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System Identification for Robust Control

by

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System Identification for Robust Control

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Abstract

In the design of a robust control system, one needs a nominal model together with a quantitative bound on the uncertainty that results from under-modeling and disturbances. Thus, common control-oriented system identification methods deliver an optimal model, meanwhile the uncertainty is characterized with a bound which is usually rather conservative. Alternatively, in this thesis we do not intentionally seek a nominal model and a quantitative bound, instead, the uncertainty is directly parameterized so that the resulting uncertain model family can be characterized by means of a real parameter vector with at most unit length. In other words, the family can be characterized in terms of an ellipsoid.

This is an innovative approach to the control-oriented system identification, since it is not in accordance with the general philosophy of robust identification. However, it is applicable to the robust synthesis problem by taking advantage of a convex parameterization of robust controllers that simultaneously stabilize the uncertain models in the family. The robust performance problem becomes tractable since it can be converted into a quasi-convex optimization problem with Linear Matrix Inequality (LMI) constraints. The relation between the optimal robust performance and the uncertainty is studied by analyzing the explicit bounds of the maximal robust margin, whereas the bounds are given in terms of the volume, the radius or the mean square root radius of the ellipsoid.

Model (in)validation is a complement to system identification. In our approach it is an integral ingredient of the process of obtaining robust control-oriented sys-
tern models. A single model is not invalidated if it is inside the ellipsoid, and thus the intersection of the ellipsoids is not invalidated. In order to make the unfalsified model set (the intersection) fit in our framework, we can compute an optimal ellipsoid bounding the intersection of the ellipsoids. This approximation is conservative, compared to the original unfalsified model set. However, this scheme improves the optimal robust performance.
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To my darling
Xiaohui
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\section*{Notations}

\begin{itemize}
  \item $\mathbb{R}$  \hspace{1cm} field of real numbers
  \item $\mathbb{C}$  \hspace{1cm} field of complex numbers
  \item $\text{Re}\, \alpha$ \hspace{1cm} real part of $\alpha \in \mathbb{C}$
  \item $||x||$ \hspace{1cm} Euclidean norm of vector $x$
  \item $[a_{ij}]$ \hspace{1cm} a matrix with $a_{ij}$ as its $i$-th row and $j$-th column element
  \item $A^T$ \hspace{1cm} transpose of $A$
  \item $A^*$ \hspace{1cm} complex conjugate transpose of $A$
  \item $A^{-1}$ \hspace{1cm} inverse of $A$
  \item $A^+$ \hspace{1cm} pseudo-inverse of $A$
  \item $\det A$ \hspace{1cm} determinant of $A$
  \item $\text{Tr. } A$ \hspace{1cm} trace of $A$
  \item $\text{rank } A$ \hspace{1cm} rank of $A$
  \item $\lambda(A)$ \hspace{1cm} eigenvalue of $A$
  \item $\rho(A)$ \hspace{1cm} spectral radius of $A$
  \item $\sigma_{\text{min}}(A)$ \hspace{1cm} the smallest singular value of $A$
  \item $\sigma_{\text{max}}(A)$ \hspace{1cm} the largest singular value of $A$
  \item $\sigma_i(A)$ \hspace{1cm} the $i$-th singular value of $A$
  \item $||A||$ \hspace{1cm} 2-induced norm of $A$
  \item $I$, $I_n$ \hspace{1cm} identity matrix
  \item $\text{diag}(a_1, \ldots, a_n)$ \hspace{1cm} diagonal matrix with $a_i$'s on the diagonal
  \item $\hat{\mathcal{M}}$ \hspace{1cm} data matrix constructed from data measurements
  \item $\mu(x, \mathcal{M})$ \hspace{1cm} misfit function of $x$ with data matrix $\mathcal{M}$
  \item $\mathcal{E}$ \hspace{1cm} ellipsoid
\end{itemize}
\( \mathcal{F}_b \)  
model family in behavioral framework

\( \mathcal{F} \)  
model family in classical framework

\( \mathcal{F}_l(G, K) \)  
Lower LFT

\( \mathcal{F}_u(G, \Delta) \)  
Upper LFT

:=  
defined as

\( >, \geq \)  
positive definite, positive semidefinite of matrices

\( \text{arg min } f(x) \)  
value of \( x \) that minimizes \( f(x) \)

\( \blacksquare \)  
end of proof
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Chapter 1

Introduction

1.1 Motivation and Background

1.1.1 Dynamical systems

The purpose of this thesis is to develop methods for identifying uncertain dynamical systems for robust control design. In this subsection the fundamental concepts related to dynamical systems are briefly reviewed.

![Dynamical system diagram](image)

**Figure 1.1** Dynamical system.

A dynamical system can be described as depicted in Figure 1.1. The system is driven by external signals where the input $u$ can be manipulated and the disturbance $\xi$ can only be observed through their influence on the output $y$. Input $u$, output $y$ and disturbance $\xi$ are time signals which are, intrinsically continuous functions of time variable, however, usually observed in discrete time form. An essential property of dynamical systems is that the output at any time instant depend on the past and not just on the present inputs. Hence, the past behavior of a dynamical system influences
the future as a result of the memory length and its initial conditions. Some main properties can be distinguished when describing a dynamical system [30, 11, 17].

- **Causal vs. non-causal:** The current behavior of causal systems does not depend on the future but only on current and the past information which is always the case for physical systems.

- **Linear vs. nonlinear:** Linear systems possess a superposition property which simplifies the mathematical analysis. Nevertheless, practical systems are in general nonlinear.

- **Time-invariant vs. time-varying:** A dynamical system is time-invariant if its properties do not change with time.

- **Continuous-time vs. discrete-time:** For many physical system it is natural to work with a continuous-time representation. However, since measurements are only available at discrete time instants, discrete-time models are usually used for many applications, e.g. control design.

- **Single-input single-output (SISO) vs. multi-input multi-output (MIMO):** The identification of SISO system has been extensively studied in many textbooks [34] for its simplicity. Most practical systems however have a multivariable character for which the identification is much more complicated and less extensively explored.

Throughout the thesis we assume the input-output data is generated by a general dynamical causal SISO system. The system may be a continuous-time system, however, since we only have the discrete input-output measurements. We want to model the data as generated by a linear time-invariant discrete-time causal SISO system together with the additive perturbation.
1.1.2 System modeling

In order to be able to design a controller for the system under concern, we must know which system we are dealing with. Two different methods exist to obtain a model: physical modeling and identification.

Physical modeling

One way to obtain a model of a given plant is to use known physical laws and relationships that describe the working principle of the system. This method of applying physical laws and taking measurements is most successful in electromechanical systems, such as aerospace vehicles and robots. Some systems are difficult to model in this way, either because they are too complex or because their governing laws are unknown.

System Identification

A completely different way of obtaining a system model is to use identification methods. This is called "black box" modeling, since one is not interested anymore in all internal working mechanisms, but only in input-output relationships. System identification is a well established area of research. Indeed, there is a large body of literature on system identification. There are several excellent books on this subject which describe the accomplishments during the last decades. See, for example, the book by Ljung [34].

Any theory of identification has (at least) three main ingredients:

(i) A priori knowledge or assumptions about the unknown systems, the noise, and the model set from which the identified model is chosen by a potential identification algorithm:

(ii) Measured data; and

(iii) Measure of performance of the identification algorithm.
There are many different choices for these three ingredients and each choice gives rise to a different identification theory. They also determine the scope and nature of results one can logically expect to obtain. Thus, it is important to fully understand the various issues which should be considered in making these choices.

The unknown system is most often taken to be a linear time-invariant system. One may take the noise to be a stochastic process with some particular characteristics. In this case, one can not expect to derive hard deterministic bounds on the modeling error, if that is taken to be a measure of performance of the identification algorithm. However, it may be possible to obtain probabilistic error bounds for a given confidence level. The other choice is to take the noise to be a bounded signal with a known bound. This can be followed by either a worst-case or an average-case analysis. In the worst-case analysis, one may reasonably expect to arrive at hard bounds on the modeling error.

Measured data may be taken to be either time-domain input-output measurements or frequency domain measurements. It should be recognized that one can not get frequency domain measurements directly without a time-domain experiment. However, in some cases, for example if the inputs are sinusoid, it may be more efficient to represent the measured data to be frequency domain measurements. The model set may be a parameterized set of systems. On the other hand, one may let the model set be the set of all stable systems as in non-parametric identification. The former choice leads to restricted complexity identification.

The performance of an identification algorithm depends on the intended use of the identified model. Thus, if a model is sought for the purpose of prediction, then prediction error is a natural performance measure. On the other hand, if the purpose of the model is control design and analysis, then the performance measure must reflect this objective.
1.1.3 Robust identification

Classical system identification methods deliver a model in which the uncertainty is often characterized in terms of the noise. Whereas this is suitable for open-loop problems like filtering, estimation, and prediction, it is our contention that those models are inappropriate for closed-loop problems like robust feedback control. In control theory, modeling uncertainty arises due to incomplete and inexact experimental data, simplifying approximations, neglected high frequency dynamics, etc.

In modern robust control, the starting point for control system analysis and design is nominal plant and (norm) bounds on model uncertainty. This has fueled a renewed interest in worst-case deterministic formulations of the system identification problem. These worst-case deterministic formulations of the system identification problem are motivated by a perceived need to develop a theory of system identification that is compatible with modern robust control. As a result, system identification techniques should be required to provide guaranteed error bounds in addition to a candidate system model, or more generally a set of systems as the model. Recently, some papers that take a deterministic approach to the identification problem have appeared. See for example, the survey [31] and the references therein. Many of the papers deal with what may be broadly termed as “robust identification” [35]. Another approach to the interaction between robust control and identification is via iterative identification and control designs. Another direction of research activity is the so-called identification in $\mathcal{H}_\infty$ problem which is one particular formulation of an identification problem for robust control. The problem of identification in $\mathcal{H}_\infty$ was first formulated by Helmicki et al. (see the essay [31] and the references therein for details).

A common traditional approach to system identification has been to take a stochastic problem formulation. This has led to a fairly well developed theory of system identification and identification algorithms. Until recently, the issue of obtaining bounds on the resulting model error had not been a focal point of research. However, in some recent papers, results on the size of the model error from using classical identification
algorithms have been obtained. This is a subject of much current research interest. The survey paper [23] gives a nice and readable account of certain aspects of this line of research.

Set membership identification

Set membership identification theory aims to characterize the membership set of the parameters space which is consistent with the data, the model structure and the error bounds. Error bounded identification usually delivers a set of uncertain models which are all consistent with the data measurements. Set membership identification is not intended to describe the uncertain models for robust control, however, the description of the system parameters, or further, the uncertain models using set membership identification methods is quite useful for robust control design. A good survey on set membership identification and error bounded identification can be found in the thesis [17]. In Appendix B we obtain a new approach to the error bounded identification problem.

Model invalidation

Model (in)validation is a complement to identification. An identification algorithm leads to a model and bounds on model-plant mismatch. Now one can take this as a postulated model for the system to be identified. Then one can ask the question whether this postulated model correctly describes the system. This question can be addressed by performing new experiments on the system or using the old experimental data which was not used for constructing the model. The problem is then to decide whether this new data contradicts the postulated model. If it does contradict the postulated model then the one should declare that the postulated model is invalidated; otherwise, the postulated model is not an invalidated model of the system under consideration. Model invalidation has been an important component of classical identification theory and is usually performed by residual analysis as described
in the book by Ljung [34]. It is also an important aspect of the model building in the "behavior framework" in the work of Willems [68, 69, 70, 71]. Recently, model invalidation problem in the setting of robust control models have been investigated in [48, 32].

Model validation is one potential bridge that may connect stochastic identification theory with deterministic robust control. One may take the model that results from the traditional identification techniques with uncertainty bounds obtained by confidence interval methods as a robust control model. A good model validation test then may give the robust control designer the confidence needed to use this model for robust control design.

1.2 Overview and Contributions

1.2.1 Contributions

In the design of a robust control system, one needs a nominal model together with a quantitative bound on the uncertainty that results from under-modeling and disturbances. Thus, common control-oriented system identification methods deliver an optimal model, meanwhile the uncertainty is characterized with a bound which is usually rather conservative. Alternatively, in this thesis we do not intentionally seek a nominal model and a quantitative bound, instead, the uncertainty is directly parameterized so that the resulting uncertain model family can be characterized by means of a real parameter vector with at most unit length. In other words, the family can be characterized in terms of an ellipsoid.

This is an innovative approach to the control-oriented system identification, since it is not in accordance with the general philosophy of robust identification. However, it is applicable to the robust synthesis problem by taking advantage of a convex parameterization of robust controllers that simultaneously stabilize the uncertain models in the family. The robust performance problem becomes tractable since it can be converted into a quasi-convex optimization problem with Linear Matrix Inequality
(LMI) constraints. The relation between the optimal robust performance and the uncertainty is studied by analyzing the explicit bounds of the maximal robust margin, whereas the bounds are given in terms of the volume, the radius or the mean square root radius of the ellipsoid.

Model (in)validation is a complement to system identification. In our approach it is an integral ingredient of the process of obtaining robust control-oriented system models. A single model is not invalidated if it is inside the ellipsoid, and thus the intersection of the ellipsoids is not invalidated. In order to make the unfalsified model set (the intersection) fit in our framework, we can compute an optimal ellipsoid bounding the intersection of the ellipsoids. This approximation is conservative, compared to the original unfalsified model set, however, this scheme improves the optimal robust performance.

1.2.2 Overview

In the design of a robust control system, one needs a nominal model together with quantitative bounds on the uncertainty that results from under-modeling and disturbances. Namely, an uncertain model family centered at a nominal model is needed for the purpose of the robust control design. Alternatively, in this thesis we will not intentionally seek the nominal model and the bound, instead, an uncertain model family is directly characterized based on some knowledges of the perturbation within the measurements.

Our approach takes a deterministic problem formulation where the measured data is taken to be time-domain input-output measurements. It is assumed to have been generated by some discrete dynamical system. The perturbation within the measurements is assumed to be additive. Our goal is to model the data as generated by a linear time-invariant discrete-time system together with an additive perturbation.
Chapter 2: Model family description

In this chapter we first describe the uncertain model family in several different but equivalent forms (Section 2.1), then characterize the model family in both behavioral framework and the classical framework by studying the quadratic inequalities. Explicit parameterizations are obtained in terms of the singular values and the singular vectors of the data matrix. Finally, we notice that in simple cases the model family in behavioral framework or in classical framework is characterized respectively by a convex cone, or by an ellipsoid (section 2.2). In both cases, the uncertain models can be parameterized so that a real parameter vector of at most unit length linearly appear in both the numerators and the denominators of the transfer functions of all uncertain models (Section 2.3).

Chapter 3: Model family revisited

In this chapter we study several discrete topics which are closely related to the model family and hopefully helpful for a good understanding of the model family. In Section 3.1 several structure properties of model families are considered. The convexity of the uncertain model family is usually desired for robust control design, whereas by any means the measures of the size of the model family give metrics of the uncertainty within the modeling. The denseness problem of the model family and the data matrix approximation problem are helpful for us to understand how well the approximate family can describe the true models. In addition, these two problems are also closely related to the model validation problem discussed in Chapter 5. Section 3.2 deals with the model family where the data is generated by an LTI system. A classical identification problem, order determination problem, is first addressed, and then the stochastic and deterministic perturbation is separately formulated. The third is how to estimate the perturbation bound according to the data measurements. We studied this problem in detail, and found that the perturbation bound is closely related to the geometric relationship of the uncertain model families (ellipsoids) generated
with different inputs. Section 3.3 is devoted to another method of estimating the perturbation bound. When modeling the data, we sometimes have no clue on the perturbation bound. One way of obtaining a sub-optimal estimate is to decompose the data matrix into two terms, so that one term of the decomposition looks as if it were generated by an LTI system, and the other term indicates the perturbation. In this section we develop a sub-optimal method to decompose the data matrix by an iterative composite property mapping method.

This chapter is also intended to build the theoretic foundations of some results related to the model family. So, this chapter is the result of some theoretic interests as well.

Chapter 4: Robust Synthesis problem

Under some circumstance the uncertain models are parameterized with a real uncertain parameter of at most unit length. The uncertain real parameters linearly appear in the numerator and the denominator of transfer functions of the uncertain models. In this chapter we consider the robust synthesis problem, based on this kind of parameterization (Section 4.1). By using a convex parameterization of all controllers that simultaneously stabilize the uncertain models derived in [45] (Section 4.2), a robust performance problem can be stated in terms of a quasi-convex optimization, which can be solved using Ritz approximation method (Section 4.3). Several experimental observations concerning the relation of maximal stability margin and order of Youla parameter are reported (Section 4.3). The convergence of the algorithm is reflected from three aspects: convergence of the maximal stability margin, convergence of the gain of the Youla parameter and change of the poles and zeros of the Youla parameter. We also consider the relation between the maximal stability margin and various measures of the uncertain model family (Section 4.4). In Section 4.5 we discuss the $\mathcal{H}_\infty$ synthesis problem; two methods are introduced, but the second method, taking advantage of the coprime factorizations of the uncertain transfer functions, is analyzed
in detail. The last problem we consider is the robust synthesis problem for general model family. We formulate the problem in two different methods, one is similar to the problem we have already discussed, and the other is the typical $\mu$-synthesis method.

Chapter 5: Model validation problem

The problem we discuss in this chapter is that of determining whether or not the input-output data record is consistent with the uncertainty model of the plant. In other words, the problem is to decide whether the observed data could have been produced by the model for some choice of unmodeled dynamics, initial condition, and measurement noise satisfying the given bounds. This is called the model validation problem. It turns out that in our setting a single model is invalidated if it is outside the ellipsoid, it is not invalidated if it is inside the ellipsoid. Furthermore, in the multiple sets of data case the intersection of the model families is not invalidated. In order to make the unfalsified models have the simple linear linear fractional parameterizations, we can use various method to optimally bound the intersection of the ellipsoids. Since the robust performance is inversely proportional to the size of the ellipsoid, a better robust performance is expected to be achieved.

Chapter 6: An illustrative example

In this Chapter we will use a time varying mass-spring-damper system to illustrate the main results we have discussed in this thesis. We first model the system use an LTI system, and then design robust controller and analyze the performance based on the uncertain model family. Note that in this situation the perturbation results from the unmodeled dynamics.
Appendix A: Mathematical tools

In this appendix we survey some of the important mathematical tools we are using throughout the thesis. Some of the results are directly referred, others are implicitly used.

Appendix B: Error bounded identification

In this appendix we introduce a method for the error bounded identification problem, i.e., the identification problem where the observed data and records are corrupted by the unknown noise with known bounds. We formulate the problem of computing the optimal outer ellipsoidal approximation of the uncertain parameter set as a convex optimization problem with Linear Matrix Inequalities (LMI) constraints. This makes it possible to compute the optimal approximations with respect to various measures using various efficient convex optimization techniques.

Appendix C: Proofs

All proofs for the theorems, expressions and observations are given in this appendix. Some of the proofs are incorporated into the proposition for convenient presentation.
Chapter 2

Model family characterization

In the design of a robust control system, one needs a nominal model together with quantitative bounds on the uncertainty that results from undermodeling and disturbances. Namely, an uncertain model family centered at a nominal model is needed for the purpose of the robust control design. Alternatively, in this thesis we will not intentionally seek the nominal model and the bound. instead, an uncertain model family is directly characterized based on some knowledge of the perturbation within the measurements.

In this chapter we first describe the uncertain model family in several different but equivalent forms (Section 2.1), then characterize the model family in both behavioral framework and the classical framework by studying the quadratic inequalities. Explicit parameterizations are obtained in terms of the singular values and the singular vectors of the data matrix. Finally, we notice that in simple cases the model family in behavioral framework or in classical framework is characterized respectively by a convex cone or by an ellipsoid (Section 2.2). In both cases, the uncertain models can be parameterized so that a real parameter vector of at most unit length linearly appear in both the numerators and the denominators of the transfer functions of all uncertain models (Section 2.3).

2.1 Model family descriptions

2.1.1 Exact description

The purpose of this chapter is to characterize the family of uncertain models consistent with the data measurements and the noise characteristics. A natural definition of the so-called uncertain model family is

\[ \mathcal{F} := \{ \text{all models generating the noisy data } \hat{A} \}. \]
This is in fact a very informal definition which needs much more remedy. since whether a model generates the noisy data also dependents on the characteristics of the perturbation within the measurements. Almost all usual parameter identification schemes model the noise as a random variable or a stochastic process characterized by its mean and covariance and a model of the noise autocorrelation. However, as pointed in many papers, this approach has two limitations. First, if the noise is not intrinsically random, it is difficult to give a correct statistical characterization to model the noise. Second, in most circumstance, the number of observed data is insufficient to determine the statistical description of the noise. Since in control theory modelling uncertainty arises due to various sources such as incomplete and inexact experimental data, simplifying approximations, neglected high frequency dynamics, etc. it does not make sense to model the noise as a random variable. Therefore, in this thesis we describe the perturbation in a deterministic framework, such as specifying a hard bound or a soft bound with high confidence for the perturbation. However, how to obtain the deterministic characteristics of the perturbation is usually not trivial. We will particularly address this issue in Chapter 3.

To get a concrete idea of the definition, let us consider the following input-output data records:

$\hat{A} = \{ \hat{w}_t = (\hat{u}_t, \hat{y}_t) \in \mathbb{R}^2, t = 0, 1, \ldots, N \}$. \hspace{1cm} (2.1)

generated by some discrete-time dynamical system. We seek to model the data as generated by a linear, time-invariant, discrete-time system together with an additive perturbation. This implies that the data is decomposed in the form:

$\hat{w}_t = w_t + \hat{w}_t = (u_t, y_t) + (\hat{u}_t, \hat{y}_t)$ \hspace{1cm} (2.2)

where $w_t$ is supposed to be generated by the linear system of interest and $\hat{w}_t$ represents the perturbation.

To motivate the introduction of the tools used for the rest of the thesis, assume
that the data records

\[ \mathcal{A} = \{ w_t \in \mathbb{Z}, \ t = 0, 1, \ldots, N \} \quad (2.3) \]

are produced by a linear, time-invariant, discrete-time system. Consider a model which is compatible with (i.e. may have generated) the data \( w_t \):

\[ (p(\sigma) \ q(\sigma))w_t = 0 \quad (2.4) \]

for appropriate values of \( t \), where \( \sigma \) denotes the shift operator, and

\[ p(\xi) := p_0 + p_1 \xi + \cdots + p_m \xi^m, \ p_i \in \mathbb{Z} \]

\[ q(\xi) := q_0 + q_1 \xi + \cdots + q_n \xi^n, \ q_i \in \mathbb{Z} \]

where \( m, n, (m \leq n) \) are two specified nonnegative integers. Also, the model is usually represented by its transfer function

\[ H(z) = -\frac{p_0 + p_1 z + \cdots + p_m z^m}{q_0 + q_1 z + \cdots + q_n z^n}, \quad (2.5) \]

or its coefficient vector

\[ x := (p_0, p_1, \ldots, p_m, q_0, q_1, \ldots, q_n)^T \quad (2.6) \]

Equivalently, (2.4) can be written as the following difference equation

\[ (p_0 u_k + p_1 u_{k+1} + \cdots + p_m u_{k+m}) + (q_0 y_k + q_1 y_{k+1} + \cdots + q_n y_{k+n}) = 0 \quad (2.7) \]

for \( k = 0, 1, \ldots, s := N - \max(m, n) + 1 \). Re-write the equations in matrix form. we have

\[ x^T \mathcal{M} = 0. \quad (2.8) \]

i.e., \( x \) is in the left kernel of \( \mathcal{M} \), where \( \mathcal{M} \) is a matrix with two Hankel blocks constructed from the data:

\[
\mathcal{M} := \begin{bmatrix}
[u_{i+j}]_{i,j=0}^{m-1} \\
[y_{i+j}]_{i,j=0}^{n-1}
\end{bmatrix} \in \mathbb{Z}^{r \times s} \quad (2.9)
\]
where \( r = m + n + 2 \) and the two Hankel matrices

\[
U = [u_{i+j}]_{i,j=0}^{m,r-1} \quad \text{and} \quad Y = [y_{i+j}]_{i,j=0}^{n,r-1}
\]  

(2.10)

are respectively called the input matrix and the output matrix.

Motivated by the above observation we will define the matrices \( \hat{\mathcal{M}} \) and \( \hat{\mathcal{V}} \), and study the uncertain modeling problem for the case where

\[
\hat{\mathcal{M}} = \mathcal{M} + \hat{\mathcal{M}}.
\]  

(2.11)

\( \hat{\mathcal{M}} \) has a non-trivial left kernel.

\( \hat{\mathcal{M}} \) consistent with the noise specification.  

(2.12)

The matrix \( \hat{\mathcal{M}} \) is called the data matrix corresponding to the model structure and the input-output data set \( \hat{\mathcal{A}} \). The matrix \( \hat{\mathcal{M}} \) is called the perturbation matrix. Notice that the coefficient vector \( x \) in (2.6) completely determines the model. It is, thus, convenient to use coefficient vectors to represent the models in the sequel.

Throughout this thesis the perturbation is specified in a deterministic framework, it makes sense to choose the energy-related norms such as 2-induced matrix norm or Frobenius norm for the measurement of the perturbation in the data, and assume an upper bound of such a norm of the perturbation matrix \( \hat{\mathcal{M}} \) is known, say \( \varepsilon \). In terms of the 2-induced matrix norm the model family can, therefore, be characterized precisely as follows.

\[
\mathcal{F}_2 = \left\{ x \in \mathbb{R}^r : \exists \mathcal{M} \text{ (having the same block Hankel structure as } \hat{\mathcal{M}} \text{), such that } x^T \mathcal{M} = 0 \text{ and } \| \hat{\mathcal{M}} - \mathcal{M} \|_2 \leq \varepsilon \right\}
\]  

(2.13)

It is remarked that Hankel matrices are involved in the description of the model family, whereas structured matrices such as Hankel matrices are in general very difficult to handle in most situations. Therefore, the restrictions on the structure of the data matrix will be dropped off in the sequel. However, we will be back to this point in Section 3.1.3.
2.1.2 Norm description

Dropping the restrictions on the structure of the matrices, we get a looser description of the family by 2-induced matrix norm:

\[ \mathcal{F}_2 := \left\{ x \in \mathbb{R}^r : \exists \mathcal{M}. \text{ such that } x^T \mathcal{M} = 0 \text{ and } \| \mathcal{M} - \mathcal{M} \|_2 \leq \varepsilon \right\}. \tag{2.14} \]

In the above description the data matrix \( \mathcal{M} \) is split into two parts. One is the model-related part used to determine an approximation of the nominal model, and the other is related to the noise with some upper bounded energy.

With the same bound \( \varepsilon \) the family, the set

\[ \mathcal{F}_F := \left\{ x \in \mathbb{R}^r : \exists \mathcal{M}. \text{ such that } x^T \mathcal{M} = 0 \text{ and } \| \mathcal{M} - \mathcal{M} \|_F \leq \varepsilon \right\} \tag{2.15} \]

defined in terms of Frobenius norm, is obviously a subset of the family \( \mathcal{F}_2 \) due to the fact \( \| \cdot \|_2 \leq \| \cdot \|_F \), but these two families are essentially identical. This will be presented in section 2.1.5.

2.1.3 Orthogonal decomposition description

In section 2.1.2 data matrix is decomposed as the sum of two matrices. For every decomposition (2.11) an approximation of the nominal model can be obtained by investigating the left kernel of the model-related part of the decomposition. This motivates the investigation of orthogonal decomposition of the data matrix, and the natural introduction of the following family

\[ \mathcal{F}_L := \left\{ \mathcal{M}y : y \in \mathbb{R}^m, \mathcal{M} = \mathcal{M} + \mathcal{M}_{\perp}, \langle \mathcal{M}, \mathcal{M}_{\perp} \rangle = 0, \| \mathcal{M}_{\perp} \|_2 \leq \varepsilon \right\} \tag{2.16} \]

where \( \langle X, Y \rangle := Y^T X \) is the inner product of the matrices \( X \) and \( Y \).

It is remarked that the column space of the data matrix is orthogonally decomposed in the above description. Compared to the description introduced section 2.1.2, the left kernel of the model-related matrix is explicitly expressed as the image of the noise-related matrix. Notice that the above description is given by utilizing 2-norm.
A Frobenius norm version which can be similarly given is, however, omitted here due to its identity to the above family (Identity will be shown in section 2.1.5).

2.1.4 Misfit function description

For the exact data records \( \mathcal{A} \) as \( (2.3) \) the nominal model is obviously in the left kernel of the corresponding data matrix \( \mathcal{M} \), as shown in \( (2.8) \). For measured data \( \hat{\mathcal{A}} \) an approximated model \( x \) is in general not in the left kernel of the corresponding data matrix \( \hat{\mathcal{M}} \), i.e., \( x^T \hat{\mathcal{M}} \neq 0 \). To measure of the misfit of the approximated model \( x \) with respect to the measured data matrix \( \hat{\mathcal{M}} \) it is natural to introduce the *misfit function* \([2, 3, 73]\) \( \mu \) between \( x \) and \( \hat{\mathcal{M}} \):

\[
\mu(x, \hat{\mathcal{M}}) := \frac{\| x^T \hat{\mathcal{M}} \|_2}{\| x \|_2},
\]

(2.17)

where \( x \in \mathbb{R}^r \), \( \hat{\mathcal{M}} \in \mathbb{R}^{r \times m} \). Naturally, this leads to the definition of the family

\[
\mathcal{F}_\mu := \{ x \in \mathbb{R}^r : \mu(x, \hat{\mathcal{M}}) \leq \varepsilon \}.
\]

(2.18)

which contains all models whose misfits with respect to the measured data matrix \( \hat{\mathcal{M}} \) are at most \( \varepsilon \), the perturbation bound.

It seems strange to describe the uncertain model family in different ways but with the same bound \( \varepsilon \). However, this description of the model family is actually equivalent to those descriptions discussed before. The equivalence will be given in the next subsection.

2.1.5 Equivalence of descriptions

In this subsection the equivalence of the various descriptions of the uncertain model families introduced in the previous subsections is established under the assumption

\[
\text{rank} \hat{\mathcal{M}} = r, \quad \text{i.e., } \hat{\mathcal{M}} \text{ has full row rank.}
\]

**Theorem 2.1.1** Under the above assumption the three model families \( \mathcal{F}_r \), \( \mathcal{F}_\perp \) and \( \mathcal{F}_\mu \) are identical.
The proof for establishing the equivalence of the model families $\mathcal{F}_2$ and $\mathcal{F}_2'$ also shows that the family $\mathcal{F}_F$ defined in (2.15) and the family

$$\mathcal{F}_2' = \left\{ \hat{\mathcal{M}}y: y \in \mathbb{R}^m, \hat{\mathcal{M}} = \mathcal{M} + \hat{\mathcal{M}}, \langle \hat{\mathcal{M}} \cdot \mathcal{M} \rangle = 0, \| \hat{\mathcal{M}} \|_F \leq \varepsilon \right\}$$

are equivalent. Similarly, the proof for establishing the equivalence of $\mathcal{F}_2$ and $\mathcal{F}_2'$ can be slightly modified as follows

$$\| \hat{\mathcal{M}} - \mathcal{M} \|_F = \| \hat{\mathcal{M}} \|_F = \| yz^T \|_F = \| yz^T \|_2 \leq \| y \|_2 \| z \|_2 = \| z \|_2 \leq \varepsilon$$

to establish the equivalence of $\mathcal{F}_F$ and $\mathcal{F}_2'$.

As a summary the following different descriptions of the uncertain model family has been introduced and their equivalence has been established in this section. An explicit description of the equivalent model families will be presented in the next section.

$$\mathcal{F}_2 = \left\{ x \in \mathbb{R}^r: \exists \mathcal{M}, \text{ such that } x^T \mathcal{M} = 0 \text{ and } \| \hat{\mathcal{M}} - \mathcal{M} \|_2 \leq \varepsilon \right\}$$

$$\mathcal{F}_F = \left\{ x \in \mathbb{R}^r: \exists \mathcal{M}, \text{ such that } x^T \mathcal{M} = 0 \text{ and } \| \hat{\mathcal{M}} - \mathcal{M} \|_F \leq \varepsilon \right\}$$

$$\mathcal{F}_2' = \left\{ \hat{\mathcal{M}}y: y \in \mathbb{R}^m, \hat{\mathcal{M}} = \mathcal{M} + \hat{\mathcal{M}}, \langle \mathcal{M} \cdot \hat{\mathcal{M}} \rangle = 0, \| \hat{\mathcal{M}} \|_2 \leq \varepsilon \right\}$$

$$\mathcal{F}_2' = \left\{ \hat{\mathcal{M}}y: y \in \mathbb{R}^m, \hat{\mathcal{M}} = \mathcal{M} + \hat{\mathcal{M}}, \langle \mathcal{M} \cdot \hat{\mathcal{M}} \rangle = 0, \| \hat{\mathcal{M}} \|_F \leq \varepsilon \right\}$$

$$\mathcal{F}_\mu = \left\{ x \in \mathbb{R}^r: \mu(x, \hat{\mathcal{M}}) \leq \varepsilon \right\}.$$

2.2 Explicit descriptions of model families

Recall that we represent the models consistent with the data measurements using their coefficient vectors, which are essentially the behavioral representations, since the models represented by the coefficient vectors may have nonproper transfer functions. Therefore, to get a classical model, we must pose a constraint on the coefficient vector. i.e. making the coefficient $q_m$ in the coefficient vector

$$\theta = (p_0 \quad p_1 \quad \cdots \quad p_m \quad q_0 \quad q_1 \quad \cdots \quad q_n)^T$$
be 1.

In this section we will first define the family of behavioral models and the family of classical models, which are direct extensions of the model families discussed in last section. The families of two different types of models are then explicitly characterized, based on a beforehand investigation of the quadratic inequalities.

### 2.2.1 Behavioral models and classical models

As discussed in last section the model family containing all possible behavioral models consistent with the data records (2.1) is given by

$$
\mathcal{F}_\varepsilon = \left\{ x \in \mathbb{R}^r : \mu(x..\hat{M}) \leq \varepsilon \right\} = \left\{ x \in \mathbb{R}^r : x^T A x \leq 0 \right\}. 
$$

(2.19)

where $\hat{M}$ is the data matrix (2.9), $\varepsilon$ is the bound of the norm of the perturbation, and

$$
A := \hat{M} \hat{M}^T - \varepsilon^2 I_r
$$

(2.20)

is a $r \times r$ symmetric matrix.

Due to the special restriction on the coefficient vector of the classical models, to depict the model family containing all uncertain models with proper transfer functions we need to write the data matrix $\hat{M}$ as

$$
\hat{M} = \begin{bmatrix} \hat{M}_1 \\ \hat{m}_r^T \end{bmatrix},
$$

(2.21)

where $\hat{m}_r^T$ is the last row of the $\hat{M}$. Therefore, the model family can be expressed as follows:

$$
\mathcal{F}_\varepsilon = \left\{ x_0 = \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{R}^r : \mu(x_0..\hat{M}) \leq \varepsilon \right\} = \left\{ x_0 = \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{R}^r : x \in \mathcal{E} \right\}
$$

(2.22)

where

$$
\mathcal{E} = \left\{ x \in \mathbb{R}^{r-1} : x_0^T \hat{M} \hat{M}^T x_0 \leq \varepsilon^2 x_0^T x_0 \right\} = \left\{ x \in \mathbb{R}^{r-1} : x^T A x + 2b^T x + c \leq 0 \right\}
$$
with
\[ A = \hat{\mathcal{M}}_{1} \hat{\mathcal{M}}_{1}^{T} - \varepsilon^{2} I_{r-1}, \quad b = \hat{\mathcal{M}}_{1} \hat{m}_{2}, \quad c = \hat{m}_{2}^{T} \hat{m}_{2} - \varepsilon^{2}. \]  
(2.23)

or equivalently,
\[ \begin{bmatrix} A & b \\ b^{T} & c \end{bmatrix} = \hat{\mathcal{M}}_{1} \hat{\mathcal{M}}_{1}^{T} - \varepsilon^{2} I_{r}. \]

Clearly, \( \mathcal{F}_{c} \subseteq \mathcal{M}_{b} \). For the system identification of general purpose, the two families are actually equivalent. However, for the control-oriented identification, it is worthy investigating the effects of the distinct types of set-up to the controller design problem.

2.2.2 Quadratic inequalities

Notice that the model family (2.19) of behavioral uncertain models and the family (2.22) of classical uncertain models are both expressed by means of the quadratic inequality of the form
\[ \theta^{T} A \theta + 2b^{T} \theta + c \leq 0. \]  
(2.24)

where \( A \in \mathbb{R}^{r} \) is a real nonsingular symmetric matrix, \( b \in \mathbb{R}^{r} \), \( c \in \mathbb{R} \). Therefore, to characterize model families (2.19) and (2.22), it is necessary to make a detailed discussion on quadratic inequalities.

**Theorem 2.2.1** Let \( A, b, c \) be as above. Suppose \( A \) has the following eigenvalue decomposition
\[ A = U^{T} \begin{bmatrix} D_{2} & 0 \\ 0 & -D_{1} \end{bmatrix} U^{T}. \]  
(2.25)

where \( U \) is an orthogonal matrix and \( D_{1}, D_{2} \) are two diagonal positive definite matrices
\[ D_{1} = \begin{bmatrix} D_{2}^{2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{1}^{2} \end{bmatrix}, \quad D_{2} = \begin{bmatrix} \alpha_{r}^{2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_{r+1}^{2} \end{bmatrix}. \]  
\( (\alpha_{i} > 0, \ J_{j} > 0). \)
Then \( \theta \) is a vector satisfying the inequality (2.24) if and only if \( \theta \) has the following expression

\[
\theta = -A^{-1}b + \sum_{i=1}^{k} \beta_i \gamma_i u_i + \sum_{i=k+1}^{r} \alpha_i \delta_i u_i.
\]  

(2.26)

where \( u_i \)'s are columns of the matrix \( U^* \), \( U^* = [u_r \cdots u_2 \ u_1] \), and the parameters \( \delta_i \)'s and \( \gamma_i \)'s satisfy

\[
\sum_{i=k+1}^{r} \delta_i^2 - \sum_{i=1}^{k} \gamma_i^2 \leq b^T A^{-1} b - c.
\]

(2.27)

In the special case when \( b = 0 \) and \( c = 0 \), it is readily to get the following result.

**Theorem 2.2.2** Let the assumption on \( A \) be the same, then \( \theta \) is a vector satisfying the inequality

\[
\theta^T A \theta \leq 0
\]

(2.28)

if and only if \( \theta \) has the following expression

\[
\theta = \sum_{i=1}^{k} \beta_i \gamma_i u_i + \sum_{i=k+1}^{r} \alpha_i \delta_i u_i.
\]

(2.29)

where the parameters \( \delta_i \)'s and \( \gamma_i \)'s satisfy

\[
\sum_{i=k+1}^{r} \delta_i^2 - \sum_{i=1}^{k} \gamma_i^2 \leq 0.
\]

(2.30)

Another interesting special case is that \( A \) is positive definite and \( b^T A^{-1} b - c \) is positive. It is readily to see from Appendix A that in this case the set \( \mathcal{E} \) is reduced to be an ellipsoid. In the following theorem the members of the ellipsoid are characterized, which is a straightforward corollary of the above theorem \((k = 0)\).

**Theorem 2.2.3** Let the assumption on \( A \) be the same. If \( A \) is positive definite and \( b^T A^{-1} b - c \) is positive, then \( \theta \) is a vector satisfying the inequality (2.24) if and only if \( \theta \) has the following expression

\[
\theta = -A^{-1}b + \sum_{i=1}^{r} \alpha_i \delta_i u_i.
\]

(2.31)

where the parameters \( \delta_i \)'s satisfy

\[
\sum_{i=1}^{r} \delta_i^2 \leq b^T A^{-1} b - c.
\]

(2.32)
2.2.3 Model family in behavioral framework

Since every model in the uncertain behavioral model family $\mathcal{F}_b$ in (2.19) satisfying the inequality (2.28), the family $\mathcal{F}_b$ can be easily characterized using SVD of the data matrix $\hat{\mathbf{U}}$, in terms of the results we have obtained on quadratic inequalities.

We assume that the data matrix $\hat{\mathbf{U}}$ has the singular value decomposition (SVD) as follows:

$$
\hat{\mathbf{U}} = \hat{\mathbf{U}} \hat{\Sigma} \hat{\mathbf{V}}^T = [\hat{u}_r \ \hat{u}_{r-1} \ \cdots \ \hat{u}_1 \ \hat{u}_{1}] \begin{bmatrix}
\hat{\sigma}_r & & & & \\
& \ddots & & & \\
& & \hat{\sigma}_i & & \\
& & & \hat{\sigma}_1 & \\
& 0 & & & 0
\end{bmatrix} \hat{\mathbf{V}}^T.
$$

where $\hat{\mathbf{U}} \hat{\Sigma} \hat{\mathbf{V}}^T = I_r$, $\hat{\mathbf{V}} \hat{\mathbf{V}}^T = I_s$, $\hat{\sigma}_i \leq \hat{\sigma}_{i+1}$. Therefore, according to (2.20),

$$
A = \hat{\mathbf{U}} \hat{\Sigma} \hat{\mathbf{V}}^T - \varepsilon^2 I_r
$$

$$
= \hat{\mathbf{U}} \hat{\Sigma} \hat{\mathbf{V}}^T (\hat{\Sigma} \hat{\Sigma}^T - \varepsilon^2 I_r) \hat{\mathbf{V}}^T
$$

$$
= \hat{\mathbf{U}} \begin{bmatrix}
\hat{\sigma}_r^2 - \varepsilon^2 & & & \\
& \ddots & & \\
& & \hat{\sigma}_i^2 - \varepsilon^2 & \\
& & & \hat{\sigma}_1^2 - \varepsilon^2
\end{bmatrix} \hat{\mathbf{V}}^T.
$$

Specifically, in the case $\hat{\sigma}_{k+1} > \varepsilon > \hat{\sigma}_k$, the matrix $A$ has the eigenvalue decomposition as (2.25), where the two positive definite diagonal matrices $D_1$ and $D_2$ are defined in terms of the parameters $\alpha_i$'s and $\beta_j$'s given by

$$
\alpha_i = \frac{1}{\sqrt{\hat{\sigma}_i^2 - \varepsilon^2}}, \quad \beta_j = \frac{1}{\sqrt{\varepsilon^2 - \hat{\sigma}_j^2}}, \quad j = 1, \ldots, k, \quad i = k + 1, \ldots, r. \quad (2.33)
$$

According to Theorem 2.2.2, we have

Theorem 2.2.4 If $\hat{\sigma}_{k+1} > \varepsilon > \hat{\sigma}_k$, then $x \in \mathcal{F}_b$ if and only if there exist parameters $\delta_i$'s and $\gamma_i$'s satisfying

$$
\sum_{i=k+1}^{r} \delta_i^2 - \sum_{i=1}^{k} \gamma_i^2 \leq 0.
$$
such that
\[ x = \sum_{i=1}^{k} \beta_{i} \gamma_{i} \hat{u}_{i} + \sum_{i=k+1}^{r} \alpha_{i} \delta_{i} \hat{u}_{i}, \quad (2.34) \]

where \( \alpha_{i} \)'s and \( \beta_{i} \)'s are described in (2.33).

In the special case when \( k = 1 \), i.e., the perturbation bound \( \varepsilon \) is in between the two smallest singular values \( \hat{\sigma}_{1} \) and \( \hat{\sigma}_{2} \), the above characterization of the model family \( \mathcal{F}_{\delta} \) can be simplified as follows:

\[
x = \beta_{1} \gamma_{1} \hat{u}_{1} + \sum_{i=2}^{r} \alpha_{i} \delta_{i} \hat{u}_{i}
\]

\[
= \left( \hat{u}_{1} + \sum_{i=2}^{r} \frac{\alpha_{i}}{\beta_{1}} \frac{\delta_{i}}{\gamma_{1}} \hat{u}_{i} \right) \beta_{1} \gamma_{1}.
\]

and accordingly, the constraint and parameters become

\[
\sum_{i=2}^{r} \left( \frac{\delta_{i}}{\gamma_{1}} \right)^{2} \leq 1, \quad \frac{\alpha_{i}}{\beta_{1}} = \sqrt{\frac{\varepsilon^{2} - \hat{\sigma}_{1}^{2}}{\hat{\sigma}_{i}^{2} - \varepsilon^{2}}}, \quad i = 2, \ldots, r.
\]

With a slight abuse of notation, we then have:

**Theorem 2.2.5** If \( \hat{\sigma}_{2} > \varepsilon > \hat{\sigma}_{1} \), then \( x \in \mathcal{F}_{\delta} \) if and only if there exists a constant \( c \) such that \( x = cx_{0} \), and \( x_{0} \) has the expression

\[
x_{0} = \hat{u}_{1} + \alpha_{1} \delta_{2} \hat{u}_{2} + \cdots + \alpha_{r} \delta_{r} \hat{u}_{r}.
\]

where

\[
\alpha_{i} = \sqrt{\frac{\varepsilon^{2} - \hat{\sigma}_{1}^{2}}{\hat{\sigma}_{i}^{2} - \varepsilon^{2}}}, \quad i = 2, \ldots, r.
\]

and

\[
\delta_{2}^{2} + \cdots + \delta_{r}^{2} \leq 1. \quad (2.35)
\]

It is remarked that in the case when the perturbation bound \( \varepsilon \) is in between the two smallest singular values \( \hat{\sigma}_{1} \) and \( \hat{\sigma}_{2} \) of the data matrix \( \hat{M} \), the uncertain model family in behavioral framework is a convex cone around \( \hat{u}_{1} \), with semiaxis lengths

\[
\alpha_{i} := \sqrt{\frac{\varepsilon^{2} - \hat{\sigma}_{1}^{2}}{\hat{\sigma}_{i}^{2} - \varepsilon^{2}}}, \quad i = 2, \ldots, r.
\]
The central axis \( \hat{u}_1 \) of the cone is called the nominal model. In the case when the data is generated by an LTI system, the nominal model is regarded as the best estimate of the true system, since it has the smallest misfit with respect to the measured data matrix \( \hat{\mathcal{M}} \).

### 2.2.4 Model family in classical framework

In this subsection we will make use of the results obtained in section 2.2.2. to characterize the the uncertain model family \( \mathcal{F}_c \) in the classical framework

\[
\mathcal{F}_c = \left\{ x = \begin{bmatrix} \theta \\ 1 \end{bmatrix} \in \mathbb{R}^r : \theta \in \mathcal{E} \right\} \quad \text{with} \quad \mathcal{E} = \left\{ \theta \in \mathbb{R}^{r-1} : \theta^T A \theta + 2b^T \theta + c \leq 0 \right\}
\]

in terms of the SVD of matrix \( \hat{\mathcal{M}}_1 \), where \( A, b, c, \hat{\mathcal{M}}_1 \) are defined in (2.23) and (2.21).

The singular value decomposition of \( \hat{\mathcal{M}}_1 \) is assumed to be:

\[
\hat{\mathcal{M}}_1 = U \begin{bmatrix} \sigma_{r-1} & 0_{r-1 \times (s-r+1)} \\ \vdots & \ddots \\ 0_{1 \times (s-r+1)} & \sigma_1 \\ \end{bmatrix} V^T
\]

where \( U \in \mathbb{R}^{(r-1) \times (r-1)}, V \in \mathbb{R}^{s \times s} \) are both orthogonal matrices and \( 0 \leq \sigma_1 \leq \cdots \leq \sigma_{r-1} \).

Similar to the discussions in last subsection, we get

**Theorem 2.2.6** If \( \sigma_{k+1} > \varepsilon > \sigma_k \), then \( x \in \mathcal{F}_c \) if and only if \( x = [\theta^T, 1]^T \), and \( \theta \) has the following expression

\[
\theta = -A^{-1}b + \sum_{i=1}^{k} \beta_i \gamma_i u_i + \sum_{i=k+1}^{r-1} \alpha_i \delta_i u_i.
\]

where

\[
\beta_i = \frac{1}{\sqrt{\varepsilon^2 - \sigma_i^2}}, \quad \alpha_i = \frac{1}{\sqrt{\sigma_i^2 - \varepsilon^2}}.
\]

(2.36)

and the parameters \( \delta_i \)'s and \( \gamma_i \)'s satisfy

\[
\sum_{i=k+1}^{r-1} \delta_i^2 - \sum_{i=1}^{k} \gamma_i^2 \leq b^T A^{-1} b - c.
\]

(2.37)
If, furthermore, \( z \) is different from any singular value of the data matrix \( \hat{M} \), then 
\[ b^T A^{-1} b - c \neq 0 \] and \( \alpha_i \)'s and \( \beta_i \)'s in (2.36) can be replaced by 
\[ \beta_i = \sqrt{\frac{b^T A^{-1} b - c}{\varepsilon^2 - \sigma_i^2}}, \quad \alpha_i = \sqrt{\frac{b^T A^{-1} b - c}{\sigma_i^2 - \varepsilon^2}}. \]
and the condition (2.37) can be accordingly replaced by
\[ \sum_{i=k+1}^{r-1} \delta_i^2 - \sum_{i=1}^{k} \gamma_i^2 \leq \text{sign} (b^T A^{-1} b - c) = \begin{cases} 
1, & b^T A^{-1} b - c > 0 \\
-1, & b^T A^{-1} b - c < 0 
\end{cases}.

The following is a more specific result:

**Theorem 2.2.7** If \( \sigma_{\min}(\hat{M}) < \varepsilon < \sigma_1 \). Then \( b^T A^{-1} b - c > 0 \) and the set \( \mathcal{E} \) is an ellipsoid with half axis lengths
\[ \alpha_i = \sqrt{\frac{b^T A^{-1} b - c}{\sigma_i^2 - \varepsilon^2}}, \quad i = 1, \ldots, r - 1. \tag{2.38} \]
Furthermore, \( x \in \mathcal{F}_c \) if and only if \( x = [\theta^T \ 1]^T \), and \( \theta \) has the following expression
\[ \theta = -A^{-1} b + \sum_{i=1}^{r-1} \alpha_i \delta_i u_i. \]
where
\[ \sum_{i=1}^{r-1} \delta_i^2 \leq 1. \]

According to this theorem, uncertain models in the family \( \mathcal{F}_c \) have the following parameterization:
\[ x = \begin{bmatrix} \theta \\ 1 \end{bmatrix} = \begin{bmatrix} \theta_c + \sum_{i=1}^{r-1} \alpha_i \delta_i u_i \\ 1 \end{bmatrix} = \begin{bmatrix} \theta_c \\ 1 \end{bmatrix} + \sum_{i=1}^{r-1} \alpha_i \delta_i \begin{bmatrix} u_i \\ 0 \end{bmatrix}. \tag{2.39} \]
where \( \theta_c = -A^{-1} b \) is the center of the ellipsoid and the nominal model in classical form is \( x_c = [\theta_c^T \ 1]^T \). Similar to the expression (2.34), parameterization (2.39) describes uncertain models in an affine form around the nominal model. the model with the smallest misfit with respect to the measured data matrix \( \hat{M} \).

The relationship of model family in behavioral framework and model family in classical framework is visualized in Figure 2.1. where the convex cone is the behavioral model family \( \mathcal{F}_b \) and the ellipse is the classical model family \( \mathcal{F}_c \).
Figure 2.1  Behavioral model family $\mathcal{F}_b$ and classical model family $\mathcal{F}_c$, where $\mathcal{F}_b$ is an ellipse obtained by intersecting the convex cone $\mathcal{F}_c$.

2.3 Parameterization of uncertain models

As shown in (2.34) and (2.39), the uncertain models in coefficient vector form are parameterized in an affine form. In this section we will express the uncertain models in transfer function form based on these parameterizations. It turns out that a real uncertain parameter vector with length at most unit length appears linearly in the numerators and the denominators of the transfer functions.

2.3.1 Uncertain models in behavioral framework

Recall the relation between the transfer function (2.5) and the coefficient vector (2.6). we can easily get the transfer functions from the coefficient vectors. Define the polynomials in terms of the singular vectors $\hat{u}_i$:

$$\hat{\theta}_i(z) := \begin{bmatrix} 1 & z & \cdots & z^m & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & z & \cdots & z^n \end{bmatrix} \hat{u}_i =: \begin{bmatrix} -\hat{n}_i(z) \\ \hat{d}_i(z) \end{bmatrix}.$$
Then, for a model with the parameterization (2.34) we have the corresponding polynomial
\[ \hat{\theta}_s(z) = \hat{\theta}_1(z) + \sum_{i=2}^{r} \alpha_i \delta_i \hat{\theta}_i(z). \quad (\sum_{i=2}^{r} \delta_i^2 \leq 1). \]

Let \( \hat{\theta}_s(z) = [-\hat{n}_s(z) \, \hat{d}_s(z)]^T \). Then the transfer function of the corresponding uncertain model is:
\[ \frac{\hat{n}_s(z)}{\hat{d}_s(z)} = \frac{\hat{n}_1(z) + \sum_{i=2}^{r} \alpha_i \delta_i \hat{n}_i(z)}{\hat{d}_1(z) + \sum_{i=2}^{r} \alpha_i \delta_i \hat{d}_i(z)} \]

Notice that the real parameters \( \delta_i \) appear linearly in the numerator and the denominator of the transfer function. In addition, since the denominator does not necessarily have degree not less than that of the numerator, the models in the family \( \mathcal{F}_b \) may contain nonproper models or models whose transfer functions don't exist.

### 2.3.2 Uncertain models in classical framework

Similarly, we can define the polynomials as follows:
\[ \theta_s(z) := \begin{bmatrix} 1 & z & \cdots & z^m & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & z & \cdots & z^n \end{bmatrix} \begin{bmatrix} u_i \\ \vdots \\ u_r \end{bmatrix} =: \begin{bmatrix} -n_s(z) \\ d_s(z) \end{bmatrix}. \]
\[ \begin{bmatrix} 1 & z & \cdots & z^m & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & z & \cdots & z^n \end{bmatrix} \begin{bmatrix} x_c \\ \vdots \\ x_r \end{bmatrix} =: \begin{bmatrix} -n_c(z) \\ d_c(z) \end{bmatrix}. \]

Then the model family \( \mathcal{F}_c \) parameterized as (2.39) contains uncertain plants with transfer functions
\[ \frac{n_s(z)}{d_s(z)} = \frac{n_c(z) + \sum_{i=1}^{r-1} \alpha_i \delta_i n_i(z)}{d_c(z) + \sum_{i=1}^{r-1} \alpha_i \delta_i d_i(z)} \quad (\sum_{i=1}^{r-1} \delta_i^2 \leq 1). \]

Notice that all models in the family \( \mathcal{F}_c \) have proper transfer functions. Compared to the model family in the behavior framework, models which are compatible to the given input-output data are excluded from the family \( \mathcal{F}_c \) if the denominators of their transfer functions are of degrees less than \( n \).
2.4 Conclusions

The purpose of this chapter is characterize an uncertain model family based on the knowledge of the perturbation within the measurements. We first describe the uncertain model family in several different but equivalent forms, then characterize the model family in both behavioral framework and the classical framework by studying the quadratic inequalities. Explicit parameterizations are obtained in terms of the singular values and the singular vectors of the data matrix. Finally, we notice that in simple cases the model family in behavioral framework or in classical framework is characterized respectively by a convex cone, or by an ellipsoid. In both cases, the uncertain models can be parameterized so that a real parameter vector of at most unit length linearly appear in both the numerators and the denominators of the transfer functions of all uncertain models.

The so-called linear fractional parameterization of uncertain models appears only in special case, i.e., $\hat{\sigma}_1(\mathcal{M}) < \varepsilon < \hat{\sigma}_2(\mathcal{M})$ for behavioral model family and $\sigma_{\min}(\mathcal{H}) < \varepsilon < \sigma_{\min}(\mathcal{H}_1)$ for classical model family. However, since this parameterization of uncertain models are suitable for a robust synthesis scheme, we will concentrate on the study of the model families in these two special cases.
Chapter 3

Model family revisited

In this chapter we study several discrete topics which are closely related to the model family and hopefully helpful for a good understanding of the model family. In Section 3.1 several structure properties of model families are considered. The convexity of the uncertain model family is usually desired for robust control design, whereas the measures of the size of the model family give metrics of the uncertainty within the modeling. The denseness problem of the model family and the data matrix approximation problem are helpful for us to understand how well the approximate family can describe the true models. In addition, these two problems are also closely related to the model validation problem discussed in Chapter 5. Section 3.2 deals with the model family where the data is generated by an LTI system. A classical identification problem, order determination problem, is first addressed, and then the stochastic and deterministic perturbation is separately formulated. The third is how to estimate the perturbation bound according the data measurements. We studied this problem in detail, and found that the perturbation bound is closely related to the geometric relationship of the uncertain model families (ellipsoids) generated with different inputs. Section 3.3 is devoted to another method of estimating the perturbation bound. When modeling the data, we sometimes have no clue on the perturbation bound. One way of obtaining a sub-optimal estimate is to decompose the data matrix into two terms, so that one term of the decomposition looks as if it were generated by an LTI system, and the other term indicates the perturbation. In this section we develop a sub-optimal method to decompose the data matrix by an iterative composite property mapping method.

This chapter is also intended to build the theoretic foundations of some results related to the model family. So, this chapter is the result of some theoretic interests as well.
3.1 Structure properties

3.1.1 Convexity

The purpose of robust identification is to parameterize the uncertain models for robust control analysis and design, while in analysis and design approaches the uncertain model family is always expected to be convex, so that various convex optimization techniques can be applied. In this subsection we will investigate the convexity of the model families: the model families in either behavioral framework or classical framework will be discussed based on the characterization of the model families.

Behavioral models

Let us first consider this problem for the model family (2.19) in the behavioral framework

\[ F_b = \{ x \in \mathbb{R}^r : \mu(x, \hat{\mathcal{M}}) \leq \varepsilon \} = \{ x \in \mathbb{R}^r : x^T A x \leq 0 \}. \]

where \( A = \hat{\mathcal{M}} \hat{\mathcal{M}}^T - \varepsilon^2 I_r \). According to Theorem 2.2.5, in the case where the perturbation bound \( \varepsilon \) is in between the two smallest singular values of the data matrix \( \hat{\mathcal{M}} \), i.e.,

\[ \hat{\sigma}_1 < \varepsilon < \hat{\sigma}_2. \]

\( x \in F_b \) has a general form: \( x = c_x x_0 \) for some \( c_x \), and \( x_0 \) has the expression

\[ x_0 = \hat{u}_1 + \alpha_2 \delta_2 \hat{u}_2 + \cdots + \alpha_r \delta_r \hat{u}_r. \]

where

\[ \alpha_i = \sqrt{\frac{\varepsilon^2 - \hat{\sigma}_i^2}{\hat{\sigma}_i^2 - \varepsilon^2}}, \quad i = 2, \ldots, r. \]

and \( \delta = [\delta_2 \delta_3 \cdots \delta_r] \) satisfies \( \| \delta \| \leq 1 \). Therefore, for any \( \lambda, \mu \in \mathbb{R}, x, y \in F_b \), the condition \( \lambda x + \mu y \in F_b \) is equivalent to the condition

\[ \left\| \frac{\lambda c_x}{\lambda c_x + \mu c_y} \delta_x + \frac{\mu c_y}{\lambda c_x + \mu c_y} \delta_y \right\| \leq 1. \]
Considering $||\delta_x|| \leq 1$ and $||\delta_y|| \leq 1$, we conclude that the condition (3.2) will hold if $\lambda, \mu$ have the same sign and $c_x, c_y$ have the same sign. This means $F_\delta$ is a convex cone.

**Theorem 3.1.1** $F_\delta$ is a convex cone in the case where the perturbation bound $\varepsilon$ is in between the two smallest singular values of the data matrix $\hat{M}$. i.e., (3.1) holds.

Note that in the general case, i.e., $\varepsilon > \hat{\sigma}_2$, the above argument will not hold anymore. Although $F_\delta$ doesn't have the convexity in general, to some extent, it still has near-convexity property in the special case. This will facilitate the analysis and design in various robust synthesis problems.

**Classical models**

For the family (2.22) of classical models

$$F_\varepsilon = \left\{ x = \begin{bmatrix} \theta \\ 1 \end{bmatrix} \in \mathbb{R}^r: \theta \in E \right\} \text{ with } E = \left\{ \theta \in \mathbb{R}^{r-1}: \theta^T A \theta + 2 b^T \theta + c \leq 0 \right\},$$

where $A, b, c, \hat{M}_1$ are defined in (2.23) and (2.21). An explicit description of its members is obtained in Theorem 2.2.7 in the case when

$$\sigma_{\min}(\hat{M}) < \varepsilon < \sigma_{\min}(\hat{M}_1). \quad (3.3)$$

The convexity of the family $F_\varepsilon$ can thus be obtained by a similar argument as above.

**Theorem 3.1.2** The model family $F_\varepsilon$ is convex (an ellipsoid) in the case when the perturbation bound is in between the smallest singular value of $\hat{M}$ and the smallest singular value $\hat{M}_1$. i.e., (3.3) holds.

Note that in the case when the perturbation bound is too big, i.e., $\varepsilon > \sigma_{\min}(\hat{M}_1)$, the model family is not convex. Therefore, in order to make the family convex, it is important to "decrease" the perturbation bound $\varepsilon$ so that the condition (3.3) holds.
3.1.2 Measures for the size

It is usually useful to know how the “size” of the uncertain model family is related to the input, the perturbation bound, etc., when dealing with a robust identification problem. Therefore, it is necessary to define measures for the size of the model family. Since we are mainly interested in convex model family in the sequel, only the model family in the special case when $\varepsilon$ is “small” will be considered in this subsection.

**Behavioral models**

It is known that in the case where the perturbation bound $\varepsilon$ is in between the two smallest singular values of the data matrix $\hat{\mathcal{M}}$, i.e.,

$$\hat{\sigma}_1 < \varepsilon < \hat{\sigma}_2,$$

the model family $\mathcal{F}_b$ is a convex cone whose base ellipsoid has semiaxis lengths

$$\alpha_i = \sqrt{\frac{\varepsilon^2 - \hat{\sigma}_1^2}{\hat{\sigma}_i^2 - \varepsilon^2}}, \quad i = 2, \ldots, r,$$

where $\hat{\sigma}_i$'s are singular values of $\hat{\mathcal{M}}$ and $\hat{\sigma}_i \leq \sigma_{i+1}$. A natural way of measuring the size of the convex cone is thus to evaluate the size of its base ellipsoid. According to the definitions of the measures of ellipsoids introduced in Chapter 1, we can accordingly define the following measures for the model family (convex cone)

**Volume:**

$$V(\mathcal{F}_b) := \prod_{i=2}^r \alpha_i = \prod_{i=2}^r \sqrt{\frac{\varepsilon^2 - \hat{\sigma}_1^2}{\hat{\sigma}_i^2 - \varepsilon^2}}.$$  

**Radius:**

$$R(\mathcal{F}_b) := \max_{2 \leq i \leq r} \alpha_i = \alpha_2 = \sqrt{\frac{\varepsilon^2 - \hat{\sigma}_1^2}{\hat{\sigma}_2^2 - \varepsilon^2}}.$$  

**Mean square root radius:**

$$M(\mathcal{F}_b) := \sqrt{\sum_{i=2}^r \alpha_i^2} = \sqrt{\sum_{i=2}^r \frac{\varepsilon^2 - \hat{\sigma}_1^2}{\hat{\sigma}_i^2 - \varepsilon^2}}.$$
It is remarked that for the data which is generated exactly by an LTI system (we will drop the hat from the notations in this case), we know the smallest singular value $\sigma_1$ is zero, hence, the radius of the family, $R(\mathcal{F}_b)$, indicates that the size of the family is determined by the second smallest singular value $\sigma_2$. Interestingly, this coincides with the observation in [4], where $\sigma_2$ is maximized when designing an optimal input for a known system. It is also interesting to notice that the volume of the family $V(\mathcal{F}_b)$ is determined by the products of the terms $\sigma_i^2 - \varepsilon^2$, $i = 2, \ldots, r$. When the perturbation bound $\varepsilon$ is small compared to the nonzero singular values $\sigma_i$, the volume is approximately determined by the product of the nonzero singular values. Therefore, for optimal input design for a known LTI system a simple scheme is to maximize the the product of the nonzero singular values of the matrix $\mathcal{M}$. Again, this approach coincides with the method introduced in [26], where the information matrix replaces the data matrix and the objective is to find an input function to maximize the logarithm of its determinant. In our formulation this amounts to maximizing the product of the nonzero singular values of the data matrix $\mathcal{M}$.

Classical models

Recall that in the case when

$$\sigma_{\text{min}}(\hat{\mathcal{M}}) < \varepsilon < \sigma_{\text{min}}(\hat{\mathcal{M}}_1),$$

the model family in classical framework, $\mathcal{F}_c$, is an ellipsoid with semiaxis lengths

$$\alpha_i = \sqrt{\frac{b^T A^{-1} b - c}{\sigma_i^2 - \varepsilon^2}}, \quad i = 1, \ldots, r - 1,$$

where $A, b, c, \hat{\mathcal{M}}_1$ are defined in (2.23) and (2.21). $\sigma_i$'s are singular values of $\hat{\mathcal{M}}_1$. The size of the model family is thus evaluated by means of the measures of the ellipsoid.

Volume:

$$V(\mathcal{F}_c) := \prod_{i=1}^{r-1} \alpha_i = \prod_{i=1}^{r-1} \sqrt{\frac{b^T A^{-1} b - c}{\sigma_i^2 - \varepsilon^2}}.$$
Radius:

\[ R(\mathcal{F}_c) := \max_{1 \leq i \leq r-1} \alpha_i = \alpha_1 = \sqrt{\frac{b^T A^{-1} b - c}{\sigma_i^2 - \varepsilon^2}}. \]

Mean square root radius:

\[ M(\mathcal{F}_c) := \sqrt{\sum_{i=1}^{r-1} \alpha_i^2} = \sqrt{\sum_{i=1}^{r-1} \frac{b^T A^{-1} b - c}{\sigma_i^2 - \varepsilon^2}}. \]

An alternative interpretation of the measures

In this section an alternative interpretation of the measures is given, based on some algebraic and geometric assumptions about the column (short) and row (long) spaces of the data matrix. It turns out that the volume and radius of the model family are actually closely related to the signal-to-noise ratios (SNR) defined in the paper [38]. It can, therefore, be interpreted that minimizing the measures is equivalent to maximizing SNR’s.

Suppose that the noise-free data matrix \( \mathcal{M} \) has the following SVD:

\[ \mathcal{M} = U \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} V^T = [U_2 \ u_1] \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_2^T \\ u_1^T \end{bmatrix} \]

where the partitions are conformal, \( \Sigma_2 = \text{diag}(\sigma_2, \ldots, \sigma_2) \), \( (\sigma_i \leq \sigma_{i+1}) \) is nonsingular, and \( u_1, v_1 \) are left, right singular vectors corresponding to the smallest singular value \( \sigma_1 = 0 \). The noisy data matrix \( \mathcal{\hat{M}} \) is the noise-free data matrix polluted by the additive perturbation \( \mathcal{\hat{M}} \). i.e., \( \mathcal{\hat{M}} = \mathcal{M} + \mathcal{\hat{M}} \). As shown in [38], many engineering applications leads to the following generic conditions on the matrices \( \mathcal{M} \) and \( \mathcal{\hat{M}} \).

\[ \mathcal{\hat{M}}.\mathcal{\hat{M}}^T = \varepsilon^2 I. \quad \text{and} \quad \mathcal{M}.\mathcal{\hat{M}}^T = 0 \]

Under these conditions, a SVD of the noisy data matrix \( \mathcal{\hat{M}} \) can be derived as follows:

\[
\begin{align*}
\mathcal{\hat{M}} & = U_2 \Sigma_2 V_2^T + U_2 U_2^T \mathcal{\hat{M}} + u_1 u_1^T \mathcal{\hat{M}} \\
& = [U_2 \ u_1] \begin{bmatrix} (\Sigma_2^2 + \varepsilon^2 I)^{1/2} & 0 \\ 0 & \varepsilon \end{bmatrix} \begin{bmatrix} (\Sigma_2^2 + \varepsilon^2 I)^{-1/2} (\Sigma_2 V_2^T + U_2^T \mathcal{\hat{M}}) \\ u_1^T \mathcal{\hat{M}} / \varepsilon \end{bmatrix}
\end{align*}
\]
From this SVD it is observed that there is a gap in the singular spectrum, i.e., the smallest singular value of \((\Sigma^2 + \varepsilon^2 I)^{1/2}\) is larger than \(\varepsilon\), which is identical to the smallest singular value of \(\hat{\mathcal{M}}\) and then can be interpreted as a "noise threshold". The gap in the spectrum motivates the definitions of the signal-to-noise ratio (SNR) as follows. One possible definition is

\[
\text{SNR} = \frac{\sigma_1}{\varepsilon}.
\]

and another uses all nonzero singular values of \(\mathcal{M}\).

\[
\text{SNR} = \sqrt{\frac{\sigma^2_1 + \cdots + \sigma^2_i}{\varepsilon^2}} = \frac{||\mathcal{M}||_F}{||\hat{\mathcal{M}}||_F}
\]

Clearly, the first definition for \(\text{SNR}\) corresponds closely to radius of the model family in behavioral framework; and considering the following arithmetic-geometric-mean inequality

\[
\sqrt[n]{\prod_{i=2}^{r}(\sigma^2_i - \varepsilon^2)} \leq \frac{\sum_{i=2}^{r}(\sigma^2_i - \varepsilon^2)}{r - 1},
\]

we realize that the second definition for \(\text{SNR}\) has close connection with the volume of the model family in behavioral framework. What is more, maximizing the \(\text{SNR}\)'s amounts to minimizing the measures. This gives another interpretation about the optimal input design problem.

### 3.1.3 Denseness

Recall that since we dropped the restrictions on the Hankel structure of the data matrix \(\hat{\mathcal{M}}\) during the discussion of the model family (2.14)

\[
\mathcal{F}_2 = \left\{ x \in \mathbb{R}^r; \exists \mathcal{M}, \text{ such that } x^T\mathcal{M} = 0 \text{ and } ||\hat{\mathcal{M}} - \mathcal{M}||_2 \leq \varepsilon \right\}
\]

in last chapter, \(\mathcal{F}_2\) is a loose description of the uncertain models compatible with the data measurements. Compared to the family of uncertain models of true interest:

\[
\mathcal{F}_2^* = \left\{ x \in \mathbb{R}^r; \exists \mathcal{M} \text{ (having the same block Hankel structure as } \hat{\mathcal{M}}), \text{ such that } x^T\mathcal{M} = 0 \text{ and } ||\hat{\mathcal{M}} - \mathcal{M}||_2 \leq \varepsilon \right\}.
\]
\( \mathcal{F}_2 \) is actually bigger than the family \( \mathcal{F}_2^* \), i.e., \( \mathcal{F}_2 \supseteq \mathcal{F}_2^* \). For the robust control design it does not matter with this kind of minor increase of the uncertain model family. It is however interesting to make a close comparison between the two families. We conjecture that the two families \( \mathcal{F}_2 \) and \( \mathcal{F}_2^* \) are equal, or \( \mathcal{F}_2^* \) is dense in the bigger family \( \mathcal{F}_2 \). In this section the denseness of the family \( \mathcal{F}_2^* \) in the bigger family \( \mathcal{F}_2 \) is studied.

According to the definition of the denseness *, the above denseness problem can thus be formulated as follows:

**Conjecture 1.** If for some \( x \in \mathbb{R}^r \) there exists some matrix \( \mathbf{M} \in \mathbb{R}^{r \times s} \) such that \( x^T \mathbf{M} = 0 \) and \( ||\mathbf{U} - \mathbf{M}||_2 \leq \varepsilon \) then for any \( \delta > 0 \) there must exist some \( x_0 \in \mathbb{R}^r \) and matrix \( \mathbf{M}_0 \in \mathbb{R}^{r \times s} \) having the same block Hankel structure as that of \( \mathbf{U} \) such that \( x_0^T \mathbf{M}_0 = 0 \), \( ||\mathbf{U} - \mathbf{M}_0||_2 \leq \varepsilon \) and \( ||x - x_0||_2 \leq \delta \).

Obviously, the crucial part of a proof for this proposition is to construct the matrix \( \mathbf{M}_0 \) and the vector \( x_0 \). Due to the special structure (two Hankel blocks) of \( \mathbf{M}_0 \), it is very hard to get a proof. However, this open problem can be strongly supported by studying how close the data matrix \( \mathbf{U} \) can be approximated by a true data matrix generated by a model in the family \( \mathcal{F}_2 \). This will be discussed in last subsection.

### 3.1.4 Data matrix approximation

To continue investigating the closeness of the model family \( \mathcal{F}_2^* \) to the approximated family \( \mathcal{F}_2 \), in this subsection we will consider the relationships of a particular model \( x \in \mathcal{F}_2 \) with the family \( \mathcal{F}_2^* \). by computing the minimal distance of the original data matrix \( \mathbf{U} \) and the data matrix generated by the model \( x \) with different inputs. The

---

*\( A \subseteq \mathbb{R}^n \) is called dense in \( B \subseteq \mathbb{R}^n \) if for any vector \( b \in B \) and any positive number \( \delta \) there must some vector \( a \in A \) such that \( a \) is in the \( \delta \)-neighborhood of \( b \), i.e., \( ||a - b||_2 \leq \delta \).
problem is formulated as a convex optimization problem with LMI constraints, and the result strongly supports the conjecture posed in section 3.1.2.

Let $x$ be an arbitrary model in the family $\mathcal{F}_2$. For any given input sequence $u_t \in \mathbb{R}$, $t = 0, 1, \ldots, N$ and past output sequence $y_t \in \mathbb{R}$, $t = 0, 1, \ldots, n - 1$, the model $x$ generates a sequence of input-output data $\mathcal{M}(x, u, y)$, which can be similarly arranged in a matrix $\mathcal{M}(x, u, y)$ with Hankel block structure as in (2.9). To consider the minimal distance between $\hat{\mathcal{M}}$ and $\mathcal{M}(x, u, y)$ over all possible choices of $u_t$ and $y_t$,

$$\min \| \hat{\mathcal{M}} - \mathcal{M}(x, u, y) \|_2,$$

we first notice that $\mathcal{M}(x, u, y)$ is linearly dependent on $u_t$'s and $y_t$'s

$$\mathcal{M}(x, u, y) = \sum_{t=0}^{N} A_t u_t + \sum_{t=0}^{n-1} B_t y_t. \tag{3.4}$$

where $A_t, B_t$ are $r \times s$ matrices determined by the model $x$ and the matrix structure. By introducing a slack variable $\lambda$ and using Schur complement decomposition, the problem of interest can be formulated as an optimization problem (of variables $u_t$'s and $y_t$'s)

$$\min \quad \lambda$$

subject to

$$\begin{bmatrix} \lambda I_r & \hat{\mathcal{M}} - \mathcal{M}(x, u, y) \\ \hat{\mathcal{M}}^T - \mathcal{M}(x, u, y)^T & \lambda I_s \end{bmatrix} \succeq 0.$$ 

Due to (3.4) this is obviously a Positive Definite Problem, which can very efficiently solved by using the interior point methods [6, 39].

The experiments with the above optimization problem highly support the following proposition:

**Conjecture 2.** For any model $x$ in the family $\mathcal{F}_2$ there holds

$$\inf_{u, y} \| \hat{\mathcal{M}} - \mathcal{M}(x, u, y) \|_2 = \mu(x, \hat{\mathcal{M}}). \tag{3.5}$$
It is very easy to prove the conjecture 1 based on the above statement. However, these two conjectures are actually equivalent. The proof of the equivalence is provided in Appendix C. As a summary, we strongly believe the close relationship between the two families $\mathcal{F}_2$ and $\mathcal{F}_2'$, which is supported by numerical experiments, although the two conjectures are not able to prove at this time.

3.2 Model family for LTI data

Notice that although we try to model the data measurements (2.1) with an LTI system together with the perturbation, the input-output data itself is not necessarily generated by an LTI system. The primary interest of this section is to study some topics related to the model family for LTI data records. We will address three issues in this section: the order determination problem, the perturbation formulation problem and the noise energy estimation problem.

3.2.1 Order determination

Two kinds of methods on determining the order of the LTI system generated the data records are discussed in this subsection. Recall that the order of system (2.5) is referred as to the McMillan degree of the transfer function, i.e., order = max($m, n$) = $n$. Thus, we will assume $m = n$ and only classical models are considered in this subsection. It turns out that the order can be determined by computing the rank of the data matrix, or by investigating the pole-zero cancelation of the estimated nominal model.

Rank of data matrix

Before we proceed to present the main result, we need the following concept. We say the input sequence $u_t$, $t = 0, 1, \ldots, N$ is persistently exciting of order $i + 1$ if the input covariance matrix

$$\mathcal{R}_i := uu^T$$
has full row rank, where $\mathcal{U}$ is the input matrix with row number $i + 1$:

$$
\mathcal{U} := \begin{bmatrix}
  u_0 & u_1 & \cdots & u_{N-1} & u_N \\
  u_1 & u_2 & \cdots & u_N & u_{N+1} \\
  \vdots & \ddots & \ddots & \vdots & \vdots \\
  u_{i-1} & u_i & \cdots & u_N & u_{N+i-1} \\
  u_i & u_{i+1} & \cdots & u_{N-1} & u_N \\
\end{bmatrix} \in \mathbb{R}^{(i+1) \times (N-i+1)} \quad (3.6)
$$

**Theorem 3.2.1** Suppose that a model of order $n$ is excited by an input sequence $u_k$, which is persistently exciting of order $i + 1$ (not smaller than $n$). Then the rank of the data matrix $\mathcal{M}$ with row number $2(i + 1)$ is $i + n + 1$, i.e., rank $\mathcal{M} = i + n + 1$.

Based on this theorem the order $n$ of the LTI system can be determined by $n = \text{rank } \mathcal{M} - (i + 1)$. We will illustrate this in Chapter 6 (see Figure 6.3 for details). Note that in [58, 37] the order of the model can also be directly obtained from the rank of an oblique projection of the row space of an output matrix along the row space of an input matrix to the row space of the data matrix.

**Pole-zero cancelation of nominal model**

The order can be determined by investigating the pole-zero cancelation of the nominal model obtained from the data matrix.

Let $x_0 = [p_0^T ~ q_0^T]^T$ with $p_0, q_0 \in \mathbb{R}^{n+1}$ be the true LTI model generating the data, and $\mathcal{M}$ be the data matrix with row number $2(i + 1)$ where $i$ is assumed to be an integer larger than $n$, the order of the system. As noted before, the singular vector $x$ corresponding to the smallest singular value of the data matrix $\mathcal{M}$ can be considered as a nominal model compatible with the data. Furthermore, it turns out that $x$ has
the form $x = [p^T \ q^T]^T$ where

$$p = \begin{bmatrix} 0 \\ p_0 \\ 0 \end{bmatrix} \in \mathbb{R}^{r+1}, \quad q = \begin{bmatrix} 0 \\ q_0 \\ 0 \end{bmatrix} \in \mathbb{R}^{r+1}.$$

Clearly, the vectors $p$ and $q$ determine two polynomials $p(\lambda)$ and $q(\lambda)$ which are respectively the numerator and the denominator of the transfer function of the nominal model $x$. Therefore, by considering the pole-zero cancelation of the model $x$, we are able to recover the true model $x_0$.

Note that the usual methods for the pole-zero cancelation problem is to convert it to computing the rank of a matrix determined by the polynomials $p(\lambda)$ and $q(\lambda)$. The typical matrices are Bezoutian or Resultant matrix generated by the two polynomials [33, 28].

**Rank determination of noisy matrix**

In practice, the data measurements are usually corrupted by the noise or some other source of perturbations. Therefore, when trying to determine the order of an LTI system by using the the above schemes, we have to know how to determine the rank in the presence of error.

The usual approach is to compute a rank-revealing decomposition and make the decision about the rank by examining the small elements of the decomposition. As discussed in [51], the three commonly used decompositions are the singular value decomposition, the pivoted QR decomposition, and the URV decomposition.

**3.2.2 Formulations of perturbation**

Suppose that the observed input-output data (2.1) is generated by an LTI system, where $\hat{u}_t$ is the true input $u_t$, and $\hat{y}_t$ is the polluted output

$$\hat{y}_t = y_t + \xi_t.$$
i.e., only the output is contaminated by the noise or the unmodeled dynamics. The data matrix \( \mathbf{\tilde{U}} \in \mathbb{R}^{r \times s} \) defined in (2.9) is the concatenation of two Hankel matrices: input matrix \( \mathbf{\hat{U}} \in \mathbb{R}^{(m+1) \times s} \) and output matrix \( \mathbf{\hat{Y}} \in \mathbb{R}^{(n+1) \times s} \)

\[
\mathbf{\hat{U}} = \begin{bmatrix}
\hat{u}_0 & \hat{u}_1 & \cdots & \hat{u}_{m-1} \\
\hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_m \\
\vdots & \vdots & \ddots & \vdots \\
\hat{u}_m & \hat{u}_{m+1} & \cdots & \hat{u}_{m+n-1}
\end{bmatrix}, \quad \mathbf{\hat{Y}} = \begin{bmatrix}
\hat{y}_0 & \hat{y}_1 & \cdots & \hat{y}_{m-1} \\
\hat{y}_1 & \hat{y}_2 & \cdots & \hat{y}_m \\
\vdots & \vdots & \ddots & \vdots \\
\hat{y}_m & \hat{y}_{m+1} & \cdots & \hat{y}_{m+n-1}
\end{bmatrix}.
\]

where \( r = m + n + 2 \), \( s = N - \max(m, n) + 1 \). The exact data matrix \( \mathbf{U} \) is generated in a similar way. Therefore, the noise data matrix becomes

\[
\mathbf{\tilde{U}} = \mathbf{\tilde{U}} - \mathbf{U} = \begin{bmatrix}
\mathbf{\hat{U}} - \mathbf{U} \\
\mathbf{\hat{Y}} - \mathbf{Y}
\end{bmatrix} = \begin{bmatrix}
0 \\
\mathbf{N}
\end{bmatrix}.
\]

where \( \mathbf{N} \in \mathbb{R}^{(n+1) \times s} \) is the Hankel matrix

\[
\mathbf{N} = \begin{bmatrix}
\xi_0 & \xi_1 & \cdots & \xi_{m-1} \\
\xi_1 & \xi_2 & \cdots & \xi_m \\
\vdots & \vdots & \ddots & \vdots \\
\xi_m & \xi_{m+1} & \cdots & \xi_{m+n-1}
\end{bmatrix}.
\]

Clearly, \( ||\mathbf{\tilde{U}}|| = ||\mathbf{N}|| \).

In this subsection we are concerned with the upper bound of \( ||\mathbf{N}|| \) in terms of the deterministic or stochastic nature of the single disturbance \( \xi \). Two approaches are proposed. one is the deterministic approach in which \( \xi \) is assumed to be a deterministic quantity whose magnitude is upper bounded by some known number \( \varepsilon \), i.e., \( |\xi| \leq \varepsilon \). The other approach takes the stochastic formulation. in which \( \xi \) is regarded as a (real) white Gaussian random process whose covariance matrix is \( \varepsilon I_{N+1} \), i.e., \( \mathcal{R}_{nn} = \varepsilon^2 I_{N+1} \). In the deterministic formulation we derive a hard upper bound for \( ||\mathbf{N}|| \), while in the stochastic formulation every positive number can be considered as an upper bound of \( ||\mathbf{N}|| \), however, the probability that \( ||\mathbf{N}|| \) is not larger than the bound is our concern.
For the deterministic perturbation $\xi$, in terms of the norm inequality introduced in Appendix, we have the following upper bound for $\|\mathcal{V}\|$,

$$
\|\mathcal{V}\| \leq \left\| \begin{bmatrix}
\varepsilon & \varepsilon & \cdots & \varepsilon \\
\varepsilon & \varepsilon & \cdots & \varepsilon \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon & \varepsilon & \cdots & \varepsilon \\
\end{bmatrix} \right\| = \varepsilon \|1_{n+1}\| \|1_s\| \leq \sqrt{s(n+1)} \varepsilon.
$$

where $1_k$ denotes the length $k$ vector $1_k = [1 \ldots 1]^T$. Since $s + n = \mathcal{N} + 1$, it is easy to see that

$$
K_0 := \frac{\mathcal{N} + 2}{2} \varepsilon
$$

is an upper bound of $\|\mathcal{V}\|$, which is dependent on the number of measurements.

Let us now switch our attention to the stochastic formulation. We are mainly concerned with the probability that $\|\mathcal{V}\|$ is not larger than an upper bound $K$. Apparently, an appropriate bound should be large enough to be asymptotically an upper bound with full probability. In other words, when we have a very large number of measurements, the probability that $\|\mathcal{V}\|$ is not larger than the bound is very close to 1. Namely,

$$
\lim_{\mathcal{N} \to \infty} \Pr(\|\mathcal{V}\| \leq K) = 1. \tag{3.8}
$$

The probability that $\|\mathcal{V}\|$ is not larger than the bound $K$ is estimated in Appendix C:

$$
P := \Pr(\|\mathcal{V}\| \leq K) \geq \left(1 - e^{-\frac{K^2}{4(n+1)}}\right)^{\mathcal{N}+1}. \tag{3.9}
$$

This is derived in terms of the probability of a $\chi^2$ distribution with 2 degrees of freedom. It is easy to see that if $K = O(\mathcal{N}^{1+\alpha})$ for $\alpha > 0$, then (3.8) holds. Therefore, every quantity $K$ satisfying $K = O(\mathcal{N}^{1+\alpha})$ is asymptotically an upper bound of $\|\mathcal{V}\|$. However, for finite measurements we always expect a bound with high probability $P$. For example, for the hard bound (3.7), the relation of the probability $P$ and the number of the measurements $\mathcal{N}$ is roughly plotted in Figure 3.1. Notice, for small number $\mathcal{N}$, the probability is very close to 1 (when $\mathcal{N} = 40$, $P \approx 0.9991$). This means $K_0$ is one of upper bounds with high probability.
Figure 3.1  The probability that $||\mathcal{V}||$ is not larger than $K_0$ as function of the number of measurements.

3.2.3 Perturbation bound estimation

Given a set of data measurements (2.1), which is assumed to be generated by an LTI model $x$ together with additive perturbation. The data consists of input sequence $u_t \in \mathbb{Z}$, $t = 0, 1, \ldots, N$, the past output sequence $y_t \in \mathbb{Z}$, $t = 0, 1, \ldots, n - 1$, and the noisy output measurements. Suppose $E$ is the upper bound of the perturbation within the data matrix $\mathcal{M}(u, y)$. i.e.,

$$\mathcal{M}(u, y) = \mathcal{M} + \mathcal{M}_n. \ |.\mathcal{M}|| \leq E.$$ 

where $\mathcal{M}$ is the exact data matrix. As usual, for any positive number $\varepsilon$ a model family $\mathcal{F}(u, y, \varepsilon)$ can be generated, in which the nominal model $x$ is guaranteed to be contained whenever the parameter $\varepsilon$ is not less than the corresponding misfit function $\mu(x, \mathcal{M}(u, y))$. Therefore, the nominal model $x$ is in the intersection of the families

$$\mathcal{F}(\varepsilon) = \bigcap_{u, y} \mathcal{F}(u, y, \varepsilon).$$  

(3.10)
if and only if the parameter $\varepsilon$ is not less than the constant

$$\alpha = \max_{u,y} \mu(x, \hat{\mathcal{M}}(u,y)) \tag{3.11}$$

In other words,

$$\varepsilon \geq \alpha \iff x \in \mathcal{F}(\varepsilon) \tag{3.12}$$

Define the constant

$$\beta = \min \{ \varepsilon : \mathcal{F}(\varepsilon) \neq \emptyset \}.$$  

Then, whenever the selected $\varepsilon$ is smaller than $\beta$ the intersection of the model families $\mathcal{F}(\varepsilon)$ is empty, and vice versa. That is,

$$\mathcal{F}(\varepsilon) \begin{cases} = \emptyset, & \varepsilon < \beta \\ \neq \emptyset, & \varepsilon \geq \beta \end{cases} \tag{3.13}$$

Due to (3.12) $\beta$ is obviously not greater than $\alpha$. To consider the constant $\beta$ it is necessary to exclude the trivial case when some single family $\mathcal{F}(u,y,\varepsilon)$ in (3.10) is empty. Therefore, $\beta$ should not be less than the constant

$$\gamma = \max_{u,y} \sigma_{\min}(\hat{\mathcal{M}}(u,y)). \tag{3.14}$$

Clearly, $\gamma$ is closely related to the nominal model and the noise. The relations of the three parameters $\alpha$, $\beta$ and $\gamma$ are summarized as follows.

$$\alpha \geq \beta \geq \gamma. \tag{3.15}$$

In this subsection we will investigate under what condition the intersection (3.10) is empty. In other words, we want to study the relationship between the constant $\beta$ and the perturbation bound $\varepsilon$, and expect to obtain a way of estimating the perturbation bound. Due to the relation (3.15), a natural way of doing this is to investigate the two constants $\alpha$ and $\gamma$ first.

We will adopt a convention throughout this section: all data matrices discussed in this section are assumed generated by a same LTI system with possibly different input-output data, yet with the same perturbation bound $\varepsilon$. An according convention is proposed for model families.
Maximal misfit

Let the nominal model $x$ have the partition

$$x = [x_1^T ~ x_2^T]^T,$$

which is conformal to the definition of the data matrix $\mathcal{M}$ as in (2.9). In the following we will compute the constant $\alpha$ defined in (3.11), i.e., computing the maximal misfit of the nominal model $x$ with respect to each possibly observed data matrix $\mathcal{M}$. This result is potentially useful for the estimation of the upper bound $E$. In the rest of this section $\mathcal{H}$ is always referred to as a Hankel matrix of size $s \times (n + 1)$.

In order to derive our main result, let us first notice

$$||x||_2 = \max ||\mathcal{H}x||_2, \text{ subject to } \mathcal{H} : \text{ Hankel. } ||\mathcal{H}||_2 \leq 1. \quad (3.16)$$

Actually, this can be easily seen by considering that the flip matrix $P = [\delta_{i+j-n-1}]$ is a Hankel matrix and $||Px||_2 = ||x||_2$ for any $x \in \mathbb{R}^{n+1}$. Therefore, it follows

$$||x||_2 = \frac{1}{E} \max ||\mathcal{H}x||_2, \text{ subject to } \mathcal{H} : \text{ Hankel. } ||\mathcal{H}||_2 \leq E, \forall x \in \mathbb{R}^{n+1}. \quad (3.17)$$

Write

$$\mathcal{M} = \begin{bmatrix} 0 \\ \mathcal{M}_2 \end{bmatrix}, \quad \mathcal{M}_2 \in \mathbb{R}^{(n+1) \times s}.$$

Then,

$$\alpha = \max_{\mathcal{M}} \mu(x, \mathcal{M}) = \max_{\mathcal{M}} \mu(x, \mathcal{M})$$

$$= \frac{||x_1||}{||x||} \max_{||\mathcal{M}_2|| \leq E} \frac{||\mathcal{M}^T x_2||_2}{||x_2||_2}$$

$$= \frac{||x_2||}{||x||} E.$$

Hence.

**Theorem 3.2.2** Under the assumption explained as above, we have.

$$\alpha = \max_{\mathcal{M}} \mu(x, \mathcal{M}) = \frac{||x_2||_2}{||x||_2} E \quad (3.18)$$
Maximal smallest singular value

A roundabout way of considering the constant $\beta$ is to study the relation between $\alpha$ and $\gamma$, since we have the relation (3.15), $\alpha \geq \beta \geq \gamma$. Although we are able to prove the relation

$$\alpha = \gamma = \max_{u,y} \sigma_{\min}(\mathcal{M}(u,y)) \quad (3.19)$$

for matrices $\mathcal{M}$ without structure (the proof is provided in Appendix B), it seems not easy to prove it for the data matrices $\mathcal{M}$ with Hankel block structure. However, we will establish the following result:

**Theorem 3.2.3** Let $\alpha$, $\gamma$ be defined in (3.11) and (3.14). Then

$$\lim_{E \to 0} (\alpha - \gamma) = 0. \quad (3.20)$$

This theorem indicates that in small disturbance case the three constants are very close. In other words, if the upper bound of the perturbation is very small, we have $\alpha \approx \beta \approx \gamma$. This can be seen from Figure 3.2, in which the upper horizontal-level straight line is the constant $||x_2||/||x||E$. The middle curve represents the constant $\alpha$ and the lower curve represents the constant $\gamma$. As the experiment times increases, the two constants becomes very close. This means that the constant $\alpha$ is the best choice for the parameter $\varepsilon$ when generating a model family. Note that an variant of the result (3.18) is

$$E = \frac{||x||_2}{||x_2||_2} \max_{\mathcal{M}} \mu(x, \mathcal{M}). \quad (3.21)$$

This suggests a possible method to estimate the upper bound $E$.

**Bound estimation**

In this subsection we will discuss the problem of estimating the perturbation bound $\varepsilon$ based on the conclusions we have developed for the three constants. This is a problem of great importance in practice, since a perturbation bound with high fidelity is not always available. We will justify and exemplify a method of obtaining a bound directly.
Figure 3.2  Maximal misfit ($\alpha$), maximal smallest singular value ($\gamma$) and experiment times

from the experiments. It turns out that the geometric relationship between the model families, usually visualized by ellipsoids, play an important role in the our method.

As we have discussed, the constant $\alpha$ is the best choice for the parameter $\varepsilon$ when generating a model family, and the three constants are very close in small disturbance case. So, we can obtain an estimate of the constant $\alpha$ by discussing the constant $\gamma$. The following theorem gives a way to estimate $\gamma$ by examining whether there exist the disjoint ellipsoids.

**Theorem 3.2.4** In small disturbance case, there exist two data matrices such that the corresponding two model families can be characterized by two disjoint ellipsoids if and only if $\varepsilon < \gamma$.

This theorem implies that the constant $\gamma$ can be estimated by investigating the relative geometric relations between the ellipsoids. In the 2-dimension case, we can visualize the relations by plotting the ellipses. In Figure 3.3 we illustrate this in two
Figure 3.3  Disjoint ellipses exist when $\varepsilon < \gamma$ (right), and no disjoint ellipses exist when $\varepsilon > \gamma$ (left). The asterisk represents the nominal model.

typical cases: the left figure shows the case when no disjoint ellipses exist, i.e., $\varepsilon > \gamma$, the right figure depicts the contrary case when there exist two disjoint ellipses, i.e., $\varepsilon < \gamma$. In the high dimension case, ellipsoids can not be visualized, however, the feasibility of the LMI constraints representing the ellipsoids indicates the existence of disjoint ellipsoids. An alternative method of examining the ellipsoids is to compute the optimal inner ellipsoidal approximation of the intersections of the ellipsoids. It turns out that the existence of disjoint ellipsoids is equivalent to the existence of the optimal inner approximation. (This is shown in Figure 3.4). We remark that the feasibility problem of LMI and the optimal inner ellipsoidal approximation problem have been intensively discussed in [6, 39].

More specifically, let us consider model families generated by using sinusoid inputs with different frequencies, i.e. each input sequence has the following form

$$u_i = \cos(i\omega). \ i = 0, 1, \ldots, N.$$ 

where $\omega$ is the frequency for a single set of input. We can generate different sets of inputs by choosing different frequencies in the interval $[0, \pi]$. To illustrate the
constant \( \gamma \) which is of great importance to us. we plot the smallest singular values of the data matrices determined by the cosine inputs. As shown in Figure, the smallest singular values (solid curves in (a) and (b)) form a continuous curve with respect to the frequencies, and it is a concave function with a unique maximal value \( \gamma \), which is very close to the constant \( \alpha \) depicted as a horizon level straight line (dotted) in the same figure. In the case when the perturbation bound \( \varepsilon \) is smaller than \( \gamma \), yet bigger than the smallest value among the left branch of the concave curve (see (b)), there will exist two disjoint ellipsoids. since the horizon level straight line with value \( \varepsilon \) intersects the concave curve with two distinct points (see (b)), which means \( \varepsilon \) equals the smallest singular values of the two data matrices, and then two degenerate ellipsoids (points) are generated. By continuity, this implies the existence of two disjoint ellipsoids (see (c)). In the case when \( \varepsilon \) is larger than \( \gamma \), as shown in Figure 3.5, no ellipsoids exist and the intersection of the ellipsoids is a nonempty set (see (d)).

The parallel consideration is also made for exponential inputs

\[ u_i = \lambda^i, \quad i = 0, 1, \ldots, N. \]

where \( \lambda \in [-1, 1] \) can be varied to generate different sets of exponential inputs.
Figure 3.5 Sinusoid inputs and disjoint ellipsoids. (a) Smallest singular value of $\mathcal{V}_1$ (dashed), constant $\alpha$ (dotted), perturbation bound $\varepsilon$ (dashdot) and smallest singular value of $\mathcal{V}$ as functions of frequency $[0, \pi]$. (b) A close look of (a). (c) Disjoint ellipses exist when $\varepsilon < \gamma$. (d) Disjoint ellipses do not exist when $\varepsilon > \gamma$, and the intersection is non-empty. In (c) and (d) the asterisk represents the nominal model.

Similar observations and analysis can be made, and the results are shown in Figures 3.6. We finally remark that through the experiments we found that exponential inputs is advantageous compared to other inputs, since the phenomenon that disjoint ellipsoids appear is much more easily observed when using exponential inputs.
Figure 3.6  Exponential inputs and disjoint ellipsoids. (a) Smallest singular value of $\hat{\mathcal{M}}_1$ (dashed). constant $\alpha$ (dotted). perturbation bound $\varepsilon$ (dash-dot) and smallest singular value of $\mathcal{M}$ as functions of base of the exponential ($[-1,1]$). (b) A close look of (a). (c) Disjoint ellipses exist when $\varepsilon < \gamma$. (d) Disjoint ellipses do not exist when $\varepsilon > \gamma$, and the intersection is non-empty. In (c) and (d) the asterisk represents the nominal model.

3.3 Data matrix decomposition

3.3.1 Motivation and formulation

Given the observed input-output data $(\hat{u}_i, \hat{y}_i)$, $i = 0, 1, \ldots, N$, the (raw) data matrix $\hat{\mathcal{M}}$ is defined as the concatenation of the input matrix $\hat{U}$ and the output matrix $\hat{Y}$.

$$\hat{U} = (\hat{u}_{i,j})_{i,j=0}^{m,s-1} \in \mathbb{R}^{(m+1) \times s}, \quad \hat{Y} = (\hat{y}_{i,j})_{i,j=0}^{n,s-1} \in \mathbb{R}^{(n+1) \times s}.$$
which are Hankel matrices constructed respectively by the input sequence and the output sequence. More specifically,

\[ \hat{\mathcal{M}} = \begin{bmatrix} \hat{U} \\ \hat{Y} \end{bmatrix} \in \mathbb{R}^{r \times s}. \]

where \( r = m + n + 2, s = N - \max(m, n) + 1 \). As observed in Chapter 2, if the data is exactly generated by an LTI model, then the true model must be in the left kernel of the data matrix \( \hat{\mathcal{M}} \). This motivates us to seek the decomposition

\[ \hat{\mathcal{M}} = \mathcal{M} + \hat{\mathcal{M}}, \quad (3.22) \]

where \( \mathcal{M} \) and \( \hat{\mathcal{M}} \) have the same Hankel block structure as \( \hat{\mathcal{M}} \) and \( \mathcal{M} \) has a non-trivial left kernel, and \( \hat{\mathcal{M}} \) is consistent with the specification of the additive perturbation within the data measurements. Since perturbation is assumed only in the output, the matrix \( \mathcal{M} \) in decomposition (3.22) should have the following structure

\[ \mathcal{M} = \begin{bmatrix} U' \\ Y \end{bmatrix}, \quad U' \in \mathbb{R}^{(m+1) \times s}, \quad Y \in \mathbb{R}^{(n+1) \times s}, \quad (3.23) \]

with three properties:

1. \( U' = \hat{U} \).
2. \( \mathcal{M} \) has a non-trivial left kernel; and
3. \( U' \) and \( Y \) are Hankel matrices:

Clearly, the perturbation matrix \( \hat{\mathcal{M}} \) in decomposition (3.22) should have a special structure

\[ \hat{\mathcal{M}} = \begin{bmatrix} 0 \\ \hat{Y} \end{bmatrix}, \quad \hat{Y} \in \mathbb{R}^{(n+1) \times s}. \]

where \( \hat{Y} \) is a Hankel matrix. Besides, \( \hat{\mathcal{M}} \) should be consistent with the characteristics assumed on the perturbation beforehand. In our setting, the norm of \( \hat{\mathcal{M}} \) should not be greater than a perturbation bound \( \varepsilon \), provided the bound is known. Unfortunately,
in many situations the perturbation bound \( \varepsilon \) is unknown. It is therefore desired to get an estimate of the bound based on all knowledges we have on the data and the nominal model. One possible way of doing this is to seek an optimal decomposition (3.22), where the optimality means the norm of the perturbation \( \mathcal{M} \) is minimized. This analysis suggests an optimization problem formulated as follows:

\[
\text{minimize} \quad ||\mathcal{M} - \mathcal{M}|| = ||\hat{Y} - Y|| = ||\hat{Y}||
\]

(3.24)

subject to \quad \text{Properties of } \mathcal{M} \colon \ (1), \ (2), \ (3).

This kind of matrix approximation problem is generally hard to solve due to the complexity of the constraints. namely. the optimizer has to satisfy several properties simultaneously. However. a “divide-and-conquer” scheme has been proposed [8]. which solves the problem sequentially and iteratively. In other words. we seek an object which sequentially possesses each of the individual properties. and use iterations to find an object simultaneously satisfying the properties. Obviously. the approximation obtained in this way is just a suboptimal solution.

In this section we apply the so-called composite property mapping algorithm to the above approximation problem. and an iterative algorithm of obtaining a sub-optimal decomposition (3.22) is proposed and its convergence is examined.

3.3.2 Property mappings

Let us consider the metric space \((X, d)\). where \(X\) is the space of matrices and the metric \(d\) is Frobenius norm of matrices. Let \(P\) be a property of the elements of \(X\). and the corresponding property set \(S\) be specified by

\[ S = \{x \in X : x \text{ possesses property } P\}. \]  

(3.25)

Let us consider the task of finding those elements in the set \(S\) which lie closest to an arbitrary element \(x \in X\). A solution to this problem entails finding elements contained in \(S\) which solve the optimization problem

\[ \inf_{y \in S} ||x - y||_F \]
The set of elements contained in property set $S$ which solve this optimization problem is called the \emph{solution set} and is denoted by the symbol $F(x)$. It is convenient to interpret $F(\bullet)$ as a mapping (an operator) that maps $x$ into the solution set $F(x)$, that is, $F : x \rightarrow F(x)$. This mapping is called a \emph{property mapping}. [8] It could be a point-to-set mapping.

A straightforward example of the property mapping is the input-fixed mapping, which maps a matrix $\mathcal{M}_* = [\mathcal{U}_*^T \ Y_*^T]^T$ to a matrix $\mathcal{M}_# = [\mathcal{U}_#^T \ Y_#^T]^T$. The matrix $\mathcal{M}_#$ is required to satisfy property (1) and have the minimal distance from $\mathcal{M}_*$. Obviously, the property set is
\begin{equation}
S_i = \{ \mathcal{M}_# \in \mathbb{R}^{r \times s} : \text{first } m + 1 \text{ rows of } \mathcal{M}_# \text{ is } \hat{\mathcal{U}} \},
\end{equation}
and $\mathcal{U}_# = \hat{\mathcal{U}}$ and $Y_# = Y_*$. Therefore, the input-fixed mapping is defined as a mapping $F_i$ from $\mathbb{R}^{r \times s}$ to $S_i$ by
\begin{equation}
\mathcal{M}_* = \begin{bmatrix} \mathcal{U}_* \\ Y_* \end{bmatrix} \rightarrow \mathcal{M}_# = F_i(\mathcal{M}_*) = \begin{bmatrix} \hat{\mathcal{U}} \\ Y_* \end{bmatrix}.
\end{equation}

In what follows we will discuss matrix rank deficiency mapping and Hankel mapping, which respectively correspond to the properties (2) and (3) that an optimal $\mathcal{M}$ should possess.

**Matrix rank deficiency mapping**

It is required that the matrix $\mathcal{M} \in \mathbb{R}^{r \times s}$ in decomposition (3.22) has to have a nontrivial left kernel, in other words, its rank is at most $r - 1$. The property set determined by this property is
\begin{equation}
S_r = \{ \mathcal{M} \in \mathbb{R}^{r \times s} : \text{rank} \mathcal{M} \leq r - 1 \}.
\end{equation}
In order to define the property mapping $F_r$ deduced from the property, we need to consider the optimization problem (3.25), i.e.,
\begin{equation}
\begin{align*}
\text{minimize} & \quad \|\hat{\mathcal{M}} - \mathcal{M}\| \\
\text{subject to} & \quad \text{rank} \mathcal{M} \leq r - 1
\end{align*}
\end{equation}
The well-known Schmidt-Mirsky theorem solves this problem.

**Theorem 3.3.1** Let the matrix $\hat{\mathcal{M}}$ have the following singular value decomposition:

$$
\hat{\mathcal{M}} = U \begin{bmatrix}
\hat{\sigma}_1 & 0 \\
\hat{\sigma}_2 & 0 \\
\vdots & \\
\hat{\sigma}_r & 0
\end{bmatrix} V^T
$$

(3.30)

where $\hat{\sigma}_i$'s are the singular values of $\hat{\mathcal{M}}$, and assume $\hat{\sigma}_i > \hat{\sigma}_{i+1}$. Then the unique solution $\mathcal{M}_*$ to optimization problem (3.29) is obtained by truncating the SVD (3.30) to contain the first $r-1$ singular values:

$$
\mathcal{M}_* = U \begin{bmatrix}
\hat{\sigma}_1 & 0 \\
\vdots & \\
\hat{\sigma}_{r-1} & 0 \\
0 & 0
\end{bmatrix} V^T
$$

(3.31)

According to this theorem, for every $\hat{\mathcal{M}} \in \mathbb{F}^{r \times s}$ there exists a unique $\mathcal{M}_*$ solving the optimization problem (3.29). This introduces a mapping $F_r$ from $\mathbb{F}^{r \times s}$ to $S_r$ by

$$
\hat{\mathcal{M}} \rightarrow \mathcal{M}_* = F_r(\hat{\mathcal{M}}).
$$

(3.32)

$F_r$ is called a *matrix rank deficiency mapping*.

**Hankel mapping**

The Hankel mapping is related to the problem of approximating an arbitrary matrix $X$ by a Hankel matrix of the same size in the sense that the Frobenius norm of the difference is minimized, i.e.,

$$
\text{minimize} \quad ||X - H||
$$

subject to $H$ is Hankel

(3.33)

As introduced in Appendix A, the solution to this problem relies on the following theorem.
**Theorem 3.3.2** The optimal Hankel matrix approximant \( H_* = [h_{i+j}] \) of an arbitrary matrix \( X = [x_{i,j}] \) in the Frobenius norm sense is obtained by averaging the elements on the same anti-diagonal of \( X \), i.e.

\[
h_{i+j} = \frac{x_{1,i+j-1} + x_{2,i+j-2} + \cdots + x_{i+j-1,1}}{i + j - 1}.
\]

According to this theorem, for every \( X \) there exists a unique Hankel matrix \( H_* \) solving the optimization problem (3.33). This introduces a mapping \( F_h \) by

\[
X \rightarrow H_* = F_h(X).
\]

(3.34)

\( F_h \) is called a **Hankel mapping**.

Note that the solution \( \mathcal{M} \) to the problem (3.24) is a concatenation of two Hankel matrices, so the operator that defines an arbitrary matrix of size \( r \times s \) to be an \( \mathcal{M} \)-like matrix can be regarded as a "concatenation" of two Hankel mapping. More specifically, the modified Hankel mapping, still denoted by \( F_h \), is defined as follows:

\[
X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \rightarrow \begin{bmatrix} F_h(X_1) \\ F_h(X_2) \end{bmatrix}.
\]

(3.35)

**3.3.3 A composite property mapping algorithm**

By introducing the input-fixed mapping (3.27), the matrix rank deficiency mapping (3.32) and the Hankel mapping (3.35) we know that the solution to the problem (3.24) must be in the intersection of the corresponding property sets \( S_\mathcal{I} \), \( S_r \), and \( S_h \), where \( S_\mathcal{I} \), \( S_r \) are defined in (3.26) and (3.28) and \( S_h \) is the set of all matrices of the form (3.23). As introduced before, a "divide and conquer" scheme leads to the introduction of a **composite property mapping** \( F \):

\[
F = F_h \circ F_r \circ F_\mathcal{I}.
\]

Clearly, the image of the operator \( F \) at \( \mathcal{M} \) is not necessarily the solution to the optimization problem (3.24). However, its fixed point\(^\dagger\) must satisfy properties (1).

\(^\dagger\) The matrix \( A \) is a **fixed point** of the mapping \( F \) if \( A = F(A) \).
(2) and (3). This observation leads us to considering an iterative algorithm:

\[ \mathcal{M}_{k+1} = F(\mathcal{M}_k), \quad k \geq 0. \quad (3.36) \]

where we choose the initial matrix \( \mathcal{M}_0 = \mathcal{N} \).

For consideration of finding a sub-optimal solution to (3.24) the only thing left is the convergence of the iteration (3.36). It is well known that the iteration is convergent if and only if the operator \( F \) is strictly contractive. We have established the strict contractivity of \( F \) in Appendix C. Therefore, the iteration (3.36) converges. and then a suboptimal solution to the problem (3.24) will be found. The algorithm is summarized as follows:

Choose an error tolerance bound \( \gamma > 0 \); initial \( \mathcal{M}_0 = \mathcal{N} \); and \( i = 0 \);

\[ \text{do}\]

(1) \( \mathcal{M}_* = F_*(\mathcal{M}_i) \), i.e., truncate the SVD (3.30) of \( \mathcal{M}_i \) to get \( \mathcal{M}_* \) as in (3.31);

(2) \( \mathcal{M}_{\#} = F_{\#}(\mathcal{M}_*) \), i.e., get \( \mathcal{M}_{\#} \) by replacing the first \( m + 1 \) rows of \( \mathcal{M}_* \) by the original input matrix \( \mathcal{L} \);

(3) \( \mathcal{M}_{i+1} = F_h(\mathcal{M}_{\#}) \), i.e., make the last \( n + 1 \) rows of \( \mathcal{M}_{\#} \) be a Hankel matrix by averaging elements along the same anti-diagonal;

(4) increment \( i \) by 1;

\[ \text{while} \ (\| \mathcal{M}_{i+1} - M_i \| \geq \gamma) \]

\[ \text{do}\]

3.4 Conclusions

In this chapter we studies several discrete topics which are closely related to the model family and hopefully helpful for a good understanding of the model family. The convexity of the uncertain model family holds only in special case. This inspires us to focus on the the case when the model family is a convex cone for behavioral models, or the case when the model family is an ellipsoid for classical models. Three measures
of the family size are introduced, which are closely related to the optimal input design problem, and we will say in next chapter, they are also good metrics for evaluating the robust performance. Although the denseness problem is still open, the experiments we designed strongly support our conjectures. In the case where the data is generated by an LTI system, the order of the system is determined by the rank of the data matrix, or by investigating the pole-zero cancelation of the nominal model. By studying the geometric relationship between the uncertain model families (ellipsoids) generated with different inputs, we can estimate the perturbation bound by "observing" the existence of the disjoint ellipsoids. In fact, we can discern the existence of the disjoint ellipsoids by computing the optimal inner ellipsoidal approximation of the intersection of the ellipsoids. We finally develop an iterative algorithm to compute a suboptimal decomposition using composite property mapping. The "divide-and-conquer" scheme makes the optimization problem solvable, however, the algorithm can not recover the true LTI system in general.
Chapter 4

Robust Synthesis Problem

Recall under some circumstance the uncertain models are parameterized with a real uncertain parameter of at most unit length. The uncertain real parameters linearly appear in the numerator and the denominator of transfer functions of the uncertain models. In this chapter we consider the robust synthesis problem, based on this kind of parameterization (Section 4.1). By using a convex parameterization of all controllers that simultaneously stabilize the uncertain models derived in [45] (Section 4.2), a robust performance problem can be stated in terms of a quasi-convex optimization, which can be solved using Ritz approximation method (Section 4.3). Several experimental observations concerning the relation between maximal stability margin and order of Youla parameter are reported (Section 4.3). The convergence of the algorithm is reflected from three aspects: convergence of the maximal stability margin, convergence of the gain of the Youla parameter and change of the poles and zeros of the Youla parameter. We also consider the relation between the maximal stability margin and various measures of the uncertain model family (Section 4.4). In Section 4.5 we discuss the $H_\infty$ synthesis problem: two methods are introduced, but the second method, taking advantage of the coprime factorizations of the uncertain transfer functions, is analyzed in detail. The last problem we consider is the robust synthesis problem for general model family. We formulate the problem in two different methods, one is similar to the problem we have already discussed, and the other is the typical $\mu$-synthesis method.

4.1 Ellipsoid and linear fractional parameterization

Recall that under certain condition the family of uncertain models compatible with the measured data records (2.1) is characterized by an ellipsoid (for classical models).
As presented in Chapter 2, the model family is described as

\[ \mathcal{F}_c = \left\{ x = \begin{bmatrix} \theta \\ 1 \end{bmatrix} \in \mathbb{R}^r : \theta \in \mathcal{E} \right\} \text{ with } \mathcal{E} = \left\{ \theta \in \mathbb{R}^{r-1} : \theta^T A \theta + 2b^T \theta + c \leq 0 \right\} \]

where \( A, b, c, \mathcal{U}_1 \) are defined in (2.23), (2.21). Denote the singular values of \( \mathcal{U}_1 \) by \( \sigma_i (\sigma_i < \sigma_{i+1}) \), and the corresponding left singular vectors by \( u_i, i = 1, \ldots, r - 1 \). Then every model \( x = [\theta^T \ 1]^T \in \mathcal{F}_c \) can be parameterized in terms of

\[ \theta = \theta_c + \sum_{i=1}^{r-1} \alpha_i \delta_i u_i. \]

where \( \sum_{i=1}^{r-1} \delta_i^2 \leq 1 \), and \( \theta_c = -A^{-1}b \) is the center of the ellipsoid, and the constants

\[ \alpha_i = \sqrt{\frac{b^T A^{-1} b - c}{\sigma_i^2 - \delta_i^2}}, \quad i = 1, \ldots, r - 1 \]

are the semiaxis lengths of the ellipsoid.

To characterize the system parameters using an ellipsoid is not a rare case in set membership identification theory, where the aim is to characterize the membership set of the parameter space consistent with the data, the model structure and the error bounds. For the purpose of computational simplicity, one may approximate the membership set by some simple-shaped set such as ellipsoid. Using similar notations, we assume the system parameters are characterized in terms of an ellipsoid \( \mathcal{E} \).

\[ \mathcal{E} = \left\{ \theta \in \mathbb{R}^{r-1} : \theta = \theta_c + P u, \quad \| u \|_2 \leq 1 \right\}. \]

where \( \theta_c \) is its the center. Assume the positive definite matrix \( P \) has the following SVD

\[ P = U \begin{bmatrix} \alpha_{r-1} & \cdots & \\ & \ddots & \\ & & \alpha_1 \end{bmatrix} U^T. \]

Write \( U' = [u_{r-1}, \ldots, u_1] \), then every element of \( \mathcal{E} \) has an affine expression in terms of
the singular values and singular vectors of \( P \):

\[
\theta = \theta_c + Pu = \theta_c + U \begin{bmatrix}
\alpha_{r-1} \\
\vdots \\
\alpha_1
\end{bmatrix} \delta
\]

\[
= \theta_c + \sum_{i=1}^{r-1} \alpha_i \delta_i u_i.
\]

where \( \delta = \ell^T u = [\delta_{r-1} \cdots \delta_1]^T \) has length no more than 1. Thus.

\[
x = \begin{bmatrix}
\theta \\
1
\end{bmatrix} = \begin{bmatrix}
\theta_c + \sum_{i=1}^{r-1} \alpha_i \delta_i u_i \\
1
\end{bmatrix} = \begin{bmatrix}
\theta_c \\
1
\end{bmatrix} + \sum_{i=1}^{r-1} \alpha_i \delta_i \begin{bmatrix}
u_i \\
0
\end{bmatrix}.
\]

Define the polynomials

\[
\begin{bmatrix}
-n_i(z) \\
d_i(z)
\end{bmatrix} := \begin{bmatrix}
1 & z & \cdots & z^n & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & z & \cdots & z^n
\end{bmatrix} \begin{bmatrix}
u_i \\
0
\end{bmatrix},
\]

\[
\begin{bmatrix}
-n_c(z) \\
d_c(z)
\end{bmatrix} := \begin{bmatrix}
1 & z & \cdots & z^n & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & z & \cdots & z^n
\end{bmatrix} \begin{bmatrix}
\theta_c \\
1
\end{bmatrix}.
\]

Then the transfer functions of the uncertain models can be parameterized as follows:

\[
G_\delta(z) = \frac{n_c(z) + \sum_{i=1}^{r-1} \alpha_i \delta_i n_i(z)}{d_c(z) + \sum_{i=1}^{r-1} \alpha_i \delta_i d_i(z)}. \tag{4.3}
\]

We remark that the uncertain real parameters \( \delta_i \)'s linearly appear in the numerator and the denominator of transfer function (4.3). This is called the linear fractional parameterization of the uncertain models. As shown above, once uncertain parameters can be described in terms of an ellipsoid, uncertain models have linear fractional parameterizations. Based on this characteristic, a convex parameterization of all controllers that simultaneously stabilize the uncertain model (4.3) for all \( \delta \in \mathbb{R}^{r-1} \) with \( ||\delta|| \leq 1 \) has been derived [45, 1]. The advantage of this convex parameterization is that a robust performance problem can be stated in terms of a quasi-convex optimization. By the way, this indicates another advantage of characterizing system parameters using ellipsoid in set membership identification theory.
4.2 Controller parameterization

As discussed in last section, uncertain models with linear fractional parameterizations can be expressed with the following transfer function:

\[
G_{\delta}(z) = \frac{\hat{N}(z) + \delta^T \hat{N}_{\delta}(z)}{\hat{M}(z) + \delta^T \hat{M}_{\delta}(z)}
\]  \hspace{1cm} (4.4)

where $\hat{M} \in \mathbb{R}$, $\hat{N} \in \mathbb{R}$, $\hat{M}_{\delta} \in \mathbb{R}^{r-1}$, $\hat{N}_{\delta} \in \mathbb{R}^{r-1}$ are all fixed known stable transfer function matrices, and $\delta \in \mathbb{R}^{r-1}$ with $||\delta|| \leq 1$ is a real vector representing uncertainty. In this section we will introduce a convex parameterization of all controllers that simultaneously stabilize the uncertain model (4.4) for all $\delta \in \mathbb{R}^{r-1}$ with $||\delta|| \leq 1$. Compared to the parameterization in the case of general dynamic uncertainty, this parameterization bears the advantage that a robust performance problem can be formulated as a quasi-convex optimization.

**Figure 4.1** Block diagram of the uncertain plants, where $\delta$ is a constant parameter vector, $||\delta|| \leq 1$.

The uncertain plants with linear fraction transfer function (4.4) can be represented in the diagram shown in figure 4.1, where the transfer function

\[
G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \in \mathbb{R}^{r \times 2}.
\]
with \( G_{11} \in \mathbb{R}^{r-1} \), \( G_{12} \in \mathbb{R}^{r-1} \), \( G_{21} \in \mathbb{R} \), \( G_{22} \in \mathbb{R} \) can be determined explicitly. In Appendix C an expression for \( G \) is derived

\[
G = \begin{bmatrix}
-\frac{\bar{N}_\delta}{\bar{M}} & \frac{\bar{N}_\delta \bar{M}_\delta - \bar{N}_\delta \bar{Y}}{M^2} \\
\frac{\bar{M}_\delta}{M} & \bar{Y}_\delta \\
1 & \bar{Y}_\delta 
\end{bmatrix}
\]  

(4.5)

In the sequel we seek to design robust controllers for this configuration. The uniqueness of the diagram for uncertain models is that the uncertainty results from a real parameter vector. Compared to the general case where dynamic uncertainty presents, this is much restrictive and much more specific parameterization of the controllers is thus expected. In what follows we will presents the results for general case and the restrictive case.

Assume \( \bar{M} \) and \( \bar{N} \) to be coprime, and define the square matrix

\[
W = \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = \begin{bmatrix} \bar{X} & -\bar{Y} \\ -\bar{N} & \bar{M} \end{bmatrix}^{-1}
\]  

(4.6)

for suitable choice of stable matrix functions \( M \), \( N \), \( X \), \( \bar{X} \), and \( \bar{Y} \). Regardless of the realness nature of the uncertainty in the diagram, we have the standard parameterization of all controllers that stabilize the uncertain model for all \( \delta \) with \( ||\delta||_\infty \leq 1 \).

**Theorem 4.2.1** The rational controller \( K \) stabilizes \( G_\delta \) in (4.4) for general uncertainty \( \delta \) with \( ||\delta||_\infty \leq 1 \) if and only if it can be written in the form

\[
K = (Y + MQ)(X + NQ)^{-1}
\]  

(4.7)

where \( Q \in \mathbb{R} \) is a stable function and

\[
\left\| \begin{bmatrix} -\bar{N}_\delta & \bar{M}_\delta \end{bmatrix} W \begin{bmatrix} Q \\ 1 \end{bmatrix} \right\|_\infty \leq 1.
\]  

(4.8)

The controllers are parameterized in terms of the stable function \( Q \), which is usually called the Youla parameter. In the case where the uncertainty stems from the real parameter vector, as shown in Figure 4.2, we have the following parallel results, in which the Youla parameter has a special form: \( Q = 3/\alpha \).
Lemma 4.2.1 [45] The rational controller $K$ stabilizes $G_\delta$ in (4.4) for $||\delta|| \leq 1$ if and only if it can be written in the form

$$K = (Y\alpha + M\beta)(X\alpha + N\beta)^{-1}$$  \hspace{1cm} (4.9)

where $\alpha, \beta$ are stable scalar functions and

$$\left\| \text{Re} \left( \begin{bmatrix} -\dot{\mathcal{N}} & \dot{\mathcal{M}} \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} e^{i\omega} \right) \right\| \leq \text{Re} \alpha(e^{i\omega}). \ \forall \omega \in [0,2\pi]$$  \hspace{1cm} (4.10)

Compare the two conditions (4.8) and (4.10). the former is given in terms of an $H_\infty$ norm which is in general intractable, while the latter can be easily converted into a convex constraint. the advantage of which will be reflected in a coming robust performance problem.

For future use we establish the following result:

Theorem 4.2.2 The parameterization (4.9) of the controller $K$ is independent of the coprime factorization of the nominal transfer function $\tilde{M}^{-1}\tilde{N}$ and the choice of the matrix $W$ in (4.6).
4.3 Robust performance problem

Recall rational controllers that stabilize the uncertain model $G_\delta$ in (4.4) are parameterized in terms of (4.9) with the constraint (4.10), i.e.,

$$\left\| \operatorname{Re} \left( \left[ \begin{array}{c} \tilde{N}_\delta \\ \tilde{M}_\delta \end{array} \right] W \left( \begin{array}{c} J \\ \alpha \end{array} \right) \right) e^{i\omega} \right\| \leq \operatorname{Re} \alpha e^{i\omega}. \quad \forall \omega \in [0, 2\pi]$$

To seek the optimal controllers for the uncertain system, it is desired to consider the so-called robust performance problem. In other words, we are interested in finding controllers that achieve the best robust performance. Considered the constraint (4.10) posed on all controllers, a natural measure of the robust performance of a controller $K$ determined by the Youla parameter $Q = J/\alpha$ is thus defined in terms of (4.10):

$$\lambda(\alpha, J) := \inf_{\omega} \frac{\operatorname{Re} \alpha e^{i\omega}}{||C(\omega)||}.$$  \quad (4.11)

where $\alpha(z)$, $J(z)$ are two stable functions, and

$$C(\omega) = \operatorname{Re} \left( \left[ \begin{array}{c} \tilde{N}_\delta \\ \tilde{M}_\delta \end{array} \right] W \left( \begin{array}{c} J \\ \alpha \end{array} \right) \right) e^{i\omega}. \quad (4.12)$$

$\lambda(\alpha, J)$ is also called the stability robustness, or stability margin corresponding to the controller $K$. Since the condition (4.10) is, as claimed in Theorem 4.2.2, independent of the coprime factorization of the nominal transfer function $\tilde{M}^{-1}\tilde{N}$ and the choice of the matrix $W$ in (4.6), the measure $\lambda(\alpha, J)$ is well posed. In addition, the quantity

$$\lambda_* := \max_{\alpha, J} \lambda(\alpha, J)$$  \quad (4.13)

is referred to as the maximal stability margin (robustness), and the corresponding function $Q = J/\alpha$ is called the optimal Youla parameter. With the optimal Youla parameter, the optimal controller can be constructed by means of (4.9).

In this section we will consider the problem of computing the maximal stability margin and the optimal Youla parameter. In other words, we will study the problem of maximizing the stability margin $\lambda(\alpha, J)$ over all stable functions $\alpha$ and $J$, i.e., the
problem (4.13). It turns out that the optimization problem (4.13) can be converted into a quasi-convex optimization problem which can be solved by checking the feasibility of a set of Linear Matrix Inequalities (LMI). Another formulation is presented in [44]. Throughout this section functions $\alpha$, $\beta$ are always assumed to be stable, and $\omega$ is assumed in the interval $[0,2\pi]$.

### 4.3.1 General formulation

It is readily to notice that maximizing the stability margin amounts to minimizing the following quantity

$$\phi(\alpha, \beta) = \sup_{\omega} \frac{||C(\omega)||}{\Re \alpha(\epsilon^\omega)},$$

(4.14)

Define

$$\phi_* = \min_{\alpha, \beta} \phi(\alpha, \beta).$$

(4.15)

Then $\lambda = \phi_*^{-1}$. For this reason we will focus on the approach of the optimization problem (4.15) in the remaining part of this section.

**Lemma 4.3.1** Let $D$ be a set and $f(x)$ be real function defined on $D$. Define

$$\Omega = \{(x,t): t \geq f(x), x \in D\}.$$  

Then the two optimization problems

$$\alpha = \min_{x \in D} f(x)$$

$$\beta = \min_{x \in \Omega} t.$$  

are equivalent in the sense that

1. $x_0 = \arg \min_{x \in D} f(x) \iff (x_0, f(x_0)) = \arg \min_{x \in \Omega} t$: and

2. $\alpha = \beta = f(x_0)$. 


According to this statement, the optimization problem in question can be converted into the following equivalent forms

$$\varphi = \min_{\alpha, \beta} \varphi(\alpha, \beta) = \min_{\alpha, \beta, t} t \quad \text{subject to} \quad \varphi(\alpha, \beta) \leq t$$

$$= \min_{\alpha, \beta, t} t \quad \text{subject to} \quad \|C(\omega)\| \leq t \Re \alpha(e^{i\omega}) \quad (4.16)$$

According to the result introduced in Appendix A, the constraint (4.16) can be further written in a matrix form

$$F = \begin{bmatrix} t \Re \alpha(e^{i\omega}) & C(\omega) \\ C(\omega)^T & t \Re \alpha(e^{i\omega}) \end{bmatrix} \geq 0. \quad (4.17)$$

i.e., the matrix $F$ is positive semidefinite. Therefore, the optimization problem of interest can be equivalently described as

$$\varphi = \min_{\alpha, \beta, t} t \quad \text{subject to} \quad F \geq 0. \quad (4.18)$$

This optimization problem is actually a quasi-convex optimization problem (recall Appendix A). Namely, for every given $t > 0$ the projection set

$$\mathcal{F} = \{(\alpha, \beta, \omega): F \geq 0\}$$

of the corresponding feasible set $\{(\alpha, \beta, \omega, t): F \geq 0\}$ is convex. Thus, for every given $t$, whether it is feasible can be determined by checking the feasibility of the constraint (4.17). This can be solved by formulating a convex optimization problem. Consequently, the original optimization problem can be solved by using bisection in the positive number $t$ and checking the feasibility of the corresponding convex optimization problem. In this way, the problem can be approximately tackled by searching the minimal feasible number $t$.

One of the reasons that make the problem of checking the feasibility of (4.17) is hard is that in the function $F$ the real variable $\omega$ and the functional variables $\alpha$ and $\beta$ appear in a non-linear way. What is more, the feasible set of the variables in the constraint (4.17) is in an infinite-dimensional space, i.e., the problem is an infinite-dimensional optimization problem. It turns out that Ritz approximation method
for solving infinite-dimensional optimization problems can be employed to overcome these difficulties. The key idea of this method is to approximately solve the problem by solving a sequence of finite-dimensional optimization problems, where the choice of the basis functions for the finite dimension approximate subspace is one of the important issues. In addition, to reduce the difficulty stemming from the variable $\omega$, we can discretize it over $[0, 2\pi]$. In other words, we use a large number of discrete points in $[0, 2\pi]$ to represent the continuous variable $\omega$. The details will be presented in last subsection.

4.3.2 Ritz method approach

With a chosen basis for the infinite-dimensional space, the whole space can be approximated by subspaces spanned by some finite number, say $K$, of elements in the basis. The constraint (4.17) can, therefore, be converted into a Linear Matrix Inequality (LMI) of some discrete variables and a continuous variable $\omega \in [0, 2\pi]$. By discretizing the variable $\omega$ over $[0, 2\pi]$ by an integer $N$, we can finally get $N$ LMIs. Therefore, the so-called Positive Definite Programming (PDP) methods can be used to solve the simplified optimization problems. To improve the accuracy of the solution, we can either solve the problem in subspaces of larger dimension $K$ for a fixed discretization number $N$, or solve the problem in the fixed finite-dimensional subspaces for a larger discretization number $N$.

The Ritz method consists of solving the problem over larger and larger finite-dimensional subspaces, so that the obtained suboptimal values more and more accurately approximate the global optimal value. Suppose $\{f_k, k = 0, 1, \ldots\}$ is a chosen basis for the infinite-dimensional space, where $f_k(z), k = 0, 1, \ldots$, are stable functions. For some chosen positive number $K$, our search for the optimal solution is restricted in the subspace spanned by the first $K + 1$ basis functions. Here, the Ritz method means that we may minimize $\phi(\alpha, \beta)$ over subspaces spanned by the
first $K + 1$ basis functions, i.e., the functions $\alpha$, $\beta$ in (4.12) and (4.17) have the form:

$$\alpha = \sum_{k=0}^{K} \alpha_k f_k, \quad \beta = \sum_{k=0}^{K} \beta_k f_k, \quad \alpha_k, \beta_k \in \mathbb{R}. \quad (4.19)$$

In this way, the functional variables $\alpha$, $\beta$ are replaced by real (discrete) variables $\alpha_k$, $\beta_k$, $k = 0, \ldots, K$. Furthermore, the new variables linearly appear in the constraint.

The optimization problem (4.18) with functional variables $\alpha$, $\beta$ being replaced by (4.19) can be converted into a convex optimization problem with LMI constraints in two steps. The first step is to express the matrix $F$ into the form

$$F = \sum_{k=0}^{K} F_{\alpha k} \alpha_k + \sum_{k=0}^{K} F_{\beta k} \beta_k. \quad (4.20)$$

where $F_{\alpha k}$, $F_{\beta k}$ are symmetric matrix-valued functions of continuous variable $\omega$ over the interval $[0, 2\pi]$. This has been done in Appendix C. The second step is to discretize the variable $\omega$ so that the new constraint

$$F = \sum_{k=0}^{K} F_{\alpha k} \alpha_k - \sum_{k=0}^{K} F_{\beta k} \beta_k \geq 0 \quad (4.21)$$

can be approximately replaced by its discrete forms where the values of $\omega$ are taken at different points in $[0, 2\pi]$. A usual way is to uniformly discretize the variable $\omega$ with some chosen integer $N$, i.e., constraint (4.21) can be replaced by $N$ LMI constraint with $\omega$ being taken at

$$\omega_l = 2l\pi/N, \quad l = 0, 1, \ldots, N - 1. \quad (4.22)$$

More specifically, the optimization problem (4.18) becomes

$$\phi = \min_{\alpha, \beta, \omega, t} t \quad (4.23)$$

subject to

$$F(\omega_l) = \sum_{k=0}^{K} F_{\alpha k}(\omega_l) \alpha_k + \sum_{k=0}^{K} F_{\beta k}(\omega_l) \beta_k \geq 0,$$

$$l = 0, 1, \ldots, N - 1.$$

This is a typical Positive Definite Programming (PDP) problem [6, 39, 59] and thus can be solved in very efficient convex programming methods.
We remark that if the functions $\tilde{N}_\delta$, $\tilde{M}_\delta$ and $W$ can be further linearly expressed by the chosen basis functions, we have a more specific formula for the above simplified constraint. This will be elaborated in next section.

4.3.3 Basis selection

For the Ritz approximation in our problem where functions are stable, we can choose the following typical infinite basis (stable function sequence)

$$ f_0(z) = 1, \quad f_1(z) = z^{-1}, \quad \ldots, \quad f_N(z) = z^{-K}, \quad \ldots. $$

Therefore, (4.19) becomes

$$ \alpha(z) = \sum_{k=0}^{K} \alpha_k z^{-k}, \quad \lambda(z) = \sum_{k=0}^{K} \lambda_k z^{-k}, \quad \alpha_k, \lambda_k \in \mathbb{R}. \quad (4.24) $$

Since $\tilde{N}_\delta$, $\tilde{M}_\delta$ and $W$ can all be linearly expressed in terms of finite elements of the basis $f_k(z)$, we will be able to derive more specific formulas for the matrices $F_{\alpha k}, F_{\lambda k}$ in the constraint (4.21). Please refer to Appendix C for details.

Note that the order of the Youla parameter $Q(z) = \lambda(z)/\alpha(z)$ is $K$, identical to the dimension of the subspace in which we are searching for the suboptimal solution. Thus, in order to get more accurate approximation of the maximal stability margin by solving the optimization problem in a subspace with a larger dimension, we have to increase the order of the Youla parameter, which means we will have to have an optimal controller with high order (recall controller can be construct by the Youla parameter according to (4.9)). However, for the purpose of easy implementation and convenient analysis, we always prefer low-order controllers. Therefore, in practice, various techniques in model reduction [24, 74] can be used to get a satisfactory controller. In Figure 4.3 the relation between the sub-maximal stability margin and the order of the Youla parameter is shown: as the order increase, or equivalently, the dimension of the searching subspace increases, the sub-maximal stability margin becomes bigger, indicating that a more accurate approximation is obtained. We also
notice that the sub-maximal stability margin does not have a significant improvement when the order of the Youla parameter is higher than 6. This tells us sixth-order Youla parameter is actually a relatively satisfactory approximation.

\[ \text{Figure 4.3} \quad \text{Approximated maximal stability margins and orders of Youla parameters} \]

### 4.3.4 Details of the algorithm

The algorithm described above is summarized as follows:

1. obtain linear fractional parameterization (4.3) in terms of the uncertain parameter set (4.2), and compare (4.3) with (4.4) to get stable matrices $\tilde{M}$, $\tilde{N}$, $\tilde{M}_s$, $\tilde{N}_s$, which are polynomials of $z^{-1}$;

2. determine the matrix function $W'$ in terms of (4.6); and compute the matrix functions $A_k, B_k$ for $k = 0, 1, \ldots, K'$, in terms of (C.12), where $K'$ is the chosen degree for the Youla parameter $Q$:
(3) construct the coefficient matrices $F_{a,k}, F_{3,k}$ in terms of (C.11) and (C.10), and form the constraint (4.21);

(4) choose an integer $V$ and uniformly discretize the interval $[0, 2\pi]$ to get discrete points $\omega_l$ as (4.22); and form the optimization problem (4.23);

(5) solve the quasi-convex optimization problem (4.23) via feasibility (Details are introduced in Appendix A or [7]); the Youla parameter $Q(z) = \beta(z)/\alpha(z)$ is obtained in terms of (4.24) and the approximated maximal stability margin is the reciprocal of the optimal value;

(6) Construct the optimal controller in terms of (4.9).

We remark that among the variables $\alpha_0, \ldots, \alpha_K, \beta_0, \ldots, \beta_K$ we can normalize a variable. In our experiments we set $\alpha_0 = 1$. Therefore, the optimization problem has $2K + 1$ variables which can be arranged in a vector

$$x = (\alpha_1, \alpha_2, \ldots, \alpha_K, \beta_0, \beta_1, \ldots, \beta_K).$$

Obviously, the value of $\beta_0$ represents the gain of the Youla parameter, and thus it will appropriately reflect the the convergence of the approximation when increasing the order of the Youla parameter.

We also note that there are two Matlab routines called \texttt{feasp} and \texttt{mincx} which can be used to solve the optimization problem. Given a feasibility problem of the form

$$L(x) < R(x). \quad (4.25)$$

\texttt{feasp} solves the auxiliary convex program:

$$\text{minimize} \quad t$$
$$\text{subject to} \quad L(x) < R(x) + t \ast I.$$

The system of LMI (4.25) is feasible if and only if the global minimum is negative. To use \texttt{mincx} solve the feasibility problem. we arbitrarily choose a vector $c$ to form
the optimization problem

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad (4.25).
\end{align*}
\]

The system of LMI (4.25) is feasible if and only if there exists a solution to problem (4.26).

4.3.5 Analysis of the optimal Youla parameter

Several experimental observations concerning the relation between maximal stability margin and order of Youla parameter are reported in this subsection. The convergence of the algorithm is reflected from three aspects: convergence of the maximal stability margin, convergence of the gain of the Youla parameter and change of the poles and zeros of the Youla parameter.

As noted in subsection 4.3.3, as the order of Youla parameter increases, the maximal stability margin becomes bigger, which means that more accurate result is obtained. This is shown in Figure 4.3, where the convergence is easily observed as well.

In Table 4.1 we present the optimal Youla parameters \( Q(z) = \beta(z)/\alpha(z) \) by listing the values of the coefficients of \( \beta(z) \) and \( \alpha(z) \):

\[ a_1, a_2, \ldots, a_k, \beta_0, \beta_1, \ldots, \beta_k. \]

As analyzed before, \( \beta_0 \) indicates the gain of the Youla parameter. From the table it is easy to notice the convergence of \( \beta_0 \).
<table>
<thead>
<tr>
<th></th>
<th>order = 1</th>
<th>order = 3</th>
<th>order = 5</th>
<th>order = 7</th>
<th>order = 9</th>
<th>order = 11</th>
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<td>( \alpha_0 )</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
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<td>1.0000</td>
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<tr>
<td>( \alpha_1 )</td>
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<td>0.4527</td>
<td>0.5273</td>
<td>0.5712</td>
<td>0.5988</td>
<td>0.6227</td>
</tr>
<tr>
<td>( \alpha_2 )</td>
<td>0.2707</td>
<td>0.6205</td>
<td>0.7720</td>
<td>0.8521</td>
<td>0.9057</td>
<td></td>
</tr>
<tr>
<td>( \alpha_3 )</td>
<td>0.0590</td>
<td>0.1514</td>
<td>0.2633</td>
<td>0.3269</td>
<td>0.4124</td>
<td></td>
</tr>
<tr>
<td>( \alpha_4 )</td>
<td>0.0982</td>
<td>0.3354</td>
<td>0.4662</td>
<td>0.6456</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \alpha_5 )</td>
<td>0.0442</td>
<td>0.0696</td>
<td>0.1279</td>
<td>0.3003</td>
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<td></td>
</tr>
<tr>
<td>( \alpha_6 )</td>
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<td>0.1768</td>
<td>0.4186</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>( \alpha_7 )</td>
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<td>0.0067</td>
<td>0.1827</td>
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<tr>
<td>( \alpha_8 )</td>
<td>0.0083</td>
<td>0.2162</td>
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<td></td>
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<tr>
<td>( \alpha_9 )</td>
<td>0.0319</td>
<td>0.0636</td>
<td></td>
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</tr>
<tr>
<td>( \alpha_{10} )</td>
<td>0.0224</td>
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<tr>
<td>( \alpha_{11} )</td>
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<td><strong>0.5891</strong></td>
<td><strong>0.5962</strong></td>
<td><strong>0.6059</strong></td>
<td><strong>0.6123</strong></td>
</tr>
<tr>
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<td>0.2894</td>
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<td>0.7239</td>
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<tr>
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<td>0.3099</td>
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<td>0.5073</td>
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<tr>
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<td></td>
</tr>
<tr>
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<td>0.0196</td>
<td>0.1235</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( j_7 )</td>
<td>0.0081</td>
<td>0.0962</td>
<td>0.2672</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( j_8 )</td>
<td>0.0705</td>
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<td></td>
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</tr>
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<td>( j_9 )</td>
<td>0.0066</td>
<td>0.1196</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( j_{10} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>( j_{11} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.0116</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1  Coefficients of optimal Youla parameters
Also, how the optimal Youla parameter changes with the order is analyzed by considering the pole-zero cancelation. To get a direct feeling of the positions of the poles and the zeros, we plot in Figure 4.4 the pole-zero maps of the optimal Youla parameters of orders 4, 6, 10 and 11. It is easy to see the pole-zero cancelations from the maps, and we conclude that the optimal Youla parameters can be well approximated by a second-order transfer function. Another way of analyzing the pole-zero cancelation is to consider the common roots of the denominator and the numerator of the optimal Youla parameter. In our experiments we use Bezoutians [33] generated by the denominator and the numerator to analyze this. It is well known that two polynomials have \( n \) roots in common if and only if the Bezoutian generated by the two polynomials have \( n \) zero singular values. By continuity we know if the roots are not exactly the same, then we have relatively small singular values. In Figure 4.5 we plot the singular values of the corresponding Bezoutians for the optimal Youla parameters of order 4, 6, 10 and 11. We can observe that the first two singular values are significantly larger than the rest singular values. This implies the same result as we mentioned above.

4.4 Maximal stability margin and measures of the ellipsoid

As we know the uncertain models that can be described in terms of an ellipsoid \( \mathcal{E} \) have linear fractional parameterizations

\[
G_i(z) = \frac{n_{c}(z) + \delta_1 \alpha_1 n_1(z) + \cdots + \delta_{r-1} \alpha_{r-1} n_{r-1}(z)}{d_c(z) + \delta_1 \alpha_1 d_1(z) + \cdots + \delta_{r-1} \alpha_{r-1} d_{r-1}(z)},
\]

where the functions \( n_i(z), d_i(z) \) are stable, \( \alpha_i \)'s are the semiaxis lengths of the ellipsoid, \( \alpha_1 \leq \cdots \leq \alpha_{r-1} \), and the vector \( \delta = (\delta_1 \ \delta_2 \ \cdots \ \delta_{r-1})^T \) satisfies \( ||\delta|| \leq 1 \). In this section we will consider the relation between the maximal stability margin with respect to the uncertain parameter \( \delta \) and various measures of the uncertain parameter set, the ellipsoid \( \mathcal{E} \).
Figure 4.4 Pole-zero maps of optimal Youla parameters of different orders

Three measures are usually used for ellipsoids, which are the volume, the radius and the mean square root radius.

Volume: \[ V(\mathcal{E}) := \prod_{i=1}^{r-1} \alpha_i. \]
Radius: \[ R(\mathcal{E}) := \max_{1 \leq i \leq r-1} \alpha_i = \alpha_{r-1}. \]
Mean square root radius: \[ M(\mathcal{E}) := \sqrt{\alpha_1^2 + \cdots + \alpha_{r-1}^2}. \]

To derive the inequalities indicating the relations, let us start with the following observation on a new expression of the function $\phi(\alpha, J)$ defined in (4.14). Clearly,

\[
\phi(\alpha, J) = \sup_{\omega} \left| \frac{\text{Re} \left( \begin{bmatrix} -\alpha_1 n_1 & \alpha_1 d_1 \\ \vdots & \vdots \\ -\alpha_{r-1} n_{r-1} & \alpha_{r-1} d_{r-1} \end{bmatrix} W \begin{bmatrix} J \\ \alpha \end{bmatrix} e^{i\omega} \right)}{\text{Re} \alpha e^{i\omega}} \right|
\]

\[
= \sup_{\omega} \frac{||D_{\alpha, J}(\omega)||}{b_{\alpha}(\omega)}. \tag{4.27}
\]
where \( b_\alpha(\omega) = \text{Re} \alpha(e^{i\omega}) \).

\[
D = \begin{bmatrix}
\alpha_1 \\
\vdots \\
\alpha_{r-1}
\end{bmatrix}, \quad a_{\alpha_\beta}(\omega) = \text{Re} \left( \begin{bmatrix}
-n_1 & d_1 \\
\vdots & \vdots \\
-n_{r-1} & d_{r-1}
\end{bmatrix} W \begin{bmatrix}
\mathcal{J} \\
\alpha
\end{bmatrix} \right)(e^{i\omega}).
\]

In terms of the expression (4.27) the investigation of the relations between the maximal stability margin with respect to the uncertain parameter \( \delta \) and various measures of the ellipsoid are essentially the derivation of various inequalities about the quantity:

\[
\varphi(\alpha, \beta, \omega) := \frac{||Da_{\alpha_\beta}(\omega)||}{b_\alpha(\omega)}.
\]

The result is presented in the following theorem.

**Theorem 4.4.1** For the maximal stability margin \( \lambda_* \) with respect to the real uncertainty \( \delta \), the following inequalities hold

\[
\lambda_* \leq \gamma V(\mathcal{E})^{-\frac{1}{n-1}}. \tag{4.28}
\]
\[ \lambda_0 R(\mathcal{E})^{-1} \leq \lambda_* \leq \lambda_0 \alpha_1^{-1}. \] (4.29)

\[ \lambda_* \leq \kappa M(\mathcal{E})^{-1}. \] (4.30)

where \( \lambda_0, \gamma, \kappa \) are constants independent of the semiaxis lengths of the ellipsoid \( \mathcal{E} \).

**Figure 4.6** Maximal stability margin and its upper bound and lower bound change with \( \varepsilon \), the size of the uncertainty

The upper bounds in the above inequalities about the maximal stability margin allows us to obtain a close observation on the relation between the performance of the optimal controller and the size of the uncertainty. To some extent, this reflects a straightforward relation between the performance of the optimal controller and the size of the uncertainty. In Figure 4.6, the upper bound (dotted curve) and lower bound (dashdot curve) are given in terms of the smallest semiaxis length an the radius of the ellipsoid \( \mathcal{E} \). the approximated maximal stability margin is also shown (solid curve). It turns out that the robust performance is inversely proportional to the size of the uncertainty.
According to the theorem we have a comment on the optimal input design in the system identification for robust control. The purpose of controller design is to seek optimal controller to achieve the maximal stability margin, therefore, the relation between the maximal stability margin and the measure of the uncertain model family and the relation between the measure of the uncertain model family and the input for generating the model family are important for us to seek an optimal input. Due to the absence of the of analytic formula for the maximal stability margin, an appropriate upper bound will be a very good tool for us to analyze the maximal stability margin.

The lower bound in (4.29) suggests that choosing input to minimize the radius of the ellipsoid is a good way to improve the maximal stability margin. However, it follows from (4.1) that the radius is determined by the smallest singular value of $\mathcal{U}_1$, the first $r - 1$ rows of the data matrix $\mathcal{U}$, and the bigger the the smallest singular value, the smaller the radius of the ellipsoid. Hence, the optimal input design problem can be formulated as a problem of maximizing the smallest singular value of $\mathcal{U}_1$.

### 4.5 $\mathcal{H}_\infty$ robust synthesis problem

In this section we will consider the $\mathcal{H}_\infty$ robust control problem for the uncertain models (4.3)

$$G_\delta(z) = \frac{n(z) + \sum_{i=1}^{r-1} \alpha_i \delta_i n_i(z)}{d(z) + \sum_{i=1}^{r-1} \alpha_i \delta_i d_i(z)}.$$ 

where $\delta$ is the parametric uncertainty satisfying $||\delta|| \leq 1$. The diagram for the uncertain models is shown in Figure 4.1. One way of considering the problem is to consider the real parameter uncertainty $\delta$ as general dynamic uncertainty, i.e., $||\delta||_\infty \leq 1$. This method has been discussed in section 4.2 and the result is presented in Theorem 4.2.1.

In this section we will introduce another method, in which we will take advantage of the coprime factorization (4.3). We will express the uncertain model in coprime factor uncertainty as shown in Figure 4.7. Namely, the uncertain model $G_\delta(z)$ is
expressed as the form:

$$G_d = \frac{\hat{N} + \hat{N}_N}{\hat{M} + \hat{M}_M}.$$  \hspace{1cm} (4.31)

where the functions are stable and $\hat{N}$ and $\hat{M}$ are coprime. An natural choice for the matrices is

$$\hat{M}(z) = \frac{d_c(z)}{z^n}, \quad \hat{N}(z) = \frac{n_c(z)}{z^n}. \hspace{1cm} (4.32)$$

$$\hat{N}_N(z) = \sum_{i=1}^{r-1} \alpha_i \delta_i \frac{n_i(z)}{z^n}, \quad \hat{M}_M(z) = \sum_{i=1}^{r-1} \alpha_i \delta_i \frac{d_i(z)}{z^n}. \hspace{1cm} (4.33)$$

where $n$ is the order of the uncertain models. Obviously, $\hat{N}_N$ and $\hat{M}_M$ are respectively uncertainty in the numerator and the denominator of the transfer function of the model. Define

$$\Delta(z) = \begin{bmatrix} \Delta_N(z) \\ \Delta_M(z) \end{bmatrix}. \hspace{1cm} (4.34)$$

Then we have an upper bound for the $H_\infty$ norm of the overall uncertainty $\Delta$.

$$||\Delta||_\infty \leq \sqrt{n + 1}M(\mathcal{E}). \hspace{1cm} (4.35)$$

where $M(\mathcal{E})$ is the mean square root radius of the uncertain parameter set $\mathcal{E}$. 

---

**Figure 4.7** Coprime factor perturbed systems
Therefore, we have the following result for robust controllers that stabilize the uncertain models (4.3):

**Theorem 4.5.1** The controller $\mathcal{K}$ stabilizes the uncertain model (4.3) for all $\delta$ with $||\delta|| \leq 1$ if

$$
\left\| \begin{bmatrix} K & I \\ M + \mathcal{N}K \end{bmatrix} \right\|_{\infty} \leq \frac{1}{\sqrt{n + 1M(\mathcal{E})}},
$$

(4.36)

where $\mathcal{M}(z)$, $\mathcal{N}(z)$ are given in (4.32).

4.6 Robust synthesis problem: general model family

Recall that the synthesis problem we have considered so far is based on the fact that the uncertain model family can be characterized by an ellipsoid. More specifically, only in the case when the perturbation bound $\varepsilon$ is in between the smallest two singular values of $\mathcal{M}_1$ can we describe the uncertain models in terms of an ellipsoid. Therefore, when $\varepsilon$ is larger than the second smallest singular value of $\mathcal{M}_1$, the theory we have developed is not applicable. In this section we will study the synthesis problem in this general uncertainty case. For convenience we will use notations which are a little bit different from, however, comparable with the previous notations.

A direct idea of dealing with the problem is to split the perturbed output into three parts: the exact output and two sources of perturbations. One part of the perturbations is assumed to have the property that when it is incorporated into the exact output, the resulted uncertain model family can be described in terms of an ellipsoid: the remaining part of the perturbation is attributed to the dynamical uncertainty. This idea inspires the diagram shown in Figure 4.8 for uncertain models.

In the diagram the measured output $y$ is considered as the sum of $y_1$ and $y_2$, where $y_1$ is the output of the uncertain system $\mathcal{F}_u(G, \delta^T)^*$, the nominal system $G$ perturbed.
by real parameter $\delta$ ($||\delta|| \leq 1$), and $y_2$ is the output of the dynamic uncertainty $\Delta$. A similar problem for this diagram is considered in [18]. When the uncertain model family can not be characterized in terms of an ellipsoid, this kind of decomposition is always possible. The only problem remained is how to optimally split the output so that an optimal controller can be designed to achieve the best robust performance. This is still an open problem, however, it does not affect our studying robust synthesis problem for this diagram.

Assume, as usual, we have the following expression for $\mathcal{F}_u(G, \delta)$.

$$\mathcal{F}_u(G, \delta) = \frac{\dot{\bar{N}} + \delta^T \bar{N}_{\delta}}{\bar{M} + \delta^T \bar{M}_{\delta}}.$$ 

where $\delta$ is real parameter vector, and $||\delta|| \leq 1$. According to the diagram, we thus have the following new parameterization for the uncertain model

$$G_{\delta, \Delta}(z) = \mathcal{F}_u(G(z), \delta^T) + \Delta(z) = \frac{\dot{\bar{N}}(z) + \delta^T \bar{N}_{\delta}(z)}{\bar{M}(z) + \delta^T \bar{M}_{\delta}(z)} + \Delta(z)$$

$$= \frac{\dot{\bar{N}}(z) + \delta^T \bar{N}_{\delta}(z) + [\Delta(z) \quad \Delta(z)\delta^T] \begin{bmatrix} \dot{\bar{M}}(z) \\ \bar{M}_{\delta}(z) \end{bmatrix}}{\bar{M}(z) + \delta^T \bar{M}_{\delta}(z)}$$
\[
\hat{Y}(z) + \delta^T \hat{Y}_\delta(z) + \Delta_{\text{new}}(z) \hat{Y}_\Delta(z) = \frac{\hat{M}(z) + \delta^T \hat{M}_\delta(z) + \Delta_{\text{new}}(z) \hat{M}_\Delta(z)}{M(z) + \delta^T M_\delta(z) + \Delta_{\text{new}}(z) M_\Delta(z)}. \tag{4.37}
\]

where

\[\Delta_{\text{new}}(z) := [\Delta(z), \Delta(z)\delta^T], \quad \hat{Y}_\Delta(z) := \begin{bmatrix} \hat{M}(z) \\ \hat{M}_\delta(z) \end{bmatrix}, \quad \hat{M}_\Delta(z) := 0\]

Clearly, the structured new uncertainty \( \Delta_{\text{new}} \) has an \( \mathcal{H}_\infty \) upper bound given by

\[||\Delta_{\text{new}}||_\infty \leq \sqrt{2}||\Delta||_\infty.\]

where the \( ||\Delta||_\infty \) can be given in terms of the input sequence \( u(0), u(1), \ldots, u(N) \)

and the corresponding output sequence \( y_2(0), y_2(1), \ldots, y_2(N) \). It turns out that \( ||\Delta||_\infty \) is equal to the maximal \( \gamma > 0 \) satisfying

\[U^*U^T \gamma^2 \leq Y_2 Y_2^T.\]

where \( U, Y_2 \) are two lower-triangular Toeplitz matrices constructed by \( u(n) \) and \( y_2(n) \)

\[
U^* = \begin{bmatrix}
u(0) \\
u(1) & u(0) \\
\vdots & \ddots & \ddots \\
u(N) & \cdots & u(1) & u(0)
\end{bmatrix}, \quad \begin{bmatrix}
y_2(0) \\
y_2(1) & y_2(0) \\
\vdots & \ddots & \ddots \\
y_2(N) & \cdots & y_2(1) & y_2(0)
\end{bmatrix}.
\]

In the sequel we set \( \gamma_0 := ||\Delta||_\infty. \) Then, \( ||\Delta_{\text{new}}||_\infty \leq \sqrt{2} \gamma_0. \)

We have the following theorem characterizing robust controllers that stabilize the uncertain models in diagram 4.8.

**Lemma 4.6.1** [45] The rational controller \( K \) stabilizes \( G_{\delta, \Delta} \) in (4.37) for \( ||\delta|| \leq 1 \)

and \( ||\Delta||_\infty \leq \gamma_0 \) if it can be written in the form

\[K = (Y\alpha + M\beta)(X\alpha + N\beta)^{-1}\]

where \( \alpha, \beta \) are stable scalar functions and for all \( \omega \in [0, 2\pi] \).

\[
\left| \mathbf{Re} \left( [\hat{Y}_\delta \hat{Y}_\delta] W \begin{bmatrix} \beta \\ \alpha \end{bmatrix} (e^{i\omega}) \right) + \sqrt{2} \gamma_0 \right| \left| \mathbf{Re} \left( [\hat{Y}_\delta \hat{Y}_\delta] W \begin{bmatrix} \beta \\ \alpha \end{bmatrix} (e^{i\omega}) \right) \right| \leq \mathbf{Re} \left( (e^{i\omega}) \right), \tag{4.38}\]
Figure 4.9  Parametric uncertainty $\delta$ with $||\delta|| \leq 1$ and dynamical uncertainty $\Delta$ with $||\Delta||_{\infty} \leq \gamma_0$.

Similar to the discussion in section 4.2, we can propose a robust performance problem in terms of the condition (4.38): maximizing the stability robustness with respect to $\Delta_{\text{new}}$ for fixed range of $\delta$. ($||\delta|| \leq 1$).

$$\lambda_* = \max_{\alpha, \beta} \lambda(\alpha, \beta),$$

where

$$\lambda(\alpha, \beta) = \inf_{\omega} \left[ \frac{\text{Re} \alpha(e^{i\omega}) - \left\| \text{Re} \left( \begin{bmatrix} \tilde{N}_\delta & \tilde{M}_\delta \end{bmatrix} W \begin{bmatrix} 3 \\ \alpha \end{bmatrix} (e^{i\omega}) \right) \right\|}{\sqrt{2} \gamma_0 \left\| \begin{bmatrix} -\tilde{N}_\Delta & \tilde{M}_\Delta \end{bmatrix} W \begin{bmatrix} 3 \\ \alpha \end{bmatrix} (e^{i\omega}) \right\|} \right].$$

This is an optimization problem which can be solved in the same method introduced in section 4.3. The detailed discussion will be omitted here.

Next we will introduce another method of analyzing the robust synthesis problem concerning the diagram 4.9. First, we change the diagram to diagram in Figure 4.10.
by using linear fractional transformation. Assume
\[
\begin{bmatrix}
  z_1 \\
  y_1
\end{bmatrix} = \begin{bmatrix}
  G_{11} & G_{12} \\
  G_{21} & G_{22}
\end{bmatrix} \begin{bmatrix}
  w_1 \\
  u
\end{bmatrix}
\]

where the partition are conformal. Therefore,
\[
\begin{bmatrix}
  z_1 \\
  z_2 \\
  y
\end{bmatrix} = \begin{bmatrix}
  G_{11} & 0 & G_{12} \\
  0 & 0 & 1 \\
  G_{21} & 1 & G_{22}
\end{bmatrix} \begin{bmatrix}
  w_1 \\
  w_2 \\
  u
\end{bmatrix}
\]

Therefore, we have the following standard uncertain model diagram, where there exists the structured uncertainty, the real parameter uncertainty \( \delta (||\delta|| \leq 1) \) and the dynamical uncertainty \( \Delta (||\Delta||_\infty \leq \gamma_0) \). This is a typical \( \mu \)-problem, and well developed techniques can be used to analyze the problem.

In the structured uncertainty case, the problem can be solved using standard \( \mu \)-analysis methods [74]. However, we expect a method to explore the special structure of the uncertainty (mixed parametric uncertainty and dynamical uncertainty are presented).

### 4.7 Conclusions

We have studied that in Chapters 2 and 3 under some circumstance the uncertain models are parameterized with a real uncertain parameter of at most unit length.

The uncertain real parameters linearly appear in the numerator and the denominator of transfer functions of the uncertain models. In this chapter we consider the robust synthesis problem, based on this kind of parameterization. By using a convex parameterization of all controllers that simultaneously stabilize the uncertain models, a robust performance problem can be stated in terms of a quasi-convex optimization, which can be solved using Ritz approximation method. Several experimental observations concerning the relation between maximal stability margin and order of Youla
Figure 4.10  Linear fractional transformation expression of the uncertain system for $\mu$-synthesis

parameter are reported. The convergence of the algorithm is reflected from three aspects: convergence of the maximal stability margin, convergence of the gain of the Youla parameter and change of the poles and zeros of the Youla parameter. We also consider the relation between the maximal stability margin and various measures of the uncertain model family. The $\mathcal{H}_\infty$ synthesis problem is briefly discussed for comparison: two methods are introduced, but the second method, taking advantage of the coprime factorization, is analyzed in detail. The last problem we considered is the robust synthesis problem for general model family. We formulate the problem in two different methods, one is similar to the problem we have already discussed, and the other is the typical $\mu$-synthesis method.
Chapter 5

Model validation problem

The problem we wish to address in this chapter is that of determining whether or not the input-output data record is consistent with the uncertainty model of the plant. In other words, the problem is to decide whether the observed data could have been produced by the model for some choice of unmodeled dynamics, initial condition, and measurement noise satisfying the given bounds. This is called the model validation problem. A model is said to be validated if it could have produced the data, or speaking more precisely, a model is said to be invalidated if the model is not consistent with the data. In essence, model validation is the assessment of the quality of a given model with respect to experimental data.

It turns out that in our setting a single model is invalidated if it is outside the ellipsoid, it is not invalidated if it is inside the ellipsoid. Furthermore, in the multiple sets of data case the intersection of the model families is not invalidated. In order to make the unfalsified models have the simple linear linear fractional parameterizations, we can use various method to optimally bound the intersection of the ellipsoids. Since the robust performance is inversely proportional to the size of the ellipsoid, a better robust performance is expected to be achieved.

Actually, one can never validate models because of the impossibility of testing all experimental conditions and inputs. If the model is not consistent with the data, then one can definitely conclude that the model is not a representation of the physical system. On the other hand, if a particular uncertainty model is consistent with the data, one can only conclude that the corresponding model is not contradicted by the given data, i.e., it is not invalidated. This does not mean that the model is a correct description of the physical system. Thus, validation is perhaps a misnomer, and unfalsification should be a more precise term, as suggested by R.L. Kosut [32].
However, we will not strictly distinguish the two terms in this note.

Model validation problems have been becoming an important issue in control-oriented system identification [47]. Model validation is widely regarded as an integral ingredient of the entire process of obtaining robust control-oriented system models, and it is preceded by system analysis and understanding, physical modeling, and identification. If the uncertainty model is invalidated by the input-output data record, then it becomes necessary to revisit the identification step by using additional data, or/and reformulating the identification scheme. This introduces an iterative identification scheme for robust control design.

5.1 Validation of a single model

For every set of input-output data the robust control-oriented system identification methods usually deliver either a nominal model together with guaranteed bounds on the unmodeled dynamics, initial condition uncertainty, and measurement noise, or a family of uncertainty models which is guaranteed to contain the true model. As a matter of fact, in both case a model family is obtained, which describes the uncertainty models consistent with the input-output data. In the field of parameter estimation, such a family of the system parameters is referred to as the membership set. Here we use the same concept for the family of the uncertainty models consistent with the data. Clearly, for every set of input-output data and the description or assumption of the uncertainty (e.g., the error bound of every single measurement in the bounded error case), there corresponds to a unique membership set in which the models or their parameters are explicitly characterized.

With the introduction of the concept of membership set the validation problem is equivalent to determining whether the particularly specified model is in the membership set corresponding to model uncertainty and the set of data used to validate the specified model. The model is validated if and only if it is in the membership set. Therefore, the validation problem is essentially to construct the membership set
corresponding to the data and model uncertainty description.

In our setting the membership set is characterized by the misfit function of the model and the data matrix constructed by the input-output data. Thus.

**Theorem 5.1.1** Assume the provided input-output data for validating the model \( x \) is given by (2.1), and upper error bound of the noisy data matrix is \( \varepsilon \). Then the model \( x \) is validated if and only if

\[
x \in \mathcal{F} = \left\{ x \in \mathbb{R}^r : \mu(x, \hat{\mathbf{U}}) \leq \varepsilon \right\}
\]

where the matrix \( \hat{\mathbf{U}} \) is data matrix constructed as in (2.9) by the input-output data measurements.

**Figure 5.1** Single model invalidation: any model inside the ellipse (the circle) is not invalidated, and any model outside the ellipse (the asterisk) is invalidated

Notice that the model can be conveniently validated or invalidated according to the above theorem, since the involved computation is extremely simple (See Figure 5.1). However, recall that the description of the membership set using the misfit function is actually an approximate description, since the Hankel structure of the data matrix is ignored. If the Hankel structure of the data matrix is considered, the
validation problem becomes to determine whether the model is a member of the exact membership set

\[ \mathcal{F} = \left\{ x \in \mathbb{R}^r : \exists \tilde{\mathcal{M}} \text{ (having the same block Hankel structure as } \tilde{\mathcal{M}} \text{) } \text{ such that } x^T \tilde{\mathcal{M}} = 0 \text{ and } ||\tilde{\mathcal{M}} - \mathcal{M}||_2 \leq \varepsilon \right\} \]

In this case, the computation becomes much more expensive. However, it is not intractably hard. An straightforward way is to formulate the problem as an optimization problem with LMI constraints. Then, the validation problem is essentially the feasibility problem of the optimization problem. The model \( x \) is validated if and only if the corresponding optimization problem is feasible. We remark that this is exactly what we have done in Section 3.2.3, in which data matrix approximation problem is studied in detail, and thus the results are omitted here.

5.2 Validation of a model family

A straightforward extension of the single model validation problem is that of validating a family of models, i.e., the problem of validating every single model in the family. The formulation is similar to that of single model validation problem, where a set of input-output data and an upper bound of the noise data matrix are provided.

**Theorem 5.2.1** Assume the provided input-output data for validating the model family \( \mathcal{E} \) is given by (3.1), and upper error bound of the noisy data matrix is \( \varepsilon \). Then the model \( x \in \mathcal{E} \) is validated if and only if \( x \) is in the intersection of the model family \( \mathcal{E} \) and the membership set \( \mathcal{F} \) corresponding to given data and the uncertainty bound, namely

\[ x \in \mathcal{E} \cap \mathcal{F} = \left\{ x \in \mathcal{E} : \mu(x, \tilde{\mathcal{M}}) \leq \varepsilon \right\} \]

where the matrix \( \tilde{\mathcal{M}} \) is the data matrix constructed as in (2.9) by the input-output data measurements.

This theorem is illustrated in Figure 5.2, where the shaded area is unfalsified, since the intersection of two families is not reciprocally invalidated.
A direct application of the model family validation problem is the determination of the uncertainty model family corresponding to multiple sets of data. This happens when simply multiple sets of data are available, or the data is cut into pieces for some optimization strategy. For every set of data an uncertainty model family can be formed as explained previous chapters. This model family is actually nothing but the corresponding membership set when the data is used to validate other uncertainty model family. Therefore, each model family is, on one hand, an uncertain family needed to be validated. On the other hand, it is a membership set which can be employed to validate other model families. This observation leads to the fact that the uncertainty model family corresponding to overall data set is identical to the intersection of every single uncertainty model family (See Figure 5.3).

**Theorem 5.2.2** Suppose multiple sets of input-output data \((u_{ki}, y_{ki})\), \(i = 0, 1, \ldots, V_k - 1\) and the upper error bound of the corresponding noisy data matrices \(z_k\) are available. Then the corresponding uncertainty model family \(\mathcal{F}\) is the intersections of the single
uncertainty model families \( F_k \), i.e.,

\[
F = \bigcap_k F_k = \bigcap_k \{ x \in \mathbb{R}^r : \mu(x, \hat{\mathcal{M}}_k) \leq z_k \}
\]

where the matrix \( \hat{\mathcal{M}}_k \) is data matrix constructed as in (2.9) by the input-output data.

**Figure 5.3** Model family of multiple sets of data: intersection of all single model families.

The strategy of intersecting the model families bears the advantage of reducing the size of the uncertainty model family. This is extremely useful for robust control design, since, as shown in Chapter 4, better robust performance can be achieved with small size of uncertain model family.

### 5.3 Optimal ellipsoidal bounding of the intersection of ellipsoids

To reduce the size of the uncertainty by intersecting the uncertain model families makes the models lose the simple linear fractional parameterization. To remedy this we propose to seek an ellipsoid to bound the intersection in some optimal sense. In
our case, the model family is characterized in terms an ellipsoid for every set of data. Therefore, the membership set corresponding to multiple sets of data is in effect the intersections of several ellipsoids (See Figure 5.4). We realize the uncertain models in the intersection of the model families will lose the linear fractional parameterization since the intersection is not an ellipsoid. A natural method to remedy this is bound the intersection in some optimal sense using an ellipsoid. The optimality here means minimal volume, minimal radius or minimal mean square root radius (See Appendix A for the concepts). Once an optimal ellipsoid is computed, the theories throughout the thesis can be applied to this new uncertain model family.

Reliable and efficient algorithms have been developed [6, 39] for computing various suboptimal outer ellipsoidal approximation of the intersection of ellipsoids. by formulating the problem as convex optimization problems with Linear Matrix Inequality (LMI) constraints. However, algorithm for computing the global minimum volume ellipsoid containing the intersection of ellipsoids is not available. In this section, different algorithms are briefly introduced. Readers are referred to the book [6, 60] for details.

![Figure 5.4](image)

**Figure 5.4** Optimal ellipsoidal bounding of the intersection of four ellipsoids.
5.4 Conclusions

As we know the family of the models consistent with the input-output data record can be approximately characterized as an ellipsoid under some condition. In the case when several sets of data are available, model validation problem is addressed to achieve a more compact and more strict description of the model family. This means that the updated description results from the intersection of the ellipsoids. If all of the available family are not invalidated, the new model family is the intersection of ellipsoids, and it is a convex set. Otherwise, the new family might not be a convex set. In either case, in order to make use of the above-introduced framework for robust control design, we need to approximate the new model family by an ellipsoid which is optimal in some sense. Since the best performance of the optimal controller is closely related to the volume of the ellipsoid of the model family, the minimum volume ellipsoid containing the new family is what we want.
Chapter 6

An illustrative example

In this Chapter we will use a time varying mass-spring-damper system to illustrate the main results we have discussed in this thesis. We first model the system as an LTI system, and then design robust controller and analyze the performance based on the uncertain model family. Note that in this situation the perturbation results from the unmodeled dynamics.

6.1 Introduction

The time-varying mass-spring-damper system [12] is shown in Figure 6.1. To design robust controller for the slowly time-varying system and analyze the performance of the controller, we first need to model the input-output data generated by the system as generated by an linear time-invariant system with additive perturbation. While the data can be obtained by inciting the discretized system with a chosen input. The order of the linear time-invariant system chosen for modeling the data can be determined either by computing the rank of the data matrices with different numbers of rows, or by studying the pole-zero cancelation of the nominal models determined by the data matrices (Chapter 3). The perturbation bound resulting from the modeling can be obtained by considering the data matrix decomposition problem (chapter 3). Once a model family is formed and characterized in terms of an ellipsoid (Chapter 2), the uncertain models consistent with the data can be characterized so that the transfer functions are linear fractional parameterizations of a real parameter vector with up to a unit length. The robust synthesis problem can be performed based on the parameterization of the uncertain models, and the performance of optimal controller is evaluated (Chapter 4). Furthermore, the model validation problem can be considered
by analyzing multiple sets of input-output data: the optimal ellipsoidal bounding of the ellipsoids determined by the data can be computed using convex optimization techniques, and the robust performance of the controller will be significantly improved (Chapter 5).

\[ m \quad \dot{y} \]
\[ b \quad c \]

Figure 6.1 Mass-spring-damper system.

Assume that the equation of motion is described by

\[ \frac{d}{dt} \left( m(t) \frac{dy}{dt} \right) + d(t) \frac{dy}{dt} + b(t)y(t) = f(t). \]

This can be written as

\[ \frac{d^2 y(t)}{dt^2} + \frac{c(t)}{m(t)} \frac{dy(t)}{dt} + \frac{b(t)}{m(t)} y(t) = \frac{f(t)}{m(t)}. \quad (6.1) \]

where

\[ c(t) = \frac{dm(t)}{dt} + d(t). \]

The parameters \( m \), \( b \) and \( c \) are assumed to be slowly varying mass, spring, and damping constants.

\[ m(t) = m_0 + \delta_m \sin(\omega_m t). \]
\[ b(t) = b_0 + \delta_b \sin(\omega_b t). \]
\[ c(t) = c_0 + \delta_c \sin(\omega_c t). \]
where \( m_0, b_0 \) and \( c_0 \) are values representing the nominal values of \( m, b \) and \( c \). \( \omega_m, \omega_b, \omega_c \) are frequencies, and \( \delta_m, \delta_b, \delta_c \) are numbers representing the maximal uncertainties of the three parameters. For the consideration in this chapter we assume the frequencies \( \omega_m, \omega_b, \omega_c \) are all small.

### 6.2 Generation of input-output data

We have two ways of generating the discrete input-output data by the continues-time system (6.1). One is to discretize the chosen analytic input function \( f(t) \) and the resulting output function, i.e., the solution of the difference equation (6.1). Since the coefficients of the linear time-varying system are functions of the time variable, it is usually almost impossible to get such a solution \( y(t) \). Therefore, instead of using this method, in this section we first discretize the continuous system using two approximation transformations, and computing the discrete output data \( y_k \)'s using the discretized equation with some chosen discrete input data \( f_k \)'s.

By choosing a small sampling time \( \tau \), we can discretize equation (6.1) using the following approximations

\[
\frac{d^2 y(t)}{dt^2} \approx \frac{y(t + 2\tau) - 2y(t + \tau) + y(t)}{\tau^2}, \quad \frac{dy(t)}{dt} \approx \frac{y(t + \tau) - y(t)}{\tau}.
\]

Therefore, equation (6.1) leads to

\[
\frac{y(t + 2\tau) - 2y(t + \tau) + y(t)}{\tau^2} + \frac{c(t)}{m(t)} \frac{y(t + \tau) - y(t)}{\tau} + \frac{b(t)}{m(t)} y(t) = \frac{f(t)}{m(t)}.
\]

or equivalently,

\[
y(t + 2\tau) + \left( \frac{c(t)}{m(t)} \tau - 2 \right) y(t + \tau) + \left( \frac{b(t)}{m(t)} \frac{\tau^2}{m(t)} - \frac{c(t)}{m(t)} \tau + 1 \right) y(t) = \frac{1}{m(t)} \tau^2 f(t).
\]

For some initial time \( t_0 \), we define \( t_k = t_0 + k\tau \) and

\[
y_k = y(t_k), \quad f_k = f(t_k),
\]

\[
m_k = m(t_k), \quad b_k = b(t_k), \quad c_k = c(t_k).
\]
Therefore, equation (6.1) becomes
\[ y_{k+2} + q_{1k} y_{k+1} + q_{0k} y_k = p_{0k} f_k, \quad k = 0, 1, \ldots \]  
(6.2)

where
\[ p_{0k} = \frac{1}{m_k} \tau^2, \quad q_{0k} = \frac{b_k}{m_k} \tau^2 - \frac{c_k}{m_k} \tau + 1, \quad q_{1k} = \frac{c_k}{m_k} \tau - 2. \]  
(6.3)

Figure 6.2  Two different inputs used in the experiment.

With the numerical values
\[ m_0 = 5, \quad b_0 = 3, \quad c_0 = 7, \quad \delta_m = 0.45, \quad \delta_n = 0.35, \quad \delta_s = 0.4, \quad \omega_m = 0.25 \pi, \quad \omega_n = 0.115 \pi, \quad \omega_s = 0.105 \pi, \quad \tau = 0.01, \quad t_0 = 0. \]

we can record the locus of the point \((-p_{0k}, q_{0k}, q_{1k})\) with discrete time \(k\) by computing the values of the parameters \(p_{0k}, q_{0k},\) and \(q_{1k}\). By choosing input vector \(f\) of length \(N = 10000\) (see Figure 6.2) and unit energy, and arbitrary initial values \(y_0 = y(t_0), \quad y_1 = y(t_0 + \tau)\), we can recursively compute a set of input-output data using equation (6.2).
6.3 Data modeling

Next we want to model the resulting input-output data as generated by a linear time-invariant, discrete-time system. The first problem arising is how to choose an appropriate order for the LTI system. The problem followed is to determine an appropriate upper bound for the perturbation resulting from modeling the data with an LTI system the order of which is selected in advance.

6.3.1 Order selection

In this subsection we use the method introduced in Chapter 3 to select an order. In other words, the order can be chosen in terms of the dominant singular values of the data matrix with a high number of rows.

![Graph showing singular values](image)

**Figure 6.3** Singular values of a data matrix with 42 rows: there are 23 most significant singular values. Since the input sequence is persistently exciting of order 21 (i.e., the input matrix in the data matrix has 21 rows), the order for an appropriate LTI approximation of the original system should be 23-21=2.

The singular values of the data matrix with 42 rows are shown in Figure 6.3. The
curve indicates that it is appropriate to chose 2 as the order of the LTI model. Let the second order linear time-invariant, discrete-time model has the following form

\[ y_{k+2} + q_1 y_{k+1} + q_0 y_k = p_0 f_k, \quad k = 0, 1, \ldots \] (6.4)

where \( p_0, q_0, q_1 \) are constants. Obviously, the transfer function of the model is

\[ H(z) = \frac{p_0}{z^2 + q_1 z + q_0}. \]

In the sequel we will use the coefficient vector

\[ x = (-p_0, q_0, q_1)^T \]

to represent the uncertain model. In addition, we call \( \theta = (-p_0, q_0, q_1)^T \) the uncertain parameter vector.

### 6.3.2 Determination of perturbation bound

In this subsection we will determine an appropriate upper bound for the perturbation resulting from modeling the data with a second order LTI system like (6.4). A conservative method is to select a bound according to the smallest singular values of the data matrices \( \hat{\Phi} \) and its submatrix \( \hat{\Phi}_1 \). In our experiment we also use the data matrix decomposition method discussed in Chapter 3 to determine the bound \( \varepsilon \).

For each chosen \( \varepsilon \) which is bigger than \( \hat{\sigma}_1 = 0.053 \) and smaller than \( \sigma_1 = 0.835 \), the uncertain model family is an ellipsoid (See Figure 6.1). However, it is problematic that for the chosen \( \varepsilon \), the set of the uncertain parameter vector (ellipsoid) contains the locus of the parameters \( (-p_0, q_0, q_1) \) in (6.3).

### 6.4 Robust synthesis problem

For each chosen \( \varepsilon \) the optimal robust controller that stabilizes the whole ellipsoid can be obtained, and the optimal performance can be computed, using the method introduced in Chapter 4. In Figure 6.5, the relation between the maximal stability
Figure 6.4  Locus of the parameters \((-p_{0k}, q_{0k}, q_{1k})\) of the time varying system (left) and the uncertain parameter set, an ellipsoid (right). Notice from the right figure that the locus (looks like a straight line) is contained in the ellipsoid.

margin and the size of the uncertainty, the quantity \(\varepsilon\) is depicted, where the curve indicates that better optimal performance can be achieved with a smaller \(\varepsilon\). Detailed analysis has been presented in Chapter 4.

6.5 Model validation problem

Model family problem is also considered for the example. As discussed in Chapter 5, a direct application of the theory to this example is to consider the modeling problem and control problem when multiple sets of data are available. The intersection of two model families is shown in Figure fig:contain. We consider the optimal ellipsoidal bounding of the intersection of the ellipsoids as an improved uncertain model family. According to this description of the uncertain models, we expect to design an optimal controller that achieves a better performance. Readers are referred to the discussion
Figure 6.5  The relation between the maximal stability margin and the quantity $\varepsilon$ is depicted, where the curve indicates that better optimal performance can be achieved with a smaller $\varepsilon$.

in Chapter 5 for details.

6.6 Conclusions

In this chapter we illustrate the theory developed in this thesis by modeling a slowly time-varying system as a linear time-invariant system together with additive perturbation. By choosing the perturbation bound the model family can be characterized by means of an ellipsoid, and then the transfer functions of the uncertain models can be parameterized so that the numerators and the denominators are linearly expressed. The robust synthesis problem is also performed based on the parameterization of the uncertain models, and the performance of optimal controller is evaluated by solving a convex optimization problem with LMI constraints. Furthermore, the model validation problem is considered by analyzing multiple sets of input-output data; the optimal ellipsoidal bounding of the ellipsoids determined by the data can be computed
Figure 6.6  The locus of the parameters of the time-varying system is depicted together with two uncertain parameter sets (ellipsoids). Notice that compared to the ellipsoids, the locus is actually small. Hence the usefulness of the intersection is obvious.

using convex optimization techniques.
Appendix A

Mathematical Tools

In this appendix we survey some of the important mathematical tools we are using throughout the thesis. Some of the results are directly referred, others are implicitly used.

A.1 Vectors and matrices

A.1.1 Notation

Let \( \mathbb{R} \) denote the real scalar field and \( \mathbb{C} \) the complex scalar field. Let \( \mathbb{R}^n \) and \( \mathbb{C}^n \) respectively denote the real and complex vector space of dimension \( n \). Similarly, let \( \mathbb{R}^{m \times n} \) and \( \mathbb{C}^{m \times n} \) denote the linear space of all real \( m \times n \) and complex \( m \times n \) matrices, respectively. By default, vectors are always referred to as column vectors. For simplicity, a vector \( x \) with \( x_i \) as its \( i \)-th component is sometimes denoted as \( x = [x_i] \), whereas a matrix \( A \) with \( a_{ij} \) as its \( i \)-th row and \( j \)-th column's element is written as \( A = [a_{ij}] \). Similarly, a matrix \( A \) with \( a_j \) as its \( j \)-th column, \( 1 \leq j \leq n \) is usually written as \( A = [a_1 \ a_2 \ \cdots \ a_n] \).

A real (complex) \( m \times n \) matrix \( A = [a_{ij}] \) is a linear transformation from \( \mathbb{R}^n (\mathbb{C}^n) \) to \( \mathbb{R}^m (\mathbb{C}^m) \). The image of \( A \), \( \text{Im} A \), is defined as the subspace of \( \mathbb{R}^m (\mathbb{C}^m) \) spanned by its columns. The kernel of \( A \), \( \ker A \), is defined as the solution subspace of the linear equation \( Ax = 0 \). Let \( A^T = [a_{ji}] \) denote the transpose, and \( A^* = [\bar{a}_{ji}] \) the complex conjugate transpose of \( A \). Then the left kernel of \( A \) is referred to as the kernel of \( A^T \). The rank of \( A \), \( \text{rank} A \), is defined as the number of its independent columns, i.e., the dimension of its image.

A square matrix \( A \in \mathbb{C}^{n \times n} \) is nonsingular if its rank equals \( n \). The identity matrix, \( I_n \), is a typical nonsingular matrix. The invertibility of a square matrix \( A \) is identical
to its nonsingularity. The unique inverse of a nonsingular matrix \( A \) is denoted by \( A^{-1} \). The trace of \( A \), \( \text{Tr}(A) \), is defined as the sum of its all diagonal elements.

Let \( A \in \mathbb{C}^{n\times n} \), then the eigenvalues of \( A \) are the numbers \( \lambda \in \mathbb{C} \) such that \( \text{rank}(\lambda I_n - A) < n \). The maximal modulus of the eigenvalues is called the spectral radius, denoted by \( \rho(A) = \max_{1 \leq i \leq n} |\lambda_i| \). For each eigenvalue \( \lambda \) of \( A \), the nonzero vectors \( x \in \mathbb{C}^n \) that satisfy

\[ Ax = \lambda x \]

are referred to as eigenvectors.

In order to avoid excessive notational clutter when dealing with vectors and matrices, dimensions are only rarely mentioned explicitly, unless confusion may happen. Whenever a sum of matrices such as \( A + B \) appears, it is assumed that the dimensions are compatible for addition. A similar assumption is made in the case of matrix products. When an inverse such as \( A^{-1} \) is written, it is assumed that the matrix \( A \) is square and that the inverse exists. When \( I \) is used to denote an identity matrix, its dimension should be clearly determined from the context.

### A.1.2 Matrix inversion formula

Let \( A \) be a square matrix partitioned as follows

\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\]

where \( A_{11} \) and \( A_{22} \) are also square matrices. Now suppose \( A_{11} \) is nonsingular. Then \( A \) has the following decomposition:

\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
A_{21}A_{11}^{-1} & I
\end{bmatrix} \begin{bmatrix}
A_{11} & 0 \\
0 & \Delta
\end{bmatrix} \begin{bmatrix}
I & A_{11}^{-1}A_{12} \\
0 & I
\end{bmatrix}
\]

(A.1)

with \( \Delta = A_{22} - A_{21}A_{11}^{-1}A_{12} \) and \( A \) is nonsingular if and only if \( \Delta \) is nonsingular.

Dually, if \( A_{22} \) is nonsingular, then

\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} = \begin{bmatrix}
I & A_{12}A_{22}^{-1} \\
0 & I
\end{bmatrix} \begin{bmatrix}
\Delta & 0 \\
0 & A_{22}
\end{bmatrix} \begin{bmatrix}
I & 0 \\
A_{22}^{-1}A_{21} & I
\end{bmatrix}
\]

(A.2)
with $\tilde{\Delta} = A_{11} - A_{12}A_{22}^{-1}A_{21}$ and $A$ is nonsingular if and only if $\tilde{\Delta}$ is nonsingular. $\Delta$, $\tilde{\Delta}$ are, respectively, called the Schur complements of $A_{11}$ and $A_{22}$ in matrix $A$, and the decomposition formulas are referred as the Schur complement formulas.

It is worthy of noting that an explicit formula for the inverse of $A$ can be readily obtained from the above decompositions, provided that $A$ is nonsingular. Moreover, as a by-product, the following identity is usually quite useful: suppose both $A_{11}$ and $A_{22}$ are nonsingular, then

$$(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} = A_{11}^{-1} + A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1}. \tag{A.3}$$

In particular, for $x, y \in \mathbb{C}^n$ satisfying $y^TA^{-1}x \neq -1$, we have

$$(A + xy^T)^{-1} = A^{-1} - \frac{A^{-1}xy^TA^{-1}}{1 + y^TA^{-1}x}. \tag{A.4}$$

More specifically,

$$(I + xy^T)^{-1} = I - \frac{xy^T}{1 + y^Tx} \tag{A.5}$$

if $y^Tx \neq -1$.

### A.1.3 Generalized inverse and least squares solution

It is known that a square matrix has a unique inverse if and only if it is nonsingular. In the case when a matrix $A$ is rectangular or doesn't have a full rank, a generalized inverse called pseudo-inverse or Moore-Penrose inverse is quite useful. The unique pseudo-inverse, conventionally denoted by $A^*$, satisfies the following conditions:

(i) $AA^*A = A$;

(ii) $A^*AA^* = A^*$;

(iii) $(AA^*)^* = AA^*$;


An explicit formula of the pseudo-inverse \( A^\dagger \) can be obtained in terms of the singular value decomposition of \( A \). In the special case when \( A \in \mathbb{C}^{m \times n} \) has full row rank, i.e., rank \( (A) = m \), the pseudo-inverse

\[
A^\dagger = A^* (A A^*)^{-1},
\]

is also called the right inverse of \( A \). Similarly, if rank \( (A) = n \), then the pseudo-inverse and the left inverse of \( A \) is

\[
A^\dagger = (A^* A)^{-1} A^*.
\]

The Moore-Penrose inverse has an important application in the study of the linear equation

\[
Ax = b.
\]

where \( A \in \mathbb{C}^{m \times n} \) and \( b \in \mathbb{C}^n \), and we wish to solve for \( x \in \mathbb{C}^n \). Since \( b \) may not be in the image of \( A \), \( \text{Im} A \), the exact solution to the equation does not necessarily exist. However, it turns out that the vector \( x_0 = A^\dagger b \) is the best approximate solution of the equation, in the sense that \( x_0 \) has the minimal 2-norm among all vectors \( x \) for which \( \|Ax - b\|_2 \) is minimal, i.e.,

\[
x_0 = \arg \min \{ \|x\|_2 : \|Ax - b\|_2 = \min_{x \in \mathbb{C}^n} \|Ax - b\|_2 \}.
\]

\( x_0 = A^\dagger b \) is usually called the least squares solution of the equation.

### A.1.4 Vector norms and matrix norms

A norm of a vector is a measure of the vector "length" or the vector "size". Let \( x \in \mathbb{C}^n \). Then the vector \( p \)-norm of \( x \) is defined as

\[
\|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \quad \text{for } 1 \leq p \leq \infty.
\]

In particular, when \( p = 1, 2, \infty \) we have

\[
\|x\|_1 = \sum_{i=1}^{n} |x_i|,
\]
\[ \|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}; \]
\[ \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|. \]

Similarly, we can introduce some kind of measure for a matrix. Let \( A = [a_{ij}] \in \mathbb{C}^{m \times n} \), then the matrix norm induced by a vector \( p \)-norm is defined as

\[ \|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}. \]

In particular, for \( p = 1, 2, \infty \), the corresponding induced matrix norm can be computed as

\[ \|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}| \quad \text{(row sum)}; \]
\[ \|A\|_2 = \sqrt{\rho(A^*A)}; \]
\[ \|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}| \quad \text{(column sum)}. \]

Another often used matrix norm is the so-called Frobenius norm. It is defined as

\[ \|A\|_F = \sqrt{\text{Tr}(A^*A)} = \sqrt{\text{Tr}(AA^*)} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}. \]

We shall adopt the following convention throughout the thesis for vector and matrix norms unless otherwise specified: let \( x \in \mathbb{C}^n \) and \( A \in \mathbb{C}^{m \times n} \), then we shall denote the Euclidean 2-norm of \( x \) simply by

\[ \|x\| = \|x\|_2 \]

and the induced 2-norm of \( A \) by

\[ \|A\| = \|A\|_2. \]

**Lemma A.1.1 [74]** Let \( A \) be a block partitioned matrix with

\[
A = \begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1q} \\
A_{21} & A_{22} & \cdots & A_{2q} \\
\vdots & \vdots & \ddots & \vdots \\
A_{p1} & A_{p2} & \cdots & A_{pq}
\end{bmatrix} =: [A_{ij}]
\]
and each is an appropriately dimensioned matrix. Then for any \( r \)-induced matrix norm
\[
\| A \|_r \leq \left\| \begin{bmatrix} |A_{11}|_r & |A_{12}|_r & \cdots & |A_{1q}|_r \\ |A_{21}|_r & |A_{22}|_r & \cdots & |A_{2q}|_r \\ \vdots & \vdots & \ddots & \vdots \\ |A_{p1}|_r & |A_{p2}|_r & \cdots & |A_{pq}|_r \end{bmatrix} \right\|_r.
\] (A.6)

Further, the inequality becomes equality if Frobenius norm is used.

A.2 Special matrices

A.2.1 Semidefinite matrices

A matrix \( A \) is called symmetric if \( A = A^T \), and Hermitian if \( A = A^* \). All eigenvalues of a Hermitian matrix are real. Therefore, an Eigenvalue Decomposition (EVD) of a Hermitian matrix \( A \) may have the form:
\[
A = UDU^*.
\]

where \( D \) is a real diagonal matrix, and \( U \) is a unitary matrix, i.e., \( U^*U = UU^* = I \). If \( A \in \mathbb{R}^{n \times n} \), \( U \) can be chosen as an orthogonal matrix, i.e., \( U^TU = UU^T = I \).

A Hermitian matrix \( A \) is said to be positive definite (semidefinite) if \( x^*Ax > 0 \) (\( \geq 0 \)) for all \( x \neq 0 \), and is denoted by \( A > 0 \) (\( \geq 0 \)). An important property of positive definite (semidefinite) matrices is that their eigenvalues are all positive (nonnegative). Therefore, a singular value decomposition of a positive definite or positive semidefinite matrix is also its EVD. i.e., the singular values and the eigenvalues coincide, and the singular vectors and the eigenvectors are the same as well. One another property of a positive definite (semidefinite) matrix \( A \) is that it has a unique positive definite (semidefinite) square root \( A^{1/2} \), i.e., \( A^{1/2} > 0 \) (\( A^{1/2} \geq 0 \)) and \( (A^{1/2})^2 = A \).

The following is a typical result for determining the positive semidefiniteness of a matrix with blocks, using a submatrix and its Schur complement. It is remarked that the positive semidefiniteness can be replaced by the positive definiteness.
Lemma A.2.1 [6] Let $A, B, C$ be all real matrices of appropriate sizes, and $A$ and $C$ be symmetric. Then the following two conditions are equivalent:

1) \[
\begin{bmatrix}
A & B \\
B^T & C
\end{bmatrix} \succeq 0:
\]

2) $A \succeq 0$. $C - B^T A^+ B \succeq 0$. $B^T (I - AA^+) = 0$.

where $A^+$ denotes the Moore-Penrose inverse of $A$.

A more specific result on the positive semidefiniteness of a Hermitian matrix is provided as follows.

Lemma A.2.2 Let $A$ be an $m \times n$ matrix. Then the following are equivalent:

1) \[
\begin{bmatrix}
I_m & A \\
A^* & I_n
\end{bmatrix} \succeq 0:
\]

2) $I_m - AA^* \succeq 0$:

3) $I_n - A^* A \succeq 0$:

4) $\|A\|_2 \leq 1$.

A.2.2 Linear structured matrices and matrix approximation

In numerous signal processing and system theory applications, the structural composition of a given class of matrices is of fundamental importance. For example, Hankel matrices, Toeplitz matrices and Vandermonde matrices are important tools in the system theory, especially in realization, system identification and model approximation. Intuitively, the notion of matrix structure suggests that the elements of a matrix are interrelated in some prescribed fashion. A formal definition of what we mean by the concept of matrix structure is now given as follows.

Definition A.2.1 [8] Let $f_i(\theta_1, \theta_2, \ldots, \theta_p)$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ be a given set of functions which depends on the parameters $(\theta_1, \theta_2, \ldots, \theta_p)$ in which $p \leq mn$. 
Furthermore, let the class \( \mathcal{S} \) consist of all \( m \times n \) matrices \( A = [a_{ij}] \) whose components are governed by the functional relationship

\[
a_{ij} = f_{ij}(\theta_1, \theta_2, \cdots, \theta_p) = f_{ij}(\theta) \quad \text{for} \ 1 \leq i \leq m \ \text{and} \ 1 \leq j \leq n \quad (A.7)
\]

for specific choices of the \( p \) parameters \((\theta_1, \theta_2, \cdots, \theta_p)\). The matrix class \( \mathcal{S} \) is said to have a structure induced by the functions \( f_{ij}(\theta) \) and to have \( p \) degrees of freedom. If the functions \( f_{ij}(\theta) \) are all linear, the matrix class \( \mathcal{S} \) is said to have a linear structure.

A number of prominent matrix classes are characterized by a functional relationship of the form \((A.7)\). For example, the class of \( m \times n \) Vandermonde matrices is identified by the nonlinear functions \( f_{ij}(\theta) = \theta_j^{i-1} \) in which the degree of freedom is \( n \). On the other hand, the class of \( m \times n \) Hankel matrices, \( \mathcal{H}^{m \times n} \), and the class of \( m \times n \) Toeplitz matrices, \( \mathcal{T}^{m \times n} \), are both linear, which can be observed from the corresponding functional relationships:

For Hankel matrix class,

\[
f_{ij}(\theta) = \theta_{i+j-1}, \ 1 \leq i \leq m, \ 1 \leq j \leq n;
\]

For Toeplitz matrix class,

\[
f_{ij}(\theta) = \theta_{i-j+n}, \ 1 \leq i \leq m, \ 1 \leq j \leq n.
\]

\[
H = \begin{bmatrix}
\theta_1 & \theta_2 \\
\theta_2 & \theta_3 \\
\theta_3 & \theta_4 \\
\end{bmatrix} \quad \text{Hankel matrix.} \quad T = \begin{bmatrix}
\theta_4 & \theta_1 & \theta_2 & \theta_1 \\
\theta_5 & \theta_4 & \theta_3 & \theta_2 \\
\theta_6 & \theta_5 & \theta_4 & \theta_3 \\
\end{bmatrix} \quad \text{Toeplitz matrix} \quad (A.8)
\]

In a similar fashion, it can be established that the classes of symmetric, Hermitian, diagonal, tri-diagonal, lower/upper triangular, Hessenberg, circulant, block Hankel, block Toeplitz, etc., matrices each have a linear structure.

The linear transformation \( T \) that concatenates the columns of a matrix \( A = [a_1 \ a_2 \ \cdots \ a_n] \) into a long vector \( T(A) = [a_1^T \ a_2^T \ \cdots \ a_n^T]^T \) is called a reordering
operator. This is a one-to-one correspondence between the linear spaces $\mathbb{C}^{m \times n}$ and $\mathbb{C}^{m \times n}$. More importantly, it preserves norms in the sense that

$$||A||_F = ||T(A)||_F.$$  \(\text{(A.9)}\)

**Lemma A.2.3** \[8\] A matrix class $S$ in $\mathbb{C}^{m \times n}$ has linear structure with $p$ degrees of freedom if and only if there exists a matrix $\Gamma \in \mathbb{C}^{m \times p}$ of rank $p$ such that

$$T(S) = \Gamma \mathbb{C}^p = \{\Gamma \theta \in \mathbb{C}^{m \times n} : \theta \in \mathbb{C}^p\}$$  \(\text{(A.10)}\)

where $T$ is the reordering operator defined above.

The matrix $\Gamma$ is called the characteristic matrix of the linear structured matrix class. To illustrate this concept, let us consider the class of $3 \times 2$ Hankel matrices as shown in (A.8). Clearly,

$$T(H) = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \Gamma \theta.$$  \(\text{(A.11)}\)

With the above concepts in mind, a straightforward procedure for determining the matrix contained in a specified linear structured matrix subspace that lies closest to a general matrix is available. Suppose we want to find the best approximant $A$ of an arbitrary matrix $X \in \mathbb{C}^{m \times n}$ in a linear structured matrix class $S$ whose characteristic matrix is $\Gamma \in \mathbb{C}^{m \times p}$. Here the optimality of the approximation is specified for continence in terms of the Frebenius norm:

$$A_{\text{opt}} = \arg \min_{\lambda \in S} ||X - \lambda||_F.$$  \(\text{(A.12)}\)
It follows from (A.9), Lemma 1.2.4 and the standard techniques for least squared problem that

\[ A_{\text{opt}} = \arg \min_{A \in S} ||X - A||_F \]

\[ = \arg \min_{A \in S} ||T(X) - T(A)||_2 \]

\[ = T^{-1}(\Gamma \arg \min_{\theta \in \mathbb{C}^n} ||\Gamma \theta - T(X)||_2) \]

\[ = T^{-1}(\Gamma \Gamma^T T(X)) \]

\[ = T^{-1}(\Gamma [\Gamma^* \Gamma]^{-1} \Gamma^* T(X)). \]

Continue considering the above example, let us find the best Hankel approximant to an arbitrary \(3 \times 2\) matrix \(X = [x_{ij}]\). The characteristic matrix \(\Gamma\) has been computed in (A.11), then

\[ \Gamma [\Gamma^* \Gamma]^{-1} \Gamma^* T(X) = \begin{bmatrix} x_{11} & x_{12} + x_{21} \frac{1}{2} & x_{22} + x_{31} \frac{1}{2} & x_{12} + x_{21} \frac{1}{2} & x_{22} + x_{31} \frac{1}{2} & x_{32} \end{bmatrix}^T. \]

Therefore, the closed Hankel matrix to \(X\) in the Frobenius norm sense is given by

\[ H_{\text{opt}} = \begin{bmatrix} x_{11} & \frac{x_{12} + x_{21}}{2} & \frac{x_{22} + x_{31}}{2} & \frac{x_{12} + x_{21}}{2} & \frac{x_{22} + x_{31}}{2} & x_{32} \end{bmatrix}. \]

Compared to the original matrix \(X\), the approximant \(H_{\text{opt}}\) is obtained by averaging the elements on the same anti-diagonal of \(X\). Actually, this is a generic observation:

**Theorem A.2.1** The optimal Hankel matrix approximant of an arbitrary matrix \(X\) in the Frobenius norm sense is obtained by averaging the elements on the same anti-diagonal of \(X\).

### A.3 Singular value decomposition

The singular value decomposition (SVD) is one of the most important tools in modern numerical linear algebra and numerical analysis. Owing to the linear algebraic
nature of many control problems and signal processing problems, the singular value decomposition has found its way into control and system theory, and digital signal processing and communication theory. The aim of this section is to introduce the singular value decomposition and to examine some of the properties of singular values.

A.3.1 Existence of the singular value decomposition

For any \( m \times n \) \((m \leq n)\) matrix \( A \), there exist \( m \times m \) and \( n \times n \) unitary matrices \( U \) and \( V \), and a real matrix \( \Sigma \), such that

\[
A = U [\Sigma \ 0] V^*.
\]  \hspace{1cm} (A.13)

in which \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_m) \) with \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0 \). When \( A \) is real, \( U \) and \( V \) may be chosen orthogonal.

Expression (A.13) is called a singular value decomposition (SVD) of \( A \). All the nonnegative square roots of the eigenvalues of \( AA^* \):

\[
\sigma_1, \sigma_2, \ldots, \sigma_m \geq 0.
\]

are called the singular values of \( A \). The columns of \( U \) and \( V \).

\[
U = [u_1 \ u_2 \ \cdots \ u_m], \ V = [v_1 \ v_2 \ \cdots \ v_n]
\]  \hspace{1cm} (A.14)

are called left and right singular vectors. These singular vectors are the eigenvectors of \( AA^* \) and \( A^*A \).

Since the smallest singular value and the second smallest singular value are of great importance in our analysis, we sometimes use special notations \( \sigma_1 \) and \( \sigma_2 \) to denote them (this should be easily distinguished from the context). Occasionally, \( \sigma_{\text{min}} \) is also used to represent the smallest singular value.

A.3.2 Several properties

(1) The rank of \( A \) is identical to the number of its nonzero singular values.
(2) SVD can be written equivalently in the form of the dyadic expansion:

\[ A = \sigma_1 u_1 v_1^* + \sigma_2 u_2 v_2^* + \cdots + \sigma_m u_m v_m^*. \tag{A.15} \]

(3) The maximum singular value \( \sigma \) and the minimum singular value \( \bar{\sigma} \) are closely related to the matrix norms:

\[
\sigma = \max_{||x||_2=1} ||Ax||_2 = ||A||_2.
\]

\[
\bar{\sigma} = \min_{||x||_2=1} ||Ax||_2 = \text{Tr}(A^*A)^{-1/2} = \sqrt{\sigma_1^2 + \cdots + \sigma_m^2}.
\]

(4) (Schmidt-Mirsky.) With the notation introduced in last subsection

\[
\min_{X, \text{ rank}(X) = k} ||A - X||_2 = \sigma_{k+1}(A). \tag{A.16}
\]

A (non-unique) minimizer \( X_* \) is obtained by truncating the dyadic expansion (A.15) to contain the first \( k \) terms:

\[ X_* = \sigma_1 u_1 v_1^* + \sigma_2 u_2 v_2^* + \cdots + \sigma_k u_k v_k^*. \tag{A.17} \]

### A.3.3 Singular value inequalities

Singular value inequalities are usually useful when estimating singular values in terms of some other known singular values.

**Theorem A.3.1** [27] Denote respectively by \( \sigma \) and \( \bar{\sigma} \) the maximum singular value and the minimum singular value of a matrix. For the matrices \( A, B \) of appropriate sizes, we have

\[
\sigma(A) - \sigma(B) \leq \sigma(A + B) \leq \sigma(A) + \sigma(B) \tag{A.18}
\]

\[
\bar{\sigma}(A) - \bar{\sigma}(B) \leq \bar{\sigma}(A + B) \leq \bar{\sigma}(A) + \bar{\sigma}(B) \tag{A.19}
\]

\[
\sigma(A)\bar{\sigma}(B) \leq \bar{\sigma}(AB) \leq \sigma(A)\bar{\sigma}(B) \tag{A.20}
\]

\[
\bar{\sigma}(A)\sigma(B) \leq \sigma(AB) \leq \bar{\sigma}(A)\sigma(B). \tag{A.21}
\]
Theorem A.3.2 [52] Let \( A = [A_1^T \ A_2^T]^T \) be an \( m \times n \) matrix with singular values \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \), and \( A_1 \in M^{(m-k)\times n}(C) \) be its submatrix with singular values \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{m-k} \). Then

\[
\sigma_i \geq \mu_i \geq \sigma_{i+k}, \quad i = 1, 2, \ldots, n - k. \tag{A.22}
\]

A.3.4 Singular value perturbation

SVD is usually computed from a matrix consisting of both signal and noise. It is therefore important to be able to assess the effects of the noise on the singular values, which is a typical problem in classical perturbation theory. There are two ways [52] of considering the perturbation of the singular values: by exhibiting either a perturbation bound or a perturbation expansion. A perturbation bound gives an upper bound on the difference between the perturbed quantity and its original in terms of a norm of the perturbation. A perturbation expansion seeks to approximate the singular value as a function of the perturbation. Perturbation bounds are ideal when one has a crude bound on the error, while perturbation expansions are most useful when one tries to know the asymptotic property of the singular values.

Theorem A.3.3 (Perturbation Bound. Mirsky). Let \( \hat{A} = A + E \) be a perturbed matrix of \( A \in M^{m \times n}(C) \). Denote by

\[
\hat{\sigma}_1 \geq \hat{\sigma}_2 \geq \cdots \geq \hat{\sigma}_m \\
\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m
\]

the singular values of \( \hat{A} \) and \( A \). Then

\[
|\hat{\sigma}_i - \sigma_i| \leq ||E||_2 \quad i = 1, 2, \ldots, m. \tag{A.23}
\]

It is remarked that in terms of the theorem the SVD of a matrix is well-conditioned with respect to perturbations of its entries.
Theorem A.3.4 (Second Order Expansion. Stewart). Let \( A \) be a \( m \times n \) (\( m \leq n \)) matrix with singular value decomposition

\[
A = U [\Sigma \ 0] V^T.
\]

Partition \( U \), \( V \) and \( \Sigma \) conformally

\[
U = [u_1 \ u_2], \quad \Sigma = \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix}, \quad V = [v_1 \ v_2 \ v_3].
\]

where \( \sigma_1 \) is the smallest singular value of \( A \) and \( u_1, v_1 \) are the corresponding left, right singular vectors, and \( V_3 \) is of size \( n \times (n - m) \). Let \( \hat{\sigma}_1 \) be the smallest singular value of a perturbation \( \hat{A} = A + E \) of \( A \). Then as \( E \to 0 \), there exists the following second order perturbation expansion for \( \hat{\sigma}_1 \):

\[
(1) \quad \hat{\sigma}_1 = \sigma_1 + u_1^T E v_1 + O(||E||^2). \tag{A.24}
\]

provided that \( \sigma_1 \) is large compared with \( E \); and

\[
(2) \quad \hat{\sigma}_1^2 = (u_1^T E v_1)^2 + ||u_1^T E v_3||^2 + O(||E||^3). \tag{A.25}
\]

provided that \( \sigma_1 \) is zero.

A.4 Ellipsoid and its measures

In set membership identification theory one may approximate the membership set by some simple-shaped set. Ellipsoid is a kind of frequently-chosen set due to its simple descriptions and its good approximation property. In this section the different types of descriptions of ellipsoids are reviewed and three measures of the ellipsoid size are introduced.
A.4.1 Descriptions of ellipsoids

We describe an ellipsoid $\mathcal{E}$ in two different ways. The first description uses convex quadratic functions:

$$T(x) = x^T A x + 2b^T x + c. \tag{A.26}$$

where $A$ is a real symmetric matrix, $b, x$ are vectors and $c$ is a constant. A matrix version of this function is given by:

$$T(x) = [x^T \ 1] G \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad \text{where } G = \begin{bmatrix} A & b \\ b^T & c \end{bmatrix}.$$ 

Lemma A.4.1 [6] The following two conditions are equivalent:

(i) $\mathcal{E}$ is an ellipsoid:

(ii) there exists a convex quadratic function $T(x)$ as (A.26) with $A > 0$ and $c - b^T A^{-1} b < 0$ such that

$$\mathcal{E} = \{ x : T(x) \leq 0 \}. \tag{A.27}$$

Note that the description (A.27) is homogeneous, i.e., we can scale $A, b$ and $c$ by any positive factor without affecting $\mathcal{E}$. Therefore, we may sometimes enforce the condition $c - b^T A^{-1} b = -1$ when describing an ellipsoid. Additionally, we use the condition $c - b^T A^{-1} b < 0$ instead of the condition with a non-strict inequality to ensure that the ellipsoid is not degenerate, i.e., it is not a single point.

We also describe an ellipsoid as the image of the unit ball under an affine mapping with positive definite matrix:

Lemma A.4.2 [6] The following two conditions are equivalent:

(i) $\mathcal{E}$ is an ellipsoid:

(ii) there exist a unique positive definite matrix $P$ and a unique vector $x_c$ such that

$$\mathcal{E} = \{ Pz + x_c : \|z\| \leq 1 \}. \tag{A.28}$$
or equivalently,
\[ \mathcal{E} = \{ x : (x - x_c)^T P^{-2}(x - x_c) \leq 1 \}. \]

Note that the vector \( x_c \) is the Chebyshev center of the ellipsoid, i.e.,
\[ x_c = \arg \min_{x \in \mathcal{E}} \max_{y \in \mathcal{E}} \| x - y \|. \]

whereas the positive matrix \( P \) determines the range of the ellipsoid \( \mathcal{E} \).

Each of the two descriptions of \( \mathcal{E} \) is readily converted into the other. Given the quadratic description (A.27) we may form the representation (A.28) of \( \mathcal{E} \) with
\[ P = \sqrt{b^T A^{-1} b - c} A^{-1/2} \quad x_c = -A^{-1} b. \]

Given the affine mapping description (A.28) we may construct a representation of the form (A.27) of \( \mathcal{E} \) with
\[ A = P^{-2} \quad b = -P^{-2} x_c \quad c = x_c^T P^{-2} x_c - 1. \]

### A.4.2 Measures of ellipsoids

Different measures are defined in the literatures for the size of the ellipses. We usually use the volume, the radius and the mean square root radius to measure the size of the ellipsoids.

Suppose an ellipsoid \( \mathcal{E} \) has a representation as (A.28). Without some constant factors the essential part of the three abovementioned measures are all determined by the eigenvalues of the positive definite matrix \( P \in \mathbb{R}^n \):
\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n. \]

The three measures are defined as follows:

**Volume:**
\[ V(\mathcal{E}) := \prod_{i=1}^{n} \lambda_i = \det(P); \]
Radius:

\[ R(\mathcal{E}) := \lambda_1 = \rho(P) : \]

Mean square root radius:

\[ M(\mathcal{E}) := \sqrt{\sum_{i=1}^{n} \lambda_i^2} = \sqrt{\text{Tr}(P^2)}. \]

It is remarked that the trace of the matrix \( P \) is sometimes used as a measure of the ellipsoid as well. In addition, for the ellipsoid \( \mathcal{E} \) with representation (A.27), the three measures are given as follows:

\[ V(\mathcal{E}) = \sqrt{\det((b^T A^{-1} b - c) A^{-1})} : \]

\[ R(\mathcal{E}) = \sqrt{\rho((b^T A^{-1} b - c) A^{-1})} : \]

\[ M(\mathcal{E}) = \sqrt{\text{Tr}((b^T A^{-1} b - c) A^{-1})}. \]

A.4.3 An example

Consider the ellipse governed by the inequality

\[ \frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} \leq 1 \]

where \( a, b, x_0, y_0 \in \mathbb{R} \) and \( a > 0, b > 0 \). Written in the quadratic inequality form (A.27), the representation is

\[
\begin{bmatrix}
    x \\
    y \\
    1
\end{bmatrix}
\begin{bmatrix}
    \frac{1}{a^2} & 0 & -\frac{x_0}{a^2} \\
    0 & \frac{1}{b^2} & -\frac{y_0}{b^2} \\
    -\frac{x_0}{a^2} & -\frac{y_0}{b^2} & \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - 1
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    1
\end{bmatrix} \leq 0.
\]

A straightforward transformation of coordinates leads to the affine mapping representation:

\[
\begin{bmatrix}
    x \\
    y
\end{bmatrix} =
\begin{bmatrix}
    a \\
    b
\end{bmatrix}
\begin{bmatrix}
    \rho \cos(\theta) \\
    \rho \sin(\theta)
\end{bmatrix} + \begin{bmatrix}
    x_0 \\
    y_0
\end{bmatrix}, \quad |\rho| \leq 1, \quad \theta \in [0, 2\pi).
\]

According to our definitions, the three measures of the ellipse are respectively \( ab \), \( \max(a, b) \) and \( \sqrt{a^2 + b^2} \).
A.5 Convex optimization with LMI constraints

A.5.1 Convex optimization problems

$S \subseteq \mathbb{R}^n$ is a convex set if

$$\lambda x + (1 - \lambda)y \in S \quad \text{for any } x, y \in S, \quad \lambda \in [0, 1].$$

and $S$ is called a convex cone if

$$\lambda x + \mu y \in S \quad \text{for any } x, y \in S, \quad \lambda \geq 0, \mu \geq 0.$$

A real-valued function $f$ defined on a convex set $C$ is called convex if the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

holds for any $x, y \in S, \lambda \in [0, 1]$. The function $f$ is called quasi-convex if for every $\alpha \in \mathbb{R}$ the set $S_{\alpha} = \{x : f(x) \leq \alpha\}$ is convex. Obviously, a convex function is quasi-convex.

Many problems in engineering analysis and design can be cast as convex optimization problems[7], i.e.,

$$\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in C
\end{align*}$$

where $f$ is a convex function over the convex set $C \subseteq \mathbb{R}$. It is widely known that such problems can be solved numerically with great efficiency. Furthermore, convex optimization problems are, roughly speaking, fundamentally tractable, both in theory and in practice.

A.5.2 Linear matrix inequalities

A linear matrix inequality (LMI) has the form

$$F(x) = F_0 + x_1 F_1 + \cdots + x_m F_m \geq 0. \quad (A.29)$$
where \( x = [x_1] \in \mathbb{R}^n \) is the variable and the symmetric matrices \( F_i \in \mathbb{R}^{n \times n} \), \( i = 0, \ldots, m \) are given. The inequality sign in \((A.29)\) means that \( F(x) \) is positive semidefinite. The LMI \((A.29)\) is a convex constraint on \( x \), i.e., the set \( \{ x : F(x) \geq 0 \} \) is convex. Although the LMI \((A.29)\) may seem to have a specialized form, it can represent a wide variety of convex constraints on \( x \). In particular, linear inequalities, quadratic inequalities, matrix norm inequalities, eigenvalue inequalities, and many types of constraints that arise in control theory, such as Lyapunov inequalities, can all be cast in the form of an LMI. Nonlinear (convex) inequalities can usually be converted into LMI form by using some standard techniques such as Schur complements, the \( S \)-procedure, etc.

Throughout this thesis we will encounter the following three types of convex optimization problems with LMI constraints \([6, 39]\). For a positive definite matrix variable \( X \), the convex functions of matrices such as the trace, \( \text{Tr}(X) \), the largest eigenvalue, \( \rho(X) \), and the log-determinant of its inverse, \( \log \det X^{-1} \), are frequently-used objective functions of optimization problems with LMI constraints. When the objective function is a linear function of the variables as follows.

\[
\min c^T x \quad \text{(A.30)}
\]

\[
s.t. \quad F(x) = F_0 + x_1 F_1 + \cdots + x_m F_m \geq 0
\]

the problem is called a positive definite program or PDP. Similarly, the following convex optimization problem with LMI constraints

\[
\text{minimize} \quad c^T x + \log \det G(x)^{-1} \quad \text{(A.31)}
\]

\[
\text{subject to} \quad G(x) > 0
\]

\[
F(x) \geq 0. \quad \text{(A.32)}
\]

is referred to as a maxdet-program. If \( x \in \mathbb{R} \) satisfies a specified LMI, we say \( x \in \mathbb{R}^n \) is feasible with respect to the LMI. The feasibility problem of an LMI is one of examining the existence of the feasible points. It can be regarded as a convex optimization problem with a constant objective function.
Most of the convex optimization problems encountered in engineering can be written in one of the above three types of problems. More importantly, these problems can be solved very efficiently using interior-point methods, which were first introduced for linear programming by Karmarkar and then generalized by Nesterov and Nemirovsky [39] to general convex programming.

A.5.3 (Quasi-)convex optimization via feasibility

Let \( C \in \mathbb{R}^n \) be the set of all feasibility points of LMI constraints of the convex or quasi-convex optimization problem

\[
\min f(x) \\
s.t. \quad F(x) = F_0 + x_1 F_1 + \cdots + x_m F_m \geq 0.
\]

and \( f^* \) the minimal value of \( f(x) \) over the set \( C \). The optimization problem can usually be solved via solving the feasibility problems of LMI, in the view of the observation that for any \( \gamma > 0 \), \( f^* \leq \gamma \) holds if and only if the condition

\[
f(x) \leq \gamma, \quad x \in C
\]

is feasible. Virtually, the optimal value \( f^* \) can be computed by using bisection in \( \gamma \). By initially choosing a upper bound \( u \), a lower bound \( l \) for \( f^* \), and a tolerance limit \( \varepsilon \) for the approximation, the procedure goes as follows [7]:

While \((u - l > \varepsilon)\) {

\[
\gamma := (u + l)/2
\]

if \( x \in C \), \( f(x) \leq \gamma \) feasible, \( u := \gamma \)

else \( l := \gamma \)
}
A.5.4 Ellipsoid approximation

The problem of approximating some set $S \subseteq \mathbb{R}^n$ with an ellipsoid in some optimal sense arises in many fields. It is a typical problem that can be cast into a convex optimization problem with LMI constraints, and thus can be attacked with an optimal or a suboptimal solution. For the interest of this thesis we will introduce the problem of optimal ellipsoid bounding of the intersection of ellipsoids. The global optimal solution to this problem is very hard to find, thus we will just introduce a method of computing a suboptimal ellipsoid by solving a maxdet-problem.

Let the quadratic functions $T_i(x) = x^T A_i x + 2b_i^T x + c_i$ be associated with the ellipsoids $E_i$, $i = 1, \ldots, p$. Let the quadratic function $T_0(x) = x^T A_0 x + 2b_0^T x + c_0$ be associated with the ellipsoid $E_0$ we are looking for, i.e.,

$$E_0 \supseteq \bigcap_{i=1}^p E_i.$$  \hfill (A.33)

We can normalize the representation of $E_0$ so that $b_0^T A_0^{-1} b_0 - c_0 = 1$ holds. Therefore, its volume is given by $V(E_0) = \sqrt{\det A_0^{-1}}$.

From the $S$-procedure [6] we obtain the following sufficient condition for (A.33): there exist positive scalars $\tau_1, \ldots, \tau_p$ such that

$$\begin{bmatrix} A_0 & b_0 & 0 \\ b_0^T & -1 & b_0^T \\ 0 & b_0 & -A_0 \end{bmatrix} - \sum_{i=1}^p \begin{bmatrix} A_i & b_i & 0 \\ b_i^T & -1 & b_i^T \\ 0 & b_i & -A_i \end{bmatrix} \leq 0.$$ \hfill (A.34)

Based on this sufficient condition, we can compute a suboptimal ellipsoid with the (relatively) smallest volume, by solving the following maxdet-problem:

$$\begin{align*}
\text{minimize} & \quad \log \det A_0^{-1} \\
\text{subject to} & \quad A_0 > 0, \quad \tau_1 \geq 0, \ldots, \tau_p \geq 0. \quad (A.35)
\end{align*}$$
Appendix B

Error bounded identification

In this appendix we introduce a method for the error bounded identification problem, i.e., the identification problem where the observed data and records are corrupted by the unknown noise with known bounds. We formulate the problem of computing the optimal outer ellipsoidal approximation of the uncertain parameter set as a convex optimization problem with Linear Matrix Inequalities (LMI) constraints. This makes it possible to compute the optimal approximations with respect to various measures using various efficient convex optimization techniques.

B.1 Introduction

In modern robust control, the starting point for control system analysis and design is a nominal plant and (norm) bounds on model uncertainty. As a result, system identification techniques should be required to provide guaranteed error bounds in addition to a candidate system model, or more generally a set of systems as the model. For this purpose, it seems more appropriate to assume that the observed data and records are corrupted by the unknown noise with known bounds. This approach was first suggested by Witsenhausen [72] and Schewepe [46] for the state estimation problem. Currently, this approach is known under the generic name of error bounded identification, which can be cast into a more general category of identification theory called set membership identification. The aim of set membership identification theory is to characterize the membership set of the parameters space which is consistent with the data, the model structure and the error bounds. For the models which are linear in their parameters, several techniques [19, 20, 36, 13, 43, 5] have been presented to parameterize the membership set. An exact parameterized expression has been ob-
tained in [65], where the membership set is a bounded or unbounded polyhedron and is determined by recursively computing its vertices. For the purpose of computational simplicity, one may approximate the membership set by some simple-shaped set, such as ellipsoid [19, 20, 13, 43], polytope[36], orthotope, parallelootope or rectangular box. See [40] for a survey on this topic.

In this chapter the problem of computing the optimal ellipsoid bounding of the membership set by using Linear Matrix Inequalities (LMI) is formulated. This makes it possible to compute the optimal approximations with respect to various measures using various efficient convex optimization techniques.

B.2 Basic formulations for ellipsoid approximation

Assume that the system compatible with the input-output data

\[ u_t \in \mathbb{R}, \quad y_t \in \mathbb{R}, \quad t = 0, 1, \ldots, N - 1 \]

and the bounded noise

\[ \omega_t \in \mathbb{R}, \quad |\omega_t| \leq z, \quad t = 0, 1, \ldots, N - 1 \]

can be described by the pair of polynomials

\[ p(z) = \sum_{i=0}^{m} p_i z^i, \quad q(z) = \sum_{i=0}^{n} q_i z^i. \]

with \( p_i, q_i \in \mathbb{R}, \quad q_n = 1, \) and \( n \geq m. \) Namely,

\[ y_{t+n} = -\sum_{k=0}^{n-1} q_k y_{i+k} + \sum_{k=0}^{m} p_k u_{i+k} + w_{i+n}, \quad i = 0, 1, \ldots, N - n - 1. \]

Denote \( r = m + n + 1. \) and

\[ \theta = [p_0 \ p_1 \ \ldots \ p_m \quad -q_0 \quad -q_1 \ \ldots \ -q_{n-1}]^T \in \mathbb{R}^r. \]

\[ X_{i+n} = [u_i \ u_{i+1} \ \ldots \ u_{i+m} \ y_i \ y_{i+1} \ \ldots \ y_{i+n-1}]^T \in \mathbb{R}^r. \]

Then

\[ y_k = X_k^T \theta + \omega_k, \quad k = n, n + 1, \ldots, N - 1. \]
Therefore, the exact membership set is
\[
\Omega = \{ \theta \in \mathbb{R}^r : |y_k - X_k^T \theta| \leq \varepsilon, \ k = n, n + 1, \ldots, N - 1 \} = \bigcap_{k=n}^{N-1} \Omega_k
\]  
\hspace{1cm} \text{(B.1)}

where
\[
\Omega_k = \{ \theta \in \mathbb{R}^r : |y_k - X_k^T \theta| \leq \varepsilon \}.
\]

It is well known that the simplest ellipsoid containing the set $\Omega$ is given by
\[
E_\varepsilon = \left\{ \theta \in \mathbb{R}^r : \sum_{k=n}^{N-1} q_k(y_k - X_k^T \theta)^2 \leq \varepsilon^2 \sum_{k=n}^{N-1} q_k \right\}
\]
\hspace{1cm} \text{(B.2)}

where $q_k \geq 0$. Computing the optimal ellipsoid outer-approximation of this type amounts to selecting the nonnegative numbers $q_k$'s. In [20] a recursive algorithm, called Optimal Bounding Ellipsoid (OBE) algorithm, is developed for computing the optimal ellipsoid outer-approximation of the type (B.2) with respect to two different measures. Another recursive algorithm, called Optimal Volume Ellipsoid (OVE) algorithm, is derived in [13] to find a minimal volume ellipsoid bounding the membership set based on an ellipsoid algorithm for solving the linear programming problem.

The problem can be more compactly formulated in matrix form. Denote
\[
y = [y_n \ y_{n+1} \ldots \ y_{N-1}]^T, \ X = [X_n \ X_{n+1} \ldots \ X_{N-1}], \ \omega = [\omega_n \ \omega_{n+1} \ldots \ \omega_{N-1}]^T.
\]

For any nonnegative numbers $q_k$, $k = n, n + 1, \ldots, N - 1$, define the diagonal matrix $Q = \text{diag}(q_k)$, and vector norm
\[
|x|_q = \sqrt{x^T Q x}, \quad x \in \mathbb{R}^{N-n}.
\]

Then for any $\theta \in \mathbb{R}^r$ in the membership set (B.1), we have
\[
y = X^T \theta + \omega.
\]

and then
\[
\|y - X^T \theta\|_q = \|\omega\|_q.
\]

\[(y - X^T \theta)^T Q (y - X^T \theta) = \omega^T Q \omega \leq \varepsilon^2 \varepsilon^T Q \varepsilon.
\]

\[\theta^T (XQX^T) \theta - 2(y^T X^T \theta + y^T Q y - \varepsilon^2 \varepsilon^T Q \varepsilon) \leq 0.
\]

\[\theta^T A_\eta \theta + 2b_\eta^T \theta + c_\eta \leq 0
\]
where $A_\eta = XQX^T$, $b_\eta = -XQy$, $c_\eta = y^TQy - \varepsilon^2e^TQe$, and $e = [1 \ 1 \ \ldots \ 1]^T$.

Therefore, the ellipsoid

$$\mathcal{E}_\eta = \{ \theta \in \mathbb{R}^r : \theta^T A_\eta \theta + 2b_\eta \theta + c_\eta \leq 0 \} \quad \text{(B.3)}$$

contains the membership set $\Omega$, and the optimal outer-approximations aims to minimizing the volume $V(\mathcal{E}_\eta)$, the radius $R(\mathcal{E}_\eta)$, or the mean square root radius $M(\mathcal{E}_\eta)$ (recall these measures are introduced in Chapter 1):

$$V(\mathcal{E}_\eta) = \sqrt{\det((b_\eta^TA_\eta^{-1}b_\eta - c_\eta)A_\eta^{-1})},$$

$$R(\mathcal{E}_\eta) = \sqrt{\rho((b_\eta^TA_\eta^{-1}b_\eta - c_\eta)A_\eta^{-1})},$$

$$M(\mathcal{E}_\eta) = \sqrt{\text{Tr}(((b_\eta^TA_\eta^{-1}b_\eta - c_\eta)A_\eta^{-1})).$$

The optimization problem is well formulated in this way, however, it is intractable due to the nonlinearity of the objective functions (the measures). In next section, equivalent formulations are given by using LMI, so that the optimization problem can be efficiently solved by various convex algorithms.

### B.3 LMI formulation

The following theorem is important for us to formulate the problem by LMI. It is based on the so-called S-procedure [6].

**Lemma B.3.1** Let $\mathcal{E}_i = \{ \theta \in \mathbb{R}^r : \theta^T A_i \theta + 2b_i^T \theta + c_i \leq 0 \}$, $i = 1, 2$, be two ellipsoids. Then $\mathcal{E}_1 \subseteq \mathcal{E}_2$ if and only if there exists a nonnegative scalar $q$ such that

$$\begin{bmatrix}
A_2 & b_2 \\
b_2^T & c_2
\end{bmatrix} - q
\begin{bmatrix}
A_1 & b_1 \\
b_1^T & c_1
\end{bmatrix} \leq 0.$$

According to this theorem, any ellipsoid $\mathcal{E} = \{ \theta \in \mathbb{R}^r : \theta^T A \theta + 2b^T \theta + c \leq 0 \}$ containing an ellipsoid $\mathcal{E}_\eta$ as (B.3) satisfies

$$\begin{bmatrix}
A & b \\
b^T & c
\end{bmatrix} \leq q
\begin{bmatrix}
A_\eta & b_\eta \\
b_\eta^T & c_\eta
\end{bmatrix} = \sum_{k=n}^{N-1} q q_k
\begin{bmatrix}
A_k & b_k \\
b_k^T & c_k
\end{bmatrix} \quad \text{(B.4)}$$
where \( q \) is some nonnegative scalar, and

\[
A_k = X_k X_k^T, \quad b_k = -X_k y_k, \quad c_k = y_k^T y_k - \varepsilon^2.
\]

Since the representation of \( \mathcal{E} \) is homogeneous, the matrices \( A, b, c \) can be normalized in a convenient way such that \( b^T A^{-1} b - c = 1 \). In other words, we set

\[
c = b^T A^{-1} b - 1
\]

and parameterize \( \mathcal{E} \) by \( A \) and \( b \) alone. Thus the condition (B.4) becomes

\[
\begin{bmatrix}
A & b \\
b^T & b^T A^{-1} b - 1
\end{bmatrix} - \sum_{k=n}^{N-1} \tau_k \begin{bmatrix}
A_k & b_k \\
b_k^T & c_k
\end{bmatrix} \leq 0
\]

(B.6)

where \( \tau_k = q q_k \). Using Schur complements, we obtain the equivalent LMI

\[
\begin{bmatrix}
A & b & 0 \\
b^T & b^T A^{-1} b - 1 & 0 \\
0 & b & -A
\end{bmatrix} - \sum_{k=n}^{N-1} \tau_k \begin{bmatrix}
A_k & b_k & 0 \\
b_k^T & c_k & 0 \\
0 & 0 & 0
\end{bmatrix} \leq 0.
\]

(B.7)

with variables \( A, b, \) and \( \tau_n, \ldots, \tau_{N-1} \).

Obviously, the family of ellipsoid \( \mathcal{E} = \{ \theta \in \mathbb{R}^r : \theta^T A \theta + 2 b^T \theta + c \leq 0 \} \) satisfying the condition (B.7) contains the family

\[
\Omega_q = \{ \mathcal{E}_q ; q = [q_1, q_2, \ldots, q_{N-n}] \in \mathbb{R}^{N-n}, q_j \geq 0 \}.
\]

(B.8)

which includes all ellipsoids of the form (B.3). Therefore, the optimal ellipsoids over the two ellipsoid families are identical. This means that the original optimization problem can be equivalently replaced by the optimization problems with LMI constraints (B.7).

Next we will deal with the objective functions. With the normalization (B.5), the volume of \( \mathcal{E} \) is: \( V(\mathcal{E}) = \sqrt{\det A^{-1}} \). the radius of \( \mathcal{E} \) is: \( R(\mathcal{E}) = \sqrt{\rho(A^{-1})} \). and the mean square root radius of \( \mathcal{E} \) is: \( M(\mathcal{E}) = \sqrt{\text{Tr} A^{-1}} \). Thus the original minimum volume
ellipsoid problem is equivalent to the following MAXDET (maximizing determinant) problem (with variables $A$, $b$, and $\tau_n, \ldots, \tau_{N-1}$)

\[
\begin{align*}
\text{minimize} & \quad \log \det A^{-1} \\
\text{subject to} & \quad A > 0, \quad \tau_n \geq 0, \ldots, \tau_{N-1} \geq 0 \quad \text{and} \quad (B.7) \quad (B.9)
\end{align*}
\]

It is readily to see that the minimum radius ellipsoid problem is equivalent to the following EVP (eigenvalue minimalization problem with variables $A$, $b$, and $\lambda, \tau_n, \ldots, \tau_{N-1}$)

\[
\begin{align*}
\text{maximize} & \quad \lambda \\
\text{subject to} & \quad A > \lambda I, \quad \lambda > 0, \quad \tau_n \geq 0, \ldots, \tau_{N-1} \geq 0 \quad \text{and} \quad (B.7) \quad (B.11)
\end{align*}
\]

For the minimum mean radius ellipsoid problem, a little effort is needed to express the objective function $M(E)$ as a linear function of the variables with a new LMI constraint. Let us introduce an extra symmetric matrix variable $T = (t_{ij})$, satisfying the constraint

\[
A^{-1} < T. \quad (B.13)
\]

It can be easily written as an LMI (with variables $A$ and $T$)

\[
\begin{bmatrix} A & I \\ I & T \end{bmatrix} > 0. \quad (B.14)
\]

Since $A$ and $T$ are both positive definite matrices, the inequality (B.13) implies

\[
\text{Tr} A^{-1} < \text{Tr} T.
\]

Therefore, minimizing the mean square root radius of $E$ is equivalent to minimizing $t_{11} + \cdots + t_{rr} = \text{Tr} (T)$. The minimum mean radius ellipsoid problem is thus formulated as follows

\[
\begin{align*}
\text{minimize} & \quad \text{Tr} (T) \\
\text{subject to} & \quad A > 0, \quad \tau_n \geq 0, \ldots, \tau_{N-1} \geq 0 \quad \text{and} \quad (B.7) \quad \text{and} \quad (B.14). \quad (B.15)
\end{align*}
\]
Figure B.1 An example of the minimum volume ellipsoid bounding of the membership set determined by the parallel hyperplanes. In 2-dimension case an ellipse with minimum area is obtained containing the region determined by the parallel lines.

B.4 Conclusions

In this appendix we introduce a method for the error bounded identification problem, i.e., the identification problem where the observed data and records are corrupted by the unknown noise with known bounds. We formulate the problem of computing the optimal outer ellipsoidal approximation of the uncertain parameter set as a convex optimization problem with Linear Matrix Inequalities (LMI) constraints. This makes it possible to compute the optimal approximations with respect to various measures using various efficient convex optimization techniques. In Figure B.1, we show that the minimum area (volume) ellipse is obtained using the algorithm developed in this appendix. Notice that in 2-dimension case the bounded error parameters determine a band region bounded with two parallel lines.
Appendix C

Proofs

C.1 Proofs in Chapter 2

C.1.1 Equivalence of $\mathcal{F}_2$ and $\mathcal{F}_\perp$

$\mathcal{F}_2 \supseteq \mathcal{F}_\perp$: Suppose there exist matrices $\hat{\mathcal{M}}$ and $\mathcal{M}$ such that $\hat{\mathcal{M}} = \mathcal{M} + \hat{\mathcal{M}}$. $\langle \mathcal{M}, \hat{\mathcal{M}} \rangle = 0$ and $||\hat{\mathcal{M}}||_2 \leq \varepsilon$. Then it follows from the orthogonality that

$$(\hat{\mathcal{M}}y)^T \mathcal{M} = y^T \hat{\mathcal{M}}^T \mathcal{M} = y^T \langle \mathcal{M}, \hat{\mathcal{M}} \rangle = 0,$$

holds for any $y \in \mathbb{R}^m$. This means $\mathcal{F}_2 \supseteq \mathcal{F}_\perp$.

$\mathcal{F}_2 \subseteq \mathcal{F}_\perp$: For any $x \in \mathcal{F}_2$ there exists $\mathcal{M}_1$ such that $x^T \mathcal{M}_1 = 0$ and $||\hat{\mathcal{M}} - \mathcal{M}_1||_2 \leq \varepsilon$. Denote by $S$ the subspace spanned by the columns of the matrix $\mathcal{M}_1$, and by $S^\perp$ the orthogonal complement subspace of $S$. Let $\hat{\mathcal{M}}_1$ be any matrix whose columns form an orthonormal basis of $S^\perp$. Clearly, any basis of $S^\perp$ can be extended to a basis of the whole space by including any basis of $S$, therefore, the following decomposition

$$\hat{\mathcal{M}} = \mathcal{M}_1 P + \hat{\mathcal{M}}_1 Q$$

holds for two suitable square matrices $P$ and $Q$. Denote $\mathcal{M} = \mathcal{M}_1 P$ and $\hat{\mathcal{M}} = \hat{\mathcal{M}}_1 Q$. Clearly, $\langle \hat{\mathcal{M}}, \mathcal{M} \rangle = (\mathcal{M}_1 P)^T (\hat{\mathcal{M}}_1 Q) = P^T (\mathcal{M}_1^T \hat{\mathcal{M}}_1) Q = 0$. Since $\hat{\mathcal{M}}$ has full row rank, this means the columns of matrix $\hat{\mathcal{M}}$ can form a basis for the left kernel of the matrix $\mathcal{M}$. Therefore, $x$ can be expressed as $x = \hat{\mathcal{M}} y$ for some vector $y$. The remaining condition on the norm of the difference is verified as follows:

$$||\hat{\mathcal{M}}||_2 \leq \sqrt{||\mathcal{M}_1(P - I)||_2^2 + ||\hat{\mathcal{M}}||_2^2} = ||\mathcal{M}_1(P - I) + \hat{\mathcal{M}}||_2 = ||\mathcal{M} + \hat{\mathcal{M}} - \mathcal{M}_1||_2 = ||\hat{\mathcal{M}} - \mathcal{M}_1||_2 \leq \varepsilon.$$
This concludes the proof.

\section*{C.1.2 Equivalence of $F_2$ and $F_\mu$}

$F_2 \subseteq F_\mu$: For any nonzero $x \in F_2$ there exists $M$ such that $x^T M = 0$ and $||\mathbf{\hat{M}} - M||_2 \leq \varepsilon$. Therefore,

$$\mu(x, \mathbf{\hat{M}}) = \frac{||x^T \mathbf{\hat{M}}||_2}{||x||_2} = \frac{||x^T (\mathbf{\hat{M}} - M)||_2}{||x||_2} \leq ||\mathbf{\hat{M}} - M||_2 \leq \varepsilon.$$ 

i.e., $x \in F_\mu$.

$F_2 \supseteq F_\mu$: For any $x \in F_\mu$ define $y = x/||x||_2$, $z = \mathbf{\hat{M}}^T y$. Clearly,

$$||z||_2 = ||\mathbf{\hat{M}}^T y||_2 = \frac{||x^T \mathbf{\hat{M}}||_2}{||x||_2} = \mu(x, \mathbf{\hat{M}}) \leq \varepsilon$$

Set $\mathbf{\hat{M}} = yz^T$ and $M = \mathbf{\hat{M}} - \mathbf{\hat{M}}$. Then

$$x^T M = ||x||_2 y^T M = ||x||_2 (y^T \mathbf{\hat{M}} - y^T \mathbf{\hat{M}}) = ||x||_2 (z^T - y^T y z^T) = 0$$

and

$$||\mathbf{\hat{M}} - M||_2 = ||\mathbf{\hat{M}}||_2 = ||yz||_2 \leq ||y||_2 ||z||_2 = ||z||_2 \leq \varepsilon$$

This means $x \in F_2$.

\section*{C.1.3 Proof of Theorem 2.2.1}

Define

$$\begin{bmatrix} b_2 \\ b_1 \end{bmatrix} = \ell^T b, \quad \begin{bmatrix} \theta_2 \\ \theta_1 \end{bmatrix} = \ell^T \theta, \ \forall \theta \in \mathbb{R}^r.$$ 

where the partitions are conformed to the decomposition (2.25) of the matrix $A$.

Therefore, we have the following identical expressions:

$$\theta^T A \theta + 2b^T \theta + c$$

$$= \theta^T \begin{bmatrix} D_2 \\ -D_1 \end{bmatrix} \ell^T \theta + 2b^T \ell \cdot \ell^T \theta + c$$
\[
= \begin{bmatrix} \theta_2 \\ \theta_1 \end{bmatrix}^T \begin{bmatrix} D_2 \\ -D_1 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_1 \end{bmatrix} + 2 \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}^T \begin{bmatrix} \theta_2 \\ \theta_1 \end{bmatrix} + c
\]

\[
= (\theta_2^T D_2 \theta_2 + 2b_2^T \theta_2) - (\theta_1^T D_1 \theta_1 - 2b_1^T \theta_1) + c
\]

\[
= (D_2^{\frac{1}{2}} \theta_2 \mid D_2^{-\frac{1}{2}} b_2)^T (D_2^{\frac{1}{2}} \theta_2 + D_2^{-\frac{1}{2}} b_2) - (D_1^{\frac{1}{2}} \theta_1 - D_1^{-\frac{1}{2}} b_1)^T (D_1^{\frac{1}{2}} \theta_1 - D_1^{-\frac{1}{2}} b_1)
\]

\[
+ c - b_2^T D_2^{-1} b_2 + b_1^T D_1^{-1} b_1
\]

\[
= (D_2^{\frac{1}{2}} \theta_2 + D_2^{-\frac{1}{2}} b_2)^T (D_2^{\frac{1}{2}} \theta_2 + D_2^{-\frac{1}{2}} b_2) - (D_1^{\frac{1}{2}} \theta_1 - D_1^{-\frac{1}{2}} b_1)^T (D_1^{\frac{1}{2}} \theta_1 - D_1^{-\frac{1}{2}} b_1)
\]

\[
+ c - \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}^T \begin{bmatrix} D_2 \\ -D_1 \end{bmatrix}^{-1} \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}
\]

\[
= (D_2^{\frac{1}{2}} \theta_2 + D_2^{-\frac{1}{2}} b_2)^T (D_2^{\frac{1}{2}} \theta_2 + D_2^{-\frac{1}{2}} b_2) - (D_1^{\frac{1}{2}} \theta_1 - D_1^{-\frac{1}{2}} b_1)^T (D_1^{\frac{1}{2}} \theta_1 - D_1^{-\frac{1}{2}} b_1) + c - b^T A^{-1} b.
\]

Denote

\[
\gamma = \begin{bmatrix} \gamma_k \\ \vdots \\ \gamma_1 \end{bmatrix} := D_2^{\frac{1}{2}} \theta_1 - D_1^{-\frac{1}{2}} b_1, \quad \delta = \begin{bmatrix} \delta_r \\ \vdots \\ \delta_{k+1} \end{bmatrix} := D_2^{\frac{1}{2}} \theta_2 + D_2^{-\frac{1}{2}} b_2.
\]

then for any \( \theta \in \mathbb{R}^r \), on one hand, we conclude that \( \theta^T A \theta + 2b^T \theta + c \leq 0 \) if and only if \( \delta^T \delta - \gamma^T \gamma + c - b^T A^{-1} b \leq 0 \), or equivalently, the condition (2.27) holds: on the other hand, we have

\[
\theta = \begin{bmatrix} \theta_2 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} D_2^{-\frac{1}{2}} \delta - D_2^{-1} b_2 \\ D_1^{-\frac{1}{2}} \gamma + D_1^{-1} b_1 \end{bmatrix}
\]

\[
= \begin{bmatrix} D_2^{-\frac{1}{2}} \\ D_1^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \delta \\ \gamma \end{bmatrix} - \begin{bmatrix} D_2 \\ -D_1 \end{bmatrix}^{-1} \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}
\]

\[
= [u_r \cdots u_2 u_1] \begin{bmatrix} D_2^{-\frac{1}{2}} \\ D_1^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \delta \\ \gamma \end{bmatrix} - A^{-1} b.
\]

This gives the expression (2.26).
C.1.4 Proof of Theorem 2.2.5

Let \( z \in \mathbb{R}^r \) be an arbitrary nonzero model in the family \( \mathcal{F}_n \). Denote \( y = \hat{\ell}^T x = [y_r \ldots y_2 y_1]^T \). Then \( x^T \hat{\mathbf{A}} \hat{\mathbf{A}}^T x \leq z^2 x^T x \) implies

\[
y^T \begin{bmatrix}
\hat{\sigma}_r^2 & \ddots & \\
& \ddots & \\
& & \hat{\sigma}_1^2
\end{bmatrix} y \leq z^2 y^T y.
\]

or equivalently,

\[
\sum_{i=2}^{r} (\hat{\sigma}_i^2 - \hat{\sigma}_1^2) y_i^2 \leq (z^2 - \hat{\sigma}_1^2) y_1^2.
\]

or,

\[
\begin{bmatrix}
\sqrt{\hat{\sigma}_1^2 - \hat{\sigma}_2^2} \\
\vdots \\
\sqrt{\hat{\sigma}_2^2 - \hat{\sigma}_1^2}
\end{bmatrix}
\begin{bmatrix}
y_r \\
\vdots \\
y_2
\end{bmatrix}
\leq \sqrt{z^2 - \hat{\sigma}_1^2} |y_1|.
\]

Therefore, the vector

\[
\delta := \frac{1}{\sqrt{z^2 - \hat{\sigma}_1^2} y_1} \begin{bmatrix}
\sqrt{\hat{\sigma}_1^2 - \hat{\sigma}_2^2} \\
\vdots \\
\sqrt{\hat{\sigma}_2^2 - \hat{\sigma}_1^2}
\end{bmatrix}
\begin{bmatrix}
y_r \\
\vdots \\
y_2
\end{bmatrix}
=: \begin{bmatrix}
\delta_r \\
\vdots \\
\delta_2
\end{bmatrix}
\]

satisfies (2.35), and then

\[
x = \hat{\ell}^T y = \hat{u}_1 y_1 + [\hat{u}_r \hat{u}_{r-1} \ldots \hat{u}_2]
\begin{bmatrix}
y_r \\
\vdots \\
y_2
\end{bmatrix}
\]

\[
= \hat{u}_1 y_1 + [\hat{u}_r \hat{u}_{r-1} \ldots \hat{u}_2]
\begin{bmatrix}
\sqrt{\hat{\sigma}_1^2 - \hat{\sigma}_2^2} \\
\vdots \\
\sqrt{\hat{\sigma}_2^2 - \hat{\sigma}_1^2}
\end{bmatrix}
\begin{bmatrix}
\delta_r \\
\vdots \\
\delta_2
\end{bmatrix}
\]

\[
= \left( \hat{u}_1 + \sqrt{\frac{z^2 - \hat{\sigma}_1^2}{\hat{\sigma}_2^2 - \hat{\sigma}_1^2}} \delta_2 \hat{u}_2 + \cdots + \sqrt{\frac{z^2 - \hat{\sigma}_1^2}{\hat{\sigma}_2^2 - \hat{\sigma}_1^2}} \delta_r \hat{u}_r \right) y_1.
\]

This completes the proof of one part, the other part can be proved by reversing the above process.
C.1.5 Proof of Theorem 2.2.6

The main part of the result comes directly from Theorem 2.2.1. We just need to prove \( b^T A^{-1} b - c \neq 0 \) if \( \varepsilon \) is different from any singular value of the data matrix \( \hat{\mathcal{M}} \). Use Schur complement decomposition introduced in Chapter 1. we have

\[
\hat{\mathcal{M}} \hat{\mathcal{M}}^T - \varepsilon^2 I_r = \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} = \begin{bmatrix} I_{r-1} & A \\ b^T A^{-1} & c - b^T A^{-1} b \end{bmatrix} \begin{bmatrix} I_{r-1} & A^{-1} b \\ & 1 \end{bmatrix}.
\]

So, when \( \varepsilon \) is not a singular value of the data matrix \( \hat{\mathcal{M}} \), the matrix \( \hat{\mathcal{M}} \hat{\mathcal{M}}^T - \varepsilon^2 I_r \) must be nonsingular, then we have \( b^T A^{-1} b - c \neq 0 \).

C.1.6 Proof of Theorem 2.2.7

To prove \( b^T A^{-1} b - c > 0 \), we need the following preliminary concept and results. The \textit{inertia} of a square matrix \( A \in \mathbb{R}^r \), written \( \ln(A) \), is the triple of integers

\[ \ln(A) = \{ \pi(A), \nu(A), \delta(A) \}. \]

where \( \pi(A), \nu(A) \) and \( \delta(A) \) denote the number of eigenvalues of \( A \), counted with their algebraic multiplicities, lying in the open right half plane, in the open left half-plane, and on the imaginary axis, respectively. In the particular case of a Hermitian matrix \( H \), the number \( \pi(H) \) (respectively, \( \nu(H) \)) merely denotes the number of positive (respectively, negative) eigenvalues counted with their multiplicities.

\textbf{Lemma C.1.1 (Sylvester's law of inertia).} Congruent Hermitian matrices have the same inertia characteristics.

\textbf{Lemma C.1.2 (Interlacing eigenvalues theorem for bordered matrices).} Let \( A \in \mathbb{R}^{r \times r} \) be a given Hermitian matrix, let \( y \in \mathbb{R}^r \) be a given vector, and let \( a \in \mathbb{R} \) be a
given real number. Let $\hat{A} \in \mathbb{R}^{(r+1) \times (r+1)}$ be the Hermitian matrix obtained by bordering $A$ with $y$ and $a$ as follows:

$$
\hat{A} = \begin{bmatrix}
    A & y \\
    y^T & a
\end{bmatrix}.
$$

Let the eigenvalues of $A$ and $\hat{A}$ be denoted by $\lambda_i$ and $\hat{\lambda}_i$, respectively, and assume that they have been arranged in increasing order $\lambda_1 \leq \cdots \leq \lambda_r$ and $\hat{\lambda}_1 \leq \hat{\lambda}_2 \leq \cdots \leq \hat{\lambda}_r \leq \hat{\lambda}_{r+1}$. Then

$$
\hat{\lambda}_1 \leq \lambda_1 \leq \hat{\lambda}_2 \leq \lambda_2 \leq \cdots \leq \lambda_{r-1} \leq \hat{\lambda}_r \leq \lambda_r \leq \hat{\lambda}_{r+1}.
$$

(C.3)

Proof: According to the interlacing eigenvalues theorem for bordered matrices, the smallest singular value $\sigma_1$ of the matrix $\hat{\mathbf{U}}_1$ must be not bigger than the second smallest singular value $\hat{\sigma}_2(\hat{\mathbf{U}})$ of $\hat{\mathbf{U}}$. Therefore, when $\hat{\sigma}_{\min}(\hat{\mathbf{U}}) < \varepsilon < \sigma_1$, we have $\hat{\sigma}_1(\hat{\mathbf{U}}) < \varepsilon < \hat{\sigma}_2(\hat{\mathbf{U}})$, and hence, the inertia of the Hermitian matrix $\hat{\mathbf{U}}.\hat{\mathbf{U}}^T - \varepsilon^2 I_r$ is $\{r-1,1,0\}$. It follows from the identity (C.2) and the Sylvester's law of inertia that the matrix

$$
\begin{bmatrix}
    A \\
    c - b^T A^{-1} b
\end{bmatrix}
$$

has the same inertia. However, due to $\varepsilon < \sigma_1$, the matrix $A = \hat{\mathbf{U}}_1.\hat{\mathbf{U}}_1^T - \varepsilon^2 I_{r-1}$ is positive definite, i.e., all its eigenvalues are positive. Therefore, the number $c - b^T A^{-1} b$ has to be negative. This means that the set $\mathcal{E}$ is an ellipsoid in terms of the results introduced in Chapter 1. The remaining results follows from Theorem 2.2.3.

C.2 Proofs in Chapter 3

C.2.1 Equivalence of two conjectures

To see the equivalence, we just need to prove conjecture 2 using conjecture 1. Clearly, the optimization problem (3.5) amounts to the following problem:

$$
\inf ||\hat{\mathbf{U}} - \mathbf{M}(x,u,y)||_2
$$

s.t. $\mathbf{M}(x,u,y)$: block Hankel. $x^T \mathbf{M}(x,u,y) = 0$
If the family $\mathcal{F}_2'$ is **dense** in the family $\mathcal{F}_2$ or $\mathcal{F}_2 = \mathcal{F}_2^n$, the above problem can be readily reduced to the following equivalent problem:

$$
\begin{align*}
d &= \inf ||\hat{U} - \mathcal{M}(x, u, y)||_2 = \min ||\hat{U} - \mathcal{M}(x, u, y)||_2 \\
s &\quad \text{subject to} \quad x^T \mathcal{M}(x, u, y) = 0
\end{align*}
$$

Clearly, for any $\mathcal{M}(x, u, y)$ satisfying $x^T \mathcal{M}(x, u, y) = 0$, we know $||\hat{U} - \mathcal{M}(x, u, y)||_2 \geq \mu(x, \hat{U} - \mathcal{M}(x, u, y)) = \mu(x, \hat{U})$, and then $d \geq \mu(x, \hat{U})$. On the other hand, due to the fact

$$
\mathcal{F}_\mu = \{y: \mu(y, \hat{U}) \leq \mu(x, \hat{U})\} = \{y: \exists \mathcal{M} \text{ such that } ||\hat{U} - \mathcal{M}||_2 \leq \mu(x, \hat{U})\} = \mathcal{F}_2,
$$

we know $d \leq \mu(x, \hat{U})$. This concludes the proof. ■

**C.2.2 Proof of Theorem 3.2.1**

In this proof we assume large enough amounts of data is available, i.e., $N >> n$, so we can assume the data matrices with different row numbers have the same number of columns. Similar to the definition of the input matrix in (3.6), an output matrix $\mathcal{Y}$ can be defined which has the exactly same size as $\mathcal{U}$. Therefore, the data matrix $\mathcal{M}$ is just a concatenation of an input matrix $\mathcal{U}$ and an output matrix $\mathcal{Y}$. Let $x = [\theta^T 1]^T \in \mathbb{R}^{2n+2}$ be the model generating the input-output data $(u_t, y_t)$. Due to the persistently exciting property of the input sequence $u_t$, the rank of the data matrix $\mathcal{M}_0$ with row number $2n + 2$ is obviously $2n + 1$, i.e., the $(n + 1)$-th row of $\mathcal{Y}$ is linearly dependent on the first $n + 1$ rows of $\mathcal{U}$ and the first $n$ rows of $\mathcal{Y}$. Since each row in matrices $\mathcal{U}$ and $\mathcal{Y}$ is just a shift of the previous row, the above argument holds for $\mathcal{Y}$'s last $i - n + 1$ rows. Therefore, the rank of $\mathcal{M}$ is $(2i + 2) - (i - n + 1) = i + n + 1$. ■

---

*If $A \subseteq \mathbb{R}^r$ is **dense** in $B \subseteq \mathbb{R}^r$ and $A \subseteq B$, then for any continuous function $f$ in $B$ there holds $\inf f(A) = \inf f(B)$. 
C.2.3 Proof of (3.9)

Define

\[ \xi = [\xi_0 \; \xi_1 \; \cdots \; \xi_N]^T, \quad \alpha = \frac{1}{N + 1} W^* \xi. \]

where \( W \) is the DFT matrix of order \( N + 1 \). Denote by \( W_k \) the matrix formed by the first \( k \) rows of \( W \). Therefore, \( \xi = W \alpha \), and

\[ \mathcal{N} = W_{n+1} \text{diag}(\alpha_k) W_s^T \]

\[ \mathcal{R}_{\alpha \alpha} = \frac{1}{(N + 1)^2} W^* \mathcal{R}_{nn} W = \frac{\varepsilon^2}{N + 1} I_{N+1}. \]

i.e., \( \alpha_k \) is complex white Gaussian, its variance is \( \varepsilon / \sqrt{N + 1} \). Write \( \alpha_k = \tau_k + j \eta_k \), \( \tau_k, \eta_k \in \mathbb{R} \). Then \( \tau_k, \eta_k \) are both real Gaussian variables with zero mean and variance \( \sigma = \varepsilon / \sqrt{2(N + 1)} \), i.e.,

\[ \frac{\tau_k^2}{\sigma^2} \sim \mathcal{N}(0, 1), \quad \frac{\eta_k^2}{\sigma^2} \sim \mathcal{N}(0, 1). \]

Subsequently,

\[ \rho_k := \frac{|\alpha_k|^2}{\sigma^2} = \frac{\tau_k^2}{\sigma^2} + \frac{\eta_k^2}{\sigma^2} \sim \chi^2(2). \]

i.e., \( \rho_k \) has the \( \chi^2 \) distribution with 2 degrees of freedom. Its probability density function is given by

\[ f(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}}, & x > 0 \\ 0, & x \leq 0. \end{cases} \]

Given a bound \( K > 0 \), then

\[ p := \Pr(|\alpha_k| \leq \frac{K}{\sqrt{N + 1}}) = \Pr(\frac{|\alpha_k|^2}{\sigma^2} \leq \frac{K^2}{(N + 1)^2 \sigma^2}) \]

\[ = \Pr(\rho_k \leq \frac{K^2}{(N + 1)^2 \sigma^2}) = \Pr(\rho_k \leq \frac{2K^2}{(N + 1)\varepsilon^2}) \]

\[ = \int_0^{\frac{2K^2}{(N + 1)\varepsilon^2}} f(x) dx \]

\[ = 1 - e^{-\frac{K^2}{(N + 1)\varepsilon^2}}. \]

Therefore, in terms of the independence of random variables \( \alpha_k \)'s, and

\[ W_k W_k^T = (N + 1) I_k, \quad k = 1, 2, \ldots, N + 1. \]
we know

$$\Pr(\|\mathbf{V}\| \leq K) = \Pr(\|\mathbf{W}_{n+1} \text{diag}(\alpha_k) \mathbf{W}_s^T\| \leq K)$$

$$\geq \Pr(\|\alpha\|_\infty \leq \frac{K}{N+1})$$

$$= \Pr(\max_{0 \leq k \leq N} |\alpha_k| \leq \frac{K}{N+1})$$

$$= \prod_{0 \leq k \leq N} \Pr(|\alpha_k| \leq \frac{K}{N+1})$$

$$= p^{N+1}.$$

This gives the formula (3.9). 

\[\square\]

### C.2.4 Proof of Theorem 3.2.3

Let the nominal model $x$ as usual have the partition $x = [x_1^T \ x_2^T]^T$, where $x_1 \in \mathbb{R}^{m+1}, x_2 \in \mathbb{R}^{n+1}$ are nonzero vectors. To get the equality (3.20) we need to first prove the existence of the Hankel matrices $\mathcal{M}_1 \in \mathbb{R}^{(m+1) \times s}, \mathcal{M}_2 \in \mathbb{R}^{(n+1) \times s}$ such that

$$x_1^T \mathcal{M}_1 + x_2^T \mathcal{M}_2 = 0.$$  \hfill (C.4)

Write $\mathcal{M}_1 = [h^{(1)}_{i+j}], \mathcal{M}_2 = [h^{(2)}_{i+j}]$, and let $h_1 = [h^{(1)}_{k}], h_2 = [h^{(2)}_{k}] \in \mathbb{R}^{m+s+2}$ be the vectors formed respectively by the entries of the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$. Thus, the equation (C.1) can be easily written as a linear equation $T[h_1^T \ h_2^T]^T = 0$, where $T \in \mathbb{R}^{s \times (m+n+2s+4)}$ is a full row rank matrix, which ensures the existence of $\mathcal{M}_1$ and $\mathcal{M}_2$. Define

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix}.$$ 

and let $y_2 \in \mathbb{R}^s$ be a nonzero vector in the right kernel of $\mathcal{M}$. A similar argument as above ensures the existence of a Hankel matrix $\hat{\mathcal{M}}_2 \in \mathbb{R}^{(n+1) \times s}$ such that

$$y_2^T = x_2^T \hat{\mathcal{M}}_2.$$ \hfill (C.5)

Again, define

$$y = \frac{y_2}{\|y_2\|}, \hat{\mathcal{M}} = \begin{bmatrix} 0 \\ \hat{\mathcal{M}}_2 \end{bmatrix}, \mathcal{M} = \mathcal{M} + \hat{\mathcal{M}}.$$
Then, the vectors
\[ y = \frac{\mathbf{\hat{M}}^T x}{||\mathbf{\hat{M}}^T x||} \]
and \( x \) are (unimodular) right, left singular vectors of \( \mathbf{M} \) corresponding to its zero singular value. According to the second order expansion formula (A.25), as \( ||\mathbf{\hat{M}}|| \to 0 \), there exists the following perturbed expansion for the smallest singular value of \( \mathbf{\hat{M}} \)
\[
\hat{\sigma}_{\text{min}}(\mathbf{\hat{M}})^2 = (x^T \mathbf{\hat{M}} y)^2 + ||x^T \mathbf{\hat{M}} V_3||_2^2 + O(||\mathbf{\hat{M}}||_2^3)
\]
\[
= \mu(x, \mathbf{\hat{M}})^2 + O(||\mathbf{\hat{M}}||_2^3)
\]
where \( V_3 \) is a matrix whose columns form the right singular subspace orthogonal to \( y \). Therefore, in terms of the definitions (3.11) and (3.14), we get (3.20).

C.2.5 Strict contractivity of the composite mapping in section 3.3.3

The contractivity depends on the contractivity of two mappings \( F_h \) and \( F_i \circ F_r \). More specifically, if one of two operators \( F_h \) and \( F_i \circ F_r \) is strictly contractive, then for any \( X \in \mathbb{F}^{r \times s} \) we have
\[
||F(X)|| = ||(F_h \circ F_i \circ F_r)(X)|| = ||F_h((F_i \circ F_r)(X))|| \leq ||F_h(X)|| \leq ||X||. \quad (C.6)
\]
i.e., \( ||F(X)|| < ||X|| \). Thus, \( F \) is strictly contractive. As a matter of fact, the composite property mapping \( F \) is indeed strictly contractive, since both \( F_i \circ F_r \) and \( F_h \) are in general strictly contractive.

Next we will separately establish the strict contractivity of the two operators. For the operator \( F_h \) the result can be easily obtained by considering the following inequality:
\[
\frac{a_1^2 + \cdots + a_n^2}{n} \geq \left( \frac{a_1 + \cdots + a_n}{n} \right)^2, \quad a_i \in \mathbb{F}, \quad i = 1, 2, \ldots, n.
\]
where the equality holds if and only if all \( a_i \)'s are identical. i.e., in our case the block itself is a Hankel matrix.
To consider the contractivity of $F_r \circ F_r$, let us compare the two SVD (3.30) and (3.31). Denote

$$
\Sigma = \text{diag} (\sigma_1, \sigma_2, \ldots, \sigma_r), \quad \Sigma_r = \text{diag} (\sigma_1, \sigma_2, \ldots, \sigma_{r-1}, 0).
$$

and write $L^* = [L^T_1 L^T_2]^T$ conformally with the partition (3.23). Then

$$
||\hat{\mathcal{M}}||^2 - ||F_r \circ F_r (\hat{\mathcal{M}})||^2 = ||\hat{\mathcal{M}}||^2 - ||\mathcal{M}_\#||^2 = \text{Tr} (\hat{\mathcal{M}}^T \hat{\mathcal{M}}) - \text{Tr} (\mathcal{M}_\#^T \mathcal{M}_\#)
$$

$$
= \text{Tr} (\hat{\mathcal{Y}} \hat{\mathcal{Y}}^T - \mathcal{Y} \mathcal{Y}^T) = \text{Tr} (L^T_2 (\Sigma^2 - \Sigma_r^2) L^T_2) \geq 0.
$$

where the equality holds if and only if $\hat{\sigma}_r = 0$, i.e., the matrix has already had a non-trivial left kernel. Therefore, the mapping $F_r$ is strictly contractive.  

\[\blacksquare\]

C.3 Proofs in Chapter 4

C.3.1 Proof of expression (4.5)

According the linear fractional transformation (LFT) formula and matrix inversion formula (A.5), we know

$$
G_3(z) = \mathcal{F}_4(G, \delta^T) = G_{22} + G_{21} \delta^T (I_{r-1} - G_{11} \delta^T)^{-1} G_{12}
$$

$$
= G_{22} + G_{21} \delta^T \left( I_{r-1} + \frac{G_{11} \delta^T}{1 - \delta^T G_{11}} \right) G_{12}
$$

$$
= \frac{G_{22} + \delta^T (G_{21} G_{12} - G_{22} G_{11})}{1 - \delta^T G_{11}}. \tag{C.7}
$$

Comparing the two expressions (4.4) and (C.7), we obtain an expression for $G$

$$
G = \begin{bmatrix}
-\frac{\hat{\mathcal{Y}}_1}{M} & \frac{\hat{\mathcal{Y}}_1 \mathcal{M} - \hat{\mathcal{Y}}_1 \hat{\mathcal{Y}}}{M^2} \\
1 & \frac{\hat{\mathcal{Y}}}{M}
\end{bmatrix}.
$$

\[\blacksquare\]

C.3.2 Proof of Theorem 4.2.1

According to the Small Gain Theorem [16, 27, 64], $K^*$ stabilizes $G_3$ if and only if

$$
||G_{11} + G_{12} Q G_{21}||_\infty \leq 1. \tag{C.8}
$$
where
\[ Q = K(1 - G_{22}K)^{-1} = (1 - G_{22}K)^{-1}K \]  \hspace{1cm} (C.9)
is the Youla parameter. Therefore, from (C.9) \( K \) can be expressed in terms of \( Q \).
This leads to (4.7). (4.8) is another form of (C.8).

\[ \text{C.3.3 Proof of Theorem 4.2.2} \]

Suppose \( \hat{M}_1^{-1}\hat{N}_1 \) is another coprime factorization of the nominal transfer function \( \hat{M}^{-1}\hat{N} \), and \( W_1 \) is a corresponding choice of matrix \( W \), i.e.,
\[ W_1 = \begin{bmatrix} M_1 & Y_1 \\ N_1 & X_1 \end{bmatrix} = \begin{bmatrix} \hat{X}_1 & -\hat{Y}_1 \\ -\hat{X}_1 & \hat{M}_1 \end{bmatrix}^{-1} \]
holds for some matrices \( M_1, N_1, X_1, Y_1, \hat{X}_1, \) and \( \hat{Y}_1 \). Then, in terms of the coprime-
ness of the pair \((\hat{N}, \hat{M})\) and the pair \((\hat{N}_1, \hat{M}_1)\), we have
\[ W_1 = \begin{bmatrix} lM & Y/k \\ lN & X/k \end{bmatrix} = \begin{bmatrix} \hat{X}/l & -\hat{Y}/l \\ -k\hat{X} & k\hat{M} \end{bmatrix}^{-1}. \]
where \( k \) is some nonzero scalar, and \( l \) is some scalar function such that both \( l \) and \( 1/l \) are stable. Therefore, for \( \beta_1 = \beta/(kl) \), \( \alpha_1 = \alpha \), there holds
\[ K = (Y\alpha + M\beta)(X\alpha + N\beta)^{-1} = (Y_1\alpha_1 + M_1\beta_1)(X_1\alpha_1 + N_1\beta_1)^{-1}. \]
Also, it is readily to see that the inequality (4.10) holds for \( W_1 \) and \( \beta_1, \alpha_1 \).

\[ \text{C.3.4 Proof of Lemma 4.3.1} \]

To prove the equivalence of the two optimization problem, we assume \( x_0 = \arg \min_{x \in \Omega} f(x) \).
This means for any pair \((x, t) \in \Omega \) we have \( t \geq f(x) \geq f(x_0) \), which implies \( J \geq f(x_0) \).
On the other hand, since \((x_0, f(x_0)) \in \Omega \), we know \( J \geq f(x_0) \). Thus, \( J = f(x_0) \), and then \((x_0, f(x_0)) = \arg \min_{x \in \Omega} t \). This concludes the proof of one direction. The proof
for the other direction is similar, and is thus omitted.
C.3.5 Proof of expression (4.20)

Denote

\[ [B_r(\omega) + iB_i(\omega) \ A_r(\omega) + iA_i(\omega)] := \left( [-\tilde{N}_\delta \ \tilde{M}_\delta] W \right) (e^{i\omega}) \quad g_k(\omega) + ih_k(\omega) := f_k(e^{i\omega}). \]

Then

\[
\text{Re} \ \alpha(e^{i\omega}) = \text{Re} \ \sum_{k=0}^{K} \alpha_k f_k(e^{i\omega}) = \text{Re} \ \sum_{k=0}^{K} \alpha_k (g_k(\omega) + ih_k(\omega)) = \sum_{k=0}^{K} \alpha_k g_k(\omega).
\]

and

\[
C(\omega) = \text{Re} \ \left( [-\tilde{N}_\delta \ \tilde{M}_\delta] W \ \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \right) (e^{i\omega}) = \text{Re} \ \left( \left( [-\tilde{N}_\delta \ \tilde{M}_\delta] W \right) (e^{i\omega}) \right) \\
= \text{Re} \ \left( \left[ B_r + iB_i \quad A_r + iA_i \right] \begin{bmatrix} \sum_{k=0}^{K} \alpha_k g_k \\ \sum_{k=0}^{K} \alpha_k f_k \end{bmatrix} \right) \\
= \sum_{k=0}^{K} \text{Re} \ \left[ (A_r g_k - A_i h_k) \alpha_k + (B_r g_k - B_i h_k) \beta_k \right] \\
= \sum_{k=0}^{K} (A_k \alpha_k + B_k \beta_k).
\]

where \( A_k := A_r g_k - A_i h_k \), \( B_k := B_r g_k - B_i h_k \). Therefore,

\[
F = \begin{bmatrix} t \text{Re} \ \alpha(e^{i\omega}) I & C(\omega) \\ C(\omega)^T & t \text{Re} \ \alpha(e^{i\omega}) \end{bmatrix} = \begin{bmatrix} t \sum_{k=0}^{K} \alpha_k g_k I & \sum_{k=0}^{K} (A_k \alpha_k + B_k \beta_k) \\ \sum_{k=0}^{K} (A_k^T \alpha_k + B_k^T \beta_k) & t \sum_{k=0}^{K} \alpha_k g_k \end{bmatrix} = \sum_{k=0}^{K} \begin{bmatrix} t \alpha_k g_k I & A_k \alpha_k + B_k \beta_k \\ A_k^T \alpha_k + B_k^T \beta_k & t \alpha_k g_k \end{bmatrix}.
\]
\[
\sum_{k=0}^{K} \begin{bmatrix}
 t g_k & \bar{A}_k \\
 \bar{A}_k^T & t g_k
\end{bmatrix} \alpha_k + \sum_{k=0}^{K} \begin{bmatrix}
 0 & B_k \\
 B_k^T & 0
\end{bmatrix} \beta_k
\]

\[
= \sum_{k=0}^{K} F_{0k} \alpha_k + \sum_{k=0}^{K} F_{jk} \beta_k.
\]

where

\[
F_{sk} = \begin{bmatrix}
 t g_k & \bar{A}_k \\
 \bar{A}_k^T & t g_k
\end{bmatrix}, \quad F_{jk} = \begin{bmatrix}
 0 & B_k \\
 B_k^T & 0
\end{bmatrix}.
\] (C.11)

This completes the proof.

### C.3.6 Proof of the specific formulas for \( F_{sk} \) and \( F_{jk} \) in section 4.3.3

Suppose

\[
\tilde{\mathcal{N}}_s(z) = \sum_{i=0}^{n} G_{1i} z^{-i}, \quad \tilde{\mathcal{M}}_s(z) = \sum_{i=0}^{n} G_{2i} z^{-i}, \quad W(z) = \sum_{j=0}^{m} W_j z^{-j}.
\]

Then, in terms of (C.10),

\[
[B_r(\omega) + i B_i(\omega) \quad A_r(\omega) + i A_i(\omega)] = \left( [\tilde{\mathcal{N}}_s(z) \quad \tilde{\mathcal{M}}_s(z)] W(z) \right) (e^{i \omega})
\]

\[
= \left( \sum_{i=0}^{n} \sum_{j=0}^{m} [G_{1i} \quad G_{2i} W_j z^{-(i+j)}] \right) (e^{i \omega})
\]

\[
= \left( \sum_{i=0}^{m} \sum_{j=0}^{n} G_{i} W_j z^{-i} \right) (e^{i \omega})
\]

\[
= \sum_{i=0}^{m} \sum_{j=0}^{n} (G_{i} W_j e^{-i \omega}).
\]

where \( G_i = [-G_{1i} \quad G_{2i}] \). Therefore,

\[
[B_k(\omega) \quad A_k(\omega)] = [B_r(\omega) g_k(\omega) - B_i(\omega) h_k(\omega) \quad A_r(\omega) g_k(\omega) - A_i(\omega) h_k(\omega)]
\]

\[
= \text{Re} \left( [B_r(\omega) + i B_i(\omega) \quad A_r(\omega) + i A_i(\omega)] (g_k(\omega) + i h_k(\omega)) \right)
\]

\[
= \text{Re} \left( \sum_{i=0}^{m} \sum_{j=0}^{n} G_{i} W_j e^{-i \omega} e^{-ik \omega} \right)
\]

\[
= \sum_{i=0}^{m} \sum_{j=0}^{n} G_{i} W_j \cos((k + l) \omega). \quad \text{(C.12)}
\]

The above formula for \( A_k \) and \( B_k \) can be incorporated into the matrices \( F_{0k}, F_{jk} \) in (C.11) to simplify the constraint of (4.23).
C.3.7 Proof of Theorem 4.4.1: upper bound by volume

In this subsection we will derive an upper bound for the maximal stability margin in terms of the volume $V(\mathcal{E})$ of the ellipsoid $\mathcal{E}$. For convenience we use $V$ represents $V(\mathcal{E})$ in this subsection. Write

$$a_{\alpha,j}(\omega) = \begin{bmatrix} a_{\alpha,j}^{(1)}(\omega) \\ \vdots \\ a_{\alpha,j}^{(r-1)}(\omega) \end{bmatrix}.$$ 

Then, using the arithmetic-geometric-mean inequality, we have

$$||Da_{\alpha,j}(\omega)|| = \sqrt{\sum_{j=1}^{r-1} |a_{\alpha,j}^{(j)}(\omega)|^2} \geq \sqrt{(r-1) \left( \prod_{j=1}^{r-1} |a_{\alpha,j}^{(j)}(\omega)|^2 \right)^{\frac{1}{r-1}}} = (r-1)^{\frac{1}{2}} \left( \prod_{j=1}^{r-1} \alpha_j \right)^{\frac{1}{r-1}} \left( \prod_{j=1}^{r-1} |a_{\alpha,j}^{(j)}(\omega)| \right)^{\frac{1}{r-1}}.$$ 

Therefore,

$$\phi(\alpha,\beta,\omega) = \frac{||Da_{\alpha,j}(\omega)||}{b_{\alpha}(\omega)} \geq (r-1)^{\frac{1}{2}} V^{\frac{1}{r-1}} \left( \prod_{j=1}^{r-1} \frac{|a_{\alpha,j}^{(j)}(\omega)|}{b_{\alpha}(\omega)} \right)^{\frac{1}{r-1}}$$

$$\phi(\alpha,\beta) = \sup_{\omega} \phi(\alpha,\beta,\omega) \geq (r-1)^{\frac{1}{2}} V^{\frac{1}{r-1}} \left( \sup_{\omega} \prod_{j=1}^{r-1} \frac{|a_{\alpha,j}^{(j)}(\omega)|}{b_{\alpha}(\omega)} \right)^{\frac{1}{r-1}}$$

$$\phi = \min_{\alpha,\beta} \phi(\alpha,\beta) \geq (r-1)^{\frac{1}{2}} V^{\frac{1}{r-1}} \left( \min_{\alpha,\beta} \sup_{\omega} \prod_{j=1}^{r-1} \frac{|a_{\alpha,j}^{(j)}(\omega)|}{b_{\alpha}(\omega)} \right)^{\frac{1}{r-1}}$$

$$= (r-1)^{\frac{1}{2}} \gamma^{\frac{1}{r-1}} V^{\frac{1}{r-1}}$$

where

$$\gamma = \min_{\alpha,\beta} \sup_{\omega} \prod_{j=1}^{r-1} \frac{|a_{\alpha,j}^{(j)}(\omega)|}{b_{\alpha}(\omega)}.$$
Clearly, \( \gamma \) is a positive number, and therefore, for the maximal stability margin \( \lambda_* \), we have

\[
\lambda_* = \frac{1}{\varphi_*} \leq (r - 1)^{-\frac{1}{r}} \frac{1}{\varphi} V^{-\frac{1}{r}}.
\]

(4.28) is thus proved. 

\[\blacksquare\]

C.3.8 **Proof of Theorem 4.4.1: lower bound and upper bound by semiaxis lengths**

In this section we give a lower bound and an upper bound of the maximal stability margin in terms of, respectively, the largest and the smallest semiaxis lengths of the ellipsoid. Clearly,

\[
||D a_{\alpha, j}(\omega)|| = ||D^{-1} D a_{\alpha, j}(\omega)|| \leq ||D^{-1}||||Da_{\alpha, j}(\omega)||.
\]

Therefore,

\[
\frac{1}{||D^{-1}||} ||a_{\alpha, j}(\omega)|| \leq ||Da_{\alpha, j}(\omega)|| \leq ||D||||a_{\alpha, j}(\omega)||.
\]

Define

\[
\varphi_0(\alpha, j) = \sup_{\omega} \frac{||a_{\alpha, j}(\omega)||}{b_{\alpha}(\omega)}. \text{ and } \varphi_0 = \min_{\alpha, j} \varphi_0(\alpha, j).
\]

Thus,

\[
\frac{1}{||D^{-1}||} \varphi_0(\alpha, j) \leq \varphi(\alpha, j) = \sup_{\omega} \phi(\alpha, j, \omega) \leq ||D||\varphi_0(\alpha, j).
\]

and then

\[
\frac{1}{||D^{-1}||} \varphi_0 \leq \varphi_* = \min_{\alpha, j} \varphi(\alpha, j) \leq ||D||\varphi_0.
\]

or

\[
\alpha_1 \varphi_0 \leq \varphi_* \leq \alpha_{r-1} \varphi_0.
\]

For the maximal stability margin \( \lambda_* \), this becomes

\[
\frac{1}{\alpha_1} \frac{1}{\varphi_0} \geq \lambda_* = \frac{1}{\varphi_*} \geq \frac{1}{\alpha_{r-1}} \frac{1}{\varphi_0}.
\]

i.e., (4.29) holds. 

\[\blacksquare\]
C.3.9 Proof of Theorem 4.4.1: upper bound by mean square root radius

Similar to the previous subsections, we will derive an upper bound for the maximal stability margin by means of the mean square root radius of the ellipsoid $\mathcal{E}$. Clearly,

$$
||Da_{\omega,\beta}(\omega)|| = \sqrt{\sum_{j=1}^{r-1} |\alpha_j a_{\omega,\beta}^{(j)}(\omega)|^2} \\
\geq \sqrt{\left(\prod_{j=1}^{r-1} \alpha_j^2\right) \min_{1 \leq j \leq r-1} |a_{\omega,\beta}^{(j)}(\omega)|^2} \\
= \sqrt{\prod_{j=1}^{r-1} \alpha_j^2 \min_{1 \leq j \leq r-1} |a_{\omega,\beta}^{(j)}(\omega)|}.
$$

Therefore,

$$
\phi(\alpha, \beta) = \sup_\omega \frac{||Da_{\omega,\beta}(\omega)||}{b_3(\omega)} \geq M(\mathcal{E}) \sup_\omega \min_{1 \leq j \leq r-1} \frac{|a_{\omega,\beta}^{(j)}(\omega)|^2}{b_3(\omega)} \\
\phi_* = \min_{\alpha, \beta} \phi(\alpha, \beta) \geq M(\mathcal{E}) \min_{\alpha, \beta} \sup_\omega \min_{1 \leq j \leq r-1} \frac{|a_{\omega,\beta}^{(j)}(\omega)|^2}{b_3(\omega)} \\
= M(\mathcal{E}) \kappa,
$$

where $\kappa$ is a positive constant:

$$
\kappa = \min_{\alpha, \beta} \sup_\omega \min_{1 \leq j \leq r-1} \frac{|a_{\omega,\beta}^{(j)}(\omega)|^2}{b_3(\omega)}.
$$

Therefore, for the maximal stability margin $\lambda_*$, we have

$$
\lambda_* = \frac{1}{\phi_*} \leq \frac{1}{\kappa} \frac{1}{M(\mathcal{E})},
$$

i.e., (4.30) holds.

C.3.10 Proof of (4.35) and (4.32)

Recall that the polynomials $d_i(z), n_i(z)$ are determined by the $i$-th singular value $u_i$ of the data matrix $\mathcal{N}_1$ for $i = 1, \ldots, r-1 = 2n+1$. Denote $u_i$ by $u_i = [u_{ik}] = [u_{ia}^T u_{ib}]^T$. 

where \( u_{ia} \in \mathbb{R}^{n+1}, u_{ib} \in \mathbb{R}^n \). then for any \( \omega \in \mathbb{R} \) we have

\[
\left\| \begin{bmatrix} n_i(e^{j\omega}) \\ d_i(e^{j\omega}) \end{bmatrix} \right\| = \sqrt{|n_i(e^{j\omega})|^2 + |d_i(e^{j\omega})|^2} \\
= \sqrt{\left| \sum_{k=0}^{n} u_{ik} e^{j\omega_k} \right|^2 + \left| \sum_{k=n+1}^{2n+1} u_{ik} e^{j\omega_k} \right|^2} \\
\leq \sqrt{n+1} ||u_{ia}||^2 + (n+1) ||u_{ib}||^2 \\
\leq \sqrt{n+1} ||u_i|| \\
= \sqrt{n+1}.
\]

Consequently, we have

\[
||\Delta(z)||_\infty = \max_{\omega \in [0,2\pi]} \left\| \begin{bmatrix} \Delta_N(e^{j\omega}) \\ \Delta_M(e^{j\omega}) \end{bmatrix} \right\| \\
= \max_{\omega \in [0,2\pi]} \left[ \sum_{i=1}^{r-1} \delta_i \alpha_i \left| \begin{bmatrix} n_i(e^{j\omega}) \\ d_i(e^{j\omega}) \end{bmatrix} \right| \right] \\
\leq \max_{\omega \in [0,2\pi]} \left[ \sum_{i=1}^{r-1} \delta_i |\alpha_i| \sqrt{n+1} \right] \\
= \sqrt{n+1} \sum_{i=1}^{r-1} |\delta_i| |\alpha_i| \\
\leq \sqrt{n+1} ||\delta|| \sqrt{\sum_{i=1}^{r-1} |\alpha_i|^2} \\
= \sqrt{n+1} M(\mathcal{E}).
\]

This concludes the proof of (4.35). The proof of (4.32) is directly from [74] (pp. 225).
Bibliography


