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Integral Manifold Based Control Design of an Electromechanical System

by

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A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

Master of Science

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Abstract

The control problem of an electromechanical system becomes more complex when the flexibility and friction are taken into account in the system model. To approach this problem, an electromechanical system is studied in this research as a system model which is composed of a motor driving a load through a flexible belt. The work of this thesis proposes a new integral manifold control design to achieve desired system performance based on the integral manifold control techniques. The compensation of friction is performed with a new friction compensation component in the integral manifold control algorithm. A detailed derivation of the integral manifold control law is given for regulation systems with a method of calculating control gains in the final control algorithm. A further study on the regulation system is conducted by selecting a set of control parameters based on the system characteristic equation, while a switching integral manifold phenomena is illustrated for tracking systems by exploiting system performance in state space. Simulation and experimental results are presented that show a consistent improvement in the system performance with the integral manifold control design strategy.
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Chapter 1

Introduction

The control design for flexible drive systems is a very active research area. The focus has developed because many mechanical systems consist of power transmission elements such as belts, gear trains, shafts, and harmonic drives, which often introduce compliance into system dynamics. As the system operational speed increases and control accuracy becomes more important, the deformation of these transmission elements may not be neglected in the modeling, simulation and control design processes. This requires that flexibility should be taken into account in system dynamics and control strategy, especially if control accuracy is of high importance as in the area of industrial and space systems. The goals of the control research of flexible systems are to improve the tracking trajectory and dynamic performance. However, with the presence of such physical phenomena as flexibility and friction, the design of control algorithms becomes a much more difficult and challenging task. Various experimental and simulation studies [4], [27], have shown that the elasticity and the friction of the transmission elements have a relevant influence on system dynamics.

Although many earlier strategies have been proposed for rigid systems, their applicability to the system with flexibility has not been directly resolved and extended. With the consideration of only flexibility, several advanced control approaches have been proposed in the literature, based on the concept of integral manifold [23], [24], singular perturbation theory [25], or feedback linearization techniques [26]. One of the potential advantages of the integral manifold control method compared to other control methods is that it may insure global exponential stability by appropriately
designing an asymptotically stable fast off-manifold subsystem and an exponentially stable on-manifold subsystem. The theory and application of the integral manifold control design based on an ideal system model will be discussed in Chapter 2 as background.

After reviewing current research in flexible system dynamics and control, the following objectives are identified:

- To validate integral manifold techniques [24] to a flexible system with friction ignored (an ideal model) using simulation and experimental results. An electro-mechanical system [37] in Figure 1.1 is used as a practical testbed, with the illustration of the system configuration shown in Figure 1.2.

![Figure 1.1: ECP System used in Experiments](image-url)
Figure 1.2: Configuration of ECP system

- To investigate the effects of friction on an ideal control scheme. The simulation results are used to investigate the effect of changing friction on a system performance.

- To develop a new control scheme which improves the system performance with the presence of flexibility and friction. Techniques for designing a new control structure to improve performance without the identification of frictions are investigated for the system model. An efficient control algorithm is to be developed based on the integral manifold techniques and demonstrated by both simulation and experimental results, while the results are compared between the system's observed behavior and that predicted by the simulation.

The modeling of the electromechanical system is discussed in Chapter 3. Through using the Lagrange method, a complete system model is obtained with flexibility
and friction in terms of system dynamics. The motivation for including friction in the system model and in the design process of the control scheme is that friction is universally present in the motion of systems. Inaccurate performance of a controlled system is often caused by the presence of friction, which may result in both steady-state error and tracking lags. In some cases, the friction may even generate oscillatory behaviors because of overcompensation or undercompensation.

In Chapter 4, a new control algorithm is developed that provides improvement to achieve system performance through the use of integral manifold techniques with the existence of flexibility and friction.

When kinetic friction is considered in the system model, the system equations become discontinuous, where all the existing design methods for the continuous system become invalid. However, after analyzing the system equations of motion, we find that the system model is piecewise continuous in each of the two dynamic domains ($\dot{q} > 0$ and $\dot{q} < 0$), which inspires us to utilize the integral manifold method in each of these dynamic domains. This results in a new integral manifold control design with friction compensation for regulation systems.

The aim of the integral manifold control design in this thesis is to improve the system positioning performance by cancelling the friction term in the control algorithm to prevent inaccurate friction compensation and to find a consistent control algorithm for each dynamic domain. The regulation systems are studied by adopting the new integral manifold control design, where the detailed computations of the control gains are given. Based on the previous work discussed in Chapter 2, and to make the system behavior satisfy one dynamic domain assumption, we study the system characteristic equation to select a set of control parameters which are tested in simulation and experiments. The performance of the controlled tracking system is briefly discussed at the end of the chapter.
To verify the effectiveness of the integral manifold strategy, in Chapter 5, the system parameters are first identified by experiments. Based on the identification results, the new integral manifold control algorithm is applied in the simulation and then implemented in the real system experiments. The results are compared with PD control and the integral manifold control without friction compensation, which show significant improvements in the system performance.

The conclusion and future work are presented in Chapter 6.
Chapter 2

Background

A basic flexible system model with its PD control design and its integral manifold control design developed by Ghorbel et al. [24] are reviewed in Section 2.1 and Section 2.2 as the background of this thesis work.

2.1 A Basic Flexible System Model

A basic mechanical system model in Figure 2.1 consists of a motor, a load and an elastic shaft [24]. The motor drives the load by applying a torque through the shaft. When the torsional deformation of the shaft is supposed to be linear, a model of the above system can be given as:

\[ I \ddot{q}_1 + k(q_1 - q_2) = 0. \]  \hspace{1cm} (2.1)

\[ J \ddot{q}_2 - k(q_1 - q_2) = u. \]  \hspace{1cm} (2.2)

![Figure 2.1: Basic System Model](image.jpg)
where $I$ is the inertia of the load, $J$ is the inertia of the motor, and $k$ is the shaft stiffness by assuming a linear torsional deformation of the flexible shaft.

The motor is controlled by the torque input $u$. $q_1 \in R$ is the load angular position, $\dot{q}_1$ is the load angular velocity. Similarly, $q_2 \in R$ is the motor angular position, and $\dot{q}_2$ is the motor angular velocity.

The control objective of the basic system model is to design a control law to make the load angular position $q_1$ and the angular velocity $\dot{q}_1$ track a desired profile with $u$ as the input to the system. The main challenge of the control design comes from the presence of the shaft elasticity.

One supposed way to solve this problem is to adopt PD control. In the first kind of PD control, $q_1$ can be used as the feedback, which is shown in Figure 2.2.

![Figure 2.2: PD Control System with Feedback from $q_1$](image)

Designing PD controller as $u = k_p(q_d - q_1) + k_d(\dot{q}_d - \dot{q}_1)$, the system Equation (2.1) and (2.2) becomes:

$$I\ddot{q}_1 - k_1(q_1 - q_2) = 0,$$

$$J\ddot{q}_2 - k_1q_1 - q_2 = k_p(q_d - q_1) + k_d(\dot{q}_d - \dot{q}_1).$$
Taking Laplace transform on both sides of the equations, we obtain:

\[ I q_1(s)s^2 + k(q_1(s) - q_2(s)) = 0. \]
\[ J q_2(s)s^2 - k(q_1(s) - q_2(s)) = k_p(q_d(s) - q_1(s)) + k_4(q_d(s)s - q_1(s)s). \]

Therefore, the transfer function of the complete system is:

\[ G_c(s) = \frac{q_1(s)}{q_4(s)} = \frac{k(k_p + k_4s)}{IJs^4 + (kI + kJ)s^2 + kk_4s + kk_p}. \]

The characteristic equation of the system is governed by:

\[ IJs^4 + (kI + kJ)s^2 + kk_4s + kk_p = 0. \]

The root-locus of the controlled system is shown in Figure 2.3, by adopting the identification values for the system parameters (see Chapter 5 for reference).

From Figure 2.3, we observe that a part of the root locus lies in the right half of the complex plane. Therefore, the system is an unstable system.

The second kind of PD control can be designed as using \( q_2 \) as the feedback, which is shown in Figure 2.4.

Hence, the control algorithm becomes:

\[ u = k_p(q_d - q_2) + k_4(\dot{q}_d - \dot{q}_2). \]

In this case, the complete system equation becomes:

\[ I \ddot{q}_1 + k(q_1 - q_2) = 0. \]
\[ J \ddot{q}_2 - k(q_1 - q_2) = k_p(q_4 - q_2) + k_4(\dot{q}_4 - \dot{q}_2). \]

Similarly, taking Laplace transform on both sides of the equations, the complete transfer function is described by:

\[ G_c(s) = \frac{q_1(s)}{q_4(s)} = \frac{k(k_p + k_4s)}{IJs^4 + k_4I s^3 + (kI - kJ - k_p I)s^2 + kk_4s + kk_p}. \]
The corresponding characteristic equation is obtained as:

\[ IJs^4 + k_d Ls^3 + (kI + kJ + k_p L)s^2 + k_4ks + kk_p = 0. \]

The root locus for the second PD control system is shown in Figure 2.5.

In Figure 2.5, all the root locus lies in the left half of the complex plane. Therefore, the system is stable.

However, in Figure 2.4, it is shown that the system loop is closed only at the motor side, while our principal concern is the system response on the load side which is actually open. In this situation, the system performance may deteriorate if disturbances such as friction exist in the link between the motor disk and the load disk. Therefore, PD control cannot insure the good system performance.
Figure 2.4: PD Control System with Feedback from $q_2$

Figure 2.5: Root Locus of PD Control System with Feedback from $q_2$
2.2 Integral Manifold Control

Based on the above system equation and the control consideration, one of the control strategies for the ideal model (2.1) and (2.2) is the integral manifold control design studied by Ghorbel et al. [24].

In this paper, the author studied the integral manifold approach by using only positions and velocities as feedback, rather than using accelerations, while improving the system performance.

In [24], the author utilized the integral manifold techniques (refer to Appendix A) and developed a complete design procedure which leads to an integral manifold control design for the control algorithm $u$.

To insure the exponential stability of the system equilibrium, $u$ is designed as:

$$u = k_v(q_1 - \dot{q}_2) + f_r(q_d - q_1) + f_d(\dot{q}_d - \dot{q}_1) + (I + J)\ddot{q}_d + \frac{IK_v}{K}q_d^{(3)} + \frac{IJ}{K}q_d^{(4)}.$$  (2.3)

In the above control design, the integral manifold techniques are used to obtain the control solution which consists a fast component for damping the fast dynamics and a slow component chosen by a detailed design procedure. Lyapunov function method is used in [24] to prove that the global exponential stability of the whole system is insured for an explicit range of stiffness of the flexible shaft (see Theorem 2.1 in [24] for reference).

The work in [24] deals with an ideal system model without frictions, which is the basis of this thesis work. To achieve more precise system performance, the friction becomes an important issue.

In the literature [4], kinetic friction is described as a constant friction torque opposite to the motion of the body when the velocity is different from zero. For zero velocity case, the system movement is totally governed by the static friction which will oppose all motions as long as the input torque is smaller in magnitude than the
stiction torque. Another kind of friction is viscous friction which is in proportion to velocity. In the work of this thesis, we utilize a common model of friction covering most of these components, which is illustrated in Figure 2.6 and expressed in Equation (2.4):

\[ T_f(\dot{q}) = [c\dot{q} + f_c \cdot \text{sign}(\dot{q})]. \]  

(2.1)

where \( c \) and \( f_c \) are positive constants.

The contemporary study of friction modeling is widely utilized in the control literature. Brandenburg et al. [6] have studied mechanisms with coulomb friction.
backlash and flexibility between the actuator and load. The stability of systems with kinetic friction and PD control is discussed by Kubo et al. [28]. Although the experimental results of the above studies show significant improvement in system tracking performance, their friction compensation methods either employ various estimation algorithms or adopt fixed friction parameters which are obtained by identification experiments. In these cases, the repeatability of friction forces is an obstacle to the control design as well as such concerns as the different friction forces for positive and negative motion directions to which no explanation for the behavior is available.

In this thesis work, the friction compensation issue will be taken into account in our control design, and an effective friction compensation method will be studied in the control algorithm.

Summary

Although the integral manifold control design presented in Section 2.1 produces a very good performance for the ideal system model in simulation, a performance deterioration is observed if the friction in systems can’t be ignored. The friction compensation issues introduce difficulties to the control system design but must be considered if the system control accuracy is of high importance.
Chapter 3

System Modeling and Analysis

3.1 System Modeling

Consider the electromechanical system shown in Figure 3.1, where a motor drives a load disk through a speed reduction (SR) assembly. This system configuration models the mechanical systems with flexibility caused by transmission devices. The drive motor is coupled via a rigid timing belt to a drive disk. Another rigid timing belt connects the drive disk to the SR assembly. The flexibility in the system is introduced by a flexible belt which completes the drive train to the load disk. The general operation objective of the electromechanical system is to control the angular position and velocity of the load disk to track the desired trajectory.

The following table gives the pertinent variables and parameters of the electromechanical system, which is illustrated in Figure 3.1 and will be used in developing the plant dynamics:

\[
\begin{align*}
T_D(t) & : [\text{Nm}] \quad \text{torque generated by the motor (input)} \\
\theta_1(t) & : [\text{rad}] \quad \text{angular position of drive shaft} \\
\theta_2(t) & : [\text{rad}] \quad \text{angular position of load shaft (output)} \\
J_d & : [\text{Nms}^2/\text{rad}] \quad \text{drive disk inertia} \\
J_l & : [\text{Nms}^2/\text{rad}] \quad \text{load disk inertia} \\
J_p & : [\text{Nms}^2/\text{rad}] \quad \text{pulley inertia}
\end{align*}
\]
Figure 3.1: System Variables & Parameters

\( J_{wd} \) \([\text{Nms}^2/\text{rad}]\) inertia associated with the brass weights at the drive disk

\( J_{wl} \) \([\text{Nms}^2/\text{rad}]\) inertia associated with the brass weights at the load disk

\( r_1, v_2 \) \([\text{Nms}/\text{rad}]\) viscous frictional coefficient

\( k_i \) \([\text{N/m}]\) linear spring constant

\( r_d, r_{p1}, r_{p2}, r_l \) \([\text{m}]\) pulley radii

\( f_{c1}, f_{c2} \) \([\text{Nm}]\) coulomb friction

From the system configuration and the system parameters shown in Figure 3.1, we obtain the dynamic equations of the motion by following a Lagrangian approach.

First we define:

\[
y_r = \frac{\theta_1}{\theta_2} = \frac{r_1 r_{p1}}{r_{p2} r_l}.
\]
$$g_r' = \frac{\theta_1}{\theta_p} = \frac{r_{p1}}{r_d}.$$ 

which represent the gear ratio between the drive disk and the load disk, and that between the motor disk and the pulley.

The system energy can be analyzed by:

- **Kinetic Energy**:

  $$T = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_p \dot{\theta}_p^2 + \frac{1}{2} J_4 \dot{\theta}_4^2,$$
  $$= \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_p \left( \frac{\dot{\theta}_1}{g_r'} \right)^2 + \frac{1}{2} J_4 \dot{\theta}_4^2.$$ 

- **Potential Energy**:

  $$V = \frac{1}{2} k_i (r_1 \theta_2 - r_{p2} \theta_p)^2 = k_i (r_1 \theta_2 - r_{p2} \frac{\theta_1}{g_r'})^2.$$ 

- **Nonconservative Forces**:

  Frictions are the main nonconservative forces in this system. The models of friction in a mechanical system have been extensively discussed in the literatures [4] and [10]. For this system, we employ a common friction model: viscous plus kinetic (coulomb) friction model:

  $$Q_i = -[v_i \dot{\theta}_i + f_\alpha \text{sign}(\dot{\theta}_i)] \quad i = 1, 2.$$ 

Using Euler-Lagrange’s equation.

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j.$$ 

we obtain.

- For \(\theta_1\):

  $$\left( J_p \frac{\dot{\theta}_1}{g_r'^2} + J_4 \right) \ddot{\theta}_1 - 2k_i (r_1 \theta_2 - r_{p2} \frac{\theta_1}{g_r'}) \frac{r_{p2}}{g_r'} = T_D - [v_1 \dot{\theta}_1 + f_{\alpha1} \text{sign}(\dot{\theta}_1)].$$
Defining \( J_d = \frac{J_0}{g_{r_{T_d}}} + J_d \), we obtain.

\[
J_d \ddot{\theta}_1 - 2k_l r_l^2 \left( \frac{r_{p2} r_{d}^2}{r_{p1}} \theta_2 - \frac{r_{p2}^2 r_d^2}{r_{p1}^2} \theta_1 \right) = T_D - [v_1 \dot{\theta}_1 + f_{c1} \text{sign}(\dot{\theta}_1)].
\]

Defining \( k = 2k_l r_l^2 \) and recalling that \( g_r = \frac{\theta_1}{\theta_2} = \frac{r_{p1}}{r_{p2} r_d} \), the first equation of motion is obtained by,

\[
J_d \ddot{\theta}_1 + k(g_r^{-2} \theta_1 - g_r^{-1} \theta_2) = T_D - [v_1 \dot{\theta}_1 + f_{c1} \text{sign}(\dot{\theta}_1)]. \tag{3.1}
\]

- For \( \theta_2 \):

\[
J_l \ddot{\theta}_2 + 2k_l r_l^2 (r_1 \theta_2 - r_{p2} \frac{\theta_1}{g_r'}) = -[v_2 \dot{\theta}_2 + f_{c2} \text{sign}(\dot{\theta}_2)].
\]

Recalling \( g_r' = \frac{\theta_1}{\theta_2} = \frac{r_{p1}}{r_{p2} r_d} \), we obtain.

\[
J_l \ddot{\theta}_2 + 2k_l r_l^2 \theta_2 - 2k_l r_l^2 \frac{r_{p2} r_d}{r_{p1}} \theta_1 + v_2 \dot{\theta}_2 + f_{c2} \text{sign}(\dot{\theta}_2) = 0.
\]

For the substituting purpose, the above equation is rewritten as.

\[
J_l \ddot{\theta}_2 + 2k_l r_l^2 \theta_2 - 2k_l r_l^2 \frac{r_{p2} r_d}{r_{p1}} \theta_1 + v_2 \dot{\theta}_2 + f_{c2} \text{sign}(\dot{\theta}_2) = 0.
\]

Recalling that \( k = 2k_l r_l^2 \) and \( g_r = \frac{\theta_1}{\theta_2} = \frac{r_{p1}}{r_{p2} r_d} \), the second equation of motion is obtained by.

\[
J_l \ddot{\theta}_2 + k(\theta_2 - g_r^{-1} \theta_1) + v_2 \dot{\theta}_2 + f_{c2} \text{sign}(\dot{\theta}_2) = 0. \tag{3.2}
\]

Therefore, the system equations of motion are:

\[
J_d \ddot{\theta}_1 + k(g_r^{-2} \theta_1 - g_r^{-1} \theta_2) = T_D - [v_1 \dot{\theta}_1 + f_{c1} \text{sign}(\dot{\theta}_1)]. \tag{3.3}
\]

\[
J_l \ddot{\theta}_2 + v_2 \dot{\theta}_2 + k(\theta_2 - g_r^{-1} \theta_1) + f_{c2} \text{sign}(\dot{\theta}_2) = 0. \tag{3.4}
\]

Since the coulomb friction in the motor side is very small according to the identification results (see Chapter 5 for reference), and its compensation has been widely
studied in the literatures, we will ignore it in Equation (3.3) and simplify our system model. In this thesis we will focus our effort on the compensation of the friction in the load side, which is more important because on one hand, the friction in the load side affects the system output directly; On the other hand, it is more difficult to be compensated via the system input even if the friction is supposed to be known.

Therefore, Equation (3.3) becomes:

\[ J_d^* \ddot{\theta}_1 + k(g_r^{-2}\theta_1 - g_r^{-1}\theta_2) = T_D - v_1 \dot{\theta}_1. \]

To simplify the notation, we introduce a new system input \( u = T_d - v_1 \dot{\theta}. \) therefore.

\[
\begin{align*}
J_d^* \ddot{\theta}_1 + k(g_r^{-2}\theta_1 - g_r^{-1}\theta_2) &= u, \\
J_l \ddot{\theta}_2 + v_2 \dot{\theta}_2 + k(\theta_2 - g_r^{-1}\theta_1) + f_{c2} \text{sign}(\dot{\theta}_2) &= 0.
\end{align*}
\]

For further notational simplicity, we rewrite the system equation as:

\[
\begin{align*}
J_d^* g_r(g_r^{-1}\ddot{\theta}_1) - kg_r^{-1}(\theta_2 - g_r^{-1}\theta_1) &= u, \\
J_l g_r^{-1} \ddot{\theta}_2 + v_2 g_r^{-1} \dot{\theta}_2 + kg_r^{-1}(\theta_2 - g_r^{-1}\theta_1) + g_r^{-1} f_{c2} \text{sign}(\dot{\theta}_2) &= 0.
\end{align*}
\]

and introduce new variable and parameter definitions as following.

\[
\begin{align*}
I &= J_l g_r^{-1}, \\
J &= J_d^* g_r, \\
q_1 &= \theta_2, \\
q_2 &= g_r^{-1} \theta_1, \\
c_1 &= g_r^{-1} v_2, \\
F_c &= g_r^{-1} f_{c2}, \\
\kappa &= g_r^{-1} k.
\end{align*}
\]
Therefore, the system equations become.

\[ I\ddot{q}_1 + c_1\dot{q}_1 + K(q_1 - q_2) + F_c \text{sign}(\dot{q}_1) = 0. \quad (3.5) \]

\[ J\ddot{q}_2 - K(q_1 - q_2) = u. \quad (3.6) \]

### 3.2 Analysis

The control objective of the above system model is to design a control algorithm to make the load disk angular position and velocity track a desired trajectory. The difficulty of this control system design stems from the flexibility phenomena caused by the flexible belt and the discontinuity introduced by the kinetic friction. If the kinetic friction is not supposed to be known, we have further difficulties because the control law must be designed to guarantee stabilization or tracking.
Chapter 4

Integral Manifold Control Design

In this chapter we present an integral manifold control design to deal with the control problem of the system model with flexibility and friction as described in Equation (3.5) and (3.6). The control algorithm designed in this chapter adopts a new control structure to improve the system performance and to achieve the cancelation of the friction term in the final controller. The new control design has advantages that the friction is compensated without the identification and the control structure has a unifying form for both positive and negative velocity dynamics. The regulation system design is discussed in details. The tracking performance is described for the system model to complete the section.

4.1 Integral Manifold Control Design for Regulation Systems

In this section, we propose a new integral manifold control structure for regulation systems based on the full system model to achieve the friction cancelation in the final control algorithm.

From Equation (3.5) and (3.6), the full system dynamics is governed by:

\[ I \ddot{q}_1 + c_1 \dot{q}_1 + K(q_1 - q_2) + F_c \text{sign}(\dot{q}_1) = 0. \]  \hspace{1cm} (4.1)

\[ J \ddot{q}_2 - K(q_1 - q_2) = u. \] \hspace{1cm} (4.2)

Since the friction term \( F_c \text{sign}(\dot{q}_1) \) introduces discontinuity to the system, the control problem becomes more complex. There is, however, a potential approach for the
control design. In view of our previous discussion in Figure 1.1, Coulomb friction is described by:

\[
F_c \text{sign}(\dot{q}_1) = \begin{cases} 
F_c & \text{if } \dot{q} > 0 \\
0 & \text{if } \dot{q} = 0 \\
-F_c & \text{if } \dot{q} < 0 
\end{cases}
\]

To be able to utilize the integral manifold technique on this discontinuous system (piecewise continuous system), we first study the system in one dynamic domain: \( \dot{q}_1 > 0 \), by assuming the system dynamic remains in this dynamic domain for all \( t \geq 0 \). Following the same design strategy, we also give the design for \( \dot{q}_1 < 0 \) dynamic domain. Then a further design is tested for the regulation system to show that the dynamics of the regulation system satisfies the assumption.

4.1.1 Integral Manifold Control Design in One Dynamic Domain

(1) Integral manifold control design in the dynamic domain \( \dot{q}_1 > 0 \):

The system equations of motion in the domain \( \dot{q}_1 > 0 \) are given by:

\[
I \ddot{\theta}_1 + c_1 \dot{\theta}_1 + K(q_1 - q_2) + F_c = 0 \quad \text{(for } \dot{q}_1 > 0) \quad (4.3)
\]

\[
J \ddot{\theta}_2 - K(q_1 - q_2) = u. \quad (4.4)
\]

Following the integral manifold control design approach introduced in [24], we define \( z = K(q_2 - q_1) \) to represent the torque transmitted through the flexible belt between the motor disk and the load disk, therefore,

\[
I \ddot{\theta}_1 + c_1 \dot{\theta}_1 + F_c = z.
\]

\[
\frac{J}{K}(\ddot{z} + K \ddot{\theta}_1) + z = u.
\]

After substituting \( \ddot{\theta}_1 \), we obtain:

\[
I \ddot{\theta}_1 + c_1 \dot{\theta}_1 + F_c = z.
\]

\[
\frac{J}{K} \ddot{z} + \left(1 + \frac{J}{I}\right)z - \frac{Jc_1}{I} \dot{\theta}_1 - \frac{J}{I} F_c = u.
\]
To ease the system analysis, we define a new system input,

\[ u' = u + \frac{J}{I} c_1 \dot{q}_1. \]

Therefore, the system model becomes:

\[ I \ddot{\tilde{q}}_1 + c_1 \dot{q}_1 + F_c = \ddot{z}. \]
\[ \frac{J}{K} \ddot{z} + (1 + \frac{J}{I})z = u' + \frac{J}{I} F_c. \]

Assuming \( K \) is large and defining \( K' = k_1/\epsilon^2 \), where \( k_1 = \mathcal{O}(1) \) and \( \epsilon \) is a small parameter, we obtain a singular perturbation model.

\[ I \ddot{q}_1 + c_1 \dot{q}_1 + F_c = \ddot{z}. \]
\[ \epsilon^2 J \ddot{z} + k_1 (1 + \frac{J}{I})z = k_1 u' + \frac{k_1 J}{I} F_c. \]

In this thesis, a new composite control law is designed as:

\[ u' = u_s + u_f + z. \] (4.5)

Where \( u_f = k_v (\dot{q}_1 - \dot{q}_2) \) is a fast component and \( u_s \) is a slow component (the advantage of this control design will be discussed later).

Assuming \( k_v = k_2/\epsilon \), a singular perturbation form is obtained as:

\[ I \ddot{q}_1 + c_1 \dot{q}_1 + F_c = \ddot{z}. \] (4.6)
\[ \epsilon^2 J \ddot{z} + k_2 \dot{z} + k_1 \frac{J}{I} \ddot{z} = k_1 u_s + \frac{k_1 J}{I} F_c. \] (4.7)

Writing the above equations in the state space form, we obtain:

\[ \dot{x} = A_{11} x + A_{12} u_c + B_1 (\frac{F_c}{I} + \dot{q}_d + \frac{c_1}{I} \dot{q}_d). \] (4.8)
\[ \epsilon \dot{u} = A_{22} u_c - B_2 (u_s + \frac{J}{I} F_c). \] (4.9)
where,

\[
\begin{align*}
    x &= \begin{bmatrix} q_d - q_1 \\ \dot{q}_d - \dot{q}_1 \end{bmatrix}, \\
    w &= \begin{bmatrix} z \\ \epsilon \hat{z} \end{bmatrix}, \\
    A_{11} &= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{f} \end{bmatrix}, \\
    A_{12} &= \begin{bmatrix} 0 & 0 \\ -\frac{1}{f} & 0 \end{bmatrix}, \\
    B_1 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
    A_{22} &= \begin{bmatrix} 0 & 1 \\ -\frac{k_s}{f} & -\frac{k_2}{f} \end{bmatrix}, \\
    B_2 &= \begin{bmatrix} 0 \\ \frac{k_s}{f} \end{bmatrix}.
\end{align*}
\]

Our next task is to adopt a design procedure developed in [24] to design the integral manifold controller for this system.

**Step 1: Compute the integral manifold of the homogeneous system.**

From Lemma A.1 in Appendix A, we obtain that,

\[
\begin{bmatrix} \dot{x} \\ \epsilon \dot{w} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix},
\]

has an integral manifold \( w = Lx \), where \( L = 0 \).

**Step 2: Design the slow control component \( u_s \).**

We adopt the following structure for the slow control component \( u_s \):

\[
u_s = Fx + u_{st},\]

where \( F = [f_p \ f_d] \), with \( f_p \) and \( f_d \) being the positive scalars.

Therefore, from Equation (4.8) and (4.9) the system singular perturbation form is governed by,

\[
\begin{bmatrix} \dot{x} \\ \epsilon \dot{w} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ B_2 F & A_{22} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} \frac{\epsilon}{f} + \dot{q}_d + \frac{\epsilon}{f} \dot{q}_d \\ u_{st} + \frac{f}{f} F \end{bmatrix}.
\]

**Step 3: Design the feedforward component \( u_{st} \).**
By Corollary A.1 in Appendix A with L=0, Equation (4.10) has an integral manifold:

\[ w = Mx + n. \]

where \( n \) satisfies:

\[ \epsilon \dot{n} = (A_{22} - \epsilon M A_{12})n + \begin{bmatrix} [0 & B_2] & -\epsilon M [B_1 & 0] \end{bmatrix} \begin{bmatrix} F_c + \ddot{q}_d + \frac{J}{I} \dot{q}_d \\ u_{st} + \frac{J}{I} F_c \end{bmatrix}. \] (4.11)

From Equation (4.10) we obtain:

\[ \dot{x} = (A_{11} + A_{12} M)x + A_{12} n + B_1 \left( \frac{F_c}{I} + \ddot{q}_d + \frac{c_1}{I} \dot{q}_d \right). \] (4.12)

In Equation (4.12), it can be seen that \( x=0 \) is an exponentially stable equilibrium if the following two conditions are satisfied:

1. \( A_{12} n + B_1 \left( \frac{F_c}{I} + \ddot{q}_d + \frac{c_1}{I} \dot{q}_d \right) = 0. \) (4.13)

2. \( (A_{11} + A_{12} M) \) is Hurwitz.

Combining Equation (4.13) with (4.11) yields.

\[ A_{12} n + B_1 \left( \frac{F_c}{I} + \ddot{q}_d + \frac{c_1}{I} \dot{q}_d \right) = 0. \]

\[ \epsilon \dot{n} = A_{22} n + B_2 \left( u_{st} + \frac{J}{I} F_c \right). \]

Solving for \( u_{st} \) and \( n \) we obtain:

\[ n = \begin{bmatrix} F_c + I \ddot{q}_d + c \dot{q}_d \\ \epsilon I q_d^{(3)} + c \dot{q}_d^{(2)} \end{bmatrix}. \]

\[ u_{st} = \frac{J c_1}{I} \ddot{q}_d + \left( J + \frac{k_v C_1}{K} \right) \dot{q}_d + \left( J \frac{k_v}{K} + \frac{I k_v}{K} q_d^{(3)} \right) + \frac{J}{I} q_d^{(4)} + \left[ \frac{J}{I} F_c - \frac{J}{I} F_c \right] \]

\[ = \frac{J c_1}{I} \ddot{q}_d + \left( J + \frac{k_v C_1}{K} \right) \dot{q}_d + \left( J \frac{k_v}{K} + \frac{I k_v}{K} q_d^{(3)} \right) + \frac{J}{K} q_d^{(4)} \]

\[ = f(q_d^{(4)}, q_d^{(3)}, \ddot{q}_d, \dot{q}_d). \]
It is worth noting that the friction term \( \int F_c \) is completely canceled in the above control calculation. Otherwise, a control law with friction term will require a friction identification which may result in an undercompensated or an overcompensated system.

Recalling the definition of \( u \), we obtain:

\[
\begin{align*}
    u &= u' - \frac{J}{I} c_1 \dot{q}_1 \\
    &= Fx + u_s + u_f + z - \frac{J}{I} c_1 \dot{q}_1 \\
    &= Fx + f(q_a^{(4)}, q_d^{(3)}, \ddot{q}_d, \dot{q}_d) + k_v(\dot{q}_1 - \dot{q}_2) + K(q_2 - q_1) - \frac{Jc_1}{I} \dot{q}_1.
\end{align*}
\]

Hence, we obtain a final control algorithm with the complete cancelation of friction. This complete cancelation is based on our previous assumption and achieved by adopting the new control design \( u' = u_s + u_f + z \) where \( z \) is an important component introduced for the cancelation purpose.

Now, to determine the feedback gain \( F \) to satisfy the second condition: \( (A_{11} + A_{12}M) \) is Hurwitz. we start from Corollary A.1, where \( M \) satisfies.

\[
(B_2 - \epsilon LB_1)F + (A_{22} - \epsilon LA_{12})M = \epsilon M(A_{11} + A_{12}L + A_{12}M). \tag{4.14}
\]

Since \( L = 0 \), by assuming \( M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \) and \( F = [f_p, f_d] \), we obtain:

\[
\begin{align*}
    m_{21} &= -\frac{\epsilon m_{11}m_{12}}{I} \\
    m_{22} &= \epsilon(m_{11} - \frac{m_{12}^2}{I}).
\end{align*}
\]

\[
\begin{align*}
    f_p &= \frac{(I + J)m_{11}}{I} - \frac{k_v m_{11} m_{12}}{IK} - \frac{J m_{11}(I m_{11} - c_1 m_{12} - m_{12}^2)}{I^2 K} \\
    f_d &= \frac{(I + J)m_{12}}{I} - \frac{k_v(I m_{11} - m_{12} c_1 - m_{12}^2)}{IK} - \frac{J m_{12}(2 I m_{11} - m_{12}^2)}{I^2 K}. \tag{4.15} \tag{4.16}
\end{align*}
\]

It can be verified easily that \( A_{11} + A_{12}M \) is Hurwitz if \( m_{11} > 0 \) and \( m_{12} > 0 \).
Therefore, we obtain the complete control algorithm as,

\[
    u = Fx + f(q_d^{(4)}, q_d^{(3)}, \ddot{q}_d, \dot{q}_d) + k_v(\dot{q}_1 - \dot{q}_2) + K(q_2 - q_1) - \frac{Jc_1}{I} \dot{q}_1.
\]

\[
    = f_p(q_d - q_1) + f_d(\ddot{q}_d - \dot{q}_1) + f(q_d^{(4)}, q_d^{(3)}, \ddot{q}_d, \dot{q}_d)
    + k_v(\dot{q}_1 - \dot{q}_2) + K(q_2 - q_1) - \frac{Jc_1}{I} \dot{q}_1. \tag{4.17}
\]

(2) Integral manifold control design in the dynamic domain \(\dot{q}_1 < 0\):

Following a similar design procedure, a control algorithm for \(\dot{q}_1 < 0\) domain is obtained as:

\[
    u = f_p(q_d - q_1) + f_d(\ddot{q}_d - \dot{q}_1) + f(q_d^{(4)}, q_d^{(3)}, \ddot{q}_d, \dot{q}_d)
    + k_v(\dot{q}_1 - \dot{q}_2) + K(q_2 - q_1) - \frac{Jc_1}{I} \dot{q}_1. \tag{4.18}
\]

It can be observed that a unifying controller is obtained for \(\dot{q}_1 > 0\) and \(\dot{q}_1 < 0\).

system dynamic domain. which is also due to the complete friction cancelation in the final control algorithm. Therefore, when the above integral manifold controller is applied to the system, there is no need to identify the friction or switch the controller between the two dynamic domains caused by the friction term in the controller. which is of high importance to the system control performance.

4.1.2 Further Design for Regulation Systems

The objective of this section is to further design the integral manifold control gain based on Equation (4.18) so that the regulation system time response to a step input is critically damped or slightly overdamped for initial condition \(q_1(0) = q_{10}\) and \(\dot{q}_1(0) = 0^+\). in other words. the system velocity remains in one dynamic domains \(\dot{q}_1 > 0\) (or \(\dot{q}_1 < 0\)) for all \(t \geq 0\) as in our previous assumption. The characteristic equation is studied in this section for choosing control parameters to satisfy the assumption.
First we consider the system Equation (4.19) and (4.20) which represent the system behavior in the dynamic domain \( q_1 > 0 \):

\[
I \ddot{q}_1 + c_1 \dot{q}_1 + K(q_1 - q_2) + F_c = 0. \tag{4.19}
\]

\[
J \ddot{q}_2 - K(q_1 - q_2) = u. \tag{4.20}
\]

For the regulation problem, the integral manifold control design in Equation (4.18) is simplified by \( \dot{q}_d = 0 \) for \( t \geq 0 \) as:

\[
\begin{align*}
  u &= Fx + k_v(\dot{q}_1 - \dot{q}_2) + K(q_2 - q_1) - \frac{Jc_1}{I} \dot{q}_1, \\
  &= f_p(q_d - q_1) - f_d \dot{q}_1 + k_v(\dot{q}_1 - \dot{q}_2) + K(q_2 - q_1) - \frac{Jc_1}{I} \dot{q}_1. \tag{4.21}
\end{align*}
\]

Taking (4.21) into (4.19) and (4.20), we obtain:

\[
I \ddot{q}_1 + c_1 \dot{q}_1 + K(q_1 - q_2) + F_c = 0. \tag{4.22}
\]

\[
J \ddot{q}_2 - K(q_1 - q_2) = f_p(q_d - q_1) + f_d(\dot{q}_d - \dot{q}_1) + k_v(\dot{q}_1 - \dot{q}_2) + K(q_2 - q_1) - \frac{Jc_1}{I} \dot{q}_1. \tag{4.23}
\]

From (4.22) we have:

\[
q_2 = \frac{1}{K}(I \ddot{q}_1 + c_1 \dot{q}_1 + Kq_1 + F_c). \tag{4.24}
\]

Taking (4.24) into (4.23), the system equation becomes:

\[
\frac{IJ}{K} q_1^{(4)} + \left( \frac{Jc_1}{K} + \frac{Ik_v}{K} \right) q_1^{(3)} + \left( J + \frac{k_v c_1}{K} \right) q_1^{(2)} + (f_d + \frac{Jc_1}{I}) \dot{q}_1 + f_p q_1 = f_p q_d.
\]

Therefore, the system transfer function is given by:

\[
\frac{q_1(s)}{q_d(s)} = \frac{f_p}{\frac{IJ}{K} s^4 + \left( \frac{Jc_1}{K} + \frac{Ik_v}{K} \right) s^3 + \left( J + \frac{k_v c_1}{K} \right) s^2 + (f_d + \frac{Jc_1}{I}) s + f_p}.
\]
which is a non-zero fourth order system.

The characteristic equation is therefore obtained:

$$s^4 + \left( \frac{c_1}{l} + \frac{k}{J} \right)s^3 + \left( \frac{K}{l} + \frac{k_{c_1}}{lJ} \right)s^2 + \left( \frac{Kf_d}{lJ} + \frac{K_c}{lJ^2} \right)s + \frac{Kf_p}{lJ} = 0. \quad (4.25)$$

To insure the step response of the above system to be critically damped or slightly overdamped, we first utilize the root-locus techniques to determine a set of numerical control parameters in simulation and experiments which is given in Table 4.1 as in Chapter 5. The simulation and experimental results show that the system response is slightly overdamped, which satisfies our assumption.

<table>
<thead>
<tr>
<th>( m_{12} )</th>
<th>( m_{22} )</th>
<th>( f_p )</th>
<th>( f_d )</th>
<th>( k_v )</th>
<th>Roots of the Transfer Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0056</td>
<td>0.14</td>
<td>0.14</td>
<td>0.02</td>
<td>0.06</td>
<td>-9.1627 ± 24.0584j, -10.8561 ± 3.7884j</td>
</tr>
</tbody>
</table>

Table 4.1: Control Gains for the Integral Manifold Control

If gains can be chosen to lead to a critically damped or overdamped response, a stability analysis can be performed as given in the Appendix to give a range of stiffness for which asymptotic stability is guaranteed.

### 4.2 Tracking Performance

In this section, the tracking performance is studied for the system (4.1) and (4.2) with integral manifold design as presented in (4.18). In the tracking problem, depending on the desired trajectory, the system behavior will most probably evolve from one domain to the other, and verse visa.

In the integral manifold control design (1) and (2), we conclude that the flexible system has an integral manifold \( w = Mx + u \), where.
for the dynamic domain $\dot{q}_1 > 0$:
\[
\mathbf{n} = \begin{bmatrix}
F_c + I \ddot{q}_d + c \dot{q}_d \\
\epsilon I q_d^{(3)} + \epsilon c q_d^{(2)}
\end{bmatrix}
\]

while for the dynamic domain $\dot{q}_1 < 0$:
\[
\mathbf{n} = \begin{bmatrix}
-F_c + I \ddot{q}_d + c \dot{q}_d \\
\epsilon I q_d^{(3)} + \epsilon c q_d^{(2)}
\end{bmatrix}
\]

Therefore, it yields that there may exist a manifold switching when the system dynamics moves from domain $\dot{q}_1 > 0$ to domain $\dot{q}_1 < 0$. Figure 4.1 illustrates the two integral manifolds.

For tracking systems, we apply the same integral manifold control design as in Section 4.1 for each one of the dynamic domain $\dot{q}_1 > 0$ and $\dot{q}_1 < 0$ even though this controller was based on the steady state operation. The system performance is studied in simulation and experiments in Chapter 5. Further stability study of this discontinuous tracking system is expected in future work.

Summary

In this section, an integral manifold control algorithm is designed for regulation systems based on one dynamic domain assumption (subspace). We further utilize the characteristic equation to a set of control parameters to satisfy the assumption. The performance issue of the controlled tracking system is also briefly explained. The advantage of this control algorithm is that the friction is compensated without iden-
Figure 4.1: Two Integral Manifolds

tification, and the control law has a unifying form for each of the dynamic domain
\( \dot{q}_1 > 0 \) and \( \dot{q}_1 < 0 \).
Chapter 5

Evaluation

This chapter presents the simulation and experimental results of the full system model (3.3) and (3.4) based on the integral manifold control design in the Equation (4.18). The controlled system performances are compared with that of the PD control system. The purpose of the simulation and the experiments is to examine the effectiveness of the integral manifold control design on the flexible system with friction. For this purpose, various regulation and tracking trajectories are tested in both simulation and experiments.

5.1 System Model

The integral manifold control algorithm is tested experimentally on the electromechanical system in our laboratory (See Figure 1.1). The system is modeled as in Equation (3.3) and (3.4):

\[ J_2 \ddot{\theta}_2 + v_2 \dot{\theta}_2 + k(\theta_2 - g_r^{-1} \theta_1) + f_c sign(\dot{\theta}_2) = 0. \]

\[ J_1 \ddot{\theta}_1 + k(g_r^{-2} \theta_1 - g_r^{-1} \theta_2) = T_D - [v_1 \dot{\theta}_1 + f_c sign(\dot{\theta}_1)]. \]

5.2 System Identification

Identification experiments are performed to obtain the physical parameters of the system which are used for simulation and for control algorithm design. Table 5.1 and Table 5.2 present the identification results of the system parameters. Table 5.1 includes the true values of the system parameters while Table 5.2 presents the nominal
values of the system parameters which is defined in Chapter 3 and used in the control
design of Chapter 4.

5.3 Simulation

To study the control efficiency of the integral manifold control design in Chapter 4
and to compare the performance of the integral manifold control system with that
of the PD control system, computer simulations are first performed on the system
model described by Equation (3.3) and (3.4).

The structure of the PD control system is shown in Figure 5.1, where the control
algorithm is defined by:

\[ u = k_p(\theta_1 - \theta_2) + k_d(\dot{\theta}_1 - \dot{\theta}_2). \]

For this PD control system, regulation problem is first studied and the control gain
values are selected as in Table 5.3 to represent the three typical cases for a regulation
system: underdamped, critically damped and overdamped system. In Simulation 1.
the desired trajectory of the regulation study is a step 90 degree rotation of the load
disk.

For the purpose of comparison, in Simulation 2, an integral manifold controller is
applied to the same system model. The structure of the integral manifold control law
is shown in Figure 5.2, where \( u \) is given in Equation (4.18) as:

\[ u = f_p(q_d - q_1) + f_d(\dot{q}_d - \dot{q}_1) + f(q_d^{(4)}, q_d^{(3)}, \dot{q}_d, \ddot{q}_d) + k_v(\dot{q}_1 - \dot{q}_2) + K(q_2 - q_1) - \frac{Jc_1}{I} \dot{\theta}_1. \]

The integral manifold control gains are calculated by Equation (4.15) and (4.16) and
given in Table 5.4. The desired trajectory of the integral manifold control system is
the same 90 degree rotation of the load disk.

A further study is given to the tracking performance of the system with a pure
sinusoidal trajectory in Simulation 3 and 4. The amplitude of the sinusoidal input is
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_r$: gear ratio of drive disk vs. load disk</td>
<td>4</td>
</tr>
<tr>
<td>$g'_r$: gear ratio of motor disk vs. pulley</td>
<td>2</td>
</tr>
<tr>
<td>$J_d$: drive disk inertia</td>
<td>0.000398 [Nms²/rad]</td>
</tr>
<tr>
<td>$J_l$: load disk inertia</td>
<td>0.0073 [Nms²/rad]</td>
</tr>
<tr>
<td>$J_p$: pulley inertia</td>
<td>$7.8 \times 10^{-5}$[Nms²/rad]</td>
</tr>
<tr>
<td>$\nu_1$: viscous frictional coefficient of the drive disk</td>
<td>0.004 [Nms/rad]</td>
</tr>
<tr>
<td>$\nu_2$: viscous frictional coefficient of the load disk</td>
<td>0.03 [Nms/rad]</td>
</tr>
<tr>
<td>$k$: flexible belt spring constant</td>
<td>7.63 [Nm/rad]</td>
</tr>
<tr>
<td>$f_{c1}$: coulomb friction of the motor disk</td>
<td>$5.77 \times 10^{-4}$[Nm]</td>
</tr>
<tr>
<td>$f_{c2}$: coulomb friction of the load disk</td>
<td>0.176 [Nm]</td>
</tr>
</tbody>
</table>

**Table 5.1**: True System Parameters
<table>
<thead>
<tr>
<th>Nominal Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I = J_1 g_r^{-1}$</td>
<td>0.001825 [Nms²/rad]</td>
</tr>
<tr>
<td>$J = J_d g_r = J_d + \frac{J_p}{g_r c_a}$</td>
<td>0.00167 [Nms²/rad]</td>
</tr>
<tr>
<td>$c_1 = g_r^{-1} v_2$</td>
<td>0.0075 [Nms/rad]</td>
</tr>
<tr>
<td>$F_c = g_r^{-1} J_{c2}$</td>
<td>0.044[Nm]</td>
</tr>
<tr>
<td>$K = g_r^{-1} k$</td>
<td>1.9075[Nm/rad]</td>
</tr>
</tbody>
</table>

Table 5.2: Nominal System Parameters

A=11.25 degree, and the frequency is $w = \frac{5\pi}{8}$ (rad/s). PD control system and integral manifold control system are studied for the tracking performance in Simulation 3 and Simulation 4 respectively. The control gains are selected as in the previous regulation study.

To test the integral manifold control system on a fast frequency response, we increase the frequency of the sinusoidal input by 10 times in Simulation 5, in which the frequency becomes $w = \frac{5\pi}{4}$ (rad/s). This represents a situation in which a quick tracking response is required for the controlled system.

Recalling in Chapter 2, the integral manifold controller is designed for an ideal system model by ignoring the friction in the load side, while the integral manifold controller designed in Chapter 4 is based on the full system model with consideration of friction. The purpose of Simulation 6 is to compare the effect of friction on the
Figure 5.1: The Structure of the PD Control System

system performance of these two integral manifold control systems. The same desired step response with a 90 degree rotation of the load disk is tested on both integral manifold control designs, while the friction value in the load disk is intended to be increased from $f_{c2} = 0.176 (Nm)$ to $f_{c2} = 0.473 (Nm)$.

5.4 Analysis of the Simulation Results

The regulation performances of the three PD controllers in Simulation 1 are shown in Figure B.1(a), B.2(a) and B.3(a). The specification values of the time response are measured and given in Table 5.5, where,

$M_p$: Maximum overshoot
$t_d$: Delay time
$t_s$: Settling time
$\epsilon_{ss}$: Steady state error.
From the simulation results it can be observed that the PD control system doesn’t go to its equilibrium for all of the three cases in the observed time period because of the existence of the friction.

Figure B.4(a) presents the regulation performance of the integral manifold controller with control gains given in Table 5.4. The time response is evaluated in Table 5.6.

Comparing each time response specification in Simulation 1 and Simulation 2, we observe that in Table 5.6, the integral manifold control system not only goes to its equilibrium in the observed time period but also exhibits a better performance without steady state error, while in Table 5.4, there is steady state error for each of the PD control systems. Therefore, it can be concluded that for the same regula-
Figure 5.2: The Structure of the Integral Manifold Control System

tion requirement, the integral manifold control can give a better system performance compared to the PD control system.

For the tracking system, the performances of the PD control systems are given in Figure B.5(a), Figure B.6(a), and Figure B.7(a), with the following error shown in the third subplot. From Figure B.5(a), Figure B.6(a), and Figure B.7(a), large following error can be observed in the tracking performance of the load side. It shows that PD controlled systems can't give a satisfactory tracking result when friction exists in the system.

The performance of the integral manifold controlled tracking system is given in Figure B.8(a). Here a better tracking performance with smoother response and much
Table 5.5: Time Response Evaluation for the PD Controller

<table>
<thead>
<tr>
<th>Case</th>
<th>$M_p$</th>
<th>$t_d(s)$</th>
<th>$t_s(s)$</th>
<th>$e_{ss}(Degree)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>34.9%</td>
<td>0.056</td>
<td>0.29</td>
<td>3.3</td>
</tr>
<tr>
<td>2</td>
<td>0%</td>
<td>0.25</td>
<td>0.93</td>
<td>11.5</td>
</tr>
<tr>
<td>3</td>
<td>0%</td>
<td>0.13</td>
<td>0.69</td>
<td>6.9</td>
</tr>
</tbody>
</table>

Table 5.6: Time Response Evaluation for the Integral Manifold Controller

<table>
<thead>
<tr>
<th>$M_p$</th>
<th>$t_d(s)$</th>
<th>$t_s(s)$</th>
<th>$e_{ss}(Degree)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>0.17</td>
<td>0.36</td>
<td>0</td>
</tr>
</tbody>
</table>

less following error is observed. It can be also observed that the system has zero following error in the dynamic domain $\dot{q}_1 > 0$ and $\dot{q}_1 < 0$, this verifies the performance analysis in Chapter 4 about the the system behavior in one of the two dynamic domains.

With the same integral manifold control structure and same control gain choice, a faster desired tracking trajectory is also tested on the system model in Simulation 5. The fast frequency response of the integral manifold control system is shown in
Figure B.9(a), where it can be observed that the system response remains good even the desired frequency is increased by 10 times.

In Simulation 6, the effects of the friction compensation on the integral manifold control systems are studied. Figure B.10 (a) and (c) show the step response of the integral manifold control system based on the ideal model when friction is increased from $f_{c2} = 0.176 (Nm)$ to $f_{c2} = 0.473 (Nm)$. It can be observed that in Figure B.10 (a) and (c), when friction increases, the steady state error increases. This is because the friction is not compensated in the integral manifold control design based on the ideal model. The same situation is tested on the integral manifold control design based on the full system model. It can be also observed that in Figure B.11 (a) and (c), the steady state error still remains zero with large changes of friction value. Therefore, we can conclude that the integral manifold control design presented in Chapter 4 exhibits a better robustness than the integral manifold controller based on the ideal system model when the friction changes in the system. In most cases, friction change is unavoidable in a real control system. To justify our control design, from Equation (4.20), it can be understood that the friction compensation is actually performed by the term $K(q_2 - q_1)$. Therefore, when the friction increases, to achieve the friction compensation, $q_2$ should also increase. This increase of $q_2$ can be observed in Figure B.12(a) compared with (c).

5.5 Experimental Results and Analysis

Corresponding to each of the simulations in Section 5.3, experiments are performed and analyzed on the real system in Figure 1.1. In Experiment 1, the regulation response of the PD control system is shown in Figure B.1(b), Figure B.2(b) and Figure B.3(b). From the time response specification measured in Table 5.7, it is observed that the experimental results and the simulation results (Table 5.5) have
good agreements, which verifies the accuracy of the system model and the parameter identification.

The experimental result of the integral manifold control for the regulation problem is given in Figure B.4(b) and Table 5.8 in Experiment 2, which shows that in the real system, the integral manifold control algorithm results in a better system performance than PD control when the positioning issues are concerned. This is the same result as what the simulation anticipates.

PD control and integral manifold control for the tracking system are tested in Experiment 3 and Experiment 4, with the same control gains and the same desired sinusoidal trajectory as in simulations. The results are shown in Figure B.5 (b), Figure B.6 (b), Figure B.7 (b), and Figure B.8 (b) respectively. From the real time response, it can be observed that the integral manifold control gives much better tracking performance than that of the PD control.

<table>
<thead>
<tr>
<th>Case</th>
<th>$M_p$</th>
<th>$t_d(s)$</th>
<th>$t_s(s)$</th>
<th>$e_{ss}(Degree)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>34%</td>
<td>0.074</td>
<td>0.44</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>0%</td>
<td>0.27</td>
<td>1.05</td>
<td>10.4</td>
</tr>
<tr>
<td>3</td>
<td>0%</td>
<td>0.18</td>
<td>0.46</td>
<td>6.4</td>
</tr>
</tbody>
</table>

**Table 5.7:** Time Response Evaluation for the PD Controller
<table>
<thead>
<tr>
<th></th>
<th>( t_d(s) )</th>
<th>( t_s(s) )</th>
<th>( e_{2a}(\text{Degree}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>0.21</td>
<td>0.41</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 5.8:** Time Response Evaluation for the Integral Manifold Controller

For the fast frequency response, Experiment 5 shows that the integral manifold control system exhibits a very smooth frequency response with small following error in Figure B.9(b).

In Experiment 6, the effect of the friction compensation on the integral manifold control system is tested on the real system, corresponding to the situations in Simulation 6. From Figure B.10 and Figure B.11, we can observe that in case \( f_{c2} = 0.176 (Nm) \), both the integral manifold controller without friction compensation and with friction compensation give a good performance. After a coulomb friction device is adjusted to increase the friction to \( f_{c2} = 0.473 (Nm) \) (measured by identification experiments), the same two integral manifold controllers are applied to the system. The corresponding experimental results in Figure B.11 compared to Figure B.10 show that the integral controller with friction compensation has a better robustness than the controller based on the ideal system model when friction changes in the real system.

In Figure B.13, the following errors in PD control and in integral manifold control are compared. It shows that the integral manifold control can approximately give a factor of 6 improvement. The control effort (motor torque) comparison is given in Figure B.14. From these two Figures, we can conclude that the integral manifold
control can produce much better tracking performance than PD control while the control effort is even less.

5.6 Summary

The integral manifold control design based on the full system model is tested in this chapter with both simulation and experimental results. From the system response anticipated in simulation and that obtained in the real system experiments, it can be concluded that the integral manifold control design can produce better performance in regulation and in tracking systems than PD control. Furthermore, the integral manifold control design with friction compensation exhibits better robustness than the integral manifold control design based on the ideal system model when the friction changes in systems.
Chapter 6

Conclusion

Flexible systems are used to perform various tasks in the real world. With non-conventional system structure (such as harmonic drives and flexible coupling devices) adopted in electromechanical systems, the flexibility and friction may not be neglected in the system modeling, simulation, and control design processes, especially when positioning and tracking accuracy are of high importance.

In this thesis, we have addressed the control problem of a flexible system with friction compensation. The application of the integral manifold approach to a basic flexible-shaft drive-load system as presented in [24] is first reviewed. The simulation and experimental results show that the system performance is quite satisfactory as long as the friction effects are minimal. Otherwise, a performance degradation may become evident by the absence of the friction compensation.

The main contribution of this thesis is to present a new integral manifold control design for regulation systems with friction compensation achieved independently of the friction identification. Although the full system model is a discontinuous model, we applied the integral manifold technique by adopting a subspace-based approach, in which the subspace is determined by the sign of the velocity of the load disk (two dynamic domains). A new control structure is designed for regulation systems to achieve the complete cancelation of the friction term in the control algorithm. In our final control design, it would be very interesting to observe that the friction compensation is actually performed by the torque transmitted through the flexible shaft, which produces deformation because of the friction. It is also interesting to note...
that when the integral manifold control algorithm is calculated, no knowledge of the friction is necessary and no switching of the control component is required. To satisfy one dynamic domain assumption, we further gave a set of control parameters based on the system characteristic equation and tested it in the simulation and experiments. A brief performance analysis was also given to the tracking systems.

To verify the integral manifold control design in this work, simulation and experiments were performed on a real system. First, a real analogy between the simulation and the experimental results was observed, which predicted the accuracy of the model. Second, the integral manifold control design with friction compensation results in a much better system performance (exponentially stable) compared to that of the PD control with the existence of friction. Third, the integral manifold control algorithm with friction compensation produces a more robust system than the integral manifold control algorithm without friction compensation.

In this thesis work, the coulomb friction in the motor side is small enough to be ignored in the stage of the integral manifold control design. The simulation and experimental results in Chapter 5 justify this ignore of the small friction.

Considering some other systems, the friction in the motor side may be significant and may affect the system performance evidently. Hence, to improve the system performance a further study of the friction compensation in the motor side is suggested.

The integral manifold controller can be also applied to the tracking systems as presented in Chapter 4. Although in Chapter 5, the performance of the tracking systems illustrates the efficiency of this integral manifold controller, in future work a mathematical proof of the tracking system stability would be expected by utilizing discontinuous system analysis techniques.
Appendix A

Definition and Properties of Integral Manifold

The following theory is about the integral manifold of linear systems (See [24] for reference).

Consider the homogeneous (no external input) linear time invariant singular perturbation system:

\[
\begin{bmatrix}
\dot{x} \\
\epsilon w
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
x \\
w
\end{bmatrix}.
\]  \hspace{1cm} (A.1)

where \(x \in \mathbb{R}^n, w \in \mathbb{R}^m\).

\(A_{ij}, i, j=1, 2\) are constant matrices with appropriate dimensions.

Definition 2.1

With respect to the homogeneous linear time invariant system (A.1), the set

\[
M = (x, w) \in \mathbb{R}^n \times \mathbb{R}^m : w = Lx, L \in \mathbb{R}^{m \times n}
\]

is said to be an integral manifold if for \((x(0), w(0)) \in M\) (i.e. \(w(0) = Lx(0)\)), the solution \((x(t), w(t))\) is in \(M\), that is \(w(t) = Lx(t)\) for all \(t \geq 0\). Hence, if a solution of (A.1) starts on the integral manifold, it stays there thereafter.

The properties of integral manifolds are originally introduced in [34], [25] and extended in [24].

Fact A.1:

Suppose that a constant matrix \(L \in \mathbb{R}^{m \times n}\) satisfies:

\[
A_{21} + A_{22}L = \epsilon L(A_{11} + A_{12}L).
\]
then the homogeneous linear time invariant system (A.1) has a linear integral manifold

\[ w = Lx \]

**Fact A.2:**

If the homogeneous system (A.1) has the linear integral manifold \( w = Lx \), then the system

\[
\begin{bmatrix}
    \dot{x} \\
    \epsilon \dot{w}
\end{bmatrix} =
\begin{bmatrix}
    A_{11} & A_{12} \\
    0 & A_{22}
\end{bmatrix}
\begin{bmatrix}
    x \\
    w
\end{bmatrix} +
\begin{bmatrix}
    B_1 \\
    B_2
\end{bmatrix} u.
\]

(A.2)

has a linear integral manifold given by

\[ w = Lx + p. \]

and \( p(t) \) satisfies

\[ \epsilon \dot{p} = (A_{22} - \epsilon LA_{12})p + (B_2 - \epsilon LB_1)u, \quad p(0) = 0. \]

**Fact A.3:**

Suppose the homogeneous system (A.1) has a linear integral manifold \( w = Lx \). Then System (A.2) with input \( u = kx \) has an integral manifold given by

\[ w = (L + M)x. \]

where \( M \) satisfies:

\[ \hat{B}K + \hat{A}M = \epsilon M (A_1 + B_1K + A_{12}M). \]

(A.3)
Corollary A.1:

Suppose the homogeneous system (A.1) has a linear integral manifold \( w = Lx \), then system (A.2) with input \( u = kx + f(t) \) has an integral manifold given by:

\[
    w = (L + M)x + n.
\]

where \( M \) satisfies (A.3) and \( n \) is given by:

\[
    en = \bar{A}n + \bar{B}f(t),
\]

and

\[
    \bar{A} = A_{22} - \epsilon(L + M)A_{12},
\]

\[
    \bar{B} = B_2 - \epsilon(L + M)B_1.
\]

Lemma A.1:

The homogeneous part of system:

\[
    \begin{bmatrix}
    \dot{x} \\
    \epsilon \dot{w}
    \end{bmatrix} = \begin{bmatrix}
    A_{11} & A_{12} \\
    0 & A_{22}
    \end{bmatrix}
    \begin{bmatrix}
    x \\
    w
    \end{bmatrix}.
\]

has an integral manifold \( w = Lx \) where the unique matrix \( L = O(1) \) is given by \( L = 0 \).
Appendix B

Simulation and Experimental Results
Figure B.1: PD Control for the Regulation System. kp=2, kd=0.05. (a) Simulation Results. (b) Experimental Results.
Figure B.2: PD Control for the Regulation System, kp=0.1, kd=0.03.
(a) Simulation Results. (b) Experimental Results
Figure B.3: PD Control for the Regulation System, kp=0.3, kd=0.06.
(a) Simulation Results. (b) Experimental Results
Figure B.4: Integral Manifold Control for the Regulation System. (a) Simulation Results. (b) Experimental Results
Figure B.5: PD Control for the Tracking System, kp=2, kd=0.05. (a) Simulation Results. (b) Experimental Results
Figure B.6: PD Control for the Tracking System. \( kp=0.1, kd=0.03 \). (a) Simulation Results. (b) Experimental Results
Figure B.7: PD Control for the Tracking System, kp=0.3, kd=0.06. (a) Simulation Results. (b) Experimental Results
Figure B.8: Integral Manifold Control for the Tracking System. (a) Simulation Results. (b) Experimental Results
Figure B.9: Integral Manifold Control for the Fast Tracking System. (a) Simulation Results. (b) Experimental Results
Figure B.10: Integral Manifold Control without Friction Compensation.
(a) Simulation Result for Friction = 0.176 Nm (b) Experimental Result for Friction = 0.176 Nm (c) Simulation Result for Friction = 0.473 Nm (d) Experimental Result for Friction = 0.473 Nm
Figure B.11: Integral Manifold Control with Friction Compensation. (a) Simulation Result for Friction = 0.176 Nm (b) Experimental Result for Friction = 0.176 Nm (c) Simulation Result for Friction = 0.473 Nm (d) Experimental Result for Friction = 0.473 Nm
Figure B.12: Motor Disk Angular Position of Integral Manifold Control with Friction Changing From 0.176 Nm to 0.473 Nm
Figure B.13: Following Error in PD and in Integral Manifold Control
Figure B.14: Control Effort of PD Control and Integral Manifold Control
Appendix C

Stability Analysis for Regulation Systems

In Chapter 4, we have chosen a set of control parameters for the regulation system which shows that the velocity remains the same sign for initial condition $q_1(0) = q_{10}$ and $\dot{q}_1(0) = 0^+$ in the whole time domain. To give the stability analysis for system (4.19) and (4.20), we study the continuous system:

$$I\ddot{q}_1 + c_1\dot{q}_1 + K(q_1 - q_2) + F_c = 0. \tag{C.1}$$

$$J\ddot{q}_2 - K(q_1 - q_2) = u. \tag{C.2}$$

For the above continuous system, the integral manifold control algorithm was designed in Equation (4.21), which is:

$$u = f_p(q_d - q_1) - f_d\dot{q}_1 + k_\nu(\dot{q}_1 - \dot{q}_2) + K(q_2 - q_1) - \frac{Jc_1}{I}\dot{q}_1. \tag{C.3}$$

Next, we will extend the Theorem 2.1 as in [24] to Corollary 4.1, and then give the stability analysis of this continuous system.

Corollary 4.1:

For the electromechanical system described by Equation (C.1) and (C.2) with the composite control law designed in Equation (C.3), there exists an explicit range of stiffness $(K^*, \infty]$ such that for $K$ in this range and for initial condition $q_1(0) = q_{10}, \dot{q}_1(0) = 0^+$, the integral manifold of the system is exponentially attractive, which means $\nu$ converges to $Mx + u$ locally and exponentially, and the convergence of $q_1$ to
$q_d$ and that of $\dot{q}_1$ to $\dot{q}_d$ is local and exponential for some selected control parameters as presented in Chapter 4.

Proof of Corollary 4.1:

Let $\eta = w - (Mx + n)$ represent the off-manifold term. Since the singular perturbation form of the system (C.1) and (C.2) is the same as that in Equation (4.10), to get a complete controlled system, we start from Equation (4.10) and obtain:

$$\begin{align*}
\dot{x} &= A_{11}x + A_{12}w + B_1\left(\frac{F_c}{I} + \ddot{q}_d + \frac{c_1}{I}\dot{q}_d\right), \\
\epsilon \dot{w} &= B_2Fx + A_{22}w + B_2(u_{st} + \frac{J}{I}F_c).
\end{align*}$$

(C.4)  
(C.5)

where $x, w, A_{11}, A_{12}, A_{22}, B_1, B_2$ are defined as in Equation (4.8) and (4.9).

Taking $w = \eta + (Mx + n)$ into (C.4), we obtain:

$$\dot{x} = (A_{11} + A_{12}M)x + A_{12}\eta + A_{12}n + B_1\left(\frac{F_c}{I} + \ddot{q}_d + \frac{c_1}{I}\dot{q}_d\right).$$

Recalling in the control design (1) for dynamic domain $\dot{q}_1 > 0$, $u_{st}$ is designed so that:

$$A_{12}n + B_1\left(\frac{F_c}{I} + \ddot{q}_d + \frac{c_1}{I}\dot{q}_d\right) = 0.$$ 

Therefore, we obtain:

$$\dot{x} = (A_{11} + A_{12}M)x + A_{12}\eta.$$ 

Similarly, taking $w = \eta + (Mx + n)$ into (C.5), combined with (4.11) and (4.14) we obtain after some algebra:

$$\epsilon \dot{\eta} = (A_{22} - \epsilon A_{12})\eta.$$ 

Hence, the complete system is governed by:

$$\begin{align*}
\dot{x} &= D_1x + A_{12}\eta, \\
\epsilon \dot{\eta} &= (A_{22} - \epsilon D_2)\eta.
\end{align*}$$

(C.6)
where,

\[
D_1 = \begin{bmatrix}
0 & -\frac{1}{\eta_1} \\
\frac{1}{\eta_1} & -\frac{1}{\eta_2} + \alpha_1
\end{bmatrix}.
\]

\[
D_2 = \begin{bmatrix}
0 & 0 \\
\frac{1}{\epsilon^2} & 0
\end{bmatrix}.
\]

First, let's consider the subsystem \( \dot{x} = D_1 x \). Since \( D_1 \) is Hurwitz, for any symmetric positive definite matrix \( Q_1 \), the Lyapunov matrix \( D_1^T R_1 + R_1 D_1 = -Q_1 \) has a unique symmetric positive definite solution \( R_1 \). Therefore, Lyapunov function \( V_1 = x^T R_1 x \) satisfies \((\frac{\partial V_1}{\partial x})^T D_1 x = -x^T Q_1 x \leq -\alpha_1 \|x\|^2\), where \( \alpha_1 \) is the minimum eigenvalue of \( Q_1 \).

Similarly, for subsystem \( \dot{\eta} = A_{22} \eta \), since \( A_{22} \) is Hurwitz, there exist positive definite matrix \( R_2 \) and \( Q_2 \) so that Lyapunov function \( V_2 = \eta^T R_2 \eta \) satisfies \((\frac{\partial V_2}{\partial \eta})^T D_1 \frac{1}{\epsilon} A_{22} \eta \leq -\frac{1}{\epsilon} \eta^T Q_2 \eta \leq -\frac{1}{\epsilon} \alpha_2 \|\eta\|^2\), where \( \alpha_2 \) is the minimum eigenvalue of \( Q_2 \).

Now, we consider the composite Lyapunov equation as introduced in [24].

\[
V = (1 - d) V_1 + d V_2.
\]

where \( 0 < d < 1 \).

The time derivative of \( V \) along the solution trajectories of (C.6) is given by:

\[
\dot{V} = (1 - d)(\frac{\partial V_1}{\partial x})^T (D_1 x + A_{12} \eta) + d(\frac{\partial V_2}{\partial \eta})^T (\frac{1}{\epsilon} A_{22} \eta - D_2 \eta).
\]

\[
\leq - (1 - d) \alpha_1 \|x\|^2 + (1 - d) \beta (\|x\| \|\eta\|) - \frac{d}{\epsilon} \alpha_2 \|\eta\|^2 + dr \|\eta\|^2.
\]

\[
= - \left[ \begin{array}{c} \|x\| \\ \|\eta\| \end{array} \right] P \left[ \begin{array}{c} \|x\| \\ \|\eta\| \end{array} \right].
\]

where \( \beta = \|R_1 A_{12}\|, r = \|R_2 D_2\|, \) and \( P = \begin{bmatrix}
(1 - d) \alpha_1 & \frac{1}{\epsilon^2} (1 - d) \beta \\
-\frac{1}{\epsilon^2} (1 - d) \beta & \frac{d}{\epsilon^2} \alpha_2 - dr
\end{bmatrix} \).

For \( P \) to be positive definite, a range for \( \epsilon \) is obtained by:

\[
\epsilon < \frac{\alpha_1 \alpha_2}{\alpha_1 r + \frac{(1 - d) \beta^2}{4d}} = \epsilon^*.
\]
Therefore, for \( \epsilon \in [0, \epsilon^*), \dot{V} \) is negative definite. Hence \( x=0 \) and \( \eta = 0 \) are locally exponentially stable equilibria.

To give an explicit expression for \( \epsilon^* \), we solve for \( \beta \) and \( r \) in the following way:

(1) \( \beta = \| R_1 A_{12} \|: \)

Taking \( D_1 = \begin{bmatrix} 0 & \frac{1}{f} \\ -\frac{m_1}{f} & -(m_1+c_1) \end{bmatrix} \) and \( Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) into \( D_1^T R_1 + R_1 D_1 = -Q_1 \), we obtain:

\[
R_1 = \begin{bmatrix}
\frac{m_1^2 \gamma + (m_1 c_1 + c_1)^2 + f m_1}{2 m_1 (m_1 + c_1)} & \frac{1}{2 m_1} \\
\frac{f}{2 m_1} & \frac{(f + m_1) f}{2 m_1 (m_1 + c_1)}
\end{bmatrix}
\]

Therefore,

\[
\beta = \| R_1 A_{12} \| = \left\| \begin{bmatrix} -\frac{1}{2 m_1} & 0 \\ -\frac{(f + m_1) f}{2 m_1 (m_1 + c_1)} & 0 \end{bmatrix} \right\| = \sqrt{\left(\frac{1}{2 m_1}\right)^2 + \frac{f}{2 m_1 (m_1 + c_1)^2}}.
\]

(2) \( r = \| R_2 D_2 \|: \)

Taking \( A_{22} = \begin{bmatrix} 0 & 1 \\ -\frac{k_1}{f} & -\frac{k_2}{f} \end{bmatrix} \) and \( Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) into \( A_{22}^T R_2 + R_2 A_{22} = -Q_2 \), we obtain:

\[
R_2 = \begin{bmatrix}
\frac{1}{2 k_1} & \frac{k_1 J}{2 f k_2} + \frac{J}{2 k_2} & \frac{J}{2 k_1} \\
\frac{J}{2 k_1} & \frac{J}{2 k_2} (1 + \frac{J}{k_1})
\end{bmatrix}
\]

Therefore,

\[
r = \| R_2 D_2 \| = \left\| \begin{bmatrix} -\frac{1}{2 k_1} & 0 \\ -\frac{J}{2 f k_2} (1 + \frac{J}{k_1}) & 0 \end{bmatrix} \right\| = \sqrt{\frac{1}{4 k_1^2} + \frac{J}{2 f k_2} + \frac{J}{2 k_1 k_2}}^2.
\]
Therefore, an explicit expression for $\epsilon^*$ can be obtained,

$$
\epsilon^* = \frac{\alpha_1 \alpha_2}{\alpha_1 r + \frac{(1-d)\beta^2}{4d}},
$$

$$
= \frac{1}{\sqrt{\frac{1}{4k_1^2} + \left(\frac{J}{2k_1} + \frac{J}{2k_2} + \frac{J}{2k_2}\right)^2 + \frac{1-d}{4d}\left[\left(\frac{1}{2m_{11}}\right)^2 + \left(\frac{l + m_{11}}{2m_{11}(m_{12} + c_1)}\right)^2\right]^2}}.
$$

Recalling $K = k_1/\epsilon^2$, we obtain:

$$
K^* = k_1 \left\{ \sqrt{\frac{1}{4k_1^2} + \left(\frac{J}{2k_1} + \frac{J}{2k_2} + \frac{J}{2k_2}\right)^2 + \frac{1-d}{4d}\left[\left(\frac{1}{2m_{11}}\right)^2 + \left(\frac{l + m_{11}}{2m_{11}(m_{12} + c_1)}\right)^2\right]^{\frac{1}{2}}} \right\}^2.
$$

Consequently, we conclude that there is a stiffness range $[K^*, \infty]$ for which the stability conclusion for the system (4.19) and (4.20) is satisfied.

Similarly, for the continuous system in the dynamic domain $\dot{q}_1 < 0$:

$$
I \ddot{q}_1 + c_1 \dot{q}_1 + K(q_1 - q_2) - F_c = 0,
$$

$$
J \ddot{q}_2 - K(q_1 - q_2) = u.
$$

it is easy to show that the similar conclusion can be obtained with the same stiffness range.
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