INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6” x 9” black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor MI 48106-1346 USA
313/761-4700 800/521-0600
RICE UNIVERSITY

Average Anti-plane Motion in an Elastic Solid Containing a Layer of Randomly Distributed Cracks

by

Yuri K. Koba

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

Doctor of Philosophy

Approved, Thesis Committee:

Y. C. Angel, Chairman
Associate Professor
Mechanical Engineering and Materials Science

C.-C. Wang
Professor
Mechanical Engineering and Materials Science

P. E. Pfeiffer
Professor
Computational & Applied Mathematics

Houston, Texas

April 1996
Average Antiplane Motion in an Elastic Solid Containing a Layer of Randomly Distributed Cracks

Yuri K. Koba

Abstract

The propagation of antiplane waves in an elastic solid containing a random distribution of cracks that are parallel to each other, and oriented at an arbitrary angle relative to the direction of the incident wave, is considered. The approach, which is new, is based on the system of integral equations that describes the motion of a solid containing $N$ cracks. When the cracks are randomly distributed inside a rectangular box, the average wave motion at any point in the cracked solid is obtained. Next, the length of the box is increased to infinity. By taking the limit of the preceding result, keeping the crack density constant, the wave motion corresponding to a cracked layer is obtained. In particular, the wave motion reflected by the layer is evaluated in terms of frequency, crack density, layer thickness, and crack orientation. There is also a transmitted wave motion that propagates into the solid on the other side of the layer. Reflection and transmission coefficients are plotted versus frequency, crack density, layer thickness, and crack orientation. Inside the layer, for small values of the crack density and of the layer thickness, it is shown that the attenuation and the speed of the wave motion reduce to those obtained by assuming the existence of a complex-valued wave number.
Acknowledgments

The author would like to acknowledge the support and coordination of Professor Y.C. Angel.
# Table of Contents

Abstract ii

Acknowledgments iii

List of Figures vi

1 Introduction ................................................. 1
2 Definition of the Problem .................................. 2
3 Scattering Operator for a Single Crack .................. 4
4 Deterministic N-crack Problem ............................. 9
5 Alternative Formulation of the Single Crack Problem .... 14
6 Alternative Formulation of the N-crack Problem ............. 19
7 Deterministic Two-Crack Problem ........................... 21
8 Measurable Spaces ........................................... 25
9 Integration on Borel Sets ................................. 28
10 Probability Concepts ....................................... 32
11 Restrictions on the Probability Density Function ........... 38
12 Uniform Probability Density Function for a Configuration of N Circles 43
13 Construction of a Uniform Distribution of Circles in a Layer .... 46
14 Probabilistic N-crack Problem ............................. 55
15 Probabilistic Problem in a Cracked Layer .................. 64
16 Probabilistic Equation of Motion for N Cracks ............. 67
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>Probabilistic Equation of Motion for a Cracked Layer</td>
<td>73</td>
</tr>
<tr>
<td>18</td>
<td>Average Total Field in a Plane Containing a Cracked Layer</td>
<td>84</td>
</tr>
<tr>
<td>19</td>
<td>Speed and Attenuation of the Total Field</td>
<td>88</td>
</tr>
<tr>
<td>20</td>
<td>Dimensionless Equations for the Total Field</td>
<td>92</td>
</tr>
<tr>
<td>21</td>
<td>Evaluation of the Total Field</td>
<td>100</td>
</tr>
<tr>
<td>22</td>
<td>Numerical Results</td>
<td>105</td>
</tr>
<tr>
<td>23</td>
<td>Evaluation of the Speed and Attenuation of the Total Field</td>
<td>111</td>
</tr>
<tr>
<td>24</td>
<td>Conclusions and Future Work</td>
<td>115</td>
</tr>
</tbody>
</table>

References 117
List of Figures

2.1 Incident wave on an unbounded cracked region of thickness $2(h + a)$. 3

3.1 Obliquely incident antiplane wave on a crack of width $2a$ in an unbounded elastic solid. 5

4.1 Incident wave on a set of $N$ non-intersecting cracks in an unbounded solid. 10

7.1 Incident wave on a set of 2 non-intersecting cracks in an unbounded solid. 22

8.1 An element of the set $\sigma(\tau_2)$ constructed with open rectangles. 26

9.1 An element of the set $\sigma(\tau_2)$ constructed with open rectangles. 31

10.1 $N$ points in the two-dimensional plane. 33

10.2 Rectangle $B_{2N}$ of dimension $2N$ in the space $\mathbb{R}^{2N}$. 34

12.1 Collection of $N$ circles contained in a rectangle. 43

13.1 The infinite strip $V^h_\infty$. 50

13.2 The rectangular box $V^h_{4N}$. 52

14.1 Incident wave on a set of $N$ non-intersecting cracks in an unbounded solid. 56

14.2 Collection of $N$ circles surrounding cracks contained in a rectangle. 60

15.1 Collection of an infinite number of circles surrounding cracks contained in an infinite strip. 65
17.1 Three coordinate systems in the $\mathbb{R}^2$ plane. ........................................ 76
17.2 Infinite layer viewed from a rotated coordinate system. ................... 77
17.3 Region of the $(y_2, \nu)$ plane where $b(\nu, y_2 - \nu \cos \theta_0)$ is non-zero. . . . 84
22.1 Modulus of the dimensionless crack-opening area versus frequency. . . 106
22.2 Phase of the dimensionless crack-opening area versus frequency for a
mid-layer crack. ................................................................. 107
23.1 Response of a cracked layer to an incident field. ......................... 113
22.3 Modulus of the transmission coefficient versus frequency for normal
incidence, $\tilde{h} = 1$, and three crack densities. ........................... 120
22.4 Modulus of the transmission coefficient versus frequency for normal
incidence, $\tilde{h} = 3$, and three crack densities. ........................... 121
22.5 Modulus of the reflection coefficient versus frequency for normal
incidence and $\tilde{h} = 1$. .......................................................... 122
22.6 Modulus of the reflection coefficient versus frequency for normal
incidence and $\tilde{h} = 3$. .......................................................... 123
22.7 Modulus of the transmission coefficient versus layer thickness for
normal incidence, $\kappa = 1$, and three crack densities. ................... 124
22.8 Modulus of the transmission coefficient versus layer thickness for
normal incidence, $\kappa = 5$, and three crack densities. ................... 125
22.9 Modulus of the transmission coefficient versus crack density for
normal incidence, $\kappa = 1$, and three thicknesses. ........................ 126
22.10 Modulus of the transmission coefficient versus crack density for
normal incidence, $\kappa = 5$, and three thicknesses. ........................ 127
22.11 Modulus of the transmission coefficient versus frequency for 45° incidence, $\hat{h} = 1$, and three crack densities. 128

22.12 Modulus of the transmission coefficient versus frequency for 45° incidence, $\hat{h} = 3$, and three crack densities. 129

22.13 Modulus of the reflection coefficient versus frequency for 45° incidence and $\hat{h} = 1$. 130

22.14 Modulus of the reflection coefficient versus frequency for 45° incidence and $\hat{h} = 3$. 131

22.15 Modulus of the transmission coefficient versus layer thickness for 45° incidence, $\kappa = 1$, and three crack densities. 132

22.16 Modulus of the transmission coefficient versus layer thickness for 45° incidence, $\kappa = 5$, and three crack densities. 133

22.17 Modulus of the transmission coefficient versus crack density for 45° incidence, $\kappa = 1$, and three thicknesses. 134

22.18 Modulus of the transmission coefficient versus crack density for 45° incidence, $\kappa = 5$, and three thicknesses. 135
1 Introduction

In a solid containing a distribution of obstacles (such as cracks), plane waves are subjected to multiple scattering. For point-like obstacles, the scattering was examined in the theoretical work of Waterman and Truell [25], who showed that both the attenuation and the speed can be calculated in terms of the frequency and of the density of scatterers. The approach of these authors is of a statistical type, since the positions of the scatterers are known only in a probabilistic sense.

The basic ideas of [25] have been applied to scatterers of finite size (cracks, fluid-filled cracks, cavities, fluid-filled cavities), as for example in Angel and Achenbach [2], Angel and Koba [4], Zhang and Gross [27], McCarthy and Carroll [15], Sobczyk [22], and Varadan [24]. It has not been shown, however, that it is legitimate to use for scatterers of finite size results that were established for point scatterers only.

In this work, we consider the propagation of antiplane waves in a linear elastic solid containing a random distribution of cracks that are parallel to each other. The approach, which is new, is based on the system of integral equations that describes the motion of a solid containing $N$ cracks. The solution of the $N$-crack problem for an arbitrary configuration and orientation of the cracks is presented in Sections 2 - 7. Then, in Sections 8 - 16, we determine the average wave motion at any point in the cracked solid when the cracks are randomly distributed inside a rectangular box. Next, by increasing the length of the box to infinity and by keeping the crack density constant, we find in Sections 17 - 18 the wave motion corresponding to a cracked layer.

One important question is whether the attenuation and speed of the average wave motion inside the layer can be described by a complex-valued wave number that
depends on frequency but not on position. In Section 19, we define the speed and attenuation of the average wave motion, which will be used in Section 23 in the discussion of the concept of the complex-valued wave number.

In Sections 20-22, we discuss the numerical approach and the numerical results of the probabilistic cracked-layer problem. We compute and plot the transmission and reflection coefficients, and discuss how they depend on frequency, crack density, layer thickness, and crack orientation.

2 Definition of the Problem

We consider the propagation of antiplane waves in an unbounded, linearly elastic, homogeneous, and isotropic solid that contains a distribution of flat cracks. The cracks have width 2a, lie in planes orthogonal to the \((y_1, y_2)\) plane, and extend to infinity in the \(\pm y_3\) directions. Let \(V^h_\infty\) denote the region defined by

\[
V^h_\infty \equiv \{ |y_1| < \infty, |y_2| < h \}. \tag{2.1}
\]

The center of each crack is located within the region \(V^h_\infty\), and the solid is uncracked outside the region \(V^{h+a}_\infty\), as shown in Fig. 2.1.

Let \((\zeta^i_1, \zeta^i_2)\) denote the location of the center of the \(i\)-th crack in the \((y_1, y_2)\) coordinate system. A local coordinate system \((x^i_1, x^i_2)\) is attached to the \(i\)-th crack in such a way that the axis \(x^i_2\) is perpendicular to the crack faces and the transformation from the local system to the \((y_1, y_2)\) system is given by

\[
\begin{pmatrix}
  x^i_1 \\
  x^i_2
\end{pmatrix} =
\begin{pmatrix}
  \sin \theta^i & \cos \theta^i \\
  -\cos \theta^i & \sin \theta^i
\end{pmatrix} \begin{pmatrix}
  y_1 - \zeta^i_1 \\
  y_2 - \zeta^i_2
\end{pmatrix}, \quad x^i_3 = y_3, \quad 0 \leq \theta^i < \pi. \tag{2.2}
\]
Fig. 2.1 Incident wave on an unbounded cracked region of thickness 2(h + a).

The angle $\theta^i$ is the angle between the axes $y_2$ and $x_1^i$, measured from $y_2$ in the counterclockwise direction.

Let $\rho$ and $\mu$ denote, respectively, the mass density and the shear modulus of the solid. Then, the speed $c_T$ and the slowness $s_T$ of transverse waves are given by

$$c_T = \mu/\rho, \quad s_T = 1/c_T.$$  \hspace{1cm} (2.3)

An antiplane wave is incident on the cracks in the cracked solid along the $y_2$ direction. We shall call $u_0$ and $\omega$, respectively, the amplitude and the frequency of the incident wave. The time-harmonic factor $\exp(-i\omega t)$, which is common to all field variables in a steady-state regime, is omitted throughout this work. The displacement $u^{inc}$ generated by the incident wave is written in the form

$$u^{inc}(y_2) = u_0 \exp(i\omega s_T y_2).$$  \hspace{1cm} (2.4)
The cracks contained in $V^{k+a}_\infty$ scatter the incident wave \((2.4)\). As soon as the unperturbed incident wave encounters a crack, a first order scattered wave is generated. This field propagates in all directions away from the crack and encounters the other cracks present, which generate second order scattered fields. Further reflections produce higher-order scattered fields. The scattered fields of all orders up to infinity propagate away from the cracks.

The total field in the cracked solid is the sum of the incident wave \((2.4)\) and of all the waves scattered by the cracks. For each fixed configuration of cracks, the total field can be evaluated exactly. When the positions and orientations of the cracks are known in a probabilistic sense only, one can evaluate the averaged total field. In the following sections, we determine the averaged total field in the unbounded solid containing the cracked region $V^{k+a}_\infty$, and we examine the properties of this field. We consider in detail the special case where the cracks are distributed uniformly in $V^{k+a}_\infty$.

3 Scattering Operator for a Single Crack

We consider an unbounded, linearly elastic, homogeneous, and isotropic solid containing one crack, as shown in Fig. 3.1. The crack has width $2a$, lies in a plane orthogonal to the $(y_1, y_2)$ plane, and extends to infinity in the $\pm y_3$ directions. We assume that the center of the crack is at the position $(\zeta_1, \zeta_2)$ in the $(y_1, y_2)$ coordinate system. Let $(x_1, x_2)$ be a coordinate system attached to the crack, such that the center of the crack is at the origin of the $(x_1, x_2)$ coordinate system and $x_2$ is perpendicular to the crack faces. The angle between the axes $y_2$ and $x_1$ is denoted $\theta$ and is chosen to lie in the interval $[0, \pi]$, where $\theta$ is measured from $y_2$ in the counterclockwise direction.
Fig. 3.1 Obliquely incident antiplane wave on a crack of width $2a$ in an unbounded elastic solid.

The transformation from the system $(x_1, x_2)$ to the system $(y_1, y_2)$, according to (2.2) is given by

$$
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix} = \begin{pmatrix}
  \sin \theta & \cos \theta \\
  -\cos \theta & \sin \theta
\end{pmatrix} \begin{pmatrix}
  y_1 - \zeta_1 \\
  y_2 - \zeta_2
\end{pmatrix}, \quad 0 \leq \theta < \pi. \quad (3.1)
$$

Assume that an exciting antiplane time-harmonic motion $u$ of the form

$$
u(x_1, x_2, t) = u_0(x_1, x_2) \exp(-i\omega t) \quad (3.2)$$

is incident on the crack. This exciting motion is scattered by the crack, and the total motion generated in the solid is antiplane. The governing equation for antiplane steady-state motions can be written in the form

$$u_{,\omega\omega}(x_1, x_2) + k^2 u(x_1, x_2) = 0, \quad (3.3)$$

where the wavenumber $k$ is defined by

$$k = \omega s_T. \quad (3.4)$$
In (3.3), $\alpha = 1, 2$, and a summation is implied over the repeated index. The partial derivatives in (3.3) are taken with respect to the $x$ coordinates. We assume that the exciting field $u^e$ of (3.2) satisfies the equation of motion (3.3) everywhere in an uncracked solid.

The linearity of the governing equation allows us to decompose the total field $u^T$ in the solid into the sum of the exciting field in an uncracked solid and the scattered field $u^{sc}$ in the cracked solid. Thus, one has

$$u^T(x_1, x_2) = u^e(x_1, x_2) + u^{sc}(x_1, x_2).$$  (3.5)

The crack faces are free of traction. Thus, the stress $\sigma^T_{o3}$ satisfies the equation

$$\sigma^T_{o3} n_{\alpha} = 0, \quad |x_1| < a, \quad x_2 = 0,$$  (3.6)

where

$$\sigma^T_{o3}(x_1, x_2) = \mu \frac{\partial u^T(x_1, x_2)}{\partial x_{\alpha}}, \quad \alpha = 1, 2.$$  (3.7)

In (3.6), $\alpha = 1, 2$, and a summation is implied over the repeated index. The normal $n$ to the crack face is given in the $(x_1, x_2)$ and $(y_1, y_2)$ systems of coordinates of Fig. 3.1, respectively, by

$$n = e_2 = -\cos \theta i + \sin \theta j,$$  (3.8)

where $(e_1, e_2)$ are unit vectors along the axes $(x_1, x_2)$ and $(i, j)$ are unit vectors along the axes $(y_1, y_2)$. One infers from (3.5)-(3.8) that the boundary condition for the scattered problem takes the form

$$\lim_{x_2 \to 0^+, 0^-} \frac{\partial u^{sc}(x_1, x_2)}{\partial x_2} = -\frac{\partial u^e(x_1, x_2)}{\partial x_2} \bigg|_{x_2 = 0}, \quad |x_1| < a.$$  (3.9)
Observe that (3.9) should be valid in the limit as the left-hand side approaches zero both from above and below. Thus, equation (3.9) represents two distinct boundary conditions which account for the stress free condition on the two faces of the crack.

By assumption, the exciting field $u^e$ satisfies the equation of motion (3.3) everywhere in an uncracked solid, and the total field $u^T$ satisfies the equation of motion (3.3) everywhere in the cracked solid except on the crack faces. Therefore from (3.5) and (3.3) it follows that the scattered field also satisfies the equation of motion (3.3) everywhere except on the crack faces. Thus, one has

$$u_{\alpha\alpha}^e(x_1, x_2) + k^2 u^e(x_1, x_2) = 0. \quad (3.10)$$

It is shown in Angel [3] that a general solution of (3.9) and (3.10) can be written in the form

$$u^e(x_1, x_2) = \frac{\text{sgn}(x_2)}{\pi} \int_{0}^{\infty} d\xi \int_{-a}^{a} b(\nu)L(x_1 - \nu, x_2, \xi) d\nu, \quad (3.11)$$

where sgn is the sign function and

$$L(x_1 - \nu, x_2, \xi) = \frac{\sin[\xi(x_1 - \nu)]}{\xi} \exp(-\beta|x_2|). \quad (3.12)$$

The function $\beta$ in (3.12), which ensures that the scattered field $u^e$ propagates in all directions away from the crack, is defined by

$$\beta^2 = \xi^2 - k^2, \quad \text{Im}(\beta) \leq 0. \quad (3.13)$$

From (3.11), it is easy to show that

$$u^e(x_1, 0^+) = \frac{1}{2} \int_{-a}^{a} b(\nu)\text{sgn}(x_1 - \nu) d\nu = \begin{cases} C, & x_1 > a, \\ -C, & x_1 < -a, \\ -\int_{x_1}^{a} b(\nu) d\nu + C, & |x_1| < a, \end{cases} \quad (3.14)$$
where

\[ C = \frac{1}{2} \int_{-a}^{a} b(\nu) \, d\nu. \]  

(3.15)

Since the solid is continuous everywhere except along the crack faces (|x_1| < a, x_2 = 0), and the scattered field is antisymmetric with respect to the plane of the crack, we infer from (3.14) and (3.15) that the function b must satisfy the condition

\[ C = \frac{1}{2} \int_{-a}^{a} b(\nu) \, d\nu = 0. \]  

(3.16)

Thus, (3.14) yields

\[ u^c(x_1, 0^+) = \begin{cases} 
-\int_{-a}^{a} b(\nu) \, d\nu, & |x_1| < a, \\
0, & |x_1| \geq a.
\end{cases} \]  

(3.17)

The function b is called the dislocation density function. It represents the derivative of the displacement on the crack faces and it is determined from the boundary condition (3.9). Combining (3.9) and (3.11), we find that the function b satisfies the integral equation

\[ \lim_{x_2 \to 0^+} \frac{\partial}{\partial x_2} \left\{ \text{sgn}(x_2) \int_{0}^{\infty} d\xi \int_{-a}^{a} b(\nu) L(x_1 - \nu, x_2, \xi) \, d\nu \right\} = -\pi \frac{\partial u^E}{\partial x_2}(x_1, x_2) \left|_{x_2 = 0} \right., \]

\[ |x_1| < a. \]  

(3.18)

Writing \text{sgn}(x_2) = 1 in the left-hand side of (3.18), differentiating under the integral sign, and setting \(x_2 = 0\), one finds that (3.18) yields

\[ \int_{0}^{\infty} \frac{\beta}{\xi} d\xi \int_{-a}^{a} b(\nu) \sin[\xi(\nu - x_1)] \, d\nu = -\pi \frac{\partial u^E}{\partial x_2}(x_1, x_2) \left|_{x_2 = 0} \right., \]

\[ |x_1| < a. \]  

(3.19)

Following the same procedure as in Angel [3], we can write (3.19) in the form

\[ \int_{-a}^{a} b(\nu) \left[ \frac{1}{\nu - x_1} + P(\nu - x_1) \right] d\nu = -\pi \frac{\partial u^E}{\partial x_2}(x_1, x_2) \left|_{x_2 = 0} \right., \]

\[ |x_1| < a, \]  

(3.20)
where
\[
P(y) = \int_0^\infty \left( \frac{\beta}{\xi} - 1 \right) \sin(\xi y) \, d\xi. \tag{3.21}
\]

The integral in the left-hand side of (3.20) is defined in the principal-value sense. The solution of (3.20) is unique, and can be obtained numerically by using the method of Erdogan and Gupta [10].

4 Deterministic \( N \)-crack Problem

Consider an unbounded, linearly elastic, homogeneous, and isotropic solid that contains \( N \) non-intersecting cracks, as shown in Fig. 4.1. The \( i \)-th crack, \( i = 1, \ldots, N \), has width \( 2a \). The cracks lie in planes orthogonal to the \( (y_1, y_2) \) plane, and extend to infinity in the \( \pm y_2 \) directions. The center of the \( i \)th crack is located at
\[
\zeta^i = \zeta_1^i \mathbf{i} + \zeta_2^i \mathbf{j},
\tag{4.1}
\]
where \( \mathbf{i} \) and \( \mathbf{j} \) are unit vectors along the \( y_1 \) and \( y_2 \) axes, respectively.

Let \( (x_1^i, x_2^i) \) denote a local coordinate system attached to the \( i \)th crack, so that the transformation of coordinates from the \( (x_1^i, x_2^i) \) system to the \( (y_1, y_2) \) system is given by (2.2). The angle \( \theta^i \), which defines the orientation of the \( i \)th crack, is the angle between the axes \( y_2 \) and \( x_1^i \), and is measured in the counterclockwise direction from the \( y_2 \) axis. To specify the position and orientation of the \( i \)th crack, we define the triplet \( r^i \) such that
\[
r^i = (\zeta_1^i, \zeta_2^i, \theta^i), \quad 0 \leq \theta^i < \pi. \tag{4.2}
\]

To specify the configuration of the cracks, we define the notation \( \Omega^N \) by
\[
\Omega^N = (r^1, \ldots, r^N). \tag{4.3}
\]
Fig. 4.1 Incident wave on a set of $N$ non-intersecting cracks in an unbounded solid.

We denote by $n^i$ the unit normal on the faces of the $i$th crack. Thus, if $(e^i_1, e^i_2)$ are unit vectors along the axes $(x^i_1, x^i_2)$, respectively, one has

$$n^i = e^i_2 = -\cos \theta^i i + \sin \theta^i j.$$  \hfill (4.4)

An antiplane wave $u^{\text{inc}}$, as in (2.4), is incident on the cracks along the $y_2$ direction. The incident wave and the scattering from the other cracks cause a wave to be scattered by the $i$th crack. This wave has the form

$$u^{sc}(x^i_1, x^i_2; r^i | \Omega^N) = \bar{u}^{sc}(y_1, y_2; r^i | \Omega^N) \equiv u^{sc}_i(y_1, y_2),$$  \hfill (4.5)

where $(x^i_1, x^i_2)$ denotes the field point of evaluation in the local coordinate system attached to the crack, $(y_1, y_2)$ denotes the same point in the $(y_1, y_2)$ coordinate system,
$i$ is the number of the crack scattering the wave, $\Omega^N$ indicates the dependence of the scattered wave on the configuration of $N$ cracks, and $u^{sc}_i$ is a short notation for the scattered field from the $i$th crack.

The total field $u^T$ in the cracked solid can be decomposed into the sum of the field $u^{sc}_i$ scattered by the $i$th crack and the exciting field $u^E_i$ acting on the $i$th crack. Thus, one can write

$$u^T(y_1, y_2|\Omega^N) = \tilde{u}^{sc}(y_1, y_2; r^i|\Omega^N) + \tilde{u}^E(y_1, y_2; r^i|\Omega^N), \quad (4.6)$$

where

$$\tilde{u}^E(y_1, y_2; r^i|\Omega^N) = u^E(x^i_1, x^i_2; r^i|\Omega^N) \equiv u^E_i(y_1, y_2). \quad (4.7)$$

In (4.7) and (4.5), the coordinates $(y_1, y_2)$ and $(x^i_1, x^i_2)$ are related by (2.2).

The exciting field $u^E_i$ acting on the $i$th crack is the sum of the incident wave $u^{inc}$ and of the waves scattered by all the cracks other than the $i$th. Thus, one has

$$\tilde{u}^E(y_1, y_2; r^i|\Omega^N) = u^{inc}(y_2) + \sum_{j=1, j\neq i}^{N} \tilde{u}^{sc}(y_1, y_2; r^j|\Omega^N). \quad (4.8)$$

Combining (4.8) and (4.6), we can write the total field in the form

$$u^T(y_1, y_2|\Omega^N) = u^{inc}(y_2) + \sum_{i=1}^{N} \tilde{u}^{sc}(y_1, y_2; r^i|\Omega^N). \quad (4.9)$$

The governing equation for antiplane steady-state motions can be written in the form

$$u_{,\alpha\alpha}(y_1, y_2) + k^2 u(y_1, y_2) = 0, \quad (k = \omega s_T), \quad (4.10)$$

where $\alpha = 1, 2$, and a summation is implied over the repeated index. The partial derivatives in (4.10) are taken with respect to the $y_1$ and $y_2$ coordinates.

The total field $u^T$ of (4.6) satisfies (4.10) everywhere in the solid except on the crack faces. The scattered field $u^{sc}$ corresponding to the $i$th crack is defined as a
solution of the governing equation (4.10). This solution is valid everywhere in the
solid except on the faces of the ith crack and it must correspond to a wave motion that
propagates in all directions away from the ith crack. Using the scattering operator
(3.11) of the preceding section, we write the wave scattered by the ith crack in the
form
\[
u^s(x_1^i, x_2^i; r^i | \Omega^N) = \frac{\text{sgn}(x_2^i)}{\pi} \int_0^\infty d\xi \int_{-\infty}^\infty b(\nu; r^i | \Omega^N) L(x_1^i - \nu, x_2^i; \xi) d\nu, \tag{4.11}
\]
where the function \(b\), as in (3.16), satisfies the condition
\[
\int_{-\infty}^\infty b(\nu; r^i | \Omega^N) d\nu = 0. \tag{4.12}
\]
In (4.11), the function \(L\) is given by (3.12). It follows from (4.8) and (4.11)-(4.12)
that the exciting field \(u^e_i\) on the ith crack satisfies the equation of motion (4.10) in the
same region as the total field \(u^T\) and also on the faces of the ith crack. The boundary
conditions for the total field are such that the stresses vanish on the faces of all the
cracks. This can be written, as in equation (3.6), in the form
\[
\sigma^T_{\alpha \beta} n^i_\alpha = 0, \quad |x_1^i| < a, \quad x_2^i = 0, \quad i = 1, \ldots, N, \tag{4.13}
\]
where the unit normal \(n^i\) is defined in (4.4).

Using (4.4)-(4.7), we find that the boundary condition (4.13) for the ith crack
yields the equation
\[
\lim_{x_2^i \to 0^+} \frac{\partial}{\partial x_2^i} u^s(x_1^i, x_2^i; r^i | \Omega^N) = -\frac{\partial}{\partial x_2^i} u^e(x_1^i, x_2^i; r^i | \Omega^N) \bigg|_{x_2^i = 0}, \quad |x_1^i| < a, \quad i = 1, \ldots, N. \tag{4.14}
\]
Next, with the \(N\) boundary conditions (4.14), we can determine the \(N\) unknown
functions \(b_i(\nu)\) of equations (4.11)-(4.12), \(i = 1, \ldots, N\), where \(b_i\) denotes the function
\( b(\nu; r^i|\Omega^N) \). Combining (4.11) and (4.14), one finds that the \( N \) functions \( b_i \) are determined by

\[
\lim_{x_2 \to 0^+} \frac{\partial}{\partial x_2} \{ \text{sgn}(x_2) \int_0^\infty d\xi \int_{-a}^a b(\nu; r^i|\Omega^N) L(x_1^i - \nu, x_2^i, \xi) d\nu \} = \]

\[
-\pi \frac{\partial}{\partial x_2} u^e(x_1^i, x_2^i; r^i|\Omega^N) \bigg|_{x_2^i = 0}, \quad (4.15)
\]

\(|x_1^i| < a, \quad i = 1, \ldots, N.\)

Following the same procedure as in the previous section, we can simplify the system (4.15) and obtain a system of \( N \) singular integral equations for the \( N \) functions \( b_i \). This system takes the form

\[
\int_{-a}^a b(\nu; r^i|\Omega^N) \left[ \frac{1}{\nu - x_1^i} + P(\nu - x_1^i) \right] d\nu = -\pi \frac{\partial}{\partial x_2} u^e(x_1^i, x_2^i; r^i|\Omega^N) \bigg|_{x_2^i = 0},
\]

\(|x_1^i| < a, \quad i = 1, \ldots, N, \quad (4.16)\)

where the function \( P \) is defined in (3.21). Using (4.7), (4.8), (4.11) and (4.5), one can see that equation (4.16) represents a system of \( N \) coupled equations for the \( N \) unknowns \( b_i \). Each equation contains the \( N \) functions \( b_i \). The solution of the system (4.16) should also satisfy the conditions (4.12).

The system (4.16) can be solved numerically by using an iterative procedure. To start the procedure, we take \( b_i = 0 \) for all the cracks. Then, from (4.8) and (4.11), we find that \( u^i = u^{inc} \) for all the cracks. Introducing this value into (4.16), we solve the system for the \( N \) functions \( b_i \). This first iteration corresponds to the first-order term of the solution for the multiple scattering problem. Next, the calculated values of the \( N \) functions \( b_i \) can be substituted into (4.8) and (4.11) to find new values of \( u^i \). The new values of \( u^i \) are used to solve (4.16), the solution of which gives the second-order term of the solution for the multiple scattering problem. Higher-order terms can be
obtained by repeating the same procedure as many times as desired. Each iteration gives rise to an additional term of the solution for the multiple scattering problem. It is expected on physical grounds that increasing the number of iterations will give an increasingly accurate solution.

5 Alternative Formulation of the Single Crack Problem

In Section 3 we discussed the solution of a single crack problem for the unbounded, linearly elastic, isotropic, and homogeneous solid containing a crack. The solution of the problem was given by equations (3.5), (3.11), (3.16) and (3.20). Observe that the scattered field which is given by (3.11) is discontinuous across the crack faces. By direct substitution, assuming the legitimacy of interchanging the order of integration and differentiation, it can be shown that (3.11) satisfies the governing equation (3.3) identically everywhere except on the faces of the crack.

We now recall the basic equations that lead to the solution (3.11) for the scattered field in an elastic solid containing a single crack of finite width. These equations are the equation of motion (3.3), which is valid everywhere in the solid away from the crack faces, the boundary condition (3.6), and the definition (3.5) for the scattered field. The solution (3.11), for any exciting field $u^e$, is antisymmetric with respect to the plane $x_2 = 0$, where the crack lies. It can be derived by formulating the scattering problem, with the proper boundary conditions, in the half-plane $x_2 > 0$. Then, by using the antisymmetry of the scattered displacement $u^{sc}$, one obtains the form (3.11). In this section, we show that the single crack problem can be formulated directly by considering the entire plane $(x_1, x_2)$. This new formulation will lead automatically
to a solution of the scattered field that is antisymmetric with respect to the plane \( x_2 = 0 \).

Consider a single crack as in Section 3, and let \((x_1, x_2)\) be the coordinate system that is attached to the crack as in Fig. 2. For an arbitrary steady-state motion \( u \), we define the jump \([[u]]\) across the plane \( x_2 = 0 \) by

\[
[[u]](x_1,0) = \begin{cases} 
  u(x_1,0^+) - u(x_1,0^-), & |x_1| \leq a, \\
  0, & |x_1| > a.
\end{cases}
\]  

(5.1)

With this, we show that the solution (3.11) of the scattered problem can be obtained directly in the entire plane \((x_1, x_2)\) from the following governing equation and the boundary condition

\[
u_{\alpha\alpha}(x_1, x_2) + k^2 u(x_1, x_2) = \delta'(x_2) [[u]](x_1, 0), \quad \text{in } \mathbb{R}^2,
\]  

(5.2)

\[
\sigma_{\alpha 3}^T n_\alpha = 0, \quad |x_1| < a, \quad x_2 = 0.
\]  

(5.3)

Assume that an exciting antiplane time-harmonic motion \( u \), of the form

\[
u(x_1, x_2, t) = u^E(x_1, x_2) \exp(-i\omega t),
\]  

(5.4)

is incident on the crack. This exciting motion is scattered by the crack, and the total motion generated is antiplane. The linearity of the equation (5.2) allows us to decompose the total field \( u^T \) in the solid into the sum of the exciting field \( u^E \) in an uncracked solid and the scattered field \( u^{sc} \) in the cracked solid. Thus, one has

\[
u^T(x_1, x_2) = u^E(x_1, x_2) + u^{sc}(x_1, x_2).
\]  

(5.5)

The exciting field \( u^E \) is continuous across the faces of the crack. Therefore, the jump \([[u^E]]\) of the field \( u^E \) vanishes, and one has

\[
[[u^E]](x_1, 0) = 0, \quad \text{for all } x_1.
\]  

(5.6)
It follows that \( u^e \) satisfies the equation of motion (5.2) with the right-hand side set equal to zero. We infer from the preceding discussion that the governing equation and the boundary condition for the scattered field are given by

\[
 u^{sc}_{\alpha\alpha}(x_1, x_2) + k^2 u^{sc}(x_1, x_2) = \delta'(x_2)[[u^{sc}]](x_1, 0), \quad \text{in } \mathbb{R}^2, \quad (5.7)
\]

\[
 \lim_{x_2 \to 0^+, 0^-} \frac{\partial u^{sc}(x_1, x_2)}{\partial x_2} = -\frac{\partial u^{sc}(x_1, x_2)}{\partial x_2} \bigg|_{x_2 = 0}, \quad |x_1| < a. \quad (5.8)
\]

Observe that (5.8) should be valid in the limit as the left-hand side approaches zero both from above and below. Thus, equation (5.8) represents two distinct boundary conditions which account for the stress free condition on the two faces of the crack.

Next, we find the solution of (5.7)-(5.8) by using the Fourier transform method. We define the Fourier transform \( \hat{f}(\xi, x_2) \) relative to the variable \( x_1 \) of a function \( f(x_1, x_2) \) by

\[
 \hat{f}(\xi, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x_1, x_2) \exp(i\xi x_1) \, dx_1. \quad (5.9)
\]

The corresponding inverse relation is

\[
 f(x_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi, x_2) \exp(-i\xi x_1) \, d\xi. \quad (5.10)
\]

Taking the Fourier transform of (5.7), one obtains an inhomogeneous second order ordinary differential equation

\[
 \ddot{u}^{sc}_{\xi\xi}(\xi, x_2) - \beta^2 \ddot{u}^{sc}(\xi, x_2) = \delta'(x_2)[[u^{sc}]](\xi, 0), \quad (5.11)
\]

where \( \beta \) is given by (3.13). The solution of (5.11), which is bounded everywhere in the solid and which corresponds to a wave motion that propagates away from the crack, can be written as

\[
 \ddot{u}^{sc}(\xi, x_2) = \text{sgn}(x_2)A(\xi) \exp(-\beta|x_2|), \quad (5.12)
\]
where
\[
A(\xi) = \frac{1}{2}[[u^{sc}]](\xi, 0). \tag{5.13}
\]

Observe from (5.12) that the transformed displacement \(\tilde{u}^{sc}\), and consequently the displacement \(u^{sc}\) itself, are antisymmetric with respect to the plane \(x_2 = 0\). Thus, we can write
\[
u^{sc}(x_1, x_2) = -u^{sc}(x_1, -x_2). \tag{5.14}
\]

It follows from the antisymmetry that the two equations of (5.8) are identical.

Recall that the exciting field \(u^e\) on the crack is continuous everywhere in the solid, and the total field \(u^T\) in the solid is discontinuous only across the faces of the crack. Thus, the scattered field \(u^{sc}\) is discontinuous only across the faces of the crack and, according to (5.14), one has
\[
u^{sc}(x_1, 0) = 0, \quad |x_1| > a, \tag{5.15}
\]
\[
[[u^{sc}]](x_1, 0) = 2u^{sc}(x_1, 0^+). \tag{5.16}
\]

Combining (5.16) and (5.13), we infer that
\[
A(\xi) = \tilde{u}^{sc}(\xi, 0^+). \tag{5.17}
\]

Next, we define a function \(b\) in the interval \((-a, a)\) by
\[
u^{sc}(x_1, 0^+) = \begin{cases} 
- \int_{x_1}^{0} b(\nu) \, d\nu, & |x_1| < a, \\
0, & |x_1| \geq a,
\end{cases} \tag{5.18}
\]

Together with
\[
\int_{-a}^{a} b(\nu) \, d\nu = 0. \tag{5.19}
\]
For every integrable function $b$ in the interval $[-a,a]$, $u^{sc}(x_1, 0^+)$ is continuous in $(-\infty, \infty)$. Taking the Fourier transform of (5.18), interchanging the order of integration, and using condition (5.19), one infers from (5.15) that

$$A(\xi) = \frac{i}{\xi \sqrt{2\pi}} \int_{-a}^{a} b(\nu) \exp(i \xi \nu) d\nu. \quad (5.20)$$

By taking the inverse Fourier transform of (5.12), we find that

$$u^{sc}(x_1, x_2) = \frac{\text{sgn}(x_2)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\xi) \exp(-\beta |x_2| - i \xi x_1) d\xi. \quad (5.21)$$

Combining (5.21) and (5.20), we find that $u^{sc}$ can be written in terms of $b$ as

$$u^{sc}(x_1, x_2) = \frac{\text{sgn}(x_2)}{\pi} \int_{0}^{\infty} d\xi \int_{-a}^{a} b(\nu) L(x_1 - \nu, x_2, \xi) d\nu, \quad (5.22)$$

where the function $L$ is given by (3.12). Observe that $u^{sc}$ in (5.22) has the same form as $u^{sc}$ in (3.11). To find the function $b$ in (5.22), we use the boundary condition (5.8).

Following the same procedure as described after equation (3.17), we find that the function $b$ in (5.22) is the solution of the singular integral equation (3.20) together with (3.16). We conclude from this discussion that the formulation of sections 3 and 5 are equivalent.

We infer from (5.22) and (5.19) that

$$u^{sc}(x_1, 0^+) = \frac{1}{2} \int_{-a}^{a} b(\nu) \text{sgn}(x_1 - \nu) d\nu. \quad (5.23)$$

Thus, using (5.23), (5.1), and (5.14), one finds that the governing equation (5.2) can be written in the form

$$u_{,\alpha\alpha}(x_1, x_2) + k^2 u(x_1, x_2) = \delta'(x_2) \int_{-a}^{a} b(\nu) \text{sgn}(x_1 - \nu) d\nu, \quad \text{in } \mathbb{R}^2, \quad (5.24)$$

where the function $b$ is the solution of the singular integral equation (3.20) that satisfies the additional condition (5.19).
6 Alternative Formulation of the $N$-crack Problem

In Section 4, we discussed the problem of the scattering of an antiplane wave by a distribution of $N$ non-intersecting cracks in an unbounded elastic solid. The solution of the problem is given by the system of equations (4.16), (4.11),(4.12), and (4.5)-(4.9).

The basic equations that lead to this solution are the equation of motion (4.10), which is valid everywhere in the solid away from the crack faces, the $N$ boundary conditions (4.13) on the crack faces, and the definitions (4.5)-(4.9) for the scattered fields.

Consider the scattered field $u_i^{sc}$ corresponding to the $i$th crack as in (4.5). The scattered field $u_i^{sc}$ is defined as the solution of the governing equation (4.10), which is valid everywhere in the solid except on the faces of the $i$th crack, and of the boundary condition (4.14) on the crack faces. For convenience, we rewrite here these two equations in the form

$$
\bar{u}_{i,\infty}^{sc}(y_1, y_2; r^i | \Omega^N) + k^2 \bar{u}^{sc}(y_1, y_2; r^i | \Omega^N) = 0,
$$

(6.1)

$$
\lim_{x_2 \to 0^+} \frac{\partial}{\partial x_2} u^{sc}(x_1^i, x_2^i; r^i | \Omega^N) = -\frac{\partial}{\partial x_2} u^{sc}(x_1^i, x_2^i; r^i | \Omega^N) \Bigg|_{x_2^i = 0},
$$

(6.2)

$$
|x_1^i| < a.
$$

Using the derivation of Section 5, we infer that equation (6.1) for the $i$th crack scattering problem can be replaced by an equation of the type (5.7). In the case of (6.1), the alternative equation has the form

$$
\bar{u}_{i,\infty}^{sc}(y_1, y_2; r^i | \Omega^N) + k^2 \bar{u}^{sc}(y_1, y_2; r^i | \Omega^N) = \delta'(x_2^i)[[u^{sc}]](x_1^i, 0; r^i | \Omega^N), \quad \text{in } \mathbb{R}^2,
$$

(6.3)
where the jump \([u^s]\) is defined as in (5.1), and the coordinates \((x_1^i, x_2^i)\) are related to the coordinates \((y_1, y_2)\) as in (2.2) by the formulas

\[
x_1^i = \sin \theta^i(y_1 - \zeta_1^i) + \cos \theta^i(y_2 - \zeta_2^i),
\]

\[
x_2^i = -\cos \theta^i(y_1 - \zeta_1^i) + \sin \theta^i(y_2 - \zeta_2^i).
\]

(6.4)  

(6.5)

We infer from (4.11) and (4.12) that

\[
u^s(x_1^i, 0^+; r^i|\Omega^N) = \frac{1}{2} \int_{-a}^{a} b(\nu; r^i|\Omega^N) \text{sgn}(x_1^i - \nu) d\nu.
\]

(6.6)

Thus, using (6.6), definition (5.1), and the antisymmetry (5.14), we find that the governing equation (6.3) can be written in the form

\[
\delta'(x_2^i) \int_{-a}^{a} b(\nu; r^i|\Omega^N) \text{sgn}(x_1^i - \nu) d\nu, \quad \text{in } \mathbb{R}^2.
\]

(6.7)

Now recall that the incident wave \(u^{inc}\) satisfies the equation of motion (4.10)

\[
u^{inc}(y_2) + k^2 u^{inc}(y_2) = 0, \quad \text{in } \mathbb{R}^2.
\]

(6.8)

The total displacement \(u^r\) in the cracked solid is given in terms of \(u^{inc}\) and the \(N\) scattered fields \(u^{sc}_i\) by the sum (4.9). Likewise, the exciting field \(u^{exc}_i\) for the \(i\)th crack is given in terms of \(u^{inc}\) and \((N - 1)\) scattered fields by the sum (4.8). Using (6.7), (6.8), (4.8), and (4.9), we infer that the total field and the exciting field on the \(i\)th crack satisfy, respectively, the equations

\[
u^{sc}(y_1, y_2|\Omega^N) + k^2 u^{sc}(y_1, y_2|\Omega^N) = 2 \sum_{i=1}^{N} \delta'(x_2^i) u^{sc}(x_1^i, 0^+; r^i|\Omega^N) =
\]

\[
\sum_{i=1}^{N} \delta'(x_2^i) \int_{-a}^{a} b(\nu; r^i|\Omega^N) \text{sgn}(x_1^i - \nu) d\nu, \quad \text{in } \mathbb{R}^2.
\]

(6.9)
and
\[\tilde{u}_{\alpha\alpha}^{\Omega}(y_1, y_2; r^i | \Omega^N) + k^2 \tilde{u}^{\Omega}(y_1, y_2; r^i | \Omega^N) = \]
\[\sum_{j=1, j \neq i}^{N} \delta'(x^i_2) \int_{-\infty}^{\infty} b(\nu; r^i | \Omega^N) \text{sgn}(x^i_1 - \nu) d\nu, \quad \text{in } R^2, \]
where the functions \(b_j\) are the solutions of the system of \(N\) coupled singular integral equations (4.16) together with the additional conditions (4.12).

7 Deterministic Two-Crack Problem

Consider an unbounded, linearly elastic, homogeneous, and isotropic solid that contains two cracks, as shown in Fig. 7.1. The cracks have width 2\(a\), lie in planes orthogonal to the \((y_1, y_2)\) plane, and extend to infinity in the \(\pm y_3\) directions. The center of the \(i\)th crack, \(i = 1, 2\), is located at
\[\zeta^i = \zeta^i_1 \mathbf{i} + \zeta^i_2 \mathbf{j}, \quad i = 1, 2, \]
where \(\mathbf{i}\) and \(\mathbf{j}\) are unit vectors along the \(y_1\) and \(y_2\) axes, respectively. Let \((x^i_1, x^i_2)\) denote a local coordinate system attached to the \(i\)th crack, so that the transformation of coordinates from the \((x^i_1, x^i_2)\) system to the \((y_1, y_2)\) system is given by
\[x^i_1 = y_1 - \zeta^i_1, \]
\[x^i_2 = y_2 - \zeta^i_2.\]

An antiplane wave \(u^{\text{inc}}\), as in (2.4), is incident on the cracks along the \(y_2\) direction. This wave has the form
\[u^{\text{inc}}(y_2) = u_0 \exp(iky_2).\]
Fig. 7.1 Incident wave on a set of 2 non-intersecting cracks in an unbounded solid.

The incident wave and the scattering from the second crack cause a wave \( u_1^{sc} \) to be scattered by the first crack. Likewise, the incident wave and the scattering from the first crack cause a wave \( u_2^{sc} \) to be scattered by the second crack. The total field \( u^T \) in the cracked solid can be decomposed into the sum of the field \( u_1^{sc} \) and the exciting field \( u_1^E \) acting on the first crack, or into the sum of the field \( u_2^{sc} \) and the exciting field \( u_2^E \) acting on the second crack. Thus, one can write

\[
 u^T = u_1^{sc} + u_1^E = u_2^{sc} + u_2^E. \tag{7.5}
\]

The exciting field \( u_1^E \) acting on the first crack is the sum of the incident wave \( u^{inc} \) and of the scattered field \( u_2^{sc} \). Likewise, the exciting field \( u_2^E \) acting on the second crack is the sum of the incident wave \( u^{inc} \) and of the scattered field \( u_1^{sc} \). Thus, one
has
\[ u^p_i = u^{inc} + u^{sc}_2, \]  
\[ u^c_i = u^{inc} + u^{sc}_1. \]  
(7.6)  
(7.7)

Combining (7.5)-(7.7), we can write the total field in the form
\[ u^x = u^{inc} + u^{sc}_1 + u^{sc}_2. \]  
(7.8)

We infer now, by using an argument similar to that which leads to equation (4.11), that the scattered field \( u^{sc}_i \) has the form
\[ u^{sc}_i(y_1, y_2) = \frac{\text{sgn}(y_2 - \zeta_i^2)}{\pi} \int_0^\infty d\xi \int_{-a}^a b_i(\nu)L(y_1 - \zeta_1^i - \nu, y_2 - \zeta_2^i, \xi) d\nu, \quad i = 1, 2, \]  
(7.9)

where the function \( b_i \) satisfies the condition
\[ \int_{-a}^a b_i(\nu) d\nu = 0, \quad i = 1, 2. \]  
(7.10)

In (7.9)-(7.10), the function \( L \) is given by (3.12). The boundary conditions for the total field are such that the stresses vanish on the faces of the two cracks. This can be written in the form
\[ \sigma_{23}^x = 0, \quad \text{on} \quad y_2 = \zeta_i^2, \quad |y_1 - \zeta_i^1| < a, \quad i = 1, 2. \]  
(7.11)

The two unknown functions \( b_i \) of (7.9) are determined by using the boundary conditions (7.11). This procedure is discussed in Section 4 between equations (4.14) and (4.16). For the case of the two-crack problem discussed in this section, we obtain a system of coupled singular integral equations for the functions \( b_1 \) and \( b_2 \). The system
of equations has the form
\[
\int_{-a}^{a} b_i(\nu) \left[ \frac{1}{\nu - (y_1 - \zeta_i^1)} + P(\nu - y_1 + \zeta_i^1) \right] d\nu = -\pi \frac{\partial}{\partial y_2} u_i^{sc}(y_1, y_2) \bigg|_{y_2 = \zeta_2^i}, \\
|y_1 - \zeta_i^1| < a, \quad i = 1, 2.
\] (7.12)

In (7.12), the exciting field in the right-hand side is given, according to (7.6)-(7.7) and (7.4), by
\[
u_i(y_1, y_2) = u_0 \exp(iky_2) + u_0^{sc}(y_1, y_2), \quad i = 1, 2
\] (7.13)
\[
u_2^{sc}(y_1, y_2) = u_0 \exp(iky_2) + u_0^{sc}(y_1, y_2),
\] (7.14)

where \(u_1^{sc}\) and \(u_2^{sc}\) are defined in (7.9). Substituting (7.13) and (7.14) into (7.12), and defining a new variable \(x\) to replace \(y_1 - \zeta_1^1\) in (7.12), one finds that the system of equations for \(b_i\) is given by
\[
\int_{-a}^{a} b_1(\nu) \left[ \frac{1}{\nu - x} + P(\nu - x) \right] d\nu = -i\pi ku_0 \exp(ik\zeta_2^1) + \\
\int_0^\infty d\xi \int_{-a}^{a} b_2(\nu) \frac{\beta}{\xi} \sin[\xi(x + \zeta_1^1 - \zeta_2^1 - \nu)] \exp(-\beta|\zeta_2^1 - \zeta_2^2|) d\nu,
\]
\(|x| < a, \quad (7.15)\)
\[
\int_{-a}^{a} b_2(\nu) \left[ \frac{1}{\nu - x} + P(\nu - x) \right] d\nu = -i\pi ku_0 \exp(ik\zeta_2^2) + \\
\int_0^\infty d\xi \int_{-a}^{a} b_1(\nu) \frac{\beta}{\xi} \sin[\xi(x + \zeta_1^2 - \zeta_2^1 - \nu)] \exp(-\beta|\zeta_2^2 - \zeta_2^1|) d\nu,
\]
\(|x| < a. \quad (7.16)\)

In the derivation that yields the right-hand sides of (7.15) and (7.16), we have used the result that the derivative of the sign function is two times the Dirac delta function, and the condition (7.10) imposed on the integrals of the functions \(b_1\) and \(b_2\). Also, we have used the condition that the two cracks do not overlap. The condition of
non-overlapping can be written in the mathematical form

\[
\text{non-overlapping} \rightarrow \begin{cases} 
|\zeta_1^1 - \zeta_1^2| > 2a, & \text{if } \zeta_2^2 - \zeta_2^1 = 0, \\
\text{all } \zeta_1, & \text{otherwise.}
\end{cases}
\] (7.17)

8 Measurable Spaces

The following concepts of the theory of probability were taken in parts from Pfeiffer [19], Pfeiffer [20], Chow and Teicher [9], Billingsley [8], Ash [5], Kingman and Taylor [13], and Riesz and Nagy [21]. We first discuss the basic operations between subsets of a given set \( \Omega \). Let \( A \) and \( B \) be two subsets of a set \( \Omega \). By \( A \cup B \) and \( A \cap B \) we denote the union and intersection of \( A \) and \( B \), respectively. The sets \( A \) and \( B \) are called disjoint if their intersection is an empty set. By \( A \setminus B \) we denote all elements of \( A \) that are not in \( B \). The complement \( \Omega \setminus A \) of a set \( A \) is denoted by \( A^c \).

Next, consider a special set \( \tau_2 \) made of a collection of open rectangles of the type \((a, b) \times (c, d)\) in the two-dimensional Euclidian space \( \mathbb{R}^2 \), where \( a, b, c, d \in \mathbb{R} \), and \( b > a, d > c \). Observe that the set \( \tau_2 \) is not closed under complementation, union or intersection, since if \( A, B \in \tau_2 \), then \( A^c, A \cup B \) and \( A \cap B \) are not necessarily rectangles. However, we can construct from the set \( \tau_2 \) a new set \( \sigma(\tau_2) \) which is closed under the operations of complementation, union, or intersection. The set \( \sigma(\tau_2) \) is defined by specifying the elements that are contained in it. The elements of \( \sigma(\tau_2) \) are

1) all elements of \( \tau_2 \);

2) the complements of all elements of \( \tau_2 \);

3) the countable union of elements specified in 1) and 2).
Fig. 8.1 An element of the set $\sigma(\tau_2)$ constructed with open rectangles.

It follows from the above definition that the set $\sigma(\tau_2)$ is closed under complementation and under countable union of its elements. In addition, as it is shown in Ash [5] (p. 4), $\sigma(\tau_2)$ is closed under a countable intersection of its elements. Indeed, one has

$$\bigcap_{i=1}^{\infty} A_i = \left( \bigcup_{i=1}^{\infty} A_i^c \right)^c \in \sigma(\tau_2).$$

(8.1)

Hence, we infer from (8.1) that any half-open or closed rectangle also is an element of $\sigma(\tau_2)$. This follows from the results

$$[a, b] \times [c, d] = \bigcap_{i=1}^{\infty} \left[ a - \frac{1}{i}, b + \frac{1}{i} \right] \times \left[ c - \frac{1}{i}, d + \frac{1}{i} \right],$$

$$[a, b] \times (c, d] = \bigcup_{i=1}^{\infty} \left[ a, b + \frac{1}{i} \right] \times \left( c, d + \frac{1}{i} \right).$$

(8.2)

In general, we conclude from (8.2) that $\sigma(\tau_2)$ generated by the set $\tau_2$ according to (1) – (3) can be constructed from a collection $\tau_2$ of either half-opened, or opened, or closed rectangles. Now, by using expansions analogous to those of (8.2), we can show that

$$(-\infty, \infty) \times (c, d], \ (a, b] \times (-\infty, \infty), \ (-\infty, \infty) \times (-\infty, \infty) \in \sigma(\tau_2).$$

(8.3)
Following this preliminary discussion, we define an algebraic structure in $\Omega$ called a $\sigma$-algebra.

**Definition 8.1 (\(\sigma\)-algebra)**

Let $\mathcal{F}$ be a collection of subsets of a set $\Omega$. Then $\mathcal{F}$ is called a $\sigma$-algebra in $\Omega$ iff $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under complementation and countable union, that is,

a) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$;

b) if $A_1, \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

**Definition 8.2 (Measurable space)**

The couple $(\Omega, \mathcal{F})$ made of the space $\Omega$ and of the $\sigma$-algebra $\mathcal{F}$ in $\Omega$ is called the measurable space corresponding to the space $\Omega$ and the $\sigma$-algebra $\mathcal{F}$.

It is clear from definition (8.1) that the set $\sigma(\tau_2)$ in $\mathbb{R}^2$ which is described above in (1) – (3) is a $\sigma$-algebra. The couple $(\mathbb{R}^2, \sigma(\tau_2))$ is by definition 8.2 the measurable space corresponding to the space $\mathbb{R}^2$ and the $\sigma$-algebra $\sigma(\tau_2)$. Consider now the $\sigma$-algebra $\sigma(\tau_n)$ in $\mathbb{R}^n$ that is generated by $n$-dimensional open rectangles. The elements of the set $\tau_n$ are rectangles of the form $(a_1, b_1) \times \ldots \times (a_n, b_n)$, where $a_i$ and $b_i$ are real numbers. The elements of the $\sigma$-algebra $\sigma(\tau_n)$ are

1) all elements of $\tau_n$;

2) the complements of all elements of $\tau_n$;

3) the countable union of elements specified in (1) and (2).
The couple defined by \((\mathbb{R}^n, \sigma(\tau_n))\) is the measurable space corresponding to the space \(\mathbb{R}^n\) and the \(\sigma\)-algebra \(\sigma(\tau_n)\). The elements of the \(\sigma\)-algebra \(\sigma(\tau_n)\) are called Borel sets in \(\mathbb{R}^n\).

Next, we discuss the \(\sigma\)-algebra \(\sigma(\tau_\infty)\) which is constructed from open rectangles in \(\mathbb{R}^\infty\). The definition of \(\sigma(\tau_\infty)\) is analogous to that of \(\sigma(\tau_n)\) since there is no difficulty in constructing rectangles, unions of rectangles, complements of rectangles on the infinite-dimensional Euclidean space \(\mathbb{R}^\infty\). The space \(\mathbb{R}^\infty\) and the \(\sigma\)-algebra \(\sigma(\tau_\infty)\) form a couple \((\mathbb{R}^\infty, \sigma(\tau_\infty))\) which is called the infinite-dimensional product measurable space corresponding to the space \(\mathbb{R}^\infty\) and the \(\sigma\)-algebra \(\sigma(\tau_\infty)\).

9 Integration on Borel Sets

In this section, we consider Borel-measurable functions from \(\mathbb{R}^n\) to \(\mathbb{R}\), which are defined as follows.

**Definition 9.1 (Borel-measurable functions)**

Let \(h : \mathbb{R}^n \to \mathbb{R}\) be a function of \(n\) variables. Then, \(h\) is a Borel-measurable function iff for any Borel set \(B \in \mathbb{R}\) the inverse image \(h^{-1}(B) \in \mathbb{R}^n\) is a Borel set in \(\mathbb{R}^n\).

A simple example of a Borel-measurable function \(h : \mathbb{R}^n \to \mathbb{R}\) is a continuous function since the inverse of each open region of \(\mathbb{R}\) is an open region in \(\mathbb{R}^n\). Another example of a Borel-measurable function is a step function since any closed region can be represented as a countable union or intersection of open regions. A complex-valued function \(h : \mathbb{R}^n \to \mathbb{C}\) is called a Borel-measurable function if \(\text{Re}(h)\) and \(\text{Im}(h)\) are Borel-measurable functions.
Next, we define a measure $\mu$ for each element of $\sigma(\tau_n)$. First, we give a general definition of a measure $\mu$ on a measurable space $(\Omega, \mathcal{F})$. Then, we define $\sigma$--finite and finite measures.

**Definition 9.2 (Measure)**
Consider a measurable space $(\Omega, \mathcal{F})$. A measure $\mu$ on the elements of the $\sigma$--algebra $\mathcal{F}$ is defined as a real-valued non-negative function such that

i) $\mu(\emptyset) = 0$;

ii) $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$, if $A_i \cap A_j = \emptyset$, for $i \neq j$, and $A_i \subset \Omega$ for all $i$.

**Definition 9.3 ($\sigma$--finite and finite measure)**
The measure $\mu$ is called $\sigma$--finite if there exists a disjoint countable partition $A_i$ of the space $\Omega$ such that

1) $\bigcup A_i = \Omega$, where $A_i \cap A_j = \emptyset$, if $i \neq j$;

2) $\mu(A_i) < \infty$.

If $\mu(\Omega) < \infty$, then the measure $\mu$ is called finite.

The requirement that the measure be $\sigma$--finite allows us to define a meaningful integration procedure for Borel-measurable functions. For the case when $\Omega = \mathbb{R}^n$, for each Borel set $B_n \in \sigma(\tau_n)$, we define its Lebesgue measure $\mu_n(B_n)$ to be the volume of the element $B_n$. Hence, one has

$$\mu_n(B_n) = \int_{B_n} d\xi, \quad (9.1)$$
where $d\zeta$ denotes the $n$-dimensional volume element at the location $\zeta = (\zeta_1, \ldots, \zeta_n)$, and has the form
\[ d\zeta = d\zeta_1 \ldots d\zeta_n. \] (9.2)

The length element $d\zeta_i$ is taken in the $i$th direction of the space $\mathbb{R}^n$.

It is clear that the function $\mu_n$ of (9.1) is a measure according to Definition 9.2, since volumes satisfy conditions (i) and (ii). Observe that $\mu_n(\mathbb{R}^n) = \infty$, from which we infer that the measure $\mu_n$ is not finite. But the measure $\mu_n$ is $\sigma$-finite, since we can construct a disjoint partition $A_i$ of the space $\mathbb{R}^n$ such that $\mu_n(A_i) < \infty$ for each value of $i$. If in (9.1) $B_n = B_1^1 \times \ldots \times B_1^n$, where $B_i^i$ are Borel sets in $\mathbb{R}$, then the measure of the set $B_n$ in $\mathbb{R}^n$ is the product of the measures of each of the $B_i^i$, so that
\[ \mu_n(B_n) = \mu_1(B_1^1) \cdot \ldots \cdot \mu_1(B_1^n), \quad \text{where } \mu_1(B_i^i) = \int_{B_i^i} d\zeta_i. \] (9.3)

We define the integral of a Borel-measurable function $h : \mathbb{R}^n \to \mathbb{R}$ on the Borel set $B_n$ as the Lebesgue integral of $h$ over $B_n$. The Riemann integral of $h$ over $B_n$, if it exists, is equal to the Lebesgue integral of $h$ over $B_n$. In this work, we consider only the class of Borel-measurable functions that are Riemann integrable, and we write
\[ \int_{B_n} h \, d\zeta = \int_{B_n} h \, d\zeta_1 \ldots d\zeta_n. \] (9.4)

The integral in (9.4) can be evaluated as a repeated integral according to Fubini's theorem, which can be formulated as follows.

**Theorem 9.1 (Fubini's theorem)**

If $h : \mathbb{R}^2 \to \mathbb{R}$ is a Borel-measurable function, which is integrable, and $B_2 = B_1^1 \times B_1^2$, then
\[ \int_{B_1^i} h \, d\zeta_i, \quad i = 1, 2, \] (9.5)
is a Borel-measurable function, which is integrable, and

\[ \int_{B_2} h \, d\zeta = \int_{B_1^1} d\zeta_1 \int_{B_1^2} h \, d\zeta_2 = \int_{B_1^1} d\zeta_2 \int_{B_1^1} h \, d\zeta_1. \]  
(9.6)

For the \( n \)-dimensional case, if \( h : \mathbb{R}^n \to \mathbb{R} \) is a Borel-measurable function and \( B_n = B_1^1 \times \ldots \times B_1^n \), then

\[ \int_{B_n} h \, d\zeta = \int_{B_1^1} d\zeta_1 \ldots \int_{B_1^n} d\zeta_n = \int_{B_i_1} d\zeta_{i_1} \ldots \int_{B_i^n} h \, d\zeta_{i_n}, \]  
(9.7)
where \( \{i_k\} \) denotes an arbitrary permutation of \( \{1, \ldots, n\} \).

The proof of Fubini's theorem can be found in Chow and Teicher [9] (p. 177). A consequence of Fubini's theorem for a function \( h : \mathbb{R}^n \to \mathbb{R} \) is that the integral of \( h \) in \( \mathbb{R}^n \) can be represented in one of the several possible forms given by

\[ \int_{B_n} h \, d\zeta = \int_{B_n} h \, d\zeta_{i_1} \ldots d\zeta_{i_n}. \]  
(9.8)
10 Probability Concepts

In the previous section, we discussed that the volume of a Borel set $B_n$ in $\mathbb{R}^n$ is a measure in the measurable space $(\mathbb{R}^n, \sigma(\tau_n))$, where $\sigma(\tau_n)$ is the $\sigma-$algebra generated by the set of open rectangles $\tau_n$. The weighted volume of Borel sets, together with a normality condition, defines various measures, called probability measures, in the measurable space $(\mathbb{R}^n, \sigma(\tau_n))$. In this work, we consider only those probability measures $P$, in the measurable space $(\mathbb{R}^n, \sigma(\tau_n))$, which can be defined in terms of a probability density function $p$ as follows.

**Definition 10.1 (probability density function)**

A probability density function $p : \mathbb{R}^n \to \mathbb{R}$ of an $n-$dimensional random variable is a Borel-measurable integrable non-negative function such that

\[
\int_{\mathbb{R}^n} p(\zeta) \, d\zeta = 1. \quad (10.1)
\]

**Definition 10.2 (probability measure)**

Let $p : \mathbb{R}^n \to \mathbb{R}$ be a probability density function of an $n-$dimensional random variable. Then, the probability measure $P$ corresponding to $p$ is defined for each Borel set $B_n$ in $\mathbb{R}^n$ by

\[
P(B_n) = \int_{B_n} p(\zeta) \, d\zeta. \quad (10.2)
\]

It is easy to see that the mapping $P$ of (10.2) is a measure in the measurable space $(\mathbb{R}^n, \sigma(\tau_n))$, since the two conditions of Definition 9.2 are satisfied. Further, it follows from Definition 9.3 and the normality condition (10.1) that $P$ is a finite measure.
Fig. 10.1  $N$ points in the two-dimensional plane.

In this section, we discuss probability concepts relative to the positions in the plane $\mathbb{R}^2$ of $N$ point-like objects, as shown in Fig. 10.1. Since two scalar variables are needed to define the position of a point in $\mathbb{R}^2$, the probability density function for the $N$ points is a function of $2N$ random variables. The configuration $(\zeta^1, \ldots, \zeta^N)$ of $N$ positions in $\mathbb{R}^2$ can be represented as a single point $\zeta$ in $\mathbb{R}^{2N}$ such that

$$\zeta = (\zeta^1, \zeta^2, \ldots, \zeta^N) = (\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_1, \zeta_2).$$  \hfill (10.3)

For each configuration of $N$ positions $\zeta^1, \ldots, \zeta^N$ in $\mathbb{R}^2$, we assign $N$ area elements $d\zeta^1, \ldots, d\zeta^N$, where $d\zeta^i$ is the area element at $\zeta^i$ in $\mathbb{R}^2$, and we define a $2N$-dimensional volume element $d\zeta$ in $\mathbb{R}^{2N}$ such that

$$d\zeta = d\zeta^1 d\zeta^2 \ldots d\zeta^N = d\zeta_1^1 d\zeta_2^1 \ldots d\zeta_1^N d\zeta_2^N.$$  \hfill (10.4)

The positions of the $N$ points in Fig. 10.1 are known only in a probabilistic sense. The probability $P$ that the point $\zeta$ be inside the volume element $d\zeta$ is defined in terms of a probability density function $p$ by the two equations

$$P(d\zeta^1, \ldots, d\zeta^N) = p(\zeta) d\zeta,$$  \hfill (10.5)
Fig. 10.2 Rectangle $B_{2N}$ of dimension $2N$ in the space $\mathbb{R}^{2N}$.

\[ \int_{B_{2N}} p(\zeta) \, d\zeta = 1. \quad (10.6) \]

Now, consider $N$ arbitrary two-dimensional Borel sets $B_2^1, B_2^2, \ldots, B_2^N$ in the two-dimensional plane $\mathbb{R}^2$, and define a rectangle $B_{2N}$ in the space $\mathbb{R}^{2N}$, as shown in Fig. 10.2, by the Cartesian product

\[ B_{2N} = B_2^1 \times B_2^2 \times \ldots \times B_2^N. \quad (10.7) \]

The probability $P(B_{2N})$ of having the representative point of equation (10.3) in the rectangle $B_{2N}$ of Fig. 10.2 is given by

\[ P(B_{2N}) = \int_{B_{2N}} p(\zeta) \, d\zeta. \quad (10.8) \]
We now examine the normality condition (10.6). Using Fubini’s theorem (Theorem 9.1), one finds that (10.6) yields

$$\int_{\mathbb{R}^2} d\zeta^1 \int_{\mathbb{R}^2} d\zeta^2 \cdots \int_{\mathbb{R}^2} p(\zeta^1, \ldots, \zeta^N) d\zeta^N = 1.$$  \hspace{1cm} (10.9)

Then, for each value of $i$ ($i = 1, \ldots, N$), we define the marginal probability density function $p_i : \mathbb{R}^2 \to \mathbb{R}$ as follows.

**Definition 10.3 (marginal probability density)**

Let $p : \mathbb{R}^{2N} \to \mathbb{R}$ be a probability density function of $n = 2N$ random variables, as described in Definition 10.1. Then, the marginal probability density $p_i$ corresponding to the index $i$ ($i = 1, \ldots, N$) is the non-negative Borel-measurable function $p_i : \mathbb{R}^2 \to \mathbb{R}$ such that

\begin{align*}
i) \quad & p_i(\mathbf{x}) = \int_{\mathbb{R}^{2N-2}} p(\zeta^1, \ldots, \zeta^{i-1}, \mathbf{x}, \zeta^{i+1}, \ldots, \zeta^N) d\zeta^1 \cdots d\zeta^{i-1} d\zeta^{i+1} \cdots d\zeta^N; \\
\Rightarrow & \int_{\mathbb{R}^2} p_i(\mathbf{x}) d\mathbf{x} = 1. \hspace{1cm} (10.10)
\end{align*}

Equation (10.11) follows from the normality condition (10.6), (10.10), and Fubini’s theorem. Since the functions $p$ and $p_i$ are non-negative, it follows from Definition 10.3 that $p_i(\mathbf{x})$ vanishes for all $\mathbf{x}$ where $p(\cdot, \mathbf{x}, \cdot) : \mathbb{R}^{2N-2} \to \mathbb{R}$ vanishes. Thus, we write

$$p_i(\mathbf{x}) = 0 \Leftrightarrow p(\cdot, \mathbf{x}, \cdot) = 0, \quad (\mathbf{x} \text{ in } i\text{th position}). \hspace{1cm} (10.12)$$

Given $p$ and a fixed index $i$, we define for each value of $\mathbf{x}$ in $\mathbb{R}^2$ the conditional probability density function $p_i(\cdot | \mathbf{x}) : \mathbb{R}^{2N-2} \to \mathbb{R}$ as follows.
Definition 10.4 (conditional probability density)

Let $p : \mathbb{R}^{2N} \to \mathbb{R}$ be a probability density function of $n = 2N$ random variables, as described in Definition 10.1, and let $p_i : \mathbb{R} \to \mathbb{R}$ be the marginal probability density function, as in Definition 10.3. Then, for each value of $x$ in $\mathbb{R}^2$ and for each index $i$ ($i = 1, \ldots, N$), the conditional probability density function is the non-negative Borel-measurable function $p_i(\cdot|x) : \mathbb{R}^{2N-2} \to \mathbb{R}$ such that

i) if $p(\cdot, x, \cdot) \neq 0$ ($x$ in $i$th position), then

$$p(\xi^1, \ldots, \xi^{i-1}, x, \xi^{i+1}, \ldots, \xi^N) = p_i(x)p_i(\xi^1, \ldots, \xi^{i-1}, \xi^{i+1}, \ldots, \xi^N|x);$$  \hspace{1cm} (10.13)

ii) if $p(\cdot, x, \cdot) = 0$ ($x$ in $i$th position) then, for all $(\xi^1, \ldots, \xi^{i-1}, \xi^{i+1}, \ldots, \xi^N)$ in $\mathbb{R}^{2N-2}$, the value

$$p_i(\xi^1, \ldots, \xi^{i-1}, \xi^{i+1}, \ldots, \xi^N|x)$$ is not defined.

Observe that the condition on $p(\cdot, x, \cdot)$ in (i) and (ii) of Definition 10.4 is equivalent, by equation (10.12), to a condition on the marginal probability density function $p_i$. Now we consider a function $f : \mathbb{R}^{2N} \to \mathbb{R}$ of $2N$ real and random variables. Then, we define the average $< f >$ of the function $f$ relative to a probability density $p$ as follows.

Definition 10.5 (average or expectation of a function $f$)

Let $f : \mathbb{R}^{2N} \to \mathbb{R}$ be a Borel-measurable function of $2N$ real and random variables, and let $p : \mathbb{R}^{2N} \to \mathbb{R}$ be a probability density function as in Definition 10.1. Then, the average $< f >$ of the function $f$ is defined by

$$< f > = \int_{\mathbb{R}^{2N}} f(\xi)p(\xi) \, d\xi.$$  \hspace{1cm} (10.14)
Using the conditional probability density function $p_i(\cdot|\mathbf{x})$ of Definition 10.4, we now define the partial average (expectation) $< f >_i (\cdot) : \mathbb{R}^2 \to \mathbb{R}$ of the function $f$ relative to the index $i$ as follows.

**Definition 10.6 (partial average or conditional expectation of a function $f$)**

Let $f : \mathbb{R}^{2N} \to \mathbb{R}$ be a Borel-measurable function of $2N$ real and random variables, and for each $\mathbf{x}$ in $\mathbb{R}^2$ and each index $i$ ($i = 1, \ldots, N$) let $p_i(\cdot|\mathbf{x})$ be the conditional probability density function, as in Definition 10.4. Then, the partial average $< f >_i : \mathbb{R}^2 \to \mathbb{R}$ of the function $f$ is defined by

$$< f >_i (\mathbf{x}) = \int_{\mathbb{R}^{2N-2}} f(\zeta^1, \ldots, \zeta^{i-1}, \mathbf{x}, \zeta^{i+1}, \ldots, \zeta^N)$$

$$p_i(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^N|\mathbf{x}) d\zeta^1 \ldots d\zeta^{i-1} d\zeta^{i+1} \ldots d\zeta^N,$$

(10.15)

for all $\mathbf{x}$ in $\mathbb{R}^2$ such that the probability density function $p(\cdot, \mathbf{x}, \cdot)$ with $\mathbf{x}$ in the $i$th position does not vanish.

If $D_i$ is the set of all points $\mathbf{x}$ in $\mathbb{R}^2$ such that $p(\cdot, \mathbf{x}, \cdot)$ with $\mathbf{x}$ in the $i$th position does not vanish, as in (i) of Definition 10.4, then one has

$$D_i = \{ \mathbf{x} \in \mathbb{R}^2 | p(\cdot, \mathbf{x}, \cdot) \neq 0 \}, \text{ with } \mathbf{x} \text{ in the } i \text{th position } (i = 1, \ldots, N).$$

(10.16)

Note that, since the probability density function $p$ is Borel measurable, the set $D_i$ is a Borel measurable set. In view of (10.16) and of the above remark, we conclude that there is a Borel set $D_1 \times D_2 \ldots \times D_N$ in $\mathbb{R}^{2N}$, which may be the entire space $\mathbb{R}^{2N}$, such that the probability density function vanishes outside of this Borel set. Thus, we write

$$p(\cdot) = 0, \text{ outside } D_1 \times D_2 \ldots \times D_N.$$  

(10.17)
Now, it follows from the preceding three definitions and from Fubini’s theorem that the average \( < f > \) of the function \( f \) and the partial average \( < f >_i \) are related by

\[
< f > = \int_{R^2} dx \int_{R^{2N-2}} f(\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^N) \\
p(\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^N) d\zeta^1 \ldots d\zeta^{i-1} d\zeta^{i+1} \ldots d\zeta^N =
\int_{D_i} dx \int_{R^{2N-2}} f(\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^N) \\
p(\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^N) d\zeta^1 \ldots d\zeta^{i-1} d\zeta^{i+1} \ldots d\zeta^N =
\int_{D_i} < f >_i(x)p_i(x) dx,
\]

where \( p_i(\cdot) \) denotes the marginal probability density function of equation (10.13).

11 Restrictions on the Probability Density Function

We assume in the following that, in the probability density function \( p \) of (10.5) and (10.6), the ordering of the variables \( \zeta^1, \ldots, \zeta^N \) is immaterial. This means that the \( N \) points of Fig. 10.1 are not to be labeled in any particular way. We assume for our purposes that each one of them plays the same role as any of the \((N - 1)\) other points. Thus, one has

\[
(A) \quad p(\zeta^1, \ldots, \zeta^N) = p(\zeta^{i_1}, \ldots, \zeta^{i_N}),
\]

where \( \{i_1, \ldots, i_N\} \) is an arbitrary permutation of the indices \( \{1, \ldots, N\} \). If the random vectors \( \zeta^1 \ldots \zeta^N \) are such that (11.1) is satisfied, then these \( N \) vectors are called exchangeable random vectors. Next, let \( i \) and \( j \) be two arbitrary indices. Then, it follows from (11.1) that

\[
p(\zeta^1, \ldots, \zeta^i, \ldots, \zeta^j, \ldots, \zeta^N) = p(\zeta^1, \ldots, \zeta^j, \ldots, \zeta^i, \ldots, \zeta^N).
\]
Therefore, using (10.10) and (11.2), one can write the marginal probability density function \( p_i \) in the form

\[
p_i(\mathbf{x}) = \int_{\mathbb{R}^{2N-2}} p(\zeta^1, \ldots, \zeta^{i-1}, \zeta^i, \zeta^{i+1}, \ldots, x_i, \ldots, \zeta^N) \, d\zeta^1 \ldots d\zeta^{i-1} d\zeta^{i+1} \ldots d\zeta^N,
\]

(11.3)

where \( \mathbf{x} \) occupies the \( j \)th position in the list of arguments of \( p \). By Fubini’s theorem (9.8), we can rearrange the order of the differential elements in (11.3) in any order without changing the value of the integral. For our purposes, the new order of these elements is \( d\zeta^1 \ldots d\zeta^{i-1} d\zeta^{i+1} \ldots d\zeta^{j-1} d\zeta^{j+1} \ldots d\zeta^N \). Then, it follows from (11.3), from Definition 10.3 for the marginal probability density function, and from Fubini’s theorem that

\[
(B) \quad p_i(\mathbf{x}) = p_j(\mathbf{x}), \text{ for all } i, j = 1, \ldots, N, \text{ and all } \mathbf{x} \in \mathbb{R}^2.
\]

(11.4)

Further, we infer from (10.16), (10.12), and (11.4) that the sets \( D_i \) are identical for all indices \( i \). Thus, one has

\[
(C) \quad D_i = D_j, \text{ for all } i, j = 1, \ldots, N.
\]

(11.5)

We now examine equation (10.13), which defines the conditional probability density function \( p_i(\cdot|\mathbf{x}) \). Let \( \{k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_N\} \) be an arbitrary permutation of the \( (N - 1) \) indices \( \{1, \ldots, i - 1, i + 1, \ldots, N\} \). Then, by (11.1) we can write

\[
p(\zeta^1, \ldots, \zeta^{i-1}, x_i, \zeta^{i+1}, \ldots, \zeta^N) = p(\zeta^{k_1}, \ldots, \zeta^{k_{i-1}}, x_i, \zeta^{k_{i+1}}, \ldots, \zeta^{k_N}),
\]

(11.6)

where \( \mathbf{x} \) occupies the \( i \)th position in the list of arguments. Using Definition 10.4 for the conditional probability density function \( p_i(\cdot|\mathbf{x}) \), one infers that the right-hand side of (11.6) for \( \mathbf{x} \in D_i \) can be written in the form

\[
p(\zeta^{k_1}, \ldots, \zeta^{k_{i-1}}, x_i, \zeta^{k_{i+1}}, \ldots, \zeta^{k_N}) = p_i(\mathbf{x}) p_i(\zeta^{k_1}, \ldots, \zeta^{k_{i-1}}, \zeta^{k_{i+1}}, \ldots, \zeta^{k_N}|\mathbf{x}), \quad \mathbf{x} \in D_i.
\]

(11.7)
Comparing (11.7) and (10.13), and using (11.6), one infers that
\[
p_i(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^N | \mathbf{a}) = \\
p_i(\zeta^{k_1}, \ldots, \zeta^{k_{i-1}}, \zeta^{k_{i+1}}, \ldots, \zeta^{k_N} | \mathbf{a}), \quad \mathbf{a} \in D_i,
\]
for all $i = 1, \ldots, N$ and for all permutations of the indices $\{1, \ldots, i-1, i+1, \ldots, N\}$. Equation (11.8) says that the ordering of the $(N-1)$ arguments of the conditional probability density function $p( \cdot | \mathbf{a})$ is immaterial.

For an arbitrary point $\mathbf{a}$ in $D_i$ ($i = 1, 2, \ldots, N$) and for an arbitrary point $\mathbf{y}$ in $\mathbb{R}^2$, one can use equation (10.13), where the conditional probability density function $p_i( \cdot | \mathbf{a})$ is defined, to obtain
\[
p(\zeta^1, \ldots, \zeta^{i-1}, \mathbf{a}, \zeta^{i+1}, \ldots, \zeta^{j-1}, \mathbf{y}, \zeta^{j+1}, \ldots, \zeta^N) = \\
p_i(\mathbf{a})p_i(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^{j-1}, \mathbf{y}, \zeta^{j+1}, \ldots, \zeta^N | \mathbf{a}), \quad \mathbf{a} \in D_i.
\]
Using (11.1), we can switch the variables $\mathbf{a}$ and $\mathbf{y}$ in the left-hand side of (11.9), and using (11.4), we can replace $p_i(\mathbf{a})$ by $p_j(\mathbf{a})$ in the right-hand side of (11.9). With these two changes, (11.9) yields
\[
p(\zeta^1, \ldots, \zeta^{i-1}, \mathbf{y}, \zeta^{i+1}, \ldots, \zeta^{j-1}, \mathbf{a}, \zeta^{j+1}, \ldots, \zeta^N) = \\
p_j(\mathbf{a})p_i(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^{j-1}, \mathbf{y}, \zeta^{j+1}, \ldots, \zeta^N | \mathbf{a}), \quad \mathbf{a} \in D_i.
\]
Next, the conditional probability density function $p_j( \cdot | \mathbf{a})$ for the index $j$ is defined by equation (10.13) of Definition 10.4 as
\[
p(\zeta^1, \ldots, \zeta^{i-1}, \mathbf{y}, \zeta^{i+1}, \ldots, \zeta^{j-1}, \mathbf{a}, \zeta^{j+1}, \ldots, \zeta^N) = \\
p_j(\mathbf{a})p_j(\zeta^1, \ldots, \zeta^{i-1}, \mathbf{y}, \zeta^{i+1}, \ldots, \zeta^{j-1}, \zeta^{j+1}, \ldots, \zeta^N | \mathbf{a}), \quad \mathbf{a} \in D_j.
\]
Comparing (11.10) and (11.11), since $D_i = D_j$ according to (11.5), one infers that
\[
p_i(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^{j-1}, \mathbf{y}, \zeta^{j+1}, \ldots, \zeta^N | \mathbf{a}) = \\
p_j(\zeta^1, \ldots, \zeta^{i-1}, \mathbf{y}, \zeta^{i+1}, \ldots, \zeta^{j-1}, \zeta^{j+1}, \ldots, \zeta^N | \mathbf{a}), \quad \mathbf{a} \in D_i = D_j.
\]
Observe that the lists of arguments in the functions $p_i( \cdot | \mathbf{a})$ and $p_j( \cdot | \mathbf{a})$ of (11.12) are identical, except for the ordering. Thus, one can use the result of (11.8) to rewrite
(11.12) in the form
\[
p_i(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^{j-1}, y, \zeta^{j+1}, \ldots, \zeta^N|\alpha) = \\
p_j(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^{j-1}, y, \zeta^{j+1}, \ldots, \zeta^N|\alpha), \quad \alpha \in D_i = D_j,
\]
for all \(i, j = 1, \ldots, N\). Now, the lists of arguments in the function \(p_i(\cdot|\alpha): \mathbb{R}^{2N-2} \to \mathbb{R}\) and \(p_j(\cdot|\alpha): \mathbb{R}^{2N-2} \to \mathbb{R}\) are identical in (11.13), and they are also arranged in the same order. Hence, we conclude that
\[
(D) \quad p_i(\cdot|\alpha) = p_j(\cdot|\alpha), \text{ for all } i, j = 1, \ldots, N, \text{ and all } \alpha \text{ in } D_i = D_j. \tag{11.14}
\]

The conditional probability density functions \(p_i(\cdot|\alpha)\) and \(p_j(\cdot|\alpha)\) of (11.14) can be used to evaluate the partial averages \(<f>_i\) and \(<f>_j\) of a function \(f: \mathbb{R}^{2N} \to \mathbb{R}\) as in Definition 10.6. For each point \(\alpha\) in \(D_i = D_j\) (by equation (11.5)), one has
\[
<f>_i(\alpha) = \int_{\mathbb{R}^{2N-2}} f(\zeta^1, \ldots, \zeta^{i-1}, \alpha, \zeta^{i+1}, \ldots, \zeta^N) \\
p_i(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^N|\alpha) \, d\zeta^1 \ldots d\zeta^{i-1} d\zeta^{i+1} \ldots d\zeta^N,
\]
\[
<f>_j(\alpha) = \int_{\mathbb{R}^{2N-2}} f(\zeta^1, \ldots, \zeta^{j-1}, \alpha, \zeta^{j+1}, \ldots, \zeta^N) \\
p_j(\zeta^1, \ldots, \zeta^{j-1}, \zeta^{i+1}, \ldots, \zeta^N|\alpha) \, d\zeta^1 \ldots d\zeta^{j-1} d\zeta^{j+1} \ldots d\zeta^N. \tag{11.16}
\]

Without loss of generality, assume that \(i < j\) and make the following change of variables in (11.16)
\[
(\zeta^1, \ldots, \zeta^{i-1}, \zeta^i, \zeta^{i+1}, \ldots, \zeta^{j-1}, \zeta^{j+1}, \ldots, \zeta^N) \rightarrow \quad \tag{11.17}
(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^{j-1}, \zeta^j, \zeta^{j+1}, \ldots, \zeta^N).
\]

Then, using (11.17), the integral of (11.16) can be written in the form
\[
<f>_j(\alpha) = \int_{\mathbb{R}^{2N-2}} f(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^{j-1}, \alpha, \zeta^{j+1}, \ldots, \zeta^N) \\
p_j(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^N|\alpha) \, d\zeta^1 \ldots d\zeta^{i-1} d\zeta^{i+1} \ldots d\zeta^N. \tag{11.18}
\]
We consider now the special case where the ordering of the variables $\zeta^1, \ldots, \zeta^N$ is immaterial in the list of arguments of the function $f$. Functions of this type are called symmetric functions. Thus, for a symmetric function, one has

$$f(\zeta^1, \ldots, \zeta^N) = f(\zeta^{i_1}, \ldots, \zeta^{i_N}),$$  \hspace{1cm} (11.19)

where $\{i_1, \ldots, i_N\}$ is an arbitrary permutation of the indices $\{1, \ldots, N\}$. Then, assuming that (11.19) holds, one infers that

$$f(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^j, \zeta^j, \zeta^{j+1}, \ldots, \zeta^N) =$$

$$f(\zeta^1, \ldots, \zeta^{i-1}, \zeta^j, \zeta^{i+1}, \ldots, \zeta^j, \zeta^{j+1}, \ldots, \zeta^N),$$  \hspace{1cm} (11.20)

for all $i,j = 1, \ldots, N$. Now we conclude from (11.18), (11.15), (11.20), and from the result (11.14) that the conditional probability density function does not depend on the choice of the index. Thus, one has

\begin{equation}
\langle f \rangle_i (\mathbf{x}) = \langle f \rangle_j (\mathbf{x}), \text{ for all } i,j = 1, \ldots, N, \text{ all } \mathbf{x} \text{ in } \mathcal{D}_i = \mathcal{D}_j, \text{ and for all symmetric functions } f. \tag{E}
\end{equation}  \hspace{1cm} (11.21)

We summarize the important results of this chapter as follows.

(A) $p(\zeta^1, \ldots, \zeta^N) = p(\zeta^{i_1}, \ldots, \zeta^{i_N})$, for any permutation of $\{1, \ldots, N\}$;

(B) $p_i(\mathbf{x}) = p_j(\mathbf{x})$, for all $i,j = 1, \ldots, N$, and all $\mathbf{x}$ in $\mathbb{R}^2$, if (A) holds;

(C) $\mathcal{D}_i = \mathcal{D}_j$, for all $i,j = 1, \ldots, N$, if (A) holds;

(D) $p_i(\cdot|\mathbf{x}) = p_j(\cdot|\mathbf{x})$, for all $i,j = 1, \ldots, N$, and all $\mathbf{x}$ in $\mathcal{D}_i = \mathcal{D}_j$, if (A) holds;

(E) $\langle f \rangle_i (\mathbf{x}) = \langle f \rangle_j (\mathbf{x})$, for all $i,j = 1, \ldots, N$, and all $\mathbf{x}$ in $\mathcal{D}_i = \mathcal{D}_j$, if (A) holds and $f$ is symmetric.
12 Uniform Probability Density Function for a Configuration of $N$ Circles

In this section, we define a uniform probability density function $p : \mathbb{R}^{2N} \to \mathbb{R}$ corresponding to the configuration of $N$ impenetrable circles of radius $b$ and centers $\zeta^1, \ldots, \zeta^N$, respectively, in the two-dimensional plane $\mathbb{R}^2$. We assume that these $N$ circles are contained in a rectangular region $V_{d+b}^{h+b}$, as shown in Fig. 12.1. To motivate the following mathematical discussion, we first discuss a simple experiment that illustrates the physical meaning of the uniform probability density function.

Consider $N$ identical circles of radius $b$ that are contained in the box $V_{d+b}^{h+b}$, where $b$ is much smaller than $d$ and $h$. We imagine that a stirring mechanism allows us to move the $N$ circles randomly inside the box $V_{d+b}^{h+b}$. Note that keeping the $N$ circles inside $V_{d+b}^{h+b}$ means that the $N$ centers are inside the dashed rectangle $V_d^h$ of Fig. 12.1.
We assign to each configuration a probability density function $p : \mathbb{R}^{2N} \to \mathbb{R}$. Since the circles are all identical, we infer that the probability density function $p$ as a function of the $N$ positions $\zeta^1, \ldots, \zeta^N$ is invariant under an arbitrary reordering of its arguments.

For each time $t$ in the stirring process, one can see a particular configuration of the $N$ circles. If an infinite number of such configurations have been observed, and if the center of the $i$th circle ($i = 1, \ldots, N$) occupies with equal probability each location $\zeta^i$ in the box $V_d^h$, then we say that the probability density function $p$ corresponding to the configurations of the $N$ circles is a uniform probability density function. With this background, we now state the following definition.

**Definition 12.1 (uniform probability density function for the configuration of $N$ circles inside a finite region in $\mathbb{R}^2$)**

Consider $N$ circles of radius $b$ centered at $\zeta^1, \ldots, \zeta^N$, respectively, in the plane $\mathbb{R}^2$, where $\zeta^i$ ($i = 1, \ldots, N$) is constrained to lie inside the rectangular region $V_d^h$ of sides $2d$ and $2h$. We say that $p : \mathbb{R}^{2N} \to \mathbb{R}$ is the uniform probability density function corresponding to the $N$ circles if $p$ is non-negative, Borel-measurable, integrable, and if

1) **constraint on the location**

$$p(\zeta^1, \ldots, \zeta^N) = 0, \quad \text{if } \zeta^i \notin V_d^h \text{ for at least one } i; \quad (12.1)$$

2) **non-penetrability**

$$p(\zeta^1, \ldots, \zeta^N) = 0, \quad \text{if } |\zeta^i - \zeta^j| < 2b \text{ whenever } i \neq j; \quad (12.2)$$

3) **exchangeability**
\[ p(\zeta^1, \ldots, \zeta^N) = p(\zeta^{i_1}, \ldots, \zeta^{i_N}), \quad (12.3) \]

where \( \{i_1, \ldots, i_N\} \) is an arbitrary permutation of the indices \( \{1, \ldots, N\} \);

4) normality

\[ \int_{\mathbb{R}^{2N}} p(\zeta^1, \ldots, \zeta^N) \, d\zeta^1 \ldots d\zeta^N = 1; \quad (12.4) \]

5) uniformity

\[ \int_{\mathbb{R}^{2N-2}} p(\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^N) \, d\zeta^1 \ldots d\zeta^{i-1} d\zeta^{i+1} \ldots d\zeta^N = p_i(x), \quad (12.5) \]

where \( p_i \) is given by

\[ p_i(x) = \begin{cases} \frac{1}{Ahd^i}, & \text{if } x \in V_d^h, \\ 0, & \text{if } x \notin V_d^h. \end{cases} \quad (12.6) \]

Equation (12.5) contains only the statistical information about the \( i \)th circle, since the \((N - 1)\)-fold integration process eliminates the statistical information about the \((N - 1)\) other circles. Observe that the function \( p_i \) defined in (12.5) is the marginal probability density function corresponding to the position of the \( i \)th circle.

We showed earlier in (11.1)-(11.4) that the exchangeability condition (12.3) implies that the marginal probability density function is independent of the index \( i \). Thus, one can write

\[ p_i(x) = p_j(x), \quad \text{for all } i, j = 1, \ldots, N, \text{ and all } x \in \mathbb{R}^2. \quad (12.7) \]

We also observe that by integrating the marginal probability density function \( p_i \) of (12.6) over \( \mathbb{R}^2 \), one finds that the normality condition (10.11) is satisfied.

Based on the definition of the uniform probability density function given by Definition 12.1 and on the definition of the conditional probability density function
given by Definition 10.4, we define a uniform conditional probability density function for the configuration of $N$ circles as follows.

**Definition 12.2** (uniform conditional probability density function for the configuration of $N$ circles inside a finite region in $\mathbb{R}^2$)

Consider $N$ circles of radius $b$ centered at $\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^N$, respectively, inside the rectangle $V_d^h$ of height $2h$ and width $2d$ in the plane $\mathbb{R}^2$. For $x \in V_d^h$, we define the conditional probability density function $p_{i}(\cdot | x) : \mathbb{R}^{2N-2} \rightarrow \mathbb{R}$ corresponding to the uniform probability density function $p : \mathbb{R}^{2N} \rightarrow \mathbb{R}$ of Definition 12.1 by

$\begin{equation}
\begin{align*}
\text{i)} & \quad p_{i}(\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^N | x) = 4hdp(\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^N), \\
& \quad x \in V_d^h, \ i = 1, \ldots, N; \\
\text{ii)} & \quad p_{i}(\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^N | x) \text{ is not defined if } x \text{ is outside } V_d^h.
\end{align*}
\end{equation}$

13 Construction of a Uniform Distribution of Circles in a Layer

Let $p : \mathbb{R}^{2N_0} \rightarrow \mathbb{R}$ be the uniform probability density function for the configurations of $N_0$ circles centered inside the rectangle $V_{d_0}^h$. The properties of $p$ are described in Definition 12.1 (for $N = N_0$ and $d = d_0$). The set $D_i$ of all points $x$ in $\mathbb{R}^2$ such that $p(\cdot, x, \cdot)$ does not vanish, with $x$ in the $i$th position, is given by (10.16), the exchangeability condition (12.3), and (11.5) by

$\begin{equation}
D_i = V_{d_0}^h, \ \text{for all } i = 1, \ldots, N_0.
\end{equation}$
We define the density \( n(x) \) of the circles in the region \( V_{d_0}^h \) at the position \( x \) by
\[
  n(x) = \sum_{i=1}^{N_0} p_i(x). \tag{13.2}
\]
Since the circles are uniformly distributed inside the region \( V_{d_0}^h \), it follows from (12.6) and (13.2) that inside the region \( V_{d_0}^h \) the density \( n \) is independent of the value of \( x \), and one has
\[
  n = \frac{N_0}{4hd_0}, \quad \text{at each } x \text{ in } V_{d_0}^h. \tag{13.3}
\]
The uniform conditional probability density function \( p_i(\cdot | x) : \mathbb{R}^{2N_0-2} \to \mathbb{R} \) is given according to (13.3) and Definition 12.2 by
\[
  p_i(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^{N_0} | x) = \frac{N_0}{n} p(\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^{N_0}),
\]
for \( x \in V_{d_0}^h \).

Let \( f : \mathbb{R}^{2N_0} \to \mathbb{R} \) be a Borel-measurable integrable function of \( 2N_0 \) variables. Then, using Definition 10.5 and Definition 10.6 for the average \( <f> \) and the partial average \( <f>_i \), respectively, together with (12.6), (13.4), (13.1), and (10.18), one has
\[
  <f> = \int_{\mathbb{R}^{2N_0}} f(\zeta^1, \ldots, \zeta^{N_0}) p(\zeta^1, \ldots, \zeta^{N_0}) d\zeta^1 \ldots d\zeta^{N_0} = \frac{n}{N_0} \int_{V_{d_0}^h} <f>_i(x) dx. \tag{13.5}
\]
In (13.5), the partial average \( <f>_i \), from (10.15) and (13.4), is given by
\[
  <f>_i(x) = \frac{N_0}{n} \int_{\mathbb{R}^{2N_0-2}} f(\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^{N_0})
  \left. \right| p(\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^{N_0}) d\zeta^1 \ldots d\zeta^{i-1} d\zeta^{i+1} \ldots d\zeta^{N_0}. \tag{13.6}
\]
Now, we increase the length \( d_0 \) of the region \( V_{d_0}^h \), keeping the height \( h \) fixed, and we increase the number of circles \( N_0 \) at the same time in such a way that the density \( n \) remains unchanged. Let \( N \) denote the increased number of circles, \( N > N_0 \). Hence,
the length $d_N$ of the larger rectangle $V_{d_N}^h$ is given in terms of $N$ by

$$d_N = \frac{N}{4hn} = \frac{N}{N_0}d_0,$$

(13.7)

where $n$ is defined by (13.3). If the number $N$ of circles becomes infinitely large, we can see from (13.7) that the width $d_N$ also becomes infinitely large. In the limit, we write

$$V_{\infty}^h = \lim_{d_N \to \infty} V_{d_N}^h = \{x \in \mathbb{R}^2 : |x_2| < h\},$$

(13.8)

and $V_{\infty}^h$ is identical to the strip of equation (2.1). For each value of $N$, we associate to the $N$ circles that occupy the positions $\zeta^1, \ldots, \zeta^N$ in the rectangle $V_{d_N}^h$ a uniform probability density function $p : \mathbb{R}^{2N} \to \mathbb{R}$ with the properties described above in Definition 12.1 (for $d = d_N$), and we allow $N$ to increase to infinity.

Let $f_N : \mathbb{R}^{2N} \to \mathbb{R}$ be a Borel-measurable integrable function that describes some physical property associated with the $N$ circles. Assume that the ordering of the variables is immaterial in the list of arguments of the function $f_N$. Then, the average $\bar{f}_N$ and the partial average $\hat{f}_N$ are defined according to (13.5) and (13.6), respectively, by

$$\bar{f}_N = \frac{n}{N} \int_{V_{d_N}^h} \hat{f}_N(x) \, dx,$$

(13.9)

and

$$\hat{f}_N(x) = \frac{N}{n} \int_{\mathbb{R}^{2N-2}} f_N(\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^N)$$

$$p(\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^N) \, d\zeta^1 \ldots d\zeta^{i-1} \, d\zeta^{i+1} \ldots d\zeta^N.$$  

(13.10)

In (13.10), the index $i$ can be chosen arbitrarily in the range $1, \ldots, N$, and $p$ vanishes if at least one of its arguments is outside $V_{d_N}^h$. Thus, we conclude that the region of integration in (13.10) can be written as the $(N - 1)$-fold Cartesian product $V_{d_N}^h \times \ldots \times V_{d_N}^h$, which is a bounded rectangle in $\mathbb{R}^{2N-2}$. It follows from this discussion,
since \( f_N : \mathbb{R}^{2N} \to \mathbb{R} \) and \( p : \mathbb{R}^{2N} \to \mathbb{R} \) are Borel-measurable and integrable, that the function \( \hat{f}_N : \mathbb{R}^2 \to \mathbb{R} \) of (13.10) is Borel-measurable and integrable in \( \mathbb{R}^2 \) for each \( N \geq N_0 \).

Now, we examine the limit of the sequence of functions \( \hat{f}_N : \mathbb{R}^2 \to \mathbb{R} \) as \( N \to \infty \). Observe from (13.10) and the constraint on the location (12.1) for the probability density function \( p \) that the function \( \hat{f}_N \) for all \( N \) vanishes outside \( V_N^h \). Moreover, for all \( N \) the function \( \hat{f}_N \) vanishes in \( V_N^h \setminus V_N^\epsilon \), where the region \( V_N^h \setminus V_N^\epsilon \) is represented by the shaded area in Fig. 13.1. The region where the function \( \hat{f}_N \) may not vanish is \( V_N^\epsilon \). In the limit as \( N \to \infty \) the region \( V_N^h \) approaches \( V_N^\epsilon \), and hence in the limit as \( N \to \infty \) \( \hat{f}_N \) may not vanish for \( x \in V_N^\epsilon \) of (13.8). We say that the sequence of functions \( \{\hat{f}_N\} \) converges uniformly to \( \hat{f}_\infty \) in \( V_N^\epsilon \) if all the conditions of the following definition are satisfied.

**Definition 13.1 (uniformly convergent sequence in \( V_N^\epsilon \))**

Let \( \hat{f}_N : V_N^h \to \mathbb{R} \) be a sequence of Borel-measurable integrable functions, where \( V_N^h \) is a closed rectangle of height \( 2h \) and width \( 2d_N = N/(2hn) \), \( n \) is a constant, \( N \) is an integer, and \( V_N^h \) approaches the infinite two-dimensional layer \( \mathbb{R}^2 : |x_N| < h \) as \( N \to \infty \). The sequence of functions \( \{\hat{f}_N\} \) is said to be uniformly convergent in \( V_N^\epsilon \) if there exists a Borel-measurable integrable function \( \hat{f}_\infty : V_N^\epsilon \to \mathbb{R} \) such that for any \( \epsilon > 0 \) and for any positive integer \( T \) one can find an integer \( Q(\epsilon, T) \) so that in the rectangle \( V_T^h \), which corresponds to \( T \), one has

\[
|\hat{f}_N(x) - \hat{f}_\infty(x)| < \epsilon, \quad \text{for all } N \geq Q(\epsilon, T) \text{ and for all } x \in V_T^h.
\]

(13.11)
Since the function $\hat{f}_\infty$ of Definition 13.1 is Borel-measurable and integrable in $V^h_\infty$, we can find for any $\epsilon > 0$ a number $M(\epsilon)$ such that

$$\left| \int_{V^h_\infty \setminus V^h_{d_N}} \hat{f}_\infty(x) \, dx \right| < \epsilon, \quad \text{for all } N > M(\epsilon). \quad (13.12)$$

The region $V^h_\infty \setminus V^h_{d_N}$ of (13.12) is represented by the shaded area in Fig. 13.1. The condition (13.12) implies that the dominant part of the function $\hat{f}_\infty$ is distributed within the rectangle $V^h_{d_M}$.

Along with the condition of uniform convergence of the sequence $\{\hat{f}_N\}$ in $V^h_\infty$, we say that the sequence $\{\hat{f}_N\}$ is a fundamental sequence in $V^h_\infty$ if all the conditions of the following definition are satisfied.
Definition 13.2 (fundamental sequence in $V_{\infty}^h$)

Let $\hat{f}_N : V_{d_N}^h \to \mathbb{R}$ be a sequence of Borel-measurable integrable functions defined for each $N$ in the corresponding rectangular region $V_{d_N}^h$ as in Definition 13.1. This sequence is said to be a fundamental sequence in $V_{\infty}^h$ if

1) the sequence $\{\hat{f}_N\}$ is uniformly convergent in $V_{\infty}^h$ as in Definition 13.1;

2) for any $\epsilon > 0$ there exists an integer $L(\epsilon)$ for which

$$\left| \int_{V_{d_N}^h \setminus V_{d_L}^h} \hat{f}_N(x) \, dx \right| < \epsilon, \text{ for all } N \geq L(\epsilon).$$

(13.13)

The region $V_{d_N}^h \setminus V_{d_L}^h$ of (13.13) is represented by the shaded area in Fig. 13.2. The condition (13.13) implies that the dominant part of the function $\hat{f}_N$, for all $N$ greater than some number $L$, is distributed within the region $V_{d_L}^h$. Equation (13.13) can be replaced by an equivalent equation according to the following lemma.

Lemma 13.1 (alternative condition (2) in Definition 13.2)

Condition (2) in the definition of a fundamental sequence (Definition 13.2) is equivalent to the following condition.

For any $\epsilon > 0$ there exists an integer $P(\epsilon)$ such that

$$\left| \int_{V_{d_N}^h \setminus V_{d_R}^h} \hat{f}_N(x) \, dx \right| < \epsilon, \text{ for all } N \text{ and } R, \text{ where } N \geq R \geq P(\epsilon).$$

(13.14)

Indeed, (13.13) follows from (13.14) by taking $R = P(\epsilon)$ and $L(\epsilon) = P(\epsilon)$. Thus, we have to prove that (13.14) follows from (13.13). From (13.13), one infers that

$$\left| \int_{V_{d_N}^h \setminus V_{d_L}^h} \hat{f}_N(x) \, dx \right| < \frac{\epsilon}{2}, \text{ for all } N \geq L\left(\frac{\epsilon}{2}\right).$$

(13.15)
Now, choose an integer $R$ greater than $L(\epsilon/2)$ and less than some integer $N$. Using the triangular inequality in the region $V_N^h \setminus V_N^k = V_N^h \setminus V_N^k \cup V_N^k \setminus V_N^h$, and the condition (13.15), we infer that

$$\left| \int_{V_N^h \setminus V_N^k} \hat{f}_N(x) \, dx - \int_{V_N^k \setminus V_N^h} \hat{f}_N(x) \, dx \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for all $N$ and $R$, where $N \geq R \geq L(\epsilon/2)$.

Equation (13.16), with $P(\epsilon) = L(\epsilon/2)$ proves the result (13.14) and Lemma 13.1. Let $\{I_N\}$ be the sequence of integrals of the functions $\hat{f}_N$ such that

$$I_N = \int_{V_N^h} \hat{f}_N(x) \, dx.$$  

(13.17)
By Lemma 13.1, for any $\epsilon > 0$, there exists a number $P(\epsilon)$ such that the condition (13.14) can be written in the form

$$|I_N - I_R| < \epsilon, \text{ for all } N \text{ and } R, \text{ where } N, R \geq P(\epsilon). \quad (13.18)$$

Thus, the sequence $\{I_N\}$ is a Cauchy sequence and, consequently, it converges to a limit as $N \to \infty$. We show in the following that the limit of $I_N$ is equal to the integral of the limiting function $\hat{f}_\infty$ over the infinite strip $V^h_\infty$.

**Theorem 13.1 (integral of a fundamental sequence in $V^h_\infty$)**

Let $\hat{f}_N : V^h_{d_N} \to \mathbb{R}$ be a fundamental sequence of functions in $V^h_\infty$, and let $\hat{f}_\infty : V^h_\infty \to \mathbb{R}$ be the limit of this sequence, as in Definition 13.2. Then, the sequence of integrals of the functions $\hat{f}_N$ has a limit $I_\infty$ which is given by

$$I_\infty = \lim_{N \to \infty} \int_{V^h_{d_N}} \hat{f}_N(x) \, dx = \int_{V^h_\infty} \hat{f}_\infty(x) \, dx. \quad (13.19)$$

To prove Theorem 13.1, we proceed as follows. For any $\epsilon > 0$, we need to find a number $K(\epsilon)$ such that for all $N > K(\epsilon)$ one has

$$\left| \int_{V^h_{d_N}} \hat{f}_N(x) \, dx - \int_{V^h_\infty} \hat{f}_\infty(x) \, dx \right| < \epsilon, \text{ for all } N \geq K(\epsilon). \quad (13.20)$$

Choose a number $\epsilon > 0$ and recall that $\hat{f}_\infty$ is Borel-measurable and integrable in $V^h_\infty$. Hence, by (13.12) there exists a number $M(\epsilon)$ such that

$$\left| \int_{V^h_\infty \setminus V^h_{d_N}} \hat{f}_\infty(x) \, dx \right| < \frac{\epsilon}{3}, \text{ for all } N \geq M(\epsilon/3). \quad (13.21)$$

According to the condition (13.14) in Lemma 13.1, which is equivalent to (13.13) of Definition 13.2, there is a number $P(\epsilon)$ such that

$$\left| \int_{V^h_{d_N} \setminus V^h_T} \hat{f}_N(x) \, dx \right| < \frac{\epsilon}{3}, \text{ for all } N \text{ and } T, \text{ where } N \geq T \geq P(\frac{\epsilon}{3}). \quad (13.22)$$
Since in (13.22) $T$ can be any integer such that $T \geq P(\varepsilon/3)$, let $T$ be the greater of the two numbers $M$ and $P$ specified in (13.21) and (13.22). Thus, let

$$T(\varepsilon) = \max \left[ M(\frac{\varepsilon}{3}), P(\frac{\varepsilon}{3}) \right].$$  \hspace{1cm} (13.23)

For $T(\varepsilon)$ specified in (13.23), the inequality (13.22) yields

$$\left| \int_{V^h_{\sigma_T} \setminus V^h_{\sigma_T}} \hat{f}_N(\mathbf{x}) \, d\mathbf{x} \right| < \frac{\varepsilon}{3}, \text{ for all } N \geq T(\varepsilon).$$  \hspace{1cm} (13.24)

Since (13.21) holds for all $N \geq M(\varepsilon/3)$, from equations (13.21) and (13.23) we infer that

$$\left| \int_{V^h_{\sigma_T} \setminus V^h_{\sigma_T}} \hat{f}_\infty(\mathbf{x}) \, d\mathbf{x} \right| < \frac{\varepsilon}{3}.$$  \hspace{1cm} (13.25)

Since the sequence $\{\hat{f}_N\}$ is uniformly convergent in $V^h$, there exists by (13.11) a number $Q(\frac{en}{3T(\varepsilon)}, T(\varepsilon))$ such that

$$|\hat{f}_N(\mathbf{x}) - \hat{f}_\infty(\mathbf{x})| < \frac{en}{3T(\varepsilon)}, \text{ for all } N \geq Q\left(\frac{en}{3T(\varepsilon)}, T(\varepsilon)\right) \text{ and for all } \mathbf{x} \in V^h_{\sigma_T}.$$  \hspace{1cm} (13.26)

We define the number $K(\varepsilon)$ as the maximum of the numbers $T$ in (13.24) and $Q$ in (13.26). Thus, one has

$$K(\varepsilon) = \max \left[ T(\varepsilon), Q\left(\frac{en}{3T(\varepsilon)}, T(\varepsilon)\right) \right].$$  \hspace{1cm} (13.27)

Then, for any arbitrary number $N$ such that $N \geq K(\varepsilon)$, both (13.24) and (13.26) are satisfied. Therefore, using the triangular inequality, together with (13.24)-(13.26),
and recalling that the area of $V^h_{d_T}$ is $4h d_T = T(\epsilon)/n$, one finds that

$$\left| \int_{V^h_{d_T}} \hat{f}_N(x) \, dx - \int_{V^h_{d_T}} \hat{f}_\infty(x) \, dx \right| = \left| \int_{V^h_{d_T} \setminus V^h_{d_T}} \hat{f}_N(x) \, dx + \int_{V^h_{d_T}} \hat{f}_N(x) \, dx \right.$$ 

$$- \int_{V^h_{d_T} \setminus V^h_{d_T}} \hat{f}_\infty(x) \, dx - \int_{V^h_{d_T}} \hat{f}_\infty(x) \, dx \right| \leq$$

$$\left| \int_{V^h_{d_T}} [\hat{f}_N(x) - \hat{f}_\infty(x)] \, dx \right| + \left| \int_{V^h_{d_T} \setminus V^h_{d_T}} \hat{f}_N(x) \, dx \right| + \left| \int_{V^h_{d_T} \setminus V^h_{d_T}} \hat{f}_\infty(x) \, dx \right| < \epsilon,$$

for all $N \geq K(\epsilon)$.  

Equation (13.28) proves Theorem 13.1.

### 14 Probabilistic $N$-crack Problem

Consider an unbounded, linearly elastic, homogeneous, and isotropic solid that contains $N$ non-intersecting parallel cracks, as shown in Fig. 14.1. The cracks lie in planes orthogonal to the $(y_1, y_2)$ plane and extend to infinity in the $\pm y_3$ directions.

The $i$th crack has width $2a$ and the center is located at

$$\zeta^i = \zeta_1^i i + \zeta_2^i j, \quad i = 1, \ldots, N,$$

where $i$ and $j$ are unit vectors along the $y_1$ and $y_2$ axes, respectively.

Let $(x_1^i, x_2^i)$ denote a local coordinate system attached to the $i$th crack, so that the transformation of coordinates from the $(x_1, x_2)$ system to the $(y_1, y_2)$ system is given by (2.2). The angle $\theta^i$, which defines the orientation of the $i$th crack, is the angle between the axes $y_2$ and $x_1$, and is measured in the counterclockwise direction from the $y_2$ axis. Since all the cracks are parallel, let $\theta_0$ be the angle such that

$$\theta^i = \theta_0, \quad 0 \leq \theta_0 < \pi, \quad \text{for } i = 1, \ldots, N.$$
Fig. 14.1 Incident wave on a set of $N$ non-intersecting cracks in an unbounded solid.

Therefore the transformation of coordinates from the $(x^i_1, x^i_2)$ system to the $(y_1, y_2)$ system can be written as

$$
\begin{pmatrix}
  x^i_1 \\
  x^i_2
\end{pmatrix} =
\begin{pmatrix}
  \sin \theta_0 & \cos \theta_0 \\
  -\cos \theta_0 & \sin \theta_0
\end{pmatrix}
\begin{pmatrix}
  y_1 - \zeta^i_1 \\
  y_2 - \zeta^i_2
\end{pmatrix}, \quad x^i_3 = y_3, \quad 0 \leq \theta_0 < \pi. \quad (14.3)
$$

To specify the configuration of $N$ parallel cracks for a given $\theta_0$, we define, as in (4.3), the symbol $A^N$ by

$$
A^N = (\zeta^1, \ldots, \zeta^N). \quad (14.4)
$$

An antiplane wave $u^{\text{inc}}$, as in (2.4), is incident on the cracks along the $y_2$ direction. This wave has the form

$$
u^{\text{inc}}(y_2) = u_0 \exp (iky_2), \quad (14.5)$$

where the wave number $k$ is given by

$$
k = \omega s_T. \quad (14.6)$$
The incident wave and the scattering from the other cracks cause a wave to be scattered by the $i$th crack. This wave is given by (4.5) for \( r^i = (\zeta_1^i, \zeta_2^i, \theta_0) \), $i = 1, \ldots, N$. Since the $N$ cracks are parallel, we omit the presence of \( \theta_0 \) in the list of arguments of the function $u^{sc}_i$, but we keep in mind that for different values of $\theta_0$ the function $u^{sc}_i$ is different. This allows us to write the field $u^{sc}_i$ in the form

$$u^{sc}(x_1^i, x_2^i; \zeta^i | A^N) = \tilde{u}^{sc}(y_1, y_2; \zeta^i | A^N) \equiv u^{sc}_i(y_1, y_2), \quad (14.7)$$

where $(x_1^i, x_2^i)$ denote the coordinates of the point of evaluation in the local coordinate system attached to the crack, $(y_1, y_2)$ denote the coordinates of the same point in the fixed coordinate system, $i$ is the number of the crack scattering the wave, and $A^N$ indicates the dependence of the scattered wave on the configuration of the $N$ parallel cracks. The total field $u^\tau$ in the cracked solid is given by equation (4.6) for $r^i = (\zeta_1^i, \zeta_2^i, \theta_0)$. Thus, using the notation of equation (14.7), we write

$$u^\tau(y_1, y_2 | A^N) = \tilde{u}^{sc}(y_1, y_2; \zeta^i | A^N) + \tilde{u}^{e}(y_1, y_2; \zeta^i | A^N), \quad i = 1, \ldots, N, \quad (14.8)$$

where, as in (4.7), the exciting field $\tilde{u}^{e}$ has the form

$$\tilde{u}^{e}(y_1, y_2; \zeta^i | A^N) = u^{e}(x_1^i, x_2^i; \zeta^i | A^N) \equiv u^{e}_i(y_1, y_2), \quad i = 1, \ldots, N. \quad (14.9)$$

In (14.7) and (14.9) the coordinates $(y_1, y_2)$ and $(x_1^i, x_2^i)$ are related by (14.3).

In terms of the incident field $u^{inc}$ and of the scattered fields $u^{sc}_i$ ($i = 1, \ldots, N$) the exciting field $u^{e}_i$ acting on the $i$th crack is given by (4.8) for $r^i = (\zeta_1^i, \zeta_2^i, \theta_0)$. Thus, using the notation (14.7), we write

$$\tilde{u}^{e}(y_1, y_2; \zeta^i | A^N) = u^{inc}(y_2) + \sum_{j=1, j \neq i}^{N} \tilde{u}^{sc}(y_1, y_2; \zeta^j | A^N), \quad i = 1, \ldots, N. \quad (14.10)$$
Combining (14.10) and (14.8), we can write the total field \( u^T \) in the form

\[
  u^T(y_1, y_2|A^N) = u^\text{inc}(y_2) + \sum_{i=1}^{N} \tilde{u}^\text{sc}(y_1, y_2; \zeta_i|A^N) = \]

\[
  u^\text{inc}(y_2) + \sum_{i=1}^{N} u_i^\text{sc}(y_1, y_2),
\]

which agrees with (4.9) for \( r^i = (\zeta_1^i, \zeta_2^i, \theta_0) \). The scattered field \( \tilde{u}^\text{sc} \) from the crack centered at \( x \in \mathbb{R}^2 \), when the remaining \( (N - 1) \) cracks are centered at \( \zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^{j-1}, y, \zeta^{j+1}, \ldots, \zeta^N \), is given as in (14.7) by

\[
  \tilde{u}^\text{sc}(y_1, y_2; x|\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^{j-1}, y, \zeta^{j+1}, \ldots, \zeta^N). \tag{14.12}
\]

Since the numbering of the cracks is immaterial, we can interchange the \( x \) and \( y \) vectors after the vertical slash and we infer that

\[
  \tilde{u}^\text{sc}(y_1, y_2; x|\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^{j-1}, y, \zeta^{j+1}, \ldots, \zeta^N) = \]

\[
  \tilde{u}^\text{sc}(y_1, y_2; x|\zeta^1, \ldots, \zeta^{i-1}, y, \zeta^{i+1}, \ldots, \zeta^{j-1}, x, \zeta^{j+1}, \ldots, \zeta^N), \tag{14.13}
\]

for any pair of distinct indices \( i, j = 1, \ldots, N \). From this discussion, we infer the following properties of the scattered, exciting, and total fields.

**Properties 14.1**

Let \( \{j_1, \ldots, j_N\} \) denote an arbitrary permutation of \( \{1, \ldots, N\} \). Then,

a) the field \( \tilde{u}^\text{sc} \) scattered by the crack positioned at \( x \in \mathbb{R}^2 \) remains unchanged if the indices in the list of arguments of \( \tilde{u}^\text{sc} \) are reordered after the vertical slash.

Thus, one has

\[
  \tilde{u}^\text{sc}(y_1, y_2; x|\zeta^1, \ldots, \zeta^N) = \tilde{u}^\text{sc}(y_1, y_2; x|\zeta^{j_1}, \ldots, \zeta^{j_N}), \tag{14.14}
\]

where \( x \) denotes one of the \( \zeta^i, i = 1, \ldots, N \);

b) the exciting field \( \tilde{u}^E \) on the crack positioned at \( x \in \mathbb{R}^2 \) and the total field \( u^T \) also remain unchanged if the indices in the list of arguments of \( \tilde{u}^E \) and \( u^T \) are reordered after the vertical slash. Thus, one has
\[ \bar{u}^{\nu}(y_1, y_2; \mathbf{x} | \zeta^1, \ldots, \zeta^N) = \bar{u}^{\nu}(y_1, y_2; \mathbf{x} | \zeta^{j_1}, \ldots, \zeta^{j_N}), \]
\[ \text{where } \mathbf{x} \text{ denotes one of the } \zeta^i, \ i = 1, \ldots, N; \]
\[ u^T(y_1, y_2 | \zeta^1, \ldots, \zeta^N) = u^T(y_1, y_2 | \zeta^{j_1}, \ldots, \zeta^{j_N}). \]  

The properties (14.15) and (14.16) for \( \bar{u}^{\nu} \) and \( u^T \), respectively, follow directly from the property (14.14) for the scattered fields, the expression (14.10) for the exciting fields, and the expression (14.11) for the total field.

We now assume that the configuration of the cracks is known in a probabilistic sense, and that for each possible configuration \( \zeta^1, \ldots, \zeta^N \) of the \( N \) cracks there exists a number \( b > a \) such that the \( N \) circles of radius \( b \) centered at \( \zeta^1, \ldots, \zeta^N \) do not intersect. This restriction on the configuration implies that the \( N \) cracks do not intersect (see Fig. 14.2). The centers \( \zeta^1, \ldots, \zeta^N \) of the \( N \) cracks are assumed to be distributed inside the rectangle \( V_{d_N}^h \) of height \( 2h \) and width \( 2d_N \). The width \( 2d_N \) depends on the number of cracks \( N \), according to (13.7).

We assume that the probability density function \( p \) of the crack distribution inside the rectangle \( V_{d_N}^h \) is uniform, and that the numbering of the cracks is immaterial. By uniformity, we mean that each crack can be centered with equal probability at each point of the rectangle \( V_{d_N}^h \). Thus, the probability density function \( p : \mathbb{R}^{2N} \rightarrow \mathbb{R} \) of the configuration of \( N \) non-intersecting cracks (circles) is given by Definition 12.1 for \( d = d_N \), and the marginal probability density function \( p_i : \mathbb{R}^2 \rightarrow \mathbb{R} \) for the \( i \)th crack is given by
\[
\begin{align*}
p_i(\mathbf{x}) &= \begin{cases} 
\frac{1}{4hd_N}, & \text{if } \mathbf{x} \in V_{d_N}^h, \\
0, & \text{if } \mathbf{x} \notin V_{d_N}^h.
\end{cases} \quad \text{for } i = 1, \ldots, N.
\end{align*}
\]
Fig. 14.2 Collection of $N$ circles surrounding cracks contained in a rectangle.

If $n$ is the constant number of cracks per unit area defined as in (13.2)-(13.3) by

$$n = \frac{N}{4hd_N}, \quad \text{at each } x \text{ in } V_{d_N}^h,$$  \hfill (14.18)

then the marginal probability density function $p_i$ of (14.17) can be written as

$$p_i(x) = \begin{cases} \frac{n}{N}, & \text{if } x \in V_{d_N}^h, \\
0, & \text{if } x \notin V_{d_N}^h, \end{cases} \quad \text{for } i = 1, \ldots, N. \hfill (14.19)$$

The uniform conditional probability density function $p_i(\cdot | x) : \mathbb{R}^{2N-2} \to \mathbb{R}$ of the configuration of $N$ non-intersecting cracks (circles) is given by Definition 12.2 for $d = d_N$. Thus, using (12.8) for $d = d_N$ and (14.18), one has

$$p_i(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^N | x) = \frac{N}{n} p(\zeta^1, \ldots, \zeta^{i-1}, x, \zeta^{i+1}, \ldots, \zeta^N),$$  \hfill (14.20)\quad i = 1, \ldots, N, \quad x \in V_{d_N}^h.$$

In order to define the configurational and the partial average of the total field, we adopt the following convention.
Convention 14.1 If the probability density function $p$ for the position vectors $\zeta^1, \ldots, \zeta^N$ vanishes, then we make the convention that the solid is uncracked and that the conditional probability density function $p_i(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^N|\mathbf{x})$ for $i = 1, \ldots, N$ vanishes. Hence, the scattered fields $u_i^s$ vanish, and it follows from the expressions for the exciting field (14.10) and the total field (14.11) that the exciting fields $u_i^e$ and the total field $u^r$ are equal to the incident field. Thus, one has

$$p(\zeta^1, \ldots, \zeta^N) = 0 \Rightarrow u^r = u_i^e = u_i^s = 0, \quad \text{for } i = 1, \ldots, N; \quad (14.21)$$

$$p(\zeta^1, \ldots, \zeta^{i-1}, \mathbf{x}, \zeta^{i+1}, \ldots, \zeta^N) = 0 \Rightarrow p_i(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^N|\mathbf{x}) = 0,$$

$$\quad \text{for } i = 1, \ldots, N, \text{ and } \mathbf{x} \in \mathbb{R}^2. \quad (14.22)$$

Convention 14.1 can be used to define the uniform conditional probability density function $p_i(\cdot|\mathbf{x})$ for $\mathbf{x} \notin V_{d_N}^h$ in the case of a uniform density of cracks. Since, by (12.1), $p(\zeta^1, \ldots, \zeta^{i-1}, \mathbf{x}, \zeta^{i+1}, \ldots, \zeta^N)$ vanishes when $\mathbf{x} \notin V_{d_N}^h$, it follows from (14.22) that the uniform conditional probability density function $p_i(\cdot|\mathbf{x})$ vanishes when $\mathbf{x} \notin V_{d_N}^h$.

From this discussion, we infer that the result (14.20) can be extended to the entire plane $\mathbb{R}^2$, and we write

$$p_i(\zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^N|\mathbf{x}) = \frac{N}{n} p(\zeta^1, \ldots, \zeta^{i-1}, \mathbf{x}, \zeta^{i+1}, \ldots, \zeta^N), \quad (14.23)$$

$$\quad i = 1, \ldots, N, \quad \mathbf{x} \in \mathbb{R}^2.$$

For a given configuration $\zeta^1, \ldots, \zeta^N$ of non-intersecting cracks, the scattered field $u_i^s$ is continuous everywhere in $\mathbb{R}^2$ except on the faces of the $i$th crack, where it has a finite discontinuity. Hence, from the expression (14.10), we infer that the exciting field $u_i^e$ on the $i$th crack is continuous across the faces of the $i$th crack, and is discontinuous across the faces of the remaining $(N-1)$ cracks. Further, we infer from the expression
(14.11) that the total field \( u^T \) is continuous everywhere in \( \mathbb{R}^2 \) except on the faces of the \( N \) cracks.

This discussion together with (14.21) of Convention 14.1 implies that the functions \( u^i_{sc} \), \( u^i_{r} \), and \( u^T \) for a fixed observation point \((y_1, y_2)\) are Borel-measurable in \( \mathbb{R}^{2N} \). Since the uniform probability density function \( p \) is Borel-measurable in \( \mathbb{R}^{2N} \), it follows that the product functions \( u^i_{sc}p \), \( u^i_{r}p \), and \( u^T p \) are Borel-measurable in \( \mathbb{R}^{2N} \). Further, by Convention 14.1, these product functions vanish when \( p \) vanishes.

Next, we consider equation (14.11), which defines the total field as the sum of the incident field and the scattered fields from the \( N \) cracks. We multiply each term in (14.11) by the uniform probability density function \( p : \mathbb{R}^{2N} \rightarrow \mathbb{R} \) of (14.17) - (14.23) and we integrate over the region \( \mathbb{R}^{2N} \). Then, for each point \((y_1, y_2)\), one finds that

\[
< u^T >_N (y_1, y_2) = u^{inc}(y_2) + \sum_{i=1}^{N} < \tilde{u}^i_{sc} >^i_N (y_1, y_2),
\]

where \( < u^T >_N \) and \( < \tilde{u}^i_{sc} >^i_N \) are the configurational averages of the fields \( u^T \) and \( u^i_{sc} \), respectively. By (10.14) of Definition 10.5, one has for the total field

\[
< u^T >_N (y_1, y_2) = \int_{\mathbb{R}^{2N}} u^T(y_1, y_2; \zeta^1, \ldots, \zeta^N) p(\zeta^1, \ldots, \zeta^N) \, d\zeta^1 \ldots d\zeta^N.
\]

Using Fubini's theorem (Theorem 9.1) for the scattered field, one has

\[
< \tilde{u}^i_{sc} >^i_N (y_1, y_2) = \int_{\mathbb{R}^{2N}} \tilde{u}^i_{sc}(y_1, y_2; \zeta^1, \ldots, \zeta^N) p(\zeta^1, \ldots, \zeta^N) \, d\zeta^1 \ldots d\zeta^N
\]

\[
= \frac{n}{N} \int_{\mathbb{R}^2} < \tilde{u}^i_{sc} >^i_{N-1} (y_1, y_2; \zeta) \, d\zeta, \quad i = 1, \ldots, N.
\]

In (14.26), \( < \tilde{u}^i_{sc} >^i_{N-1} \) is the partial average of the scattered field \( u^i_{sc} \) with the position \( \zeta \) of the \( i \)th crack held fixed, and is given by

\[
< \tilde{u}^i_{sc} >^i_{N-1} (y_1, y_2; \zeta) = \frac{N}{n} \int_{\mathbb{R}^{2(N-2)}} \tilde{u}^i_{sc}(y_1, y_2; \zeta^1, \ldots, \zeta^{i-1}; \zeta, \zeta^{i+1}, \ldots, \zeta^N) p(\zeta^1, \ldots, \zeta^{i-1}, \zeta, \zeta^{i+1}, \ldots, \zeta^N) \, d\zeta^1 \ldots d\zeta^{i-1} d\zeta^{i+1} \ldots d\zeta^N.
\]

(14.27)
If in (14.27) we make the following change of integration variables

\[(\zeta^1, \ldots, \zeta^i, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^N) \to (\zeta^1, \ldots, \zeta^{j-1}, \zeta^{j+1}, \ldots, \zeta^i, \ldots, \zeta^N),\]  

(14.28)

and recall the property that the density function \(p\) is unchanged when the indices are reordered, together with (14.16) of Properties 14.1 that allows one to reorder the position vectors after the slash in the scattered field \(\bar{u}^{sc}\), then using Fubini's theorem (Theorem 9.1) we find that

\[\langle \bar{u}^{sc} \rangle_{N-1}^i (y_1, y_2; \zeta) = \langle \bar{u}^{sc} \rangle_{N-1}^j (y_1, y_2; \zeta), \text{ for } i, j = 1, \ldots, N.\]  

(14.29)

In view of the result (14.29) and equation (14.26) that defines \(\langle \bar{u}^{sc} \rangle_{N-1}^i\) in terms of \(\langle \bar{u}^{sc} \rangle_{N}^i\), we write

\[
\begin{align*}
\langle \bar{u}^{sc} \rangle_{N-1} (y_1, y_2; \zeta) & \equiv \langle \bar{u}^{sc} \rangle_{N-1}^i (y_1, y_2; \zeta), \text{ for all } i = 1, \ldots, N, \\
\langle \bar{u}^{sc} \rangle_{N} (y_1, y_2) & \equiv \langle \bar{u}^{sc} \rangle_{N}^i (y_1, y_2), \text{ for all } i = 1, \ldots, N.
\end{align*}
\]

(14.30)

(14.31)

Since \(p\) vanishes for \(\zeta \not\in V_{dN}^h\), one infers from (14.27) that

\[\langle \bar{u}^{sc} \rangle_{N-1} (y_1, y_2; \zeta) = 0, \text{ for } \zeta \not\in V_{dN}^h.\]  

(14.32)

Since the configurational average of the scattered field is independent of \(i\) by (14.31), the field \(\langle u^T \rangle_N (y_1, y_2)\) of (14.24) can be written in the form

\[\langle u^T \rangle_N (y_1, y_2) = u^{inc}(y_2) + N \langle \bar{u}^{sc} \rangle_N (y_1, y_2).\]  

(14.33)

By substituting the configurational average of the scattered field (14.26) into (14.33), and using (14.32) which says that \(\langle \bar{u}^{sc} \rangle_{N-1} (y_1, y_2; \zeta)\) vanishes for \(\zeta \not\in V_{dN}^h\), one finds that the configurational average of the total field can be written as

\[\langle u^T \rangle_N (y_1, y_2) = u^{inc}(y_2) + n \int_{V_{dN}^h} \langle \bar{u}^{sc} \rangle_{N-1} (y_1, y_2; \zeta) d\zeta.\]  

(14.34)
15 Probabilistic Problem in a Cracked Layer

In this section, we consider the limit of the \( N \)-crack problem as the number of cracks \( N \) tends to infinity, while the crack density \( n \) remains constant (see Fig. 15.1). In the limit, the finite cracked rectangle \( V_{4N}^h \) is transformed into the infinite strip

\[
V^h_\infty = \{ \zeta \in \mathbb{R}^2 \mid |\zeta_2| < h \}.
\]  

(15.1)

The configurational average \(< u^T >_N \) of the total field \( u^T \) for the \( N \)-crack probabilistic problem is given by (14.33) or (14.34). On physical ground, we assume that the limit of \(< u^T >_N \), as \( N \) increases to infinity and the density \( n \) remains constant, exists and is Borel-measurable. This is a reasonable assumption, since for each finite value of \( N \) the average \(< u^T >_N \) can be calculated as in (14.25) and, as \( N \) approaches infinity with the density \( n \) kept constant, the values \(< u^T >_N \) should converge. Likewise, we assume that the limit of \(< \bar{u}^{sc} >_{N-1} \) as \( N \) approaches infinity exists and is Borel-measurable. In addition, we assume that the limit of \(< \bar{u}^{sc} >_{N-1} \) is reached uniformly in the sense of Definition 13.1, and that the integral of \(< \bar{u}^{sc} >_{N-1} \) over the finite rectangle \( V_{4N}^h \) has a limit as \( N \) approaches infinity. We summarize these assumptions as follows.

**Assumptions (limit of the wave motion)**

1) \( \lim_{N \to \infty} < u^T >_N (y_1, y_2) = < u^T >_\infty (y_1, y_2), \)

where \(< u^T >_\infty \) is Borel-measurable;

2) \( \lim_{N \to \infty} < \bar{u}^{sc} >_{N-1} (y_1, y_2; \zeta) = < \bar{u}^{sc} >_\infty (y_1, y_2; \zeta), \)

where \(< \bar{u}^{sc} >_\infty \) is Borel-measurable.
Fig. 15.1 Collection of an infinite number of circles surrounding cracks contained in an infinite strip.

3) the limit \( \bar u^{sc} \) is reached uniformly; thus, for any \( \epsilon > 0 \) and any positive integer \( T \) we can find an integer \( Q(\epsilon, T, y_1, y_2) \) so that in the rectangle \( V^h_{d_2} \), which corresponds to \( T \), one has

\[
| < \bar u^{sc} >_{N-1} (y_1, y_2; \zeta) - < \bar u^{sc} >_{\infty} (y_1, y_2; \zeta) | < \epsilon,
\]

for all \( N \geq Q(\epsilon, T, y_1, y_2) \) and for all \( \zeta \in V^h_{d_2} \). \hspace{1cm} (15.4)

4) \(
\lim_{N \to \infty} \int_{V^h_{d_N}} < \bar u^{sc} >_{N-1} (y_1, y_2; \zeta) d\zeta \) exists. \hspace{1cm} (15.5)

By using Theorem 13.1, we now infer from (15.3) - (15.5) that the limit of the sequence of integrals in (15.5) is given by

\[
\lim_{N \to \infty} \int_{V^h_{d_N}} < \bar u^{sc} >_{N-1} (y_1, y_2; \zeta) d\zeta = \int_{V^h_{d_2}} < \bar u^{sc} >_{\infty} (y_1, y_2; \zeta) d\zeta.
\] \hspace{1cm} (15.6)

Observe that the assumption (15.2) is a consequence of the assumption (15.5), of equation (14.34), and also of the assumption that the density \( n \) remains constant. Thus, the assumption (15.2) is not an independent assumption. In fact, using (14.34)
and (15.6), one finds that
\[ < u^T >_\infty (y_1, y_2) = u^{inc}(y_2) + n \int_{V^h_\infty} < \tilde{u}^{sc} >_\infty (y_1, y_2; \zeta) \, d\zeta. \]  \hfill (15.7)

We now use the symmetry of the problem. In the limit of an infinite strip, the total field \( < u^T >_\infty (y_1, y_2) \) evaluated at \((y_1, y_2)\) is identical to the field \( < u^T >_\infty (y_1 + q, y_2) \) evaluated at \((y_1 + q, y_2)\), where \( q \) is any real number. Thus, one has
\[ < u^T >_\infty (y_1, y_2) = < u^T >_\infty (y_1 + q, y_2), \quad \text{for all } q \in \mathbb{R}. \]  \hfill (15.8)

Also, the scattered field \( < \tilde{u}^{sc} >_\infty (y_1, y_2; \zeta) \) evaluated at \((y_1, y_2)\) when the crack is fixed at \( \zeta \) is identical to the field \( < \tilde{u}^{sc} >_\infty (y_1 + q, y_2; \zeta + q \imath) \) evaluated at \((y_1 + q, y_2)\) when the crack is fixed at \( \zeta + q \imath \), where \( q \) is any real number. Thus, one has
\[ < \tilde{u}^{sc} >_\infty (y_1, y_2; \zeta) = < \tilde{u}^{sc} >_\infty (y_1 + q, y_2; \zeta + q \imath), \quad \text{for all } q \in \mathbb{R}. \]  \hfill (15.9)

In view of the results (15.8) and (15.9) we choose \( q = -y_1 \) and we adopt the following notations
\[ < u^T >_\infty (y_2) \equiv < u^T >_\infty (y_1, y_2), \]  \hfill (15.10)
\[ < \tilde{u}^{sc} >_\infty (y_2; \zeta - y_1 \imath) \equiv < \tilde{u}^{sc} >_\infty (y_1, y_2; \zeta). \]  \hfill (15.11)

It follows from (15.11), after performing a change of integration variables along the \( y_1 \) direction of the unbounded rectangle \( V^h_\infty \), that the integral in (15.7) can be written in the form
\[ \int_{V^h_\infty} < \tilde{u}^{sc} >_\infty (y_1, y_2; \zeta) \, d\zeta = \int_{V^h_\infty} < \tilde{u}^{sc} >_\infty (y_2; \zeta - y_1 \imath) \, d\zeta = \int_{V^h_\infty} < \tilde{u}^{sc} >_\infty (y_2; \zeta) \, d\zeta. \]  \hfill (15.12)

Finally, we combine (15.7), (15.10), and (15.12) to infer that the limit of (14.34) as \( N \) approaches infinity is given by
\[ < u^T >_\infty (y_2) = u^{inc}(y_2) + n \int_{V^h_\infty} < \tilde{u}^{sc} >_\infty (y_2; \zeta) \, d\zeta. \]  \hfill (15.13)
16 Probabilistic Equation of Motion for \( N \) Cracks

In this section, we consider the average of the governing equation for the total field in the solid when the incident wave is an antiplane wave of the type (2.4) and when the solid contains \( N \) cracks uniformly distributed inside the rectangle \( V_{dN}^h \). The cracks are parallel with an angle \( \theta_0 \) measured from the \( y_2 \) axis, as in Section 15, and the uniform distribution is constructed as in Section 14. We recall that the crack centers occupy the \( N \) positions \( \zeta^1, \ldots, \zeta^N \), and we use the notation

\[
A^N = (\zeta^1, \ldots, \zeta^N).
\]

The deterministic total field \( u^T \) in the cracked plane \( \mathbb{R}^2 \) is given by equation (6.9). For convenience, we rewrite (6.9) in the form

\[
u^T_{\alpha\alpha}(y_1, y_2|A^N) + k^2 u^T(y_1, y_2|A^N) = 2 \sum_{i=1}^{N} \delta'(x_2^i) u^{sc}(x_1^i, 0^+; \zeta^i|A^N).
\]

The scattered field \( u^{sc} \) in the right-hand side of (16.2) is determined by the differential equation (6.7) and the boundary condition (6.2). We write (6.7) in the form

\[
u^{sc}_{\alpha\alpha}(y_1, y_2; \zeta^i|A^N) + k^2 u^{sc}(y_1, y_2; \zeta^i|A^N) = 2 \delta'(x_2^i) u^{sc}(x_1^i, 0^+; \zeta^i|A^N),
\]

\((y_1, y_2) \in \mathbb{R}^2.\)

In (16.3) the field \( u^{sc} \) is defined as in (14.7) by

\[
u^{sc}(x_1^i, x_2^i; \zeta^i|A^N) = \tilde{u}^{sc}(y_1, y_2; \zeta^i|A^N),
\]

where the transformation from the \( (x_1^i, x_2^i) \) coordinate system to the \( (y_1, y_2) \) coordinate system is given by (14.3). Since the scattered field corresponding to the \( i \)th crack is antisymmetric with respect to the plane of the crack, it must vanish along the axis
\( x_2^i = 0 \) excluding the crack faces. In fact, we can see that (6.6) and (4.12) yield
\[
\begin{align*}
\text{\( u^{sc}(x_1^i, 0^+; \zeta^i | A^N) = 0, \quad \text{for } |x_1^i| > a. \) (16.5)}
\end{align*}
\]

The boundary condition (6.2) on the faces of the \( i \)th crack, which is positioned at \( \zeta^i \), has the form
\[
\begin{align*}
\frac{\partial}{\partial x_2} u^{sc}(x_1^i, x_2^i; \zeta^i | A^N) \bigg|_{x_2^i = 0^+} = -\frac{\partial}{\partial x_2} u^p(x_1^i, x_2^i; \zeta^i | A^N) \bigg|_{x_2^i = 0},
\end{align*}
\]
where \( u^p \) is given, according to (14.9) and (14.10), by
\[
\begin{align*}
u^p(x_1^i, x_2^i; \zeta^i | A^N) = \bar{u}(y_1, y_2, \zeta^i | A^N) = u^p(y_1, y_2 | A^N) - \bar{u}^{sc}(y_1, y_2; \zeta^i | A^N) = \
\text{\( u^{inc}(y_2) \) + \sum_{j=1, j \neq i}^{N} \bar{u}^{sc}(y_1, y_2; \zeta^i | A^N). \) (16.7)}
\end{align*}
\]

Now we take the average, according to the procedure defined by (14.25), of the governing equation (16.2) for the total field \( u^T \). The total average of the right-hand side term in equation (16.2) is evaluated by using the transformation of coordinates (14.3), the uniform conditional probability density function (14.23), and Fubini’s theorem (Theorem 10.1). The result is
\[
\begin{align*}
\int_{R^2} \delta'(x_2^i) u^{sc}(x_1^i, 0^+; \zeta^i | A^N) p(\zeta^1, \ldots, \zeta^N) \, d\zeta^1 \ldots d\zeta^N = \\
\frac{n}{N} \int_{R^2} \delta'(x_2^i) < u^{sc} >_{N-1} (x_1^i, 0^+; \zeta^i) \, d\zeta^i, \quad (16.8)
\end{align*}
\]
where
\[
\begin{align*}
x_1^i &= \sin \theta_0(y_1 - \zeta_1^i) + \cos \theta_0(y_2 - \zeta_2^i), \\
x_2^i &= -\cos \theta_0(y_1 - \zeta_1^i) + \sin \theta_0(y_2 - \zeta_2^i). \quad (16.9)
\end{align*}
\]
and the partial average \( < u^sc >_{N-1}^i \) of the scattered field \( u^sc \) with the position \( \zeta^i \) of the \( i \)th crack held fixed is given by

\[
< u^sc >_{N-1}^i (x_1^i, 0^+; \zeta^i) = \frac{N}{n} \int_{R^{2N-2}} u^sc(x_1^i, 0^+; \zeta^i|\mathbf{A}^N)p(\zeta^1, \ldots, \zeta^N) \, d\zeta^1 \ldots d\zeta^{i-1} d\zeta^{i+1} \ldots d\zeta^N. \tag{16.11}
\]

Now, we remove the superscripts in equations (16.9) and (16.10), and we define the positions \( x_1 \) and \( x_2 \) by

\[
x_1 = \sin \theta_0(y_1 - \zeta_1) + \cos \theta_0(y_2 - \zeta_2), \tag{16.12}
\]

\[
x_2 = -\cos \theta_0(y_1 - \zeta_1) + \sin \theta_0(y_2 - \zeta_2). \tag{16.13}
\]

Then, replacing the integration variable \( \zeta^i = (\zeta_1^i, \zeta_2^i) \) in (16.8) with \( \zeta = (\zeta_1, \zeta_2) \), and using the definitions (16.12) and (16.13), we can write

\[
\int_{R^{2N}} \delta'(x_2^i) u^sc(x_1^i, 0^+; \zeta^i|\mathbf{A}^N)p(\zeta^1, \ldots, \zeta^N) \, d\zeta^1 \ldots d\zeta^N = \frac{n}{N} \int_{R^2} \delta'(x_2^i) < u^sc >_{N-1}^i (x_1^i, 0^+; \zeta^i) \, d\zeta. \tag{16.14}
\]

In (16.14), the partial average \( < u^sc >_{N-1}^i \) is given by (16.11). For clarity, we rewrite this expression in the form

\[
< u^sc >_{N-1}^i (x_1, 0^+; \zeta) = \frac{N}{n} \int_{R^{2N-2}} u^sc(x_1, 0^+; \zeta|\mathbf{A}^N)p(\zeta^1, \ldots, \zeta^{i-1}, \zeta, \zeta^{i+1}, \ldots, \zeta^N) \, d\zeta^1 \ldots d\zeta^{i-1} d\zeta^{i+1} \ldots d\zeta^N. \tag{16.15}
\]

Using the expression (16.15), which is identical in form to (14.27), we can show by using the procedure outlined between (14.28) and (14.30) that the partial average \( < u^sc >_{N-1}^i \) is independent of the choice of the index \( i \). Thus, we now write

\[
< u^sc >_{N-1}^i (x_1, 0^+; \zeta) \equiv < u^sc >_{N-1}^i (x_1, 0^+; \zeta), \quad \text{for all } i = 1, \ldots, N. \tag{16.16}
\]
Substituting (16.5) into (16.15), we find that the partial average \( < u^s >_{N-1}^{i} \) vanishes on the axis \( x_2 = 0 \), excluding the crack faces. Thus, using the notation (16.16), one has

\[
< u^s >_{N-1}^{i} (x_1, 0^+; \zeta) = 0, \quad \text{for} \ |x_1| > a. \tag{16.17}
\]

Also, since the probability density function \( p \) in (16.15) vanishes for \( \zeta \notin V^h_{dn} \), we infer that

\[
< u^s >_{N-1}^{i} (x_1, x_2; \zeta) = 0, \quad \text{for} \ \zeta \notin V^h_{dn}. \tag{16.18}
\]

Assuming that it is permissible to change the orders of integration and differentiation, we infer by taking the total average of (16.2), and using (16.14) and (16.16), that the governing equation for \( < u^T >_N \) is given by

\[
< u^T >_{N,\alpha\alpha} (y_1, y_2) + k^2 < u^T >_N (y_1, y_2) = 2n \int_{\mathbb{R}^2} \delta'(x_2) < u^s >_{N-1}^{i} (x_1, 0^+; \zeta) d\zeta. \tag{16.19}
\]

In (16.19), the scattered field in the \((x_1, x_2)\) coordinate system is related to the scattered field in the \((y_1, y_2)\) coordinate system by the relation

\[
< u^s >_{N-1}^{i} (x_1, x_2; \zeta) = < \bar{u}^s >_{N-1}^{i} (y_1, y_2; \zeta), \tag{16.20}
\]

where the transformation from the \((x_1, x_2)\) coordinate system to the \((y_1, y_2)\) coordinate system is given by (16.12) and (16.13). The result (16.20) follows directly from the definition (16.15) of the partial average and from the definition (16.4) of the function \( u^s \) in terms of the function \( \bar{u}^s \).

To find the field \( < u^s >_{N-1} \), which appears in (16.19), we consider the partial average of the governing equation (16.3) for the scattered field \( u^s_i \) with the position \( \zeta^i \) of the \( i \)th crack held fixed. Assuming the legitimacy of interchanging the orders of
integration and differentiation in the left-hand side of equation (16.3), we infer that
the governing equation for the field \( < u^{sc} >_{N-1} \) can be written in the form
\[
< \tilde{u}^{sc} >_{N-1,\alpha\alpha} (y_1, y_2; \zeta^i) + k^2 < u^{sc} >_{N-1} (y_1, y_2; \zeta^i) = 2 \delta'(x^i_2) < u^{sc} >_{N-1} (x^i_1, 0^+; \zeta^i), \tag{16.21}
\]
i = 1, \ldots, N.

In (16.21), we have used the notation (14.30) for the average \( < \tilde{u}^{sc} >_{N-1} \) together
with the definition (16.20) of the field \( < u^{sc} >_{N-1} \). Observe that the definition of the
field \( u^{sc} \) in terms of \( \tilde{u}^{sc} \) in (16.4), and the transformation of coordinates (16.9) and
(16.10), imply that the Laplacian of \( u^{sc} \) with respect to \( (x^i_1, x^i_2) \) and that of \( \tilde{u}^{sc} \) with
respect to \( (y_1, y_2) \) are identical. Thus, one can write
\[
\tilde{u}^{sc}_{\alpha\alpha}(y_1, y_2; \zeta^i|A^N) = u^{sc}_{\gamma\gamma}(x^i_1, x^i_2; \zeta^i|A^N), \tag{16.22}
\]
where the differentiations with respect to \( (y_1, y_2) \) are denoted by an index \( \alpha \), and
those with respect to \( (x^i_1, x^i_2) \) by an index \( \gamma \). Taking the partial average of (16.22),
and assuming that the orders of integration and differentiation can be interchanged,
one infers that the Laplacians of \( < \tilde{u}^{sc} >_{N-1} \) and \( < u^{sc} >_{N-1} \) are equal. Thus, one has
\[
< \tilde{u}^{sc} >_{N-1,\alpha\alpha} (y_1, y_2; \zeta^i) =< u^{sc} >_{N-1,\gamma\gamma} (x^i_1, x^i_2; \zeta^i). \tag{16.23}
\]
It follows from (16.23) and (16.4) that the differential equation (16.21) can be written
in the equivalent form
\[
< u^{sc} >_{N-1,\gamma\gamma} (x^i_1, x^i_2; \zeta^i) + k^2 < u^{sc} >_{N-1} (x^i_1, x^i_2; \zeta^i) =
2 \delta'(x^i_2) < u^{sc} >_{N-1} (x^i_1, 0^+; \zeta^i), \tag{16.24}
i = 1, \ldots, N.
\]

The boundary condition for \( < \tilde{u}^{sc} >_{N-1} \) is obtained by taking the partial average
of the boundary condition (16.6) with the position \( \zeta^i \) of the \( i \)th crack held fixed.
By (16.7), the exciting field $u^e_i$ is expressed in terms of the total field and the field scattered from the $i$th crack. Thus, the partial average $<u^e>_N^{-1}$ is independent of the index $i$, as in the result (16.16) for the scattered field. Assuming the legitimacy of interchanging the orders of integration and differentiation, one finds that (16.6) yields

$$
\frac{\partial}{\partial x_2} <u^{sc}>_{N-1}(x_1^i, x_2^i; \zeta^i) \bigg|_{x_2^i = 0^+} = -\frac{\partial}{\partial x_2^i} <u^e>_{N-1}(x_1^i, x_2^i; \zeta^i) \bigg|_{x_2^i = 0^+},
$$

$$
|x_1^i| < a, \quad i = 1, \ldots, N.
$$

(16.25)

In (16.25) the field $<u^e>_N^{-1}$, according to (16.7) and the definition given by equation (14.27), can be written in the form

$$
<u^e>_N^{-1}(x_1^i, x_2^i; \zeta^i) = <\bar{u}^e>_N^{-1}(y_1, y_2; \zeta^i) - u^{inc}(y_2|\zeta^i) + (N-1) <\bar{u}^{sc}>_{N-1}(y_1, y_2||\zeta^i),
$$

(16.26)

where $<u^e>_N^{-1}$ is the partial average of the total field with the position $\zeta^i$ of the $i$th crack held fixed. The field $u^{inc}(y_2|\zeta^i)$ is defined by

$$
u^{inc}(y_2|\zeta^i) = \begin{cases} u^{inc}(y_2), & \text{if } \zeta^i \in V^h_{dN}, \\ 0, & \text{if } \zeta^i \notin V^h_{dN}. \end{cases}
$$

(16.27)

In (16.26), the field $<\bar{u}^{sc}>_{N-1}(y_1, y_2||\zeta^i)$ is defined by

$$
<\bar{u}^{sc}>_{N-1}(y_1, y_2||\zeta^i) \equiv <\bar{u}^{sc}>_{N-1}^{-1}(y_1, y_2||\zeta^i) = \frac{N}{n} \int_{R_{2N-2}} \bar{u}^{sc}(y_1, y_2; \zeta^1, \ldots, \zeta^{i-1}, \zeta^i, \zeta^{i+1}, \ldots, \zeta^N) p(\zeta^1, \ldots, \zeta^{i-1}, \zeta^i, \zeta^{i+1}, \ldots, \zeta^N) d\zeta^1 \ldots d\zeta^{i-1} d\zeta^{i+1} \ldots d\zeta^N,
$$

$$
j = 1, \ldots, i - 1, i + 1, \ldots, N, \text{ and } i = 1, \ldots, N.
$$

(16.28)
We show next that the right-hand side of (16.28) is independent of the values of \( i \) and of the values of \( j \) for a given \( i \). Indeed, since the integration in (16.28) is performed over the variables \( \zeta^1, \ldots, \zeta^{i-1}, \zeta^{i+1}, \ldots, \zeta^N \), and \( \zeta^j \) is one of these \( N - 1 \) integration variables, it follows that \( \zeta^j \) is a dummy variable. Further, using the exchangeability condition (12.3) for the probability density function \( p \), the property (14.14) that allows us to reorder the position vectors after the slash in \( \bar{u}^{sc} \), and Fubini's theorem (Theorem 9.1), we conclude that the right-hand side of (16.28) for a given \( i \) is independent of the value of \( j \).

Also, using the exchangeability condition (12.3) for the probability density function \( p \), the property (14.14) that allows us to reorder the position vectors after the slash in \( \bar{u}^{sc} \), and Fubini's theorem (Theorem 9.1), we infer that the right-hand side of (16.28) is independent of the value of \( i \). The preceding argument is analogous to that of (14.28) and (14.29).

It follows from the independence of the right-hand side of (16.28) on the value of \( j \) that the partial averages of the \( (N - 1) \) terms in the summation of (16.7) are all identical. This yields the \( (N - 1) \) factor in (16.26).

17 Probabilistic Equation of Motion for a Cracked Layer

In this section, we consider the limit of the governing equation (16.19) as the number of cracks \( N \) tends to infinity, and the bounded region \( V^k_{dN} \) approaches the layer \( V^k_{\infty} \). First, we recall the assumptions (15.2) and (15.3) on the limit of the wave motion. According to these assumptions, the averages \( < u^T >_N \) and \( < \bar{u}^{sc} >_{N-1} \)
converge as $N$ approaches infinity, and one has

$$< u^T >_N (y_1, y_2) = \lim_{N \to \infty} < u^T >_N (y_1, y_2),$$

$$< \bar{u}^{sc} >_N (y_1, y_2; \zeta) = \lim_{N \to \infty} < \bar{u}^{sc} >_{N-1} (y_1, y_2; \zeta).$$

Next, we define the average $< u^{sc} >_\infty$ in terms of $< \bar{u}^{sc} >_\infty$ by taking the limit of (16.20) and using (17.1). Thus, one has

$$< u^{sc} >_\infty (x_1, x_2; \zeta) = \lim_{N \to \infty} < u^{sc} >_{N-1} (x_1, x_2; \zeta) = < \bar{u}^{sc} >_\infty (y_1, y_2; \zeta),$$

where the transformation of coordinates between $(x_1, x_2)$ and $(y_1, y_2)$ is given by (16.12) and (16.13). Now, taking the limit of (16.17) and (16.18), and using the definition (17.3) for $< u^{sc} >_\infty$, we infer that

$$< u^{sc} >_\infty (x_1, 0^+; \zeta) = 0, \quad \text{for } |x_1| > a,$$

$$< u^{sc} >_\infty (x_1, x_2; \zeta) = 0, \quad \text{for } \zeta \notin V^h.$$ 

Next, we recall the symmetry properties of the averages $< u^T >_\infty$ and $< \bar{u}^{sc} >_\infty$. It was shown in (15.10) and (15.11) that $< u^T >_\infty$ is a function of $y_2$ only, and $< \bar{u}^{sc} >_\infty$ depends only on $y_2$, provided that the position of the fixed crack relative to which the average is taken is translated along the $y_1$ axis together with the observation point. Thus, one has

$$< u^T >_\infty (y_2) \equiv < u^T >_\infty (y_1, y_2),$$

$$< \bar{u}^{sc} >_\infty (y_2; \zeta - y_1) \equiv < \bar{u}^{sc} >_\infty (y_1, y_2; \zeta).$$

Now, we show that the field $< u^{sc} >_\infty$ in (17.3) is independent of $\zeta_1$. Indeed, by using the transformation of coordinates given by (16.12)-(16.13) and the property of
\[ < u^{sc} >_{\infty} (x_1, x_2; \zeta) = < \tilde{u}^{sc} >_{\infty} (\zeta_1 + q + x_1 \sin \theta_0 - x_2 \cos \theta_0, \zeta_2 + x_1 \cos \theta_0 + x_2 \sin \theta_0; \zeta + q, \theta_0), \]

for all \( q \in \mathbb{R} \).

For \( q = -\zeta_1 \), (17.8) yields

\[ < u^{sc} >_{\infty} (x_1, x_2; \zeta) = < \tilde{u}^{sc} >_{\infty} (x_1 \sin \theta_0 - x_2 \cos \theta_0, \zeta_2 + x_1 \cos \theta_0 + x_2 \sin \theta_0; \zeta_2). \]

(17.9)

Since \( \zeta_1 \) does not appear in the right-hand side of (17.9), we adopt the following notation

\[ < u^{sc} >_{\infty} (x_1, x_2; \zeta_2) \equiv < u^{sc} >_{\infty} (x_1, x_2; \zeta). \]

(17.10)

Now, we examine the limit of the differential equation (16.19) for the average total field \( < u^T >_N \). Assuming that the limit operation can be interchanged with the differentiations in the left-hand side, and with the integration in the right-hand side, and using the symmetry properties (17.6) and (17.10), we infer that (16.19) yields

\[ < u^T >_{\infty, 22} (y_2) + k^2 < u^T >_{\infty} (y_2) = 
2n \int_{\mathbb{R}^2} \mathcal{B}'(x_2) < u^{sc} >_{\infty} (x_1, 0^+; \zeta_2) d\zeta_1 d\zeta_2, \]

(17.11)

where \( x_1 \) and \( x_2 \) are related to \( \zeta_1 \) and \( \zeta_2 \) by (16.12) and (16.13). It is convenient in (17.11) to perform the change of integration variables from \((\zeta_1, \zeta_2)\) to \((\bar{\zeta}_1, \bar{\zeta}_2)\) such that

\[ \zeta_1 = y_1 - \bar{\zeta}_1 \sin \theta_0 + \bar{\zeta}_2 \cos \theta_0, \]

(17.12)

\[ \zeta_2 = y_2 - \bar{\zeta}_1 \cos \theta_0 - \bar{\zeta}_2 \sin \theta_0. \]

(17.13)

In the transformation (17.12) and (17.13), the point \((y_1, y_2)\) is to be viewed as a fixed parameter. The transformation is illustrated graphically in Fig 17.1.
Fig. 17.1 Three coordinate systems in the $\mathbb{R}^2$ plane.

(17.12) and (17.13), we infer that the Jacobian of the transformation is unity. Hence, according to (17.12) and (17.13), equation (17.11) can be rewritten in the form

$$< u^T >_{\infty, 22} (y_2) + k^2 < u^T >_{\infty} (y_2) =$$

$$2n \int_{R^2} \delta'(\zeta_2) < u^{sc} >_{\infty} (\zeta_1, 0^+; y_2 - \zeta_1 \cos \theta_0 - \zeta_2 \sin \theta_0) d\zeta_1 d\zeta_2. \quad (17.14)$$

The region of integration in (17.14) is in fact the infinite layer $V^h_{\infty}$ of Fig. 17.2, since by (17.5) $< u^{sc} >_{\infty}$ vanishes when the fixed point $\zeta$ is outside $V^h_{\infty}$. Figure 17.2 applies when the angle $\theta_0$ is such that $0 < \theta_0 < \pi / 2$. When $\pi / 2 < \theta_0 < \pi$, the layer is inclined in the opposite direction relative to the $\zeta_2$ axis. When $\theta_0 = \pi / 2$, the layer is horizontal and parallel to the $\zeta_1$ axis. Finally, when $\theta_0 = 0$, the layer is vertical and parallel to the $\zeta_2$ axis. Now we recall that the Dirac delta function has the property

$$\int_{\alpha}^{\beta} \delta'(\zeta_2) f(\zeta_2) d\zeta_2 = -f'(0), \quad (\alpha < 0, \beta > 0), \quad (17.15)$$

where the prime superscript denotes the derivative. Thus, using the limits of integration $a(\zeta_2) = -\zeta_2 \tan \theta_0 - (h - y_2) / \cos \theta_0$ and $b(\zeta_2) = -\zeta_2 \tan \theta_0 + (h + y_2) / \cos \theta_0$,
(when $\theta_0 \neq \pi/2$), one finds that

$$
\int_{\gamma_{h}^{\beta}} \delta'(\tilde{\zeta}_2) < \nu^{\infty} >_{\infty} (\zeta_1, 0^+; y_2 - \tilde{\zeta}_1 \cos \theta_0 - \tilde{\zeta}_2 \sin \theta_0) \, d\tilde{\zeta}_1 \, d\tilde{\zeta}_2 = 
$$

$$
- \left\{ \frac{\partial}{\partial \tilde{\zeta}_2} J(\tilde{\zeta}_2, y_2) \right\}_{\tilde{\zeta}_2 = 0}. 
$$

The integral $J(\tilde{\zeta}_2, y_2)$ in (17.16) is defined by

$$
J(\tilde{\zeta}_2, y_2) = \text{sgn}(\cos \theta_0) \int_{\zeta_{1}(\tilde{\zeta}_2)}^{\tilde{\zeta}(\tilde{\zeta}_2)} < \nu^{\infty} >_{\infty} (\zeta_1, 0^+; y_2 - \tilde{\zeta}_1 \cos \theta_0 - \tilde{\zeta}_2 \sin \theta_0) \, d\zeta_1. 
$$

(17.17)

In equation (17.17), we change the integration variable $\tilde{\zeta}_1$ into $v$ such that

$$
v = \tilde{\zeta}_1 + \tilde{\zeta}_2 \tan \theta_0, \quad dv = d\tilde{\zeta}_1. 
$$

(17.18)
Thus, one finds that

\[ J(\tilde{\zeta}_2, y_2) = \text{sgn}(\cos \theta_0) \int_{(y_2-h)/\cos \theta_0}^{(y_2+h)/\cos \theta_0} < u^{sc} > \infty (v - \tilde{\zeta}_2 \tan \theta_0, 0^+; y_2 - v \cos \theta_0) \, dv. \]  

(17.19)

Differentiating (17.19) with respect to \( \tilde{\zeta}_2 \) and evaluating the result at \( \tilde{\zeta}_2 = 0 \), one has

\[ \frac{\partial}{\partial \tilde{\zeta}_2} J(\tilde{\zeta}_2, y_2) \bigg|_{\tilde{\zeta}_2 = 0} = -|\tan \theta_0| \int_{(y_2-h)/\cos \theta_0}^{(y_2+h)/\cos \theta_0} \left\{ \frac{\partial}{\partial \tilde{\zeta}_1} < u^{sc} > \infty (\tilde{\zeta}_1, 0^+; y_2 - v \cos \theta_0) \right\} \tilde{\zeta}_1 = v \, dv. \]  

(17.20)

Equation (17.20) can be written in the equivalent form

\[ \frac{\partial}{\partial \tilde{\zeta}_2} J(\tilde{\zeta}_2, y_2) \bigg|_{\tilde{\zeta}_2 = 0} = -\tan \theta_0 \int_{\frac{y_2}{\cos \theta_0}}^{\frac{y_2}{\cos \theta_0} + \frac{h}{|\cos \theta_0|}} \left\{ \frac{\partial}{\partial \tilde{\zeta}_1} < u^{sc} > \infty (\tilde{\zeta}_1, 0^+; y_2 - v \cos \theta_0) \right\} \tilde{\zeta}_1 = v \, dv. \]  

(17.21)

First, consider the case when \( \cos \theta_0 > 0 \). For \( v > \frac{y_2 + h}{\cos \theta_0} \) we have \( y_2 - v \cos \theta_0 < -h \). For \( v < \frac{y_2 - h}{\cos \theta_0} \) we have \( y_2 - v \cos \theta_0 > h \). Second, consider the case when \( \cos \theta_0 < 0 \). For \( v > \frac{y_2 - h}{\cos \theta_0} \) we have \( y_2 - v \cos \theta_0 > h \). For \( v < \frac{y_2 + h}{\cos \theta_0} \) we have \( y_2 - v \cos \theta_0 < -h \).

Recall that \( < u^{sc} > \infty (\tilde{\zeta}_1, 0^+; v) \), according to (17.5) and (17.10), vanishes for \( |v| > h \).

It follows from the preceding remarks that the limits of integration in (17.21) can be replaced by \( -\infty \) and \( +\infty \), respectively. Thus, one has

\[ \frac{\partial}{\partial \tilde{\zeta}_2} J(\tilde{\zeta}_2, y_2) \bigg|_{\tilde{\zeta}_2 = 0} = -\tan \theta_0 \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial \tilde{\zeta}_1} < u^{sc} > \infty (\tilde{\zeta}_1, 0^+; y_2 - v \cos \theta_0) \right\} \tilde{\zeta}_1 = v \, dv. \]  

(17.22)
Also recall that \( u^{sc} > \infty (\zeta_1, 0^+; v) \), according to (17.4) and (17.10), vanishes for \( |\zeta_1| > a \). Hence, the integrand in (17.22) vanishes for all values of \( v \) such that \( |v| > a \).

Thus, the integral (17.22) can be written in the form

\[
\frac{\partial}{\partial \zeta_2} J(\zeta_2, y_2) \Bigg|_{\zeta_2 = 0} =
-\tan \theta_0 \int_{-a}^{a} \left\{ \frac{\partial}{\partial \zeta_1} < u^{sc} > \infty (\zeta_1, 0^+; y_2 - v \cos \theta_0) \right\} \zeta_1 = v \quad dv.
\]

(17.23)

We now combine (17.16) and (17.23), and we substitute the result into equation (17.14). This gives the following differential equation for the average total field \( u^T > \infty \) in the case \( \theta_0 \neq \pi/2 \)

\[
<u^T>_{\infty, 22} (y_2) + k^2 < u^T > \infty (y_2) =
2n \tan \theta_0 \int_{-a}^{a} \left\{ \frac{\partial}{\partial \zeta_1} < u^{sc} > \infty (\zeta_1, 0^+; y_2 - v \cos \theta_0) \right\} \frac{\partial}{\partial \zeta_1} = 0 \quad dv.
\]

(17.24)

When the cracks inside the layer are parallel to the \( y_1 \) axis of Fig. 17.1, then \( \theta = \pi/2 \) and the layer of Fig. 17.2 is horizontal. Then, one has

\[
\int_{V_2} \xi'(\zeta_2) < u^{sc} > \infty (\zeta_1, 0^+; y_2 - \zeta_2) \, d\zeta_1 \, d\zeta_2 = - \left\{ \frac{\partial}{\partial \zeta_2} J^{(\zeta_2, y_2)} \right\} \zeta_2 = 0.
\]

(17.25)

The integral \( J^{(\zeta_2, y_2)} \) in (17.25) is defined by

\[
J^{(\zeta_2, y_2)} = \int_{-\infty}^{\infty} < u^{sc} > \infty (\zeta_1, 0^+; y_2 - \zeta) \, d\zeta_1.
\]

(17.26)

Differentiating (17.26) with respect to \( \zeta_2 \) and evaluating the result at \( \zeta_2 = 0 \), one has

\[
\frac{\partial}{\partial \zeta_2} J^{(\zeta_2, y_2)} \Bigg|_{\zeta_2 = 0} = - \frac{\partial}{\partial y_2} \int_{-\infty}^{\infty} < u^{sc} > \infty (\zeta_1, 0^+; y_2) \, d\zeta_1 =
- \frac{\partial}{\partial y_2} \int_{-a}^{a} < u^{sc} > \infty (\zeta_1, 0^+; y_2) \, d\zeta_1,
\]

(17.27)
where we have used the result (17.4) according to which the field \( < u^{sc} >_{\infty} (\zeta_1, 0^+; y_2) \) vanishes for \( |\zeta_1| > a \). We now combine (17.25) and (17.27), and we substitute the result into equation (17.14). This gives the following differential equation for the average total field \( < u^T >_{\infty} \) in the special case \( \theta_0 = \pi/2 \)

\[
< u^T >_{\infty, 22} (y_2) + k^2 < u^T >_{\infty} (y_2) = 2n \frac{\partial}{\partial y_2} \int_{-a}^{a} < u^{sc} >_{\infty} (x_1, 0^+; y_2) \, dx_1.
\]

(17.28)

The field \( < u^{sc} >_{\infty} \) in the right-hand sides of (17.24) and (17.28) can be found by taking the limit of equation (16.24) and the boundary condition (16.25) as the number \( N \) goes to infinity. Assuming that the limit operation can be interchanged with the differentiations, and using the definition (17.3) for \( < u^{sc} >_{\infty} \) and the symmetry property (17.10), we infer that (16.24) yields

\[
< u^{sc} >_{\infty, \gamma \gamma} (x_1^i, x_2^i; \zeta_2^i) + k^2 < u^{sc} >_{\infty} (x_1^i, x_2^i; \zeta_2^i) = 2 \delta'(x_2^i) < u^{sc} >_{\infty} (x_1^i, 0^+; \zeta_2^i),
\]

(17.29)

\( i = 1, \ldots, N \).

Now recall that the partial average \( < u^E >_{N-1} \) of the exciting field is defined by (16.26). We assume the existence of the limits \( < u^E >_{\infty} \) and \( \tilde{u}^E >_{\infty} \) such that

\[
< \tilde{u}^E >_{\infty} (y_1, y_2; \zeta^i) = \lim_{N \to \infty} < \tilde{u}^E >_{N-1} (y_1, y_2; \zeta^i) = \lim_{N \to \infty} < u^E >_{N-1} (x_1^i, x_2^i; \zeta^i).
\]

(17.30)

As in the case of the scattered field \( \tilde{u}^{sc} >_{\infty} \), the physical symmetry of the infinite-layer problem implies that the exciting field \( < u^E >_{\infty} \) depends only on \( y_2 \), provided that the position of the fixed crack relative to which the average is taken is translated along the \( y_1 \) axis together with the observation point. Thus, one has

\[
< \tilde{u}^E >_{\infty} (y_2; \zeta^i - y_1 i) \equiv < \tilde{u}^E >_{\infty} (y_1, y_2; \zeta^i).
\]

(17.31)
Then, we can show, by using a line of arguments similar to that of (17.8)-(17.10), that \( < u^e >_\infty \) in (17.30) is independent of \( \zeta_1^i \). Thus, we adopt the notation
\[
< u^e >_\infty (x_1^i, x_2^i, \zeta_2^i) \equiv < u^e >_\infty (x_1^i, x_2^i, \zeta^i).
\] (17.32)

We take the limit as \( N \) goes to infinity of the boundary condition (16.25). Assuming that the limit operation can be interchanged with the differentiations, and using the definition (17.30) for \( < u^e >_\infty \) and the symmetry properties (17.32) and (17.10), we infer that (16.25) yields
\[
\frac{\partial}{\partial x_2^i} < u^{sc} >_\infty (x_1^i, x_2^i, \zeta_2^i) \bigg|_{x_2^i = 0^+} = -\frac{\partial}{\partial x_2^i} < u^e >_\infty (x_1^i, x_2^i, \zeta_2^i) \bigg|_{x_2^i = 0},
\]
\[
|x_1^i| < a, \quad i = 1, \ldots, N.
\] (17.33)

Equations (17.29) and (17.33) are the two equations that determine the scattered field \( < u^{sc} >_\infty \) that appears in (17.24) and (17.28). These two equations can be written without the superscripts \( i \). Thus, one has
\[
< u^{sc} >_{\gamma \gamma} (x_1, x_2; \zeta_2) + k^2 < u^{sc} >_\infty (x_1, x_2; \zeta_2) =
2 \delta(x_2) < u^{sc} >_\infty (x_1, 0^+; \zeta_2),
\] (17.34)
\[
\frac{\partial}{\partial x_2} < u^{sc} >_\infty (x_1, x_2; \zeta_2) \bigg|_{x_2 = 0^+} = -\frac{\partial}{\partial x_2} < u^e >_\infty (x_1, x_2; \zeta_2) \bigg|_{x_2 = 0},
\]
\[
|x_1| < a.
\] (17.35)

We recall from (4.11) that the scattered field \( u^{sc} \) is antisymmetric with respect to the \( x_1 \) axis. Hence, the partial average \( < u^{sc} >_{N-1} \), defined as in (14.27), and the limit \( < u^{sc} >_\infty \) defined in (17.3), are also antisymmetric with respect to the \( x_1 \) axis. Thus, using (17.4) and the notation (17.10), one has
\[
< u^{sc} >_\infty (x_1, 0; \zeta_2) = 0, \quad |x_1| \geq a,
\] (17.36)
\[ < u^{sc} >_{\infty} (x_1,0^+;\zeta_2) = 2 < u^{sc} >_{\infty} (x_1,0^-;\zeta_2), \]  
\[ |x_1| < a. \]  
(17.37)

It follows from Section 5 that the solution of (17.34) and (17.35) can be written in the form

\[ < u^{sc} >_{\infty} (x_1,x_2;\zeta_2) = \frac{\sgn(x_2)}{\pi} \int_{0}^{\infty} d\xi \int_{-\infty}^{\infty} b(\nu;\zeta_2) L(x_1 - \nu,x_2,\xi) d\nu. \]  
(17.38)

The function \( L \) of (17.38) is defined in (3.12), and the function \( b \) satisfies the singular integral equation (3.20). For convenience, we write (3.20) in the form

\[ \int_{-\infty}^{\infty} b(\nu;\zeta_2) \left[ \frac{1}{\nu - x_1} + P(\nu - x_1) \right] d\nu = -\pi \frac{\partial}{\partial x_2} < u^{sc} >_{\infty} (x_1,x_2;\zeta_2) \bigg|_{x_2 = 0}, \]  
(17.39)

where the function \( P \) is defined in (3.21). In addition, the function \( b \) satisfies the continuity condition (5.19) and is related to the scattered field on the crack faces by (5.23). We write these two equations in the form

\[ \int_{-\infty}^{\infty} b(\nu;\zeta_2) d\nu = 0, \]  
(17.40)

\[ < u^{sc} >_{\infty} (x_1,0^+;\zeta_2) = \frac{1}{2} \int_{-\infty}^{\infty} b(\nu;\zeta_2) \sgn(x_1 - \nu) d\nu. \]  
(17.41)

Now it follows from (17.36), (17.40), and (17.41) that

\[ < u^{sc} >_{\infty} (x_1,0;\zeta_2) = 0, \quad |x_1| \geq a, \]  
(17.42)

\[ < u^{sc} >_{\infty} (x_1,0^+;\zeta_2) = \int_{-\infty}^{x_1} b(\nu;\zeta_2) d\nu, \quad |x_1| < a. \]  
(17.43)

The exciting field \( < u^{e} >_{N-1} (x_1,x_2;\zeta) \), according to (16.26), vanishes when \( \zeta \) is outside \( \mathcal{V}_{d_0}^h \). Thus, by taking the limit of (16.26) as \( N \) approaches infinity, and using the definition (17.30) and the symmetry property (17.32), one finds that
\(< u^\alpha \rangle_\infty (x_1, x_2; \zeta_2) \) vanishes when \( \zeta = (\zeta_1, \zeta_2) \) is outside \( V^h_\infty \). In this case, the right-hand side of the singular integral equation (17.39) vanishes, and the function \( b(\nu; \zeta_2) \) must also vanish. From this discussion, we infer that

\[
b(\nu; \zeta_2) = 0, \text{ when } |\zeta_2| > h.
\] (17.44)

Combining (17.24) and (17.43), we infer that the governing equation of motion for \( < u^\alpha >_\infty \) when \( \theta_0 \neq \pi/2 \) can be written in the form

\[
< u^\alpha >_{\infty, 22} (y_2) + k^2 < u^\alpha >_\infty (y_2) = 2n \tan \theta_0 \int_a^a b(\nu; y_2 - v \cos \theta_0) \, dv.
\] (17.45)

For the case when \( \theta_0 = \pi/2 \), we substitute (17.43) into (17.28). Thus, one has

\[
< u^\alpha >_{\infty, 22} (y_2) + k^2 < u^\alpha >_\infty (y_2) = 2n \frac{\partial}{\partial y_2} \int_a^a \int_a^{x_1} b(\nu; y_2) \, dv \, dx_1.
\] (17.46)

Changing the order of integration in (17.46), and using (17.40), we find that the governing equation of motion for \( < u^\alpha >_\infty \) when \( \theta_0 = \pi/2 \) is given by

\[
< u^\alpha >_{\infty, 22} (y_2) + k^2 < u^\alpha >_\infty (y_2) = -2n \frac{\partial}{\partial y_2} \int_a^a \nu b(\nu; y_2) \, dv.
\] (17.47)

In (17.47), the values of \( y_2 \) for which the integrand vanishes are obtained directly from (17.44). These values lie in the interval \( |y_2| > h \). In (17.45), the values of \( y_2 \) for which the integrand vanishes depend on the value of \( v \). Since \( v \) is constrained by the condition \( |v| < a \) and (17.44) holds, one can see that the integrand \( b(\nu; y_2 - v \cos \theta_0) \) of (17.45) vanishes everywhere in the \((y_2, v)\) plane except in the shaded region of Fig. 17.3. Figure 17.3 corresponds to the case \( \cos \theta_0 > 0 \). If \( \cos \theta_0 < 0 \), the inclined line \( y_2 = v \cos \theta_0 + h \) has a negative slope; it still intersects the \( y_2 \) axis at \( y_2 = h \), but the intersection with the \( v \) axis is on the side of \( v > 0 \) at \( v = -h/\cos \theta_0 \). Likewise, the inclined line \( y_2 = v \cos \theta_0 - h \) has a negative slope.
Fig. 17.3 Region of the \((y_2, v)\) plane where \(b(v, y_2 - v \cos \theta_0)\) is non-zero.

18 Average Total Field in a Plane Containing a Cracked Layer

In this section, we discuss the solution of the system of equations given by (17.45), (17.39), and (17.40) for oblique incidence, and (17.47), (17.39), and (17.40) for normal incidence. The general solution of (17.45) can be written in the form

\[
< u^T > \infty = C_1 \exp(i k y_2) + C_2 \exp(-i k y_2) + K(y_2),
\]  

where \(C_1\) and \(C_2\) are two constants. In (18.1), the function \(K(y_2)\) is defined by

\[
K(y_2) = 2 n \tan \theta_0 \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\alpha} b(v; \zeta - v \cos \theta_0) \, dv \right\} G(y_2, \zeta) \, d\zeta,
\]  

(18.2)
where the Green’s function $G(y_2, \zeta)$ is the solution of the equation

$$\frac{d^2G}{dy_2^2} + k^2G = \delta(y_2 - \zeta).$$  \hspace{1cm} (18.3)

The function $G$ is given by

$$G(y_2, \zeta) = -\frac{i}{2k} \exp(ik|y_2 - \zeta|).$$  \hspace{1cm} (18.4)

Changing the order of integration in (18.2), and recalling that $b(\nu; \zeta - \nu \cos \theta_0)$ vanishes outside the shaded region of the plane $(\zeta, \nu)$ in Fig. 17.3, one finds that

$$K(y_2) = 2n \tan \theta_0 \int_{-a}^{a} \left\{ \int_{\nu \cos \theta_0 - h}^{\nu \cos \theta_0 + h} b(\nu; \zeta - \nu \cos \theta_0)G(y_2, \zeta) \, d\zeta \right\} \, d\nu.$$  \hspace{1cm} (18.5)

Substituting (18.4) into (18.5) and performing the change of integration variables given by

$$\xi = \zeta - \nu \cos \theta_0,$$  \hspace{1cm} (18.6)

we can write (18.5) in the form

$$K(y_2) = -\frac{in \tan \theta_0}{k} \int_{-a}^{a} \left\{ \int_{-h}^{h} b(\nu; \xi) \exp(ik|y_2 - \xi - \nu \cos \theta_0|) \, d\xi \right\} \, d\nu.$$  \hspace{1cm} (18.7)

The integral in (18.7) for $|y_2 > h + a| \cos \theta_0|$ represents a wave that propagates with wave number $k$ either in the positive $y_2$ direction or in the negative $y_2$ direction, depending on the sign of the quantity $(y_2 - \xi - \nu \cos \theta_0)$ for $\xi$ in the range $|\xi| < h$ and $\nu$ in the range $|\nu| < a$. In fact, if $y_2 > h + a| \cos \theta_0|$, the quantity inside the absolute value signs is always positive; if $y_2 < -h - a| \cos \theta_0|$, the quantity inside the absolute value signs is always negative. Thus, one has

i) $y_2 > h + a| \cos \theta_0|$

$$K(y_2) = D_1 \exp(iky_2),$$  \hspace{1cm} (18.8)

$$D_1 = -\frac{in \tan \theta_0}{k} \int_{-a}^{a} \left\{ \int_{-h}^{h} b(\nu; \xi) \exp[-ik(\xi + \nu \cos \theta_0)] \, d\xi \right\} \, d\nu;$$  \hspace{1cm} (18.9)
ii) $y_2 < -h - a|\cos \theta_0|$

$$K(y_2) = D_2 \exp(-iky_2), \quad (18.10)$$

$$D_2 = -\frac{i n \tan \theta_0}{k} \int_{-a}^{a} \left\{ \int_{-h}^{h} b(\nu; \xi) \exp[ik(\xi + \nu \cos \theta_0)] d\xi \right\} d\nu. \quad (18.11)$$

When $|y_2| < h + a|\cos \theta_0|$, we can write $K(y_2)$ in the form

$$K(y_2) = D_3(y_2) \exp(i ky_2) + D_4(y_2) \exp(-iky_2), \quad (18.12)$$

where the quantities $D_3$ and $D_4$ depend on $y_2$, and are determined by the integrals in the plane $(\xi, \nu)$ of the function $(-in \tan \theta_0/k)b(\nu; \xi) \exp[\pm ik(\xi + \nu \cos \theta_0)]$ over the two parts of the rectangle $[-h, h] \times [-a, a]$ cut by the straight line of equation $\xi + \nu \cos \theta_0 = y_2$. From (18.1), (18.8), (18.10), and (18.12), we infer that the average total field in the solid has the form

$$<u^T>_{\infty} = C_1 \exp(i ky_2) + C_2 \exp(-iky_2) +$$

$$\begin{cases} 
D_1 \exp(i ky_2), & y_2 > h + a|\cos \theta_0|, \\
D_2 \exp(-iky_2), & y_2 < -h - a|\cos \theta_0|, \\
D_3(y_2) \exp(i ky_2) + D_4(y_2) \exp(-iky_2), & |y_2| < h + a|\cos \theta_0|. 
\end{cases} \quad (18.13)$$

The constants $C_1$ and $C_2$ in (18.13) can be obtained from the physics of the problem. In the region $y_2 > h + a|\cos \theta_0|$, the average field is expected to have the form $A \exp(i ky_2)$, where $A$ is a complex-valued constant. From this and (18.13), we infer that $C_2$ vanishes. In the region $y_2 < -h - a|\cos \theta_0|$, the average field is expected to have the form $u_0 \exp(i ky_2) + B \exp(-iky_2)$, where the first term is the incident field (2.4) and the second term is the average reflection caused by the presence of the cracks, and $B$ is a complex-valued constant. From this and (18.13), we infer that $C_1$ is equal to $u_0$. Thus, using (18.1) and (18.7), together with $C_1 = u_0$ and $C_2 = 0$, we
write the average total field for $\cos \theta_0 \neq 0$ in the form

$$< u^T >_\infty = u_0 \exp(iky_2) - \frac{in \tan \theta_0}{k} \int_{-h}^{h} \left\{ \int_{-a}^{a} b(\nu; \xi) \exp(ik|y_2 - \xi - \nu \cos \theta_0|) \, d\nu \right\} \, d\xi. \tag{18.14}$$

For normal incidence, the solution of (17.47) can be written in the form

$$< u^T >_\infty = C_1 \exp(iky_2) + C_2 \exp(-iky_2) + K_{\frac{x}{2}}(y_2), \tag{18.15}$$

where $C_1$ and $C_2$ are two constants. In (18.15) the function $K_{\frac{x}{2}}(y_2)$ is defined by

$$K_{\frac{x}{2}}(y_2) = -2n \int_{-\infty}^{\infty} \frac{\partial}{\partial \xi} \left\{ \int_{-a}^{a} \nu b(\nu; \xi) \, d\nu \right\} G(y_2, \xi) \, d\xi, \tag{18.16}$$

where the Green's function $G(y_2, \xi)$ is defined by (18.4). Substituting (18.4) into (18.16), one finds that

$$K_{\frac{x}{2}}(y_2) = \frac{in}{k} \int_{-\infty}^{\infty} \frac{\partial}{\partial \xi} \left\{ \int_{-a}^{a} \nu b(\nu; \xi) \, d\nu \right\} \exp(ik|y_2 - \xi|) \, d\xi. \tag{18.17}$$

Integrating (18.17) by parts and recalling from (17.44) that $b(\nu; \xi)$ vanishes for $|\xi| > h$, we find that

$$K_{\frac{x}{2}}(y_2) = n \int_{-h}^{h} \left\{ \int_{-a}^{a} \nu b(\nu; \xi) \, d\nu \right\} \text{sgn}(\xi - y_2) \exp(ik|y_2 - \xi|) \, d\xi. \tag{18.18}$$

The integral in (18.18) for $|y_2| > h$ represents a wave that propagates with wave number $k$ either in the positive $y_2$ direction or in the negative $y_2$ direction, depending on the sign of the quantity $(y_2 - \xi)$ for $\xi$ in the range $|\xi| < h$. In fact, one has

i) $y_2 > h$

$$K_{\frac{x}{2}}(y_2) = D_5 \exp(iky_2), \tag{18.19}$$

$$D_5 = -n \int_{-h}^{h} \left\{ \int_{-a}^{a} \nu b(\nu; \xi) \, d\nu \right\} \exp(-ik\xi) \, d\xi. \tag{18.20}$$
\text{ii) } y_2 < -h

\begin{equation}
K_{\xi}(y_2) = D_0 \exp(-iky_2),
\end{equation}

\begin{equation}
D_0 = n \int_{-h}^{h} \left\{ \int_{-a}^{a} \nu b(\nu; \xi) \, d\nu \right\} \exp(ik\xi) \, d\xi.
\end{equation}

When \(|y_2| < h\), we can write \(K_{\xi}(y_2)\) in the form

\begin{equation}
K_{\xi}(y_2) = D_7(y_2) \exp(iky_2) + D_8(y_2) \exp(-iky_2),
\end{equation}

where the quantities \(D_7\) and \(D_8\) depend on \(y_2\), and are determined by the integrals in the plane \((\xi, \nu)\) of the function \(n\nu b(\nu; \xi) \text{sgn}(\xi - y_2) \exp(\pm ik\xi)\) over the two parts of the rectangle \([-h, h] \times [-a, a]\) cut by the horizontal straight line of equation \(\xi = y_2\).

Following the same line of arguments as for the case \(\cos \theta_0 \neq 0\), we infer that the constants in (18.15) take the values \(C_1 = u_0\) and \(C_2 = 0\). Thus, it follows from (18.15) and (18.18) that the average total field for \(\cos \theta_0 = 0\) is given by

\begin{equation}
< u_T >_\infty = u_0 \exp(iky_2) + n \int_{-h}^{h} \left\{ \int_{-a}^{a} \nu b(\nu; \xi) \, d\nu \right\} \text{sgn}(\xi - y_2) \exp(ik|y_2 - \xi|) \, d\xi.
\end{equation}

19 \text{ Speed and Attenuation of the Total Field}

In this section, we discuss the phase velocity and the attenuation of the field \(< u_T >_\infty\). In order to define these quantities, it is convenient to write \(< u_T >_\infty\) of (18.14) or (18.24) in the form

\begin{equation}
< u_T >_\infty (y_2) = u_0 \exp[iK(y_2)] = u_0 \exp[-Q(y_2)] \exp[i\omega P(y_2)],
\end{equation}

where \(Q\) and \(P\) are real-valued functions such that

\begin{equation}
K(y_2) = \omega P(y_2) + iQ(y_2).
\end{equation}
The function $K$ in (19.1) is defined by

$$K(y_2) = \frac{1}{i} \ln \frac{<u^r>_{\infty}}{u_0} = \frac{1}{i} \ln \frac{<u^r>_{\infty}}{|<u^r>_{\infty}|} - i \ln \frac{|<u^r>_{\infty}|}{u_0}. \quad (19.3)$$

In (19.3), the imaginary part of the logarithm is chosen in the range $[0,2\pi)$. Observe that the first term in the right-hand side of (19.3) is real valued, since the logarithm of a unimodular complex number is purely imaginary. From (19.2) and (19.3), we infer that the functions $P$ and $Q$ are given by

$$\omega P(y_2) = \frac{1}{i} \ln \frac{<u^r>_{\infty}}{|<u^r>_{\infty}|}, \quad (19.4)$$

$$Q(y_2) = -\ln \frac{|<u^r>_{\infty}|}{u_0}. \quad (19.5)$$

Next, we define the phase velocity and the attenuation of the wave motion (19.1). The attenuation $\alpha(y_2)$ is defined as the derivative of $Q(y_2)$. Hence, using (19.5), we write

$$\alpha(y_2) = Q'(y_2) = -\frac{d}{dy_2} \ln |<u^r>_{\infty}| = -\frac{(2|<u^r>_{\infty}<u^r>^*_\infty)'}{2|<u^r>_{\infty}|^2}. \quad (19.6)$$

We now consider curves where the phase of the motion (19.1) is constant for each time $t$. Using the time factor $\exp(-i\omega t)$, we recall from (19.1) that a curve of constant phase $C$ is described by an equation of the form

$$P(y_2) - t = C. \quad (19.7)$$

If (19.7) can be solved uniquely for $y_2$ for each $t$ in some time interval, then one can define a curve $y_2(t)$ of constant phase $C$ for this time interval. The phase velocity $c$ at $y_2(t)$ is the speed of the curve of constant phase $y_2(t)$ and is defined by

$$c[y_2(t)] = \frac{dy_2(t)}{dt}. \quad (19.8)$$
By differentiating (19.7) with respect to time, we obtain

\[ c[y_2(t)] = \frac{1}{P'(y_2(t))}. \] (19.9)

From (19.9), we see that \( c[y_2(t)] \) is independent of the phase \( C \) given in (19.7). It follows from (19.9) that

\[ c(y_2) = \frac{1}{P'(y_2)}. \] (19.10)

From (19.4), we infer that the derivative \( P'(y_2) \) in the denominator of (19.10) can be expressed in the form

\[ i\omega P'(y_2) = \frac{< u^T \to \infty >}{< u^T \to \infty >} - \frac{1}{< u^T \to \infty >} = \frac{1}{2} \left( \frac{< u^T \to \infty > - < u^T \to \infty >}{< u^T \to \infty >} \right) \left( < u^T \to \infty > - < u^T \to \infty > \right). \] (19.11)

Hence, combining (19.10) and (19.11), we can write \( c(y_2) \) in the form

\[ c(y_2) = \frac{2i\omega}{< u^T \to \infty >} \frac{< u^T \to \infty >^2}{< u^T \to \infty > - < u^T \to \infty >} \frac{< u^T \to \infty >}{\text{Im}(< u^T \to \infty > - < u^T \to \infty >)}. \] (19.12)

As an example, to illustrate the formulas for the attenuation and the speed, we consider a wave of the form

\[ u = u_0 \exp(iky_2) + Au_0 \exp(-i(ky_2 + \phi)), \] (19.13)

where \( A \) and \( \phi \) are, respectively, a constant real-valued amplitude and a constant real-valued phase. To compute the attenuation \( \alpha \) of (19.13), we use the result (19.6). First, we find that the modulus \( |u| \) of (19.13) is given by

\[ \frac{|u|^2}{u_0^2} = \frac{uu^*}{u_0^2} = 1 + A^2 + 2A \cos(2ky_2 + \phi). \] (19.14)

Now, using the result (19.6), we infer that the attenuation \( \alpha \) is given by

\[ \alpha(y_2) = -\frac{2A k \sin(2ky_2 + \phi)}{1 + A^2 + 2A \cos(2ky_2 + \phi)}. \] (19.15)
It is not difficult to show that

\[(1 - |A|)^2 \leq 1 + A^2 + 2A \cos(2ky_2 + \phi) \leq (1 + |A|)^2. \tag{19.16}\]

The equal sign in the upper bound of (19.16) is valid when \(A \geq 0\) and \(\cos(2ky_2 + \phi) = 1\) or \(A \leq 0\) and \(\cos(2ky_2 + \phi) = -1\). The equal sign in the lower bound of (19.16) is valid when \(A \geq 0\) and \(\cos(2ky_2 + \phi) = -1\) or \(A \leq 0\) and \(\cos(2ky_2 + \phi) = 1\).

From (19.15) and (19.16), we infer that the attenuation can be positive or negative. When the attenuation is negative, the wave attenuates in the negative \(y_2\) direction or, equivalently, it amplifies in the positive direction. To compute the speed \(c\) of the wave (19.13), we first find from (19.13) that

\[
\text{Im}(u' u^*) = ku_0^2(1 - A^2). \tag{19.17}
\]

Hence, the speed \(c\) is given according to (19.12), (19.17), and (19.14) by

\[
c(y_2) = c_t \frac{1 + A^2 + 2A \cos(2ky_2 + \phi)}{1 - A^2}, \tag{19.18}
\]

where \(c_t\) is given by

\[
c_t = \frac{\omega}{k}. \tag{19.19}
\]

Observe from (19.18) and (19.16) that when \(A > 1\) the speed \(c(y_2)\) of the wave (19.13) is negative and, hence, the wave propagates in the negative \(y_2\) direction. Further, if \(0 < A < 1\), the speed \(c(y_2)\) is greater than \(c_t\) for all values of \(y_2\) that satisfy the condition \(\cos(2ky_2 + \phi) = 1\). As another example, we consider a wave of the form

\[
u = u_0 \exp(-Ay_2) \exp(iky_2), \tag{19.20}
\]

where \(A\) is a real-valued constant. The attenuation \(\alpha\) and the speed \(c\), according to (19.6) and (19.12), respectively, are given by

\[
\alpha(y_2) = A, \quad c(y_2) = \frac{\omega}{k}. \tag{19.21}
\]
20 Dimensionless Equations for the Total Field

In the preceding sections, we formulated the equations that describe the infinite-layer probabilistic problem. For convenience, we rewrite here the basic equations of the problem. The total field is given, according to (18.14) and (18.24), by

i) \( \theta_0 \neq \pi/2 \)

\[
\begin{align*}
< u^r >_\infty &= u_0 \exp(iky_2) - \\
&\frac{in \tan \theta_0}{k} \int_{-h}^{k} \left\{ \int_{-\alpha}^{\alpha} b(\nu; \zeta_2) \exp(ik|y_2 - \zeta_2 - \nu \cos \theta_0|) \, d\nu \right\} \, d\zeta_2; \\
\end{align*}
\]  

(20.1)

ii) \( \theta_0 = \pi/2 \)

\[
\begin{align*}
< u^r >_\infty &= u_0 \exp(iky_2) + \\
&n \int_{-h}^{k} \left\{ \int_{-\alpha}^{\alpha} b(\nu; \zeta_2) \, d\nu \right\} \, d\zeta_2. \\
\end{align*}
\]  

(20.2)

The derivatives \( < u^r >'_\infty \) for oblique and normal incidence are given by

i) \( \theta_0 \neq \pi/2 \)

\[
\begin{align*}
< u^r >'_\infty &= iku_0 \exp(iky_2) + n \tan \theta_0 \\
&\int_{-h}^{k} \left\{ \int_{-\alpha}^{\alpha} b(\nu; \zeta_2) \, d\nu \right\} \, d\zeta_2; \\
\end{align*}
\]  

(20.3)

ii) \( \theta_0 = \pi/2 \)

\[
< u^r >'_\infty = iku_0 \exp(iky_2) - 2n \int_{-\alpha}^{\alpha} b(\nu; y_2) \, d\nu - \\
ink \int_{-h}^{k} \left\{ \int_{-\alpha}^{\alpha} b(\nu; \zeta_2) \, d\nu \right\} \, d\zeta_2.
\]  

(20.4)

The solution \( < u^r >_\infty \) given by (20.1) and (20.2) is valid for all \( y_2 \in \mathbb{R} \). In the region \( y_2 > h + a| \cos \theta_0 | \) the fields \( < u^r >_\infty \) and \( < u^r >'_\infty \) can be written in the form
i) $\theta_0 \neq \pi/2$, $y_2 > h + a|\cos\theta_0|$

\[< u^r >_\infty = u_0 T_{\theta_0} \exp(iky_2);\]  
(20.5)

\[< u^r >'_\infty = iku_0 T_{\theta_0} \exp(iky_2);\]  
(20.6)

ii) $\theta_0 = \pi/2$, $y_2 > h$

\[< u^r >_\infty = u_0 T_{\pi/2} \exp(iky_2);\]  
(20.7)

\[< u^r >'_\infty = iku_0 T_{\pi/2} \exp(iky_2).\]  
(20.8)

In (20.5)-(20.8), the constants $T_{\theta_0}$ and $T_{\pi/2}$ are given by

\[T_{\theta_0} = 1 - \frac{in \tan \theta_0}{u_0 k} \int_{-h}^{h} \left\{ \int_{-a}^{a} b(\nu; \zeta_2) \exp[-ik(\zeta_2 + \nu \cos \theta_0)] d\nu \right\} d\zeta_2,\]  
(20.9)

\[T_{\pi/2} = 1 - \frac{n}{u_0} \int_{-h}^{h} \left\{ \int_{-a}^{a} \nu b(\nu; \zeta_2) d\nu \right\} \exp(-ik\zeta_2) d\zeta_2.\]  
(20.10)

In the region $y_2 < -h - a|\cos\theta_0|$, the fields $< u^r >_\infty$ and $< u^r >'_\infty$ can be written in the form

i) $\theta_0 \neq \pi/2$, $y_2 < -h - a|\cos\theta_0|$

\[< u^r >_\infty = u_0 \exp(iky_2) + u_0 R_{\theta_0} \exp(-iky_2);\]  
(20.11)

\[< u^r >'_\infty = iku_0 \exp(iky_2) - iku_0 R_{\theta_0} \exp(-iky_2);\]  
(20.12)

ii) $\theta_0 = \pi/2$, $y_2 < -h$

\[< u^r >_\infty = u_0 \exp(iky_2) + u_0 R_{\pi/2} \exp(-iky_2);\]  
(20.13)

\[< u^r >'_\infty = iku_0 \exp(iky_2) - iku_0 R_{\pi/2} \exp(-iky_2).\]  
(20.14)
In (20.11)-(20.14), the constants $R_{\delta_0}$ and $R_{\pi/2}$ are given by

$$R_{\delta_0} = \frac{in \tan \theta_0}{u_0 k} \int_{-h}^{h} \left\{ \int_{-a}^{a} b(\nu; \zeta_2) \exp[i k (\zeta_2 + \nu \cos \theta_0)] d\nu \right\} d\zeta_2, \quad (20.15)$$

$$R_{\pi/2} = \frac{n}{u_0} \int_{-h}^{h} \left\{ \int_{-a}^{a} \nu b(\nu; \zeta_2) d\nu \right\} \exp(i k \zeta_2) d\zeta_2. \quad (20.16)$$

In (20.1)-(20.16), the function $b$ is the solution of the singular integral equation (17.39). This equation is

$$\int_{-a}^{a} b(\nu; \zeta_2) \left[ \frac{1}{\nu - x_1} + P(\nu - x_1) \right] d\nu = -\pi \frac{\partial}{\partial x_2} \begin{vmatrix} u^e_{\infty}(x_1, x_2; \zeta_2) \end{vmatrix}_{x_2 = 0}, \quad (20.17)$$

$$|x_1| < a,$$

where the function $P$ is defined in (3.21). In addition, the function $b$ satisfies the continuity condition

$$\int_{-a}^{a} b(\nu; \zeta_2) d\nu = 0. \quad (20.18)$$

The exciting field $< u^e >_{\infty}$ in (20.17) is defined in (17.30) and (17.32) as the limit of $< u^e >_{N-1}$ when $N$ approaches infinity. The field $< u^e >_{N-1}$ can be written, according to (16.26), either in terms of the partial average field $< u^r >_{N-1}$ with one crack held fixed, or in terms of the incident field $u^{inc}$. Now we recall from (17.30), (17.32), (17.3), (17.10) that we made assumptions for the existence of the following limits

$$< u^e >_{\infty}(x_1, x_2; \zeta_2) = \lim_{N \to \infty} < u^e >_{N-1}(x_1, x_2; \zeta_1, \zeta_2), \quad (20.19)$$

$$< u^{inc} >_{\infty}(x_1, x_2; \zeta_2) = \lim_{N \to \infty} < u^{inc} >_{N-1}(x_1, x_2; \zeta_1, \zeta_2). \quad (20.20)$$

Thus, we conclude from (20.19), (20.20), and (16.26) that $< u^r >_{N-1}(y_1, y_2; \zeta)$ and $(N - 1) < \bar{u}^{inc} >_{N-1}(y_1, y_2; ||\zeta||)$ have limits as $N$ approaches infinity. We define these
limits to be \( < u^T >_\infty (y_1, y_2 | \zeta) \) and \( < \bar{u}^{sc} >_\infty (y_1, y_2 || \zeta) \), and we write

\[
< u^T >_\infty (y_1, y_2 | \zeta) = \lim_{N \to \infty} < u^T >_{N-1} (y_1, y_2 | \zeta),
\]

(20.21)

\[
< \bar{u}^{sc} >_\infty (y_1, y_2 || \zeta) = \lim_{N \to \infty} (N - 1) < \bar{u}^{sc} >_{N-1} (y_1, y_2 || \zeta).
\]

(20.22)

Using (20.19)-(20.22) and (16.26), we conclude that \( < u^E >_\infty \) is given by

\[
< u^E >_\infty (x_1, x_2; \zeta_2) = < \bar{u}^{E} >_\infty (y_1, y_2; \zeta) = < u^T >_\infty (y_1, y_2 | \zeta) - < \bar{u}^{sc} >_\infty (y_1, y_2 | \zeta) = u^{inc}(y_2 | \zeta_2) + < \bar{u}^{sc} >_\infty (y_1, y_2 || \zeta).
\]

(20.23)

In (20.23), the field \( u^{inc}(y_2 | \zeta_2) \) follows from (16.27) in the form

\[
u^{inc}(y_2 | \zeta_2) = \nu^{inc}(y_2) H(h - |\zeta_2|),
\]

(20.24)

where the function \( H \) is the Heaviside step function, and the field \( < \bar{u}^{sc} >_\infty (y_1, y_2; \zeta) \), according to (17.3), (17.10), and (17.38), is given by

\[
< \bar{u}^{sc} >_\infty (y_1, y_2; \zeta) = < u^{sc} >_\infty (x_1, x_2; \zeta_2) = \frac{\text{sgn}(x_2)}{\pi} \int_{0}^{\infty} d\xi \int_{-a}^{a} b(\nu; \zeta_2) L(x_1 - \nu, x_2, \xi) d\nu.
\]

(20.25)

In (20.25), the transformation of coordinates between \((x_1, x_2)\) and \((y_1, y_2)\) is given by (16.12) and (16.13). In (20.25), the function \( L \) is given by (3.12). For completeness, we recall that the right-hand side of (20.21) is given as in (14.27) by

\[
< u^T >_{N-1} (y_1, y_2 | \zeta) = \frac{N}{n} \int_{\mathbb{R}^{2-N-2}} u^T(y_1, y_2 | \zeta^i, \ldots, \zeta^{i-1}, \zeta, \zeta^{i+1}, \ldots, \zeta^N) p(\zeta^1, \ldots, \zeta^{i-1}, \zeta, \zeta^{i+1}, \ldots, \zeta^N) d\zeta^1 \ldots d\zeta^{i-1} d\zeta^{i+1} \ldots d\zeta^N,
\]

(20.26)
and the right-hand side of (20.22) is given by (16.28) in the form
\[
< \tilde{u}^{sc}_{N-1}(y_1, y_2|\zeta) = \frac{N}{n} \int_{\mathbb{R}^{2N-2}} \tilde{u}^{sc}(y_1, y_2; \zeta^i_1, \zeta^i_2, \cdots, \zeta^{i-1}_1, \zeta_i, \zeta^{i+1}_1, \cdots, \zeta^N) \nu(\zeta^i_1, \cdots, \zeta^{i-1}_1, \zeta_i, \zeta^{i+1}_1, \cdots, \zeta^N) d\zeta_1^i \cdots d\zeta^{i-1}_i d\zeta^{i+1}_i \cdots d\zeta^N, \\
\quad j = 1, \ldots, i - 1, i + 1, \ldots, N. \tag{20.27}
\]

Now, we write equations (20.1)-(20.18) in dimensionless form. For this purpose, we introduce the following notations
\[
\tilde{y}_1 = y_1/a, \quad \tilde{y}_2 = y_2/a, \quad \tilde{x}_1 = x_1/a, \quad \tilde{x}_2 = x_2/a, \tag{20.28}
\]
\[
\tilde{\zeta} = \zeta/k, \quad \tilde{\zeta}_1 = \zeta_1/a, \quad \tilde{\zeta}_2 = \zeta_2/a, \tag{20.29}
\]
\[
\tilde{\nu} = \nu/a, \quad \kappa = ka, \quad \tilde{h} = h/a, \quad \epsilon = na^2, \quad \tilde{u}_0 = u_0/a, \tag{20.30}
\]
\[
< \tilde{u}^{r}_{\infty}(\tilde{y}_2) =< u^{r}_{\infty}(a\tilde{y}_2)/u_0, \quad \tilde{u}^{inc}(\tilde{y}_2) = u^{inc}(a\tilde{y}_2)/u_0, \tag{20.31}
\]
\[
\tilde{b}(\tilde{\nu}; \tilde{\zeta}_2) = b(a\tilde{\nu}; a\tilde{\zeta}_2)/(ku_0), \quad \tilde{P}(\tilde{\nu} - \tilde{x}_1) = aP(a\tilde{\nu} - a\tilde{x}_1), \tag{20.32}
\]
\[
< \tilde{u}^{e}_{\infty}(\tilde{x}_1, \tilde{\zeta}_2) =< u^{e}_{\infty}(a\tilde{x}_1, a\tilde{\zeta}_2)/u_0, \tag{20.33}
\]
\[
\tilde{u}^{inc}(\tilde{y}_2|\tilde{\zeta}_2) = u^{inc}(a\tilde{y}_2|a\tilde{\zeta}_2)/u_0, \tag{20.34}
\]
\[
< \tilde{u}^{sc}_{\infty}(\tilde{y}_1, \tilde{y}_2|\tilde{\zeta}_1, \tilde{\zeta}_2) =< \tilde{u}^{sc}_{\infty}(a\tilde{y}_1, a\tilde{y}_2|a\tilde{\zeta}_1, a\tilde{\zeta}_2)/u_0, \tag{20.35}
\]
\[
< \tilde{u}^{sc}_{\infty}(\tilde{y}_1, \tilde{y}_2|\tilde{\zeta}_1, \tilde{\zeta}_2) =< \tilde{u}^{sc}_{\infty}(a\tilde{y}_1, a\tilde{y}_2|a\tilde{\zeta}_1, a\tilde{\zeta}_2)/u_0, \tag{20.36}
\]
\[
< \tilde{u}^{r}_{\infty}(\tilde{y}_1, \tilde{y}_2|\tilde{\zeta}_1, \tilde{\zeta}_2) =< u^{r}_{\infty}(a\tilde{y}_1, a\tilde{y}_2|a\tilde{\zeta}_1, a\tilde{\zeta}_2)/u_0. \tag{20.37}
\]

From (20.34), (20.24) and (14.5), we infer that \( \tilde{u}^{inc} \) is given by
\[
\tilde{u}^{inc}(\tilde{y}_2|\tilde{\zeta}_2) = \exp(\imath \kappa \tilde{y}_2) H(\tilde{h} - |\tilde{\zeta}_2|). \tag{20.38}
\]

Observe that in (20.32) the dislocation density function \( \tilde{b} \) is the ratio of the function \( b \) to the amplitude \( u_0 \) and the wave number \( k \). Hence, equations (20.1) and (20.2) yield
i) \( \theta_0 \neq \pi/2 \)

\[
< \bar{u}^T >_{\infty} (\bar{y}_2) = \exp(i \kappa \bar{y}_2) - 
\]

\[
\i \epsilon \tan \theta_0 \int_{-\hat{h}}^{\hat{h}} \left\{ \int_{-1}^{1} \hat{b}(\hat{v}; \hat{\zeta}_2) \exp(i \kappa |\hat{y}_2 - \hat{\zeta}_2 - \hat{v} \cos \theta_0|) \, d\hat{v} \right\} \, d\hat{\zeta}_2; \tag{20.39}
\]

ii) \( \theta_0 = \pi/2 \)

\[
< \bar{u}^T >_{\infty} (\bar{y}_2) = \exp(i \kappa \bar{y}_2) + 
\]

\[
\epsilon \kappa \int_{-\hat{h}}^{\hat{h}} \left\{ \int_{-1}^{1} \hat{\nu} \hat{b}(\hat{v}; \hat{\zeta}_2) \, d\hat{v} \right\} \text{sgn}(\hat{\zeta}_2 - \bar{y}_2) \exp(i \kappa |\hat{y}_2 - \hat{\zeta}_2|) \, d\hat{\zeta}_2. \tag{20.40}
\]

The dimensionless derivative \( < \bar{u}^T >'_{\infty} \) of the field \( < \bar{u}^T >_{\infty} \) is defined by

\[
< \bar{u}^T >'_{\infty} (\bar{y}_2) = \frac{d}{d\bar{y}_2} < \bar{u}^T >_{\infty} (\bar{y}_2) = \frac{1}{\bar{u}_0} \left. \frac{d}{dy_2} < u^T >_{\infty} (y_2) \right|_{y_2 = \alpha \bar{y}_2}. \tag{20.41}
\]

Thus, from (20.3)-(20.4) and (20.41), we infer that \( < \bar{u}^T >'_{\infty} \) is given by

i) \( \theta_0 \neq \pi/2 \)

\[
< \bar{u}^T >'_{\infty} (\bar{y}_2) = i \kappa \exp(i \kappa \bar{y}_2) + \epsilon \kappa \tan \theta_0 
\]

\[
\int_{-\hat{h}}^{\hat{h}} \left\{ \int_{-1}^{1} \hat{b}(\hat{v}; \hat{\zeta}_2) \text{sgn}(\hat{y}_2 - \hat{\zeta}_2 - \hat{v} \cos \theta_0) \exp(i \kappa |\hat{y}_2 - \hat{\zeta}_2 - \hat{v} \cos \theta_0|) \, d\hat{v} \right\} \, d\hat{\zeta}_2; \tag{20.42}
\]

ii) \( \theta_0 = \pi/2 \)

\[
< \bar{u}^T >'_{\infty} (\bar{y}_2) = i \kappa \exp(i \kappa \bar{y}_2) - 2 \epsilon \kappa \int_{-1}^{1} \hat{\nu} \hat{b}(\hat{v}; \bar{y}_2) \, d\hat{v} - 
\]

\[
i \epsilon \kappa^2 \int_{-\hat{h}}^{\hat{h}} \left\{ \int_{-1}^{1} \hat{\nu} \hat{b}(\hat{v}; \hat{\zeta}_2) \, d\hat{v} \right\} \exp(i \kappa |\hat{y}_2 - \hat{\zeta}_2|) \, d\hat{\zeta}_2. \tag{20.43}
\]

From (20.5)-(20.8), (20.31), and (20.41), we infer that in the region \( \hat{y}_2 > \hat{h} + |\cos \theta_0| \)
the fields \( < \bar{u}^T >_{\infty} \) and \( < \bar{u}^T >'_{\infty} \) can be written in the form
i) $\theta_0 \neq \pi/2, \tilde{y}_2 > \tilde{h} + |\cos \theta_0|$

\[
< \tilde{u}^\tau >_{\infty} = T_{\theta_0} \exp(i\kappa \tilde{y}_2); \tag{20.44}
\]

\[
< \tilde{u}^\tau >'_\infty = i\kappa T_{\theta_0} \exp(i\kappa \tilde{y}_2); \tag{20.45}
\]

ii) $\theta_0 = \pi/2, \tilde{y}_2 > \tilde{h}$

\[
< \tilde{u}^\tau >_{\infty} = T_{\pi/2} \exp(i\kappa \tilde{y}_2); \tag{20.46}
\]

\[
< \tilde{u}^\tau >'_\infty = i\kappa T_{\pi/2} \exp(i\kappa \tilde{y}_2). \tag{20.47}
\]

In (20.44)-(20.47), the constants $T_{\theta_0}$ and $T_{\pi/2}$, according to (20.9) and (20.10), are given by

\[
T_{\theta_0} = 1 - i\epsilon \tan \theta_0 \int_{-\tilde{h}}^{\tilde{h}} \left\{ \int_{-1}^{1} \tilde{b}(\tilde{\nu}; \tilde{\z}_2) \exp[-i\kappa (\tilde{\z}_2 + \tilde{\nu} \cos \theta_0)] \, d\tilde{\nu} \right\} \, d\tilde{\z}_2, \tag{20.48}
\]

\[
T_{\pi/2} = 1 - \epsilon \int_{-\tilde{h}}^{\tilde{h}} \left\{ \int_{-1}^{1} \tilde{\nu} \tilde{b}(\tilde{\nu}; \tilde{\z}_2) \, d\tilde{\nu} \right\} \exp(-i\kappa \tilde{\z}_2) \, d\tilde{\z}_2. \tag{20.49}
\]

From (20.11)-(20.14), (20.31), and (20.41), we infer that in the region $\tilde{y}_2 < -\tilde{h} - |\cos \theta_0|$ the fields $< \tilde{u}^\tau >_{\infty}$ and $< \tilde{u}^\tau >'_\infty$ can be written in the form

i) $\theta_0 \neq \pi/2, \tilde{y}_2 < -\tilde{h} - |\cos \theta_0|$

\[
< \tilde{u}^\tau >_{\infty} = \exp(i\kappa \tilde{y}_2) + R_{\theta_0} \exp(-i\kappa \tilde{y}_2); \tag{20.50}
\]

\[
< \tilde{u}^\tau >'_\infty = i\kappa \exp(i\kappa \tilde{y}_2) - i\kappa R_{\theta_0} \exp(-i\kappa \tilde{y}_2); \tag{20.51}
\]

ii) $\theta_0 = \pi/2, \tilde{y}_2 < -\tilde{h}$

\[
< \tilde{u}^\tau >_{\infty} = \exp(i\kappa \tilde{y}_2) + R_{\pi/2} \exp(-i\kappa \tilde{y}_2); \tag{20.52}
\]

\[
< \tilde{u}^\tau >'_\infty = i\kappa \exp(i\kappa \tilde{y}_2) - i\kappa R_{\pi/2} \exp(-i\kappa \tilde{y}_2). \tag{20.53}
\]
In (20.50)-(20.53), the constants $R_{\theta_0}$ and $R_{\pi/2}$, according to (20.15) and (20.16), are given by

$$R_{\theta_0} = -i\epsilon \tan \theta_0 \int_{-\hat{h}}^{\hat{h}} \left\{ \int_{-1}^{1} \bar{b}(\hat{\nu}; \hat{\zeta}_2) \exp[i\kappa(\hat{\zeta}_2 + \hat{\nu} \cos \theta_0)] \, d\hat{\nu} \right\} \, d\hat{\zeta}_2,$$  \hspace{1cm} (20.54)

$$R_{\pi/2} = \epsilon \kappa \int_{-\hat{h}}^{\hat{h}} \left\{ \int_{-1}^{1} \hat{\nu} \bar{b}(\hat{\nu}; \hat{\zeta}_2) \, d\hat{\nu} \right\} \exp(i\kappa \hat{\zeta}_2) \, d\hat{\zeta}_2.$$  \hspace{1cm} (20.55)

The singular integral equation (20.17) is given in dimensionless form, according to (20.33), by

$$\int_{-1}^{1} \tilde{b}(\hat{\nu}; \hat{\zeta}_2) \left[ \frac{1}{\hat{\nu} - \hat{x}_1} + \hat{P} (\hat{\nu} - \hat{x}_1) \right] \, d\hat{\nu} = -\frac{\pi}{\kappa} \frac{\partial}{\partial \hat{x}_2} \left. < \hat{u}^e >_{\infty} (\hat{x}_1, \hat{x}_2; \hat{\zeta}_2) \right|_{\hat{x}_2 = 0},$$  \hspace{1cm} (20.56)

and the function $\tilde{b}$ satisfies the condition

$$\int_{-1}^{1} \tilde{b}(\hat{\nu}; \hat{\zeta}_2) \, d\hat{\nu} = 0.$$  \hspace{1cm} (20.57)

In equation (20.56), the function $\hat{P}(\hat{\nu} - \hat{x}_1)$, according to (3.21) and (20.32), is given by

$$\hat{P}(\hat{\nu} - \hat{x}_1) = aP(a\hat{\nu} - a\hat{x}_1) = \kappa \int_0^\infty (\hat{\beta} - 1) \sin(\kappa \hat{\xi} (\hat{\nu} - \hat{x}_1)) \, d\hat{\xi},$$  \hspace{1cm} (20.58)

where $\hat{\beta}$, according to (3.13) and (20.29), is defined by

$$\hat{\beta}^2 = \hat{\xi}^2 - 1, \hspace{1cm} \text{Im}(\hat{\beta}) \leq 0.$$ \hspace{1cm} (20.59)

In equation (20.56), the function $< \hat{u}^e >_{\infty}$ is given, according to (20.23) and (20.33) - (20.37), by

$$< \hat{u}^e >_{\infty} (\hat{x}_1, \hat{x}_2; \hat{\zeta}_2) = < \hat{u}_t >_{\infty} (\hat{y}_1, \hat{y}_2; \hat{\xi}_1, \hat{\zeta}_2) - < \hat{u}^{ac} >_{\infty} (\hat{y}_1, \hat{y}_2; \hat{\xi}_1, \hat{\zeta}_2)$$

$$= \hat{u}^{inc}(\hat{y}_2; \hat{\zeta}_2) + < \hat{u}^{inc} >_{\infty} (\hat{y}_1, \hat{y}_2; \hat{\xi}_1, \hat{\zeta}_2),$$ \hspace{1cm} (20.60)
where the transformation of coordinates from \((\tilde{x}_1, \tilde{x}_2)\) to \((\tilde{y}_1, \tilde{y}_2)\), according to (16.12)-(16.13) and (20.28)-(20.29), is defined by

\[
\tilde{x}_1 = \sin \theta_0 (\tilde{y}_1 - \tilde{\zeta}_1) + \cos \theta_0 (\tilde{y}_2 - \tilde{\zeta}_2),
\]

\[
\tilde{x}_2 = -\cos \theta_0 (\tilde{y}_1 - \tilde{\zeta}_1) + \sin \theta_0 (\tilde{y}_2 - \tilde{\zeta}_2).
\]

(20.61)

(20.62)

21 Evaluation of the Total Field

The fields \(<\tilde{u}^x>_{\infty}\) of (20.39) and (20.40) are given in terms of the dislocation density function \(\tilde{b}\), which is the solution of the singular integral equation (20.56). To solve (20.56), one needs to know the derivative of the field \(<\tilde{u}^x>_{\infty}\). Because \(<\tilde{u}^x>_{\infty}\) is expressed in (20.60) in terms of the unknown average \(<\tilde{u}^{ac}>_{\infty} (\tilde{y}_1, \tilde{y}_2|\tilde{\zeta}_1, \tilde{\zeta}_2)\), we need to make an assumption that will close the system of equations. As a first approximation, which is expected to be appropriate for small crack densities, we neglect the scattered field \(<\tilde{u}^{ac}>_{\infty} (\tilde{y}_1, \tilde{y}_2|\tilde{\zeta}_1, \tilde{\zeta}_2)\) in the expression for the exciting field \(<\tilde{u}^x>_{\infty}\) given by (20.60). Thus, according to (20.38), (20.61), and (20.62), one has

\[
\frac{\partial}{\partial \tilde{x}_2} <\tilde{u}^x>_{\infty} (\tilde{x}_1, \tilde{x}_2; \tilde{\zeta}_2) \bigg|_{\tilde{x}_2 = 0} \approx \frac{\partial}{\partial \tilde{x}_2} \tilde{u}^{nc} (\tilde{y}_2|\tilde{\zeta}_2) \bigg|_{\tilde{x}_2 = 0} =
\]

\[
H(\tilde{h} - |\tilde{\zeta}_2|) \frac{\partial}{\partial \tilde{x}_2} \exp[i\kappa(\tilde{\zeta}_2 + \tilde{x}_1 \cos \theta_0 + \tilde{x}_2 \sin \theta_0)] \bigg|_{\tilde{x}_2 = 0} =
\]

\[
i\kappa \sin \theta_0 H(\tilde{h} - |\tilde{\zeta}_2|) \exp[i\kappa(\tilde{\zeta}_2 + \tilde{x}_1 \cos \theta_0)].
\]

(21.1)
Based on this discussion, we introduce the following assumption.

**Assumption 21.1 (closure assumption)**

For small crack densities, we assume that the derivative of the exciting field \( \tilde{u}^B >_\infty \) on the faces of a fixed crack is given by

\[
\frac{\partial}{\partial \tilde{x}_2} < \tilde{u}^B >_\infty (\tilde{x}_1, \tilde{x}_2; \tilde{\zeta}_2) \bigg|_{\tilde{x}_2 = 0} = \frac{\partial}{\partial \tilde{x}_2} \tilde{u}^{\text{inc}}(\tilde{y}_2; \tilde{\zeta}_2) \bigg|_{\tilde{x}_2 = 0}
\]

\[
= i \kappa \sin \theta_0 H(\tilde{h} - |\tilde{\zeta}_2|) \exp[i \kappa (\tilde{\zeta}_2 + \tilde{x}_1 \cos \theta_0)],
\]

for \( |\tilde{x}_1| < 1 \).

The following derivation is based on the closure assumption stated above. It should be stressed that the closure assumption is an assumption on the derivative of the exciting field on the faces of a fixed crack, rather than on the exciting field itself.

With (21.1), equation (20.56) can be written in the form

\[
\int_{-1}^{1} \tilde{b}(\tilde{\nu}; \tilde{\zeta}_2) \left[ \frac{1}{\tilde{\nu} - \tilde{x}_1} + \tilde{P}(\tilde{\nu} - \tilde{x}_1) \right] d\tilde{\nu} = -i \pi \sin \theta_0 H(\tilde{h} - |\tilde{\zeta}_2|) \exp[i \kappa (\tilde{\zeta}_2 + \tilde{x}_1 \cos \theta_0)],
\]

for \( |\tilde{x}_1| < 1 \).

The singular integral equation (21.3) is valid for all values of \( \theta_0 \). From (21.3) and (20.57), we infer that the function \( \tilde{b}(\tilde{\nu}; \tilde{\zeta}_2) \) can be written as the product of a function \( \tilde{b}(\tilde{\nu}) \), which depends only on \( \tilde{\nu} \), and a combination of terms that depend on \( \tilde{\zeta}_2 \). The result is

\[
\tilde{b}(\tilde{\nu}; \tilde{\zeta}_2) = -i \sin \theta_0 H(\tilde{h} - |\tilde{\zeta}_2|) \exp(i \kappa \tilde{\zeta}_2) \tilde{b}(\tilde{\nu}),
\]
where the function \( \hat{b}(\check{\nu}) \) is the solution of the singular integral equation

\[
\int_{-1}^{1} \hat{b}(\check{\nu}) \left[ \frac{1}{\check{\nu} - \check{x}_1} + \check{P}(\check{\nu} - \check{x}_1) \right] d\check{\nu} = \pi \exp(i\kappa \check{x}_1 \cos \theta_0), \quad |\check{x}_1| < 1,
\]

(21.5)

and satisfies the condition

\[
\int_{-1}^{1} \hat{b}(\check{\nu}) d\check{\nu} = 0.
\]

(21.6)

The system of equations (21.5)-(21.6) is closed and can be solved for the function \( \hat{b}(\check{\nu}) \). The total field \( \check{u}^T >_\infty \) for oblique and normal incidences, according to (20.39), (20.40), and (21.4), can now be written as

i) \( \theta_0 \neq \pi/2 \)

\[
\check{u}^T >_\infty (\check{y}_2) = \exp(i\kappa \check{y}_2) - \\
\epsilon \tan \theta_0 \sin \theta_0 \int_{-\check{h}}^{\check{h}} \left\{ \int_{-1}^{1} \hat{b}(\check{\nu}) \exp[i\kappa(\check{\zeta}_2 + |\check{y}_2 - \check{\zeta}_2 - \check{\nu} \cos \theta_0|)] d\check{\nu} \right\} d\check{\zeta}_2;
\]

(21.7)

ii) \( \theta_0 = \pi/2 \)

\[
\check{u}^T >_\infty (\check{y}_2) = \exp(i\kappa \check{y}_2) + \\
\epsilon \kappa \left( \int_{-1}^{1} \check{\bar{b}}(\check{\nu}) d\check{\nu} \right) \int_{-\check{h}}^{\check{h}} \text{sgn}(\check{y}_2 - \check{\zeta}_2) \exp[i\kappa(\check{\zeta}_2 + |\check{y}_2 - \check{\zeta}_2|)] d\check{\zeta}_2.
\]

(21.8)

The field \( \check{u}^T >'_\infty \) for oblique and normal incidences, according to (20.42), (20.43), and (21.4), can be written as

i) \( \theta_0 \neq \pi/2' \)

\[
\check{u}^T >'_\infty (\check{y}_2) = i\kappa \exp(i\kappa \check{y}_2) - i\epsilon \kappa \tan \theta_0 \sin \theta_0 \\
\int_{-\check{h}}^{\check{h}} \left\{ \int_{-1}^{1} \check{b}(\check{\nu}) \text{sgn}(\check{y}_2 - \check{\zeta}_2 - \check{\nu} \cos \theta_0) \exp[i\kappa(\check{\zeta}_2 + |\check{y}_2 - \check{\zeta}_2 - \check{\nu} \cos \theta_0|)] d\check{\nu} \right\} d\check{\zeta}_2;
\]

(21.9)
\[<\tilde{u}^T>_{\infty} (\tilde{y}_2) = i\kappa \exp(i\kappa \tilde{y}_2) + 2\epsilon \kappa H(\tilde{h} - |\tilde{y}_2|) \exp(i\kappa \tilde{y}_2) \int_{-1}^{1} \tilde{b}(\tilde{\nu}) d\tilde{\nu} - \epsilon \kappa^2 \left( \int_{-1}^{1} \tilde{b}(\tilde{\nu}) d\tilde{\nu} \right) \int_{-\tilde{h}}^{\tilde{h}} \exp[i\kappa(\tilde{\zeta}_2 + |\tilde{y}_2 - \tilde{\zeta}_2|)] d\tilde{\zeta}_2. \]  

(21.10)

In the regions \(|\tilde{y}_2| > \tilde{h} + |\cos \theta_0|\), by performing the integration with respect to \(\tilde{\zeta}_2\) in (21.7) and (21.9), one can write the fields \(<\tilde{u}^T>_{\infty}\) and \(<\tilde{u}^T>_{\infty}'\) in the form

a) \(\theta_0 \neq \pi/2, \tilde{y}_2 > \tilde{h} + |\cos \theta_0|\)

\[<\tilde{u}^T>_{\infty} (\tilde{y}_2) = T_{\theta_0} \exp(i\kappa \tilde{y}_2), \]  

(21.11)

\[<\tilde{u}^T>_{\infty}' (\tilde{y}_2) = i\kappa T_{\theta_0} \exp(i\kappa \tilde{y}_2); \]  

(21.12)

b) \(\theta_0 \neq \pi/2, \tilde{y}_2 < -\tilde{h} - |\cos \theta_0|\)

\[<\tilde{u}^T>_{\infty} (\tilde{y}_2) = \exp(i\kappa \tilde{y}_2) + R_{\theta_0} \exp(-i\kappa \tilde{y}_2), \]  

(21.13)

\[<\tilde{u}^T>_{\infty}' (\tilde{y}_2) = i\kappa \exp(i\kappa \tilde{y}_2) - i\kappa R_{\theta_0} \exp(-i\kappa \tilde{y}_2). \]  

(21.14)

In (21.11)-(21.14), the coefficients \(T_{\theta_0}\) and \(R_{\theta_0}\) are given by

\[T_{\theta_0} = 1 - 2\tilde{h} \epsilon \tan \theta_0 \sin \theta_0 \int_{-1}^{1} \tilde{b}(\tilde{\nu}) \exp(-i\kappa \tilde{\nu} \cos \theta_0) d\tilde{\nu}, \]  

(21.15)

\[R_{\theta_0} = -\epsilon \tan \theta_0 \sin \theta_0 \frac{\sin(2\kappa \tilde{h})}{\kappa} \int_{-1}^{1} \tilde{b}(\tilde{\nu}) \exp(i\kappa \tilde{\nu} \cos \theta_0) d\tilde{\nu}. \]  

(21.16)

For the case of normal incidence, we perform the integration with respect to \(\tilde{\zeta}_2\) in (21.8) and (21.10), and we infer that the field \(<\tilde{u}^T>_{\infty}\) and the derivative \(<\tilde{u}^T>_{\infty}'\) are given by
a) $\theta_0 = \pi/2, \bar{y}_2 > \bar{h}$

$$< \bar{u}^\tau > \infty (\bar{y}_2) = T_{\pi/2} \exp(i\kappa \bar{y}_2),$$  \hspace{1cm} (21.17)

$$< \bar{u}^\tau >' \infty (\bar{y}_2) = i\kappa T_{\pi/2} \exp(i\kappa \bar{y}_2);$$  \hspace{1cm} (21.18)

b) $\theta_0 = \pi/2, \bar{y}_2 < -\bar{h}$

$$< \bar{u}^\tau > \infty (\bar{y}_2) = \exp(i\kappa \bar{y}_2) + R_{\pi/2} \exp(-i\kappa \bar{y}_2),$$  \hspace{1cm} (21.19)

$$< \bar{u}^\tau >' \infty (\bar{y}_2) = i\kappa \exp(i\kappa \bar{y}_2) - i\kappa R_{\pi/2} \exp(-i\kappa \bar{y}_2);$$  \hspace{1cm} (21.20)

c) $\theta_0 = \pi/2, \bar{y}_2 \in (-\bar{h}, \bar{h})$

$$< \bar{u}^\tau > \infty (\bar{y}_2) = (A + B\bar{y}_2) \exp(i\kappa \bar{y}_2) + C \exp(-i\kappa \bar{y}_2),$$  \hspace{1cm} (21.21)

$$< \bar{u}^\tau >'(\bar{y}_2) = (i\kappa A + B\epsilon + i\kappa B\bar{y}_2) \exp(i\kappa \bar{y}_2) - i\kappa C \exp(-i\kappa \bar{y}_2).$$  \hspace{1cm} (21.22)

In (21.17)-(21.22), the constants $A$, $B$, $C$, $T_{\pi/2}$, and $R_{\pi/2}$ are given by

$$A = 1 + (\bar{h} + \frac{1}{2i\kappa})B\epsilon, \quad B = i\kappa \int_{-\bar{h}}^{\bar{h}} \bar{v} \bar{b}(\bar{v}) d\bar{v}, \quad C = -\frac{\exp(2i\kappa \bar{h})}{2i\kappa} B\epsilon,$$  \hspace{1cm} (21.23)

$$T_{\pi/2} = 1 + 2\bar{h}B, \quad R_{\pi/2} = -\epsilon B \frac{\sin(2\kappa \bar{h})}{\kappa}.$$  \hspace{1cm} (21.24)

Recall from (17.43) that the average crack-opening displacement on a fixed crack centered at $y_2 = \zeta_2$ in the cracked layer is given by

$$< u^\infty > \infty (x_1, 0^+; \zeta_2) = \int_{-a}^{a} b(\nu; \zeta_2) d\nu, \quad |x_1| < a.$$  \hspace{1cm} (21.25)

The crack-opening area $A_c$ is the integral of the crack-opening displacement and can be written by using (17.40), (20.32), and (21.4) for normal incidence in the form

$$A_c = -\int_{-a}^{a} \nu b(\nu; \zeta_2) d\nu = -\kappa \bar{u}_0 a^2 \int_{-1}^{1} \bar{v} \bar{b}(\bar{v}; \zeta_2) d\bar{v} =$$

$$i\kappa \bar{u}_0 a^2 \exp(i\kappa \zeta_2) \int_{-1}^{1} \bar{v} \bar{b}(\bar{v}) d\bar{v}, \quad \text{for } |\zeta_2| < \bar{h}. \hspace{1cm} (21.26)$$
Combining (21.26) and (21.23), one finds that the crack-opening area $A_c$ is related to the quantity $B$ by

$$B = \frac{A_c}{\bar{u}_o a^2} \exp(-i \kappa \bar{z}).$$  (21.27)

### 22 Numerical Results

First, we discuss the singular integral equation (21.5). The solution of (21.5) will be used in (21.15), (21.16), and (21.24) to evaluate the transmission coefficient $T_{\theta_0}$ and reflection coefficient $R_{\theta_0}$ for oblique incidence, and $T_{\pi/2}$ and $R_{\pi/2}$ for normal incidence. It will also be used to compute the coefficients $A$, $B$, and $C$ of (21.23). The singular integral equation (21.5) is approximated by using the method of Erdogan and Gupta [10] for the case where the unknown function $\hat{b}$ has square-root singularities at the crack tips. The resulting linear system of equations has been solved for values of the dimensionless frequency $\kappa$ at intervals 0.01 in the range [0, 10]. For each frequency, we have solved a system of 50 complex-valued equations for 50 unknowns.

From (21.23), we observe that the constant $B$, which is related to the crack-opening area $A_c$ by (21.27), is independent of the thickness $\hat{h}$ of the layer, since the solution $\hat{b}(\hat{\nu})$ of (21.5) is independent of $\hat{h}$. Figure 22.1 shows the modulus $|B|$ versus the frequency $\kappa$. At $\kappa = 0$, the value of $|B|$ is zero. Then, as $\kappa$ increases, the value of $|B|$ increases until it reaches its absolute maximum 2.67866 at frequency $\kappa = 1.45$. Thereafter, the values of $|B|$ appear to oscillate about the horizontal asymptote $|B| = 2$, and the amplitude of the oscillations is damped as the frequency increases. Figure 22.2 shows the phase arg$(B)$ versus the frequency $\kappa$, where the range of the argument is $[0, 2\pi)$. At $\kappa = 0$, the value of arg$(B)$ is $\pi/2$. Then, as $\kappa$ increases,
the value of \( \arg(B) \) increases until it reaches a local maximum 3.06270 at frequency \( \kappa = 2.90 \). Thereafter, the values of \( \arg(B) \) appear to oscillate approaching the value \( \pi \), and the amplitude of the oscillations is damped as the frequency increases.

From Fig. 22.1 and Fig. 22.2, we assume that the coefficient \( B \) has a finite limit, as the frequency increases to infinity, and we write

\[
\lim_{\kappa \to \infty} B = -2.
\] (22.1)
Fig. 22.2 Phase of the dimensionless crack-opening area versus frequency for a mid-layer crack.

Combining (21.23), (21.24), and (22.1), we infer that the limits of $A$, $C$, $T_{\pi/2}$, and $R_{\pi/2}$ as the frequency increases to infinity are given by

$$\lim_{\kappa \to \infty} A = 1 - 2\epsilon \tilde{h}, \quad \lim_{\kappa \to \infty} C = 0,$$  \hspace{1cm} (22.2)

$$\lim_{\kappa \to \infty} T_{\pi/2} = 1 - 4\epsilon \tilde{h}, \quad \lim_{\kappa \to \infty} R_{\pi/2} = 0.$$  \hspace{1cm} (22.3)
Figures 22.3 and 22.4 show the modulus $|T_{\pi/2}|$ versus the frequency $\kappa$ for crack densities $\epsilon = 0.01, 0.02, 0.03$ and thicknesses $\bar{h} = 1, 3$, respectively. At $\kappa = 0$, the value of $|T_{\pi/2}|$ is 1 for all crack densities $\epsilon$ and thicknesses $\bar{h}$. Then, as $\kappa$ increases, the value of $|T_{\pi/2}|$ for all densities $\epsilon$ and thicknesses $\bar{h}$ decreases until it reaches its absolute minimum.

The absolute minima of $|T_{\pi/2}|$ for $\bar{h} = 1$ in Fig. 22.3 are 0.951064, 0.90244417, and 0.85417702 for the crack densities $\epsilon = 0.01$, 0.02, and 0.03, respectively. These minima are found at frequencies $\kappa = 1.72, 1.73,$ and 1.73, respectively. The absolute minima of $|T_{\pi/2}|$ for $\bar{h} = 3$ in Fig. 22.4 are 0.85417702, 0.7118191, and 0.574199 for the crack densities $\epsilon = 0.01$, 0.02, and 0.03, respectively. These minima are found at frequencies $\kappa = 1.73, 1.78,$ and 1.83, respectively. Thereafter, the values of $|T_{\pi/2}|$ appear to oscillate and the amplitude of the oscillations is damped as the frequency increases. As the frequency increases, the function $|T_{\pi/2}|$ appears to approach a limit, which depends on the crack density $\epsilon$ and on the thickness $\bar{h}$.

Figures 22.5 and 22.6 show the modulus $|R_{\pi/2}|/\epsilon$ versus the frequency $\kappa$ for thicknesses $\bar{h} = 1, 3$, respectively. At $\kappa = 0$, the value of $|R_{\pi/2}|/\epsilon$ is 0 for all crack densities $\epsilon$ and thicknesses $\bar{h}$. Then, as $\kappa$ increases, the value of $|R_{\pi/2}|/\epsilon$ increases until it reaches its absolute maximum 2.073866 for $\bar{h} = 1$ at frequency $\kappa = 0.86$, and a first local maximum 1.654306 for $\bar{h} = 3$ at frequency $\kappa = 0.27$. Thereafter, the values of $|R_{\pi/2}|/\epsilon$ have cyclic variations. For $\bar{h} = 1$, the amplitude of the oscillations is damped as the frequency increases. For $\bar{h} = 3$, the amplitude of the oscillations first reaches its absolute maximum 2.0495670 at $\kappa = 0.8$ and then is damped as the frequency increases. Combining (22.1) and (21.24), we infer that as the frequency increases, the function $|R_{\pi/2}|/\epsilon$ approaches 0.
Figures 22.7 and 22.8 show the modulus $|T_{\pi/2}|$ versus the thickness $\tilde{h}$ for crack densities $\epsilon = 0.01, 0.02, 0.03$ and frequencies $\kappa = 1, 5$, respectively. At $\tilde{h} = 0$, the value of $|T_{\pi/2}|$ is 1 for all crack densities $\epsilon$ and all frequencies $\kappa$. Then, as $\tilde{h}$ increases, the value of $|T_{\pi/2}|$ decreases monotonically. At $\tilde{h} = 5$, the values of $|T_{\pi/2}|$ are 0.954897, 0.920366, and 0.8976265 for crack densities $\epsilon = 0.01, 0.02$, and 0.03, respectively, and frequency $\kappa = 1$. At $\tilde{h} = 5$, the values of $|T_{\pi/2}|$ are 0.930062, 0.8601437, and 0.79025102 for crack densities $\epsilon = 0.01, 0.02$, and 0.03, respectively, and frequency $\kappa = 5$. The plot $|R_{\pi/2}|$ versus $\tilde{h}$ is not very informative since, according to (21.24), it is just a sine function.

Figures 22.9 and 22.10 show the modulus $|T_{\pi/2}|$ versus the crack density $\epsilon$ for thicknesses $\tilde{h} = 1, 3, 5$ and frequencies $\kappa = 1, 5$, respectively. At $\epsilon = 0$, the value of $|T_{\pi/2}|$ is 1 for all thicknesses $\tilde{h}$ and all frequencies $\kappa$. Then, as $\epsilon$ increases, the value of $|T_{\pi/2}|$ decreases monotonically. At $\epsilon = 0.03$, the values of $|T_{\pi/2}|$ are 0.9717445, 0.926364, and 0.8976265 for thicknesses $\tilde{h} = 1, 3$, and 5, respectively, and frequency $\kappa = 1$. At $\epsilon = 0.03$, the values of $|T_{\pi/2}|$ are 0.958035, 0.87412552, and 0.79025102 for thicknesses $\tilde{h} = 1, 3$, and 5, respectively, and frequency $\kappa = 5$.

Figures 22.11 and 22.12 show the modulus $|T_{\pi/4}|$ versus the frequency $\kappa$ for crack densities $\epsilon = 0.01, 0.02, 0.03$ and thicknesses $\tilde{h} = 1, 3$, respectively. At $\kappa = 0$, the value of $|T_{\pi/4}|$ is 1 for all crack densities $\epsilon$ and thicknesses $\tilde{h}$. Then, as $\kappa$ increases, the value of $|T_{\pi/4}|$ for all densities $\epsilon$ and thicknesses $\tilde{h}$ decreases until it reaches a local minimum. The first local minima of $|T_{\pi/4}|$ for $\tilde{h} = 1$ are 0.98171, 0.963580, and 0.94562254 for the crack densities $\epsilon = 0.01, 0.02$, and 0.03, respectively. These minima are found at frequencies $\kappa = 1.55, 1.56$, and 1.56, respectively. The first local minima of $|T_{\pi/4}|$ for $\tilde{h} = 3$ are 0.945622, 0.89285, and 0.8419480 for the crack densities $\epsilon =$
0.01, 0.02, and 0.03, respectively. These minima are found at frequencies $\kappa = 1.56$, 1.57, and 1.59, respectively. Thereafter, the values of $|T_{\pi/4}|$ appear to oscillate and the amplitude of the oscillations is damped as the frequency increases. As the frequency increases, there appears to be a limit, which depends on the crack density.

Figures 22.13 and 22.14 show the modulus $|R_{\pi/4}|/\epsilon$ versus the frequency $\kappa$ for thicknesses $\tilde{h} = 1, 3$, respectively. At $\kappa = 0$, the value of $|R_{\pi/4}|/\epsilon$ is 0 for all crack densities $\epsilon$ and thicknesses $\tilde{h}$. Then, as $\kappa$ increases, the value of $|R_{\pi/4}|/\epsilon$ increases until it reaches its absolute maximum 0.85872 for $\tilde{h} = 1$ at frequency $\kappa = 0.8$, and a first local maximum 0.764365 for $\tilde{h} = 3$ at frequency $\kappa = 0.27$. Thereafter, the values of $|R_{\pi/4}|/\epsilon$ have cyclic variations. For $\tilde{h} = 1$, the amplitude of the oscillations is damped as the frequency increases. For $\tilde{h} = 3$, the amplitude of the oscillations first reaches its absolute maximum 0.85804 at $\kappa = 0.79$ and then is damped as the frequency increases.

Figures 22.15 and 22.16 show the modulus $|T_{\pi/4}|$ versus the thickness $\tilde{h}$ for crack densities $\epsilon = 0.01, 0.02, 0.03$ and frequencies $\kappa = 1, 5$, respectively. At $\tilde{h} = 0$, the value of $|T_{\pi/4}|$ is 1 for all crack densities $\epsilon$ and all frequencies $\kappa$. Then, as $\tilde{h}$ increases, the value of $|T_{\pi/4}|$ decreases monotonically. At $\tilde{h} = 5$, the values of $|T_{\pi/4}|$ are 0.978030, 0.959605, and 0.944930 for crack densities $\epsilon = 0.01, 0.02$, and 0.03, respectively, and frequency $\kappa = 1$. At $\tilde{h} = 5$, the values of $|T_{\pi/4}|$ are 0.892763, 0.79710, and 0.717661 for crack densities $\epsilon = 0.01, 0.02$, and 0.03, respectively, and frequency $\kappa = 5$. The plot $|R_{\pi/4}|$ versus $\tilde{h}$ is not very informative since, according to (21.16), it is just a sine function.

Figures 22.17 and 22.18 show the modulus $|T_{\pi/4}|$ versus the crack density $\epsilon$ for thicknesses $\tilde{h} = 1, 3, 5$ and frequencies $\kappa = 1, 5$, respectively. At $\epsilon = 0$, the value
of $|T_{\pi/4}|$ is 1 for all thicknesses $\tilde{h}$ and all frequencies $\kappa$. Then, as $\epsilon$ increases, the value of $|T_{\pi/4}|$ decreases monotonically. At $\epsilon = 0.03$, the values of $|T_{\pi/4}|$ are 0.986431, 0.963058, and 0.945006 for thicknesses $\tilde{h} = 1$, 3, and 5, respectively, and frequency $\kappa = 1$. At $\epsilon = 0.03$, the values of $|T_{\pi/4}|$ are 0.934610, 0.815435, and 0.71808 for thicknesses $\tilde{h} = 1$, 3, and 5, respectively, and frequency $\kappa = 5$.

23 Evaluation of the Speed and Attenuation of the Total Field

The dimensionless speed $\tilde{c}(\tilde{y}_2)$ of a wave is defined by

$$\tilde{c}(\tilde{y}_2) = \frac{c(\alpha \tilde{y}_2)}{c_T},$$

(23.1)

where $c(y_2)$ is given by (19.12), and $c_T$ is given by (19.19). Combining (19.12), (20.30), (20.31), and (20.41), we find that the speed $\tilde{c}$ is given by

$$\tilde{c}(\tilde{y}_2) = \frac{\kappa |< \tilde{u}^T >_\infty |^2}{\text{Im}(< \tilde{u}^T >' \infty < \tilde{u}^T >_* \infty )}.$$  

(23.2)

It follows from (23.2) that

$$\frac{\kappa}{\tilde{c}(\tilde{y}_2)} = \text{Im} < \tilde{u}^T >'_\infty < \tilde{u}^T >_* \infty .$$

(23.3)

The dimensionless attenuation $\tilde{\alpha}(\tilde{y}_2)$ of a wave is defined by

$$\tilde{\alpha}(\tilde{y}_2) = \frac{2a}{\epsilon} \alpha(\alpha \tilde{y}_2),$$

(23.4)

where $\alpha(y_2)$ is given by (19.6). Combining (19.6), (20.30), (20.31), and (20.41), we find that the attenuation $\tilde{\alpha}$ is given by

$$\tilde{\alpha}(\tilde{y}_2) = -\frac{( < \tilde{u}^T >_\infty < \tilde{u}^T >_* \infty )'}{\epsilon | < \tilde{u}^T >_\infty |^2}.$$ 

(23.5)
It follows from (23.5) that
\[
-\frac{e\tilde{\alpha}(\tilde{y}_2)}{2} = \text{Re} \frac{<\tilde{u}^T >'_\infty}{<\tilde{u}^T >_\infty}.
\] (23.6)

In (23.2)-(23.6), the fields $<\tilde{u}^T >_\infty$ and $<\tilde{u}^T >'_\infty$ are those of (21.7) and (21.9) for oblique incidence, and those of (21.17)-(21.22) for normal incidence. Now, we define a complex-valued number $W$ by
\[
\frac{<\tilde{u}^T >_\infty(\tilde{h})}{\tilde{u}^{inc}(-\tilde{h})} = \exp(2iW\tilde{h}).
\] (23.7)

In (23.7), the left-hand side is the ratio of the wave motion that exits the layer at $\tilde{y}_2 = \tilde{h}$ to the incident wave motion that enters the layer at $\tilde{y}_2 = -\tilde{h}$. According to Nussenzveig[18], for an infinitely small $\tilde{h}$, the quantity $W$ is called the refractive complex-valued wave number, and the function $\exp(2iW\tilde{h})$ is the response function of the cracked layer (see Fig. 23.1). The value of $W$ can be found by taking the limit of (23.7) as $\tilde{h}$ goes to zero. Thus, one has
\[
W = \lim_{\tilde{h} \to 0} \left\{ \frac{1}{2i\tilde{h}} \ln \frac{<\tilde{u}^T >_\infty(\tilde{h})}{\tilde{u}^{inc}(-\tilde{h})} \right\}.
\] (23.8)

From (21.17) and (21.24), we infer that $<\tilde{u}^T >_\infty(\tilde{h})$ for normal incidence is given by
\[
<\tilde{u}^T >_\infty(\tilde{h}) = (1 + 2B\epsilon \tilde{h}) \exp(i\kappa \tilde{h}).
\] (23.9)

The field $\tilde{u}^{inc}(\tilde{y}_2)$ is the dimensionless form of the incident field $u^{inc}(y_2)$ given by (14.5), and is defined by
\[
\tilde{u}^{inc}(\tilde{y}_2) = u^{inc}(a\tilde{y}_2)/u_0 = \exp(i\kappa\tilde{y}_2).
\] (23.10)

Thus, combining (23.8)-(23.10), we infer that for normal incidence the number $W$ is given by
\[
W = \kappa - i\epsilon B.
\] (23.11)
Since $W$ in (23.11) represents a complex-valued wave number, we can decompose (23.11) into real and imaginary parts, and define the dimensionless speed $\hat{c}$ and dimensionless attenuation $\hat{\alpha}$ of the average wave in the infinitesimally thin layer by

$$W = \frac{\kappa}{\hat{c}} + i \frac{\varepsilon \hat{\alpha}}{2}.$$  \hfill (23.12)

Then, it follows from (23.11) and (23.12) that

$$\hat{c} = \frac{\kappa}{\kappa + \varepsilon \text{Im}(B)},$$  \hfill (23.13)

$$\hat{\alpha} = -2 \text{Re}(B).$$  \hfill (23.14)

By combining (23.14) and the definition of $B$ in (21.23), we infer that

$$\hat{\alpha} = 2 \kappa \text{Im} \left( \int_{-1}^{1} \hat{\nu} \tilde{b}(\hat{\nu}) \, d\hat{\nu} \right).$$  \hfill (23.15)
In (23.15), the function $\tilde{b}$ is the solution of equations (21.5) and (21.6). For normal incidence, these equations are such that

$$\int_{-1}^{1} \tilde{b}(\tilde{\nu}) \left[ \frac{1}{\tilde{\nu} - \tilde{x}_1} + \hat{P}(\tilde{\nu} - \tilde{x}_1) \right] d\tilde{\nu} = \pi, \quad |\tilde{x}_1| < 1,$$

$$\int_{-1}^{1} \tilde{b}(\tilde{\nu}) d\tilde{\nu} = 0. \tag{23.17}$$

The attenuation for normal incidence was also obtained in Angel and Koba [4] by using a simple energy method for an antiplane wave passing through a cracked volume element. The method of [4] ignores the reflection of the wave and is valid for thin layer thicknesses, as discussed in Nussenzveig [18]. The dimensionless attenuation $\tilde{\alpha}$ obtained in [4] is given by equations (25), and (33) - (35), and can be written as

$$\tilde{\alpha} = \frac{2}{u_0} \text{Re} \left( \int_{-1}^{1} \tilde{\nu} \tilde{b}(\tilde{\nu},\tilde{\omega},\pi/2) d\tilde{\nu} \right), \tag{23.18}$$

where the function $\tilde{b}$ is defined in equation (33) by $\tilde{b}(\tilde{\nu}) = b(a\tilde{\nu})$, and $b$ satisfies equations (8) and (9) of [4]. In dimensionless form, equation (9) reads

$$\int_{-1}^{1} \tilde{b}(\tilde{\nu},\tilde{\omega},\pi/2) \left[ \frac{1}{\tilde{\nu} - \tilde{x}_1} + \hat{P}(\tilde{\nu} - \tilde{x}_1) \right] d\tilde{\nu} = -i\pi\tilde{\kappa}u_0, \quad |\tilde{x}_1| < 1, \tag{23.19}$$

and $\tilde{b}$ satisfies the continuity condition

$$\int_{-1}^{1} \tilde{b}(\tilde{\nu},\tilde{\omega},\pi/2) d\tilde{\nu} = 0. \tag{23.20}$$

In (23.19), the function $\hat{P}$ is defined in (20.58). Observe that the dimensionless frequency $\tilde{\omega} = \omega a s_T$ in [4] is the same as the dimensionless frequency $\kappa$ used in this work. Further, we infer from (23.19) - (23.20) and (23.16) - (23.17) that the functions $\bar{b}$ and $\tilde{b}$ are related by

$$\bar{b} = -i\kappa u_0 \tilde{b}. \tag{23.21}$$
Substituting (23.21) into (23.18), one finds that $\hat{\alpha} = \bar{\alpha}$, where $\bar{\alpha}$ is the attenuation of (23.15). Thus, the attenuation obtained in the limit of small layer thicknesses by the averaging method is identical to the attenuation of [4], which is obtained by using a simple energy method.

The speed $\check{c}$ of the coherent wave in the cracked region was obtained in [4] by using the expression $\check{\alpha}$ for the attenuation coefficient and the Kramers-Kronig relations, which relate the speed of the wave for each given frequency to the attenuation for all frequencies. Numerically, for $\epsilon = 0.01$, we have verified that the speed $\check{c}$ of (23.13) takes the same values as those shown in Figure 8 of [4]. Thus, we conclude that, for small values of the crack density $\epsilon$ and for small values of the layer thickness, the speed obtained by the averaging method of this work coincides with the Kramers-Kronig speed of [4].

24 Conclusions and Future Work

In this research, we have investigated the propagation of antiplane waves in an elastic solid that contains a layer of parallel cracks. The cracks are randomly distributed inside the layer, and the incident wave propagates at a normal or an oblique angle relative to the cracks. The wave motion inside the solid is obtained by averaging the equations of the deterministic $N$-crack problem and then taking the limit as $N$ goes to infinity. In principle, the probabilistic $N$-crack problem can be solved by using successive equations corresponding to one, two, and up to $N$ cracks fixed, although for large numbers of cracks the computer time is prohibitive. In the limit when the number of cracks is infinite, a closure assumption needs to be made.
We have assumed in this investigation that for a sufficiently dilute distribution of cracks, the wave motion scattered by a crack is not affected by the presence of the other cracks, and therefore the exciting field on a crack is simply the incident field. Based on this assumption, we have evaluated the average wave in the solid and also the transmission and reflection coefficients on either side of the layer. Curves for the transmission and reflection coefficients have been presented versus frequency, layer thickness, crack density, and crack orientation.

We have also evaluated the attenuation coefficient and the speed of the average wave in the cracked region. Our investigation shows that, for an infinitesimally thin cracked layer, the attenuation and the speed are in agreement with those of other authors.

The closure assumption (Assumption 21.1) states that the exciting field on the cracks is equal to the incident field regardless of the location of the cracks in the layer. Physically, an attenuation of the exciting field with increasing depth in the layer is expected. Thus, for large thicknesses, a better assumption is needed. One assumption that will incorporate the attenuation of the exciting field can be that introduced by Waterman and Truell [25]. According to these authors, the exciting field on a given crack can be approximated by the total field that would exist if the specified crack were absent.
References


Fig. 22.3 Modulus of the transmission coefficient versus frequency for normal incidence, $\tilde{h} = 1$, and three crack densities.
Fig. 22.4 Modulus of the transmission coefficient versus frequency for normal incidence, \( \tilde{h} = 3 \), and three crack densities.
Fig. 22.5 Modulus of the reflection coefficient versus frequency for normal incidence and $\tilde{h} = 1$. 
Fig. 22.6 Modulus of the reflection coefficient versus frequency for normal incidence and $\lambda = 3$. 
Fig. 22.7 Modulus of the transmission coefficient versus layer thickness for normal incidence, $\kappa = 1$, and three crack densities.
Fig. 22.8 Modulus of the transmission coefficient versus layer thickness for normal incidence, $\kappa = 5$, and three crack densities.
Fig. 22.9 Modulus of the transmission coefficient versus crack density for normal incidence, $\kappa = 1$, and three thicknesses.
Fig. 22.10 Modulus of the transmission coefficient versus crack density for normal incidence, \( \kappa = 5 \), and three thicknesses.
Fig. 22.11 Modulus of the transmission coefficient versus frequency for 45° incidence, $\tilde{h} = 1$, and three crack densities.
Fig. 22.12 Modulus of the transmission coefficient versus frequency for 45° incidence, $\tilde{h} = 3$, and three crack densities.
Fig. 22.13 Modulus of the reflection coefficient versus frequency for $45^\circ$ incidence and $\tilde{h} = 1$. 
Fig. 22.14 Modulus of the reflection coefficient versus frequency for $45^\circ$ incidence and $\tilde{h} = 3$. 
Fig. 22.15 Modulus of the transmission coefficient versus layer thickness for $45^\circ$ incidence, $\kappa = 1$, and three crack densities.
Fig. 22.16 Modulus of the transmission coefficient versus layer thickness for $45^\circ$ incidence, $\kappa = 5$, and three crack densities.
Fig. 22.17 Modulus of the transmission coefficient versus crack density for 45° incidence, \( \kappa = 1 \), and three thicknesses.
Fig. 22.18 Modulus of the transmission coefficient versus crack density for 45° incidence, $\kappa = 5$, and three thicknesses.