with their means in each case equal to the deterministic values given above. Their standard deviations are respectively taken as $\sigma_\omega = 1.0$ [rad/sec], $\sigma_\xi = 0.01$ [-] and $\sigma_\gamma = 0.25$ [in$^{-1}$]. To study the effect of system uncertainty, three cases of gradually increasing system uncertainty were considered: (1) $\omega$ only uncertainty, (2) $\omega$ and $\xi$ uncertainty, and (3) $\omega$, $\xi$ and $\gamma$ uncertainty.

The figures that follow illustrate (1) the effect of gradually increasing the system uncertainty, (2) the “design point” excitation, (3) the “design point” system variables, and (4) the effect of the displacement threshold level. For the Bouc-Wen hysteretic oscillator, no exact solution for stationary mean out-crossing rate is available.

![Figure 3.28](image)

**Fig. 3.28** Mean Out-Crossing Rates for Deterministic and Uncertain Bouc-Wen Oscillator

From Fig. 3.28, it is clear that the system uncertainty has a dramatic effect on the mean
out-crossing rates (~490% increase for $\omega$, $\xi$ and $\gamma$ uncertainties!). The effect of uncertainty in $\omega$ outweighs the effect of uncertainty in $\xi$ and $\gamma$. The uncertainty in $\gamma$ has negligible influence on the mean out-crossing rates after the response reaches stationarity.

It is of interest to examine the “design point” excitation time history and the system variables at the design point. These quantities are plotted for the system with $\omega$, $\xi$ and $\gamma$ uncertainties. Fig. 3.29 represents the “design point” excitation while Fig. 3.30 gives the displacement response time history at the design point for an out-crossing event at $t = 11$ sec.

![Graph](image)

**Fig. 3.29** “Design Point” Excitation for Bouc-Wen Model with $\omega$, $\xi$ and $\gamma$ Uncertainties

As was the case for the Duffing Oscillator, the “design point” displacement response time history corresponds to the first out-crossing at the time of interest, namely $t = 11.0$ sec. Again, the “design point” excitation is quasi-harmonic with “exponentially” increasing amplitude when approaching the time of interest $t = 11.0$ sec and the discussion follows that given in Sec. 3.7.1.
The effect of system uncertainty is understood better by comparing the "design point" system parameters with the corresponding deterministic parameter values which coincide with the mean values of the uncertain parameters. These comparisons are plotted in Figs. 3.32 to 3.34.
Fig. 3.32 Initial Natural Circular Frequency $\omega$ at the Design Point

Fig. 3.33 Damping Ratio $\xi$ at the Design Point

The value of each of the uncertain parameter at the design point is different from the deterministic value, in this case lower. The parameter $\gamma$ at design point, though different from its deterministic value, has negligible effect on the mean out-crossing rates in the stationary phase of the response as seen from Fig. 3.28. It has some influence in the transient phase. The normalized response sensitivity $\frac{\partial\mu(t)}{\partial\gamma}\sigma_\gamma$ at the design point is extremely small
as compared to those for $\omega$ and $\xi$, which explains the negligible effect of $\gamma$ parameter uncertainty on the mean out-crossing rate after stationarity is reached.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{gamma_plot.png}
\caption{System Parameter $\gamma$ at the Design Point}
\end{figure}

To examine the dependence of the effect of system uncertainty on the displacement threshold level, the limit state function $G(x, t) = u(x, t) - 2\sigma_u$ is considered. The results are plotted in Fig. 3.35. The effect of system uncertainty is not as dramatic as in the previous case for a threshold level of $3\sigma_u$. Again, it is seen that the effect of system uncertainty is very much dependent on the response threshold level.
Fig. 3.35 Mean Out-Crossing Rates for Deterministic and Uncertain Bouc-Wen Oscillator

3.7.4 Rule-Based Bilinear Hysteretic Oscillator

The properties of the bilinear hysteretic oscillator considered are:

\[ m = 1.00 \text{ [lb]} \quad : \text{mass} \]
\[ c = 1.256637 \text{ [s}^{-1}] \quad : \text{damping coefficient} \]
\[ k_1 = 39.4784 \text{ [lb/in]} \quad : \text{pre-yield stiffness} \]
\[ k_2 = 0.01 k_1 \text{ [lb/in]} \quad : \text{post-yield stiffness} \]
\[ u_y = 0.2 \text{ [in]} \quad : \text{yield displacement} \]

The stationary mean-squared relative displacement response, \( \sigma_u^2 \), for a linear oscillator with the above deterministic system properties \( m, c, k_1 \) to white noise excitation of intensity \( \phi_0 = 1.0 \) is \( \sigma_u^2 = (0.2516)^2 \text{ [in}^2] \). The standard deviation \( \sigma_f \) of each loading normal random variable is \( \sigma_f = 0.39633 \text{ [in/s]} \) as given by (3.3) for \( \Delta t = 0.025 \text{ sec} \). The limit-state function considered here is again \( G(x, t) = u(x, t) - 3\sigma_u \). The results obtained
are based on $\Delta t = 0.025 \sec$ for the white noise discretization, $\delta t = 0.0001 \sec$ for evaluating the mean out-crossing rate and $\Delta t/32$ for integrating the equation of motion using the Newmark-$\beta$ method.

The effect of system uncertainty is examined by modeling $m$, $c$ and $u_y$ as random variables. While $m$ and $c$ are treated as lognormal variables, $u_y$ is given a gamma distribution, with their means in each case equal to the deterministic values specified above. Their standard deviations are taken as $\sigma_m = 0.1 \ [\text{lb}]$, $\sigma_c = 0.25 \ [\text{lb/sec}]$ and $\sigma_{u_y} = 0.25 \ [\text{in}]$.

Mean out-crossing rates at a few points in time are computed for a deterministic system as reported in Table 3.4. The rule-based bilinear oscillator exhibited problems in converging to the design point especially for out-crossing after a relatively long time. Due to these difficulties, mean out-crossing rate calculations were not obtained for the uncertain system. It is suspected that the convergence problem is due to the piecewise differentiable only nature of the displacement response to a train of impulses, from which the limit-state function is evaluated. Therefore, a numerically robust plasticity based formulation of the bilinear inelastic oscillator with piecewise linear interpolation of the white noise will be considered instead of the rule-based bilinear hysteretic oscillator with random impulse train discretization of the white noise excitation.
Table 3.4  Mean Out-crossing Rate Bilinear Hysteretic Oscillator

<table>
<thead>
<tr>
<th>$t$ [sec]</th>
<th>Out-crossing Rate [1/sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
<td>0.000632</td>
</tr>
<tr>
<td>1.5</td>
<td>0.00973</td>
</tr>
<tr>
<td>2.5</td>
<td>0.02635</td>
</tr>
<tr>
<td>3.5</td>
<td>0.03885</td>
</tr>
<tr>
<td>4.5</td>
<td>0.07366</td>
</tr>
</tbody>
</table>

The excitation time history, displacement response time history, and the force-displacement hysteretic response, all at the design point are shown in Figs. 3.36, 3.37 and 3.38 respectively, for out-crossing at $t = \text{4.5 sec}$.

![Graph showing excitation and thresholds](image)

**Fig. 3.36** “Design Point” Excitation for Rule-Based Bilinear Hysteretic Oscillator
Fig. 3.37 Displacement Response Time History at Design Point

Fig. 3.38 Restoring Force vs Displacement Response at the Design Point
3.8 Conclusions

This chapter presents application examples of out-crossing rate calculations. First, out-crossing rate calculations for a linear elastic SDOF oscillator subjected to white noise excitation are presented for both deterministic and uncertain system properties. A convergence study is performed for the stationary out-crossing rate with respect to the constant time step $\Delta t$ used in discretizing the white noise excitation as a random pulse train and the time increment $\delta t$ used in evaluating the mean out-crossing rates in (3.14). The effect of system uncertainty and response threshold level on the mean out-crossing rates are investigated. The "design point" excitation and "design point" system parameters are examined. This exercise is repeated for a nonlinear elastic Duffing oscillator for various values of its nonlinearity parameter $\gamma$ and for the Bouc-Wen hysteretic oscillator. The bilinear hysteretic SDOF oscillator exhibited problems of convergence to the design point.

It can be concluded that the system uncertainty has a considerable effect on the mean out-crossing rate and this effect is strongly dependent on the response threshold level considered for computing the mean out-crossing rates.
PART II: PLASTICITY BASED CONSTITUTIVE MODELS
CHAPTER 4 Response Calculation for Nonlinear Mechanics-Based Models of Dynamic Systems

4.1 Introduction

This chapter deals with the dynamic response calculation for mechanics based plasticity models of structures. Specifically, the dynamic loading is assumed to be due to a ground motion. The two steps involved in the computation of the dynamic response are reviewed, namely (1) finite element discretization (in space) of the strong form of the partial differential equation of motion written for an infinitesimal element in the continuum, to obtain the weak form or the matrix ordinary differential form of the equation of motion for the structure, and (2) discretization of the matrix equation of motion in time and solving for the structural response using a time stepping algorithm such as the Newmark-β method. The complete derivation is given for computing the internal resisting force and the consistent tangent stiffness matrix for the uni-axial \( J_2 \) plasticity model and the internal resisting force (the derivation of the tangent stiffness matrix for the cap model is borrowed from elsewhere) for the cap plasticity model required at each time step of the Newmark-β method. The return map algorithm which is used to integrate the rate constitutive equations in order to evaluate the structure tangent stiffness matrix and the internal resisting force vector is briefly described. Finally, these steps are summarized in the conclusions at the end of the chapter.

4.2 Differential Form of Equations of Motion

The equations of motion of an elastoplastic body occupying a region \( \Omega \) in \( \mathbb{R}^n \) and undergoing small (infinitesimal) deformations can be expressed as
\[ \rho \ddot{u} = \nabla \cdot \sigma + \rho b, \quad \sigma = D : (\nabla^{(s)} \dot{u} - \dot{\varepsilon}^p), \]
\[ u = \bar{u} \text{ on } \partial \Omega_u, \quad \sigma \cdot n = \tilde{t} \text{ on } \partial \Omega_\sigma \]  
(4.1)

where \( \partial \Omega_u \cup \partial \Omega_\sigma = \partial \Omega \) and \( \partial \Omega_u \cap \partial \Omega_\sigma = \emptyset \). In (4.1), \( \ddot{u} \) (\( u \)) and \( \sigma \) denote the velocity (displacement) and stress fields over \( \Omega \), \( \dot{\varepsilon}^p \) the plastic strain rates, \( \rho b \) the body forces per unit volume, \( \rho \) the density and \( D \) the elastic modulus tensor, with the usual symmetries. The symbol (:) implies the contraction \( D : \nabla^s \dot{u} \equiv D_{ijkl} \frac{1}{2} (\dot{u}_{kl} + \dot{u}_{lk}) \). Finally, \( \bar{u} \) and \( \tilde{t} \) denote the prescribed displacements and tractions over the kinematic and traction boundaries \( \partial \Omega_u \) and \( \partial \Omega_\sigma \), respectively. The symmetric part of the gradient of the displacement vector defines the small (infinitesimal) strain tensor, i.e.,

\[ \varepsilon = \nabla^{(s)} u = 1/2 [ \nabla u + (\nabla u)^T ] . \]  
(4.2)

The constitutive equation, expressed in rate form in (4.1), is also commonly expressed in the infinitesimal incremental form

\[ d\sigma = \frac{d\sigma}{dt} dt = \dot{\sigma} dt = D : (d\varepsilon - d\varepsilon^p) . \]  
(4.3)

In order to have a complete set of equations, one must supplement (4.1) with some constitutive relations for \( \dot{\varepsilon}^p \). At this point, it suffices to assume that \( \dot{\varepsilon}^p \) can be expressed as a function of the stresses and some set of plastic internal variables \( q \), i.e.,

\[ \dot{\varepsilon}^p = T(\sigma, q) . \]  
(4.4)

The internal variables \( q \) may, for example, represent the yield stress for an isotropic hardening model or the translation of the elastic domain for a kinematic hardening model. Equation (4.4) is general enough to accommodate perfect and hardening plasticity. It is
finally assumed that the evolution of the plastic internal variables is governed by kinetic equations of the form

\[ \dot{\mathbf{q}} = f(\sigma, q) \] (4.5)

Combining (4.1), (4.4), and (4.5), the following set of equations of evolution is obtained

\[ \rho \ddot{\mathbf{u}} = \nabla \cdot \sigma + \rho \mathbf{b}, \quad \sigma = D : \nabla \dot{\mathbf{u}} - D : T(\sigma, q), \quad \dot{\mathbf{q}} = f(\sigma, q), \]
\[ \mathbf{u} = \bar{\mathbf{u}} \text{ on } \partial \Omega_u, \quad \sigma \cdot \mathbf{n} = \bar{\mathbf{i}} \text{ on } \partial \Omega_\sigma \] (4.6)

which, together with the initial conditions

\[ \mathbf{u}(x, 0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(x, 0) = \dot{\mathbf{u}}_0, \quad \sigma(x, 0) = \sigma_0, \quad \mathbf{q}(x, 0) = \mathbf{q}_0, \] (4.7)

define an initial boundary value problem for the unknowns \( \mathbf{u}, \sigma, \mathbf{q} \).

4.3 Finite Element Discretization of Equations of Motion

In practice, approximate solutions of (4.6) are obtained using the finite element method for the spatial discretization combined with a time stepping scheme for the integration over time. For the sake of completeness, a brief account of the derivation of the discretized equations of motion is given below.

By applying the principle of virtual work (displacement), a weak form of the linear momentum balance equation is obtained as

\[ \int_{\Omega} [\rho (\ddot{\mathbf{u}} - \mathbf{b}) \cdot \delta \mathbf{u} + \sigma : \nabla \delta \mathbf{u}] d\Omega = \int_{\partial \Omega} \bar{\mathbf{i}} \cdot \delta \mathbf{u} \, d\Gamma \] (4.8)

in symbolic vector notation, or

\[ \int_{\Omega} \rho \ddot{\mathbf{u}} \delta \mathbf{u} d\Omega + \int_{\Omega} \sigma \delta \epsilon_{ij} d\Omega = \int_{\partial \Omega} \rho \dot{b}_i \delta u_i d\Omega + \int_{\partial \Omega} \bar{t}_i \delta u_i d\Gamma \] (4.9)
in indicial-tensor notation in which a repeated index implies summation over the range of the index, i.e., \( \rho b_i \delta u_i = \rho b_1 \delta u_1 + \rho b_2 \delta u_2 + \rho b_3 \delta u_3 \). In (4.8) and (4.9), \( \delta u_i \) and \( \delta \epsilon_{ij} \) are virtual displacement increments and virtual (small) strain increments, respectively, and they form a compatible set of deformations; and the set \( \{ \sigma_{ij}, \rho \dot{u}_i, \rho b_i, \text{ and } \tau_i \} \) is in dynamic equilibrium.

In matrix form, (4.8) or (4.9) becomes

\[
\int_{\Omega} \rho \delta \mathbf{u}^T \cdot \ddot{\mathbf{u}} \, d\Omega + \int_{\Omega} \delta \mathbf{e}^T \cdot \sigma \, d\Omega = \int_{\Omega} \delta \mathbf{u}^T \cdot \rho b \, d\Omega + \int_{\partial \Omega} \delta \mathbf{u}^T \cdot \mathbf{i} \, d\Gamma \tag{4.10}
\]

where the vectors for acceleration \( \ddot{u} \), displacement \( u \), strain \( \epsilon \), and stress \( \sigma \) are defined as

\[
\mathbf{u}^T = \{u_1, u_2, u_3\}, \quad \delta \mathbf{u}^T = \{\delta u_1, \delta u_2, \delta u_3\} \tag{4.11}
\]

\[
\mathbf{e}^T = \{\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{23}, 2\epsilon_{31}\} \tag{4.12}
\]

\[
\delta \mathbf{e}^T = \{\delta \epsilon_{11}, \delta \epsilon_{22}, \delta \epsilon_{33}, 2\delta \epsilon_{12}, 2\delta \epsilon_{23}, 2\delta \epsilon_{31}\} \tag{4.13}
\]

\[
\sigma^T = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}\} \tag{4.14}
\]

For a geometrically linear analysis, or a small-deformation analysis, the strains are linearly related to the displacement field as

\[
\epsilon = L \mathbf{u}, \quad \delta \epsilon = L \delta \mathbf{u} \tag{4.15}
\]

where \( L \) is the differential operator matrix defined as
\[
L = \begin{bmatrix}
\frac{\partial}{\partial x_1} & 0 & 0 \\
0 & \frac{\partial}{\partial x_2} & 0 \\
0 & 0 & \frac{\partial}{\partial x_3} \\
\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\
0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\
\frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1}
\end{bmatrix}_{(6 \times 3)}
\] (4.16)

The *finite element approximation* is introduced such that the displacement field \( u \) within the element \( e \) is defined as

\[
u^{(e)} = \sum_{\text{Nodes}} N_f(\xi) u^I = [N_I, N_J, \ldots] \begin{bmatrix} u_I^{(e)} \\ u_J^{(e)} \\ \vdots \end{bmatrix} = \mathbf{N} \mathbf{U}^{(e)}
\] (4.17)

in which the functions \( N_I \) are called the *shape functions* at node \( I \), \( \xi \) are the natural coordinates for the element, and \( u^I \) are the values of the displacement vector at node \( I \). The shape functions \( N_I, N_J, \ldots \), are chosen such that they give appropriate nodal displacements when the coordinates of the corresponding nodes are inserted in (4.17). In general, if all the components of the displacement field are interpolated in an identical manner, we have

\[
N_I = N_I(\xi) \mathbf{I}
\] (4.18)

where \( \mathbf{I} \) denotes the (3x3) identity matrix, and \( N_I \) is the shape function at node \( I \) chosen such that \( N_I = 1 \) at \( x_I^I, x_J^I, \) and \( x_K^I \), but zero at the other nodes of the element.

By combining (4.15) and (4.17), the strain field within the element is directly related to the
nodal displacement vectors,

$$\nabla^{(s)} \mathbf{u} = \mathbf{e}(\mathbf{u}) = \sum_{\text{Nodes}} \mathbf{B}_i \mathbf{u}' = \mathbf{B} \mathbf{U}^{(e)}$$ (4.19)

in which $\mathbf{B} = [\mathbf{B}_1, \mathbf{B}_2, \ldots]$ , and $\mathbf{B}_i$ is the strain-displacement matrix for node $i$ of the element, expressed as

$$\mathbf{B}_i = \begin{bmatrix} N_{i,1} & 0 & 0 \\ 0 & N_{i,2} & 0 \\ 0 & 0 & N_{i,3} \\ N_{i,2} & N_{i,1} & 0 \\ 0 & N_{i,3} & N_{i,2} \\ N_{i,3} & 0 & N_{i,1} \end{bmatrix}_{(6 \times 3)}$$ (4.20)

where

$$N_{i,j} = \frac{\partial N_i}{\partial x_i}. \quad \text{(4.21)}$$

The approximated displacement field over the whole region $\Omega$ is obtained by summing the approximated displacement fields over all the elements $\Omega_e$, i.e.,

$$\mathbf{u} = \sum_{e = 1}^{N_{el}} \mathbf{u}^{(e)}$$ (4.22)

Using the same shape or interpolation functions for both the real and virtual displacements and substituting (4.17) and (4.19) into (4.10), it is found that

$$\sum_{e = 1}^{N_{el}} (\delta \mathbf{U}^{(e)})^T \int_{\Omega_e} \rho \mathbf{N}^T \mathbf{N} \mathbf{d} \Omega_e \mathbf{U}^{(e)} + \sum_{e = 1}^{N_{el}} (\delta \mathbf{U}^{(e)})^T \int_{\Omega_e} \mathbf{B}^T \mathbf{\sigma} \mathbf{d} \Omega_e =$$

$$\sum_{e = 1}^{N_{el}} (\delta \mathbf{U}^{(e)})^T \int_{\Omega_e} \mathbf{N}^T \rho \mathbf{b} \mathbf{d} \Omega_e + \sum_{e = 1}^{N_{el}} (\delta \mathbf{U}^{(e)})^T \int_{\partial \Omega_e} \mathbf{N}^T \mathbf{i} \mathbf{d} \Gamma_e$$ (4.23)
The above equation can be rewritten as

\[
\sum_{e=1}^{Ne} (\delta U^{(e)})^T \left[ \int_{\Omega_e} \rho N^T N d\Omega_e \dot{U}^{(e)} + \int_{\Omega_e} B^T \sigma d\Omega_e - \right. \\
\left. \int_{\Omega_e} N^T \rho b d\Omega_e - \int_{\partial\Omega_e} N^T \dot{i} d\Gamma_e \right] = 0
\]  
(4.24)

Since the above equality is valid for any arbitrary set of virtual element nodal displacement vectors, \( \delta U^{(e)} \), the following equality must hold

\[
\sum_{e=1}^{Ne} \left[ \int_{\Omega_e} \rho N^T N d\Omega_e \dot{U}^{(e)} + \int_{\Omega_e} B^T \sigma d\Omega_e - \int_{\Omega_e} N^T \rho b d\Omega_e - \int_{\partial\Omega_e} N^T \dot{i} d\Gamma_e \right] = 0
\]  
(4.25)

Equation (4.25) represents the spatially discretized weak form of the linear momentum balance equations and expresses the dynamic equilibrium between the inertia forces (first term), the internal resisting forces (second term) and the external forces (third and fourth terms) of the whole system or structure.

The various terms in (4.25) define the following element matrices:

\[
m^{(e)} = \int_{\Omega_e} \rho N^T N d\Omega_e, \quad \text{(4.26)}
\]

\[
r^{(e)} = \int_{\Omega_e} B^T \sigma d\Omega_e, \quad \text{(4.27)}
\]

\[
f^{(e)} = \int_{\Omega_e} N^T \rho b d\Omega_e + \int_{\partial\Omega_e} N^T \dot{i} d\Gamma_e, \quad \text{(4.28)}
\]

where \( m^{(e)} \) is the element mass matrix, \( r^{(e)} \) is the element internal resisting force vector, and \( f^{(e)} \) is the element external force vector. Using the relations in (4.17) to (4.19), the above element matrices can be further expressed as an assembly of element submatrices:
\begin{align*}
\mathbf{m}^{(e)} &= \sum_l \sum_j \mathbf{m}_{ij}^{(e)} \quad \text{where} \quad \mathbf{m}_{ij}^{(e)} = \int_{\bar{n}} \rho N_i N_j d\Omega_e \\
\mathbf{r}^{(e)} &= \sum_l \mathbf{r}_i^{(e)} \quad \text{where} \quad \mathbf{r}_i^{(e)} = \int_{\bar{n}} \mathbf{B}_i^T \mathbf{\sigma} d\Omega_e \\
\mathbf{f}^{(e)} &= \sum_l \mathbf{f}_i^{(e)} \quad \text{where} \quad \mathbf{f}_i^{(e)} = \int_{\bar{n}} \mathbf{N}_i \rho \mathbf{b} d\Omega_e + \int_{\partial \bar{n}} \mathbf{N}_i \mathbf{i} d\Gamma_e
\end{align*}

(4.29) (4.30) (4.31)

In the above equations, the subscripts \( I \) and \( J \) over which the sums are carried out denote the element node numbers.

By assembling the element matrices or submatrices into the corresponding system matrices, (4.25) reduces to the following matrix differential equation or \textit{spatially discretized} equation of motion:

\[ \mathbf{M} \ddot{\mathbf{u}}(t) + \mathbf{R}(\mathbf{u}, t) = \mathbf{F}(t) \]  

(4.32)

where \( t \) denotes the time, \( \mathbf{u}(t) \) denotes the system vector of nodal displacements, \( \mathbf{M} \) is the system mass matrix, \( \mathbf{R}(\mathbf{u}, t) \) is the system internal resisting force vector, \( \mathbf{F}(t) \) is the system external load vector, and a superposed dot indicates differentiation with respect to time.

### 4.4 Damping

In (4.32), \( \mathbf{R}(\mathbf{u}, t) \) accounts for the nonlinear, hysteretic behavior of the structure under dynamic loading and hence for the hysteretic energy dissipation. However, there are several other mechanisms that induce damping and energy dissipation in a dynamical system: air resistance, fluid resistance, dry friction between components, material microcracking. The effect of these other mechanisms for energy dissipation is considered by using the model of linear viscous damping, namely a force directly proportional to the velocity. Under the assumptions of linear viscous damping, the matrix differential equation of motion becomes:
\[ M\ddot{u}(t) + C\dot{u}(t) + R(u, t) = F(t) \]  
(4.33)

where \( C \) is the system damping matrix. Unlike the mass matrix and the internal resisting force vector which are assembled from the element mass matrices and internal resisting force vectors, \( C \) is obtained directly at the system level.

One procedure for defining a physical damping matrix is to employ a particular form of damping called *Rayleigh damping*, in which the damping matrix is chosen to be a linear combination of the system mass (\( M \)) and initial tangent stiffness (\( K \)) matrices, i.e.,

\[ C = \alpha M + \beta K \]  
(4.34)

where \( \alpha \) and \( \beta \) are constants. An eigen analysis is performed based on the mass matrix \( M \) and the initial tangent stiffness matrix \( K \) to obtain the natural frequencies and mode shapes. The constants \( \alpha \) and \( \beta \) are selected to produce specified modal damping ratios for two given modes. After uncoupling the equations of motion, (4.33), by using modal coordinates, it can be shown that the modal damping ratios are given by

\[ \xi_n = \frac{1}{2}\left(\frac{\alpha}{\omega_n} + \beta \omega_n\right) \]  
(4.35)

Thus, Rayleigh damping is easy to define by choosing \( \xi_n \) for two modes and solving for \( \alpha \) and \( \beta \). The damping ratios in the remaining modes is then determined by (4.35). For this study, *mass proportional damping* (\( \beta = 0 \)) is assumed.

### 4.5 Earthquake Loading

When the external dynamic loading acting on the structure is due to a ground/support motion, the matrix equation of motion (4.33) takes the form:
\[
\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{R}(\mathbf{u}, t) = -\mathbf{M}\ddot{\mathbf{u}}_g(t)
\]  
(4.36)

where \( \mathbf{r} \) is an influence coefficient vector which represents the displacements resulting from a unit support displacement, and \( \ddot{\mathbf{u}}, \dot{\mathbf{u}} \) and \( \mathbf{u} \) are the acceleration, velocity and displacement response vectors relative to the moving ground. In (4.36), \( \ddot{\mathbf{u}}_g(t) \) represents the free-field input ground acceleration at the base of the structure. In arriving at (4.36), it is assumed that the ground motion acts simultaneously at all support points of the structure (rigid soil earthquake excitation). Moreover, only the case of a single horizontal ground motion component will be considered, i.e., \( \ddot{\mathbf{u}}_g(t) \) is a scalar valued function of time.

### 4.6 Response Calculation using Explicit Formulation of the Newmark-\( \beta \) Method

The equation of motion (4.36) is solved for the response quantities \( \mathbf{u}, \dot{\mathbf{u}} \) and \( \ddot{\mathbf{u}} \) (relative displacements, velocities and accelerations) at discrete times \( t_n = n\Delta t \) occurring at constant intervals of size \( \Delta t \). This is done by expressing the velocities and displacements at the end of time step \( [t_n, t_{n+1}] \) in terms of displacements and velocities at the beginning of the time step and the accelerations at the end of the time step:

\[
\ddot{\mathbf{u}}_{n+1} = \ddot{\mathbf{u}}_n + (1 - \alpha)\Delta t\dddot{\mathbf{u}}_n + \alpha\Delta t\dddot{\mathbf{u}}_{n+1}
\]  
(4.37)

\[
\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta t\dot{\mathbf{u}}_n + \left( \frac{1}{2} - \beta \right) (\Delta t)^2\ddot{\mathbf{u}}_n + \beta(\Delta t)^2\dddot{\mathbf{u}}_{n+1}
\]  
(4.38)

where \( \alpha \) and \( \beta \) are the parameters of the Newmark-\( \beta \) method described in section 2.2.1.

In this study, we used \( \alpha = 0.5 \) and \( \beta = 0.25 \), which correspond to the constant acceleration method. Thus, (4.37) and (4.38) reduce to:

\[
\ddot{\mathbf{u}}_{n+1} = \ddot{\mathbf{u}}_n + 0.5\Delta t(\dddot{\mathbf{u}}_n + \dddot{\mathbf{u}}_{n+1})
\]  
(4.39)
\[ u_{n+1} = u_n + \Delta t \dot{u}_n + \frac{1}{4}(\Delta t)^2(\ddot{u}_n + \dddot{u}_{n+1}) \]  \hspace{1cm} (4.40)

Expressing \( \ddot{u}_{n+1} \) in terms of \( u_{n+1} \) in (4.40) and substituting this expression for \( \ddot{u}_{n+1} \) in (4.39), both \( \ddot{u}_{n+1} \) and \( \dot{u}_{n+1} \) can be expressed in terms of \( u_{n+1} \). Substituting these expressions for \( \ddot{u}_{n+1} \) and \( \dot{u}_{n+1} \) in terms of \( u_{n+1} \) in the equation of dynamic equilibrium (4.36) expressed at \( t_{n+1} \), we obtain:

\[
\frac{4}{(\Delta t)^2}Mu_{n+1} + \frac{2}{\Delta t}Cu_{n+1} + R(u_{n+1}) = \tilde{F}_{n+1}
\]  \hspace{1cm} (4.41)

where

\[
\tilde{F}_{n+1} = -M \ddot{u}_k(t_{n+1}) + M\left(\frac{4}{(\Delta t)^2}u_n + \frac{4}{\Delta t} \dot{u}_n + \ddot{u}_n\right) + C\left(\frac{2}{\Delta t}u_n + \dot{u}_n\right)
\]

or, in residual form,

\[
\Psi(u_{n+1}) = \left(\frac{4}{(\Delta t)^2}Mu_{n+1} + \frac{2}{\Delta t}Cu_{n+1} + R(u_{n+1})\right) - \tilde{F}_{n+1} = 0.
\]  \hspace{1cm} (4.42)

In the above equation, the vector, \( \Psi(u_{n+1}) \) is the residual force for the structure at the current step. In general, it is a nonlinear function of \( u_{n+1} \) since \( R(u_{n+1}) \) is generally a nonlinear implicit function of \( u_{n+1} \). Hence, (4.42) is solved iteratively for \( u_{n+1} \) using the Newton-Raphson method. Assuming that the i-th approximation to the displacement vector, \( u_{n+1}^i \), is available and expanding the residual vector function \( \Psi(u_{n+1}) \) using the Taylor series expansion about \( u_{n+1}^i \) while neglecting the higher order terms, we have:

\[
\Psi(u_{n+1}) = \Psi(u_{n+1}^i) + \left(\frac{\partial \Psi}{\partial u}\right)_{n+1}^i (u_{n+1} - u_{n+1}^i)
\]
Hence,
\[
\left( \frac{4}{(\Delta t)^2} M + \frac{2}{\Delta t} C + \frac{\partial R}{\partial u} \bigg|_{n+1} \right) \delta u' = -\left( \frac{4}{(\Delta t)^2} M u_n' + \frac{2}{\Delta t} C u_n' + R(u_n', u_{n+1}) \right) + \tilde{F}_{n+1} \tag{4.43}
\]
where \( \delta u' = u_{n+1} - u_{n+1}' \). The term in parentheses on the left hand side of (4.43) is called the *dynamic tangent stiffness matrix*. The term \( \frac{\partial R}{\partial u} \bigg|_{n+1} \) which constitutes the structure tangent stiffness matrix is obtained as:
\[
\left( \frac{\partial R}{\partial u} \right)_{n+1}^i = \left( K_T \right)_{n+1}^i = \sum_c \int_{\Omega_c} B^T D_T(\epsilon_n'^i) B \, d\Omega_c \tag{4.44}
\]
where \( D_T \) denotes the *consistent* tangent constitutive matrix. Further, \( u_{n+1}' \) is updated as:
\[
u_{n+1}' = u_{n+1}' + \delta u' \tag{4.45}
\]
In practice, the Newton-Raphson iterations for a given time step are stopped when a convergence criterion is satisfied. These criteria are summarized below.

1. Displacement criterion:
\[
\| \delta u' \| < \varepsilon_D \| u_n' - u_n \| \tag{4.46}
\]
where \( \| \| \) denotes the Euclidean norm and \( \varepsilon_D \) is a prescribed dimensionless tolerance parameter for the displacement \( u \).

2. Force criterion:
\[
\| \Psi(u_{n+1}') \| < \varepsilon_F \| \Psi(u_{n+1}) \| \tag{4.47}
\]
where \( \varepsilon_F \) is a prescribed dimensionless tolerance parameter for the norm of the out-of-
balance force vector.

3. Internal energy criterion:

$$\left\| \left( \delta u' \right)^T \left( \Psi(u_{n}^{n+1}) \right) \right\| < \varepsilon_E \left\| \left( \delta u' \right)^T \Psi(u_{n}^{n+1}) \right\|$$  \hspace{1cm} (4.48)

where $\varepsilon_E$ is a prescribed dimensionless tolerance for the internal energy. It is the internal energy criterion which is implemented in FEAP, the finite element analysis program used in this study.

4.7 Concepts of Classical Plasticity Theory

The plasticity theory mathematically describes the nonlinear behavior of a material under irreversible straining conditions. To capture the nonlinear behavior and irreversible straining conditions, a plasticity model is defined by the following three ingredients:

(1) An initial yield criterion represented by one or more in case of multi-surface plasticity yield surface(s) in the stress space, defining the stress level at which plastic deformations begin.

(2) A flow rule which relates the increment of plastic strain tensor to the stress state on the yield surface(s) (or plastic potential).

(3) A hardening rule which defines the change of configuration of the yield surface during the plastic deformation process.

Without loss of generality, the yield function can be expressed in scalar form as:

$$F(\sigma_{ij}, \varepsilon_{ij}^p, \kappa) = f(\sigma_{ij}, \varepsilon_{ij}^p) - Y(\kappa)$$  \hspace{1cm} (4.49)

where the function $f$, which indicates the form of the yield criterion, depends on the stress tensor, $\sigma_{ij}$, and the plastic strain tensor, $\varepsilon_{ij}^p$, and the current yield stress, $Y(\kappa)$, is related
to the deformation history of the material via the hardening parameter \( \kappa \). Usually the hardening parameter is defined as either the effective plastic strain \( \dot{\varepsilon}^p \), for the strain-hardening model:

\[
\kappa = \dot{\varepsilon}^p = \int_0^t \dot{\kappa} dt \quad \text{where} \quad \dot{\kappa} = \left[ \frac{2}{3} \left( \dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ij}^p \right) \right]^{1/2}
\]  
\hspace{1cm} (4.50)

or the plastic work \( W_p \), for the work-hardening model:

\[
\kappa = W_p = \int_0^t \dot{W}_p dt \quad \text{where} \quad \dot{W}_p = \sigma_{ij} \dot{\varepsilon}_{ij}^p.
\]  
\hspace{1cm} (4.51)

The yield surface in the stress space is defined by

\[
F(\sigma_{ij}, \varepsilon_{ij}^p, \kappa) = 0
\]  
\hspace{1cm} (4.52)

By convention, the yield function is defined in such a way that the elastic range forms the interior of the yield surface, i.e., the material is elastic if

\[
f(\sigma_{ij}, \varepsilon_{ij}^p) - Y(\kappa) < 0
\]  
\hspace{1cm} (4.53)

Further, the yield function is constrained to be less than or equal to zero. Thus, the stress state cannot lie outside the yield surface. When plastic deformations occur, the yield surface expands (for hardening materials), but the stress state remains on the "expanding" yield surface. Thus, the material state is given by the value of the yield function as:

\[
F(\sigma_{ij}, \varepsilon_{ij}^p, \kappa) < 0 : \text{elastic state}
\]  
\hspace{1cm} (4.54)

\[
F(\sigma_{ij}, \varepsilon_{ij}^p, \kappa) = 0 : \text{yielding state}
\]  
\hspace{1cm} (4.55)

\[
F(\sigma_{ij}, \varepsilon_{ij}^p, \kappa) > 0 : \text{not physically possible}
\]  
\hspace{1cm} (4.56)

Given the material state and the infinitesimal increment of the yield function, \( dF \), the material state event is given by:
\[ F < 0 \text{ and } dF > 0 : \text{ elastic loading} \quad (4.57) \]
\[ F = 0 \text{ and } dF < 0 : \text{ elastic unloading from plastic state} \quad (4.58) \]
\[ F = 0 \text{ and } dF = 0 : \text{ continued yielding} \quad (4.59) \]
\[ F = 0 \text{ and } dF > 0 : \text{ not possible in the plastic regime} \quad (4.60) \]

As expressed by (4.59), in case of continued yielding, the increment of yield function, \( dF \), is constrained to be zero. Thus,

\[ dF = \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial \epsilon^p_{ij}} d\epsilon^p_{ij} + \frac{\partial F}{\partial \kappa} = 0 \quad (4.61) \]

The above relation is called the \textit{consistency condition}. The loading-unloading conditions can also be formulated in the standard \textit{Kuhn-Tucker conditions} of optimization theory

\[ d\lambda \geq 0, \quad F d\lambda = 0, \quad F(\sigma, \epsilon^p, \kappa) \leq 0 \quad (4.62) \]

If the material is assumed to be isotropic, it does not have any preferred directions and the state of stress is uniquely defined by the three principal stresses. Therefore, the yield criterion can be described only in terms of the three principal stresses rather than the six components of the stress tensor. More conveniently, the yield criterion can also be expressed in terms of the three stress invariants \( I_1, I_2 \) and \( I_3 \), where \( I_1 \) is the first invariant of the stress tensor \( \sigma_{ij} \), and \( I_2 \) and \( I_3 \) are the second and third invariants of the deviatoric stress tensor \( s_{ij} \). That is, for an isotropic material,

\[ f(\sigma_{ij}, \epsilon^p_{ij}) = f(I_1, I_2, I_3). \quad (4.63) \]

The flow rule relates the plastic strain increments to the stress state. It is generally defined by using the concept of a plastic potential function, \( \mathcal{Q}(\sigma_{ij}, \epsilon^p_{ij}, \kappa) \), to which the incremental plastic strain tensor is orthogonal. In other words, the gradient of the plastic potential
surface in the stress space, \( \frac{\partial Q}{\partial \sigma_{ij}} \), defines the direction of the incremental plastic strain tensor, i.e.,

\[
d\varepsilon^p_{ij} = d\lambda \frac{\partial Q}{\partial \sigma_{ij}}. \tag{4.64}
\]

The flow rule is said to be \textit{associative} if the plastic potential function \( Q(\sigma_{ij}, \varepsilon^p_{ij}, \kappa) \) is assumed to be identical to the yield function \( F(\sigma_{ij}, \varepsilon^p_{ij}, \kappa) \). Therefore, for an associative flow rule,

\[
d\varepsilon^p_{ij} = d\lambda \frac{\partial F}{\partial \sigma_{ij}} = d\lambda \frac{\partial f}{\partial \sigma_{ij}}. \tag{4.65}
\]

The associative flow rule implies that plastic strain increments are normal to the yield surface in the stress space. The scalar parameter \( d\lambda \) in Eqs. (4.64) and (4.65) is called the \textit{plastic consistency parameter}. This parameter is zero for elastic behavior and positive for plastic behavior.

The hardening rule controls the change of configuration of the yield surface upon subsequent yielding. Two basic models of hardening exist: (a) \textit{isotropic hardening}, and (b) \textit{kinematic hardening}. In the case of isotropic hardening, the yield surface is assumed to expand uniformly about the origin in the stress space during the plastic deformation, while in the case of kinematic hardening, the yield surface simply translates in the stress space preserving its initial shape. More complex mixed hardening rules may be obtained by combining the isotropic and the kinematic hardening rules. The concepts presented in this section are applied to specific functional forms of the yield function for the \( J_2 \) plasticity model and
the cap plasticity model.

4.8 Computational Plasticity

In Sec. 4.6, it was shown that at every time step in integrating the equation of motion (4.36) a nonlinear algebraic equation, $\Psi(u_{n+1}) = 0$ needs to be solved. The Newton-Raphson scheme used to solve this nonlinear algebraic equation requires the computation of the following integrals

$$
\left(\frac{\partial R}{\partial u}\right)_{n+1}^i = (K_T)_{n+1}^i = \sum_\epsilon \int_\Omega B^T D_T(\epsilon_{n+1}^i) B \ d\Omega_\epsilon
$$

(4.66)

$$
k(u_{n+1}^i) = \sum_\epsilon \int_\Omega B^T \sigma_{n+1}^i \ d\Omega_\epsilon
$$

(4.67)

These integrals are evaluated at the element level, using the Gauss-quadrature formula. In the process, $\sigma_{n+1}^i$ and $D_T(\epsilon_{n+1}^i)$ are computed at each Gauss point given $\epsilon(u_{n+1}^i)$ at the Gauss point of interest. The stress tensor $\sigma_{n+1}^i$ is obtained from the strain tensor $\epsilon(u_{n+1}^i)$ in an incremental form through a numerical constitutive integration algorithm, called the \textit{return map} algorithm. The specific application of the return map algorithm to compute the stress history from the strain history is presented below for the $J_2$ plasticity model and the cap plasticity model.

4.8.1 Return Map Algorithm

In the context of nonlinear finite element analysis, it is necessary to update the stresses at the Gauss points of each element for a given incremental deformation. This phase of the calculation is called \textit{state determination}. Knowing the state variables (stresses, strains and history variables) at the converged time step $t_n$ and given a total increment of the strain
tensor, $\Delta \varepsilon$, the problem consists of finding the values of the state variables at time $t_{n+1}$ such that they satisfy the constitutive equations (yield criterion, flow rule and hardening rule). The return map algorithm provides an efficient and robust integration algorithm to obtain the update of the state variables. It belongs to the family of *elastic-predictor plastic-corrector algorithms*, and hence it is a two-step algorithm (Ortiz, Pinsky and Taylor 1983). In the first step, a purely elastic trial state is computed, and if the trial state violates the yield criterion, the plastic corrector is applied such that the final state satisfies the constitutive model.

The return map algorithm is a natural consequence of the fact that the constitutive relations can be split into elastic and plastic parts (Simo and Ortiz 1985). In the elastic step, the plastic response of the material is frozen, so that all the incremental strain goes into the elastic strain. In the plastic corrector phase, however, the spatial configuration of the material remains fixed, and the constitutive laws are satisfied through a "plastic" relaxation of the elastic trial stresses towards a suitably updated yield surface. Based on the notion of the elastic-plastic operator split, a return map algorithm may be conveniently defined by first defining an elastic predictor state, and then applying the plastic corrector using the elastic predictor as an initial condition. A geometrical interpretation of the algorithm is shown in Fig. 4.1

It can be seen that the elastic predictor is returned to the updated yield surface in successive steps. Each one of these steps involves a projection of the stresses onto a linear approximation of the yield surface. For an associative flow rule, the computed return path is an approximation to the steepest descent path as defined by the tensor $C^e - C^p$ (Ortiz
and Simo 1986). In general, the return path is not known in advance, nor can it be determined analytically. It is therefore necessary to compute the return path numerically in an iterative fashion. At every iteration, the yield function is linearized at the current values of the state variables, onto which the stress point is projected to obtain the state variables for the next iteration. The iterative procedure has converged when the yield criterion is satisfied.
If the yield function is convex and the plastic flow is derived from a convex loading function, the return map algorithm can be shown to be unconditionally stable and quadratically convergent (Simo and Ortiz 1985).

The following sections are devoted to obtaining stress history from strain history and the tangent stiffness matrix, from the computational stand point, for the $J_2$ plasticity model and the cap plasticity model.

### 4.8.2 Application of the Associative $J_2$ Plasticity Model to the Truss Element

The yield function for the associative $J_2$ plasticity model has the functional form:

$$F(s, \alpha, \varepsilon^p) = \|\Sigma\| - Y(\varepsilon^p) \leq 0$$

(4.68)

where $\Sigma$ is the relative deviatoric stress tensor defined as the difference between the deviatoric stress tensor $s$ and the back stress tensor $\alpha$. Therefore,

$$\Sigma = s - \alpha.$$  

(4.69)

In (4.68), $\|\|$ denotes the Euclidean norm of a second-order tensor (or matrix), and $Y$ is the current radius of the yield surface which is represented by a cylinder centered on the hydrostatic or tridiagonal axis in the principal stress space. The current radius of the yield surface, $Y(t)$, is directly related to the current uniaxial yield stress, $\sigma_y(t)$, through

$$Y(t) = \sqrt{\frac{2}{3}} \sigma_y(t).$$

(4.70)

Assuming a linear isotropic hardening rule, the current uniaxial yield stress $\sigma_y(t)$ is given by:

$$\sigma_y(t) = (\sigma_0 + H_{iso} \varepsilon^p(t)).$$

(4.71)
where $H_{iso}$ is the isotropic hardening modulus.

Thus, the material hardening parameter $\kappa$ (introduced in (4.49)) used in the assumed linear isotropic hardening law is the accumulated effective plastic strain.

For the truss element, the stress and strain spaces are uni-dimensional. Therefore, $\sigma_{22}(t) = \sigma_{33}(t) = 0$ and $\sigma_{ij} = 0$ for $i \neq j$, $i, j = 1, 2, 3$. Thus, the deviatoric stress tensor $\mathbf{s}$ reduces to:

\[
\mathbf{s} = \begin{bmatrix}
\frac{2\sigma_{11}}{3} & 0 & 0 \\
0 & -\frac{\sigma_{11}}{3} & 0 \\
0 & 0 & -\frac{\sigma_{11}}{3}
\end{bmatrix}
\]  

(4.72)

The plastic strain rate is given by:

\[
\dot{\varepsilon}_{ij} = \dot{\lambda} \frac{\partial F}{\partial \sigma_{ij}}
\]  

(4.73)

where $\dot{\lambda}$ is a positive constant. For the functional form of the yield function given in (4.68), it can be shown that:

\[
\dot{\varepsilon}_{11}^p + \dot{\varepsilon}_{22}^p + \dot{\varepsilon}_{33}^p = 0
\]  

(4.74)

The above relation indicates that for the $J_2$ plasticity model, the plastic deformations are isochoric. Since, there are no shear stresses, it is also obvious that:

\[
\varepsilon_{ij} = 0 \quad \text{for } i \neq j, \ i, j = 1, 2, 3
\]  

(4.75)

For the one-dimensional case, due to symmetry in the 22 and 33 directions of the plastic strain, it follows from (4.74) that:
\[
\dot{\varepsilon}_{12}^p = -0.5\dot{\varepsilon}_{11}^p \\
\dot{\varepsilon}_{13}^p = -0.5\dot{\varepsilon}_{11}^p
\]  

(4.76)

For linear kinematic hardening, the center of yielding \( \alpha \) evolves according to:

\[
\dot{\alpha} = \frac{2}{3} H_{kin} \varepsilon^p
\]  

(4.77)

where \( H_{kin} \) is the kinematic hardening modulus. From (4.68), (4.73), (4.74) and (4.77), it follows that:

\[
\dot{\alpha}_{22} = -0.5\dot{\alpha}_{11} \\
\dot{\alpha}_{33} = -0.5\dot{\alpha}_{11}
\]  

(4.78)

and

\[
\dot{\alpha}_{ij} = 0 \quad \text{for } i \neq j, \quad i, j = 1, 2, 3
\]  

(4.79)

Integrating the scalar equations in (4.78), we obtain:

\[
\alpha_{22} = -0.5\alpha_{11}, \quad \text{and } \alpha_{ij} = 0 \quad \text{for } i \neq j, \quad i, j = 1, 2, 3
\]  

(4.80)

Defining \( \tilde{\alpha}_{11} = \frac{2}{3}\alpha_{11} \) and using (4.80), (4.70) and (4.72), the yield function given in (4.68), for the one-dimensional case takes the form

\[
|\sigma_{11} - \tilde{\alpha}_{11}| - \sigma_y = 0
\]  

(4.81)

Dropping the subscript on \( \sigma \) and \( \tilde{\alpha} \), we get:

\[
|\sigma - \tilde{\alpha}| - \sigma_y = 0
\]  

(4.82)

Further, it can be shown that

\[
\dot{\alpha} = H_{kin} \varepsilon^p
\]  

(4.83)

Equations (4.71), (4.82) and (4.83) together with the additive decomposition of the uni-
axial strain $\varepsilon$:

$$
\varepsilon = \varepsilon^e + \varepsilon^p \quad \text{and} \quad d\varepsilon = d\varepsilon^e + d\varepsilon^p
$$

(4.84)

are discretized in the following section to obtain the stress history from the strain history and the constitutive tangent matrix (a scalar quantity in the one dimensional case) for each truss element. In the following sections, the superscript $\sim$ on $\alpha$ is dropped for the sake of convenience.

### 4.8.2.1 Constitutive Equations in Discrete Form

A discrete solution at time $t_n$ is defined in terms of the state $\varepsilon_n$, $\sigma_n$, $\alpha_n$, $\varepsilon^p_n$, and $\bar{\varepsilon}^p_n$. The solution is then advanced to time $t_{n+1}$ by specifying the strain, $\varepsilon_{n+1}$. Applying the implicit backward Euler rule to (4.73), (4.83), and (4.84), the following incremental relations are obtained:

$$
\varepsilon^p_{n+1} = \varepsilon^p_n + \Delta \lambda_{n+1} n_{n+1}
$$

(4.85)

$$
\alpha_{n+1} = \alpha_n + H_{ki} \Delta \lambda_{n+1} n_{n+1}
$$

(4.86)

$$
\bar{\varepsilon}^p_{n+1} = \bar{\varepsilon}^p_n + \Delta \lambda_{n+1}
$$

(4.87)

In the present uni-dimensional case, the unit directional vector $n_{n+1}$ reduces to the following scalar quantity:

$$
n_{n+1} = \frac{\sigma_{n+1} - \alpha_{n+1}}{|\sigma_{n+1} - \alpha_{n+1}|} = \begin{cases} +1 & \text{if yielding in tension} \\ -1 & \text{if yielding in compression} \end{cases}
$$

(4.88)

and the incremental discrete consistency parameter $\Delta \lambda_{n+1}$ is defined as:

$$
\Delta \lambda_{n+1} = \int_{t_n}^{t_{n+1}} \dot{\lambda} \, dt = \lambda_{n+1} - \lambda_n
$$

(4.89)
The incremental discrete consistency parameter \( \Delta \lambda_{n+1} \) is zero in case of purely elastic deformations and positive in case of elastoplastic deformations. The stress quantities are also incremented as follows:

\[
\sigma_{n+1} = E(\epsilon_{n+1} - \epsilon^p_{n+1}) = E(\epsilon_{n} - \epsilon^p_{n}) - E\Delta \lambda_{n+1} n_{n+1}
\]  
(4.90)

\[
\sigma_{n+1} - \alpha_{n+1} = E(\epsilon_{n+1} - \epsilon^p_{n}) - E\Delta \lambda_{n+1} n_{n+1} - \alpha_{n} - H_{kin}\Delta \lambda_{n+1} n_{n+1}
\]  
(4.91)

\[
\sigma_{y,n+1} = \sigma_{y,n} + H_{iso}\Delta \lambda_{n+1}
\]  
(4.92)

where \( E \) denotes the elastic Young's modulus.

The yield criterion is now given by:

\[
F_{n+1} = |\sigma_{n+1} - \alpha_{n+1}| - \sigma_{y,n+1} \leq 0
\]  
(4.93)

### 4.8.2.2 Return Map Algorithm

The return map algorithm for the uni-dimensional version of the \( J_2 \) plasticity model in discrete form is presented below.

**Elastic Predictor Step:**

In the elastic predictor step, it is assumed that no plastic deformation occurs during the current time step. Therefore,

\[
\Delta \lambda_{n+1}^{Trial} = 0
\]  
(4.94)

\[
(\epsilon^p_{n+1})^{Trial} = \epsilon^p_{n}
\]  
(4.95)

\[
\alpha_{n+1}^{Trial} = \alpha_{n}
\]  
(4.96)

\[
(\tilde{\epsilon}^p_{n+1})^{Trial} = \tilde{\epsilon}^p_{n}
\]  
(4.97)

\[
\sigma_{n+1}^{Trial} = E(\epsilon_{n+1} - \epsilon^p_{n})
\]  
(4.98)
\( \sigma_{y,n+1}^{\text{Trial}} = \sigma_{y,n} \) \hspace{1cm} (4.99)

where the superscript "Trial" denotes the trial values for the elastic predictor step. Now, the trial current state of stress is checked against the yield criterion as given in (4.93). If the yield criterion is satisfied, then the trial state of stress represents the true state of stress at time \( t_{n+1} \). Otherwise, the following plastic corrector step must be applied.

**Plastic Corrector Step:**

The plastic corrector step is based upon satisfying the consistency condition in discrete form:

\[
|\sigma_{n+1} - \alpha_{n+1}| - \sigma_{y,n+1} = 0 \hspace{1cm} (4.100)
\]

where

\[
\sigma_{n+1} = E(\epsilon_{n+1} - \epsilon_{n+1}^p) = E(\epsilon_{n+1} - \epsilon_{n+1}^p - \Delta \lambda_{n+1} n_{n+1}) \hspace{1cm} (4.101)
\]

\[
\alpha_{n+1} = \alpha_n + H_{\text{kin}} \Delta \lambda_{n+1} n_{n+1} \hspace{1cm} (4.102)
\]

\[
\sigma_{y,n+1} = \sigma_{y,n} + H_{\text{str}} \Delta \lambda_{n+1} \hspace{1cm} (4.103)
\]

In (4.101) and (4.103), the "unit" directional vector \( n_{n+1} \) is given by:

\[
n_{n+1} = \frac{\sigma_{n+1} - \alpha_{n+1}}{|\sigma_{n+1} - \alpha_{n+1}|} \hspace{1cm} (4.104)
\]

By substituting (4.101) and (4.102) into (4.100) and using (4.103), it follows that:

\[
|\sigma_{n+1} - \alpha_{n+1}| n_{n+1} = (\sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}}) - (E + H_{\text{kin}}) \Delta \lambda_{n+1} n_{n+1} \hspace{1cm} (4.105)
\]

Since two terms out of three in (4.105) are in the direction \( n_{n+1} \), it follows that the third term is also in this direction. Thus,
Using the above relation, (4.105) reduces further to:

\[
|\sigma_{n+1} - \alpha_{n+1}| = \left|\sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}}\right| - (E + H_{\text{kin}})\Delta \lambda_{n+1} \tag{4.107}
\]

By substituting (4.103) and (4.106) into (4.100), the discrete consistency condition becomes:

\[
|\sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}}| - (E + H_{\text{kin}})\Delta \lambda_{n+1} - \sigma_{y,n} - H_{\text{iso}}\Delta \lambda_{n+1} = 0 \tag{4.108}
\]

The incremental discrete consistency parameter \( \Delta \lambda_{n+1} \) is obtained from the above equation:

\[
\Delta \lambda_{n+1} = \frac{|\sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}}| - \sigma_{y,n}}{E + H_{\text{iso}} + H_{\text{kin}}} \tag{4.109}
\]

Once \( \Delta \lambda_{n+1} \) has been determined, the complete solution at time \( t_{n+1} \) can be updated as follows:

\[
n_{n+1} = \frac{\sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}}}{|\sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}}|} \tag{4.110}
\]

\[
\alpha_{n+1} = \alpha_{n} + H_{\text{kin}}\Delta \lambda_{n+1}n_{n+1} \tag{4.111}
\]

\[
\varepsilon_{n+1}^{p} = \varepsilon_{n}^{p} + \Delta \lambda_{n+1}n_{n+1} \tag{4.112}
\]

\[
\bar{\varepsilon}_{n+1}^{p} = \bar{\varepsilon}_{n}^{p} + \Delta \lambda_{n+1} \tag{4.113}
\]

\[
\sigma_{y,n+1} = \sigma_{y,n} + H_{\text{iso}}\Delta \lambda_{n+1} \tag{4.114}
\]

\[
\sigma_{n+1} = E(\varepsilon_{n+1} - \bar{\varepsilon}_{n+1}^{p}) \tag{4.115}
\]

The return map algorithm for the \( J_2 \) plasticity in the uniaxial stress case, is illustrated in
Fig. 4.2 Return map algorithm in the uniaxial case with pure kinematic hardening ($H_{iso} = 0$)
Fig. 4.2 for the particular case of pure kinematic hardening (i.e., $H_{iso} = 0$).

4.8.2.3 Consistent Tangent Stiffness Matrix for the Truss Element

The tangent stiffness matrix obtained by consistent linearization of the numerical scheme used for integrating the rate constitutive equations to obtain the stress history from the strain history is called the consistent (or algorithmic) tangent stiffness matrix. The derivation of the consistent tangent stiffness matrix for uni-axial $J_2$ plasticity is given below.

From the elastic stress-strain relationship and the flow rule, we have

$$\sigma_{n+1} = E(\epsilon_{n+1}^p - \epsilon_{n+1}^p)$$

$$\epsilon_{n+1}^p = \epsilon_n^p + \Delta \epsilon_{n+1}^p = \epsilon_n^p + \Delta \lambda_{n+1} n_{n+1}.$$  \hspace{1cm} (4.116)  \hspace{1cm} (4.117)

Differentiating (4.116) and (4.117) yields

$$d \sigma_{n+1} = E(d \epsilon_{n+1}^p - d \epsilon_{n+1}^p)$$

and

$$d \epsilon_{n+1}^p = d \lambda_{n+1} n_{n+1}.$$  \hspace{1cm} (4.118)  \hspace{1cm} (4.119)

Substituting (4.119) into (4.118) gives

$$d \sigma_{n+1} = E(d \epsilon_{n+1}^p - d \lambda_{n+1} n_{n+1})$$

From the consistency condition we have

$$f_{n+1} = \|\Sigma_{n+1}\| - \sigma_{y, n+1} = 0$$

$$df_{n+1} = d\|\Sigma_{n+1}\| - d\sigma_{y, n+1} = 0.$$  \hspace{1cm} (4.120)  \hspace{1cm} (4.121)

Differentiating the above, we have

$$df_{n+1} = d\|\Sigma_{n+1}\| - d\sigma_{y, n+1} = 0.$$  \hspace{1cm} (4.122)

Recognizing that
\[ \| \Sigma_{n+1} \| = (\sigma_{n+1} - \alpha_{n+1})n_{n+1} = (\sigma_{n+1} - \alpha_{n} - H_{kin}\Delta\lambda_{n+1} n_{n+1})n_{n+1} \]  
\text{(4.123)}

and

\[ \sigma_{y,n+1} = H_{iso}\Delta\lambda_{n+1} \]  
\text{(4.124)}

we have

\[ d\| \Sigma_{n+1} \| = (d\sigma_{n+1} - H_{kin}d\lambda_{n+1} n_{n+1})n_{n+1} \]  
\text{(4.125)}

and

\[ \sigma_{y,n+1} = H_{iso}d\lambda_{n+1} \]  
\text{(4.126)}

Upon substituting (4.126) and (4.125) into (4.122) we get

\[ (d\sigma_{n+1} - H_{kin}d\lambda_{n+1} n_{n+1})n_{n+1} - H_{iso}d\lambda_{n+1} = 0 \]  
\text{(4.127)}

Substituting the value of \( d\sigma_{n+1} \) from (4.120) into (4.127) and rearranging the terms, we obtain the expression for \( d\lambda_{n+1} \):

\[ d\lambda_{n+1} = \left( \frac{E}{E + H_{kin} + H_{iso}} \right) d\varepsilon_{n+1} n_{n+1} \]  
\text{(4.128)}

In the above re-arrangement use has been made of the identity

\[ n_{n+1} \cdot n_{n+1} = 1 \]  
\text{(4.129)}

Substitution of (4.128) into (4.120) leads to

\[ d\sigma_{n+1} = E\left( d\varepsilon_{n+1} - \left( \frac{E}{E + H_{kin} + H_{iso}} \right) d\varepsilon_{n+1} \right). \]  
\text{(4.130)}

Hence,

\[ \frac{d\sigma_{n+1}}{d\varepsilon_{n+1}} = E_{alp} = E\left( 1 - \frac{E}{E + H_{kin} + H_{iso}} \right) \]  
\text{(4.131)}

where \( E_{alp} \) is the algorithmic elasto-plastic tangent modulus for the \( J_2 \) truss element. If no
yielding takes place in the current step, then:

$$\frac{d\sigma_{n+1}}{d\epsilon_{n+1}} = E$$  \hspace{1cm} (4.132)

In the case of pure kinematic hardening, (4.131) reduces to

$$\frac{d\sigma_{n+1}}{d\epsilon_{n+1}} = E_{alg}^{ep} = E\left(1 - \frac{E}{E + H_{kin}}\right).$$

It is observed that, unlike in the multi-dimensional case, the 'continuum' and algorithmic tangent moduli are identical in the uni-axial case for the \(J_2\) plasticity model. This is due to the fact that in the uni-axial case with linear isotropic and kinematic hardening laws, there is no numerical approximation involved in computing a stress increment from a total strain increment.

### 4.8.3 Constitutive Relations for the Cap Model

The cap model used in this study is a non-smooth, multisurface, rate independent, associative plasticity model (DiMaggio and Sandler 1971; Sandler et al. 1976; Simo et al. 1988a).

As shown in Fig. 4.3, it is defined by a convex yield surface, which consists of a failure surface or envelope, \(f_1(\sigma)\), a hardening elliptical cap, \(f_2(\sigma, \kappa)\), and a tension cut-off region, \(f_3(\sigma)\), where \(\sigma\) denotes the stress tensor and \(\kappa\) is the hardening parameter. Thus, the failure envelope and the tension cut-off region are modeled as ideal plasticity surfaces while the cap is modeled as a strain hardening surface.

The functional forms of \(f_1, f_2\) and \(f_3\) are:

$$f_1(\sigma) = \|s\| - F_c(I_1), \quad \text{for } T \leq I_1 \leq \kappa,$$  \hspace{1cm} (4.133)

where
Fig. 4.3  The yield surface of the cap model

\[
F_c(I_1) = \alpha - \lambda e^{-\beta I_1} + \theta I_1 ,
\]

\[
f_3(||s||, I_1, \kappa) = F_c(||s||, I_1, \kappa) - F_c(\kappa) , \quad \text{for } \kappa \leq I_1 \leq X(\kappa) \tag{4.135}
\]

where

\[
F_c(||s||, I_1, \kappa) = \sqrt{||s||^2 + \left(\frac{I_1 - L(\kappa)}{R}\right)^2} ,
\]

and

\[
f_3(\sigma) = T - I_1 , \quad \text{for } I_1 = T . \tag{4.137}
\]

In the above definitions, \(\alpha, \beta, \lambda, \theta\) and \(R\) are material parameters for the cap and failure
evelope, and \(T\) is the tension cut-off or maximum allowable hydrostatic tension, which is
a material constant.

\(I_1\) and \(||s||\) are the first invariant of the stress tensor \(\sigma\) and the norm of the deviatoric
stress tensor, respectively. The function \(L(\kappa)\) is defined as:

\[
L(\kappa) = \begin{cases} 
\kappa & \text{if } \kappa > 0 \\
0 & \text{if } \kappa \leq 0 
\end{cases} \tag{4.138}
\]
In the following analysis, compressive stresses and compressive strains (i.e., compaction) are taken to be positive. The point of intersection of the cap with the I_1-axis is defined as

$$X(\kappa) = \kappa + RF_1(\kappa)$$  \hspace{1cm} (4.139)

in which R represents the major to minor axis ratio. The deviatoric strain tensor is defined by

$$e = \epsilon - \frac{1}{3} \bar{I}_1 I$$  \hspace{1cm} (4.140)

where $\epsilon$ denotes the strain tensor and $\bar{I}_1 = \text{trace}(\epsilon)$. The hardening parameter $\kappa$ is implicitly defined in terms of the effective plastic volumetric strain, $\bar{\varepsilon}^p_v$, as

$$\bar{\varepsilon}^p_v = W(1 - e^{-DX(\kappa)})$$  \hspace{1cm} (4.141)

in which $W$ and $D$ are material parameters. The effective plastic volumetric strain, $\bar{\varepsilon}^p_v$, is a history dependent functional of the volumetric plastic strain and is defined in rate form as:

$$\dot{\bar{\varepsilon}}^p_v = \begin{cases} \dot{\bar{I}}^p_v = \dot{\varepsilon}^p_v & \text{if } \dot{I}_1 > 0 \text{, or } \kappa > 0 \text{ and } \kappa > I_1 \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (4.142)

The hardening relationship between $\bar{\varepsilon}^p_v$ and $X(\kappa)$ in (4.141) is represented graphically in Fig. 4.4. The hardening parameter $\kappa$ is thus obtained by substituting (4.142) into (4.141) and solving for $X(\kappa)$, after which (4.139) is solved for $\kappa$ given $X(\kappa)$ using a Newton iterative procedure. Therefore, according to the hardening rule defined in (4.138), (4.139), (4.141), and (4.142), the cap moves out or in as $\bar{\varepsilon}^p_v$ increases or decreases, respectively, as shown in Fig. 4.5. If the stress point reaches the failure envelope, the plastic strain rate
vector has a negative volumetric component (dilatancy), see Fig. 4.6(a). This causes the cap to move back until it reaches the stress point, thus limiting further plastic volume increases, see Fig. 4.6(b). At the compressive corner point, movement of the cap is prevented and the model behaves as in perfect plasticity, thus avoiding a softening response.

The elastic domain is defined in terms of the three yield surfaces as
Fig. 4.6 Contraction behavior of the cap

\[ f_i(\sigma) < 0, \quad i = 1, 2, 3 \]  \hspace{1cm} (4.143)

The flow rule is assumed to be associative and, in the case of multisurface plasticity, takes the generalized form:

\[ \varepsilon^p = \sum_{i=1}^{3} \dot{\gamma}_i \frac{\partial f_i}{\partial \sigma}, \]  \hspace{1cm} (4.144)

where \( \dot{\gamma}_i, i=1, 2, 3 \), are the plastic consistency parameters. Plastic loading or elastic loading/unloading can be formulated in the standard Kuhn-Tucker form as:

\[ \dot{\gamma}_i \geq 0, \quad f_i \leq 0, \quad \text{and} \quad \dot{\gamma}_i f_i = 0, \quad i=1,2,3 \]  \hspace{1cm} (4.145)

4.8.3.1 Return map Algorithm

In order to compute the internal resisting force increments for a given incremental strain tensor, the corresponding incremental stress tensor must be determined at each integration point (Gauss point) of the element. As for the \( J_2 \) single-surface plasticity model, the return map algorithm can be used to find the true internal stresses. It consists of two major steps,
the elastic-predictor step and the return mapping to the yield surface, which can be geometrically interpreted as a closest point projection of the elastic trial stress onto the yield surface, also referred to as a plastic corrector.

In the following discussion, \( \epsilon^p \), \( \epsilon^e \) and \( \epsilon \) denote the plastic strain tensor, the elastic strain tensor and the total strain tensor, respectively. Application of an implicit backward Euler integration scheme to the constitutive equations of the cap model results in the following incremental constitutive equations (Hofstetter et al. 1993):

\[
\epsilon^p_{n+1} = \epsilon^p_n + \sum_{i=1}^{3} \Delta \gamma_{i,n+1} \left( \frac{\partial f_i}{\partial \sigma} \right)_{n+1}
\]

\[
\sigma^\text{Trial}_{n+1} = \mathbf{C}(\epsilon_{n+1} - \epsilon^p_n)
\]

\[
\sigma_{n+1} = \mathbf{C}(\epsilon_{n+1} - \epsilon^p_{n+1}) = \sigma^\text{Trial}_{n+1} - \mathbf{C} \Delta \epsilon^p_{n+1}
\]

where \( \Delta \gamma_{i,n+1} = \int_{t_n}^{t_{n+1}} \dot{\gamma} \, dt \). \( \mathbf{C} \) is the elastic constitutive fourth-order tensor and the symbol (:) denotes the doubly contracted tensor product, e.g., \( (\mathbf{C} : \epsilon)_{ij} = C_{ijkl} \epsilon_{kl} \). The Kuhn-Tucker conditions in discrete form can be expressed as:

\[
\Delta \gamma_{i,n+1} \geq 0, \quad f_{i,n+1} \leq 0 \quad \text{and} \quad \Delta \gamma_{i,n+1} f_{i,n+1} = 0, \quad i = 1, 2, 3.
\]

The trial elastic deviatoric and spherical stresses are defined as:

\[
s^\text{Trial}_{n+1} = 2G(\epsilon_{n+1} - \epsilon^p_n)
\]

\[
I^\text{Trial}_{1,n+1} = 3K(\overline{I}_{1,n+1} - \overline{I}^p_{1,n+1})
\]

It can be shown (Simo and Hughes, to appear) that the convexity condition of the yield surface implies that \( f_{i,n+1} \leq f^\text{Trial}_{i,n+1} \) for \( i = 1, 2, 3 \). Hence, if \( f^\text{Trial}_{i,n+1} \leq 0 \) for \( i = 1, 2, 3 \), then
the process is elastic. Otherwise, the current step is inelastic and it is necessary to determine \( \Delta \varepsilon_{n+1}^{p} \) and \( \kappa_{n+1} \). Then, the true stresses can be obtained from the trial elastic stresses as:

\[
\begin{align*}
    s_{n+1}^{trial} & = s_{n+1}^{trial} - 2G \Delta \varepsilon_{n+1}^{p} , \\
    I_{1,n+1}^{trial} & = I_{1,n+1}^{trial} - 3K \Delta I_{1,n+1}^{p} .
\end{align*}
\]

The difficulty associated with multisurface plasticity is that the active set of yield surfaces is not known in advance, since \( f_{i,n+1}^{trial} > 0 \) does not guarantee that \( f_{1,n+1} = 0 \) if more than one yield criterion is active (Simo, Kennedy, and Govindjee, 1988b).

### 4.8.3.2 Loading in the Various Modes of the Cap Model

The following sections summarize the return mapping algorithm for the various modes of the cap model. A detailed derivation of the algorithm is available in the literature (Hofstetter, Simo and Taylor, 1993).

1. **Loading in the Failure Envelope Mode:**

   Loading in the failure envelope mode is characterized by \( f_{1,n+1}^{trial} > 0 \), \( \Delta \gamma_{1,n+1} > 0 \), \( \Delta \gamma_{2,n+1} = \Delta \gamma_{3,n+1} = 0 \). To determine the discrete consistency parameter, \( \Delta \gamma_{1,n+1} \), associated with the failure surface \( f_{1} = 0 \), the following nonlinear scalar equation must be solved for \( I_{1,n+1}^{trial} \):

\[
I_{1,n+1}^{trial} + 9K \Delta \gamma_{1,n+1} \left( \frac{dF_{e}}{dI_{1,n+1}^{trial}} \right) - I_{1,n+1} = 0
\]

where
\[ \Delta \gamma_{1,n+1} = \frac{\|s_{n+1}^{\text{trial}}\| - F_e(I_{1,n+1})}{2G} \]  

(4.155)

The above equation can be solved using a Newton iterative technique. Once \( I_{1,n+1} \) is known, \( \Delta \gamma_{1,n+1} \) is obtained from (4.155) and the true state at time \( t_{n+1} \) is determined through the following relations:

\[ \Delta I_{1,n+1}^p = -3 \Delta \gamma_{1,n+1} \frac{dF_e(I_{1,n+1})}{dI_{1,n+1}} \]  

(4.156)

\[ \Delta e_{n+1}^p = \Delta \gamma_{1,n+1} n_{n+1} \]  

(4.157)

\[ s_{n+1} = \|s_{n+1}\| n_{n+1} \]  

(4.158)

where

\[ \|s_{n+1}\| = \|s_{n+1}^{\text{trial}}\| - 2G \Delta \gamma_{1,n+1} \quad \text{and} \quad n_{n+1} = \frac{s_{n+1}^{\text{trial}}}{\|s_{n+1}\|} \]  

(4.159)

\[ \sigma_{n+1} = s_{n+1} + 1/3 I_{1,n+1} I \]  

(4.160)

(2) Loading in the Tensile Corner Region:

Loading in the tensile corner region is characterized by \( \Delta \gamma_{1,n+1} > 0, \Delta \gamma_{2,n+1} = 0 \) and \( \Delta \gamma_{3,n+1} > 0 \). Since both the failure surface and the tension cut-off surface behave as in ideal plasticity, the return point, \( \sigma_{n+1} \), is the point of intersection of these two surfaces. Thus,

\[ I_{1,n+1} = T \]  

(4.161)

\[ \|s_{n+1}\| = F_e(T) \]  

(4.162)

Solving for the discrete consistency parameters \( \Delta \gamma_{1,n+1} \) and \( \Delta \gamma_{3,n+1} \) results in:
\[ \Delta \gamma_{1,n+1} = \frac{\left\| s_{n+1}^{\text{Trial}} \right\| - F_e(T)}{2G}, \]  
\[ (4.163) \]

\[ \Delta \gamma_{3,n+1} = \frac{T - I_{1,n+1}^{\text{Trial}}}{9K} - \Delta \gamma_{1,n+1} \frac{dF_e(T)}{dI_1}. \]  
\[ (4.164) \]

Once \( \Delta \gamma_{1,n+1} \) and \( \Delta \gamma_{3,n+1} \) are known, the true state at time \( t_{n+1} \) is obtained according to the following relations:

\[ \Delta I_{1,n+1}^p = -3 \left( \Delta \gamma_{1,n+1} \frac{dF_e(T)}{dI_1} + \Delta \gamma_{3,n+1} \right) = -3 \left( \frac{T - I_{1,n+1}^{\text{Trial}}}{9K} \right) \]  
\[ (4.165) \]

\[ \Delta e_{n+1}^p = \Delta \gamma_{1,n+1} n_{n+1} \]  
\[ (4.166) \]

\[ s_{n+1} = \left\| s_{n+1} \right\| n_{n+1} \]  
\[ (4.167) \]

\[ \sigma_{n+1} = s_{n+1} + 1/3 I_{1,n+1} I \]  
\[ (4.168) \]

In (4.167), \( \left\| s_{n+1} \right\| \) and \( n_{n+1} \) are given by (4.162) and (4.159)_2, respectively.

(3) **Loading in the Cap Mode:**

Loading within the cap region is characterized by \( f_{2,n+1}^{\text{Trial}} > 0, \Delta \gamma_{2,n+1} > 0 \), and \( \Delta \gamma_{1,n+1} = \Delta \gamma_{3,n+1} = 0 \). The discrete consistency parameter \( \Delta \gamma_{2,n+1} \) is found to be the solution of the following set of nonlinear equations:

\[ \sqrt{\left( \frac{\left\| s_{n+1}^{\text{Trial}} \right\| F_e(\kappa_{n+1})}{F_e(\kappa_{n+1}) + 2G \Delta \gamma_{2,n+1}} \right)^2 + \left( \frac{I_{1,n+1}^{\text{Trial}} - \kappa_{n+1}}{R + (9K \Delta \gamma_{2,n+1})/(RF_e(\kappa_{n+1}))} \right)^2}, \]  
\[ (4.169) \]

\[ -F_e(\kappa_{n+1}) = 0 \]

\[ \Delta \gamma_{2,n+1} = \frac{R^2 H(\kappa_{n+1}) F_e(\kappa_{n+1})}{3(I_{1,n+1} - \kappa_{n+1})^3} \]  
\[ (4.170) \]

and
\[ I_{1,n+1} = I_{1,n+1}^{Trial} - 3KH(\kappa_{n+1}) \]  

where

\[ H(\kappa_{n+1}) = W(e^{-DX(\kappa_n)} - e^{-DX(\kappa_{n+1})}) \].

Substituting (4.171) into (4.170) and inserting the result into (4.169) gives a scalar nonlinear equation in terms of \( \kappa_{n+1} \), which can be solved by a Newton iteration technique. Once \( \kappa_{n+1} \) is known, \( \Delta \gamma_{2,n+1} \) and \( I_{1,n+1} \) are determined by (4.170) and (4.171), respectively.

In the case where \( I_{1,n+1} = \kappa_{n+1} \), the right-hand side of (4.170) is an indeterminate expression. However, in this case \( I_{1,n+1} = \kappa_{n+1} = I_{1,n+1}^{Trial} = \kappa_n \) and \( \Delta \gamma_{2,n+1} \) can be obtained from (4.169) as

\[ \Delta \gamma_{2,n+1} = \frac{\|s_{n+1}\| - F_e(\kappa_n)}{2G} \]  

The true state at time \( t_{n+1} \) is then obtained through the following relations:

\[ \Delta \bar{I}_{1,n+1} = 3 \Delta \gamma_{2,n+1} \frac{\partial f_{2,n+1}}{\partial I_{1,n+1}} = W(e^{-DX(\kappa_n)} - e^{-DX(\kappa_{n+1})}) = H(\kappa_{n+1}) \]  

\[ \Delta e_{n+1} = \Delta \gamma_{2,n+1} \frac{\partial f_{2,n+1}}{\partial s_{n+1}} = \Delta \gamma_{2,n+1} \frac{s_{n+1}}{F_e(\|s_{n+1}\|, I_{1,n+1}, \kappa_{n+1})} \]  

where

\[ \|s_{n+1}\| = \frac{\|s_{n+1}\|}{1 + \frac{2G\Delta \gamma_{2,n+1}}{F_e(\kappa_{n+1})}} \]  

The deviatoric stresses, \( s_{n+1} \), and total stresses, \( \sigma_{n+1} \), are obtained using (4.158), (4.159)_2, and (4.160).
(4) **Loading in the Compressive Corner Region:**

This mode is characterized by \( \Delta \gamma_{1,n+1} > 0 \), \( \Delta \gamma_{2,n+1} > 0 \) and \( \Delta \gamma_{3,n+1} = 0 \). From the hardening law, the movement of the cap is prevented if the stress point is at the compressive corner point. Thus, the model behaves as in perfect plasticity. Hence, the final stress point must lie at the intersection of the cap and failure surfaces. In this case, the discrete consistency parameters are found to be:

\[
\Delta \gamma_{1,n+1} = \frac{\kappa_n - I_{1,n+1}^\text{Trial}}{9K \frac{dF_e(\kappa_n)}{dI_1}},
\]

\[\Delta \gamma_{2,n+1} = \frac{\|s_{n+1}^\text{Trial}\| - F_e(\kappa_n)}{2G} - \Delta \gamma_{1,n+1} \]  \hspace{1cm} (4.178)

Once \( \Delta \gamma_{1,n+1} \) and \( \Delta \gamma_{2,n+1} \) are known, the complete state at time \( t_{n+1} \) can be obtained as follows:

\[
\Delta T^\rho_{1,n+1} = -3 \Delta \gamma_{1,n+1} \frac{dF_e(\kappa_n)}{dI_1} \]  \hspace{1cm} (4.179)

\[
\Delta e^\rho_{n+1} = (\Delta \gamma_{1,n+1} + \Delta \gamma_{2,n+1}) n_{n+1} \]  \hspace{1cm} (4.180)

\[
I_{1,n+1} = \kappa_n \]  \hspace{1cm} (4.181)

\[
\|s_{n+1}\| = F_e(\kappa_n) \]  \hspace{1cm} (4.182)

In (4.180), \( n_{n+1} \) is defined as in (4.159)\(_2\). The deviatoric stresses, \( s_{n+1} \), and total stresses, \( \sigma_{n+1} \), are obtained by substituting (4.181) and (4.182) into (4.158) and (4.160).

(5) **Loading in the Tension Cut-off Region:**

Loading in the tension cut-off mode is characterized by \( f_{3,n+1}^\text{Trial} > 0 \), \( \Delta \gamma_{3,n+1} > 0 \) and
\[ \Delta y_{1,n+1} = \Delta y_{2,n+1} = 0 \]. Using the geometric interpretation of the return mapping algorithm as a closest-point projection of the elastic trial state onto the yield surface, it follows that:

\[ I_{1,n+1} = T \]  \hspace{1cm} (4.183)

and

\[ s_{n+1} = s_{n+1}^{\text{Trial}}. \]  \hspace{1cm} (4.184)

The plastic deviatoric and volumetric strains are given by:

\[ \Delta e_{n+1}^p \]  \hspace{1cm} (4.185)

and

\[ \Delta \gamma_{1,n+1} = -3 \Delta y_{3,n+1} \]  \hspace{1cm} (4.186)

where

\[ \Delta y_{3,n+1} = \frac{T - I_{1,n+1}^{\text{Trial}}}{9K}. \]  \hspace{1cm} (4.187)

Finally, the return map algorithm is illustrated in Fig. 4.9 for the various regions of the cap model.

**4.8.3.3 Consistent Elastoplastic Tangent Moduli**

The consistent elastoplastic tangent moduli for the cap model have been derived by Hofstetter et al. (1993). The interested reader is referred to their paper. It should be emphasized however that the hardening law for the cap region is nonassociative leading to unsymmetric consistent tangent moduli. Therefore, the unsymmetric equation solver of FEAP invoked by the macro-command 'utan' must be used. Obviously, an unsymmetric
4.9 Conclusions

The basic principles of nonlinear finite element analysis of structures for the calculation of dynamic response to ground motions was discussed in this chapter. The concepts of plasticity theory used to mathematically model the nonlinear, hysteretic behavior of actual structures were reviewed. In particular, two plasticity models, the $J_2$ plasticity model and the cap plasticity model, were presented. From the computational standpoint, the return map algorithm using the notions of elastic-plastic operator split and closest point projection in the stress space was described. The formulation of the return map algorithm used to integrate the rate constitutive equations for the $J_2$ and cap plasticity models was presented in detail.

The $J_2$ plasticity model allows to model realistically the inelastic behavior of metals;
while the predictive ability of the cap model is very good for soil and concrete materials. These inelastic constitutive models have the advantage to predict realistically the inelastic material behavior of structures near their failure region, which is very important for the safety or reliability analysis since most structural fail in the nonlinear range. These inelastic models can be incorporated in the framework of the finite element reliability method developed to determine realistically the seismic reliability of structural systems.
CHAPTER 5 Dynamic Response Sensitivity Analysis

5.1 Introduction

This chapter deals with the calculation of response sensitivities with respect to loading and material parameters for inelastic dynamic systems based on plasticity based constitutive models. Alternative methods to constitute sensitivity equations are discussed. In formulating the sensitivity equations, questions regarding continuity of response sensitivities in time arise. A mathematical description of discontinuities (if any) together with a physical interpretation of their cause in terms of switching between material states of a plasticity model are presented. The complete set of sensitivity equations for the uniaxial $J_2$ plasticity model and the multi-axial, multi-surface cap plasticity model, obtained by differentiating exactly the numerical scheme for the response calculation are derived. Numerical examples for both single-degree-of-freedom (SDOF) and multi-degree-of-freedom (MDOF) systems incorporating the above plasticity models are presented for computation of their response sensitivities with respect to both loading and material parameters.

These response sensitivities with respect to loading and material parameters form the basic inputs into a reliability analysis of a structure with uncertain system (or material) parameters subjected to stochastic loading.

5.2 Method for Calculating the Response Sensitivities

The sensitivity of a response quantity with respect to a generic sensitivity parameter $x$ denoting either a loading or a material parameter is mathematically expressed as the partial derivative of this response quantity with respect to the variable $x$. The response sensitivity is both a function of time $t$ and the sensitivity parameter $x$ which takes a
particular value \( x_0 \) during a sensitivity analysis. Hence, by response sensitivity we actually mean the partial derivative of the response quantity with respect to \( x \) evaluated at \( x = x_0 \). Mathematically, this is denoted as \( \left. \frac{\partial w(t)}{\partial x} \right|_{x = x_0} \), where \( w(t) \) denotes a generic response quantity such as displacement, stress, strain, cumulative effective plastic strain, acceleration etc.

By definition, \( \left. \frac{\partial w(t)}{\partial x} \right|_{x = x_0} \) is continuous in time at a particular time \( t = t_0 \) if:

\[
\lim_{\Delta t_0 \to 0} \left. \frac{\partial w(t)}{\partial x} \right|_{x = x_0, t = t_0 - \Delta t_0} = \lim_{\Delta t_0 \to 0} \left. \frac{\partial w(t)}{\partial x} \right|_{x = x_0, t = t_0 + \Delta t_0} = \left. \frac{\partial w(t)}{\partial x} \right|_{x = x_0, t = t_0}
\] (5.1)

The response sensitivity of an inelastic system, \( \left. \frac{\partial w(t)}{\partial x} \right|_{x = x_0} \) is not continuous in time. The discontinuities along the time axis are due to the switchings between material states (e.g., switching between elastic and plastic states and vice-versa). While a single surface plasticity model such as \( J_2 \) plasticity, allows switchings between only two material states, elastic and plastic, multi-surface plasticity models such as the cap plasticity model allow switchings between more than two material states.

For illustration purposes, consider the semi-discretized (discretized in space using the finite element method) equation of motion for quasi-static loading. The displacement response sensitivity is obtained by differentiating the following equilibrium equation:

\[
\mathbf{R}(\mathbf{u}(t, x), x) = \mathbf{F}(t, x)
\] (5.2)

with respect to the sensitivity parameter \( x \). The left hand side of (5.2), \( \mathbf{R}(\mathbf{u}(x, t), x) \)
Fig. 5.1 Discontinuities in Displacement Response Sensitivity
**Method I**

Semi-discretized, nonlinear matrix differential equation of motion
(discretized in space and continuous in time)

Exact differentiation w.r.t 'x'

Nonlinear matrix differential equation governing the exact gradient, $\partial u(t)/\partial x$, of analytical response time history, $u(t)$,

$$(A_1)$$

Time discretization of both response and gradient eqn.

Nonlinear matrix algebraic equation governing the numerically computed response gradient,

$$(\partial u(t)/\partial x)|_{t=t_i}, i = 1, 2, ..., n$$

$$(A_2)$$

Numerical gradient of analytical response:

$$(\partial u(t)/\partial x)|_{t=t_i}, i = 1, 2, ..., n$$

**Method II**

Time discretization of response equation

Set of nonlinear, recursive algebraic equations governing numerically computed response $u_i, i = 1, 2, ..., n$

$$(B_1)$$

Exact differentiation w.r.t 'x'

Nonlinear, recursive, algebraic equations governing the exact gradient of the numerical response, $\partial u_i/\partial x, i = 1, 2, ..., n$

$$(B_2)$$

Analytical (exact) gradient of numerical response $\partial u_i/\partial x, i = 1, 2, ..., n$

---

*Fig. 5.2* Flow Chart for the Two strategies to Compute Response Sensitivities
represents the history dependent internal resisting force vector which is a continuous function in time of \( \mathbf{u}(x, t) \), the displacement vector satisfying the equilibrium equation (5.2).

The right hand side of (5.2), \( \mathbf{F}(t, x) \) is the external quasi-static loading. The displacement response vector \( \mathbf{u}(t, x) \) at a particular time \( 't' \) is a function of history variables (not explicitly shown here such as the hardening parameter, the current yield strength, and the current back stress). Assuming that \( 't' \) is the exact time of occurrence of a transition from material state A to material state B (A and B denoting generic material states, e.g., elastic and plastic state, any of the three: cap, tension cut-off, and the failure mode of the cap plasticity model), the governing sensitivity equations at \( t^- \) and \( t^+ \), are obtained by differentiating (5.2) exactly with respect to \( 'x' \) and using the chain rule of differentiation (see footnote on page 141).

\[
K_{T_\alpha} \frac{\partial \mathbf{u}(t^-, x)}{\partial x} = - \left. \frac{\partial \mathbf{R}(\mathbf{u}(t^-, x), x)}{\partial x} \right|_{\mathbf{u}} + \left. \frac{\partial \mathbf{F}(t^-, x)}{\partial x} \right|_{\mathbf{u}} \tag{5.3}
\]

\[
K_{T_\beta} \frac{\partial \mathbf{u}(t^+, x)}{\partial x} = - \left. \frac{\partial \mathbf{R}(\mathbf{u}(t^+, x), x)}{\partial x} \right|_{\mathbf{u}} + \left. \frac{\partial \mathbf{F}(t^+, x)}{\partial x} \right|_{\mathbf{u}} \tag{5.4}
\]

where \( K_{T_\alpha} \) and \( K_{T_\beta} \) denote the continuum tangent stiffness matrices for states A and B, respectively. The terms \( \left. \frac{\partial \mathbf{R}(\mathbf{u}(t^-, x), x)}{\partial x} \right|_{\mathbf{u}} \) and \( \left. \frac{\partial \mathbf{R}(\mathbf{u}(t^+, x), x)}{\partial x} \right|_{\mathbf{u}} \), which are strongly history dependent, are computed from

\[
\left. \frac{\partial \mathbf{R}(\mathbf{u}(t^-, x), x)}{\partial x} \right|_{\mathbf{u}} = \sum_{\epsilon} \int_{\Omega_\epsilon} \mathbf{B}^T \frac{\partial \mathbf{\sigma}}{\partial x} \bigg|_{\mathbf{u}} d\Omega_\epsilon,
\]

where material state A is used for computing \( \left. \frac{\partial \mathbf{\sigma}}{\partial x} \right|_{\mathbf{u}} \) and

\[
\left. \frac{\partial \mathbf{R}(\mathbf{u}(t^+, x), x)}{\partial x} \right|_{\mathbf{u}} = \sum_{\epsilon} \int_{\Omega_\epsilon} \mathbf{B}^T \frac{\partial \mathbf{\sigma}}{\partial x} \bigg|_{\mathbf{u}} d\Omega_\epsilon.
\]
where material state B is used for computing \( \frac{\partial \sigma}{\partial x} \). The displacement response sensitivities \( \frac{\partial u(t^-, x)}{\partial x} \) and \( \frac{\partial u(t^+, x)}{\partial x} \) are obtained as solutions to (5.3) and (5.4), respectively. The displacement response sensitivity, \( \frac{\partial u(t, x)}{\partial x} \) at ‘t’, the exact time of transition between two material is undefined as the constitutive relations are not uniquely defined at the exact transition point. Since \( K_{T_a} \) and \( K_{T_b} \) are different and so are the terms \( \frac{\partial R(u(t^-, x), x)}{\partial x} \) and \( \frac{\partial R(u(t^+, x), x)}{\partial x} \) due to the different constitutive relations used in computing them, \( \frac{\partial u(t^-, x)}{\partial x} \) and \( \frac{\partial u(t^+, x)}{\partial x} \) are not necessarily equal. Thus, discontinuities in the response sensitivities with respect to loading and material variables may occur a finite number of times along the time axis corresponding to transition points from one material state to another.

Under condition of dynamic loading, the equation of motion from (4.33) is:

\[
M(x)\ddot{u}(t, x) + C(x)\dot{u}(t, x) + R(u(t, x), x) = F(t, x)
\]  

(5.5)

Assuming that the transition from material state A to B occurs exactly at time ‘t’, the sensitivity equations at \( t^- \) and \( t^+ \) are obtained by differentiating (5.5) with respect to ‘x’. It is to be noted that the material constitutive relations are uniquely defined just before and after the exact material transition times.
Assuming material state A at time \( t^- \), we obtain

\[
M(x) \frac{d\bar{u}(t^-, x)}{dx} + C(x) \frac{du(t^-, x)}{dx} + K_r \frac{d\bar{u}(t^-, x)}{dx} = -\left( \frac{\partial R(u(t^-, x), x)}{\partial x} \right)_u + \frac{\partial F(t^-, x)}{dx}
\]

\[
-\frac{\partial M(x)}{dx} \bar{u}(t^-, x) - \frac{\partial C(x)}{dx} \bar{u}(t^-, x)
\]

(5.6)

Similarly, assuming material state B at time \( t^+ \):

\[
M(x) \frac{d\bar{u}(t^+, x)}{dx} + C(x) \frac{du(t^+, x)}{dx} + K_r \frac{d\bar{u}(t^+, x)}{dx} = -\left( \frac{\partial R(u(t^+, x), x)}{\partial x} \right)_u + \frac{\partial F(t^+, x)}{dx}
\]

\[
-\frac{\partial M(x)}{dx} \bar{u}(t^+, x) - \frac{\partial C(x)}{dx} \bar{u}(t^+, x)
\]

(5.7)

As in the case of quasi-static loading, the sensitivities \( \frac{\partial u(t, x)}{dx} \), \( \frac{\partial \bar{u}(t, x)}{dx} \) and \( \frac{\partial \bar{u}(t, x)}{dx} \) are defined for the material states A and B at the exact material state transition time, but undefined for the transition state, and discontinuities in time of these response sensitivities occur at the transition from A to B. Figure 5.1 illustrates the occurrence of discontinuities in displacement response sensitivity at the times of material state transition (times \( t_1 \), \( t_2 \) and \( t_3 \) in Fig. 5.2). At the transition points, the response sensitivity is undefined as shown in Figs. 5.1(b) and 5.1(c), the values of the sensitivity being different for the two material states. Hence, numerical algorithms used to compute response sensitivities must be able

---

1. Using the chain rule for differentiation, the partial derivative of the inelastic resisting force vector with respect to the sensitivity parameter 'x' is obtained as

\[
\frac{\partial R(u(t, x), x)}{\partial x} = \sum_c \int_{\Omega_c} B^T \frac{\partial \sigma}{\partial x} d\Omega_c = \sum_c \int_{\Omega_c} B^T \left( \frac{\partial \sigma}{\partial \epsilon} \right) \frac{\partial \epsilon}{\partial x} d\Omega_c + \sum_c \int_{\Omega_c} B^T \frac{\partial \sigma}{\partial x} \left|_u \right. d\Omega_c
\]

\[
= \left( \sum_c \int_{\Omega_c} B^T \left( \frac{\partial \sigma}{\partial \epsilon} \right) B d\Omega_c \right) \frac{\partial \epsilon}{\partial \xi} + \sum_c \int_{\Omega_c} B^T \frac{\partial \sigma}{\partial x} \left|_u \right. d\Omega_c
\]

\[
= K_r \frac{\partial \epsilon}{\partial \xi} + \sum_c \int_{\Omega_c} B^T \frac{\partial \sigma}{\partial \xi} \left|_u \right. d\Omega_c
\]
to handle these discontinuities.

Numerical algorithms for response sensitivity computation can be formulated in two ways as shown in Fig. 5.2. While the first method (represented by steps $A_1$ and $A_2$ in Fig. 5.2) involves obtaining the differential equations governing the response sensitivity and then discretizing them in time to numerically compute the response gradients, the second method (represented by steps $B_1$ and $B_2$ in Fig. 5.2) involves discretizing in time, the equation of motion to evaluate the response numerically and then differentiating this numerical scheme exactly (or analytically) with respect to the sensitivity parameter 'x' to obtain the numerical scheme for computing the response sensitivity. The steps for response sensitivity computation are summarized in Fig. 5.2. The question is whether the sensitivities computed from the two schemes are the same. The answer to this question in the context of this study is that it depends on the discretization schemes used to numerically compute the response and the sensitivities (Heinkenschloss 1996) and how these numerical schemes handle the discontinuities discussed before.

Consider the second method represented by steps $B_1$ and $B_2$ in Fig. 5.2. The time stepping scheme (Newmark-β method) used to integrate numerically the equation of motion (5.5) was presented in the previous chapter. These numerical equations for the response are differentiated exactly with respect to the sensitivity parameter 'x' to render the sensitivity equations. In doing so, it is assumed that the changes of material state occur within time steps and never exactly at the discretized time values. The justification is that the probability of a material state transition occurring exactly at a discretized time value $t_n$, while performing the computations using double precision arithmetic is negligible (Kleiber 1997).
This implies that at any discrete time \( t_n \), the material state and hence the tangent stiffness matrix are uniquely defined.

### 5.2.1 Derivation of Exact Gradient of Numerical Response (Method II)

The Newmark-\( \beta \) average acceleration method (cf. section 4.6) used to integrate numerically the equation of motion is:

\[
\ddot{u}_{n+1} = (u_{n+1} - u_n) \frac{4}{(\Delta t)^2} - \frac{4\dot{u}_n}{\Delta t} - \ddot{u}_n \quad (5.8)
\]

\[
\dot{u}_{n+1} = (u_{n+1} - u_n) \frac{2}{\Delta t} - \dot{u}_n \quad (5.9)
\]

\[
M\ddot{u}_{n+1} + C\dot{u}_{n+1} + R(u_{n+1}) = F(t_{n+1}) \quad (5.10)
\]

By performing algebraic manipulations on (5.8), (5.9) and (5.10), described in the previous chapter, the following nonlinear matrix algebraic equation in the unknowns \( u_{n+1} \), for the residual force in the current step is obtained.

\[
\Psi(u_{n+1}) = \left( \frac{4}{(\Delta t)^2}Mu_{n+1} + \frac{2}{\Delta t}Cu_{n+1} + R(u_{n+1}) \right) - \bar{F}_{n+1} = 0 \quad (5.11)
\]

where

\[
\bar{F}_{n+1} = -M\dot{u}_x(t_{n+1}) + M\left( \frac{4}{(\Delta t)^2}u_n + \frac{4}{\Delta t}\dot{u}_n + \ddot{u}_n \right) + C\left( \frac{2}{\Delta t}u_n + \dot{u}_n \right)
\]

Once \( u_{n+1} \) is known, \( \dot{u}_{n+1} \) and \( \ddot{u}_{n+1} \) are obtained from (5.9) and (5.8), respectively.

Assuming that \( u_{n+1} \) is the converged solution for the current step, and differentiating (5.11) with respect to 'x' using the chain rule (see footnote on page 141), we obtain:
\[ \left( \frac{4}{(\Delta t)^2} M + \frac{2}{\Delta t} C + K_T \right) \frac{\partial u_{n+1}}{\partial x} = -\sum_c \int_{\Omega_c} B^T \frac{\partial \sigma_{n+1}}{\partial x} \bigg|_{\epsilon_{n+1}} \ d\Omega_c - \frac{\partial \tilde{F}_{n+1}}{\partial x} \]  

(5.12)

where \( K_T \) is the consistent (with the constitutive law integration scheme) tangent stiffness matrix, obtained through the consistent linearization of the numerical scheme used to integrate the rate constitutive equations. As defined in the previous chapter, the term in parentheses on the left hand side of (5.12) is the dynamic tangent stiffness matrix. Notice that once the numerical response of the system at \( t_{n+1} \) is known, the matrix sensitivity equation (5.12) is linear in the response gradient \( \frac{\partial u_{n+1}}{\partial x} \) and has the same left hand side as the equation for the response, namely (4.43). Furthermore, the gradient equation (5.12) has the same left hand side as (4.43). Therefore, only the right hand side of (5.12) needs to be computed and since the factorization of the effective tangent dynamic stiffness matrix is already available at the converged time step \( t_{n+1} \), solution of (5.12) is computationally very inexpensive (only backward substitution phase). The sensitivity vector \( \frac{\partial \tilde{F}_{n+1}}{\partial x} \) requires the computation of \( \frac{\partial F_{n+1}}{\partial x} \), and \( \frac{\partial M}{\partial x} \) and \( \frac{\partial C}{\partial x} \) which is performed easily at the element level and then assembled at the structural level using the direct stiffness assembly procedure. Computation of the sensitivity vector \( \sum_c \int_{\Omega_c} B^T \frac{\partial \sigma_{n+1}}{\partial x} \bigg|_{\epsilon_{n+1}} \ d\Omega_c \) however, is much more involved due to a strong history dependence. First at the element level this vector is computed from the stress gradient \( \frac{\partial \sigma_{n+1}}{\partial x} \bigg|_{\epsilon_{n+1}} \) which depends on the constitutive law.
of the material. Then the sensitivity vector at the structural level is obtained by assembling the contributions from all the elements using the direct stiffness assembly procedure. The analytical expressions to compute the stress gradient will be presented in later sections for the uniaxial $J_2$ and the multi-axial, multi-surface cap plasticity models.

In solving a nonlinear equation of motion using an incremental/iterative procedure, in conjunction with a return map algorithm to integrate the rate constitutive equations, the exact time between discrete time values at which material state transitions occur are not explicitly solved for. If it were the case, as in the SDOF Bouc-Wen and the rule-based bilinear, hysteretic oscillators, the exact time (within a time step) at which a material state switches would be differentiated with respect to the sensitivity parameter ‘$x$’ and this would carry the discontinuity of the response sensitivity across the change of material state (cf. sections 2.3.2 and 2.4.2.). In the case of the return map constitutive integration algorithm, the conditional derivative $\left. \frac{\partial \sigma_{\alpha+1}}{\partial x} \right|_{\epsilon_{\alpha}, \ldots}$ is obtained from the unconditional derivative $\frac{\partial \sigma_{\alpha}}{\partial x}$ and the derivatives of the history variables (e.g., cumulative hardening parameter, cumulative effective plastic strain, etc.) at time ‘$t_n$’. Assuming a material state transition occurs within the time step $[t_n, t_{n+1}]$, the resulting discontinuities in the derivatives of the history variables are captured by the present differentiation algorithm, although the exact time of occurrence within the step of the material state transition event is not solved for (cf. sections 4.8.2 and 4.8.3 for the $J_2$ and cap plasticity models). Then, these discontinuities in history variables account for the discontinuity of the response sensitivity $\frac{\partial u}{\partial x}$ during the time step. Further, differentiating (5.8) and (5.9) we obtain
\[
\frac{\partial \ddot{u}_{n+1}}{\partial x} = \left(\frac{\partial u_{n+1}}{\partial x} - \frac{\partial u_n}{\partial x}\right) \frac{4}{(\Delta t)^2} - \frac{4 \ddot{u}_n}{\Delta t \partial x} - \frac{\partial \ddot{u}_n}{\partial x}
\] (5.13)

\[
\frac{\partial \dddot{u}_{n+1}}{\partial x} = \left(\frac{\partial u_{n+1}}{\partial x} - \frac{\partial u_n}{\partial x}\right) \frac{2}{\Delta t} - \frac{\partial \ddot{u}_n}{\partial x}
\] (5.14)

from which \(\frac{\partial \ddot{u}}{\partial x}\) and \(\frac{\partial \dddot{u}}{\partial x}\) are solved for. From (5.13) and (5.14), it is also clear that the discontinuities in \(\frac{\partial \ddot{u}}{\partial x}\) and \(\frac{\partial \dddot{u}}{\partial x}\) are carried across a material state transition consistently from the discontinuity in displacement sensitivity contained in the difference term

\[
\left(\frac{\partial u_{n+1}}{\partial x} - \frac{\partial u_n}{\partial x}\right).
\]

5.2.2 Derivation of Numerical Gradient of Exact Response (Method I)

Method I represented by steps A₁ and A₂ in Fig. 5.2 involves first differentiating exactly the semi-discretized, nonlinear equation of motion (5.5) with respect to the sensitivity parameter ‘x’. Thus,

\[
M(x) \frac{\partial \ddot{u}(t, x)}{\partial x} + C(x) \frac{\partial \ddot{u}(t, x)}{\partial x} + K_T(u(t, x)) \frac{\partial u(t, x)}{\partial x} = \frac{\partial F(t, x)}{\partial x} - \frac{\partial M(x)}{\partial x} \ddot{u}(t, x)
\]

\[
- \frac{\partial C}{\partial x} u(t, x) - \frac{\partial}{\partial x} R(u((t, x), x)) \bigg|_u
\]

where \(K_T\) denotes the continuum tangent stiffness matrix. To constitute the sensitivity equation, we have to change the order of differentiation with respect to ‘t’ and the sensitivity parameter ‘x’, namely, the operators \(\frac{\partial}{\partial x}\) and \(\frac{\partial}{\partial t}\), and \(\frac{\partial}{\partial x}\) and \(\frac{\partial^2}{\partial t^2}\) have to be switched.

A sufficient condition for permuting \(\frac{\partial}{\partial x}\) and \(\frac{\partial}{\partial t}\) is that the partial derivatives \(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\)
exist and be continuous in both 'x' and 't' at all points along the time axis at \( x = x_0 \), the reference value of the sensitivity parameter (Courant 1988). While \( \frac{\partial u}{\partial t} \), the velocity response is clearly continuous both in time and 'x', \( \frac{\partial u}{\partial x} \) and \( \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} \right) \) are only piecewise continuous as already discussed above. Therefore, in the intervals of continuity, the operators \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial t} \) can be reversed. Similarly, a sufficient condition for permuting the differential operators \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial t} \) when applied to the velocity response \( \dot{u} \) is that \( \frac{\partial \dot{u}}{\partial x}, \frac{\partial \dot{u}}{\partial t} \)

\( \frac{\partial}{\partial x} \left( \frac{\partial \dot{u}}{\partial t} \right) \) exist and be continuous in both 'x' and 't' at all points along the time axis at \( x = x_0 \). While \( \frac{\partial \dot{u}}{\partial t} \), the acceleration response, is clearly continuous in time and 'x' for continuous loading functions, \( \frac{\partial \dot{u}}{\partial x} \) and \( \frac{\partial}{\partial x} \left( \frac{\partial \dot{u}}{\partial t} \right) \) are only piecewise continuous in 'x' and time 't' along the time axis at \( x = x_0 \). Therefore in the intervals of continuity we have that

\[
\frac{\partial}{\partial x} \left( \frac{\partial \dot{u}}{\partial t} \right) = \frac{\partial}{\partial t} \left( \frac{\partial \dot{u}}{\partial x} \right)
\]

\[
\frac{\partial}{\partial x} \left( \frac{\partial^2 u}{\partial t^2} \right) = \frac{\partial}{\partial t} \left( \frac{\partial^2 \dot{u}}{\partial x} \right)
\]

and since \( \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} \right) = \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial x} \right) \) as discussed earlier,
\[
\frac{\partial}{\partial x} \left( \frac{\partial^2 u}{\partial t^2} \right) = \frac{\partial^2}{\partial t^2} \left( \frac{\partial u}{\partial x} \right)
\]  

(5.16)

Thus in the intervals of continuity the differential operators \( \frac{\partial}{\partial x} \) and \( \frac{\partial^2}{\partial t^2} \) can be permuted when applied to \( u \). Thus (5.15) takes the form

\[
M(x) \ddot{v}(t, x) + C(x) \dot{v}(t, x) + K_T(u(t, x))v(t, x) = \frac{\partial F(t, x)}{\partial x} - \frac{\partial M(x)}{\partial x} \ddot{u}(t, x) - \frac{\partial C(x)}{\partial x} \dot{u}(t, x) - \frac{\partial}{\partial x} R(u((t, x), x)) \bigg|_{u}
\]

(5.17)

where \( v(t, x) = \frac{\partial u(t, x)}{\partial x} \). Only now, sensitivity equation (5.17) is discretized in time to obtain a numerical scheme for calculating the response sensitivity. Assuming that the return map algorithm is used to compute the history dependent inelastic resisting force vector \( R \) and hence its derivative \( \frac{\partial R}{\partial x} \bigg|_{u} \), the times at which changes of material states occur are not explicitly solved for. In such a case, the discontinuities of the response sensitivities across state transitions are carried over through discontinuities in the history variables in turn reflected as discontinuities in the sensitivity term \( \frac{\partial R}{\partial x} \bigg|_{u} \) which acts as a loading term for the sensitivity equation. Therefore, the following stepping scheme same as for the response can be used:

\[
\ddot{v}_{n+1} = (v_{n+1} - v_n) \frac{4}{(\Delta t)^2} - \frac{4\dot{v}_n}{\Delta t} - \dot{v}_n
\]

(5.18)

\[
\dot{v}_{n+1} = (v_{n+1} - v_n) \frac{2}{\Delta t} - \dot{v}_n
\]

(5.19)
\[ M\ddot{v}_{n+1} + C\dot{v}_{n+1} + K_T v_{n+1} = \frac{\partial F_{n+1}}{\partial x} - \frac{\partial M}{\partial x} \dot{u}_{n+1} - \frac{\partial C}{\partial x} u_{n+1} - \frac{\partial R}{\partial x}_{\dot{u}_{n+1}} \]  

(5.20)

Assuming that the consistent tangent stiffness matrix is used for \( K_T \) in (5.20), instead of the continuum tangent stiffness matrix, to efficiently compute the gradients (since the factorized consistent tangent stiffness matrix is already available at the converged response solution at the current step), the same matrix sensitivity equations are obtained from methods I and II.

### 5.2.3 Response Sensitivity Analysis for the Uniaxial J_2 Plasticity Model

This section deals with computing the conditional and the unconditional derivatives of the state variables \( \epsilon_{n+1}, \sigma_{n+1}, \alpha_{n+1}, \varepsilon^p_{n+1}, \) and \( \bar{\varepsilon}^p_{n+1} \) of the \( J_2 \) plasticity model with respect to the sensitivity parameter 'x' for the current time step \([t_n, t_{n+1}]\). These derivatives of the state variables at time \( t_{n+1} \) are computed by differentiating exactly the numerical scheme used to compute them, which consists of the elastic predictor step combined with the plastic corrector step of the return map algorithm.

If no plastic deformations take place during the current time step \([t_n, t_{n+1}]\), the trial solutions for the state variables given by the elastic predictor step is also the correct solution, i.e., there is no plastic corrector step following the elastic predictor step. Hence, dropping the superscript 'trial' from equations (4.94) to (4.99) and differentiating them with respect to the sensitivity parameter 'x', we obtain:

\[ \frac{\partial \Delta \lambda_{n+1}}{\partial x} = 0 \]  

(5.21)
\[
\frac{\partial \varepsilon_{n+1}^p}{\partial x} = \frac{\partial \varepsilon^{p}_n}{\partial x} \quad (5.22)
\]

\[
\frac{\partial \alpha_{n+1}}{\partial x} = \frac{\partial \alpha_n}{\partial x} \quad (5.23)
\]

\[
\frac{\partial \varepsilon_{n+1}^p}{\partial x} = \frac{\partial \varepsilon^{p}_n}{\partial x} \quad (5.24)
\]

\[
\frac{\partial \sigma_{n+1}}{\partial x} = E \left( \frac{\partial \varepsilon_{n+1}^p}{\partial x} - \frac{\partial \varepsilon^{p}_n}{\partial x} \right) + \frac{\partial E}{\partial x} (\varepsilon_{n+1} - \varepsilon^{p}_n) \quad (5.25)
\]

\[
\frac{\partial \sigma_{y,n+1}}{\partial x} = \frac{\partial \sigma_{y,n}}{\partial x} \quad (5.26)
\]

If plastic deformations take place in the current step \([t_n, t_{n+1}]\), the elasto-plastic constitutive relations in the discrete form are differentiated exactly with respect to the sensitivity parameter ‘\(x\)’ to compute the derivatives of the state variables at \(t_{n+1}\).

Differentiating (4.101) with respect to ‘\(x\)’,

\[
\frac{\partial \sigma_{n+1}}{\partial x} = E \left( \frac{\partial \varepsilon_{n+1}^p}{\partial x} - \frac{\partial \varepsilon^{p}_n}{\partial x} \right) + \frac{\partial E}{\partial x} (\varepsilon_{n+1} - \varepsilon^{p}_n) \quad (5.27)
\]

The quantity \(\frac{\partial \varepsilon_{n+1}^p}{\partial x}\) is obtained by differentiating (4.85) exactly with respect to ‘\(x\)’:

\[
\frac{\partial \varepsilon_{n+1}^p}{\partial x} = \frac{\partial \varepsilon^{p}_n}{\partial x} + \frac{\partial \Delta \lambda_{n+1}}{\partial x} n_{n+1} + \Delta \lambda_{n+1} \frac{\partial n_{n+1}}{\partial x} \quad (5.28)
\]

By definition,

\[
n_{n+1} = \frac{\sigma_{n+1} - \alpha_{n+1}}{|\sigma_{n+1} - \alpha_{n+1}|} = \text{sgn}(\sigma_{n+1} - \alpha_{n+1}) \quad (5.29)
\]

where \(\text{sgn}(\ldots)\) is the sign function which is either +1 or -1 depending on the sign of the argument. By differentiating (5.29), we get:
\[
\frac{\partial n_{n+1}}{\partial x} = \frac{\partial (\text{sgn}(\sigma_{n+1} - \alpha_{n+1}))}{\partial x} = 0 \tag{5.30}
\]

Further, using (4.110) and (5.29), we obtain

\[
n_{n+1} = \frac{\sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}}}{\sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}}} = \text{sgn}(\sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}}) = \text{sgn}(\sigma_{n+1} - \alpha_{n+1}) \tag{5.31}
\]

Using the relations in (5.31), the incremental discrete consistency parameter \( \Delta \lambda_{n+1} \) obtained from (4.109) takes the form:

\[
\Delta \lambda_{n+1} = \frac{|\sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}}| - \sigma_{y,n}}{E + H_{\text{iso}} + H_{\text{kin}}}
\]

\[
= \frac{\left(\sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}}\right) \text{sgn}(\sigma_{n+1}^{\text{Trial}} - \alpha_{n+1}^{\text{Trial}}) - \sigma_{y,n}}{E + H_{\text{iso}} + H_{\text{kin}}}
\tag{5.32}
\]

Differentiating (5.32) with respect to \('x'\) and using (5.30), (4.96), (4.98), (5.23) and (5.25) gives:

\[
\frac{\partial \Delta \lambda_{n+1}}{\partial x} = \frac{\left(\frac{\partial \sigma_{n+1}^{\text{Trial}}}{\partial x} - \frac{\partial \alpha_{n+1}^{\text{Trial}}}{\partial x}\right) \text{sgn}(\sigma_{n+1} - \alpha_{n+1}) - \frac{\partial \sigma_{y,n}}{\partial x}}{\left(E + H_{\text{iso}} + H_{\text{kin}}\right)^2}
\]

\[
= \frac{\left(\frac{\partial E}{\partial x} + \frac{\partial H_{\text{iso}}}{\partial x} + \frac{\partial H_{\text{kin}}}{\partial x}\right)\left(\sigma_{n+1}^{\text{Trial}} - \sigma_{n+1}^{\text{Trial}}\right) \text{sgn}(\sigma_{n+1} - \alpha_{n+1}) - \sigma_{y,n}}{\left(E + H_{\text{iso}} + H_{\text{kin}}\right)^2}
\tag{5.33}
\]

where the derivative of \( \sigma_{n+1}^{\text{Trial}} \) with respect to \('x'\) is obtained as

\[
\frac{\partial \sigma_{n+1}^{\text{Trial}}}{\partial x} = E \left(\frac{\partial \varepsilon_{n+1}}{\partial x} - \frac{\partial \varepsilon_{n}}{\partial x}\right) + \frac{\partial E}{\partial x} \left(\varepsilon_{n+1} - \varepsilon_{n}\right) \tag{5.34}
\]

At this juncture it is interesting to make the following observation. Suppose that the time step \([t_n, t_{n+1}]\) contains the first yielding event such that the yielding takes place just before
time \( t_{n+1} \). In such case, \( \Delta \lambda_{n+1} = 0 \). However, \( \frac{\partial \Delta \lambda_{n+1}}{\partial x} \) is non-zero from (5.33) and hence

\[ \frac{\partial \lambda_{n+1}}{\partial x}. \]

For all times before first yield, the derivative \( \frac{\partial \lambda}{\partial x} \) was zero. Thus, \( \frac{\partial \lambda}{\partial x} \) encounters a discontinuity at the change of the material state and is able to pick up the discontinuity naturally in the numerical scheme for the response sensitivity. This discontinuity propagates as discontinuity in the history variables as in (5.28) and in the conditional derivative

\[ \left| R(u((t, x), x)) \right| \frac{\partial R}{\partial x} \]

It is worth mentioning that in the above equations, the derivatives of the state variables with respect to ‘\( x \)’ are known at \( t_n \) and they are computed at \( t_{n+1} \).

The derivatives of the history variables with respect to the sensitivity parameter ‘\( x \)’, needed for the next time step are computed by differentiating (4.113) and (4.114).

\[
\frac{\partial \bar{\varepsilon}^p_{n+1}}{\partial x} = \frac{\partial \bar{\varepsilon}^p_n}{\partial x} + \frac{\partial \Delta \lambda_{n+1}}{\partial x}
\]

\[
\frac{\partial \sigma_{y,n+1}}{\partial x} = \frac{\partial \sigma_{y,n}}{\partial x} + \frac{\partial H^{iso}_{n+1}}{\partial x} \Delta \lambda_{n+1} + H^{iso}_{n+1} \frac{\partial \Delta \lambda_{n+1}}{\partial x}
\]

The various conditional derivatives needed to compute the right-hand-side of the sensitivity equation (5.12) are obtained by substituting zero for \( \frac{\partial \varepsilon^{(e)}}{\partial x} \) in the expressions (5.21) to (5.36) for the unconditional derivatives. Equation (5.12) is then solved for \( \frac{\partial u_{n+1}}{\partial x} \) and \( \frac{\partial \varepsilon^{(e)}}{\partial x} \) for each element (e). The latter is obtained from \( \frac{\partial u_{n+1}}{\partial x} \) using the strain-displace-
ment relations as:

\[
\frac{\partial \varepsilon^{(e)}_{n+1}}{\partial x} = \frac{\partial}{\partial x}\left[ B \ u^{(e)}_{n+1} \right] = B \left[ \frac{\partial u^{(e)}_{n+1}}{\partial x} \right]
\]  \hspace{1cm} (5.37)

Once \( \frac{\partial \varepsilon^{(e)}_{n+1}}{\partial x} \) is known, all the unconditional derivatives in (5.21) to (5.36) can be evaluated.

5.2.4 Response Sensitivity Analysis for the Multi-Axial, Multi-Surface Cap Plasticity Model

This section deals with the derivation of the complete set of sensitivity equations needed to compute the derivatives of stresses, strains and history variables with respect to the sensitivity parameter ‘x’ for the current time step \([t_n, t_{n+1}]\). The derivations are given for each mode or material state of the cap model in which the current state of stress might be.

(1) Loading in the Failure Envelope Mode:

Differentiating (4.154) and (4.155) with respect to x yields:

\[
\frac{\partial I_{Trials}^{(n+1)}}{\partial x} + 9K \frac{\partial}{\partial x} \gamma_{1,n+1} \left( \frac{dF_e}{dI_1} \right)_{n+1} + 9K \frac{\partial}{\partial x} (\Delta \gamma_{1,n+1}) \left( \frac{dF_e}{dI_1} \right)_{n+1} + 9K \Delta \gamma_{1,n+1} \left( \frac{\partial}{\partial x} \left[ \frac{dF_e}{dI_1} \right] \right)_{I_{Trials}} + \frac{\partial}{\partial I_1} \left[ dF_e(I_{1,n+1}) \right] \left( \frac{dI_{1,n+1}}{dx} \right) \frac{\partial I_{1,n+1}}{\partial x} = 0
\]  \hspace{1cm} (5.38)

and
\[
\frac{\partial}{\partial x}(\Delta \gamma_{1,n+1}) = \frac{2G \left( \frac{\partial}{\partial x} \left\| s_{n+1}^{\text{Trial}} \right\| - \frac{\partial}{\partial x} F_c(I_{1,n+1}) \right|_{I_i} + \frac{\partial F_c}{\partial I_{1,n+1}} \frac{\partial I_{1,n+1}}{\partial x} \right)}{(2G)^2}
\]

(5.39)

\[
- \frac{2}{(2G)^2} \frac{\partial G}{\partial x} \left( \left\| s_{n+1}^{\text{Trial}} \right\| - F_c(I_{1,n+1}) \right)
\]

in which the gradients \( \frac{\partial I_{1,n+1}^{\text{Trial}}}{\partial x} \) and \( \frac{\partial}{\partial x} \left\| s_{n+1}^{\text{Trial}} \right\| \) are obtained by differentiating (4.151) and (4.150), respectively, as

\[
\frac{\partial I_{1,n+1}^{\text{Trial}}}{\partial x} = 3 \frac{\partial K}{\partial x} I_{1,n+1} + 3K \frac{\partial I_{1,n+1}}{\partial x} - 3 \frac{\partial K}{\partial x} \Delta I_{1,n+1}^p - 3K \frac{\partial}{\partial x} (\Delta I_{1,n+1})
\]

(5.40)

\[
\frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} = 2 \frac{\partial G}{\partial x} (e_{n+1} - e_n^p) + 2G \left( \frac{\partial e_{n+1}}{\partial x} - \frac{\partial e_n^p}{\partial x} \right).
\]

(5.41)

Substituting (5.39) into (5.38) results in a linear scalar equation in \( \frac{\partial I_{1,n+1}}{\partial x} \). Once \( \frac{\partial I_{1,n+1}}{\partial x} \) is known, \( \frac{\partial}{\partial x}(\Delta \gamma_{1,n+1}) \) can be easily obtained by evaluating (5.39). The gradient of the deviatoric stress tensor \( s_{n+1} \) can be obtained from

\[
\frac{\partial s_{n+1}}{\partial x} = \frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} - 2 \frac{\partial G}{\partial x} \Delta e_{n+1}^p - 2G \frac{\partial}{\partial x} (\Delta e_{n+1}^p)
\]

(5.42)

where

\[
\frac{\partial}{\partial x}(\Delta e_{n+1}^p) = \frac{\partial}{\partial x}(\Delta \gamma_{1,n+1}) n_{n+1} + \Delta \gamma_{1,n+1} \frac{\partial n_{n+1}}{\partial x}
\]

(5.43)

and
\[ n_{n+1} = \frac{s_{n+1}}{\|s_{n+1}\|} = \frac{s_{n+1}^{\text{Trial}}}{\|s_{n+1}^{\text{Trial}}\|} \]  

(5.44)

Differentiating (5.44) with respect to the sensitivity parameter 'x' gives

\[
\frac{\partial n_{n+1}}{\partial x} = \frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} \left( \frac{\|s_{n+1}^{\text{Trial}}\|^2}{\|s_{n+1}^{\text{Trial}}\|^2} \right) \quad (5.45)
\]

in which

\[
\frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} = \frac{s_{n+1}^{\text{Trial}}}{\|s_{n+1}^{\text{Trial}}\|} \frac{\partial s_{n+1}^{\text{Tri}}}{\partial x} \quad (5.46)
\]

The symbol (:) above denotes the dyadic product or inner product of the two tensors.

The gradient of the trial deviatoric stress tensor, \( \frac{\partial s_{n+1}^{\text{Trial}}}{\partial x} \), is given by (5.41).

\[
\frac{\partial}{\partial x} (\Delta \gamma_{1,n+1}) = -3 \left( \frac{\partial}{\partial x} (\Delta \gamma_{1,n+1}) \frac{d}{dI_1} F_e (I_1) + \Delta \gamma_{1,n+1} \left( \frac{\partial}{\partial x} \left[ \frac{d}{dI_1} F_e (I_1) \right] \bigg|_{I_1=T} + \frac{\partial}{\partial x} \left[ \frac{d}{dI_1} F_e (I_1) \right] \bigg|_{I_1} \right) \right) \quad (5.47)
\]

It is worth recalling that immediately following convergence of the response calculation at time \( t_{n+1} \), when computing the gradient of the internal resisting force vector with \( u_{n+1} \) fixed, \( \frac{\partial}{\partial x} (R(u_{n+1})) \bigg|_{u_{n+1}} = \sum \int_{\Omega} B^T \frac{\partial \sigma_{n+1}}{\partial x} \bigg|_{e_{n+1}} d\Omega \) in (5.12), the conditional gradients

\[
\frac{\partial s_{n+1}}{\partial x} \bigg|_{e_{n+1}} \quad \text{and} \quad \frac{\partial I_{1,n+1}}{\partial x} \bigg|_{e_{n+1}}
\]

are needed. They are obtained simply by substituting

\[
\frac{\partial e_{n+1}}{\partial x} \bigg|_{e_{n+1}} = \frac{\partial I_{1,n+1}}{\partial x} \bigg|_{e_{n+1}} = 0 \quad (5.48)
\]

in the above gradient equations. On the other hand, after the gradient equation at the
structural level, (5.12), is solved for $\frac{\partial u_{n+1}}{\partial x}$, the unconditional gradients $\frac{\partial s_{n+1}}{\partial x}$ and $\frac{\partial I_{1,n+1}}{\partial x}$ can be obtained by using the unconditional deviatoric and volumetric strain gradients, $\frac{\partial e_{n+1}}{\partial x}$ and $\frac{\partial I_{1,n+1}}{\partial x}$. These unconditional gradients will be needed to compute the conditional gradients at the next time step.

(2) Loading in the Tensile Corner Region:

Differentiating (4.161) with respect to the sensitivity parameter 'x' yields

$$\frac{\partial I_{1,n+1}}{\partial x} = \frac{\partial T}{\partial x} \tag{5.49}$$

The gradient of the deviatoric stress vector is obtained by differentiating (4.157) with respect to 'x':

$$\frac{\partial}{\partial x}(\Delta e_{n+1}) = \frac{\partial \Delta \gamma_{1,n+1}}{\partial x}(n_{n+1}) + \Delta \gamma_{1,n+1} \frac{\partial n_{n+1}}{\partial x} \tag{5.50}$$

The gradient $\frac{d(\Delta \gamma_{1,n+1})}{dx}$ is obtained by differentiating (4.163) with respect to 'x':

$$\frac{\partial}{\partial x}(\Delta \gamma_{1,n+1}) = \frac{2G\left(\frac{\partial}{\partial x}\|s_{n+1}\| - \frac{\partial}{\partial x}F_{e}(I_{1} = T)\right) - 2\frac{\partial G}{\partial x}\left[\|s_{n+1}\| - F_{e}(T)\]}{\left(2G\right)^{2}} \tag{5.51}$$

where

$$\frac{\partial}{\partial x}F_{e}(I_{1} = T) = \frac{\partial F_{e}}{\partial x} \bigg|_{I_{1}} + \frac{\partial F_{e}}{\partial I_{1}} \bigg|_{I_{1}} \frac{\partial I_{1}}{\partial x} =$$

$$\frac{\partial F_{e}}{\partial \alpha} \bigg|_{I_{1} = T} \frac{\partial \alpha}{\partial x} + \frac{\partial F_{e}}{\partial \lambda} \frac{\partial \lambda}{\partial x} + \frac{\partial F_{e}}{\partial \beta} \frac{\partial \beta}{\partial x} + \frac{\partial F_{e}}{\partial \theta} \frac{\partial \theta}{\partial x} + \left(\frac{\partial F_{e}}{\partial I_{1}} \frac{\partial I_{1}}{\partial x}\right) \bigg|_{I_{1} = T} \tag{5.52}$$

Finally, the gradient of the increment of volumetric plastic strain tensor is obtained by differentiating (4.165) with respect to 'x':
\[
\frac{\partial}{\partial x}(\Delta I_{n+1}) = -3\left(\frac{\partial}{\partial x}(\Delta Y_{1,n+1}) \frac{d}{dI_1} F_e(T) + \Delta Y_{1,n+1} \left( \frac{\partial}{\partial x} \left[ \frac{d}{dI_1} F_e(T) \right] \left|_{I_1} \right. \right) + \frac{\partial}{\partial x} \frac{\partial F_e(T)}{\partial x} \right)
\]

(5.53)

(3) **Loading in the Cap Mode:**

Differentiating (4.171) and (4.170) with respect to the sensitivity parameter ‘x’ gives

\[
\frac{\partial I_{n+1}}{\partial x} = \frac{\partial I_{n+1}^{T_{\text{trial}}}}{\partial x} - 3 \frac{\partial K}{\partial x} H(\kappa_{n+1}) - 3 K \frac{\partial H(\kappa_{n+1})}{\partial x}
\]

(5.54)

and

\[
\frac{\partial}{\partial x}(\Delta Y_{2,n+1}) = \frac{3(I_{1,n+1} - \kappa_{n+1}) \frac{\partial}{\partial x} [R^2 H(\kappa_{n+1}) F_e(\kappa_{n+1})]}{[3(I_{1,n+1} - \kappa_{n+1})]^2} + \frac{3R^2 H(\kappa_{n+1}) F_e(\kappa_{n+1}) (\frac{\partial I_{n+1}}{\partial x} - \frac{\partial \kappa_{n+1}}{\partial x})}{[3(I_{1,n+1} - \kappa_{n+1})]^2}
\]

(5.55)

in which

\[
H(\kappa_{n+1}) = W(e^{-DX(\kappa_n)} - e^{-DX(\kappa_{n-1})})
\]

(5.56)

\[
\frac{\partial}{\partial x} H(\kappa_{n+1}) = \frac{\partial W}{\partial x} (e^{-DX(\kappa_n)} - e^{-DX(\kappa_{n-1})}) + \frac{W[-X(\kappa_n)e^{-DX(\kappa_n)} + X(\kappa_{n+1})e^{-DX(\kappa_{n+1})}] \frac{\partial D}{\partial x}}{X(\kappa_{n+1})}
\]

(5.57)

\[
De^{-DX(\kappa_n)} \frac{\partial X(\kappa_n)}{\partial x} + De^{-DX(\kappa_{n-1})} \frac{\partial X(\kappa_{n+1})}{\partial x} X(\kappa_{n+1})
\]

The gradients \( \frac{\partial}{\partial x} X(\kappa_n) \) and \( X(\kappa_{n+1}) \) can be obtained from (4.139) as:

\[
\frac{\partial}{\partial x} X(\kappa_{n+1}) = \frac{\partial \kappa_{n+1}}{\partial x} + \frac{\partial R}{\partial x} F_e(\kappa_{n+1}) + R \frac{\partial}{\partial x} F_e(\kappa_{n+1})
\]

(5.58)
in which, from (4.134):

$$\frac{\partial}{\partial x} F_e(\kappa_{n+1}) = \frac{\partial}{\partial x} F_e(I_{1, n+1}) \bigg|_{I_{1, n+1} = \kappa_{n+1}} + \left[ \frac{\partial}{\partial x} F_e(I_{1, n+1} = \kappa_{n+1}) \right] \frac{\partial I_{1, n+1}}{\partial x} \quad (5.59)$$

The gradient $\frac{\partial}{\partial x} F_e(\kappa_{n+1})$ can also be obtained by differentiating (4.169) with respect to 'x' as follows:

$$\frac{\partial}{\partial x} F_e(\kappa_{n+1}) = \frac{\partial h}{\partial x} \bigg|_{\kappa_{n+1}} + \frac{\partial h}{\partial \|s_{n+1}^{\text{Trial}}\|} \frac{\partial \|s_{n+1}^{\text{Trial}}\|}{\partial x} + \frac{\partial h}{\partial F_e(\kappa_{n+1})} \frac{\partial F_e(\kappa_{n+1})}{\partial x} + \frac{\partial h}{\partial \kappa_{n+1}} \frac{\partial \kappa_{n+1}}{\partial x} + \frac{\partial h}{\partial (\Delta \gamma_{2, n+1})} \frac{\partial (\Delta \gamma_{2, n+1})}{\partial x} \quad (5.60)$$

in which

$$h = \sqrt{\left( \frac{\|s_{n+1}^{\text{Trial}}\|}{F_e(\kappa_{n+1})} \right)^2 + \left( \frac{I_{1, n+1} - \kappa_{n+1}}{R + (9K\Delta \gamma_{2, n+1})/(RF_e(\kappa_{n+1}))} \right)^2}$$

By substituting (5.54), (5.55) and (5.59) into (5.60), a single nonlinear scalar equation is
obtained, which can be solved for \( \frac{\partial \kappa_{n+1}}{\partial x} \) using the Newton-Raphson iteration scheme for example. Once \( \frac{\partial \kappa_{n+1}}{\partial x} \) is known, the gradients \( \frac{\partial I_{1,n+1}}{\partial x} \) and \( \frac{\partial \gamma_{2,n+1}}{\partial x} \) are obtained by simple back-substitution. in (5.54) and (5.55). Then the gradient of the deviatoric and spherical stress tensors can be obtained by differentiating the following relations with respect to 'x':

\[
\begin{align*}
\sigma_{n+1} &= 2G[e_{n+1} - (e_n^p + \Delta e_{n+1}^p)] \\
I_{1,n+1} &= 3K[\bar{I}_{1,n+1} - (\bar{I}_{1,n} + \Delta \bar{I}_{1,n+1}^p)] \\
\end{align*}
\]

(5.61)

and

\[
\begin{align*}
\Delta e_{n+1}^p &= \Delta \gamma_{2,n+1} \frac{\sigma_{n+1}}{F_c(\|\sigma_{n+1}\|, I_{1,n+1}, \kappa_{n+1})} \\
\Delta \bar{I}_{1,n+1}^p &= \Delta \gamma_{2,n+1} \frac{3(I_{1,n+1} - \kappa_{n+1})}{R^2(F_c(\|\sigma_{n+1}\|, I_{1,n+1}, \kappa_{n+1}))} \\
\end{align*}
\]

(5.52)

In the case where \( I_{1,n+1} = \kappa_{n+1} \), the gradient \( \frac{\partial \Delta \gamma_{2,n+1}}{\partial x} \) in (5.55) must be substituted by the following expression obtained by differentiating (4.173) with respect to 'x':

\[
\frac{\partial}{\partial x}(\Delta \gamma_{2,n+1}) = \frac{2G\left(\frac{\partial}{\partial x}\|s^\text{trial}_{n+1}\| - \frac{\partial}{\partial x}F_c(\kappa_{n+1})\right) - 2\frac{\partial G}{\partial x}\|s^\text{trial}_{n+1}\| - F_e(\kappa_{n+1})}{(2G)^2}
\]

(5.63)

(4) **Loading in the Compressive Corner Region:**

From (4.181), the gradient of the first invariant of the total stress tensor is given by:

\[
\frac{\partial I_{1,n+1}}{\partial x} = \frac{\partial \kappa_{n}}{\partial x},
\]

(5.64)

while the gradient of the deviatoric stress tensor is given as in (5.42). The gradient of
the incremental plastic deviatoric strain tensor is obtained by differentiating (4.180) with respect to the sensitivity parameter ‘x’:

\[
\frac{\partial}{\partial x}(\Delta e_{n+1}^p) = \left( \frac{\partial}{\partial x}(\Delta \gamma_{1,n+1}) + \frac{\partial}{\partial x}(\Delta \gamma_{2,n+1}) \right) n_{n+1} + \nabla (\Delta \gamma_{1,n+1} + \Delta \gamma_{2,n+1}) \frac{\partial n_{n+1}}{\partial x}
\]  

(5.65)

in which \( \frac{\partial n_{n+1}}{\partial x} \) is as in (5.45) and the gradients of the discrete consistency parameters \( \Delta \gamma_{1,n+1} \) and \( \Delta \gamma_{2,n+1} \) are obtained by differentiating (4.177) and (4.178) with respect to ‘x’ as:

\[
\frac{\partial}{\partial x}(\Delta \gamma_{1,n+1}) = \frac{9K}{d I_{1,n+1}} \frac{d}{dI_{1,n+1}} F_\varepsilon(\kappa_n) \left[ \frac{\partial \kappa_n}{\partial x} + \frac{\partial I_{1,n+1}^{\text{Trial}}}{\partial x} \right]
\]

\[
- \left( \frac{9K}{d I_{1,n+1}} \frac{d}{dI_{1,n+1}} F_\varepsilon(\kappa_n) \right)^2
\]

\[
[\kappa_n - I_{1,n+1}^{\text{Trial}}] \left[ 9 \left( \frac{\partial K}{\partial x} d I_{1,n+1} \right) \frac{d}{dI_{1,n+1}} F_\varepsilon(\kappa_n) + 9K \frac{\partial}{\partial x} \left( \frac{d}{dI_{1,n+1}} F_\varepsilon(\kappa_n) \right) \right]_l \cdots \right] +
\]

\[
\frac{\partial}{\partial x} \left[ \frac{d}{dI_{1,n+1}} F_\varepsilon(\kappa_n) \right] \left[ \frac{\partial I_{1,n+1}}{\partial x} \right] + \frac{\partial}{\partial I_1} \left[ \frac{d}{dI_{1,n+1}} F_\varepsilon(\kappa_n) \right] \left[ \frac{\partial I_{1,n+1}}{\partial x} \right]
\]

\[
\left( \frac{9K}{d I_{1,n+1}} \frac{d}{dI_{1,n+1}} F_\varepsilon(\kappa_n) \right)^2
\]

(5.66)

\[
\frac{\partial}{\partial x}(\Delta \gamma_{2,n+1}) = \frac{2G}{(2G)^2} \left( \frac{\partial}{\partial x} \left[ \frac{\partial I_{1,n+1}^{\text{Trial}}}{\partial x} - \frac{\partial}{\partial x} F_\varepsilon(\kappa_n) \right]_l \right) + \frac{\partial}{\partial I_1} \left[ \frac{d}{dI_{1,n+1}} F_\varepsilon(\kappa_n) \right] \left[ \frac{\partial I_{1,n+1}}{\partial x} \right]
\]

\[
- \frac{2}{(2G)^2} \frac{\partial}{\partial x} \left[ ||s_{n+1}^{\text{Trial}}|| - F_\varepsilon(\kappa_n) \right] - \frac{\partial}{\partial x}(\Delta \gamma_{1,n+1})
\]

(5.67)
The partial derivatives in the right-hand-side of the above equations are readily available.

(5) Loading in the Tension Cut-Off Region:

Differentiating equations (4.183), (4.185), and (4.186) with respect to the sensitivity parameter ‘x’, we find:

\[
\frac{\partial I_{1,n+1}}{\partial x} = \frac{\partial T}{\partial x}
\]

\[\frac{\partial}{\partial x}(\Delta \varepsilon_{n+1}^p) = 0\]  

\[\frac{\partial}{\partial x} \Delta F_{l,n+1}^p = -3 \frac{\partial}{\partial x}(\Delta \gamma_{3,n+1})\]  

in which, from (4.187)

\[
\frac{\partial}{\partial x}(\Delta \gamma_{3,n+1}) = \frac{9K \frac{\partial}{\partial x}(T - I_{1,n+1}^{Trial}) - 9 \frac{\partial K}{\partial x}(T - I_{1,n+1}^{Trial})}{(9K)^2}
\]

The gradient of the total stress tensor is then given by

\[
\frac{\partial \sigma_{n+1}}{\partial x} = \frac{\partial s_{n+1}^{Trial}}{\partial x} + \frac{1}{3} \frac{\partial I_{1,n+1}}{\partial x} I
\]

In the context of the uniaxial J2 plasticity model (cf. section 5.2.3), it was shown that \( \frac{\partial \lambda}{\partial x} \)
is able to pick up the discontinuity across a material state transition. A similar observation can be made for the derivatives of the various history variables (such as hardening parameter) of the cap model, namely that they are able to consistently capture the discontinuities across material state transitions.
5.2.5 Summary of Response Gradient Computation Method

The above derivations completes the set of equations required to compute the gradient of the internal resisting force vector with the displacements fixed, \( \frac{d}{dx} R(u_{n+1}) \bigg|_{u_*} \), for both the uniaxial J\(_2\) plasticity model and the multi-axial cap plasticity models. Once this gradient is known, the governing gradient equation (5.12) can be integrated over one time step \([t_n, t_{n+1}]\) to provide the gradient of the inelastic displacement response at \( t_{n+1} \), \( \frac{\partial u_{n+1}}{\partial x} \).

In summary, the following procedure is used to compute the response sensitivity of inelastic dynamic systems:

1. Solve the matrix equation of motion (5.10) at time \( t_{n+1} \) in conjunction with the discrete constitutive relations of the assumed plasticity model by means of the return map algorithm, and compute quantities such as \( I_{1,n+1}, e_{p,n+1}, I_{1,n+1}, e_{p,n+1}, \dot{e}_{n+1}, \kappa_{n+1}, \)
\( s_{n+1}, I_{1,n+1}, \sigma_{n+1} \).

2. At each integration point (i.e., Gauss point), compute the derivative of the stress vector with respect to the sensitivity parameter 'x' with the strains fixed at time \( t_{n+1} \), i.e.,
\[ \frac{\partial \sigma_{n+1}}{\partial x} \bigg|_{e_*} \]. All the conditional derivatives of the state and history variables are obtained simply by setting the terms \( \frac{\partial I_{1,n+1}}{\partial x} \) and \( \frac{\partial e_{n+1}}{\partial x} \) to zero in the corresponding expressions for the unconditional derivatives.

3. Compute (assemble) the partial derivative of the internal resisting force vector with the displacements maintained fixed at time \( t_{n+1} \).

4. Compute the right-hand-side vector of the gradient equation (5.12).
(5) Solve the gradient equation (5.12) for \( \partial \vec{u}_{n+1} / \partial x \) using the factorized form of the dynamic tangent stiffness matrix, available from step (1).

(6) Compute the unconditional gradient of the strain and its spherical and deviatoric components from the gradient of the displacement vector using the finite element strain-displacement relation:

\[
\frac{\partial \epsilon_{n+1}}{\partial x} = B \frac{\partial}{\partial x}(\vec{u}_{n+1})
\]  

\[
\frac{\partial I_{1,n+1}}{\partial x} = \frac{\partial \epsilon_{1,n+1}}{\partial x} + \frac{\partial \epsilon_{22,n+1}}{\partial x} + \frac{\partial \epsilon_{33,n+1}}{\partial x}
\]  

\[
\frac{\partial \epsilon_{n+1}}{\partial x} = \frac{\partial \epsilon_{n+1}}{\partial x} - \frac{1}{3} \frac{\partial}{\partial x}(I_{1,n+1}) I
\]

(7) Using the results of step 6, update all the "conditional" derivatives at time \( t_{n+1} \) to the unconditional derivatives such as \( \frac{\partial \epsilon^p_{n+1}}{\partial x} \), \( \frac{\partial I^p_{1,n+1}}{\partial x} \), \( \frac{\partial \kappa_{n+1}}{\partial x} \) for use in computing the conditional derivatives at the time step.

The formulation presented here can easily be incorporated into any existing finite element code. The return map algorithm for the truss element with \( J_2 \) plasticity model and the bilinear isoparametric plane strain element with the cap plasticity model and the subroutines needed to perform an analytical response sensitivity analysis have been implemented in the nonlinear finite element analysis program FEAP.

5.3 Application Examples

5.3.1 Single-Degree-of-Freedom Elasto-Plastic Oscillator SubJECTED TO GROUND Motion

An example of an SDOF oscillator subjected to ground motion is shown in Fig. 5.3. The
single member elasto-plastic truss with one horizontal degree of freedom acts as an elasto-plastic SDOF oscillator.

The damping of the system is modeled through a mass proportional damping matrix at the

\[
A = 4 \text{ [in}^2]\]
\[
L = 20 \text{ [in]}\]
\[
E = 300 \text{ [ksi]}\]
\[
\rho = 0.0375 \text{ [kip. s}^2\text{/in}^4]\]
\[
\sigma_{y0} = 10 \text{ [ksi]}\]
\[
H_{iso} = 0\]
\[
H_{kin} = 100 \text{ [ksi]}\]

**Fig. 5.3** SDOF system: Elasto-plastic Truss Element

element level which gives the required damping ratio in the first mode (the only mode here). Accordingly, the damping ratio for this example is chosen as 5%. The material parameters selected as sensitivity parameters are: Young’s modulus \( E \), the initial yield stress \( \sigma_{y0} \), and the mass density \( \rho \). The value of the ground acceleration at each 0.01 [sec] constitute the vector of loading variables \( x = [x_1, x_2, ..., x_n]^T \). The ground acceleration time history is linearly interpolated between consecutive \( x_i, i = 1, 2, ..., n \). The N-S component of the 1940 El Centro earthquake record shown in Fig. 5.4 forms the input ground excitation to the elasto-plastic oscillator. The equation of motion is integrated using the Newmark-\( \beta \) method with a constant integration step of \( \Delta t = 0.01 \text{[sec]} \). The relative displacement response to this ground motion and the stress-strain history are shown in Figs. 5.5 and 5.6, respectively. From the stress-strain history, it is clear that the system
has a strong hysteretic response with large plastic straining. Figures 5.7 to 5.11 show the analytically computed relative displacement response sensitivities with respect to various material and loading parameters and their comparison with corresponding finite difference results. In general, as the sensitivity parameter increment is decreased, the finite difference results converge to the analytically computed sensitivities. However, there are exceptions as in the case of the sensitivity with respect to the first loading variable, $x_1$, shown in Fig. 5.10. According to the step-size dilemma discussed earlier (cf. section 2.2.1) decreasing the increment of the sensitivity parameter beyond a certain limit may have an adverse effect (due to the round-off errors) on the convergence of finite difference results to the exact (analytical) sensitivity.

![Graph showing time versus input ground motion](image-url)

**Fig. 5.4** Imperial Valley Earthquake, May 18, 1940, El Centro Site, component S00E (N-S)
Fig. 5.5 Relative Displacement Response of the SDOF Elasto-Plastic Oscillator

Fig. 5.6 Stress-strain Response History of the Inelastic Truss Element
Fig. 5.7 Displacement Response Sensitivity with Respect to Mass Density, $\rho$

Fig. 5.8 Displacement Response Sensitivity with Respect to Initial Yield Stress, $\sigma_{y}$
Fig. 5.9 Displacement Response Sensitivity with Respect to Young's Modulus, $E$

Fig. 5.10 Displacement Response Sensitivity with Respect to Loading Variable, $x_1$
Fig. 5.11 Displacement Response Sensitivity with Respect to Loading Variable, $x_{470}$

5.3.2 Ten Member Elasto-Plastic Truss Subjected to Ground Motion

The ten member inelastic truss under consideration and its defining parameters are shown in Fig. 5.12. The input ground acceleration shown in Fig. 5.4 is scaled by a factor 2.5 to constitute the ground acceleration time history at the base of the truss. The truss system is assumed to have a mass proportional damping with 5% damping ratio in the first mode. The damping ratios in the higher modes are shown in Fig. 5.13. A constant time step of $\Delta t = 0.01$ [sec] is used to integrate the equation of motion by means of the constant-acceleration Newmark-$\beta$ method. The value of the ground acceleration time history at every 0.02 [sec] constitute the vector of loading variables $\mathbf{x} = [x_1, x_2, ..., x_n]^T$.

The horizontal relative displacement time history $u_2(t)$ of node 2 is shown in Fig. 5.14. The extent of the nonlinear, hysteretic behavior of the truss for the above input ground
For all members:

\[ E = 30000 \text{ [kN/cm}^2\text{]} \]
\[ \rho = 0.0000761 \text{ [kN/(cm.s)}^2\text{]} \]
\[ \sigma_{y0} = 12 \text{ [kN/cm}^2\text{]} \]
\[ H_{iso} = 0 \]
\[ H_{kin} = 300 \text{ [kN/cm}^2\text{]} \]
\[ A = 3 \text{ [cm}^2\text{]} \]

**Fig. 5.12**  The ten member truss under consideration

acceleration can be directly assessed from the stress-strain histories for elements 1 and 7 shown in Figs. 5.15 and 5.16, respectively.

For the sensitivity analysis, the material parameters selected as sensitivity parameters are: Young’s modulus \( E \), the initial yield stress \( \sigma_{y0} \), and the mass density \( \rho \). Figures 5.17 to 5.21 show the analytically computed sensitivity of \( u_2(t) \) with respect to the material and loading parameters and their comparison with finite difference results. It can be seen that the finite difference results converge asymptotically to the analytically computed sensitivities as the parameter increments are taken smaller and smaller, thus validating the exact gradient computation scheme and its implementation in FEAP.
Fig. 5.13 Damping Ratios of the Eight Modes of the Ten-Member Truss

Fig. 5.14 Horizontal Relative Displacement of Node 2 of the Ten-Member Truss
Fig. 5.15 Stress-Strain History of Element 1

Fig. 5.16 Stress-Strain History of Element 7
Fig. 5.17 Displacement Response Sensitivity with Respect to Mass Density, $\rho$

Fig. 5.18 Displacement Response Sensitivity with Respect to Initial Yield Stress, $\sigma_{y_0}$
Fig. 5.19 Displacement Response Sensitivity with Respect to Young's Modulus, E

Fig. 5.20 Displacement Response Sensitivity with Respect to Loading Variable, $x_1$
Fig. 5.21  Displacement Response Sensitivity with Respect to Loading Variable, $x_{125}$

5.3.3  Isoparametric Plane Strain Element with Cap Plasticity Model Subjected to Ground Motion

In this section, the response and response gradients of a single plane strain element with the cap plasticity model are analyzed for earthquake ground excitation as shown in Fig. 5.22
Fig. 5.22  Inelastic Plane Strain Element Subjected to Ground Motion

The material parameters chosen for the sensitivity analysis are $\alpha$, $\beta$, $\theta$, $T$, $R$, $W$, $\lambda$ which define the cap plasticity model. The value of the input ground acceleration at every 0.02 [sec] constitutes the vector of loading variable, $x = [x_1, x_2, ..., x_n]^T$.

The ground motion input to the plane strain element is defined by the acceleration record shown in Fig. 5.4 scaled by a factor of 0.015. The dynamic system is assumed to be undamped. The equation of motion is integrated using the average acceleration Newmark-$\beta$ method with a constant time step of $\Delta t = 0.01$ [sec].

The parameters of the cap model are given in Table 5.1.
Table 5.1 Parameters of the Cap Constitutive Model

<table>
<thead>
<tr>
<th>Parameter of the cap model</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>0.3 ksi</td>
</tr>
<tr>
<td>α</td>
<td>3.86 ksi</td>
</tr>
<tr>
<td>λ</td>
<td>1.16 ksi</td>
</tr>
<tr>
<td>W</td>
<td>0.42</td>
</tr>
<tr>
<td>β</td>
<td>0.44 (ksi)^{-1}</td>
</tr>
<tr>
<td>θ</td>
<td>0.11</td>
</tr>
<tr>
<td>R</td>
<td>4.43</td>
</tr>
<tr>
<td>D</td>
<td>0.0032 (ksi)^{-1}</td>
</tr>
<tr>
<td>κ₀</td>
<td>1.25 ksi</td>
</tr>
<tr>
<td>K</td>
<td>2100 ksi</td>
</tr>
<tr>
<td>G</td>
<td>1700 ksi</td>
</tr>
</tbody>
</table>

The horizontal displacement response, u₃(t) of node 3 is shown in Fig. 5.23. To gain more insight into the inelastic behavior of the element, the stress path at Gauss point #1 (cf. Fig. 5.22) in the I₁ - Iₚₜₗₘₚ is shown in Fig. 5.24. Comparison between the analytical and finite difference sensitivities of the displacement response sensitivities u₃(t) with respect to the material and loading parameters, α, β, θ, T, R, W, λ, x₁ and x₅₀ are shown in Figs. 5.25 to 5.33.

In accordance with the step size dilemma (see section 2.5.1), notice that for parameters α and β, the finite difference result for the response sensitivity does not converge monotonically to the exact response sensitivity as the sensitivity parameter is taken smaller and smaller. In other words, for parameters α and β, decreasing the parameter increment first
worsens the match between the finite difference and the exact (analytical) sensitivity results and then it improves.

**Fig. 5.23** Inelastic Displacement Time History of the Single Plane Strain Element

**Fig. 5.24** Stress Path in the $I_1$-lsls Space When Subjected to Ground Motion
Fig. 5.25 Displacement Response Sensitivity with Respect to Parameter \( \alpha \)

Fig. 5.26 Displacement Response Sensitivity with Respect to Parameter \( \beta \)
Fig. 5.27  Displacement Response Sensitivity with Respect to Parameter $\theta$

Fig. 5.28  Displacement Response Sensitivity with Respect to Parameter $W$
Fig. 5.29 Displacement Response Sensitivity with Respect to Parameter R

Fig. 5.30 Displacement Response Sensitivity with Respect to Parameter T
Fig. 5.31 Displacement Response Sensitivity with Respect to Parameter $\lambda$

Fig. 5.32 Displacement Response Sensitivity with Respect to Loading Variable $x_1$
Fig. 5.33  Displacement Response Sensitivity with Respect to Loading Variable $x_{50}$

5.3.4 Inelastic Dam with Cap Plasticity Model Subjected to Ground Motion

In this application example, the Pine Flat Dam on King’s River near Fresno, California, is considered as an application example for the displacement response sensitivity computation. The finite element discretization of the dam along with its dimensions are shown in Fig. 5.34. The cap constitutive parameters of the dam are tabulated in Table 5.1. It is assumed that the dam is on a rigid foundation, the reservoir is empty and the dam system is undamped.

The Pine Flat Dam is subjected to the input ground acceleration shown in Fig. 5.4 scaled by a factor of 3.0. For the response computation, the gravity load is applied prior to the seismic load. The constant-acceleration Newmark-$\beta$ method with a constant step size of $\Delta t = 0.01$ [sec] is used to integrate the equation of motion and the gradient equations.
Fig. 5.34 Finite Element Model of the Pine Flat Dam
Fig. 5.35  Displacement Response Time History at the Top of the Dam

Fig. 5.36  Plot of the Stress-Path in the $I_1$-II$_{sl}$ Space at Point A
Fig. 5.37 Plot of the Stress-path in the $I_1$-lsl Space at Point B

Fig. 5.38 Plot of the Stress-path in the $I_1$-lsl Space at Point C
Fig. 5.39  Plot of the Stress-path in the $I_1$-lsli Space at Point D

Fig. 5.40  Displacement Response Sensitivity with Respect to Parameter $\alpha$
**Fig. 5.41** Displacement Response Sensitivity with Respect to Parameter $\beta$

**Fig. 5.42** Displacement Response Sensitivity with Respect to Parameter $T$
Fig. 5.43  Displacement Response Sensitivity with Respect to Parameter R

Fig. 5.44  Displacement Response Sensitivity with Respect to Parameter W
Fig. 5.45  Displacement Response Sensitivity with Respect to Loading Variable $x_1$

Fig. 5.46  Displacement Response Sensitivity with Respect to Loading Variable $x_{50}$
The material parameters selected for the sensitivity analysis are $\alpha$, $\beta$, $T$, $R$, $W$ and the magnitude of the ground acceleration at every 0.02 [sec] constitutes the loading variables $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$. The loading parameters chosen for the sensitivity analysis are $x_1$ and $x_{50}$.

The inelastic displacement response time history at the crest of the dam (node 91 in Fig. 5.34) is shown in Fig. 5.35. Figure 5.34 also gives the location of the four points A, B, C and D at which the stress-path in the $I_1$-$\|s\|$ space is displayed in Figs. 5.36 to 5.39. The sensitivities of the displacement response at the crest of the dam with respect to material and loading parameters are shown in Figs. 5.40 to 5.46. In each case, it is observed that the finite difference results converge to the analytical sensitivity result, thus validating the analytical gradient computation method and its implementation in FEAP.

5.4 Conclusions

This chapter dealt with the calculation of response sensitivities with respect to loading and material parameters for plasticity based models of dynamic systems. The two alternative methods to formulate the sensitivity equations were discussed, compared and unified. The issue of discontinuities in the response sensitivities due to material state transitions was discussed at length. While it is mathematically consistent to use the method of differentiating exactly the numerical scheme for the response in order to obtain the response sensitivities, several heuristic arguments have to be brought in to show that the other method of discretizing in time the gradient differential equation leads to the same algebraic sensitivity equations. It was also shown that in order to derive the exact derivative of the numerical scheme for the response, the consistent (or algorithmic) tangent stiffness matrix must be
used. The complete set of algebraic sensitivity equations were derived for the uniaxial J₂ plasticity model and the multi-axial, multi-surface cap plasticity model. Insight has been gained on how the discontinuities in the internal state variables consistently carry discontinuities across the material state transition events.

Application examples were provided in the form of a single member J₂ truss acting as a SDOF elasto-plastic oscillator and a ten-member multi-degree-of-freedom J₂ truss structure. The analytical response sensitivities with respect to loading and material parameters were validated with finite difference results. Next, a single plane strain element and the Pine Flat dam both incorporating the cap plasticity material model and subjected to ground motion excitation were considered. Here again, the analytical response sensitivities were verified using finite difference result calculations thus validating the exact response sensitivity computation method and its implementation in the nonlinear finite element analysis program FEAP.
CHAPTER 6 Conclusions

5.1 Summary of Work

The present study consists of two distinct phases. In the first phase, several phenomenological (rule-based) nonlinear, hysteretic single-degree-of-freedom (SDOF) models of structures are considered while the second phase focuses on mechanics-based plasticity models of both SDOF and multi-degree-of-freedom (MDOF) structural systems. The SDOF models considered in the first phase are the linear elastic oscillator model, the Duffing oscillator model, the Bouc-Wen oscillator model and the bilinear hysteretic oscillator model. The response and response sensitivities with respect to system and loading parameters are derived for the above oscillator models and the analytically derived sensitivities are compared and validated with finite difference results.

The probability of failure due to random dynamic loading and uncertain system parameters can be obtained indirectly through the mean out-crossing rate formulation. For this purpose, the loading parameters only are modeled as random variables for deterministic systems while both loading and system parameters are modeled as random variables for uncertain systems. Application examples of mean out-crossing rate computation for the linear elastic, Duffing and the Bouc-Wen oscillator subjected to white noise base excitation are presented. These mean out-crossing rates for the deterministic systems are compared with those obtained assuming uncertain system parameters in order to assess the relative effects and importance of loading and system uncertainties.

The second phase of the study deals with the computation of the response and response sensitivities with respect to loading and material parameters for mechanics-based plastic-
ity models of dynamic systems. Issues of continuity in time of the response sensitivities in the context of material state transitions are addressed. The importance of using the consistent tangent stiffness matrix instead of the continuum tangent stiffness matrix for response sensitivity calculations is discussed. The capabilities of the finite element program FEAP are extended to incorporate the response sensitivity calculations. While routines for calculation of response sensitivity with respect to material parameters for the $J_2$ and cap plasticity models were implemented earlier (Jagannath 1995), routines for computation of sensitivities with respect to loading parameter are developed and implemented in FEAP. Application examples of response sensitivity calculations with respect to material and loading parameters are considered which include SDOF elasto-plastic oscillator and the ten member truss column for the uniaxial $J_2$ plasticity model and a single plain strain element and the Pine Flat concrete gravity dam for the cap plasticity model.

5.2 Summary of Findings
This study led to the following findings for the phenomenological and mechanics-based plasticity models of dynamic systems:

(1) The mean-out-crossing rates for white noise base excitation and hence the probability of failure is increased when system parameter uncertainties are accounted for. In a formulation, where the limit state function is expressed in terms of the displacement response, the deterministic displacement threshold whose exceedence constitutes "failure" also has a significant effect on the mean-out-crossing rates

(2) In the case of systems with strong hysteretic behavior, the limit state function in terms of the displacement response exceeding a given threshold value at a particular time
results in problems to find the design point and hence the mean out-crossing rate.

(3) The two fundamental approaches to computing the response sensitivity of hysteretic
dynamic systems, namely (1) the exact sensitivity of the numerical response, and (2)
the numerical evaluation of the exact continuum sensitivity equation, have been uni-
ified.

(4) The use of the consistent (or algorithmic) tangent stiffness matrix for computing the
exact sensitivity of the numerical response, which is what is needed in an optimization
algorithm, has been clarified.

(5) A better understanding of the discontinuities in time of the response sensitivities and
their physical interpretation in terms of material state transitions has been gained for
plasticity based models of dynamic systems.

5.3 Recommendations for Future Work

Based on the present study, the following recommendations for future research can be
made:

(1) Perform mean out-crossing rate calculation for the mechanics based models with both
deterministic and uncertain material properties.

(2) Investigate the failure regions of dynamic systems with nonlinear, hysteretic behavior
to ensure that the formulation of the limit state function results in smooth contours in
the failure space. This is important for easy convergence to the design point at every
calculation of mean out-crossing rate.

(3) Extend the present formulation of reliability analysis to a variety of other plasticity
models to represent different material behaviors and to contribute to a more realistic modeling and analysis of large and complex structural systems, especially in the failure region.

(4) Implement the assembly of the mass and stiffness matrices, factorization of the dynamic tangent stiffness matrix and dynamic response sensitivity calculation in a parallel computation environment to gain considerable savings in time for the reliability analysis of real-world problems.

(5) Create benchmark solution using Monte-Carlo simulation to investigate the accuracy of the mean out-crossing rate formulation for computing failure probabilities of nonlinear, hysteretic dynamic systems.

(6) Extend the present formulation for mean out-crossing rate and failure probability calculation to the case of nonstationary (both in intensity and frequency content) ground excitation accounting also for the uncertainty of global ground motion parameters (e.g., overall intensity, duration, overall frequency content)
References


FEAP, Finite Element Analysis Program developed by R.L. Taylor, Department of Civil Engineering, University of California, Berkeley, CA 94720.


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Simo, J.C., and Ortiz, M., "A Unified Approach to Finite Deformation Elastoplastic Analysis Based on the Use of Hyperelastic Constitutive Equations," *Comp. Meth.


STRAP, Structural Reliability Analysis Program, Version 1.0, 1996.


Appendix A: Approximation to Linearly Interpolated White Noise

Numerically, the mean-squared displacement or velocity response at stationarity is found by averaging the square of displacement or velocity at a particular time across a large number of response realizations due to a corresponding set of linear interpolated white noise realizations of the white noise.

An approximation to linearly interpolated white noise is made by fixing $\alpha_r$ deterministically to zero, where in the original linearly interpolated white noise, $\alpha_r$ is a random variable uniformly distributed between 0 and $\Delta t$, see Sec 3.2.2. By doing this, the white noise discretization is not stationary any more. To examine the effect of this approximation in practical terms, the approximate linearly interpolated white noise excitation process is also treated as being ergodic. With this assumption, the stationary mean-squared response can be calculated by averaging along the time axis the square of responses at discrete time steps, instead of averaging across the ensemble of responses corresponding to different realizations of the approximate discrete white noise. This averaging along the time axis is performed by omitting a large number of response values at discrete times $t_i$ in the transient phase of the response.

As an example, a linear SDOF oscillator with natural period $T = 1.0 \text{ sec}$ is considered. A parametric study for the numerically estimated stationary values of the mean-squared displacement and velocity is performed for $\xi = 0.05$ and $\xi = 0.10$ for various values of the time step $\Delta t$ used in discretizing the white noise. For each value of $\Delta t$ the standard deviation $\sigma_f$ of the zero mean normal random variables $f_i$ defining the discrete white noise
input (see Sec. 3.2.2) is found from (3.10) as $\frac{\sigma^2 \Delta t}{2\pi} = \phi_0$, where $\phi_0$ is the intensity of the white noise.

The equation of motion is integrated using a piecewise linear exact algorithm using the same time step $\Delta t$ as for the white noise discretization. The values of displacement and velocity at $t_i < 15T$ are omitted from the time averaging operation to determine the mean-squared responses.

![Normalized Stationary Mean-Squared Displacement Response](image)

**Fig. A.1** Normalized Stationary Mean-Squared Displacement Response
The values of the numerically estimated stationary mean-squared responses normalized with respect to the corresponding theoretical mean-squared response to white noise are plotted in Fig. A.1 and Fig. A.2. These figures show good agreement between the numerically estimated and theoretical mean-squared response quantities.

![Normalized Stationary Mean-squared Velocity Response](image)

**Fig. A.2** Normalized Stationary Mean-squared Velocity Response

These numerical tests justify that the approximate linearly interpolated white noise can be used to accurately discretize the stationary Gaussian white noise excitation.