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DETERMINISTIC AND RANDOM VIBRATIONS OF SYSTEMS WITH FREQUENCY DEPENDENT PARAMETERS OR FRACTIONAL DERIVATIVES

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE Master of Science

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Yvonne Renee U. Alsandor

Abstract

Frequency and time domain analyses of system models containing frequency dependent parameters or fractional derivatives are considered. Both model types result in integro-differential equations in the time domain, and in algebraic equations in the frequency domain. The deterministic and random vibrations of these systems are considered. A deterministic solution technique is developed and its ability to accurately approximate a system response is shown. Techniques for the random vibration analysis are presented using the Monte Carlo simulation method. It is assumed that the power spectral density of the random excitations is known. An AR method is used to generate an ensemble of random excitations records compatible with the given spectral density. The deterministic solution technique is employed to obtain the corresponding ensemble of system responses. Numerical results are presented which show that the Monte Carlo Simulation method yields good estimates of statistical information for the system response.
Acknowledgments

Thanks to the people. Glory and honor to God.

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<tr>
<td>ACF</td>
<td>Auto-Correlation Function.</td>
</tr>
<tr>
<td>AR</td>
<td>Auto Regressive.</td>
</tr>
<tr>
<td>DFT</td>
<td>Discrete Fourier Transform.</td>
</tr>
<tr>
<td>FD</td>
<td>Fractional Derivative.</td>
</tr>
<tr>
<td>FDs</td>
<td>Fractional Derivatives.</td>
</tr>
<tr>
<td>FDP</td>
<td>Frequency Dependent Parameter.</td>
</tr>
<tr>
<td>FDPs</td>
<td>Frequency Dependent Parameters.</td>
</tr>
<tr>
<td>FFT</td>
<td>Fast Fourier Transform.</td>
</tr>
<tr>
<td>FT</td>
<td>Fourier Transform.</td>
</tr>
<tr>
<td>IDE</td>
<td>Integro-Differential Equation.</td>
</tr>
<tr>
<td>IDEs</td>
<td>Integro-Differential Equations.</td>
</tr>
<tr>
<td>IDFT</td>
<td>Inverse Discrete Fourier Transform.</td>
</tr>
<tr>
<td>IFFT</td>
<td>Inverse Fast Fourier Transform.</td>
</tr>
<tr>
<td>IFT</td>
<td>Inverse Fourier Transform.</td>
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JPDF  Joint Probability Density Function.
PDF   Probability Density Function.
PSD   Power Spectral Density.
SDOF  Single Degree of Freedom.
WAWS  Weighted-Amplitude Wave Superposition.

Symbols                      Explanation

$\alpha, \beta$                Order of a Fractional Derivative.

D, L                           Derivative Operators.

$\sigma$                      Stress.

$\epsilon$                    Strain.

$\nu$                         Coefficient of Viscosity.

E                              Modulus of Elasticity.

t                              Time Variable.

$j$                            Imaginary Number, $j = \sqrt{-1}$.

$\omega$                      Frequency Variable.

$i, l, m, n$                  Integer Counters.

$\mathcal{F}$                 Fourier Transform.
<table>
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<th>Symbol</th>
<th>Description</th>
</tr>
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<tr>
<td>$\mathcal{F}^{-\infty}$</td>
<td>Inverse Fourier Transform.</td>
</tr>
<tr>
<td>$R K_r$</td>
<td>Runge-Kutta Approximation Function.</td>
</tr>
<tr>
<td>$x(t)$</td>
<td>System Response.</td>
</tr>
<tr>
<td>$y(t)$</td>
<td>System Excitation.</td>
</tr>
<tr>
<td>$x(t)$</td>
<td>Stochastic (Random) Process.</td>
</tr>
<tr>
<td>$\mu_x$</td>
<td>Mean Value of $x$.</td>
</tr>
<tr>
<td>$m_s_x$</td>
<td>Mean Square Value of $x$.</td>
</tr>
<tr>
<td>$\sigma^2_x$</td>
<td>Variance of $x$.</td>
</tr>
<tr>
<td>$R_x(t_1, t_2)$</td>
<td>Auto-Correlation Function of the Random Process $x$ for Times $t_1$ and $t_2$.</td>
</tr>
<tr>
<td>$R_x(\tau)$</td>
<td>Auto-Correlation Function of the Stationary Random Process $x$ for a Time Lag $\tau$.</td>
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<td>$H(\omega)$</td>
<td>Complex Frequency Response Function, Filter Transfer Function.</td>
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\[ \ddot{x}(t) + \frac{0.1}{|\omega|} \dot{x}(t) + 10x(t) = e^{-t} - e^{-2t}. \]

C.2 Response in Frequency Domain for Hysteretic Damped System:

\[ \ddot{x}(t) + \frac{0.1}{|\omega|} \dot{x}(t) + 10x(t) = e^{-t} - e^{-2t}. \]

C.3 Response of SDOF Hysteretic Damped System:

\[ \ddot{x}(t) + \frac{0.1}{|\omega|} \dot{x}(t) + 10x(t) = e^{-t} - e^{-2t}. \]

C.4 Transfer Function and FFT of Excitation for Hysteretic Damped System:

\[ \ddot{x}(t) + \frac{0.1}{|\omega|} \dot{x}(t) + 10x(t) = \sin(10t). \]

C.5 Response in Frequency Domain for Hysteretic Damped System:

\[ \ddot{x}(t) + \frac{0.1}{|\omega|} \dot{x}(t) + 10x(t) = \sin(10t). \]

C.6 Response of SDOF Hysteretic Damped System:

\[ \ddot{x}(t) + \frac{0.1}{|\omega|} \dot{x}(t) + 10x(t) = \sin(10t). \]
Chapter 1

Introduction

The modeling of dynamic systems is a crucial step in the analysis and design of engineering systems. Engineers and scientists gain insight into the important components of a system and their effects on system response. The first step of modeling is the development of a simplified model which resembles the physical system in its prominent features, but is more amenable to analytical studies [17]. Once this is achieved, compatibility and equilibrium relations are employed to describe the dynamics of the system mathematically, resulting in the mathematical model of the system.

Mathematical models of a real system vary in complexity based on simplifying assumptions used in the development of the physical model and simplification employed in defining the compatible and equilibrium relations. Common simplifying assumptions include: neglecting small effects, assuming the surrounding environment remains unaltered by the system, and replacing distributed characteristics with approximating lumped characteristics. Other assumptions are: assuming simple linear cause and effect relationships between physical variables, assuming physical parameters are time and frequency invariant, and neglecting uncertainty and noise [17].

Employing simplifying assumptions leads to less complexity in the dynamic equations describing a system, therefore allowing for simpler methods of solution. Often these assumptions are used to produce mathematical models which are linear time-invariant (constant parameter) deterministic ordinary differential equations which have well known and widely publicized methods of solution.
While invoking simplifying assumptions leads to the advantage of mathematical convenience, it can also bring with it the disadvantage of an inaccurate portrayal of the response of the real system under consideration. This is especially true when considering real systems with significant nonlinearities, parameters that significantly vary with time or frequency, and excitations that are significantly random. Therefore, while these simplified models may be adequate for preliminary studies, subsequent models for design calculations and decision-making require accurate representation and lead to more complex mathematical models.

Clearly, in the analysis and design of real systems, it is important that solution techniques be available for the more complex family of dynamic models. Often these models include nonlinearities, frequency or time dependent parameters and randomness of the excitations.

In this thesis, frequency dependent parameter (FDP) and fractional derivative (FD) models are considered. It is shown that both of these model forms are related, in that they both alternatively can be expressed as integro-differential equations in the time domain and algebraic equations in the frequency domain. Solution techniques are presented for these systems when subjected to deterministic and random excitations. Both of these model forms have been employed in real engineering applications to describe various physical phenomenon. By considering this class of problems, new solution techniques can be developed for complex systems, thus yielding more accurate analyses of real systems and portrayal of system response.

1.1 Frequency Dependent Parameter Models

Typically, constant parameter models are used to describe the dissipation of mechanical energy. These models are often expressed in the form

\[ M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = Ny(t) \]  

(1.1)
In this equation, $x$ is an $n$-dimensional response vector, where the dot represents differentiation with respect to time, and $(t)$ denotes it as function of time. The $y$ is an $m$-dimensional loading vector. The symbols $M$, $C$, and $K$ denote constant mass, damping, and stiffness matrices, respectively; and $N$ is the constant $n \times m$ matrix representing the influence of each loading component on the system.

When considering systems with frequency dependent parameters, a similar expression is used due to its simple form and similarity with the widely recognized model above. This model is

$$M(\omega)\ddot{x}(t) + C(\omega)\dot{x}(t) + K(\omega)x(t) = y(t),$$

(1.2)

where $M(\omega)$, $C(\omega)$, and $K(\omega)$ represent the frequency dependent parameters of the system, and $(\omega)$ is used to denote the parameter variation with frequency; $x(t)$ represents the time variant excitation, and $y(t)$ represents the corresponding time variant response of the system.

According to Crandall, [30], the above expression is a “non-equation”, in that it has no physical meaning since it is represented in both the time and frequency domain and therefore can not be solved in this form. However, this equation can be used to obtain useful equivalent forms in either the time, or frequency domains as presented in the Section 3.2.

FDP models are gaining more attention because of their increased use in a wide range of applications. From the literature review, it is seen that FDP models have been extensively employed to model systems which display hysteretic damping behavior. In many civil engineering applications, viscoelastic materials, such as asphalt, acrylic polymers, rubbers, and silicon gels, are employed in damping mechanisms which result in energy dissipation with negligible dependence on the excitation frequency. The behavior of these mechanisms is best described by models whose energy dissipation is independent of the excitation frequency. This type of dissipation is often
referred to as hysteretic damping and can be approximated using models containing frequency dependent parameters.

A more extensive review of the historical development, the use in dynamic modeling, and the associated solution techniques of FDP models is presented in Section 2.1.

1.2 Fractional Derivative Models

Dynamic models are commonly presented in the form of differential equations which must be solved to obtain the predicted dynamic response for a system. Fractional Derivative (FD) models are an expansion of the more commonly accepted ordinary differential equations (ODE). However ODE's only contain derivatives of integer order, whereas FD models employ derivatives of arbitrary (fractional) order, thus allowing for non-integer derivatives. This concept is based on the theory of fractional calculus.

Dynamic models often can be expressed in the form

\[ D[x(t)] = L[f(t)] \quad (1.3) \]

in which \( D \) and \( L \) represent a series of derivative operators acting on the response vector, \( x(t) \) and excitation vector, \( f(t) \), respectively. For most dynamic modeling purposes, the derivative operators in \( D \) and \( L \) are limited to derivatives of integer order. This limitation has resulted in dynamic models such as

\[ \sigma(t) = E_1 \varepsilon(t) + E_2 \frac{d^1 \varepsilon(t)}{dt^1} \quad (1.4) \]

which contains only an integer order derivative. The use of this limitation is often considered desirable since it results in ODE's which have well established methods of solution. However, due to an increasing need for precise modeling of complex systems, this limitation has been removed and derivatives operators have been allowed
to take on non-integer orders, sacrificing simplicity for efficiency and accuracy. This relaxation results in models containing derivatives of fractional order, such as

$$\sigma(t) = E_2 \frac{d^{\frac{1}{2}} \epsilon(t)}{dt^{\frac{1}{2}}}.$$  \hspace{1cm} (1.5)

Applications for use of fractional calculus models are in the areas of structural damping, solid mechanics, material processing, polymer science, and for determining the properties of biological materials [43]. A review of the historical development of the use of fractional derivatives in dynamic modeling along with corresponding solution techniques is presented in Section 2.2.

1.3 Equivalent Models and Related Topics

FDP and FD models have been employed in the study of many engineering systems. However, the specific models which result are often referred to by various names in literature. Therefore, to provide a good overview of FDP and FD models, one should be familiar with the associated terms used in literature to describe such models.

One of the most common FDP models used in structural engineering applications is the hysteretic damping model. Hysteretic damping is a term used to describe the damping phenomenon that results in energy dissipation independent of frequency. Hidden in this definition is the requirement of a frequency dependent damping coefficient as explained in Section 3.2.

Examples of hysteretic damping behavior in real applications are presented in the following chapter. This type of damping is also referred to as structural damping or complex stiffness in literature. The term structural damping may stem from the fact that this type of damping is found in structures and structural dissipative devices that utilize materials such as asphalt, acrylic polymers, rubber, or silicon gels. The term complex stiffness comes from consideration of the model in the frequency domain,
which leads to a stiffness factor multiplied by a complex variable, as explained in Section 3.2.

A common use for FD models is to describe the behavior of viscoelastic materials, such as polymers and geological materials which exhibit both viscous and elastic behavior. Modeling such behavior using fractional derivatives is favored because these models require relatively few variables and avoid the causality problems associated with the use of some FDP models. FD models may also be referred to as fractional calculus models, viscoelastic models, complex modulus models, or memory mechanisms. The term fractional calculus is used because these models employ FDs which are a part of fractional calculus theory. The term viscoelastic refers to the fact that these models are able to accurately describe the material behavior spectrum from purely viscous to purely elastic based on the value of the fractional derivative parameter used. The term complex modulus stems from the consideration of this model in the frequency domain, which results in a frequency dependent complex parameters as further explained in Section 3.3. The term memory mechanism is derived from the fact that these model forms are able to capture the memory characteristic associated with the behavior of some viscoelastic materials, such as stress relaxation or creep phenomena.

To thoroughly investigate FDP and FD models used in structural and mechanical engineering applications, it is necessary to be familiar with the corresponding terms used to describe such systems. In the review of literature for this work, the most common terms interchanged have been hysteretic damping, structural damping and complex stiffness to describe FDP models, and fractional calculus, viscoelastic damping, complex modulus and memory mechanism to describe FD models.
1.4 Random Vibration

Processes resulting in vibrational motion can either be classified as deterministic or random. A deterministic process is a process in which experiments repeated under identical conditions result in identical responses. A random process is a process in which experiments repeated with all controllable conditions identical result in different responses. Note that the term random vibration has often been incorrectly used to describe repeatable responses which fluctuate in time without following an obvious pattern.

If the excitation of a system is random, then the corresponding response will also be random. Therefore, it is important to understand techniques developed for determining probability structure of the excitation and response of these systems, which can be linear or nonlinear. In most cases, the statistical information of the system can be obtained using standard random vibration analysis methods. However, when considering nonlinear systems, some of these methods no longer apply or become quite intricate. Thus, Monte Carlo simulations are often employed to avoid this limitation.

Interest in quantifying uncertainty was initially prompted by games of chance which lead to the development of the classical theory of probability. Soon after, advances in science and engineering made way for a wide application of probability theory to a variety of problems as well as the mathematical development encompassing random variables and its generalization known as random processes.

The first major applications of probability theory started in physics and then broadened to astronomy. The most significant early advances in engineering occurred in communication theory to deal with problems of noise in signals. Later, the application was broadened to include structural and mechanical systems. By 1950, a new
area called random vibration had developed which used probability theory to advance
the understanding of the behavior of structural and mechanical systems [60].

The degree of randomness in a system can be considered the level of uncertainty
associated with that system. Often times this uncertainty is small and the analysis
of the system is treated as deterministic, formulated in terms of averages, neglecting
variations about the average values. In these situations, the inevitable randomness
can be accounted for through the use of safety factors, load factors or safety margins.

When the uncertainty is high and cannot be ignored, or the analysis requires a
more accurate account of true system behavior, a probabilistic approach is employed
to allow for the quantitative treatment of the uncertainty or randomness in the sys-
tem. This type of approach involves identifying sources of uncertainty, constructing
probabilistic models for each, and then incorporating these models into a problem for-
mulation. Analyzing a system in this way makes it possible to assess the sensitivity
of system behavior to variations in the level of uncertainty.

The most significant sources of uncertainty in a system usually stem from the
interaction between the system and its environment leading to random excitations. Many random excitations associated with structural or mechanical systems result
from an inability to accurately predict or model earthquake excitations, wind char-
acteristics, resulting jet noise, thrust associated with burning stages, boundary layer
turbulence, gusts during atmospheric flight, ocean wave characteristics, and weather
conditions. Other sources of randomness include uncertainty in system parameters,
uncertainty in variable constraints, and inequalities involved in system constraints.
Some examples include variations in system parameters due to manufacturing toler-
ances, uncertainty in material properties, and uncertainty arising from variations in
assembly.
1.5 Overview of This Work

This thesis focuses on linear FDP and FD models and their solutions when subjected to deterministic and random vibrations. The goals of this work was to identify literature concerning the use of these models, to show that the two model forms are related, and to obtain numerical solutions for the general FDP model as an example. In this work a survey of FDP and FD models is attempted and corresponding example models are identified. Solution techniques for deterministic and random excitations are presented and used to develope numerical algorithms for obtaining responses.

Chapter 2 contains the literature review of the models considered and related topics for this work. Mathematical concepts related to the use and solution of FDP and FD models subjected to deterministic are presented in Chapter 3. The random vibration of systems is addressed in Chapter 4. Chapter 5 contains the numerical development and analysis results for the general FDP model solved as an example. Concluding remarks and comments on future work are presented in Chapter 6. Various mathematical derivations and related work are provided in the appendices.
Chapter 2

Survey of Literature

2.1 Frequency Dependent Parameters in Modeling

The use of frequency dependent parameters in modeling has been previously considered by Crandall [30] for the damping parameter and by Spanos and Zeldin [75, 74] for the general frequency dependent parameter model subjected to Random excitations. These types of models have been used in engineering applications to describe systems with distributed parameters [82], to describe the harmonic motion of bar and beam elements [65], and to model the interaction of structures with continuous media such as fluids [26] and soils [63, 80, 81, 83]. Some models have employed both frequency dependent mass and stiffnesses to describe the vibration of structures [67]; however, the majority of frequency dependent parameter models have concentrated on describing the viscoelastic damping behavior in systems. The corresponding literature for these systems is discussed below.

A common type of frequency dependent damping behavior is known as hysteretic damping. Many model forms have been proposed in the last 50 years to describe this type of behavior. A few of these models are highlighted below. This form of frequency dependent damping is also referred to as structural damping or complex stiffness in the literature.

Linear damping in which the energy loss per cycle is independent of deformation frequency is referred to as ideal hysteretic damping. This behavior is easily defined in the frequency domain, where the damping coefficient is expressed as a function of
frequency such that a frequency independent energy loss factor results. However, a
time domain expression for hysteretic damping is required for the study of nonlin-
ear structures which employ hysteretic damping devices. Consequently, the focus of
several research endeavors has been to define hysteretic behavior in the time domain.

The theoretically correct way to obtain an equivalent expression for hysteretic
damping in the time domain is to take the inverse Fourier transform of the frequency
domain model. Unfortunately, evaluating the resulting Fourier integral is not element-
tary. Therefore, attempts have been made to defined simplified models of hysteretic
behavior which originate in the time domain. From a study of the hysteretic damping
model in the time domain has show that this ideal behavior is physically unrealizable
because it violates the principle of causality [29].

Over the years, many different time domain models have been proposed to approx-
imate the frequency dependent force-displacement relationship for hysteretic damp-
ing. The majority of them are presented as either a real-valued frequency dependent
parameter models [38, 12] or as complex-valued frequency independent parameter
models [80, 2, 27]. While several model forms have been proposed to describe hys-
teretic behavior in the time domain, the correctness of these models has been widely
debated.

Linear hysteretic damping models were proposed as early as 1937 according to
Bishop [11]. The construction of these models have engaged many researchers in
the last 50 years and has been accompanied by lively discussions on the subject
[11, 22, 25, 24, 30, 32, 39, 40, 54, 66]. Two questions widely debated are: the proper
mathematical representation of the hysteretic damping force in the time domain, and
appropriate solution techniques for systems employing hysteretic damping. Several
researchers have also proposed mixed time and frequency domain representations to
define the hysteretic damping force. However, this formulation has been shown to be mathematically incorrect [30].

In 1940, Theodorsen and Garrick [78] presented a linear damping model referred to as structural friction or structural damping which was shown to exhibit frequency independent energy dissipation per cycle. This type of model was later labeled hysteretic damping by Bishop [11]. Crandall, [31] notes that a more appropriate label for this type of damping behavior is ideal hysteretic damping.

In 1955, Bishop [11] considered the time domain expression for hysteretic damping which is defined using real frequency dependent parameters. He also introduced the fact that this model is equivalent to complex damping models. Most importantly, Bishop notes that the frequency dependent parameter expression of hysteretic damping is only valid for harmonic oscillation and cannot be used for non-harmonic excitations or free vibration, which will not be harmonic, until the present definition is extended.

In 1956, Reid [66] introduced a causal hysteretic damping model in the time domain to describe linear hysteretic damping behavior. He then attempts to use this model to describe the free vibration of a SDOF system with hysteretic damping. Unfortunately, Reid failed to notice that his model was actually nonlinear and therefore not susceptible to frequency domain analysis [22], and that according to Bishop [11], the free vibration of this system should not be assumed harmonic. This nonlinear model was later examined by Caughey, et al to assess its ability to model linear hysteretically damped behavior [22].

Two years later, Biot [10] presented a practical example of a mechanism very weakly dependent on frequency. However, according to Inaudi and Kelly [39] his proposed hysteretic damping model contains a storage moduli which increases without bound as $|\omega| \to \infty$, and this model is considered improper since under certain
parameter conditions it could not represent an element with storage and loss moduli independent of frequency.

In 1963, Crandall [29] examined the impulse response function of a SDOF system with structural damping and showed that the linear structural damping model violates the principle of causality. However, he went on to state that if the system is excited by a stationary white noise process this model can be used to obtain the exact solution of the mean square response.

In 1965, Hardin [38] proposed the use of a frequency dependent viscosity model to describe the behavior of dry sands subjected to small amplitude sinusoidal vibration over a large frequency range. This model is derived from the Kelvin-Voigt model, which additively combines purely elastic and purely viscous expressions. The viscous component in this model employed a frequency dependent viscosity parameter defined inversely proportional to frequency. The viscous component of this model is equivalent to the popular hysteretic damping model defined using real frequency dependent coefficients. Other works, such as [80], present hysteretic damping models which are defined with frequency independent complex coefficients. These two forms of the hysteretic damping model can be shown to be equivalent as explained in Section 3.2.3.

In 1970, Crandall [30] discussed the frequency dependent dashpot and showed how it can be defined such that the corresponding energy loss factor is frequency independent. Such ideal hysteretic damping behavior is physically unrealizable because it violates the principle of causality. But, according to Crandall, many mechanisms have loss factors which remain substantially constant within certain frequency ranges. Presented is the frequency domain representation for a system with a frequency dependent dashpot and the corresponding time domain representation defined as the inverse Fourier transform. Crandall notes that the mixed time-frequency domain rep-
representation employed by many frequency dependent damping models, such as that in equation (1.2), are non-equation and should not be interpreted literally.

In the mid-80's, Bishop and Price [13, 14] introduced the band-limited hysteretic damping model in an attempt to model causal hysteretic damping. The use of this model simplifies calculations by avoiding the transfer function singularity at $\omega = 0$, but it does not alter the noncausality of the hysteretic model [31].

In 1991, Crandall examined the characterization of ideal and band-limited hysteretic damping models in the time and frequency domain. He concluded that both models are flawed in that they produce non-causal responses and therefore can not be physically realized. He notes that this flaw is not fatal. The models can be used to obtain meaningful results when considering steady state oscillation at the onset of instability or stationary random vibrations. He also introduces the idea of a hidden delta function in the transfer function of hysteretic damping models which should be accounted for when transforming them into the time domain. These delta functions result from considering the non-integrable transfer functions as generalized functions. This topic has been further studied by Makris [51]. It should be noted that some researchers do not believe the use of the hysteretic damping model to be valid for random vibration applications, and it is therefore expected that future papers will be generated with regard to this topic.

In 1994, Chen, et al [25] proposed and solved a causal hysteretic damping model for general loadings in the time domain. This model was criticized by Crandall [32] because it is not equivalent to the frequency domain representation of hysteretic damping. This flaw was later acknowledged by Chen, et al in 1997 [24]. Recently, Makris has also proposed a hysteretic damping model in the time domain, which is claimed to satisfy the principle of causality. Because this model has been proposed very recently, its validity has yet to be established.
In the 1990's, a new approach to modeling hysteretic damping in the time domain using the complex parameter expression has been developed by Makris, Inaudi, and Kelly. This approach utilizes the complex frequency independent parameter expression of hysteretic damping, and eliminates the previous shortcoming of this model producing complex output for real input by formulating an expression to define the appropriate complex input to be used with the model. The imaginary counterpart to a real displacement record is defined in terms of the Hilbert transform of the displacement. This model still violates the fundamental principle of causality. However, evidence has been presented to indicate that this flaw is not fatal [31] in that it does not affect the correctness of the calculated response when the solution of the system is properly dealt with [39, 40, 49, 54]. Therefore, this approach to modeling hysteretic damping is considered useful, and, by some researchers, the mathematically correct way to replace complex stiffness parameters or frequency-dependent damping coefficients commonly used to model structures exhibiting linear hysteretic damping.

In 1994, Makris presented [49] the idea that any recorded motion has an imaginary counterpart which must be considered when complex parameter models are used. This imaginary counterpart was defined using the Hilbert transform of the real recorded motion. The validity of this procedure was demonstrated by showing that the response obtained for a hysteretic Kelvin model employing the complex stiffness form of hysteretic damping was equivalent to the response obtained from the real valued frequency dependent formulation.

In 1995, Inaudi and Kelly presented [39] a time domain equivalent expression for linear hysteretic damping using the Hilbert transform. They introduced this model as the mathematically correct way to replace the limited complex stiffness parameter and frequency dependent parameter models of hysteretic damping. They also proposed an
iterative solution technique to obtain the response of the integro-differential equation which results when this model is employed in a SDOF system.

In 1996, Inaudi and Makris [40] considered two linear hysteretic damping models and their time domain solutions. The first model employed a hysteretic Kelvin element, in which a hysteretic damper is placed in parallel with a conventional linear spring. The second employed a hysteretic Maxwell element, in which a hysteretic damper is placed in series with a conventional linear spring. Both models used the Hilbert transform in their time domain representation, which result in Integro-differential equations when real valued signals are utilized. An alternative presentation is also proposed which uses analytical signals and complex-valued coefficients to transform the integro-differential equations into differential equations for more efficient time domain solution techniques.

In 1996, Makris, Inaudi, and Kelly [54] further discuss the generation of the imaginary counterpart of the real records and the time domain representation of the linear viscoelastic model presented in their earlier works [39, 40, 49]. They also analyze the noncausality of the proposed model.

In 1997, Chen and You [24] proposed an IDE in the time domain, as an alternative to the complex stiffness model in the frequency domain. This model was used for the free vibration of a single degree-of-freedom system with hysteretic damping. The integral of the Hilbert transform was embedded in the IDE, and a direct iteration time domain technique introduced and employed to obtain the response of the system. The model proposed by Chen and You was similar to the models proposed and studied in the works by Makris, Inaudi, and Kelly.

In 1997, Makris [51] used the properties of the Hilbert Transform and Kramers-Kroning relations to show that transfer functions with singularities at \( \omega = 0 \) in their imaginary part should be corrected by adding a delta function at the origin in their
real part. The requirement of the delta function was explained by introducing the concept of a hidden delta which is present whenever $i/\omega$ appears in an impedance or flexibility transfer function. As examples, the transfer functions of classical viscoelastic models were examined which had either impedance or flexibility transfer functions with singularities at $\omega = 0$ for the imaginary part. For these models it was shown that the transfer functions should be corrected to include a delta function in the real part to ensure causality, consistent with the theory of generalized functions. Then a similar singularity was shown to exist in the impedance function of the ideal hysteretic damping model presented in literature which is noncausal. The introduction of a delta function for the real part of the transfer function was proposed, and it was noted that this correction does not change the noncausality of the ideal hysteretic model. The concept of hidden delta functions was previously considered by Crandall in 1991 [31].

In the same year, Makris [50] introduced a causal hysteretic element that is physically realizable at finite frequencies. This element was constructed with the same imaginary part as the ideal hysteretic damper, and with the real component computed such that the real and imaginary parts would satisfy the Kramers-Kroning relations which ensure its causality. In the development, Makris states that a change in the upper limit of the convolution integral of the response leads to the causality of the element, which is the only difference between the proposed model and the noncausal complex stiffness model proposed earlier [39]. Makris also states that the behavior of the proposed model was examined in the time and frequency domain and shown to be the limiting case of the viscoelastic model proposed by Biot [10] and studies by Caughey [21]. The response of a mass-hysteretic element system was also discussed based on a problem previously studied [21]. Whether or not this model stands up to the claims presented in Markis’ paper has yet to be seen. It is expected that this will
be the topic of future literature in this area. It is also expected that the debate over the proper representation of hysteretic damping in the time domain will continue.

In 1994, Makris proposed a different approach for modeling dynamic behavior using macroscopic force-displacement relations which employ coefficients and orders of time derivatives that are allowed to be complex valued [48]. He showed that these complex parameters are outcomes of rigorous non-linear regression analysis on complex-valued functions like the dynamic stiffness of hysteretic damping systems. As an example, a Kelvin type model was used which employed complex parameters and orders of differentiation to describe the viscoelastic behavior of elastic foundations under earthquake excitation. Frequency and time domain algorithms used to obtain numerical results were shown to be in good agreement with other proposed methods and analytical solutions.

Although the majority of literature surveyed for frequency dependent parameters focused on the frequency dependence of the damping parameter, other works have also employed frequency dependent stiffness and mass coefficients. Two examples are given by Przemieniecki [65] who showed that for a bar undergoing harmonic motion, the resulting equivalent mass and stiffness matrices are frequency dependent. The same was shown to be true for beam elements experiencing harmonic motion. It was stated that this result could be true for other structure or structural elements performing forced or free harmonic motion. Due to the difficulty associated with the resulting frequency dependence of these coefficient matrices, this work employed assumed approximations for the parameters considered in the presented analyses. Other works also considered the use of frequency dependent masses [26, 58], frequency dependent stiffnesses [63, 83], and frequency dependent masses and stiffnesses [67].
2.2 Fractional Derivatives in Modeling

The use of fractional calculus to model the behavior of viscoelastic materials was motivated by Nutting [61] who observed that the stress relaxation phenomenon for some materials may be accurately expressed as a fractional power law with respect to time.

The use of fractional derivatives in modeling was proposed as early as 1936 by Andrew Gemant [36] who observed that the stiffness and damping properties of viscoelastic materials were proportional to fractional powers of frequency. Gemant suggested that these fractional powers of frequency lead to fractional ordered derivatives in the time domain, and from the analysis of experimental results, justifies the use of fractional time derivative models for viscoelastic materials [37].

In 1949, Scott-Blair, et al [68], proposed the use of fractional derivatives for the stress strain relations of materials such as plastics which diverge from Nutting’s stress relaxation law [61].

In 1968, Mandelbrot, et al [55], introduced the concept of fractional Brownian motion, an extension of Brownian motion which incorporates a memory mechanism defined using the Riemann-Liouville integral, which is equivalent in expression to the use of a derivative of fractional order.

Slonimsky [69] proposed a quantitative description of relaxation phenomena in polymers based on the use of fractional integral operators. The expressions presented in his paper are equivalent to fractional derivative models proposed by others such as Smit and de Vries [70], and Calleja and Guzman [16].

Smit and de Vries [70] called attention to the usefulness of FD models in Rheology as intermediate models able to model the spectrum of viscoelastic materials, from purely elastic (for a derivative of zero order), to purely viscous (when the derivative is of order one). They introduce the basic fractional derivative equation as an al-
ternative combination of the purely elastic Hookean equation and the purely viscous
Newtonian equation, as described in Section 3.3. The use of this new intermediate FD
model is considered advantageous because of its capacity to describe both relaxation
and retardation phenomena. In this paper, Smit and de Vries discussed the behavior
of this model in relation to the known behavior of viscoelastic materials and showed
its usefulness in modeling the viscoelastic behavior of materials by analyzing experi-
mental data of synthetic fibers in terms of the intermediate model. A link between
the FD model and theory of linear viscoelasticity is also discussed and some light
is shed on the physical meaning behind the use of FDs which stems from studies in
molecular statistics.

Calleja and Guzman [16] also proposed similar FD models to describe the relax-
ation phenomenon of a viscoelastic solid. The results from their proposed fractional
derivative model are compared to existing formulations and are said to be in very
good agreement with these other formulations.

In the 1970's, Caputo and Mainardi used fractional derivatives to model what
they termed the memory mechanism component of energy dissipation which exists in
various geological materials such as metals, rocks, and sediments. This established the
usefulness of FD models in geophysics applications. In 1971, Caputo and Mainardi
[20] proposed the use of FDs to characterize the mechanical properties of metals and
rocks. Then in 1974, Caputo [18] proposed the use of FDs to model the behavior of
an infinite viscoelastic layer representative of the behavior of sediments. And later in
1976, Caputo [19] proposed the use of FD models for dissipation in elastic medium
characterized as an infinite plate with frequency independent loss factor as a useful
model for geophysics applications.

In 1983, Bagley and Torvik [7] presented a link between molecular theories and
the macroscopic fractional calculus approach to viscoelasticity. They also wrote a
series of other papers stating that the fractional calculus approach leads to well-posed problems and model forms which accurately portray the mechanical behavior of numerous materials used for structural damping or energy dissipation [3, 4, 6].

In 1983, Bagley and Torvik [5, 6] also demonstrated the adaptability of the fractional calculus model to the finite element analysis of viscoelastically damped structures. In [5] fractional derivative models were employed for the stress-strain relationships of viscoelastic materials in structures and a Laplace method was proposed and used to obtain the closed-form solutions to the equations of motion. In [6] FD constitutive relationship were used to model the behavior of a viscoelastic damping layer in a structure, and the system model used to predict the transient response of the structure to step loading. Solutions were obtained using continuum and finite element formulations. The solution technique in [5] was used to obtain the numerically predicted response. Results from the two formulations were then compared to verify the use of the FD model with the finite element approach.

Throughout their papers, Bagley and Torvik have noted the benefits of using the fractional derivative approach. Some of these benefits include: its ability to accurately describe the mechanical properties of a variety of viscoelastic materials, its requirement of few empirical parameters compared to the classical viscoelastic model, and its ability to produce well-posed equations of motion with closed form solutions. Other benefits of using the fractional derivative approach include: its link to molecular theory, its ability to predict real, continuous, responses, its usefulness with general loading conditions, and its adaptability to the finite element formulation. The adaptability to the finite element formulation results in closed-form solutions for structures of engineering interest. This approach is also consistent with physical principles, such as the second law of thermodynamics and causality.
Papers by Bagley and Torvik also mention alternative approaches to modeling the behavior of viscoelastic materials. One approach is the complex parameter (modulus) approach, which yields the simplest model. However, this approach is only valid for sinusoidal loading conditions. A second approach employs what is known as a structural damping model. This model is equivalent to the complex parameter model, and has been shown to violate the second law of thermodynamics and predict noncausal responses [33]. Another approach is the use of the classical linear viscoelastic model. However, the disadvantage of using this model is that it may require higher order derivatives, and therefore a substantial number of parameters to adequately model systems which significantly vary with frequency.

In 1984, Torvik and Bagley [79] showed that fractional derivative relationships arise naturally in the description of certain Newtonian Fluid motion attempting to further justify the use of FDs in the modeling of viscoelastic materials. From theory they derived a FD relationship between stress and velocity for a viscous fluid induced by the sinusoidal motion of a plate on its surface. They showed that theoretical consideration of this system can result in FD equations of motion for a simple mechanical system immersed in a viscous fluid. They went on to state that FD relationships should therefore be anticipated in any system characterized by localized motion in viscous fluid, such as the oscillation of polymeric materials. They concluded that this existence of a theoretical link accounts for the success behind the use of FDs in modeling these materials. Also presented was an application of a FD constitutive relationship which shows the superiority of the three parameter FD model over the Voigt solid model in predicting the transient response of an elastomer which is a frequency dependent material.

In 1984, Koeller [43] noted the equivalence between Rabotnov’s Hereditary Solid Mechanics theory and a FD relation of stress to strain. He presented expressions
for creep and relaxation functions. He also proposed the Fractional Kelvin-Voigt and Fractional Maxwell model to model viscoelastic solids and fluids, respectively. These models were developed by replacing the dashpots in the Kelvin-Voigt (spring parallel to damper) and Maxwell (spring in series with damper) models with a fractional calculus element whose stress is proportional to a fractional derivative of strain. Having the characteristics of both a spring and dashpot, he called it a spring-pot and noted that the constitutive equation form was able to span the continuum from ideal solid state to ideal fluid state as the fractional derivative operator parameter varied between zero and one.

Later in the 80's, Bagley and Torvik [9] applied the thermodynamic constraints based on nonnegative energy dissipation rates and nonnegative internal work to the five parameter FD model, and showed that the application of these constraints lead to well-posed mathematical descriptions of viscoelastic phenomenon, resulting in the prediction of real, continuous, causal responses with positive loss factors for the stress relaxation and creep functions for viscoelastic materials. It was also noted that FD models can successfully fit data obtained using concepts of thermorheologically simple materials. The data is generated by consolidating measurements taken over ranges of both temperature and frequency into a single curve representing one temperature and a broader frequency range. The general FD relationship between stress and strain, as presented in Section 3.3, is capable of describing a complex spectrum of material behaviors.

In 1990, Koh and Kelly [44] proposed the use of the FD Kelvin model for elastomeric bearings used in a base-isolation system. They presented a numerical scheme developed to solve the fractional equations in the time domain and showed that experimental results agreed well with the fractional derivative model.
In 1991 and 1992, Makris and Constantinou [47, 52, 46] proposed the use of fractional derivative Maxwell models to describe the behavior of viscous dampers used in vibration isolation systems. These models were developed and calibrated using experimental data. Numerical algorithms were developed and presented for both the time and frequency domain to obtain the response of the vibration isolation systems. The use of these fractional derivative constitutive models were shown to result in good agreement with results obtained experimentally. Around the same time, Makris and Constantinou also proposed the use of complex order derivatives to model the viscoelastic behavior of viscous dampers [47, 53]. Makris later expanded his research to include the use of frequency dependent parameters in modeling.

2.3 Survey of Solution Techniques

2.3.1 Frequency Domain

Solving the general frequency dependent parameter model presented in equation (1.2) is a straightforward procedure. Since the definition for hysteretic damping is easily defined in the frequency domain, elementary mathematical techniques are required to obtain the frequency domain expression for the solution. The only difficulty that may arise in solving this model in the frequency domain is transforming the solution of this model back into the time domain. Since the solution techniques required for this domain are straightforward, at least for the 1-dimensional case considered in this work, an extensive review of literature on frequency domain solution techniques for frequency dependent parameter models is not necessary. However, a review of the theory involved with solving this system in the frequency domain, such as the general model form in the frequency domain and executing Fourier transformations, is considered in the following chapter.
2.3.2 Time Domain

Solving the general frequency dependent parameter model in the time domain is not as straightforward as solving the equivalent model in the frequency domain. This is largely due to the complexity of the expression that results for these models in the time domain. Therefore, a brief review of solving the frequency dependent parameter model for hysteretic damping which frequently appears in literature is conducted to introduce basic ideas associated with solving these models in the time domain.

For a SDOF system with hysteretic damping mechanism, the different variations of the resulting equation of motion can be expressed as integral differential equations. Several methods have been proposed for solving these resulting equations when subjected to a deterministic excitations. In a paper by Inaudi and Kelly [39], an iterative time domain analysis was proposed. Makris et al [54] presented the tools for a frequency domain analysis. Inaudi and Markis [40] presented a state space formulation using an analytical excitation which is complementary to the effective one leading to equations of motion with complex modes that must be decoupled before being numerically integrated.

In the work by Inaudi and Kelly [39], a time domain representation for linear hysteretic damping and a corresponding time domain solution technique was presented. For the time domain solution technique, convergence of the solution obtained is analyzed and numerical examples are presented. Also considered in this paper are the use of this solution technique for extensions to multiple degree-of-freedom structures and nonlinear structures employing linear hysteretic elements.

In the work by Inaudi and Makris [40], solution techniques are presented for hysteretic models developed using the Hilbert transform to obtain the correct time domain representation. These models result in IDE when real valued signals are utilized. An alternative presentation is also presented which shows that using analyt-
ical signals and complex-valued coefficients allows the integro-differential equations to be transformed into differential equations allowing for the use of more efficient time domain solution techniques. This is possible since the use of analytic signals transforms integral operators into differential operators on analytic signals due to the commutative property of the Hilbert transform with time differentiation. The solution technique presented uses modal decomposition of state-space equations and integration of the resulting differential equations. However, since these equations contain both stable and unstable poles, integration for the stable poles is carried out forward in time while for the model coordinates with unstable poles it is carried out backward in time. The proposed integration scheme is presented as an alternative to the frequency domain solutions and is considered less costly than the time domain iterative technique proposed in [39]. Using linearization methods, this time domain technique can be employed by the authors for the study of nonlinear force-deformation problems. Numerical examples are also presented.

In 1996, Makris et al [54] examined the time domain analysis of practical constitutive dissipation models containing complex valued coefficients. The purpose of their paper was to show the relationship between the imaginary counterpart of real-valued records and their Hilbert transforms, to discuss the causality of viscoelastic models with complex-valued coefficients, and to present a newly developed real-value representation for time domain viscoelastic models with complex-valued coefficients which are needed when such a device is placed in structures exhibiting a nonlinear response.

The solution technique presented in [24] employs a direct integration scheme using the Duhamel integral formulation to solve the time domain IDE presented. Conditions for obtaining a convergent solution are also presented. It is proven that the time domain model presented is equivalent to the accepted frequency domain representation.
As expected from this model, the lack of dependence of the dissipation energy on the exciting frequency has been confirmed.

2.4 Random Vibration of FDP Models

In literature, a random process is often referred to as a random function, stochastic process, or time series.

Little has been done concerning the solution of SDOF systems with hysteretic damping excited by a random process until recent years. This topic has been considered by Floris [35], who presents two solution techniques for determining the probabilistic description of the response of a SDOF oscillator with hysteretic damping excited by a Gaussian white noise process. The solution techniques are developed using what is known as Itô’s stochastic differential calculus and numerical simulation.

It is noted that Itô’s stochastic differential calculus is only effective if a viscous dashpot acts in parallel with the linear hysteretic device, otherwise the second order moment equations are indeterminate. The numerical simulation uses a standard Monte Carlo simulation to generate the Gaussian white noise excitation through a superposition of cosine waves. By this means, it is shown that the statistical moments can be evaluated in close form if the random excitation is simulated using the Weighted-Amplitude Wave Superposition (WAWS) method. This method is considered computationally less costly than generating many excitation histories and solving the equation of motion for each using the methods in [39, 40, 54]. Formulation shows close form solution and stochastic differential calculus solution to be the same. Numerical examples show perfect agreement between the two approaches and excellent agreement with the methods developed in [39]. There are some limitations with both methods that must be considered.
The stochastic response of systems with linear hysteretic damping has also been considered by Spencer and Bergman in [77] and Spanos and Zeldin in [75, 74]. In [75] the solution of FDP systems is discussed for state space and convolution integral formulations. FD models are presented as a special of FDP models. It is also shown that the response of this type of system subjected to random excitation can be determined by a standard formula.
Chapter 3

Mathematical Background

3.1 Preliminary Concepts

Mathematically speaking, system models containing frequency dependent parameters or fractional derivatives result in integro-differential equations in the time domain. These models can also be expressed as simple algebraic equations in the frequency domain. Often, it is advantageous to solve a system of equations in the frequency domain. One advantage is that convolution integrals in the time domain often transform to simple algebraic expressions in the frequency domain. However, there are some limitations on what can be transformed into the frequency domain. First, performing calculations in the frequency domain requires a complete record of the system excitation. Therefore, systems cannot be solved in-line, that is, as the excitation is being revealed. A second and more major limitation is that Fourier analysis can only be applied to linear systems. Therefore, since non-linear models cannot be transformed into the frequency domain, we are forced to solve them in the time domain.

Solving equations in the time domain also has its advantages. First, time domain analysis can facilitate both linear and nonlinear equation forms. Second, if a system record is in the process of being revealed, a time domain solution can still be considered for the part of the response that is already known. In other words, systems can be solved in-line. Also, some kernel expressions contained in convolution integrals can be more efficiently dealt with or lead to better expressions in the time domain, especially if their Fourier transforms are not easily accessible. Therefore, the ability
to solve equations in the time domain is important. Thus, one must consider the question of how to solve the integro-differential equations resulting from the time domain analysis of systems with frequency dependent parameters or fractional derivatives.

3.1.1 Fourier Transform

The Fourier Transform (FT) of a real-valued (or complex) function $x(t)$ of the real-variable $t$ is defined as

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \quad (3.1)$$

Symbolically, this expression can be stated as

$$X(\omega) = \mathcal{F}\{x(t)\} \quad (3.2)$$

where $\mathcal{F}\{\ast\}$ is used to denote the Fourier operator acting on $\ast$. The use of the FT definition in equation (3.1) results in a complex valued function of the real variable $\omega$, used to denote frequency. The Inverse Fourier Transform (IFT), used to reverse this transformation, is defined as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega. \quad (3.3)$$

Symbolically, this expression can be stated as

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} \quad (3.4)$$

where $\mathcal{F}^{-1}\{\ast\}$ denotes the FT acting on $\ast$. Note that the infinite limits in equations (3.1) and (3.3) lead to the question of convergence, as further addressed in reference [64].

Familiarization with the properties of the FT is important in understanding the benefits of its application to system analysis. Some of the basic properties useful in this work are presented below. The notation used assumes that the FT of a real-valued function $x(t)$ over all time $t$ is defined as $\mathcal{F}\{x(t)\} = X(\omega)$. 
Several Fourier transform properties follow.

1. The Fourier transform is a linear transformation. That is,
\[ \mathcal{F}\{x(t) + y(t)\} = \mathcal{F}\{x(t)\} + \mathcal{F}\{y(t)\}. \] (3.5)

2. The Fourier transform preserves even and oddness characteristics. Specifically, if given
\[ x(t) = u(t) + jv(t), \] (3.6)
and
\[ X(\omega) = A(\omega) + jB(\omega), \] (3.7)
the following relationships result.

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3. Convolution in the time-domain transforms to multiplication in the frequency-domain. That is.
\[ x(t) = h(t) \ast y(t) = \int_{-\infty}^{\infty} h(t - \tau) y(\tau) \, d\tau, \] (3.9)
transforms to
\[ X(\omega) = \mathcal{F}\{h(t) \ast y(t)\} = \mathcal{F}\{h(t)\} \ast \mathcal{F}\{y(t)\}. \] (3.10)

4. The Fourier transform of the derivative of a function is related to the Fourier transform of the function according to the equation
\[ \mathcal{F}\left\{\frac{dx}{dt}\right\} = j\omega X(\omega). \] (3.11)

For a further understanding of the FT and its applications, refer to reference [64].
3.1.2 Integro-Differential Equations

3.1.2.1 Background

An integro-differential equation (IDE) is an equation in which the solution variable is presented in both an integral expression and a derivative expression. These equation forms arise frequently in mathematical models of various disciplines. The origin of IDEs can be traced from works of mathematicians such as Abel, Lotka, Fredholm, Malthus, Verjulst, and Volterra on problems in mechanics, mathematics, biology and economics [45]. These works have led to the theory and applications of Volterra IDEs with bounded and unbounded delays.

The basic theory behind the solution of IDEs considers equations of the form

\[ \dot{x}(t) = f(t, x(t)) + \int_{t_0}^{t} K(t, s, x(s)) \, ds \]  \hspace{1cm} (3.12)

where the initial conditions are given as \( x(t_0) = x_0 \) for \( t_0 \geq 0 \). This expression is equivalent to

\[ x(t) = x_0 + \int_{t_0}^{t} \left[ f(s, x(s)) + \int_{s}^{t} K(\sigma, s, x(s)) \, d\sigma \right] \, ds. \]  \hspace{1cm} (3.13)

Thus, if \( f \) and \( K \) are continuous in equation (3.13), then differentiating this equation yields equation (3.12) [45].

It is assumed that the integro-differential equations in question do not include the trivial set with known analytical solutions. Therefore attention is immediately turned to integro-differential equations that require computer-assisted numerical solutions.

In considering numerical methods to solve integro-differential equations, it is essential to first understand the numerical methods available for solving differential equations and integral equations. An introduction to the subject of numerical methods is beyond the scope of this work, but can be obtained by referring to [23], [42] or other introductory references on the subject. Below is a listing of a few of the fundamental numerical methods used to solve ordinary differential and integral equations.
I. Numerical Methods for Differentiation

(a) One Step Methods

i. Euler’s Self-starting Method

ii. Modified Euler Methods

iii. Runge-Kutta Methods

(b) Finite Difference Method (Formulas)

(c) Multi-Step Methods

i. Predictor-Corrector Methods

ii. Other Functions of Integration Methods

II. Numerical Methods for Integration

(a) Newton-Cotes Polynomial Formulas

i. Trapezoidal Rule

ii. Simpsons Rules (1/3rd and 3/8th)

(b) Adams Formulas

(c) Romberg Integration

(d) Gauss Quadrature

From a knowledge of numerical methods, it is clear that the application of these methods in solving differential and integral equations results in a discrete approximation of the original equation. Applying each of the various methods results in an approximation that differs from other approximations in several ways, such as in the order of the truncation error, the sensitivity to round-off error, the number of starting conditions needed, and the simplicity of expression.
Therefore, solving an IDE numerically includes, not only the idea of finding a
discrete approximation of the integral and derivative expressions respectively, but
also appropriately combining approximations with compatible traits suitable for pro-
gramming. One of the most notable traits that should be compatible is the order of
the truncation error. This, however, does not mean that the order of the integra-
tion and differentiation methods are the same. What it means is that the resulting
truncation error magnitudes are compatible. In other words, if the derivative part
of the expression is slow varying while the function integrated is of high order, then
a more accurate (higher order) integration method may be required, while using a
differentiation method of relatively low order to obtain a balance in compatibility.

The Runge-Kutta method, like all one-step methods, approximate derivative func-
tions of the form
\[
\frac{dx}{dt} = f(t, x)
\] (3.14)
by the numerical iteration form
\[
x^{i+1} = x^i + \phi(t^i, x^i, h)
\] (3.15)
where \(i\) is the discrete time index and \(h\) denotes the constant incremental step in
time. This approximation can be interpreted as

\[
\text{New value} = \text{old value} + \text{slope} \times \text{stepsize}.
\] (3.16)

In the Runge-Kutta general formula, \(\phi\) is called the increment function which can be interpreted as the average slope taken over the interval. In a 4th-order Runge-Kutta
scheme, the increment function is defined as
\[
\phi = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]
\] (3.17)
where the \(k_m\)'s are
\[
k_1 = f(t^i, x^i)
\] (3.18)
\[ k_2 = f(t^i + 0.5h, \ x^i + 0.5h \cdot k_1) \]  \hspace{1cm} (3.19)

\[ k_3 = f(t^i + 0.5h, \ x^i + 0.5h \cdot k_2) \]  \hspace{1cm} (3.20)

\[ k_4 = f(t^i + h, \ x^i + h \cdot k_3) \]  \hspace{1cm} (3.21)

and the number \( m \) denotes order of the Runge-Kutta function.

For a set of \( n \) first order differential equations, the derivative functions are defined as

\[ \frac{dx_1}{dt} = f_1(t, \ x_1, \ x_2, \ldots, \ x_n) \]  \hspace{1cm} (3.22)

\[ \frac{dx_2}{dt} = f_2(t, \ x_1, \ x_2, \ldots, \ x_n) \]  \hspace{1cm} (3.23)

\[ \vdots \]

\[ \frac{dx_n}{dt} = f_n(t, \ x_1, \ x_2, \ldots, \ x_n) \]  \hspace{1cm} (3.24)

where the \( f_i \) is the function used to define the \( i \)th differential equation. The \( n \) corresponding 4th-order Runge-Kutta increment functions are

\[ \phi_1 = \frac{1}{6}[k_{11} + 2k_{12} + 2k_{13} + k_{14}] \]  \hspace{1cm} (3.25)

\[ \phi_2 = \frac{1}{6}[k_{21} + 2k_{22} + 2k_{23} + k_{24}] \]  \hspace{1cm} (3.26)

\[ \vdots \]

\[ \phi_n = \frac{1}{6}[k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}] \]  \hspace{1cm} (3.27)

where the \( k_{lm} \)'s are defined as

\[ k_{11} = f_1(t^i, \ x^i_1, \ x^i_2, \ldots, \ x^i_n) \]  \hspace{1cm} (3.28)
\[ k_{i2} = f_i(t^i + 0.5h, x^i_1 + 0.5h \cdot k_{i1}, x^i_2 + 0.5h \cdot k_{21}, \ldots, x^i_n + 0.5h \cdot k_{n1}) \]  (3.29)

\[ k_{i3} = f_i(t^i + 0.5h, x^i_1 + 0.5h \cdot k_{i2}, x^i_2 + 0.5h \cdot k_{22}, \ldots, x^i_n + 0.5h \cdot k_{n2}) \]  (3.30)

\[ k_{i4} = f_i(t^i + h, x^i_1 + h \cdot k_{i3}, x^i_2 + h \cdot k_{23}, \ldots, x^i_n + h \cdot k_{n3}). \]  (3.31)

### 3.1.2.2 An Example Solution

This section contains a brief example of how to solve an IDE. This example does not attempt to address all of the issues associated with solving IDEs, but merely attempts to provide a brief introduction of how solutions can be approached and obtained. Therefore, consider the following example for obtaining the numerical solution of an IDE.

**Example 3.1**

**Problem Statement:** Given the following integro-differential system model, assuming \( x_0 = 0 \) and \( \dot{x}_0 = 0 \), obtain a numerical solution using a 4th order Runge-Kutta to approximate the derivative and the Trapezoidal Rule to approximate the integral.

\[ \ddot{x} + 0.1\dot{x} + x + \int_0^t x(\tau) e^{-3\tau} \sin(t - \tau) d\tau = e^{-4t} \sin(3t) \]  (3.32)

**Solution:**

Step 1. Ignore the integral expression in equation (3.32) and solve for the Runge-Kutta solution to the resulting equation.

\[ \ddot{x} + 0.1\dot{x} + x = e^{-4t} \sin(3t) \]  (3.33)

Let \( x_1 = x \) and \( x_2 = \dot{x} \), so that \( \dot{x}_1 = x_2 \) and \( \dot{x}_2 = \ddot{x} \), can be substituted in to get the following set of first order ordinary differential equations.

\[ \dot{x}_1 = x_2 \]  (3.34)
\[ \dot{x}_2 = -0.1x_2 - x_1 + e^{-4t} \sin(3t) \]  

(3.35)

From the 4th order Runge-Kutta Formulas comes the solution (see Appendix A)

\[ x^{i+1}_1 = x^i_1 + Rk_1(x^i_1, x^i_2, t^i) \]  

(3.36)

\[ x^{i+1}_2 = x^i_2 + Rk_2(x^i_1, x^i_2, t^i) \]  

(3.37)

where \( Rk_r \) represents the Runge-Kutta approximation function for \( \dot{x}_r \) above.

Step 2. Approximate the integral expression in equation (3.32) using the Trapezoidal Rule.

\[ \int_0^{t^i} x(\tau) e^{-3\tau} \sin(t - \tau) d\tau \approx \sum_{k=1}^{i} (h) \cdot \frac{1}{2} \cdot [x(t^k) e^{-3t^k} \sin(t^{i-1} - t^k)] \]  

(3.38)

where \( h = t^{k+1} - t^k = \text{constant} \), and \( i = \text{last time step in integration limit} \). Therefore, approximating the integral in this manner results in the integral expression being approximated by

\[ i = 1 : \quad (h) \cdot \frac{1}{2} \cdot [x(t^1) e^{-3t^1} \sin(t^2 - t^1)] \]  

(3.39)

\[ i \geq 2 : \quad \sum_{k=2}^{i-1} (h) \cdot [x(t^k) e^{-3t^k} \sin(t^{i-1} - t^k)] \].  

(3.40)

Note that \( x(0) = 0 \) and \( \sin(t^{i-1} - t^{i-1}) = 0 \), so the first and last expressions of the Trapezoidal approximation drop out of the expression and the above result is obtained (see Appendix B).
Step 3. Combine Steps 1 and 2 to obtain an approximate formulation for the solution of the integro-differential equation (3.32). Combining equations (3.36) and (3.39), noting that the approximation of the integral is only a function of $x_1$ and not $x_2$, gives the results in the following solution.

**Solution: Computer Logic to Solve Equation**

\[ x_1^{i+1} = x_1^i + Rk_1(x_1^i, x_2^i, t_1^i) \]  

\[ x_2^{i+1} = x_2^i + Rk_2(x_1^i, x_2^i, t_1^i) - I(i) \]  

for $i = 1$: \[ I(i) = (h) \cdot \frac{1}{2} [x_1^2 e^{-3t_2} \sin(i^2 - t_1)] \]  

for $i \geq 2$: \[ I(i) = \sum_{k=2}^{i-1} (h) \cdot [x_1^k e^{-3t_k} \sin(t_{i-1}^i - t_k)] \]

See Appendix B for the corresponding program and results.

### 3.2 Frequency Dependent Parameters

An accurate description of the behavior of structural components is important in the modeling of structures subjected to excitations such as wind loads and earthquakes. Many structural damping devices exhibit what is known as hysteretic damping behavior which is accurately described using Frequency Dependent Parameters (FDP) models. The modeling of components, such as bars and beams, have shown to result in frequency dependent mass and stiffness matrices [65]. To analyze these models, it is important to understand the proper time and frequency domain representations for FDP models and the relation of these representations to the common non-equation
form presented in Section 1.1. Therefore, the following introduction to the basic theory associated with the use of FDPs in modeling is provided. First presented is the theory which shows that the definition of hysteretic damping leads to the use of a frequency dependent damping coefficient. Then specific hysteretic model forms proposed in literature are presented and briefly discussed. Finally, the proper time domain and frequency domain representations for FDP models are extracted from the non-equation form introduced earlier.

3.2.1 Basic Concepts

FDPs are system parameters which are modeled as varying with the frequency of excitation, as shown in equation (1.2). Most common in literature is the use of a frequency dependent damping coefficient used to describe the damping behavior associated with structures, known as hysteretic damping. Frequency dependent mass and stiffness parameters have also been shown to result from the consideration of bars and beams.

Since frequency dependent damping occurs often in literature, constitutive damping model which relates the force and displacement for a damping mechanism is considered

$$F_d = C \cdot D^n[u(t)]$$

(3.45)

where $F_d$ is the damping force, $u(t)$ is the displacement, and $D^n$ is the $n$-th derivative of the displacement, commonly presented as $n = 1$. In this equation, $C$ is a model parameter which can be evaluated by plotting the relationship between the damping force and the derivative of the displacement. For simplicity, this relationship is usually approximated as linear, and $C$ is considered constant. However, when this behavior significantly deviates from the linear approximation and an accurate description of damping is desired, the parameter $C$ can no longer be considered constant, and instead
should be defined to best model the relationship between the force and derivative of displacement. Such as in the case with hysteretic damping. For this type of damping, $C$ is frequency dependent and therefore can be described by defining it as a FDP, $C(\omega)$.

For structural damping, models are typically presented as linear differential equations relating force and displacement using real-valued constant parameters. The two models most widely used are the Kelvin and Maxwell models. The Kelvin model is derived from considering a damper in parallel with a linear spring, whereas the Maxwell model is derived from considering a damper and spring in series. Though widely used, these constant parameter models are not able to adequately describe the hysteretic damping exhibited in structural applications which employ viscoelastic materials as damping mechanisms. This is because damping behavior associated with viscoelastic materials has been shown to produce hysteretic loops which implies frequency dependence. Therefore, it is best to model this type of damping behavior using a frequency dependent damping coefficient.

Recall from Section 1.3 that ideal hysteretic damping is defined as damping which results in energy dissipation independent of the frequency of excitation. To show that type of dissipation results in a frequency dependent damping coefficient, consider the following model excited by a sinusoidal excitation.

$$M \ddot{x}(t) + C \dot{x}(t) + Kx(t) = F_0 \sin(\omega t)$$  \hspace{1cm} (3.46)

In this equation, $M \ddot{x}(t)$, $C \dot{x}(t)$, and $Kx(t)$ are the inertia, damping, and storage forces, respectively, which collectively account for the effects of the forcing function on the system. Note that this model employs the Kelvin constitutive model.

Based on the excitation, it is known that the response of the system will be of the form [28]

$$x(t) = X_0 \sin(\omega t - \phi),$$  \hspace{1cm} (3.47)
where
\[ X_0 = \frac{F_0}{[(K - M\omega^2)^2 + (C\omega)^2]^2} \] (3.48)

and
\[ \tan(\phi) = \frac{C\omega}{M\omega^2 - K}. \] (3.49)

From this, the damping force can be expressed as
\[ F_d = C\dot{x} \]
\[ = C \cdot X_0 \omega \cos(\omega t - \phi) \] (3.50)

and the energy dissipated during one cycle is found to be
\[ E_d = \int F_d dx \]
\[ = \pi C X_0^2 \omega. \] (3.51)

Therefore, energy dissipated is proportional to frequency, and only becomes frequency independent if the damping coefficient varies inversely with frequency, such as
\[ C(\omega) = \frac{a_0}{|\omega|}, \] (3.52)

where \( a_0 \) is an arbitrary constant. Thus, it is shown that the definition of hysteretic damping implies the presence of a frequency dependent damping coefficient, symbolically noted as \( C(\omega) \). Note that |\( \omega \)| instead of \( \omega \) is required in equation (3.52), since the coefficient, \( C \), must be positive for system stability. Substituting equation (3.52) into equation (3.51) and assuming only positive frequencies leads to the following frequency independent expression for energy dissipation.
\[ E_d = \pi a_0 X_0^2 \] (3.53)
3.2.2 Properties

One important property of FDP models that should be noted is that they can result in models which violate the physical law of causality and therefore may be physically unrealizable. This has been shown to be true for the hysteretic damping model. The noncausality of these models results from the fact that these models are defined based on ideal cases in the frequency domain where the law of causality is not applied. These models are then transformed into the time domain to obtain time domain expressions that satisfy frequency domain behavior.

3.2.3 Model Forms

In the frequency domain, the linear hysteretic damping model is expressed as

\[ P(\omega) = [K_1 + j K_2 \text{sgn}(\omega)] u(\omega) \]  

(3.54)

where \( P(\omega) \) and \( u(\omega) \) are the Fourier transforms of the real-valued force \( P(t) \) and real-valued displacement \( u(t) \), respectively. This form of hysteretic damping in the frequency domain is largely accepted and used as the correct form of ideal hysteretic damping. However, the proper representation of hysteretic damping in the time domain has been widely debated as noted in Section 2.1.

Common simplified time domain expressions for hysteretic damping have employed real-valued frequency dependent parameters, such as [38]

\[ P(t) = K_1 u(t) + C(\omega) \dot{u}(t) \]  

(3.55)

where

\[ C(\omega) = \frac{K_2}{\omega} \]  

(3.56)

or complex-valued constant parameters, such as [2, 27, 80]

\[ P(t) = (K_1 + j K_2) u(t) \]  

(3.57)
$P(t)$ is the real-valued force, $u(t)$ is the real-valued displacement, and the $K$'s are constants independent of frequency. However, these models are considered incorrect [39] because the first is an improper mix of the time and frequency domains, and the second can improperly result in a complex force from a real deformation. Yet, this second model has been shown to take on physical meaning if the displacement is complex [49].

In 1996, a procedure was developed [54] to compute the appropriate complex displacement record from the real displacement record to be used with equation 3.57 that would result in a complex force whose real component represented the actual force response of the system. In this development, the appropriate imaginary complement of the real valued displacement was defined as the Hilbert transform of the real displacement. Using this procedure yields the following solution for the hysteretic damping model in the time domain

$$P(t) = K_1 u(t) - K_2 \dot{u}(t)$$

which is equivalent to the real part of the resulting complex force. In this equation, $\dot{u}(t)$ is the Hilbert transform of the real-valued displacement record, $u(t)$, defined as [54]

$$\dot{u}(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\tau)}{\tau - t} \, d\tau$$

where the Cauchy principle value is taken about $\tau = t$ [40]. Plugging this definition into the above equation gives an integro-differential expression for the hysteretic damping model in the time domain. This model is still noncausal, but has been shown to be useful in modeling hysteretic behavior.

For the general case, the following expression, introduced in Section 1.1 and repeated here, can be used to introduce frequency dependent parameter models.

$$M(\omega)\ddot{x}(t) + C(\omega)\dot{x}(t) + K(\omega)x(t) = y(t)$$

(3.60)
Recall that $M(\omega)$, $C(\omega)$, and $K(\omega)$ denote FDPs, and $y(t)$ and $x(t)$ are the time variant excitation and response, respectively.

As previously mentioned in Section 1.1, this expression has no physical meaning [30] since it is a hybrid representation of both the time and frequency domains and therefore cannot be solved in its present form. However, this equation can be used to obtain meaningful relationships in both the time and frequency domains.

It has been shown [75] that the proper frequency domain representation of this FDP model can be obtained by taking the Fourier transform of equation (3.60), which results in the expression

$$
[-\omega^2 M(\omega) + j \omega C(\omega) + K(\omega)]X(\omega) = Y(\omega).
$$

(3.61)

In this equation, $X(\omega)$ and $Y(\omega)$ denote the Fourier transforms of $x(t)$ and $y(t)$, respectively.

To obtain a meaningful representation of the model for analysis in the time domain, the inverse Fourier transform of the above expression can be taken [30]. This can be accomplished by first rearranging equation (3.61) into the form

$$
-\omega^2 M(\omega) X(\omega) + j \omega C(\omega) X(\omega) + K(\omega) X(\omega) = Y(\omega)
$$

(3.62)

and then applying the inverse Fourier transform, recalling that multiplication in the frequency domain transforms to convolution in the time domain. This leads to the following time domain integro-differential expression for FDP models in general.

$$
\int_0^t M(t-\tau) \dddot{x}(\tau) \, d\tau + \int_0^t C(t-\tau) \ddot{x}(\tau) \, d\tau + \int_0^t K(t-\tau) x(\tau) \, d\tau = y(t)
$$

(3.63)

Equation (3.63) gives the time-domain expression that must be solved to obtain the response of frequency-dependent parameter models for time-domain analysis.
3.3 Fractional Derivatives

An accurate description of the dynamic behavior of various complex systems, such as viscoelastic materials, can be obtained using models which employ derivatives of fractional order. To analyze these models, a basic knowledge of a few fractional calculus concepts is imperative, particularly ones related to the definition and properties of fractional derivatives (FD). For this reason, the following subsections have been provided to introduce basic theory associated with FDs and their use in modeling the behavior of mechanical systems. Presented first is an example used to visually introduce the basic idea behind the development of the formal definition for derivatives of arbitrary order. Following this is a brief review of the historical development of the FD. The formal definition and basic properties useful in the analysis of FD models are also presented, as well as common FD constitutive equations and alternative models used in the literature to describe the behavior of real systems. Finally, general equation forms for FD models in the frequency and time domains are presented. A more complete development of related fractional calculus theory can be viewed in [34. 57] or similar fractional calculus references.

3.3.1 Basic Concepts

Before introducing the use of the FD in modeling, its meaning should be explored. The following example is provided for this purpose, and should give insight into the formulation of the FD definition, which is presented later.

**Foundation**: From an understanding of derivatives of integer order, the following nth order derivative expression can be considered,

\[ D^n[x(t)] = \frac{d^n x}{dt^n}, \quad (3.64) \]
where \( D^n \) is the derivative operator of order \( n \) acting on the function \( x(t) \) which results in the \( n \)th-order derivative of \( x(t) \). Based on this notation, a fractional-order derivative can then be described as the derivative that results when \( n \) in equation (3.64) is allowed to take on fractional, or non-integer, values, such as \( 1/2 \) or \( 0.3 \).

When the derivative expression of equation (3.64) is allowed to take on a non-integer value, it will be designated by replacing the integer operator \( n \) by the arbitrary derivative operator, \( \alpha \) in this work.

A better understanding of the arbitrary derivative may be obtained through a careful examination of the following example.

**Example 3.2**

**Given:** \( y = x^m \), where \( m = \text{constant} \)

**Find:** \( \frac{d^n y}{dx^n} \)

**Solution:** By assuming that \( n = 1 \), the first-order derivative solution is known to be:

\[
\frac{d}{dx}(x^m) = m \cdot x^{m-1}
\]

(3.65)

From this it can be further concluded that for \( n = 2 \) the second-order derivative solution is:

\[
\frac{d^2}{dx^2}(x^m) = \frac{d}{dx}(m \cdot x^{m-1})
\]

(3.66)

\[
= m \cdot (m - 1) \cdot x^{m-2}.
\]

(3.67)

Taking this one step further for \( n = 3 \), the third-order derivative solution is:

\[
\frac{d^3}{dx^3}(x^m) = \frac{d}{dx}[m \cdot (m - 1) \cdot x^{m-2}]
\]

(3.68)

\[
= m \cdot (m - 1) \cdot (m - 2) \cdot x^{m-3}.
\]

(3.69)
Now generalizing this trend, the following solution can be presented for the $n$th-order derivative of $y(x)$:

$$\frac{d^n}{dx^n}(x^m) = m \cdot (m-1) \cdots (m-n+1) \cdot x^{m-n} \quad (3.70)$$

$$= \frac{m!}{(m-n)!} x^{m-n} \quad (3.71)$$

$$= \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}. \quad (3.72)$$

Therefore, the solution to this problem is:

$$\frac{d^n}{dx^n}(x^m) = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}. \quad (3.73)$$

While this example is only good for polynomial expressions, it serves as a good introduction to understanding the development of the definition for a fractional derivative. This example particularly brings into light the use of the gamma function in the definition of an arbitrary order derivative, which is presented in Section 3.3.3. Where the gamma function is the generalization of the factorial to non-integer numbers, defined by the expression [1]

$$\Gamma(m) = \int_0^\infty x^{m-1}e^{-x} \, dx, \quad k > 0. \quad (3.74)$$

Using integration-by-parts, it can also be shown that

$$\Gamma(m) = (m-1) \cdot \Gamma(m-1). \quad (3.75)$$

### 3.3.2 History and Development

In 1819, Lacroix introduced the definition for a fractional derivative based on equation (3.73). By setting $n$ in this equation to $1/2$, he introduced the following definition for a derivative of $1/2$-order acting on the polynomial expression $x^a$ [34].
\[
\frac{d^\frac{1}{2}}{dx^{\frac{1}{2}}} (x^a) = \frac{\Gamma(a + 1)}{\Gamma(a + \frac{1}{2})} x^{a-\frac{1}{2}}
\] (3.76)

For the example of \( y(x) = x^1 \), Lacroix developed the formulation

\[
\frac{d^\frac{1}{2}}{dx^{\frac{1}{2}}} (x) = \frac{2\sqrt{x}}{\sqrt{\pi}}.
\] (3.77)

However, this method for defining a derivative of arbitrary order was later discredited when the error which occurs for \( n = 0 \) was noted.

By 1832, Liouville expanded the concept of an arbitrary derivative and its application by considering functions of the form

\[
f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}
\] (3.78)

similar in form to a Fourier series expansion of a function. Using this function form, Liouville introduced the following definition for an arbitrary derivative of a function.

\[
D_x^{\alpha}[(x)] = \sum_{n=0}^{\infty} c_n a_n^\alpha e^{a_n x}
\] (3.79)

In this equation, \( \alpha \) denotes the order of the derivative and \( x \) denote that it is a derivative with respect to the variable \( x \). Equation (3.79) is considered to be the first logical definition presented for a derivative of arbitrary order [34].

Later, Liouville developed his definition of a function and its arbitrary derivative into an integral form, which was further expanded by Riemann in 1847. The works of both Liouville and Riemann are what led to the formal and widely accepted definition of an arbitrary derivative, which is named in their honor. The formal definition of an arbitrary derivative and its properties are presented and discussed in the following section.
3.3.3 Definition and Properties

Fractional derivatives are defined using the generalized definition of a differentiation. In the definition, the order of the derivative can be arbitrarily defined as an integer or fraction, as well as real or complex. The generalized derivative is defined as the inverse operation of generalized integration, whose definition was developed by Riemann and Liouville. The formal definition of integration of arbitrary order is [62]

\[ cD_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_c^t f(\tau)(t - \tau)^{\alpha-1} \, d\tau, \quad t > c, \quad \Re(\alpha) > 0, \]  

(3.80)

where \( \alpha \) is the order of integration which can be real or complex, \( t \) and \( c \) represent the upper and lower limits of integration, respectively, and \( f(t) \) is the function integrated, which must be continuous between the limits \( c \) and \( t \).

Under special consideration, this definition is used to define the generalized derivative as the inverse of generalized integration, mathematically signified by employing negative real values of \( \alpha \), say \( \alpha = -r \). Therefore, the following relationship is established between the arbitrary order derivative and integral.

\[ D^\alpha[f(t)] = \frac{d^\alpha f(t)}{dt^\alpha} = I^{-\alpha} f(t) \]  

(3.81)

From this relationship comes the formal definition of an arbitrary order (generalized) derivative [34, 62].

\[ cD_t^\alpha f(t) = \frac{1}{\Gamma(-\alpha)} \int_c^t f(\tau)(t - \tau)^{\alpha-1} \, d\tau \]  

(3.82)

In this definition, \( D^\alpha \) denotes the derivative operator of order \( \alpha \), where \( \alpha \) is an arbitrary constant (real or complex). Thus, the right side of equation (3.82) represents the \( \alpha \)-order derivative of the function \( f(t) \). In this expression, \( \Gamma \) is the gamma function and \( \tau \) is a dummy variable.

Some of the properties associated with derivatives of arbitrary order should be noted. For one, setting the the lower limit of integration, \( c \), to zero does not effect the
generality of the expression, yet provides the causality required for the system models considered. When \( c \) is zero, the operator is often written as \( D_t^\alpha \). The expression for the generalized derivative operator is usually presented as \( D^\alpha[f(t)] \), where the upper integration limit of \( t \) is implied. Other important properties useful for an understanding of derivatives of arbitrary order include [34]:

I. If \( f(t) \) is an analytical function of the variable \( t \), then the arbitrary derivative \( D^\alpha[f(t)] \), is an analytical function of \( \alpha \) and \( t \), where \( z \) can be real or complex.

II. The operation \( D^\alpha[f(t)] \) must produce:

(a) the same result as ordinary differentiation when \( \alpha \) is a positive integer

(b) \( n \)-fold integration when \( \alpha \) is a negative integer, such as \( \alpha = -n \), with

\[ cD_t^{-n}[f(t)] \] and its \( n - 1 \) derivatives vanishing at \( t = c \).

III. The operator of order zero leaves the function unchanged. \( D^0[f(t)] = f(t) \).

IV. Derivatives of arbitrary order are linear operators:

\[ D^\alpha[a \cdot f(t) + b \cdot g(t)] = a \cdot D^\alpha[f(t)] + b \cdot D^\alpha[g(t)]. \]

V. The law of exponents for differentiation holds:

\[ D^\alpha[f(t)] \cdot D^\beta[f(t)] = D^{\alpha+\beta}[f(t)]. \]

Another important relationship to note is the Fourier transform of an arbitrary derivative, defined as

\[ \mathcal{F}[D^\alpha f(t)] = (j \omega)^\alpha \mathcal{F}[f(t)] \] (3.83)

which, along with the definition in equation (3.82), is useful in the analysis of system models containing fractional derivatives.
An alternative definition of the arbitrary derivative developed by Grunwald [62] for \( f(t) = 0 \) when \( t < 0 \) is
\[
D^\alpha [f(t)] = \lim_{N \to \infty} \left\{ \left( \frac{1}{N} \right)^{-\alpha} \sum_{i=0}^{N-1} \frac{\Gamma(i - \alpha)}{\Gamma(i + 1)} f(t - i \frac{t}{N}) \right\}. \tag{3.84}
\]
This definition is preferred over the Riemann-Liouville definition in equation for numerical analysis because it only involves the evaluations of the function itself and not its derivatives or integrals [47].

### 3.3.4 Model Forms

As stated in Chapters 1 and 2, FD models are extensively used to describe the mechanical behavior of various viscoelastic media. Viscoelastic materials can be simply described as materials which display a combination of elastic and viscous behavior. This behavior is frequency dependent, therefore difficult to accurately model over a wide range of frequencies using conventional models. Several approaches have been attempted, resulting in various models used to describe viscoelastic behavior. Some of the more common models include the classical viscoelastic model, the Kelvin-Voigt model and the FD viscoelastic model.

A general form of the classical viscoelastic model can be expressed as [6]
\[
\sum_{m=0}^{M} b_m D^m \sigma(t) = \sum_{n=0}^{N} b_n D^n \epsilon(t) \tag{3.85}
\]
which uses a series of integer differential operators acting on stress and strain, where \( \sigma \) is stress, \( \epsilon \) is strain, and \( b_m \) and \( b_n \) are constant coefficients. To adequately describe the behavior of viscoelastic materials, this model requires higher order derivatives, leading to a large number of parameters which become cumbersome to evaluate.

A simple alternative model to the classical approach is obtained by additively combining the purely elastic Hookean equation
\[
\sigma(t) = E \epsilon(t) \tag{3.86}
\]
and the purely viscous Newtonian equation

$$\sigma_v(t) = \nu \dot{\epsilon}(t)$$ \hspace{1cm} (3.87)

where $\sigma$ denotes stress, $\epsilon$ denotes strain, $E$ is the modulus of elasticity, and $\nu$ is the coefficient of viscosity. Note that equation (3.87) is equivalent to the constitutive expression for mechanical damping presented in equation (1.1), also known as the constitutive Kelvin damper model. The viscoelastic model obtained by additively combining these components is known as the Kelvin-Voigt model.

$$\sigma_v(t) = E \epsilon(t) + \nu \dot{\epsilon}(t).$$ \hspace{1cm} (3.88)

An accurate and more compact viscoelastic model is the constitutive FD model. In [70], this model is introduced as an alternative combination of equations (3.86) and (3.87), defined by the following expression

$$\sigma_\alpha(t) = E \theta^\alpha \mathcal{D}^\alpha \epsilon(t)$$ \hspace{1cm} (3.89)

where $\theta = \nu / E$, and $\alpha$ is a positive real number. Note that when $\alpha = 0$ equation (3.86) results, and when $\alpha = 1$ equation (3.87) is obtained. Therefore, when $\alpha$ is allowed to vary between 0 and 1, this expression is able to model material spectrum from viscous fluid to elastic solid. This model is also referred to as an intermediate model to denote the fact that it is able to model intermediate behavior between a purely elastic and purely viscous body [70, 69].

Common forms of constitutive viscoelastic models employed in the modeling of viscoelastic materials or components are the 3-parameter and 5-parameter forms as presented below [79].

3 parameter FD model:

$$\sigma(t) = E_0 \epsilon(t) + E_1 \mathcal{D}^\alpha [\epsilon(t)]$$ \hspace{1cm} (3.90)
5 parameter FD model:

\[ \sigma(t) + bD^\alpha[\sigma(t)] = E_0\epsilon(t) + E_1D^\alpha[\epsilon(t)]. \]  

(3.91)

The 3-parameter model is the general form of a fractional derivative model used to describe the mechanical properties of various viscoelastic materials \([7, 16, 20, 18, 19, 69, 70, 79]\), in which the time dependent stress is a superposition of elastic stress and a viscoelastic term containing a fractional derivative of order \(\alpha\). In such model equations, \(\sigma(t)\) denotes stress, \(\epsilon(t)\) denotes strain, and \(E_0, E_1, \) and \(\alpha\) are considered the parameters of the system model. A 4-parameter FD model which is also employed can be obtained by applying the constraint \(\beta = \alpha\) to the above 5 parameter model in equation (3.91). The general 3-dimensional constitutive equation for a general homogeneous and isotropic viscoelastic material, as presented in \([8]\), is

\[ (1 + \sum_{k=1}^K a_k D^\beta_k)(1 + \sum_{p=1}^P c_p D^\beta_p)\sigma_{mn}(t) \]

\[ = \delta_{mn}(1 + \sum_{p=1}^P b_p D^\beta_p)(\lambda_0 - \sum_{j=1}^J \lambda_j D^\alpha_j)\epsilon_{kk}(t) \]

\[ + 2(1 + \sum_{k=1}^K a_k D^\beta_k)(\mu_0 + \sum_{l=1}^L a_p D^\beta_p)\epsilon_{mn}(t) \]  

(3.92)

which can be adequately used to describe extremely complex viscoelastic behavior.

The use of FD viscoelastic models is favored for several reasons. First, they are able to accurately portray the behavior of viscoelastic materials in the transition region. Second, they predict real, continuous, and causal responses for systems with viscoelastic components which undergo impulsive loading. This model form also produces the desired elliptic stress-strain hysteresis loops for the sinusoidal motion of viscoelastic material. A further understanding of the evolution of FDs in modeling viscoelastic behavior can be obtained from a review of the references \([70, 7, 79]\).

The general form of the FD model considered in this work is the 3-parameter model presented in equation (3.90). In the time domain, this model can be expressed
as above. However when considering the solution of this equation, the definition of
the FD should be substituted, resulting in the proper time domain integro-differential
equation form
\[ \sigma(t) = E_0 \epsilon(t) + E_1 \frac{1}{\Gamma(-\alpha)} \int_0^t \frac{\epsilon(\tau)}{(t - \tau)^{\alpha+1}} d\tau. \]  
(3.93)
Proper representation of this model in the frequency domain can be obtained by
taking the Fourier transform of equation (3.90) which leads to the algebraic equation
\[ \hat{\sigma}(\omega) = E_0 \hat{\epsilon}(\omega) + E_1 [j \omega]^\alpha \hat{\epsilon}(\omega) \]  
(3.94)
which contains fractional powers of frequency.
3.4 Solution Techniques for Deterministic Excitations

3.4.1 Frequency Domain Analysis

Considering a signal in terms of its sinusoidal frequency content, or *spectra*, has proven to be a powerful tool in the analysis of continuous and discrete-time signals. This type of analysis is known as Fourier or frequency analysis, which employs Fourier Transforms or related methods to transform systems in and out of the frequency domain. A frequency domain transformation of a signal is simply a decomposition of the signal into its sinusoidal frequency content – record of harmonic frequency components and corresponding amplitudes. According to Fourier analysis, this representation is possible for signals in general, since an arbitrary function can be represented as a series of sines and cosines. Employing this type of transformation is desirable because in the frequency domain these signals are often easier to analyze, and after the analysis the results can be transformed back into original domain. An added advantage of this representation is that an exact frequency component's contribution to the signal can be clearly visualized.

In this work, the signals considered are the excitation and response of FD or FDP models. In analyzing these models, the frequency domain representation is employed, giving a clearer indication of the significant contributions of frequency on system behavior and leading to a more feasible solution for the predicted response of systems due to the beneficial properties associated with solving in the frequency domain (see Section 3.1.1).

In [28], [41], and [56], the use of the Fourier transform in solving for the response of linear systems subjected to arbitrary excitations is clearly presented. From an understanding of this theory the following solution for the response of FD and FDP models can be derived using a frequency-domain analysis.
3.4.1.1 FDP Models

In Section 3.2.3, it was shown that the proper representation for FDP models in the frequency domain is given by

\[-\omega^2 M(\omega) + j\omega C(\omega) + K(\omega)]X(\omega) = Y(\omega).
\]  

(3.95)

In this equation, \(X(\omega)\) and \(Y(\omega)\) denote the Fourier transforms of \(x(t)\) and \(y(t)\), respectively. This frequency domain equation can be employed to obtain the predicted response of a system, assuming that the excitation and system parameters are known. This can be done by first solving equation (3.95) in terms of the response, \(X(\omega)\), which results in the expression

\[X(\omega) = [-\omega^2 M(\omega) + j\omega C(\omega) + K(\omega)]^{-1} Y(\omega)
\]

(3.96)

which can also be written as

\[X(\omega) = H(\omega) Y(\omega),
\]

(3.97)

where \(H(\omega)\) is defined as

\[H(\omega) = [-\omega^2 M(\omega) + j\omega C(\omega) + K(\omega)]^{-1}.
\]

(3.98)

This function is called the complex frequency response function, which is simply the Fourier transform of the impulse response. Equation (3.97) states that the Fourier transform of the response, \(X(\omega)\), is equal to the complex frequency response function, \(H(\omega)\), multiplied by the Fourier transform of the excitation, \(Y(\omega)\). Therefore, taking the Fourier transform of the excitation record and assuming that the frequency dependent parameters are known functions of frequency, can lead to the response expressed in the frequency domain. The time domain response of the system can then be obtained by taking the inverse Fourier transform of the resulting product, \(X(\omega)\).

\[x(t) = \mathcal{F}^{-1}\{H(\omega) Y(\omega)\}
\]

(3.99)
\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [-\omega^2 M(\omega) + j\omega C(\omega) + K(\omega)]^{-1} Y(\omega) e^{j\omega t} \, d\omega \]  

(3.101)

or

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) Y(\omega) e^{j\omega t} \, d\omega. \]  

(3.102)

Therefore, equations (3.101) and (3.102) are the predicted response of frequency dependent parameter models obtained through frequency domain analysis. Based on these equations the solution for the response of the system model can be obtained if both the excitation and complex frequency response function are known for a system, where the complex frequency response function can be generated from equation (3.98) if the parameters \( M(\omega), C(\omega), \) and \( K(\omega) \) are known.

### 3.4.1.2 FD Model

From Section 3.3.4, it was seen that the proper frequency domain representation of the general FD model for viscoelastic material behavior (equation 3.94) is

\[ \dot{\sigma}(\omega) = E_0 \dot{e}(\omega) + E_1 \cdot (j\omega)^\alpha \cdot \dot{e}(\omega). \]  

(3.103)

In this equation \( \dot{\sigma} \) and \( \dot{e} \) are the Fourier transforms of \( \sigma(t) \) and \( \epsilon(t) \), which are often used to represent stress and strain, respectively.

From this frequency domain formulation it can be stated that the system model above contains a frequency dependent complex modulus of the form

\[ \hat{E}(\omega) = E_0 + E_1 \cdot (j\omega)^\alpha \]  

(3.104)
resulting in an equation form similar to that of elastic stress,

\[ \dot{\sigma}(\omega) = \hat{E}(\omega) \dot{\epsilon}(\omega). \quad (3.105) \]

The constitutive model of equation (3.103) can be used to obtain the predicted stress response given the input strain by employing frequency domain analysis techniques. This can be done by first obtaining the Fourier transform of the strain record and then known complex modulus function and multiplying them together to obtain the stress response in the frequency domain, as shown symbolically in the equation above. The stress response in the time domain can then be obtained by taking the inverse Fourier transform of the resulting product, \( \dot{\sigma}(\omega) \).

\[
\sigma(t) = \mathcal{F}^{-1}\{\hat{E}(\omega) \dot{\epsilon}(\omega)\} \quad (3.106)
\]

\[
= \mathcal{F}^{-1}\{[E_0 + E_1 \cdot (j \omega)^\alpha] \dot{\epsilon}(\omega)\} \quad (3.107)
\]

Thus, using the definition of the inverse Fourier transform in equation (3.3) on the above expression results in the following stress response for the fractional derivative model:

\[
\sigma(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{E}(\omega) \dot{\epsilon}(\omega) e^{j\omega t} \, d\omega \quad (3.108)
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} [E_0 + E_1 (j \omega)^\alpha] \dot{\epsilon}(\omega) e^{j\omega t} \, d\omega. \quad (3.109)
\]

Therefore, equation (3.108) can be evaluated to obtain the system response for the fractional derivative model presented in equation (3.103), assuming that the excitation (in this case the strain) and the real and complex parameter functions (in this case the modulus and complex modulus) are known. Moreover, the frequency domain analysis applied to this model can be used to obtain the response of similar fractional derivative models in general.
The constitutive model of equation (3.103) can also be used to solve for unknown strain displacement when the applied stress on the system is known. This can be done by solving equation (3.103) in terms of $\varepsilon(\omega)$, which leads to the expression

$$\dot{\varepsilon}(\omega) = [E_0 + E_1(j\omega)^{\alpha}]^{-1}\dot{\sigma}(\omega). \quad (3.110)$$

This equation can be used to obtain the predicted strain response given the exciting stress by first obtaining the Fourier transform of the strain record and then known complex modulus function and multiplying them together to obtain the stress response in the frequency domain, as shown symbolically in the equation above. The strain response in the time domain can then be obtained by taking the inverse Fourier transform of the resulting product, $\hat{\varepsilon}(\omega)$.

$$\varepsilon(t) = \mathcal{F}^{-1}\{[E_0 + E_1(j\omega)^{\alpha}]^{-1}\dot{\sigma}(\omega)\} \quad (3.111)$$

Employing the definition of the inverse Fourier transform of equation (3.3) on the above expression results in the following strain response for the fractional derivative model.

$$\varepsilon(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [E_0 + E_1(j\omega)^{\alpha}]^{-1}\dot{\sigma}(\omega) e^{i\omega t} \, d\omega \quad (3.112)$$

Comparing equations (3.106) and equations (3.100) it can be stated that FD and FDP models yield similar equation forms in the frequency domain. The numerical solution of these models in the frequency domain require similar numerical techniques. Therefore, only the numerical solution of FDP models are considered in the next chapter and the techniques developed to obtain the solution are assumed representative of those required to obtain the solution for FD models as well.
3.4.2 Time Domain Analysis

3.4.2.1 FDP Models

When information about a system is given in the time domain, then the analysis and solution for the system may be more feasible in the time domain. Thus, when considering a frequency-dependent parameter model, such as the model form presented in equation (1.2), it is necessary to obtain a meaningful representation of the model for analysis in the time domain. Obtaining a time-domain representation can be accomplished by first rearranging equation (3.61) to be

\[- \omega^2 M(\omega)X(\omega) + i \omega C(\omega)X(\omega) + K(\omega)X(\omega) = Y(\omega) \]  \hspace{1cm} (3.113)

and then applying the inverse Fourier transform to transform it into the time domain. Applying the inverse Fourier transform and recalling from the transform property that multiplication in the frequency domain transforms to convolution in the time domain leads to the following integro-differential equation time-domain expression of the model presented in equation (3.60).

\[ \int_{0}^{t} M(t - \tau) \ddot{x}(\tau) \, d\tau + \int_{0}^{t} C(t - \tau) \dot{x}(\tau) \, d\tau + \int_{0}^{t} K(t - \tau) x(\tau) \, d\tau = y(t) \]  \hspace{1cm} (3.114)

Therefore, equation (3.114) is the time domain expression that must be solved to obtain the solution to the general frequency-dependent parameter model in time-domain analysis. Solving this expression requires a knowledge in solution techniques for integro-differential equations.

3.4.2.2 FD Models

Based on the similarities between fractional derivative models and frequency-dependent parameter models in the frequency domain, it can be expected that similarity will be
found between these model types in the time domain. This is because the time domain model can be obtained by taking the inverse Fourier transform of the frequency-domain model forms. These similarities in model form result from the fact that fractional derivative models and frequency-dependent parameter models are members of the same model family, and, similarly, result in integro-differential equations in the time domain.

To clarify that fractional derivative models result in integro-differential equations, consider the following model of the macroscopic viscoelastic properties of a dilute solution of coiled polymer molecules placed in a Newtonian fluid medium [7].

\[
\sigma(t) = \mu_s \frac{d\varepsilon}{dt} + \left[ \frac{3}{2} (\mu_0 - \mu_s) n k T \right]^{1/2} D^{1/2} [\varepsilon(t)]
\]  

(3.115)

This expression is similar to that in equation (3.90). The parameters of this model are \(\mu_0\) and \(\mu_s\), which denote the steady-flow viscosities of the solution and solvent; and \(T\), \(k\), and \(n\), which denote the absolute temperature, Boltzmann's constant, and number of molecules per unit solution volume, respectively. Note that an integro-differential form of this model results when the derivative operator is replaced its integral definition from equation (3.82). This substitution results in the equation

\[
\sigma(t) = \mu_s \frac{d\varepsilon}{dt} + \left[ \frac{3}{2} (\mu_0 - \mu_s) n k T \right]^{1/2} \frac{1}{\Gamma(-\frac{1}{2})} \int_0^t (t - \tau)^{-\frac{3}{2}} \varepsilon(\tau) \, d\tau.
\]  

(3.116)

Therefore, similar to the FDP model, FD models in the time domain lead to integro-differential equations which need to be solved. Solving these expression requires a background knowledge in solution techniques for integro-differential equations. One useful source for the numerical analysis of integro-differential equations is [15].
Chapter 4

Random Vibrations

The analysis of engineering systems subjected to random excitations is a topic of great importance in mechanical and structural engineering where random loadings such as winds, earthquakes, and sea waves are considered. Reliable models of engineering uncertainty also involve the use of random processes. In modeling these applications, the use of stationary random processes is favored and a probabilistic approach is required to obtain an accurate description of the system response.

The following sections provide a brief review of random theory required for the analysis of FD and FDP models using a probabilistic approach. Presented first are the basic concepts of probability, random processes and random vibrations applicable to the developments in this work. Following this is an introduction to the use of the Monte Carlo method for obtaining statistical information on the response systems subjected to random excitations. Finally, the Autoregressive method which can be used to generate excitation time histories compatible with a given power spectrum is introduced.

4.1 Background Theory

When considering a random process, the totality of all possible records that might have been produced under similar conditions must be taken into account. Therefore, a single record is not as meaningful as a statistical description of all possible records produced under similar conditions. For this reason a probabilistic approach is employed to produce a statistical description of the process.
To understand how probabilistic information can be used to characterize a random process, consider the following analogy between the description of a sinusoidal vibration and random vibration. A sinusoidal function is often characterized by its amplitude and frequency. Similarly, random vibration is characterized by an average amplitude and a decomposition in frequency. The average amplitude most often employed is the root-mean-square value, and the frequency decomposition is often indicated by the mean square spectral density. Note that additional information about a sinusoidal can be given to provide a more complete description of it, such as phase. Similarly additional statistical parameters can also be employed to provide a more complete description of a random process. Some of the important statistical parameters useful in this work are highlighted below.

4.1.1 Random Processes

A stochastic or random process, denoted as $x(t)$, does not describe just one time history resulting from a process, but a whole family or ensemble of possible time histories which could result from the same experiment. Theoretically, an ensemble contains an infinite number of time histories. Any individual time history belonging to the ensemble is called a *sample function* and can be denoted as $x^j(t)$. Figure 4.1 shows an ensemble of individual sample functions resulting from a random process. The functions are sketched as a function of time. Note that the time interval for each sample function of a single random process is the same.

In a random process, certain sample functions may be more probable to occur than others. Therefore, the use of probabilistic information is essential in the description of a random process. The probability of a favorable outcome occurring is defined as the fraction of favorable outcomes over all possible outcomes. Thus, a probability expression will be either positive or zero.
Figure 4.1: Ensemble of Time Histories for a Random Process.

A fundamental component of probabilistic information is the first order probability distribution which is defined at fixed values in time for a random process, $x(t_i)$ for the fixed value of $t = t_i$. The first order probability distribution commonly relied on is the probability density function (PDF), $p[x(t_i)]$, denoted as $p(x)$ when the point in time, $t_i$, is assumed to be understood. This function contains information on the probability distribution of individual values across the ensemble at fixed points in time. Several characteristics can be noted for the PDF, however, the only one important to this work is the fact that PDFs of homogeneous or stationary random processes can be defined independent of $t$. A stationary random process contains a shift-invariant probability structure. Further information on probability distributions can be obtained from [1, 33]

Another important probability distribution to note is the second order distribution which describes the joint distribution of probabilities for pairs of values, such as $x(t_1)$ and $x(t_2)$, which occur in the ensemble for the fixed values of time $t_1$ and $t_2$. This distribution is known as the joint probability density function (JPDF), and is denoted
as \( p[x(t_1)x(t_2)] \) or \( p(x_1, x_2) \) when the values \( t_1 \) and \( t_2 \) are assumed understood. For homogeneous or stationary random processes this function does not dependent on the values of \( t_1 \) and \( t_2 \), but on their time lag \( \tau \), defined as \( \tau = t_2 - t_1 \).

From the PDF and JPDF, useful information about the random process can be obtained. Information important to this work include the mean, mean square, variance, auto-correlation function (ACF), and power spectral density (PSD).

The mean of a random process is the expected value of that process at a fixed point in time. The mean is defined as

\[
\mu_x = \mathbb{E}[x(t)]
\]  

(4.1)

where \( \mathbb{E}[\cdot] \) is the expectation operator define for the function \( x \) as

\[
\mathbb{E}[x] = \int_{-\infty}^{\infty} x(t) \ p[x(t)] \ dt.
\]  

(4.2)

When considering discrete processes, such as those considered in this work, the mean value can be expressed as the weighed sum of the process values, where the weights are the probabilities of the occurrence of the value. When the probability distribution of values is constant this can be interpreted as the average value of \( x \) at each point in time.

The mean square of a process is the expected squared value of the process at each point in time, defined by

\[
m_{x^2} = \mathbb{E}[x^2].
\]  

(4.3)

The variance of a random process is the expected squared value of the process with respect its mean at each time point,

\[
\sigma_x^2 = \mathbb{E}[(x - \mu_x)]
\]  

(4.4)

\[
= \mathbb{E}[x^2] - (\mathbb{E}[x])^2
\]  

(4.5)
which is a measure of the fluctuation of a process about its mean value. Using equation (4.5), the variance can be derived from knowledge of the mean and mean square functions.

The expected value of the product of two points in time \( x(t_1) \) and \( x(t_1) \) for a random process is known as the auto-correlation function (ACF), which is defined by

\[
R_x(t_1, t_2) = \mathbb{E}[x_1 \ x_2]. \tag{4.6}
\]

When \( t_1 = t_2 \) the auto-correlation function is equivalent to the mean square function. Using equations (4.3), (4.5) and (4.6), the variance at any point in time of a zero mean process can be defined as

\[
\sigma_x^2(t_i) = R_x(t_i, t_i). \tag{4.7}
\]

For clarity, it should be reiterated that the mean, mean square, variance functions, as well as the ACF are all function of time, in general. However, when considering stationary random processes of interest in this work, these functions become time invariant. Therefore, the mean, mean square, and variance are constant in time, and the ACF becomes only a function of the time lag

\[
R_x(t_1, t_2) = R_x(t_2 - t_1) = R_x(\tau). \tag{4.8}
\]

For stationary random processes, a useful spectral representation can be employed. This representation is given by the power spectral density (PSD) function, related to the auto-correlation function through the Wiener-Khintchine equations

\[
S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega \tau} R_x(\tau) \ d\tau \tag{4.9}
\]

\[
R_x(\tau) = \int_{-\infty}^{\infty} e^{j\omega \tau} S_x(\omega) \ d\omega. \tag{4.10}
\]
From equations (4.7) and (4.10) a relationship for the variance of a zero mean process can be derived as a function of the PSD.

\[
\sigma_x^2(t_i) = \int_{-\infty}^{\infty} S_x(\omega) \, d\omega
\]  

(4.11)

One type of random process often employed in the study of random vibrations is the Gaussian White noise random process which makes its use desirable. The important characteristic of this process which should be noted is that it yields a constant PSD. This constant PSD results from the fact the values of this process are uncorrelated for all time lags except \( \tau = 0 \), which results in an ACF of

\[
R(\tau) = 2\pi \delta(\tau)
\]

(4.12)

with \( \delta(\tau) \) denoting the Dirac function.

### 4.1.2 Spectral Relationships

The theory of random processes provides useful analytical tools for obtaining statistical information on the response of systems subjected to random excitations.

Given the PSD of a random excitation and the frequency response function describing a system, the PSD of the system response can be obtained using the expression [59]

\[
S_x(\omega) = H^*(\omega) S_y(\omega) H(\omega)
\]

\[
= |H(\omega)|^2 S_y(\omega)
\]

(4.13)

where * denotes the complex conjugate.

Once the PSD of the excitation is known, the variance of the response can be obtained. Since the excitation is a zero mean process, the variance of the response
can be computed using equation (4.11).

\[ \sigma_z^2 = \int_{-\infty}^{\infty} S_z(\omega) \, d\omega \]  \hspace{1cm} (4.14)

Substituting in the expression for the PSD of the response presented in equation (4.13), the variance of the response can be defined in terms of the PSD of the excitation and the frequency response function of the system.

\[ \sigma_z^2 = \int_{-\infty}^{\infty} |H(\omega)|^2 S_y(\omega) \, d\omega \]  \hspace{1cm} (4.15)

Therefore, given the system model and PSD of the excitation, the variance of the response can be obtained.

## 4.2 Monte Carlo Simulations

The Monte Carlo simulation is a statistical sampling method which involves the repeated generation of random variates and the subsequent use of deterministic methods for obtaining the system response to each realization of the random variates [76]. When excitations or engineering uncertainty involve random processes, implementation of this method requires the statistical synthesis of an ensemble of random realizations, which are employed to yield estimates of response statistics.

In this work, the Monte Carlo method employed consists of three major steps. First, given PSD of the excitations, a compatible ensemble of excitation time histories is generated. This is done via an AR modeling technique as described in the following section. Second, each excitation generated is considered deterministic and the response of the system to each is obtained using deterministic solution methods. Finally, the system responses to the excitation ensemble are collected and statistically synthesized to obtain useful estimates of response statistics. A flow chart of the Monte Carlo simulation method employed in this work is provided in Figure 5.7 of
Chapter 5. The fundamental idea behind the Monte Carlo method is that an adequate sized excitation ensemble can sufficiently represent the statistics of the random process considered and thus the resulting response ensemble will sufficiently represent the statistics of the responses.

While the Monte Carlo method is quite versatile, it can also be computationally costly when applied to complex structural problems. Therefore, computational efficiency is important, but beyond the scope of this work. An excellent introduction to the Monte Carlo method is given in [71]. A more comprehensive review of the Monte Carlo method and related references is covered by Spanos and Zeldin [76].

4.3 Generating Compatible Excitations - AR Modeling

Before the Monte Carlo Simulation technique can be employed in this work, a method of generating discrete excitations compatible with a given power spectra must be identified. The method identified and used for this work is the Autoregressive(AR) method. This method generates compatible excitations by filtering a white noise input, where the coefficients of the filter are optimized to produce an output whose PSD is a discrete approximation of the target PSD given.

To understand the theoretical development of the AR-method, consider the following objective and analysis.

Objective: Given the PSD of a random process, generate compatible discrete time histories.

Analysis: This is a black box problem, in that an unknown system must be identified to generate an output which can simulate a given PSD. When the system to be identified is discrete, it is often referred to as a digital filter. In both cases, the system is described by a transfer function which provides a complete description of the system. The symbolic representation of this problem is presented in Figure 4.2,
where $W$ is the input, $Y$ is the output, and $H$ is a mathematical description of the system used to transfer $W$ into $Y$.

To solve a black box problem, the input and output of the system must be known. In this case, the input has not been specified, and only the PSD of the output is known. Since only a PSD is known and a system transfer function is desired, a relationship between the desired PSD and transfer function must be established. For linear stationary systems, the following relationship derived from equation (4.13) which relates a system transfer function to the input and output PSDs expressed in the frequency domain.

$$S_Y(\omega) = |H_{YW}(\omega)|^2S_W(\omega)$$  \hspace{1cm} (4.16)

In this equation, $H_{YW}(\omega)$ is the transfer function expressed in the frequency domain, and $S_W(\omega)$ and $S_Y(\omega)$ are the PSDs of the input and output, respectively. The symbolic representation for this system is given in Figure 4.3, where $|H_{YW}(\omega)|^2$ is the system to be identified.

If the PSDs of the input and output are known, equation (4.16) can be used to solve for the transfer function of the system. In this analysis, the desired PSD, $S_Y(\omega)$, is given. Therefore, a random process should be selected as the input with a known PSD which can be used with the transfer function to result in a PSD which is a good approximation of the target PSD. An excellent choice for the input is white

![Unknown System Diagram](image)

**Figure 4.2: Black Box Problem: Identify system $H$ that takes input $W$ and transforms it to output $Y$.**
\[ S_w(\omega) \xrightarrow{\text{\mid H(\omega)\mid}^2} S_Y(\omega) \]

Figure 4.3: Black Box Problem: Identify system \( |H(\omega)|^2 \) that takes input \( S_w(\omega) \) and transforms it to output \( S_Y(\omega) \).

noise because it is a simply defined process which can be easily generated numerically with a specified constant PSD. Using a white noise input, its PSD can be defined as \( S_w(\omega) = 1 \). Then, equation (4.16) can be used to define the desired transfer function as

\[ |H_{YW}(\omega)|^2 = S_Y(\omega). \quad (4.17) \]

Recall that the use of equation (4.16) results in a linear time-invariant system. It is also desired that the system be discrete, as mentioned in the objective, so that it can be used in the numerical analysis employed in this work. Therefore, it is important to note that a discrete \( n \)-input \( n \)-output linear, time-invariant system is defined by the equation

\[ Y_k = \sum_{l=-\infty}^{\infty} h_{k-l} W_l \quad (4.18) \]

where \( W_k \) and \( Y_k \) are \( n \)-component vectors which represent the \( k^{th} \) samples of the input and output, respectively, and \( h_k \) is an \( n \times n \) matrix representing the \( k^{th} \) sample of the impulse response sequence. This expression is given in the discrete time domain, therefore, \( h_k \) is related to \( H(\omega) \) through the discrete Fourier transform. However, in digital signal processing, discrete systems are frequently analyzed using the \( z \)-transform. This transform is defined as

\[ H(z) = \sum_{k=-\infty}^{\infty} h_k z^{-k} \quad (4.19) \]
wherever this series converges in the complex $z$ plane. On the unit circle, $|z| = 1$, the $z$-transform is equivalent to the discrete Fourier transform of $h_k$ [73].

$$H(e^{j\omega \Delta t}) = H(z)|_{z=e^{j\omega \Delta t}}$$  \hspace{1cm} (4.20)

In the above expression, $H(e^{j\omega \Delta t})$ is equivalent to the short hand expression $H(\omega)$ used throughout this work.

For a discrete system the transfer function in equation (4.18) can be described by the analytical expression

$$H(\omega) = \frac{b_0 + b_1\omega^1 + b_2\omega^2 + \ldots + b_n\omega^n}{a_0 + a_1\omega^1 + a_2\omega^2 + \ldots + a_m\omega^m}$$  \hspace{1cm} (4.21)

where the $b_i$'s and $a_i$'s are the coefficients which are used to uniquely define a system's input and output relationship.

Using the development presented above, equation (4.17) can now be employed to find $H(\omega)$ desired. First, an acceptable form for the transfer function must be determined. Therefore, $n$ and $m$ must be selected in equation(4.21) to determine number of coefficients which will be used to define the system. Then, the relationship in equation (4.17) can be used to determine the coefficients for the transfer function which best approximate the given PSD.

Consider the use of the AR filter to model the relationship between the input and output of the system which will be used to generate output compatible with a given PSD [73].

$$Y_r = - \sum_{k=1}^{m} A_k Y_{r-k} + b_0 W_r$$  \hspace{1cm} (4.22)

This represents an $n$-input $n$-output AR system of order $m$, AR($m$). For this system $Y_r$ is the $r^{th}$ sample of the response which is computed from the $m$ previous responses and $W_r$ which is the corresponding $r^{th}$ component of the input. The recursive characteristic of using past outputs to generate the current value leads to the name from this method.
The transfer function relating the input and output of this system is

\[ H(z) = \left[ I_n + \sum_{k=1}^{m} A_k z^{-k} \right]^{-1} b_0 \]  

(4.23)

where \( H(w) \) can be obtained with the substitution \( z = e^{j\omega \Delta t} \).

\[ H(\omega) = \left[ I_n + \sum_{k=1}^{m} A_k e^{-j\omega \Delta t} k \right]^{-1} b_0 \]  

(4.24)

Using this expression with the frequency domain substitution in equation (4.17) leads to the following approximation of the given PSD:

\[ \hat{S}_Y(\omega) = \frac{b_0^2}{|1 + \sum_{k=1}^{m} A_k z^{-k}|^2}. \]  

(4.25)

The parameters \( A_k \) and \( b_0 \) can be selected to give a good approximation of the desired output PSD, \( S_Y(\omega) \), by minimizing the error between the simulated PSD and the PSD given. A meaningful representation of this error is

\[ \varepsilon = \frac{b_0^2}{2\omega_c} \int_{-\omega_c}^{\omega_c} S_Y(\omega) \hat{S}_Y(\omega) d\omega. \]  

(4.26)

Substituting equation (4.25) into equation (4.26) gives

\[ \varepsilon = \frac{1}{2\omega_c} \int_{-\omega_c}^{\omega_c} S_Y(\omega) \left| 1 + \sum_{k=1}^{m} A_k z^{-k} \right|^2 d\omega. \]  

(4.27)

The minimization of \( \varepsilon \), which requires that

\[ \frac{\partial \varepsilon}{\partial A_k} = 0, \quad 0, 1 \leq k \leq m \]  

(4.28)

leads to the following relationship which can be solved for the coefficients, \( A_k \) [72].

\[ \sum_{k=1}^{m} A_k R_{l-k} = -R_l, \quad 1 \leq l \leq m \]  

(4.29)

This expression represents a system of simultaneous linear equations which must be solved to determine the coefficients \( A_k \). The \( R_l \)'s can be found from

\[ R_k = \int_{-\omega_c}^{\omega_c} S_Y(\omega) \cos(k \omega \Delta t) d\omega. \]  

(4.30)
From the requirement of equal energy in the target spectrum and the output spectrum, the following relationship is derived to obtain the coefficient $b_0$, known as the energy normalization constant [72].

$$R_0 = 2\omega_c b_0^2 - \sum_{k=1}^{m} A_k R_k \quad (4.31)$$

Therefore, once the coefficients $A_k$ are determined, the expression above can be used to determine the coefficient $b_0$.

Once the coefficients of the transfer function are determined, they can be used in equation (4.22) to generate output compatible with the target PSD. Substituting equation (4.24) into equation (4.22) gives the following time domain response of the digital recursive filter.

$$Y_r = \begin{cases} 
0 & \text{for } k < 0 \\
 b_0 W_r - \sum_{k=1}^{m} A_k Y_{r-k} & \text{for } k \geq 0 
\end{cases} \quad (4.32)$$

Once obtained, this filtered output can then be used as one record of the excitation. See references [72] and [73].
Chapter 5

Numerical Analysis

Techniques introduced in the previous chapter are used in this chapter to numerically analyze the general FDP model form introduced in this work. In Section 3.4 it has been established that FDP and FD models yield similar equation forms in the time domain and frequency domain, and therefore require similar solution techniques to obtain their responses. For this reason only the FDP model form is considered in this analysis, and the techniques employed are assumed representative of those required to obtain solutions for both FDP and FD models.

5.1 Problem Description

In the following analyses, a single degree-of-freedom system with frequency dependent parameters is considered which can be informally described by the following model form:

\[ M(\omega)\ddot{x}(t) + C(\omega)\dot{x}(t) + K(\omega)x(t) = y(t). \]  

(5.1)

It is assumed that information about the excitation, \( y(t) \), and system parameters, \( M(\omega) \), \( C(\omega) \), and \( K(\omega) \), is known and the response, \( x(t) \), is desired. The analyses are presented in two parts. The first section considers the response of this system when subjected to deterministic excitations. The second considers the response of this system when subjected to random excitations. For both cases, solutions in the frequency domain are developed. To verify the numerical results for each analysis, a simple test problem is considered and solved analytically, and the results compared with those obtained numerically. Only the numerical results with a derived analytical
solution are presented in this work as verification of the numerical techniques and correctness of the algorithm developed.

5.2 Deterministic Excitations: Frequency Domain Analysis

In the following frequency domain analysis, the excitation of the the FDP model is considered deterministic, and the numerical solution for the system model is computed. To carry out this analysis, a numerical algorithm has been developed with the assumption that the excitation, $y(t)$, is a known function of time and the system parameters, $M(\omega)$, $C(\omega)$, and $K(\omega)$, are known functions of frequency. The development of the program is described below, and followed by an example problem which is solved analytically. The algorithm is then used to simulate the system response in the example, and the results are compared to the analytical solution to demonstrate the correctness of the program developed. The derivations used in the development of the program and analytical solutions are provided in Appendix B.

5.2.1 Program Development

The objective was to develop a code to simulate the response of the system described by equation (5.1), using the assumption that the excitation and parameters are known. This program has been developed assuming that the deterministic excitation, $y(t)$, is given as a known analytical function of time and the parameters $M(\omega)$, $C(\omega)$, and $K(\omega)$ are given as known analytical functions of frequency. The system is considered to start from rest, resulting in the initial conditions, $x(0) = 0$ and $\dot{x}(0) = 0$. In the program, the known functions are discretized by defining discrete time and frequency vectors, $t_i$ and $\omega_k$, respectively, and substituting them into the given functions to obtain the corresponding discrete excitation and parameter vectors, $y(t_i)$, $M(\omega_k)$, $C(\omega_k)$, and $K(\omega_k)$. Compatibility between the time and frequency domain is ensured
using the Nyquist cut-off frequency

\[ \omega_c \leq \frac{\pi}{\Delta t} \]  \hspace{1cm} (5.2)

an imposed constraint for the discrete fourier transform centered about \( \omega = 0 \).

Following the development in Section 3.4.1.1, the Fourier transform of equation (5.1) can be expressed as

\[ X(\omega) = H(\omega) Y(\omega). \]  \hspace{1cm} (5.3)

In this equation \( Y(\omega) \) is the Fourier transform of the excitation, \( X(\omega) \) is the Fourier transform of the system response, and \( H(\omega) \) is the complex frequency response function defined as

\[ H(\omega) = \left[ -\omega^2 M(\omega) + j\omega C(\omega) + K(\omega) \right]^{-1}. \]  \hspace{1cm} (5.4)

To simulate equation (5.3) numerically, the discrete vectors \( H(\omega_k) \) and \( Y(\omega_k) \) must first be computed. \( H(\omega_k) \) is obtained by substituting the discrete parameter vectors, \( M(\omega_k), C(\omega_k), \) and \( K(\omega_k), \) into equation (5.4). For the single degree of freedom system considered, this is defined by the scheme

\[ H(\omega_k) = \frac{1}{\left[ -\omega_k^2 M(\omega_k) + j\omega_k C(\omega_k) + K(\omega_k) \right]}. \]  \hspace{1cm} (5.5)

\( Y(\omega_k) \) is obtained by taking the Discrete Fourier Transform (DFT) of the excitation vector, \( y(t_i), \) evoked in the program using a Fast Fourier Transform (FFT) subroutine.

Once \( H(\omega_k) \) and \( Y(\omega_k) \) are computed, the DFT of the response is obtained using the scheme

\[ X(\omega_k) = H(\omega_k) Y(\omega_k). \]  \hspace{1cm} (5.6)

This results in the response defined in the frequency domain. Therefore, it must be transformed into the time domain to yield the response of the system, \( x(t) \). To numerically perform this transformation, the Inverse Discrete Fourier Transform (IDFT) of
Figure 5.1: Flow Chart of the FDP Model Solution Algorithm.
\( X(\omega_k) \) is taken. This is carried out using an Inverse Fast Fourier Transform (IFFT) subroutine.

Using the steps described above, the code is able to simulate the solution for the general frequency-dependent parameter system subjected to a deterministic excitation. These steps are also summarized by the program flow chart presented in Figure 5.1. To analyze the ability of this code to produce accurate results, the results of this program are compared to those obtained analytically.

As a related work to this development, Appendix A contains the development of a generic DFT subroutine which was also employed to obtain the system response. This subroutine was derived based on a numerical approximation for the Fourier transforms. The use of this DFT code yields identical results to those obtained from the use of the FFT subroutines. However, the use of FFT subroutines were employed to obtain the results presented here because of numerical efficiency associated with their use.

### 5.2.2 Analytical Example

In this example, a single degree of freedom FDP model excited by a deterministic forcing function is considered. For this system, the excitation and model parameters are given as

\[
y(t) = \begin{cases} 
0 & \text{if } t < 0 \\
\sin(3t) & \text{if } t \geq 0
\end{cases}, \quad (5.7)
\]

\[
M(\omega) = \omega^2, \quad (5.8)
\]

\[
C(\omega) = 0.1, \quad (5.9)
\]

\[
K(\omega) = 1.0. \quad (5.10)
\]
Therefore, the system equation can be written as
\[
\omega^2 \ddot{x}(t) + 0.1 \dot{x}(t) + 1.0 x(t) = \sin(3t), \quad t \geq 0. \tag{5.11}
\]
The initial conditions for this system are given as \( x(0) = 0 \) and \( \dot{x}(t) = 0 \). From the information above, this system can be solved analytically in the frequency domain and the solution transformed into the time domain to obtain the desired response, \( x(t) \).

The theory required to analytically obtain the response of the system in equation (5.11) using a frequency domain approach was presented in Section 3.4.1.1. In this section it was concluded that the general response to equation (5.1) can be expressed as
\[
x(t) = \mathcal{F}^{-1}[X(\omega)]. \tag{5.12}
\]
where \( \mathcal{F}^{-1} \) represents the inverse of the Fourier transform, and \( X(\omega) \) is the response expressed in the frequency domain. According to equation (3.97), \( X(\omega) \) is defined as
\[
X(\omega) = H(\omega) \cdot Y(\omega), \tag{5.13}
\]
where \( H(\omega) \) is the complex frequency response function and \( Y(\omega) \) is the FT of the excitation. Applying the definition of \( H(\omega) \) in equation (3.98), to the one dimensional case, gives
\[
H(\omega) = \frac{1}{[-\omega^2 - j \omega \cdot 0.1 + 1.0]} \tag{5.14}
\]
Taking the FT of excitation defined in equation (5.7) results in
\[
Y(\omega) = \frac{j \pi}{2} [\delta(\omega + 3) - \delta(\omega - 3)] + \frac{3}{9 - \omega^2}. \tag{5.15}
\]
Now substituting equations (5.14) and (5.15) into equation (5.13) and taking the inverse transform of the product according to equation (5.12), leads to the following solution for the response of the system considered in equation (5.11).
\[
x(t) = 0.09375 \sin(t) + 0.075 e^{-t} - 0.0125 \sin(3t) - 0.00002 \cos(3t) \tag{5.16}
\]
This response equation is valid for $0 \leq t \leq \infty$ according to the problem definition. It should also be noted that this system is noncausal in that a non-zero response of $x(0) = 0.07498$ results for $t = 0$. However, the noncausality of the system does not affect the correctness of the analytical solution. A complete derivation of this solution is provided in Appendix B.1.

5.2.3 Numerical Comparison

To verify the results obtained using the numerical code, the frequency-dependent parameter problem of equation (5.11) was simulated and the numerical response compared with the analytical solution given in equation (5.16). Figures 5.2 through 5.5 show the results obtained for this example.

As shown in Figure 5.4, the numerical response gives a good prediction of the true response derived for this system. The maximum error for the simulation is 19% at the end deflection for at $t = 50$ seconds. A plot of this error is presented in Figure 5.5. This error is normalized with respect to the maximum value of the analytical response which occurs around $t = 3$ seconds. This is done to avoid numerical problems in error which occur as the response approaches zero. A further discussion of the error is presented in Appendix B.

Note that besides large error deflection at the end, the numerical simulation gives a good approximation of the response with in an error band of 0.5%. This error is largely due to the inability of the FFT to approximate the delta functions of the Fourier transform, as shown in Appendix B. The error deflection shown in the results of this example occur regardless of the cut-off time considered. From this it is concluded that the numerical program developed successfully predicts system responses for frequency-dependent parameter systems subjected to deterministic excitations when presented in a form similar to equation (5.1).
Figure 5.2: Frequency Response Function and Fourier Transform of Excitation for the system \( \omega^2 \ddot{x} + 0.1 \dot{x} + 1.0x = \sin(3t) \).

Figure 5.3: Response Expressed in the Frequency Domain for the system \( \omega^2 \ddot{x} + 0.1 \dot{x} + 1.0x = \sin(3t) \).
Figure 5.4: System Excitation and Solution Comparison for the system \( \omega^2 \dddot{x} + 0.1 \dddot{x} + 1.0 \dot{x} = \sin(3t) \).

Figure 5.5: Numerical Simulation Error for the system \( \omega^2 \dddot{x} + 0.1 \dddot{x} + 1.0 \dot{x} = \sin(3t) \).
5.3 Random Excitation: Frequency Domain Analysis

In the following analysis, the excitation in equation (5.1) is considered random. It is assumed that the PSD of the excitation, $S_y(\omega)$, and the system parameters, $M(\omega)$, $C(\omega)$, and $K(\omega)$, are known functions of frequency, and statistical information about the response is desired. To facilitate this random vibration problem the Monte Carlo method is employed. Therefore, an ensemble of excitations compatible with the given PSD is generated, each excitation treated as deterministic, and the corresponding responses obtained. The ensemble of obtained responses is then synthesized to extract the desired statistical information.

Numerical codes have been developed to perform the tasks mentioned above. Their development is described below. An example problem is considered in which the statistical information about the response is derived analytically. The programs are then used to simulate numerical results for the example, and are then compared with those obtained analytically. This is done to demonstrate the validity of the programs. The derivations used in the development of the programs and analytical solutions have been provided in Appendix B.

5.3.1 Program Development

One objective of this work was to develop a code to simulate the desired statistical information on the response of a FDP or FD system subjected to a random excitation. This code has been developed assuming that the system parameters, $M(\omega)$, $C(\omega)$, and $K(\omega)$, and PSD of the excitation, $S_y(\omega)$ are given as known analytical functions of frequency. The initial conditions of this system are taken as $x(0) = 0$ and $\dot{x}(0) = 0$. Use of the Monte Carlo Simulation method is desired to obtain statistical information on the response of the system.
To satisfy the objectives and assumptions for this analysis, a series of subroutines were developed and incorporated in a program to carry out the Monte Carlo Simulation method. The first routine allows for the PSD of the excitation to be defined and discretized. This is done by defining a discrete frequency vector $\omega_k$ and then substituting it into the analytical formula to obtain the corresponding discrete PSD of the excitation, $S_y(\omega_k)$. Then a second subroutine is employed, which takes the discretized PSD and uses it to generate a compatible excitation. This is done by employing the AR model described in Section 4.3. Once an excitation is generated it is treated as deterministic and the response of the system obtained via the deterministic program discussed in Section 5.2.1.

These subroutines were then used in the performance the Monte Carlo simulation method which repeats the generation of excitations and subsequent solution of system responses until a specified number of response records for an adequate ensemble are obtained. The response ensemble is then numerically synthesized to compute statistical information about the response, such as the mean and variance.

To generate excitations compatible with the target PSD given, an AR model program was employed. Before incorporating this program for the Monte Carlo simulations, its ability to generate excitations compatible was assessed. This was done by defining a target PSD, using it in the AR program to generate a sample excitation, calculating the PSD of the generated excitation, and then comparing it with the original target PSD. To calculate the PSD of the discrete excitation record, the Welch's averaged periodogram method was used. This method involves approximating the PSD of overlapping windowed sections of the excitation record, and then taking the average of these PSDs to obtain an approximation for the entire record. The results of this analysis are provided in Figure 5.6 and shown that the AR model program adequately produces excitation records whose PSD closely approximates the target.
PSD given. In this example and the Monte Carlo simulations an AR filter length of 10 was selected based on an analysis conducted to assess the filter error resulting from various filter lengths.

In the program, the parameters and PSD of the excitation are discretized by defining a discrete frequency vector \( \omega_k \), and substituting it into the analytical formulas to obtain the corresponding discrete excitation and parameter vectors, \( y(t_i) \), \( M(\omega_k) \), \( C(\omega_k) \), and \( K(\omega_k) \). Compatibility between the time and frequency domain is ensured by using the Nyquist cut-off frequency relationship \( \omega_c = 0.5(2\pi/\Delta t) \), a constraint imposed by the use of the discrete Fourier transform centered about \( \omega = 0 \).

To verify the numerical approximations obtained from the Monte Carlo simulations, a simple test problem was considered which allowed for the integral in equation (4.15) to be evaluated analytically. The analytical results were then compared with those obtained using the Monte Carlo simulation program.

### 5.3.2 Analytical Example

The following FDP system excited by a zero mean white noise was considered and the statistical information on the response is obtained analytically,

\[
\omega^2 \ddot{x}(t) + 0.1 \dot{x}(t) + 1.0x(t) = W(t).
\]

In this equation the FDP is \( M(\omega) = \omega^2 \), and \( W(t) \) is used to denote the white noise excitation. The initial conditions for the system are given as \( x(0) = 0 \) and \( \dot{x}(0) = 0 \). The PSD of the excitation is given as

\[
S_W(\omega) = S_0,
\]

where \( S_0 \) is a constant, which will later be set equal to 1.

The solution for the variance of the system response is obtained analytically using the theory described in Section 4.1.2. Starting from the above system equation, the
Figure 5.6: AR Model Program Comparison with Given PSD.
MONTE CARLO ALGORITHM FLOW CHART FOR RANDOM VIBRATIONS

Figure 5.7: A Flow Chart of the Monte Carlo Algorithm for the FDP Model.
complex frequency response function is defined as

$$H(\omega) = \frac{1}{[-\omega^2 \cdot \omega^2 + j\omega \cdot 0.1 + 1.0]}.$$  \hspace{1cm} (5.19)

Then applying the relationship in equation (4.13) the PSD of the system response is defined by

$$S_x(\omega) = S_0 \left| \frac{1}{[-\omega^2 \cdot \omega^2 + j\omega \cdot 0.1 + 1.0]} \right|^2.$$  \hspace{1cm} (5.20)

Since the excitation is a zero mean process, the variance of the response can be expressed as follows using the relationship in equation (4.15),

$$\sigma_x^2 = \int_{-\infty}^{\infty} S_0 \left| \frac{1}{[-\omega^2 \cdot \omega^2 + j\omega \cdot 0.1 + 1.0]} \right|^2 d\omega.$$  \hspace{1cm} (5.21)

The analytical solution of this integral is very lengthy and complex, and therefore is not provided here. For all practical purposes considered in this work, this analytical solution can be obtained by numerically approximating the integral between the finite limits $\pm \omega_c$. Since equation (4.14) states that variance is the area under the curve $S_x(\omega)$, the finite limits used should satisfy $S_x(\pm \omega_c) \approx 0$. For the system considered above with $S_0 = 1$, the limits of $\omega_c = \pm 20$ rad/s were used since $S_x(\omega)$ is essential zero beyond these points as shown in Figure 5.8.

Using the limits of $\pm 20$ rad/s produce an excellent approximation for variance which was essentially equivalent to that obtained using the limits of $\pm 200$ rad/s as shown in Figure 5.9. In this example $S_x(\omega) = 1$. Evaluating equation (5.21) with the limits of $\pm 20$ rad/s results in the variance of

$$\sigma_y^2 = 40$$  \hspace{1cm} (5.22)

for the excitation, and the variance of

$$\sigma_x^2 = 16.829$$  \hspace{1cm} (5.23)

for the corresponding response.

The details of these derivation are provided in Appendix (B.4).
Figure 5.8: PSD of System: $\omega^2 \ddot{x}(t) + 0.1 \dot{x}(t) + 1.0 x(t) = W(t)$, where $W(t) =$ white noise with $S_W = 1$.

Figure 5.9: Error in Power Spectral Density Due to Selection of Cut-off Frequency at $\pm 20 \text{ rad/s}$ Instead of $\pm 200 \text{ rad/s}$. 
5.3.3 Numerical Comparison

To verify the results obtained using the Monte Carlo simulation program described in Section 5.3.1, the example of Section 5.3.2 was solved numerically and its results compared to those obtained analytically. In the solution of this example, an AR filter of length 10 is employed and the PSD of the excitation is specified as $S_x(\omega) = 1$. The Gaussian white noise records used as input into the AR model is obtained via a zero mean random number generator which uniformly distributes numbers on the interval $(0.0, 1.0)$. Since a constant PSD is specified in the AR model program the output of the filter is a magnification of the white noise input, and therefore white noise. This redundant process of using white noise to generate the white noise is done to confirm the correctness of results obtained by AR program.

The results of this analysis are presented in the following figures. Figure 5.10 shows the PSD of the excitation defined for $\omega = \pm 20$ rad/s. According to equation (4.14), the area under this PSD curve should be equal to the variance of the excitation. Therefore the variance of the excitations generated is $\sigma = 40$. This coincides with the variance obtained in the analytical analysis.

Figure 5.11 shows the statistical information obtained from an excitation ensemble of consisting of 500 realizations. From these graphs mean and mean square values of $\mu_W = 0$ and $ms_W = 40$, respectively, can be deduced. It should be noted that the excitation is a zero mean process and therefore the mean square value is equivalent to the variance based on equation (4.5). Therefore the mean and variance of the excitation is approximated as

\[ \mu_W = 0 \quad (5.24) \]

\[ \sigma_W = 40 \quad (5.25) \]
Figure 5.10: Power Spectral Density Used for Monte Carlo Simulations.

Figure 5.11: Excitation Statistics Computed From 500 Monte Carlo Realizations. System: $\omega^2 \ddot{x} + 0.1 \dot{x} + 1.0 x = Y(t)$. $Y(t)$ is white noise.
from the ensemble of 500 excitations. These results match those stated and derived analytically. Oscillation about these values can be shown to decrease with an increase in the size of the ensemble. This correlation of result accuracy to ensemble size is as expected from theory.

Figure 5.12 shows the statistical information obtained for the response of the frequency dependent parameter system considered. From these graphs, a mean value and mean square value of $\mu_x = 0$ and $ms_x = 16.58$, respectively, can be deduced. The mean square value approximation is obtained by taking the average of the mean square values which occur for $60 \geq t$. In the analysis, this approximation was shown to improve as the time period and number of realizations increased. This correlation between an increase in the ensemble size and the statistical approximation is expected from theory. By comparing the results of Figures 5.11 and 5.12, obtained from 500 realizations, with those of Figures 5.13 and 5.14, obtained from 300 realizations, improvement due to an increase in the ensemble size can be seen. Specifically, the amplitude of oscillations were shown to decrease and improvements in the average mean and standard deviations were realized.

From the results of this analysis, it can be stated that the statistical information obtained for the response using Monte Carlo simulation compares well with the results obtained analytically. Therefore, the use of the Monte Carlo simulation program developed successfully results in system response information which accurately portrays the responses of FDP systems excited by random excitations. The techniques employed in this work for the solution of FDP models can be applied to obtain the solution of FD models as well.

The system in equation (5.17) was also solved using other PSD functions to characterize the excitations and the numerical and analytical results obtained. Similar
Figure 5.12: Response Statistics Computed From 500 Monte Carlo Response Realizations. System: $\omega^2 \ddot{x} + 0.1 \dot{x} + 1.0 x = Y(t)$. $Y(t)$ is white noise.

Figure 5.13: Excitation Statistics Computed From 300 Monte Carlo Response Realizations. System: $\omega^2 \ddot{x} + 0.1 \dot{x} + 1.0 x = Y(t)$. $Y(t)$ is white noise.
Figure 5.14: Response Statistics Computed From 300 Monte Carlo Response Realizations. System: \( \omega^2 \ddot{x} + 0.1 \dot{x} + 1.0x = Y(t) \). \( Y(t) \) is white noise.

Figure 5.15: Standard Deviation for 500 Monte Carlo Response Realizations. System: \( \omega^2 \ddot{x} + 0.1 \dot{x} + 1.0x = Y(t) \). \( Y(t) \) is white noise.
to the example presented here, the numerical results matched very closely with the analytical solutions and supported the conclusions above.
Chapter 6

Concluding Remarks

The work reported in this manuscript covers frequency dependent parameter and fractional derivative models subjected to deterministic and random excitations. In this work, a comprehensive literature survey of these model forms was attempted and topics related to the response of these models when subjected to deterministic and random excitations discussed.

Also in this work, the appropriate time and frequency domain expressions for frequency dependent parameter and fractional derivative models were presented. As a result, these model forms were shown to yield similar forms in both the time and frequency domains. In the time domain, these models were shown to result in integro-differential equations. In the frequency domain, these models were shown to result in algebraic functions of frequency. A brief discussion on possible procedures for solving integro-differential equations was presented and procedures for the frequency domain analysis have been introduced. A review of random vibration considerations was also given.

Most important in this work is the development of numerical algorithms for obtaining the response of these systems subjected to random excitations. An algorithm was first developed to obtain the response of the frequency dependent parameter model subjected to deterministic excitations. The responses were obtained via a frequency domain analysis. The results of this program were verified by comparing a simulated response with derived analytical solutions. A second algorithm employing an AR model was developed for generating excitations compatible with a power spec-
tral density. The results of this program were verified and an example presented in this work. These programs were then employed in a Monte Carlo algorithm to obtain important statistical information about the response of the frequency dependent parameter model subjected to random excitations. Results from the Monte Carlo simulation program were verified by considering an example problem with a derivable solution. As a result, the Monte Carlo algorithm was shown to yield good estimates of statistical information for the system response.

The examples considered in this work are assumed representative of solution techniques applicable to frequency dependent parameter and fractional derivative models. It is expected that future work in this area will focus on explicitly applying the techniques presented to the numerical solution of fractional derivative models. More importantly, effort should be focused on the development and refinement of time domain solution techniques. Issues of stability system for these systems should also be addressed. Finally, an expansion to multi-degree of freedom and nonlinear systems should be clearly developed with corresponding numerical examples.
Bibliography


Appendix A

Fourier Integral Numerical Approximations

A.1 Fourier Transform Approximation

The Fourier transform of an arbitrary function, \( y(t) \) of the real-variable \( t \) is defined as

\[
Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} \, dt.
\]  

(A.1)

In the applications considered in this study, \( y(t) \) is only defined for \( t \geq 0 \): therefore the Fourier transform can be more efficiently defined as

\[
Y(\omega) = \int_{0}^{\infty} y(t) e^{-j\omega t} \, dt.
\]  

(A.2)

To numerically integrate the expression in equation (A.2), a numerical integration scheme must be employed. One of the simplest integration schemes is the Trapezoidal Rule, which is defined by the following approximation formula [23].

\[
\int_{a}^{b} f(t) \, dt \approx \frac{b - a}{2n} [f(t_a) + 2 \sum_{i=1}^{n-1} f(t_i) + f(t_b)]
\]  

(A.3)

In this definition it is assumed that the variable \( t_i \) and function \( f(t_i) \) represent discrete values of \( t \) and \( f(t) \), with the integer subscripts \( *_a \) and \( *_b \) denoting the initial and final values in the variable domain, respectively. Note that the value of \( n \) can be defined as \( n = \frac{b - a}{\Delta t} \), where \( \Delta t \) is simply the uniform step size between consecutive discrete values, defined as \( \Delta t = t_{i+1} - t_i \).

This approximation can be applied to the definition of the Fourier transform in equation (A.2) to obtain a programmable numerical scheme to numerically approximate the Fourier transform of a discrete function. Before applying the approximation
to equation (A.2), the upper limit of integration must be changed to a finite value, say $t_n$. This results in the following integral equation to be approximated:

$$Y(\omega) = \int_0^{t_n} y(t) e^{-j\omega t} \, dt.$$  \hfill (A.4)

Note that using the finite value $t_n$ leads to the assumption that $y(t) \approx 0$ for $t \geq t_n$. Therefore $t_n$ should be selected with care and its effect on the accuracy of the numerical approximation should be analyzed.

Now, applying the Trapezoidal Rule in equation (A.3) to equation (A.4) leads to the numerical approximation

$$Y(\omega) \approx \frac{t_n - 0}{2n} [f(0) + 2 \sum_{i=1}^{n-1} f(t_i) + f(t_n)],$$  \hfill (A.5)

where $f(t_i) = y(t_i) e^{-j\omega t_i}$ and $n = (t_n - 0)/\Delta t$ can be substituted in to give

$$Y(\omega) \approx \frac{\Delta t}{2} [y(0) e^0 + 2 \sum_{i=1}^{n-1} y(t_i) e^{-j\omega t_i} + y(t_n) e^{-j\omega t_n}].$$  \hfill (A.6)

If $Y(\omega)$ is only defined for discrete values $\omega_k$ then equation (A.6) becomes

$$Y(\omega_k) \approx \frac{\Delta t}{2} [y(0) e^0 + 2 \sum_{i=1}^{n-1} y(t_i) e^{-j\omega_k t_i} + y(t_n) e^{-j\omega_k t_n}].$$  \hfill (A.7)

Equation (A.7) is the approximation scheme used in the programs developed for this work to obtain the Fourier transforms of the discrete functions considered.

### A.2 Inverse Fourier Transform Approximation

The inverse Fourier transform, used to transform an arbitrary complex function $X(\omega)$ of the real frequency variable $\omega$ into the time domain, is defined as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega.$$  \hfill (A.8)

To numerically integrate the expression in equation (A.8), a numerical integration scheme must be employed. The integration schemes employed in this work is the
Trapezoidal Rule, which approximates an integral as defined by the following formula.

\[
\int_{a}^{b} F(\omega) \, d\omega \approx \frac{b-a}{2n} \left[ F(\omega_a) + 2 \sum_{i=1}^{n-1} F(\omega_i) + F(\omega_b) \right]
\]  

(A.9)

In this definition it is assumed that the variable \( \omega_i \) and function \( F(\omega_i) \) represent discrete values of \( \omega \) and \( f(\omega) \), with the integer subscripts \( *_a \) and \( *_b \) denoting the initial and final values in the variable domain. Note that the value of \( n \) is defined as \( n = \frac{(b-a)}{\Delta\omega} \), where \( \Delta\omega \) is simply the uniform step size between consecutive discrete values, defined as \( \Delta\omega = \omega_{i+1} - \omega_i \).

This approximation can be applied to the definition of the inverse Fourier transform in equation (A.8) to obtain a programmable numerical scheme which will numerically approximate the inverse Fourier transform of a discrete function.

Before applying the approximation to equation (A.8), the upper and lower limits of integration must be changed to finite values, say \( +\omega_c \) and \( -\omega_c \), respectively. This results in the following integral equation to be approximated:

\[
x(t) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} X(\omega) e^{j\omega t} \, d\omega.
\]  

(A.10)

Note that using the finite values of \( \pm\omega_c \), leads to the assumption that \( X(\omega) \approx 0 \) for \( \omega \geq |\omega_c| \). Therefore, \( \omega_c \) should be selected with care and its effect on the accuracy of the numerical approximation should be analyzed. Note that for the systems considered in this work, \( \omega_c \) can be chosen based upon the criteria that the complex frequency response function, \( H(\omega) \), is approximately zero for \( \omega \geq |\omega_c| \).

Now, applying the Trapezoidal Rule to equation (A.10) leads to the approximation

\[
x(t) \approx \frac{\omega_c \pm \omega_c}{4n \pi} \left[ f(-\omega_c) + 2 \sum_{i=1}^{n-1} f(\omega_i) + f(\omega_c) \right],
\]  

(A.11)

where \( f(\omega_i) = X(\omega_i) e^{j\omega_i t} \) and \( n = (\omega_c + \omega_c)/\Delta\omega \) can be substituted in to give

\[
x(t) \approx \frac{\Delta\omega}{4\pi} \left[ X(-\omega_c) e^{-j\omega_c t} + 2 \sum_{i=1}^{n-1} X(\omega_i) e^{j\omega_i t} + X(\omega_c) e^{j\omega_c t} \right].
\]  

(A.12)
If $x(t)$ is only defined for the discrete values $t_k$, then equation (A.12) becomes

$$x(t_k) \approx \frac{\Delta \omega}{4\pi} [X(-\omega_c) e^{-j\omega_c t_k} + 2 \sum_{i=1}^{n-1} X(\omega_i) e^{j\omega_i t_k} + X(\omega_c) e^{j\omega_c t_k}]. \quad (A.13)$$

Equation (A.13) is the approximation scheme used in the programs of this work to obtain the inverse Fourier transforms of the discrete functions considered.
Appendix B

Analytical Solutions

B.1 Deterministic: Frequency Domain Analysis

Problem Statement: Given the system

\[ m(\omega)\ddot{x} + 0.1\dot{x} + x = \sin(3t) \quad (B.1) \]

defined for \(0 \leq t \leq \infty\), with the initial conditions \(x(0) = 0\) and \(\dot{x}(0) = 0\), and the frequency dependent parameter function defined as \(m(\omega) = \omega^2\); solve for the response of the system, \(x(t)\) using a Frequency Domain analysis.

Solution: Recall that the Fourier transform of a function, \(f(t)\) is

\[ F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt. \quad (B.2) \]

Applying the Fourier transform to the excitation function, which can be defined as

\[ y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \sin(3t) & \text{if } t \geq 0 \end{cases} \quad (B.3) \]

gives

\[ Y(\omega) = \int_{0}^{\infty} \sin(3t)e^{-j\omega t} \, dt \]

\[ = \frac{j\pi}{2} [\delta(\omega + 3) - \delta(\omega - 3)] + \frac{3}{9 - \omega^2}. \quad (B.4) \]

Using equation (B.4) and (3.98) applied to this system, the Fourier transform of the system in equation (B.1) above can be written as:

\[ [-\omega^2\omega^2 + 0.1j\omega + 1]X(\omega) = \frac{j\pi}{2} [\delta(\omega + 3) - \delta(\omega - 3)] + \frac{3}{9 - \omega^2}. \quad (B.5) \]
Solving this expression for $X(\omega)$ gives

$$X(\omega) = \frac{j\pi[\delta(\omega + 3) - \delta(\omega - 3)]}{2(-\omega^4 + 0.1j\omega + 1)} + \frac{3}{[-\omega^4 + 0.1j\omega + 1][9 - \omega^2]}.$$  \hspace{1cm} (B.6)

To transform this solution into the time domain response, $x(t)$, the inverse Fourier transform of equation (B.6) must be taken.

Recall that the inverse Fourier transform of a function, $F(\omega)$ is defined as

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} \, d\omega.$$  \hspace{1cm} (B.7)

Applying the inverse Fourier transform to equation (B.6), this problem can be solved in two parts by employing the property that the inverse Fourier transform of a sum of linear functions is equal to the sum of the inverse Fourier transform of each function. This can be expressed as $F^{-1}[x + y] = F^{-1}[x] + F^{-1}[y]$, where $F^{-1}$ represents the inverse Fourier transform operator. Therefore the inverse Fourier transform of equation (B.6) is the sum of the inverse transforms of each of the two fractional expressions on the right hand side of the equation.

Taking the inverse Fourier transform of the first fractional expression leads to first part of the solution, derived in the following way:

$$x_1(t) = \mathcal{F}^{-1}\left[\frac{j\pi(\delta(\omega + 3) - \delta(\omega - 3))}{2(-\omega^4 + 0.1\omega + 1)}\right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{j\pi e^{j\omega t}}{2(-\omega^4 + 0.1\omega + 1)}[\delta(\omega + 3) - \delta(\omega - 3)]d\omega$$

$$= \frac{j}{4} \left[\frac{e^{-j3t}}{(-81 - 0.3j + 1)} - \frac{e^{-j3t}}{(-81 + 0.3j + 1)}\right]$$

$$= \frac{j}{2((-80)^2 + (0.3)^2)} \left[\frac{-80(e^{-j3t} - e^{j3t})}{2} + \frac{0.3j(e^{j3t} + e^{-j3t})}{2}\right]$$

$$= \frac{j}{12800.18}[+80j\sin(3t) + 0.3j\cos(3t)]$$
\[ x_2(t) = \mathcal{F}^{-1} \left[ \frac{3}{(-\omega^4 + 0.1\omega + 1)(9 - \omega^2)} \right] \]

Taking the inverse Fourier transform of the second fractional expression in equation (B.6) leads to the second part of the solution, derived with the use of partial fraction expansion in the following way:

\[ x_2(t) = \mathcal{F}^{-1} \left[ \frac{-0.18750}{\omega^2 - 1} + \frac{0.1500}{\omega^2 + 1} + \frac{0.0375}{\omega^2 - 9} \right] \]

\[ = 0.09375\sin(t) + 0.075e^{-t} - 0.00625\sin(3t). \quad (B.9) \]

Adding the two solutions together we find that the total analytical solution for the response of the system given by taking the inverse Fourier transform of equation (B.6) is

\[ x(t) = x_1(t) + x_2(t) \]

\[ = 0.09375\sin(t) + 0.075e^{-t} - 0.0125\sin(3t) - 0.00002\cos(3t). \quad (B.10) \]

This solution is valid for \( 0 \leq t \leq \infty. \)

### B.2 Fast Fourier Transform Comparison and Error

To validate the use of the discrete Fourier transforms derived and used throughout this work, a program was written to numerically generate Fourier transforms and compare the results with known analytical transform expressions. In this program, the discrete Fourier transforms were calculated using Fast Fourier Transforms. Expressions used for the analytical transforms were either derived from equation (A.2), or obtained from Fourier transform tables.
The error resulting from the use of the Fast Fourier Transform (FFT) was analyzed. Some results from this study are presented below. In the first analysis, a sinusoidal excitation with a derivable Fourier transform (FT) was considered. The function presented here is

\[ y(t) = \sin(3t), \quad t \geq 0. \]  

(B.11)

The results of this analysis are provided in Figures B.1, B.2 and B.3. The first figure shows a comparison between the known FT and numerically computed FFT. Large error is shown to result because of the inability of the FFT to adequately approximate the delta functions and singularities of the FT. This error is depicted in the second graph of Figure B.3. In Figure B.2 a comparison between the known inverse Fourier transform (IFT) and inverse Fast Fourier Transform (IFFT) is shown. The IFFT in this figure was generated by taking the IFFT of the known FT of the sinusoid. As shown in Figure B.3, the IFFT gives an excellent simulation of the response with relatively small error. This error is due to the FFT attempting to approximate a sinusoidal function that does not die out with time.

Another example was considered where the function decays with time, depicted in Figures B.4 B.5 and B.6. The function considered here is

\[ y(t) = e^{-t} - e^{-2t}, \quad t \geq 0. \]  

(B.12)

In this example, the numerical approximation of the Fourier transform is essentially identical to its analytical expression as shown in Figure B.4 and B.5. These results and similar examples led to the justified use of the Fourier transform approximations in the development of this work. In this example, the FFT and IFFT are shown to result in excellent approximations of solutions. It is believed that this reduction in error is directly related to the fact that this function decays in time, and therefore, no significant information is lost in approximating the Fourier transform defined with infinite limits by an approximation defined at finite limits.
Figure B.1: Comparison of Fast Fourier Transform and Fourier transform of $y(t) = \sin(3t)$.

Figure B.2: Comparison of inverse Fast Fourier Transform and Fourier transform for the base function $y(t) = \sin(3t)$. 
Figure B.3: Error Incurred for the inverse Fast Fourier Transforms of $y(t) = \sin(3t)$ and Its Inverse.

Figure B.4: Comparison of Fast Fourier Transform and Fourier transform of $y(t) = e^{-t} - e^{-2t}$. 
Figure B.5: Comparison of inverse Fast Fourier Transform and Fourier transform for the base function $y(t) = e^{-t} - e^{-2t}$.

Figure B.6: Error Incurred for the inverse Fast Fourier Transforms of $y(t) = e^{-t} - e^{-2t}$ and Its Inverse.
B.3 Error from Deterministic Program

The error was calculated using the equation

\[ e = \frac{x - x_{approx}}{(x)_{max}} \cdot 100\% \]  

(B.13)

where \( x \) is the analytical solution, \( x_{approx} \) is the numerical solution, and \( (x)_{max} \) is used to normalize this error with respect to the maximum value of the analytical response which occurs near \( t = 3 \). The resulting error is plotted in Figure B.7. The maximum error was shown to occur as an end deflection of 19\%, at \( t = 50s \). In the analysis, this "edge effect" error was found to occur independent of the cutoff time considered. Other than the edge effect error, the resulting error in this analysis is relatively small. Shown in the figure, the resulting error stays within a 0.5\% error band if the edge effect error is not considered, with the maximum error occurring as an overshoot at the first peak. This error plot was generated for the response of the system using approximating step sizes \( \Delta = 0.1 \) and \( \Delta \omega = 0.01 \). A further discussion of the results obtained for this analysis can be found in Section 5.3.3.

B.4 Random Excitations: Frequency Domain Analysis

Given the following system

\[ \omega^2 \ddot{x}(t) + 0.1 \dot{x}(t) + 1.0x(t) = W(t), \]  

(B.14)

and the PSD of the white noise excitation

\[ S_W(\omega) = 1, \]  

(B.15)

the PSD of the response can be expressed as

\[ S_x(\omega) = H^*(\omega) S_y(\omega) H(\omega) \]

\[ = S_y(\omega)|H(\omega)|^2. \]  

(B.16)
Figure B.7: Error of the Numerical Approximation for the Response of the System: $\omega^2 \dddot{x}(t) + 0.1 \dot{x}(t) + 1.0x(t) = \sin(3t)$, valid for $t \geq 0$. 
For this system $H(\omega)$ is

$$H(\omega) = \frac{1}{[-\omega^2 \cdot \omega^2 + j \omega \cdot 0.1 + 1.0]},$$  \hspace{1em} (B.17)

resulting in an $S_x$ of

$$S_x(\omega) = \frac{1}{[(1 - \omega^4)^2 + (0.1\omega)^2]^{1/2}}.$$  \hspace{1em} (B.18)

For a zero mean process the variance of the response, $x(t)$ can be expressed as

$$\sigma_x^2 = \int_{-\infty}^{\infty} S_x(\omega) \, d\omega.$$  \hspace{1em} (B.19)

For the system considered, this results in the expression

$$\sigma_x^2 = \int_{-\infty}^{\infty} S_0 \left| \frac{1}{[-\omega^2 \cdot \omega^2 + j \omega \cdot 0.1 + 1.0]} \right|^2 \, d\omega.$$  \hspace{1em} (B.20)

An expression for the analytical solution of equation (B.20) was obtained using MAPLE, a symbolic mathematics package.

Evaluating this expression for the limits $\pm 20$ gives

$$\sigma_x^2 = 16.82899231.$$  \hspace{1em} (B.21)

Increasing these limits to $\pm 50$ only differs from this expression by $+0.00000002$. 

Appendix C

Other Numerical Considerations

Other deterministic numerical examples were considered using the program developed in this work. These examples, shown in Figures C.1 through C.6 are believed to have derivable solutions, however, these analytical solutions are not presented in this work. Despite the lack of a comparable solution, these results are presented here as an interesting example of hysteretic type damping behavior which can easily be solved using the program in this work.
Figure C.1: Transfer Function and FFT of Excitation for Hysteretic Damped System: \( \ddot{x}(t) + \frac{0.1}{|\omega|} \dot{x}(t) + 10x(t) = e^{-t} - e^{-2t} \).

Figure C.2: Response in Frequency Domain for Hysteretic Damped System: \( \ddot{x}(t) + \frac{0.1}{|\omega|} \dot{x}(t) + 10x(t) = e^{-t} - e^{-2t} \).
Figure C.3: Response of SDOF Hysteretic Damped System: \( \ddot{x}(t) + \frac{0.1}{\omega} \dot{x}(t) + 10x(t) = e^{-t} - e^{-2t}. \)

Figure C.4: Transfer Function and FFT of Excitation for Hysteretic Damped System: \( \ddot{x}(t) + \frac{0.1}{\omega} \dot{x}(t) + 10x(t) = \sin(10t). \)
Figure C.5: Response in Frequency Domain for Hysteretic Damped System: $\ddot{x}(t) + \frac{0.1}{|\omega|} \dot{x}(t) + 10x(t) = \sin(10t)$.

Figure C.6: Response of SDOF Hysteretic Damped System: $\ddot{x}(t) + \frac{0.1}{|\omega|} \dot{x}(t) + 10x(t) = \sin(10t)$.